STABILITY ROBUSTNESS ANALYSIS OF LINEAR SYSTEMS

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Mehmet KARAN
February 1990

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tarafından bağışlanmıştır.

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Mehmet Karan

February, 1990

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

STABILITY ROBUSTNESS ANALYSIS OF LINEAR SYSTEMS

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M. S. in Electrical and Electronics Engineering
Supervisor: Prof. Dr. M. Erol Sezer
February, 1990

In this thesis, robustness of stability of linear, time-invariant, continuous- and discrete-time systems is investigated. Only state-space models and additive perturbations are considered. Existing results concerning stability robustness of continuous-time systems, based on Liapunov approach and continuity of eigenvalues, are reviewed; and similar results for discrete-time systems under single- and multi-parameter additive perturbations are derived. An inherent difficulty which originates from mixed linear and bilinear appearance of perturbation parameters in inequalities defining robustness regions of discrete-time systems is resolved by transforming the problem to robustness of a higher order continuous-time system. Finally, stability robustness of discrete-time interconnected systems is studied, and various approaches are compared.

Keywords: Robust Stability, Discrete-time systems, Additive perturbations, Liapunov stability, Interconnected systems.

ÖZET

DOĞRUSAL SİSTEMLERİN KARARLILIĞININ GÜRBÜZLÜK AÇISINDAN İNCELENMESİ

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Ocak, 1990

Bu tezde, doğrusal, zamana göre degişmeyen, sürekli ve ayırtık zamanlı sistemlerin kararlılığının gürbüzlüğü araştırılmıştır. Yalnızca durum uzayı düşünülmüştür. Sürekli zamanlı sistemlerin gürbüz kararlılığına ilişkin varolan sonuçlar, Liapunov yaklaşımı ve özdeğerlerin sürekliliği açısından gözden geçirilmiştir. Ayrıca, tek parametreli ya da çok parametreli sistem belirsizlikleri altında ayırtık zamanlı sistemler için de benzer sonuçlar elde edilmiştir. Ayırtık zamanlı sistemlerin gürbüzlük alanlarını tanımlayan eşitsizlikler içinde belirsizlik parametrelerinin doğrusal ve ikildoğrusal gözükmelerinden kaynaklanan doğal bir zorluk da, problemi daha yüksek boyutlu sürekli zamanlı bir sistemin gürbüzlüğüne dönüştürülerek aşılmıştır. Son olarak, ayırtık zamanlı birbirine bağli sistemlerin kararlılık gürbüzlüğü çalışılmış ve değişik yöntemler karşılaştırılmıştır.

Anahtar sözcükler: Gürbüz kararlılık, Ayırtık zamanlı sistemler, Toplamsal belirsizlikler, Liapunov kararlılığı, Bağlı sistemler.

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Chapter 1

INTRODUCTION

An essential feature of complex dynamic systems is the uncertainty in the system parameters, which may arise due to modelling errors or change of operating conditions. Since stability is one of the major properties of systems, it is desirable to be able to determine to what extend a nominal system remains stable when subject to perturbations. This is the robust stability problem.

In analysis of stability robustness, perturbations can be considered as having stochastic or deterministic nature. In the case of stochastic perturbations, one attempts to obtain robustness bounds for nominal system in terms of statistical properties of perturbations such as mean and variance. Another way is to view perturbations as completely or partially unknown deterministic uncertainties. The partial information about the perturbations is usually expressed in terms of the structure of the system.

In the context of stability robustness analysis, there has been many new advances such as quantitative feedback theory (Horowitz [1]), singular value theory (Doyle and Stein [2]), H^∞ theory (Zames and Francis [3]). The recent results on the frequency domain robustness analysis are based mainly on the seminal paper of Kharitonov [4] . In this paper, Kharitonov showed that stability of a family of polynomials which correspond to a hyper-rectangle in the coefficient space is equivalent to the stability of only four extreme polynomials corresponding to the vertices of the rectangle with the assumption of independent perturbations in the coefficients of the polynomials. Later, Bartlett, Hollot and Lin [5] have established the well-known Edge Theorem which says that the strict stability of the entire family of polytopes is equivalent to the strict stability of the exposed edges. A recent paper by Šiljak [6] provides an excellent survey of parameter space methods in robustness analysis and robust control design.

The techniques of state-space robustness analysis in recent literature can be viewed from two perspectives, namely,

- Time Domain Methods
- Frequency Domain Methods

In time domain methods, Lyapunov approach is the fundamental framework, which is known to be the best approach for time-varying perturbations. In the literature, mostly the stability of a linear time-invariant system in the presence of time-invariant and completely or partially unknown perturbations has been considered. Patel and Toda [7] have presented an explicit robustness bound. Later, Yedavalli [8,9,10] provided an improved bound on structured perturbations taking into account different types of perturbations. Zhou and Khargonekar [11] gave better stability robustness bounds for systems with structured uncertainty.

Frequency domain methods are based on the transfer function representation of systems and eigenvalue type of considerations. Qiu and Davison [12] have studied the robust stability problem for a state space representation of a system using frequency domain approach. Fu and Barmish [13] obtained results which can be extended to single-parameter perturbation case easily. Later, Qiu and Davison [14,15] obtained frequency domain results with similar techniques. Hinrichsen and Pritchard [16,17] formulated the problem formally and found the distance of the system matrix to the unstable complex matrices.

As has already be mentioned, perturbations may be viewed as partially or completely unknown deterministic uncertainties. In particular, for state space robustness analysis, a physical system can be described as,

$$\dot{x}(t) = (A + A_p)\, x(t) \qquad \text{(Continuous Time)} \tag{1.1}$$

$$x_{k+1} = (\Phi + \Phi_p)\, x_k \qquad \text{(Discrete Time)} \tag{1.2}$$

where $x(t) \in R^n$ is the state of the continuous system at time t, and correspondingly $x_k \in R^n$ is the state of the discrete-time system at time k. $A \in R^{n\times n}$ and $\Phi \in R^{n\times n}$ are nominal system matrices which are assumed to be asymptotically stable, $A_p \in R^{n\times n}$ and $\Phi_p \in R^{n\times n}$ are the perturbation matrices which are completely or partially unknown. Perturbations may be classified as

- Unstructured Perturbations
- Structured Perturbations
- Parametric Perturbations

(i) Unstructured Perturbations :

No information about the perturbation exists. A stability robustness bound on either the norm of A_p [resp. Φ_p] or on its entries, is tried to be obtained.

(ii) Structured Perturbations :

In this case, we have partial information about the perturbations, i.e. the structure of the perturbations of A_p [resp. Φ_p] is prespecified, and the bounds on such structured perturbations are tried to be obtained. This structure information may source from the physical nature of the system. For example, an oscillator's motion ξ obeys the equation

$$\ddot{\xi} + a_1\dot{\xi} + a_2\xi = 0 \tag{1.3}$$

which yields the state equation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} x(t) \tag{1.4}$$

where $x(t) = [\xi \quad \dot{\xi}]^T$. Since perturbations can occur only on the oscillator parameters a_1 and a_2, A_p has a structure

$$A_p = \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}$$

(iii) Parametric Perturbations :

A_p [Φ_p] may depend on one or several parameters.In this case, we can model the perturbation matrix as

- Linear Parametric Perturbations:

$$A_p \, [\, \Phi_p \,] = \sum_{k=1}^{m} p_k E_k$$

where E_k's are known, constant, square matrices, p_k's are unknown,real parameters. Here $m = 1$ ($m > 1$) case denotes single parameter perturbation (multi-parameter perturbations).

- Polynomial Parametric Perturbations :

$$A_p \, [\, \Phi_p \,] = \sum_{k=1}^{r} \sum_{l=1}^{s} f_l(p_k) E_l$$

Here, also, E_l's are known, p_k's are unknown, f_l's are known polynomials of p_k's.

- Nonlinear Parametric Perturbations:

$$A_p \, [\, \Phi_p \,] = \sum_{k=1}^{r} \sum_{l=1}^{s} f_l(p_k) E_l$$

The same assumptions as before, but now f_l's are some nonlinear, known functions of p_k's.

So far, there has been a considerable number of results on stability robustness analysis of continuous-time systems in state-space domain. But, we felt a lack of a survey on this subject, and we devoted Chapter 2 to this purpose, where we stated the existing results in their original perturbation models. We also provided a comparison of these results using a linear parametric perturbation model, which is suitable for applications of the results reviewed in this chapter.

In Chapter 3, using the techniques in Chapter 2, we developed similar stability robustness results on discrete-time systems in state-space domain with linear parametric perturbations. For single parameter perturbation case, we developed necessary and sufficient conditions for the stability of the perturbed system. For the case of multi-parameter perturbation, sufficient conditions are derived and it is shown that stability of a nominal discrete-time system matrix under multi-parameter perturbation is equivalent to the stability of a higher dimensional continuous-time system matrix with continuous-time perturbation matrices which are obtained from the discrete-time perturbation matrices. Therefore, stability robustness analysis of discrete-time systems is reduced to that of continuous-time systems.

In Chapter 4, we applied the results of Chapter 3 to interconnected systems, where the strength of the interconnections for the stability of the overall system is a fundamental question. Vector-Liapunov functions and global Liapunov function methods can give several bounds for the strength of these interconnections. In this chapter, we compared these two methods for discrete-time systems, which have been obtained in Sezer and Šiljak [18] and in Chapter 3.

Finally, in Chapter 5, we stated several further research areas in the field of stability robustness ; and, in the Appendix A, provided some background material.

Chapter 2

STABILITY ROBUSTNESS BOUNDS FOR CONTINUOUS-TIME SYSTEMS

2.1 Robust Stability Problem

Consider a continuous-time system containing additive perturbations

$$\mathcal{S}_p : \qquad \dot{x}(t) = (A + A_p)x(t) \tag{2.1}$$

where $x(t) \in \mathcal{R}^n$ is the state of $\mathcal{S}_p$, A and A_p are constant matrices of appropriate dimensions representing the nominal system matrix and perturbations, respectively. We assume that the nominal system described by

$$\mathcal{S} : \qquad \dot{x}(t) = Ax(t) \tag{2.2}$$

is stable.

Stability robustness analysis is concerned with obtaining suitable bounds on the perturbation matrix A_p which guarantee stability of the perturbed system $\mathcal{S}_p$.

When no information about the structure of A_p is known, that is, in the case of unstructured perturbations, stability robustness bound is usually expressed in terms of the norm of A_p as

$$\mu_u = \sup\{\|A_p\| : \mathcal{S}_p \text{ is stable}\} \tag{2.3}$$

Patel and Toda [7], Yedavalli [8,9,10], and Qiu and Davison [12] have tried to maximize this bound using various techniques.

Information about the structure of A_p may be useful in obtaining improved robustness bounds, or in expressing these bounds in a different form. One way of incorporating structural information on A_p is to define a normalized perturbation matrix $U_p = (u_{ij}^p)$ as

$$u_{ij}^p = a_{ij}^p / a_{\max}^p \tag{2.4}$$

where

$$a_{\max}^p = \max_{i,j}\{|a_{ij}^p|\} \tag{2.5}$$

and write $A_p = a_{\max}^p U_p$. Now, U_p carries information about the relative values of the uncertain parameters, but more important than this, information about fixed zeros in A_p. Using U_p, the robustness bound can be defined in terms of $a_{\max}^p$ as

$$\mu_n = \sup\{a_{\max}^p : \mathcal{S}_p \text{ is stable}\} \tag{2.6}$$

Yedavalli [8,10] adopted this approach in his work on stability robustness analysis.

An alternative way of making use of structural information on A_p is to decompose it as

$$A_p = BD_pC \tag{2.7}$$

where B and C are fixed matrices, and all uncertainty is included in D_p. In this case, the robustness bound is expressed in terms of D_p as

$$\mu_d = \sup\{\|D_p\| : \mathcal{S}_p \text{ is stable}\} \tag{2.8}$$

An attractive feature of the decomposition in (2.7) is that it allows the uncertainty to be interpreted as output feedback gain D_p applied to the system (A, B, C). This way, well-known results on robustness of feedback systems can be applied directly to the system $\mathcal{S}_p$. This approach has been used by Hinrichsen and Pritchard [16,17] and Qiu and Davison [12].

Most commonly used structured perturbation models in the literature are parametric perturbations described as

$$A_p = \sum_{k=1}^{m} p_k E_k \tag{2.9}$$

where E_k are fixed, known matrices, and p_k are uncertain parameters. Note that the perturbation model in (2.6) is a special case of (2.9) corresponding to a single-parameter perturbation. In multi-parameter perturbation model, stability robustness is specified in terms of a region in the parameter space as

$$\Omega_p = \sup\{\Omega \subset \mathcal{R}^m : \mathcal{S}_p \text{ is stable}\} \tag{2.10}$$

where $\mathcal{R}^m$ is the parameter space. However, Ω_p is usually difficult to characterize in terms of the perturbation parameters. A common approach is to imbed a region into Ω_p, such as a diamond, parallelopiped or sphere, which yield

$$\text{(Diamond)} \quad \Omega_D \; : \; \sum_{k=1}^{m} \alpha_k |p_k| < 1 \tag{2.11}$$

$$\text{(Parallelopiped)} \quad \Omega_P \; : \; \|p\|_\infty = \max\{|p_k|\} < \mu_P \tag{2.12}$$

$$\text{(Sphere)} \quad \Omega_S \; : \; \|p\|_2 = (\sum_{k=1}^{m} p_k^2)^{1/2} < \mu_S \tag{2.13}$$

where $p = (p_1, p_2, \ldots, p_m)$ is the parameter vector, and α_k are real constants. Multi-parameter perturbation models have been used by Zhou and Khargonekar [11].

2.2 Liapunov Approach to Stability Robustness Analysis

The essence of Liapunov techniques in stability robustness analysis of linear systems is to construct a Liapunov function for the nominal (stable) system, and seek bounds on the perturbations to establish stability of the perturbed system using the same Liapunov function.

Let $V(x) = x^T H x$ be a quadratic Liapunov function for the nominal system $\mathcal{S}$ where H is the positive definite solution of the equation

$$A^T H + HA = -G \tag{2.14}$$

for some positive definite G.

The derivative of V along the solutions of the perturbed system $\mathcal{S}_p$ of (2.1) is computed and bounded as

$$\begin{aligned} \dot{V}(x)\,|_{\mathcal{S}_p} &= x^T[\,(A+A_p)^T \;+\; H(A+A_p)\,]x \\ &= -x^T[\,G - (A_p^T H + HA_p)\,]x \\ &= -x^T G^{1/2}[\,I - G^{-1/2}(A_p^T H + HA_p)G^{-1/2}\,]G^{1/2}x \\ &\le -(1 - \sigma_{max}[G^{-1/2}(A_p^T H + HA_p)G^{-1/2}]\,)\,\|G^{1/2}x\|^2 \end{aligned}$$

where $\sigma_{max}(\cdot)$ denotes the maximum singular value of the indicated matrix. From (2.15), a sufficient condition for the stability of $\mathcal{S}_p$ is obtained as

$$\sigma_{max}[\,G^{-1/2}(A_p^T H + HA_p)G^{-1/2}\,] < 1 \tag{2.15}$$

(2.15) can be used to derive several robustness bounds for both structured and unstructured perturbations. The most common approach is to choose $G = \bar{G} = I$ to maximize the estimate of the degree of stability of the nominal system, in which case (2.15) becomes

$$\sigma_{max}(A_p^T \bar{H} + \bar{H} A_p) < 1 \tag{2.16}$$

where $\bar{H}$ is the solution of (2.14) corresponding to $\bar{G}$.

The simplest bound for unstructured perturbations is obtained by direct majorization of (2.16) as

$$\sigma_{max}(A_p) < \frac{1}{2\sigma_{max}(\bar{H})} \triangleq \mu_{u_1} \tag{2.17}$$

which is the bound obtained by Patel and Toda [7]. Noting that

$$\sigma_{max}(A_p) \leq n a^p_{max} \;, \tag{2.18}$$

where a^p_{max} is defined in (2.5), (2.17) can be further be majorized to obtain the bound

$$a^p_{max} < \frac{1}{2n\sigma_{max}(\bar{H})} \triangleq \mu_{u_2} \tag{2.19}$$

To incorporate structural perturbations, we let $A_p = a^p_{max} U_p$, where U_p is the normalized perturbation matrix defined in (2.4). Then, (2.16) is implied by

$$a^p_{max} < \frac{1}{\sigma_{max}(U_p^T|\bar{H}| + |\bar{H}|U_p)} \triangleq \mu_{s_1} \tag{2.20}$$

where $|\cdot|$ denotes a matrix obtained by taking the absolute value of every element of the indicated matrix. The bound in (2.20), obtained by Yedavalli [8], is less conservative than μ_{u_1} and μ_{u_2}.

In the case of parametric perturbations, substituting (2.9) for A_p, (2.16) becomes

$$\sigma_{max}(\sum_{k=1}^{m} p_k \bar{F}_k) < 1 \tag{2.21}$$

where

$$\bar{F}_k = E_k^T \bar{H} + \bar{H} E_k \tag{2.22}$$

Starting from (2.21), Zhou and Khargonekar [11] obtained the following stability regions in the parameter space.

$$(i) \quad \bar{\Omega}_D \; : \; \sum_{k=1}^{m} |p_k| \sigma_{max}(\bar{F}_k) < 1 \tag{2.23}$$

$$(ii) \quad \bar{\Omega}_P \; : \; \|p\|_\infty = \max_k |p_k| < \sigma^{-1}_{max}(\sum_{k=1}^{m} |\bar{F}_k|) \tag{2.24}$$

$$(iii) \quad \bar{\Omega}_S \; : \; \|p\|_2 = (\sum_{k=1}^{m} p_k^2)^{1/2} < \lambda^{-1/2}_{\max}(\sum_{k=1}^{m} \bar{F}_k^2) \tag{2.25}$$

All the robustness bounds mentioned so far are obtained for the special choice of $\bar{G} = I$. Sezer and Šiljak [19] have pointed out that $\bar{G} = I$ is not always the best choice to use in (2.15). Leaving G free, (2.17) becomes

$$\|A_p\| < \frac{\sigma_{min}(G)}{2\sigma_{max}(H)} \triangleq \mu_{u_3} \tag{2.26}$$

Since the ratio $\sigma_{min}(G)/\sigma_{max}(H)$ is maximum for $G = \bar{G} = I$, $\mu_{u_3} \leq \mu_{u_1}$, that is, additional freedom in the choice of G does not provide any improvement in the robustness bound for unstructured perturbations. However, for structured perturbations, (2.20) becomes

$$a^p_{max} < \frac{\sigma_{min}(G)}{\sigma_{max}(U_p^T|H| + |H|U_p)} \triangleq \mu_{n_2} \tag{2.27}$$

and depending on the structure of the matrix U_p, a choice of G other than $\bar{G} = I$, may give a better bound for a^p_{max}.

In the case of parametric perturbations, for a general G, the stability regions in (2.23) - (2.25) becomes

$$\Omega_D \qquad \sum_{k=1}^{m} |p_k|\sigma_{max}(F_k) < \sigma_{min}(G) \tag{2.28}$$

$$\Omega_P \qquad \|p\|_\infty = \max|p_k| < \sigma_{min}(G)\sigma_{max}^{-1}(\sum_{k=1}^{m}|F_k|) \tag{2.29}$$

$$\Omega_S \qquad (\sum_{k=1}^{m} p_k^2)^{1/2} < \sigma_{min}(G)\sigma_{max}^{-1/2}(\sum_{k=1}^{m} F_k^2) \tag{2.30}$$

Again, depending on the structure of the perturbation matrices E_k, a suitable choice of G may result in larger stability regions than those in (2.23) - (2.25). Unfortunately, there is so far no systematic way of choosing the best G to maximize the bound in (2.27) or the stability regions in (2.28) - (2.30).

Another attempt to improve stability robustness bounds has been to use a similarity transformation

$$x = T\tilde{x} \tag{2.31}$$

which transforms the perturbed system into

$$\tilde{S}_p \ : \ \dot{\tilde{x}}(t) = (\tilde{A} + \tilde{A}_p)\tilde{x}(t) \tag{2.32}$$

where

$$\tilde{A} = T^{-1}AT, \quad \tilde{A}_p = T^{-1}A_pT \tag{2.33}$$

Then, the Liapunov equation (2.14) becomes

$$\tilde{A}^T\tilde{H} + \tilde{H}\tilde{A} = -\tilde{G} \tag{2.34}$$

Let $\tilde{\tilde{H}}$ denote the solution of (2.34) corresponding to the choice $\tilde{G} = \tilde{\tilde{G}} = I$. Then, the bound in (2.17) becomes

$$\sigma_{max}(A_p) < \frac{\sigma_{min}(T)}{2\sigma_{max}(T)\sigma_{max}(\tilde{\tilde{H}})} \triangleq \mu_{u_3} \tag{2.35}$$

Yedavalli and Liang [9] argued that a suitable choice of the transformation matrix T may give better estimate of the degree of of the nominal system, as measured by $1/\sigma_{max}(\tilde{\tilde{H}})$, which offsets the reduction in the robustness bound due to the ratio $\sigma_{min}(T)/\sigma_{max}(T)$, and resulting in $\mu_{u_1} < \mu_{u_3}$. They also suggested a procedure for computing the best diagonal T to maximize μ_{u_3}. However, as pointed out by Sezer and Šiljak [19], a comparison of (2.34) with (2.14) shows that

$$\tilde{H} = T^THT, \quad \tilde{G} = T^TGT \tag{2.36}$$

Now, using (2.33) and (2.36), $\dot{V}(\tilde{x})$ can be bounded as

$$\begin{aligned} \dot{V}(\tilde{x}) \mid_{\tilde{S}_p} &= -\tilde{x}^T\tilde{G}\tilde{x} + \tilde{x}^T(\tilde{A}_p^T\tilde{H} + \tilde{H}\tilde{A}_p)\tilde{x} \\ &\leq -(1 - \sigma_{max}[G^{-1/2}(A_p^TH + HA_p)G^{-1/2}])\|G^{1/2}T\tilde{x}\|^2 \end{aligned} \tag{2.37}$$

yielding the same stability condition as given in (2.35) This shows that the effect of a similarity transformation is equivalent to the effect of choosing a different G matrix for the original system. It also shows that finding the best transformation matrix is as difficult as finding the best G.

Before closing this section, we note that better stability robustness bounds can be obtained when A has some special properties. For example, when A is normal, that is, it satisfies $A^TA = AA^T$, using the explicit expression (A.10) for the solution of Liapunov equation and choosing $\bar{G} = I$, it follows that

A^T and $\bar{H}$ commute, so that

$$\bar{H} = -(A^T + A)^{-1} = -\frac{1}{2}A_s^{-1} \tag{2.38}$$

where $A_s = \frac{1}{2}(A^T + A)$ is the symmetric part of A. From (2.38) we obtain

$$\sigma_{max}(\bar{H}) = \frac{1}{2}\sigma_{min}(A_s) = \frac{1}{\sigma_0} \tag{2.39}$$

where σ_0 is the exact degree of stability of the nominal system. Accordingly, the bounds in (2.17) - (2.20), are modified into

$$\sigma_{max}(A_p) < \sigma_0$$

by Patel and Toda [7],

$$\sigma_{max}(A_p) < \sigma_{min}(A_s) = \sigma_0$$

by Yedavalli [10], and

$$a^p_{max} < \frac{2}{\sigma_{max}(U_p^T|A_s^{-1}| + |A_s^{-1}|U_p)} = \mu_{s_1}$$

by Yedavalli [8].

2.3 Non-Liapunov Approaches to Stability Robustness Analysis

In this section we summarize non-Liapunov methods for obtaining stability robustness bounds, which are based on continuity of eigenvalues of a matrix on its parameters or Kronecker operations on matrices. As in the Liapunov approach, stability conditions obtained through these methods are sufficient, but not necessary except in special cases.

Stability robustness bounds based on the continuity of eigenvalues make use of the fact that the system matrix $A + A_p$ of the perturbed system S_p

can be viewed as a continuous deformation of the A matrix of the nominal system $\mathcal{S}$. Since $\mathcal{S}$ is assumed to be stable,

$$det[jwI - A] \neq 0 \tag{2.40}$$

and $\mathcal{S}_p$ remains to be stable when A_p is small enough to satisfy

$$det(jwI - A - A_p) \neq 0,$$

or equivalently,

$$det[I - (jwI - A)^{-1}A_p] \neq 0 \quad w \geq 0 \tag{2.41}$$

From (2.41), a sufficient condition for $\mathcal{S}_p$ to be stable is obtained (Qiu and Davison [12]) as

$$\|A_p\| < \frac{1}{\sup_{w\geq 0} \|(jwI - A)^{-1}\|} \triangleq \mu_{u_4} \tag{2.42}$$

where $\|\cdot\|$ denotes any matrix norm which satisfies $\|AB\| \leq \|A\|\|B\|$. For spectral norm (2.42) becomes

$$\sigma_{max}(A_p) < \inf_{w\geq 0} \sigma_{min}(jwI - A) \tag{2.43}$$

In the case of structural perturbations modeled as $A_p = BD_pC$, where B and C are constant, (2.41) becomes

$$det[I - C(jwI - A)^{-1}BD_p] \neq 0 \quad w \geq 0 \tag{2.44}$$

which leads to the condition (Hinrichsen and Pritchard [17])

$$\sigma_{max}(D_p) < \frac{1}{\sup_{w\geq 0} \sigma_{max}[C(jwI - A)^{-1}B]} \triangleq \mu_{s_3} \tag{2.45}$$

For single parameter perturbations modeled as $A_p = pE$, (2.43) gives

$$|p| < \frac{\inf_{w\geq 0} \sigma_{min}(jwI - A)}{\sigma_{max}(E)} \triangleq \mu_{u_4} \tag{2.46}$$

and from (2.45) by taking $D_p = pI$, $B = E$ and $C = I$, we obtain

$$|p| < \frac{1}{\sup_{w\geq 0} \sigma_{max}[(jwI - A)^{-1}E]} \triangleq \mu_{s_3} \tag{2.47}$$

An alternative to the bound in (2.47) was obtained by Qiu and Davison [12] as

$$|p| < \frac{1}{\sup_{w\geq 0} \Pi[|C(jwI-A)^{-1}B||E|]} \triangleq \mu_{s_4} \qquad (2.48)$$

(Here, $\Pi(\cdot)$ denotes the Perron-eigenvalue of a nonnegative matrix.)

In the case of multi-parameter perturbations, (2.41) is satisfied if

$$\sup_{w\geq 0} \sigma_{max}[\sum_{k=1}^{m} p_k(jwI-A)^{-1}E_k] < 1 \qquad (2.49)$$

Following the technique of Zhou and Khargonekar ([11]), we derive the following stability regions from (2.49)

$$\Omega_D \quad \sum_{k=1}^{m} |p_k| \sup_{w\geq 0}\{\sigma_{max}[(jwI-A)^{-1}E_k]\} < 1 \qquad (2.50)$$

$$\Omega_P \ : \ \max_{(1\leq r\leq m)} |p_k| < \frac{1}{\sup_{(w\geq 0)}\{\sigma_{max}(\sum_{k=1}^{m} \mid (jwI-A)^{-1}E_k \mid)\}} \qquad (2.51)$$

$$\Omega_S \ : \ (\sum_{k=1}^{m} p_k^2)^{1/2} < \inf_{(w\geq 0)} \lambda_{\max}^{-1/2}(\sum_{k=1}^{m} E_k^T(-jwI-A^T)^{-1}(jwI-A)^{-1}E_k) \qquad (2.52)$$

Robustness bounds derived from Kronecker operations also make use of the continuity of eigenvalues. From the properties of the Kronecker sum (see the Appendix) it follows that if a matrix M has eigenvalues on the imaginary axis, then $M \oplus M$ has at least two eigenvalues at the origin. Qiu and Davison [14] used this observation to conclude that $\mathcal{S}_p$ is stable if

$$\sigma_{max}(A_p) < \min\{\sigma_{min}(A), \frac{1}{2}\sigma_{n^2-1}(A\oplus A)\} \triangleq \mu_{u_5} \qquad (2.53)$$

For single-parameter perturbation case, i.e. when $A_p = pE$, $\mathcal{S}_p$ remains to be stable for p small enough to satisfy

$$det[(A+pE)\oplus(A+pE)] \neq 0 \qquad (2.54)$$

or equivalently,

$$det[I + p(A\oplus A)^{-1}(E\oplus E)] \neq 0 \qquad (2.55)$$

Using (2.55), Fu and Barmish [13] showed that $\mathcal{S}_p$ is stable for $p \in (p_{\min}, p_{\max})$ where

$$p_{\min} = \frac{1}{\min_{1\leq i\leq n^2}(\lambda_i^{r-}[-(A\oplus A)^{-1}(E\oplus E)])} \tag{2.56}$$

$$p_{\max} = \frac{1}{\max_{1\leq i\leq n^2}(\lambda_i^{r+}[-(A\oplus A)^{-1}(E\oplus E)])} \tag{2.57}$$

where $\lambda_i^{r+}(\cdot)$ and $\lambda_i^{r-}(\cdot)$ denote respectively the positive and the negative real eigenvalues of the indicated matrix. If a bound on $|p|$ is searched, then we obtain

$$|p| < \frac{1}{\max_{1\leq i\leq n^2} \mid \lambda_i^r[(A\oplus A)^{-1}(E\oplus E)] \mid} \triangleq \mu_{s_5} \tag{2.58}$$

Also, a more conservative bound can be obtained as

$$|p| < \frac{1}{\sigma_{max}[(A\oplus A)^{-1}(E\oplus E)]} \triangleq \mu_{s_6} \tag{2.59}$$

The technique of Fu and Barmish [13] can also be applied to multi-parameter perturbations. Straightforward computations yield the following stability regions in the parameter space:

$$\Omega_D \ : \ \sum_{k=1}^{m} |p_k| \sigma_{max}[(A\oplus A)^{-1}(E_k\oplus E_k)] < 1 \tag{2.60}$$

$$\Omega_P \ : \ \max_{(1\leq k\leq m)} |p_k| < \sigma_{max}^{-1}(\sum_{k=1}^{m} \mid (A\oplus A)^{-1}(E_k\oplus E_k) \mid) \tag{2.61}$$

$$\Omega_S \ : \ (\sum_{k=1}^{m} p_k^2)^{1/2} < \lambda_{\max}^{-1/2}[\sum_{k=1}^{m}(E_k^T\oplus E_k^T)(A^T\oplus A^T)^{-1}(A\oplus A)^{-1}(E_k\oplus E_k)] \tag{2.62}$$

At this point, it is appropriate to mention the single-parameter polynomial perturbation model considered by Genesio and Tesi [20], where

$$A_p = \sum_{k=1}^{m} p^k E_k \tag{2.63}$$

This model is interesting, because, unlike other perturbation models, robustness analysis based on it can also be applied, with some modifications, to discrete-time systems as we consider in the next chapter.

For A_p of (2.63), (2.55) becomes

$$det[I + \sum_{k=1}^{m} p^k F_k] \neq 0 \tag{2.64}$$

where

$$F_k = (A \oplus A)^{-1}(E_k \oplus E_k) \tag{2.65}$$

Noting that

$$\begin{aligned} & det[I + \sum_{i=1}^{m} p^i F_i] \\ = \; & det \begin{bmatrix} I & -pI & 0 & & 0 \\ 0 & I & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & I & -pI \\ pF_m & pF_{m-1} & \dots & pF_2 & (I + pF_1) \end{bmatrix} \\ = \; & det[I + p\mathcal{F}] \end{aligned}$$

where

$$\mathcal{F} = \begin{bmatrix} 0 & -I & 0 & \dots & 0 \\ 0 & 0 & -I & 0 & \vdots \\ 0 & 0 & 0 & & 0 \\ \vdots & \dots & & 0 & -I \\ F_m & F_{m-1} & & F_2 & F_1 \end{bmatrix}$$

we obtain the robustness bound

$$|p| < \frac{1}{\min_i \mid \lambda_i^r(\mathcal{F}) \mid} \triangleq \mu_{s_7}. \tag{2.66}$$

A more conservative bound can be obtained as

$$|p| < \frac{1}{\sigma_{max}(\mathcal{F})} \triangleq \mu_{s_8} \tag{2.67}$$

which reduces to (2.59) for $m = 1$.

Note that since F_k are $n^2 \times n^2$ matrices, obtaining the bounds in (2.58) - (2.62) pose computational difficulties. However, these bounds are usually better than the ones obtained through Liapunov methods (Section 2.1), and whether the increase in computational effort is justified by the improvement in the robustness bounds depends on the particular system considered.

2.4 Summary and Examples

Before closing this chapter, we give a comparison of the robustness bounds mentioned so far. To provide a common ground for the comparison, we choose a single-parameter perturbation model, that is

$$A_p = pE,$$

where E is a constant matrix, and p is the perturbation parameter. Table 2.1 is a list of various bounds, corresponding to different majorization schemes and different choices of G by using single parameter perturbation model. Also, multi-parameter perturbation bounds are given in Table 2.2. Bounds that are obtained using the Liapunov approach correspond to different levels of majorizations. For example,

$$\sigma_{max}(U^T|\bar{H}| + |\bar{H}|U) \leq 2\sigma_{max}(|\bar{H}|)\sigma_{max}(U) \leq 2n\sigma_{max}(|\bar{H}|)$$

so that $\mu_{s_1} \leq \mu_{s_2}$ if $|\bar{H}| = \bar{H}$. Also, as given in [12] a comparison between μ_{u_1} and μ_{u_4} is available as follows: Since

$$A^T\bar{H} + \bar{H}A = -I$$

$$(-jwI - A^T)\bar{H} + \bar{H}(jwI - A) = -I$$

we have

$$\bar{H}N(jw) + N^*(jw)\bar{H} = -N(jw)N^*(jw)$$

where $N(jw) = (jwI - A)^{-1}$. Hence

$$\sigma^2_{max}(N(jw)) \leq 2\sigma_{max}(\bar{H})\sigma_{max}(N(jw))$$

so that, $\mu_{u_1} \leq \mu_{u_4}$.

In general, although eigenvalue type bounds give better robustness bounds than maximum singular value type bounds, they are not suitable when a norm type bound is searched on the perturbation matrix.

Example 2.1 Consider the motion of an oscillator described in (1.3). Let the nominal system parameters be $a_1 = 4$, $a_2 = 3$. The solution of the Liapunov equation (A.3) for $G = \bar{G} = I$, can be obtained as

$$\bar{H} = \begin{bmatrix} 7/6 & 1/6 \\ 1/6 & 1/6 \end{bmatrix} \tag{2.68}$$

If the structure information on A_P is not taken into account, we obtain

$$\sigma_{max}(A_p) < \mu_{u_1} = 0.4189 \tag{2.69}$$

$$a^p_{max} < \mu_{u_2} = 0.2095 \tag{2.70}$$

from (2.17), (2.19) and

$$\sigma_{max}(A_p) < \mu_{u_4} = \mu_{u_5} = \sigma_{min}(A) = 0.5924 \tag{2.71}$$

$$a^p_{max} < 0.2962 \tag{2.72}$$

from (2.42) and (2.53).

If the perturbations are modeled as $A_p = pE$ with

$$E = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

that is , if a single-parameter perturbation model is used then (2.23), (2.47), (2.58) and (2.59) yield the bounds

$$|p| \; < \; \mu_{s_2} = 1.5 \tag{2.73}$$

$$|p| \; < \; \mu_{s_3} = 2.1213 \tag{2.74}$$

$$|p| \; < \; \mu_{s_5} = 3 \tag{2.75}$$

$$|p| \; < \; \mu_{s_6} = 2.0371 \tag{2.76}$$

Finally, a two-parameter perturbation model, $A_p = p_1 E_1 + p_2 E_2$ with

$$E_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

results in the stability regions

$$I: \quad \begin{aligned} \Omega_D & \quad 0.4024(|p_1| + |p_2|) < 1 && (2.77) \\ \Omega_P & : \ \max\{|p_1|, |p_2|\} < 1.5 && (2.78) \\ \Omega_S & : \ (p_1^2 + p_2^2)^{1/2} < 1.8974 && (2.79) \end{aligned}$$

if (2.23) - (2.25) is used; or

$$II: \quad \begin{aligned} \Omega_D & \quad 0.3333(|p_1| + |p_2|) < 1 && (2.80) \\ \Omega_P & : \ \max\{|p_1|, |p_2|\} < 2.1213 && (2.81) \\ \Omega_S & : \ (p_1^2 + p_2^2)^{1/2} < 3 && (2.82) \end{aligned}$$

if (2.50) - (2.52) is used, or

$$III: \quad \begin{aligned} \Omega_D & \quad 0.3404|p_1| + 0.2887|p_2| < 1 && (2.83) \\ \Omega_P & \quad \max\{|p_1|, |p_2|\} < 1.9054 && (2.84) \\ \Omega_S & \quad (p_1^2 + p_2^2)^{1/2} < 2.8284 && (2.85) \end{aligned}$$

if (2.60) - (2.62) is used.

Note that, although the bounds in (II) are better than the others, they are more difficult to compute.

Finally, we note that, when $p_1 = p_2 = p$, the stability regions in (I) reduce to

$$\begin{aligned} \Omega_D^I & \rightarrow |p| < 1.2425 \\ \Omega_P^I & \rightarrow |p| < 1.5 \\ \Omega_S^I & \rightarrow |p| < 1.3416, \end{aligned}$$

all of which are worse than the bounds in (2.73) - (2.76) obtained directly for a single-parameter perturbation model. However, the bounds obtained from the stability regions in (II),

$$
\begin{aligned}
\Omega_D^{II} &\rightarrow |p| < 3 \\
\Omega_P^{II} &\rightarrow |p| < 2.1213 \\
\Omega_S^{II} &\rightarrow |p| < 2.1213
\end{aligned}
$$

are comparable to, and are better than some of the bounds in (2.73) - (2.76).

$$|p| < \frac{1}{2\,\sigma_{max}(E)\,\sigma_{max}(\bar{H})} = \mu_{u_1}$$

$$|p| < \frac{1}{2\,n\,|(E)_{ij}|_{\max}\sigma_{max}(\bar{H})} = \mu_{u_2}$$

$$|p| < \frac{\sigma_{min}(G)}{2\,\sigma_{max}(E)\,\sigma_{max}(H)} = \mu_{u_3}$$

$$|p| < \frac{1}{|(E)_{ij}|_{\max}\sigma_{max}[\,U^T|\bar{H}| + |\bar{H}|U]} = \mu_{s_1}$$

$$|p| < \frac{1}{\sigma_{max}(E^T\bar{H} + \bar{H}E)} = \mu_{s_2}$$

$$(\ A \text{ normal }) \quad |p| < \frac{1}{\sigma_{max}(E)} \min_i |Re\{\lambda_i(A)\}| = \mu_{u_1}$$

$$(\ A \text{ normal }) \quad |p| < \frac{2}{\sigma_{max}[\,U^T|A_s^{-1}| + |A_s^{-1}|U]} = \mu_{s_1}$$

$$|p| < \frac{\inf_{w\geq 0}\sigma_{min}(jwI - A)}{\sigma_{max}(E)} = \mu_{u_4}$$

$$|p| < \frac{1}{\sup_{w\geq 0}\sigma_{max}[(jwI - A)^{-1}E]} = \mu_{s_3}$$

$$|p| < \frac{1}{\sup_{w\geq 0}\Pi[|C(jwI - A)^{-1}B||E|]} = \mu_{s_4}$$

$$|p| < \frac{\min\{\sigma_{min}(A), \frac{1}{2}\sigma_{n^2-1}(A \oplus A)\}}{\sigma_{max}(E)} = \mu_{u_5}$$

$$|p| < \frac{1}{\max_{1\leq i\leq n^2} |\ \lambda_i^r[(A \oplus A)^{-1}(E \oplus E)]\ |} = \mu_{s_5}$$

$$|p| < \frac{1}{\sigma_{max}[(A \oplus A)^{-1}(E \oplus E)]} = \mu_{s_6}$$

Table 2.1. Stability robustness bounds for single-parameter perturbed Continuous-time systems

$$\Omega_D \ : \ \sum_{k=1}^{m} |p_k| \sigma_{max}(\bar{F}_k) < 1$$

$$\Omega_P \ : \ \|p\|_\infty = \max_k |p_k| < \sigma_{max}^{-1}(\sum_{k=1}^{m} \mid \bar{F}_k \mid)$$

$$\Omega_S \ : \ \|p\|_2 = (\sum_{k=1}^{m} p_k^2)^{1/2} < \lambda_{\max}^{-1/2}(\sum_{k=1}^{m} \bar{F}_k^2)$$

$$\Omega_D \ : \ \sum_{k=1}^{m} |p_k| \sup_{w \geq 0} \{\sigma_{max}[(jwI - A)^{-1} E_k]\} < 1$$

$$\Omega_P \ : \ \max_{(1 \leq k \leq m)} |p_k| < \frac{1}{\sup_{(w \geq 0)} \{\sigma_{max}(\sum_{k=1}^{m} \mid (jwI - A)^{-1} E_k \mid)\}}$$

$$\Omega_S \ : \ (\sum_{k=1}^{m} p_k^2)^{1/2} < \inf_{(w \geq 0)} \lambda_{\max}^{-1/2}(\sum_{k=1}^{m} E_k^T(-jwI - A^T)^{-1}(jwI - A)^{-1} E_k)$$

$$\Omega_D \ : \ \sum_{k=1}^{m} |p_k| \sigma_{max}((A \oplus A)^{-1}(E_k \oplus E_k)) < 1$$

$$\Omega_P \ : \ \max_{(1 \leq k \leq m)} |p_k| < \sigma_{max}^{-1}(\sum_{k=1}^{m} \mid (A \oplus A)^{-1}(E_k \oplus E_k) \mid)$$

$$\Omega_S \ : \ (\sum_{k=1}^{m} p_k^2)^{1/2} < \lambda_{\max}^{-1/2}(\sum_{k=1}^{m}(E_k^T \oplus E_k^T)(A^T \oplus A^T)^{-1}(A \oplus A)^{-1}(E_k \oplus E_k))$$

Table 2.2. Stability robustness bounds for multi-parameter perturbed Continuous-time systems

Chapter 3

STABILITY ROBUSTNESS BOUNDS FOR DISCRETE-TIME SYSTEMS

Although there has been a considerable number of results ([7] [8,9], [11], [12], [16] etc.) in the literature for stability robustness of continuous-time systems, this is hardly true for discrete-time systems. One reason for the robustness problem of discrete-time systems having been given less importance might be the widespread belief that almost all results concerning continuous-time systems can be carried over, with necessary modifications, to discrete-time systems. Stability robustness problem, however, is an example, where such a modification is not obvious. Another reason is perhaps the lack of a strong justification for any disturbance model. As an example, if a discrete-time model is obtained by sampling a continuous-time system under additive perturbations, then the perturbations enter into the system matrix nonlinearly. This raises the question of whether a discrete-time model with additive perturbations have any meaning at all. (Nevertheless, additive perturbations are not the only significant ones for continuous-time systems, and a strange perturbation model for a continuous-time system may lead to additive perturbations after sampling).

In this chapter, we aim at obtaining discrete-time counterparts of the stability robustness bounds studied in Chapter 2. We consider both unstructured (Section 3.2), and parametric (Sections 3.1 and 3.2) additive perturbation models. That is we consider a system described by

$$\mathcal{D}_p \quad : \; x_{k+1} = (\Phi + \Phi_p)x_k, \;\; k \in \mathcal{Z}_+, \tag{3.1}$$

where we assume that the nominal system

$$\mathcal{D} \quad : \; x_{k+1} = \Phi x_k, \;\; k \in \mathcal{Z}_+, \tag{3.2}$$

is stable. As in Chapter 2, we classify the analysis methods as Liapunov-type and other approaches.

3.1 Liapunov Approach to Robustness Analysis

Let $V(x) = x^T H x$ be a Liapunov function for $\mathcal{D}$, where H is the unique positive-definite solution of the discrete Liapunov equation

$$\Phi^T H \Phi - H = -G \tag{3.3}$$

for some positive-definite G.

To motivate our discussion, we start with single-parameter perturbation case, where

$$\Phi_p = pE. \tag{3.4}$$

The increment of $V(x)$ along the solutions of $\mathcal{D}_p$ is computed as

$$\begin{aligned} \Delta V(x_k)\,|_{\mathcal{D}_p} &= x_k^T[(\Phi + pE)^T H(\Phi + pE) - H]x_k \\ &= -x_k^T[G - p(E^T H \Phi + \Phi^T H E) - p^2 E^T H E]x_k \\ &= -x_k^T G^{1/2} G(p) G^{1/2} x_k \end{aligned} \tag{3.5}$$

where

$$G(p) = I - pG^{-1/2}(E^T H \Phi + \Phi^T H E)G^{-1/2} - p^2 G^{-1/2} E^T H E G^{-1/2} \tag{3.6}$$

From (3.6), a sufficient condition for stability of $\mathcal{D}_p$ is obtained as

$$|p|\sigma_{max}[G^{-1/2}(E^T H \Phi + \Phi^T H E)G^{-1/2}] + |p|^2 \sigma_{max}(G^{-1/2} E^T H E G^{-1/2}) < 1 \tag{3.7}$$

which is of the form

$$a|p|^2 + b|p| - 1 < 0 \tag{3.8}$$

where a and b are obvious from (3.7). Computing the roots of the quadratic expression in (3.8), we obtain the robustness bound

$$|p| < \frac{(4a + b^2)^{1/2} - b}{2a} \triangleq \mu_{s_1} \tag{3.9}$$

An alternative to the bound in (3.9) was obtained by Sezer and Šiljak [18] by majorizing (3.5) as

$$\begin{aligned} \Delta V(x_k) \mid_{\mathcal{D}_p} \leq & - [\sigma_{min}(G) - 2|p|\sigma_{max}^{1/2}(H - G)\sigma_{max}(E^T H E) \\ & - |p|^2 \sigma_{max}(E^T H E)] \, \|x_k\|^2 \end{aligned} \tag{3.10}$$

which leads to

$$|p| < \frac{[\sigma_{max}(H - G) + \sigma_{min}(G)]^{1/2} - \sigma_{max}^{1/2}(H - G)}{\sigma_{max}^{1/2}(E^T H E)} \triangleq \mu_{s_2} \tag{3.11}$$

An interesting property of the bound in (3.11) is that, for $G = \bar{G} = I$, a further majorization gives

$$|p| < \frac{1 - [1 - \sigma_{max}^{-1}(\bar{H})]^{1/2}}{\sigma_{max}(E)} = \frac{1 - \bar{\rho}_v}{\sigma_{max}(E)} \triangleq \mu_{u_1} \tag{3.12}$$

where $\bar{\rho}_v$ is the best estimate of the degree of stability of $\mathcal{D}$, as given by (A.17)

Another interesting result is obtained by majorizing (3.5) as

$$\begin{aligned} \Delta V(x_k) \mid_{\mathcal{D}_p} &= -x_k^T [G - p(E^T H \Phi + \Phi^T H E) - p^2 E^T H E] x_k \\ &= -x_k^T [H - \Phi^T H \Phi - p(E^T H \Phi + \Phi^T H E) - p^2 E^T H E] x_k \\ &= -x_k^T H^{1/2} \{ I - [H^{1/2}(\Phi + pE) H^{-1/2}]^T [H^{1/2}(\Phi + pE) H^{-1/2}] \} H^{1/2} x_k \\ &\leq \{1 - \sigma_{max}[H^{1/2}(\Phi + pE) H^{-1/2}]\} \, \|H^{1/2} x_k\|^2 \end{aligned} \tag{3.13}$$

From (3.13), a sufficient condition is obtained as

$$\sigma_{max}[H^{1/2}(A + pE)H^{-1/2}] < 1 \tag{3.14}$$

which is equivalent to $H^{1/2}(\Phi + pE)H^{-1/2}$ being a contraction. However, since $H^{1/2}(\Phi + pE)H^{-1/2}$ is nothing but the system matrix of an equivalent system defined by a very special similarity transformation, this is completely an expected result. Although (3.14) is even a stronger, therefore useless condition than $(\Phi + pE)$ itself being a stability matrix, it illustrates how Liapunov techniques can be both useful and conservative in robustness analysis.

A final robustness bound for single-parameter perturbation model is obtained by requiring $G(p)$ in (3.6) to be positive definite. Since $G(0) = I$ is positive definite, from the continuity of eigenvalues of $G(p)$, it follows that $\mathcal{D}_p$ is stable if $|p|$ is small enough to satisfy

$$detG(p) \neq 0 \tag{3.15}$$

Following the technique in Section 2.2, we write

$$detG(p) = det(I + p\mathcal{F}), \tag{3.16}$$

where

$$\mathcal{F} = \begin{bmatrix} 0 & H^{1/2}EG^{-1/2} \\ G^{-1/2}E^T H^{1/2} & -G^{-1/2}(E^T H\Phi + \Phi^T HE)G^{-1/2} \end{bmatrix} \tag{3.17}$$

and obtain the bound

$$|p| < \sigma_{max}^{-1}(\mathcal{F}) \triangleq \mu_{s_3} \tag{3.18}$$

As a special case, let $\bar{\mathcal{F}}$ denote the $\mathcal{F}$ matrix corresponding to $G = \bar{G} = I$, $H = \bar{H}$. Decomposing $\bar{\mathcal{F}}$ as

$$\bar{\mathcal{F}} = \begin{bmatrix} 0 & 0 \\ 0 & E^T\bar{H}^{1/2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ I & -\bar{H}^{1/2}\Phi \end{bmatrix} + \begin{bmatrix} 0 & I \\ 0 & -\Phi^T\bar{H}^{1/2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \bar{H}^{1/2}E \end{bmatrix} \tag{3.19}$$

it follows that

$$\begin{aligned}\sigma_{max}(\bar{\mathcal{F}}) &\le 2\sigma_{max}(E)\sigma_{max}^{1/2}(\bar{H})\lambda_{max}^{1/2}(I+\bar{H}^{1/2}\Phi\Phi^T\bar{H}^{1/2}) \\ &\le 2\sigma_{max}(E)\sigma_{max}^{1/2}(\bar{H})[1+\lambda_{max}(\Phi^T\bar{H}\Phi)]^{1/2} \\ &\le 2\sigma_{max}(E)\sigma_{max}(\bar{H}) \end{aligned} \tag{3.20}$$

Thus, if

$$|p| < \frac{1}{2\sigma_{max}(E)\sigma_{max}(\bar{H})} \tag{3.21}$$

then (3.18) is satisfied for $\mathcal{F} = \bar{\mathcal{F}}$. It is interesting to note that (3.21) also implies (3.12).

In the case of multi-parameter perturbations, $\Delta V(x_k)$ is computed as

$$\begin{aligned}\Delta V(x_k)\mid_{\mathcal{D}_p} &= x_k^T[(\Phi+\sum_{r=1}^{m}p_rE_r)^TH(\Phi+\sum_{r=1}^{m}p_rE_r)-H]x_k \\ &= -x_k^TG^{1/2}G(p)G^{1/2}x_k\end{aligned} \tag{3.22}$$

where

$$G(p) = I-\sum_{r=1}^{m}p_rG^{-1/2}(E_r^TH\Phi+\Phi^THE_r)G^{-1/2}-\sum_{r=1}^{m}\sum_{s=1}^{m}p_rp_sG^{-1/2}E_r^THE_sG^{-1/2} \tag{3.23}$$

It turns out that the only way to achieve a robustness bound is to use the continuity of the eigenvalues of $G(p)$, and to require $|p_r|$ to be small enough to have the inequality (3.15). Fortunately, an explicit expression can be obtained as

$$detG(p) = det[I+\sum_{r=1}^{m}p_r\mathcal{F}_r] \tag{3.24}$$

where

$$\mathcal{F}_r = \begin{bmatrix} 0 & H^{1/2}E_r G^{-1/2} \\ G^{-1/2}E_r^T H^{1/2} & -G^{-1/2}(E_r^T H\Phi + \Phi^T H E_r)G^{-1/2} \end{bmatrix} \tag{3.25}$$

Since $\mathcal{F}_r$ are symmetric, (3.15) is satisfied if

$$\sigma_{max}(\sum_{r=1}^{m} p_r \mathcal{F}_r) < 1 \tag{3.26}$$

Now, the technique of Zhou and Khargonekar [11] can be applied to (3.26) to obtain the stability regions

$$\Omega_D \qquad \sum_{r=1}^{m} |p_r| \sigma_{max}(\mathcal{F}_r) < 1 \tag{3.27}$$

$$\Omega_P \qquad \|p\|_\infty = \max_{1 \le r \le m} |p_r| < \sigma_{max}^{-1}(\sum_{r=1}^{m} | \mathcal{F}_r |) \tag{3.28}$$

$$\Omega_S \qquad \|p\|_2 = (\sum_{r=1}^{m} p_r^2)^{1/2} < \lambda_{\max}^{-1/2}(\sum_{r=1}^{m} \mathcal{F}_r^T \mathcal{F}_r) \tag{3.29}$$

in the parameter space.

3.2 Non-Liapunov Approach to Stability Robustness Analysis

A necessary and sufficient condition for the stability of $\mathcal{D}_p$ is that all eigenvalues of $(\Phi + \Phi_p)$ be within the open unit circle. Since the nominal system is stable, we have

$$det(e^{j\theta} I - \Phi) \neq 0$$

Again, using the continuity of eigenvalues, it turns out that $\mathcal{D}_p$ is stable if $\|\Phi_p\|$ is small enough to satisfy

$$det(e^{j\theta} I - \Phi - \Phi_p) \neq 0,$$

or equivalently

$$det[I - (e^{j\theta}I - \Phi)^{-1}\Phi_p] \neq 0, \quad 0 \leq \theta \leq \pi \tag{3.30}$$

Obviously, (3.30) is satisfied if

$$\|\Phi_p\| < \frac{1}{\sup_{0\leq\theta\leq\pi}\|(e^{j\theta}I - \Phi)^{-1}\|} \triangleq \mu_{u_3} \tag{3.31}$$

for any matrix norm. Using the spectral norm, (3.31) becomes

$$\sigma_{max}(\Phi_p) < \inf_{0\leq\theta\leq\pi} \sigma_{min}(e^{j\theta}I - \Phi) \triangleq \mu_{u_3} \tag{3.32}$$

In the case of a single-parameter perturbation, when $\Phi_p = pE$, (3.32) is satisfied if

$$|p| < \frac{\inf_{0\leq\theta\leq\pi} \sigma_{min}(e^{j\theta}I - \Phi)}{\sigma_{max}(E)} \triangleq \mu_{u_3} \tag{3.33}$$

Although (3.32) can also be used to obtain several stability regions in the parameter space in the case of multi-parameter perturbations, we do not pursue this point any further, because computing the expression on the right hand side of (3.32) is not an easy task except in special cases.

As in the case of continuous-time systems, where Kronecker sums are used, Kronecker products may be employed to obtain alternative robustness bounds for discrete-time systems. From the properties of Kronecker products, it follows that if a real matrix M has an eigenvalue on the unit circle, then $M \otimes M$ has two eigenvalues at $z = 1$. Applying this fact to single-parameter perturbation model of $\mathcal{D}_p$, we observe that $\mathcal{D}_p$ is stable if p is small enough to satisfy

$$det[I - (\Phi + pE) \otimes (\Phi + pE)] \neq 0 \tag{3.34}$$

Using properties of Kronecker products, and the fact that

$$det(I - \Phi \otimes \Phi) \neq 0$$

(3.34) can be rewritten as

$$det[-I + p(I - \Phi\otimes\Phi)^{-1}(E\otimes\Phi + \Phi\otimes E) + p^2(I - \Phi\otimes\Phi)^{-1}(E\otimes E)] \neq 0 \tag{3.35}$$

Although the determinant in (3.35) contains quadratic terms in p, it can be expressed as the determinant of a larger matrix in which only linear terms in p appear. This gives,

$$det(-I + p\mathcal{M}) \neq 0 \tag{3.36}$$

where

$$\mathcal{M} = \begin{bmatrix} 0 & I \\ (I - \Phi \otimes \Phi)^{-1}(E \otimes E) & -(I - \Phi \otimes \Phi)^{-1}(E \otimes \Phi + \Phi \otimes E) \end{bmatrix} \tag{3.37}$$

Although (3.36) can be used to derive several stability robustness bounds for $\mathcal{D}_p$, we observe that it is a necessary and sufficient condition for the stability of an associated continuous-time system described as

$$\mathcal{S}_p(\mathcal{D}_p) \qquad \dot{x} = (-I + p\mathcal{M})x \tag{3.38}$$

Thus, all the robustness results concerning continuous-time systems can be used to obtain bounds on $|p|$ for stability of $\mathcal{D}_p$. From (3.36), a bound on $|p|$ can be obtained as

$$|p| < \frac{1}{\max_i |\lambda_i^r(\mathcal{M})|} \tag{3.39}$$

whereas a more conservative bound can be stated as

$$|p| < \frac{1}{\sigma_{max}(\mathcal{M})} \tag{3.40}$$

Similar results can be obtained for multi-parameter perturbations. In this case, (3.34) and (3.35) become

$$det[I - (\Phi + \sum_{r=1}^{m} p_r E_r) \otimes (\Phi + \sum_{s=1}^{m} p_s E_s)] \neq 0 \tag{3.41}$$

and

$$\begin{aligned} det[\quad I \quad & -\sum_{r=1}^{m} p_r (I - \Phi \otimes \Phi)^{-1}(E_r \otimes \Phi + \Phi \otimes E_r) \\ & - \sum_{r=1}^{m}\sum_{s=1}^{m} p_r p_s (I - \Phi \otimes \Phi)^{-1}(E_r \otimes E_s)] \neq 0 \end{aligned} \tag{3.42}$$

respectively. (3.42) is equivalent to

$$det[-I + \sum_{r=1}^{m} p_r \mathcal{M}_r] \neq 0 \tag{3.43}$$

where

$$\mathcal{M}_r = \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 & E_r \otimes E_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & & 0 & 0 & 0 & & 0 & E_r \otimes E_m \\ 0 & & 0 & (I - \Phi \otimes \Phi)^{-1} & 0 & \dots & 0 & -(E_r \otimes \Phi + \Phi \otimes E_r) \end{bmatrix} \tag{3.44}$$

Thus, stability of $\mathcal{D}_p$ is equivalent to stability of the auxiliary continuous-time system

$$\mathcal{S}_p(\mathcal{D}_p) \quad : \quad \dot{x} = (-I + \sum_{r=1}^{m} p_r \mathcal{M}_r)x \tag{3.45}$$

with multi-parameter additive perturbations. From (3.45), the following stability regions in the parameter space can be obtained in a standard way:

$$\Omega_D \quad : \quad \sum_{r=1}^{m} |p_r| \sigma_{max}(\mathcal{M}_r) < 1 \tag{3.46}$$

$$\Omega_P \quad : \quad ||p||_\infty = \max_r |p_r| < \sigma_{max}^{-1}(\sum_{r=1}^{m} \mid \mathcal{M}_r \mid) \tag{3.47}$$

$$\Omega_S \quad : \quad ||p||_2 = (\sum_{r=1}^{m} p_r^2)^{1/2} < \lambda_{\max}^{-1/2}(\sum_{r=1}^{m} \mathcal{M}_r^T \mathcal{M}_r) \tag{3.48}$$

Also, if continuous-time stability robustness analysis is applied to (3.45) by using Liapunov approach, $\bar{H} = I/2$ is obtained with $G = \bar{G} = I$. Then, $\bar{F}_r = \mathcal{M}_r^T \bar{H} + \bar{H} \mathcal{M}_r = (\mathcal{M}_r)_s$ where $(\mathcal{M}_r)_s$ is the symmetric part of $\mathcal{M}_r$. From, (2.23)- (2.25) will give the following bounds,

$$\Omega_D \quad : \quad \sum_{r=1}^{m} |p_r| \sigma_{max}((\mathcal{M}_r)_s) < 1 \tag{3.49}$$

$$\Omega_P \quad : \quad ||p||_\infty = \max_r |p_r| < \sigma_{max}^{-1}(\sum_{r=1}^{m} \mid (\mathcal{M}_r)_s \mid) \tag{3.50}$$

$$\Omega_S \ : \ \|p\|_2 = (\sum_{r=1}^{m} p_r^2)^{1/2} < \lambda_{\max}^{-1/2}(\sum_{r=1}^{m}(\mathcal{M}_r)_s^2) \tag{3.51}$$

3.3 Summary and Examples

Listings of stability robustness bounds for discrete-time systems are given in Tables 3.1 and 3.2 for single- and multi- parameter perturbations, respectively.

Example 3.1 When a discrete-time system is obtained by sampling a continuous-time system, continuous-time additive perturbations appear nonlinear in discrete-time model. However, under some (perhaps, very strict) assumptions, they can also appear as additive perturbations after sampling. As an example, consider the system

$$\mathcal{S}_p : \ \dot{x}(t) = (A + pE)x$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The discrete-time model for the sampled system will then be

$$\mathcal{D}_p : \ x[(k+1)T] = (\Phi + p\Phi_p)x[kT],$$

where

$$\Phi = e^{AT} = \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix}$$

and

$$\Phi_p = \frac{1}{p}[e^{(A+pA_p)T} - e^{AT}] = \begin{bmatrix} 0 & e^{-T}(1-e^{-T}) \\ 0 & 0 \end{bmatrix}.$$

For this particular example, $\mathcal{D}_p$ is stable for all values of p as $\mathcal{S}_p$ is.

$$|p| < \frac{(4a+b^2)^{1/2}-b}{2a} = \mu_{s_1}$$

$$|p| < \frac{[\sigma_{max}(H-G)+\sigma_{min}(G)]^{1/2}-\sigma_{max}^{1/2}(H-G)}{\sigma_{max}^{1/2}(E^THE)} = \mu_{s_2}$$

$$|p| < \frac{1-[1-\sigma_{max}^{-1}(\bar{H})]^{1/2}}{\sigma_{max}(E)} = \mu_{u_1}$$

$$|p| < \frac{1}{2\sigma_{max}(E)\sigma_{max}(\bar{H})} = \mu_{u_2}$$

$$|p| < \sigma_{max}^{-1}(\mathcal{F}) = \mu_{s_3}$$

$$|p| < \frac{\inf_{0\le\theta\le\pi}\sigma_{min}(e^{j\theta}I-\Phi)}{\sigma_{max}(E)} = \mu_{u_3}$$

$$|p| < \frac{1}{\sup_{0\le\theta\le\pi}\|(e^{j\theta}I-\Phi)^{-1}E\|} = \mu_{s_4}$$

$$|p| < \frac{1}{\max_i \lambda_i^r(\mathcal{M})} = \mu_{s_5}$$

$$|p| < \frac{1}{\sigma_{max}(\mathcal{M})} = \mu_{s_6}$$

where

$$\mathcal{F} = \begin{bmatrix} 0 & H^{1/2}EG^{-1/2} \\ G^{-1/2}E^TH^{1/2} & -G^{-1/2}(E^THA+A^THE)G^{-1/2} \end{bmatrix}$$

and

$$\mathcal{M} = \begin{bmatrix} 0 & I \\ (I-\Phi\otimes\Phi)^{-1}(E\otimes E) & -(I-\Phi\otimes\Phi)^{-1}(E\otimes\Phi+\Phi\otimes E) \end{bmatrix}$$

Table 3.1. Stability robustness bounds for single-parameter perturbed Discrete-time Systems

$$\Omega_D \ : \ \sum_{r=1}^{m} |p_r| \sigma_{max}(F_r) < 1$$

$$\Omega_P \ : \ \|p\|_\infty = \max_r |p_r| < \sigma_{max}^{-1}(\sum_{r=1}^{m} \mid F_r \mid)$$

$$\Omega_S \ : \ \|p\|_2 = (\sum_{r=1}^{m} p_r^2)^{1/2} < \lambda_{\max}^{-1/2}(\sum_{r=1}^{m} F_r^T F_r)$$

$$\Omega_D \ : \ \sum_{r=1}^{m} |p_r| \sigma_{max}(\mathcal{M}_r) < 1$$

$$\Omega_P \ : \ \|p\|_\infty = \max_r |p_r| < \sigma_{max}^{-1}(\sum_{r=1}^{m} \mid \mathcal{M}_r \mid)$$

$$\Omega_S \ : \ \|p\|_2 = (\sum_{r=1}^{m} p_r^2)^{1/2} < \lambda_{\max}^{-1/2}(\sum_{r=1}^{m} \mathcal{M}_r^T \mathcal{M}_r)$$

$$\Omega_D \ : \ \sum_{r=1}^{m} |p_r| \sigma_{max}((\mathcal{M}_r)_s) < 1$$

$$\Omega_P \ : \ \|p\|_\infty = \max_r |p_r| < \sigma_{max}^{-1}(\sum_{r=1}^{m} \mid (\mathcal{M}_r)_s \mid)$$

$$\Omega_S \ : \ \|p\|_2 = (\sum_{r=1}^{m} p_r^2)^{1/2} < \lambda_{\max}^{-1/2}(\sum_{r=1}^{m} (M_r)_s^2)$$

where

$$\mathcal{F}_r = \begin{bmatrix} 0 & H^{1/2} E_r G^{-1/2} \\ G^{-1/2} E_r^T H^{1/2} & -G^{-1/2}(E_r^T H \Phi + \Phi^T H E_r) G^{-1/2} \end{bmatrix}$$

and

$$\mathcal{M}_r = \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 & E_r \otimes E_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & E_r \otimes E_m \\ 0 & \dots & 0 & (I - \Phi \otimes \Phi)^{-1} & 0 & \dots & 0 & -(E_r \otimes \Phi + \Phi \otimes E_r) \end{bmatrix}$$

$$(M_r)_s = (M_r^T + M_r)/2$$

Table 3.2. Stability robustness bounds for multi-parameter perturbed Discrete-time Systems

Example 3.2 To illustrate the computation of the stability robustness bounds for single-parameter perturbed systems, consider two perturbed systems $\mathcal{D}_{p1}$ and $\mathcal{D}_{p2}$ where

$$\Phi = \begin{bmatrix} 0.5 & 1 \\ 0 & -0.5 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The solution for the Liapunov equation for the nominal system is obtained as

$$\bar{H} = \begin{bmatrix} 20/15 & 8/15 \\ 8/15 & 36/15 \end{bmatrix}$$

An unstructured perturbation model yields the following bounds for both $\mathcal{D}_{p1}$ and $\mathcal{D}_{p2}$:

$$|p| < \mu_{u_1} = 0.2136 \quad (3.52)$$

$$|p| < \mu_{u_2} = 0.1908 \quad (3.53)$$

$$|p| < \mu_{u_3} = 0.4109 \quad (3.54)$$

On the other hand, if the structure of perturbations are taken into consideration, the bounds are modified into

For $\mathcal{D}_{p1}$		For $\mathcal{D}_{p2}$	
$\|p\| < \mu_{s_1}$	$= 0.4056$	$\|p\| < \mu_{s_1}$	$= 0.3571$
$\|p\| < \mu_{s_3}$	$= 0.3962$	$\|p\| < \mu_{s_3}$	$= 0.342$
$\|p\| < \mu_{s_4}$	$= 0.5$	$\|p\| < \mu_{s_4}$	$= 0.5$
$\|p\| < \mu_{s_5}$	$= 0.5$	$\|p\| < \mu_{s_5}$	$= \infty$
$\|p\| < \mu_{s_6}$	$= 0.3884$	$\|p\| < \mu_{s_6}$	$= 0.3489$

Now, using the same Φ, E_1 and E_2 matrices consider a system,

$$\mathcal{D}_p : \; x_{k+1} = (\Phi + p_1 E_1 + p_2 E_2) x_k$$

Then, we obtain the stability robustness regions

$$\Omega_D^I : \quad 2.3742|p_1| + 2.7742|p_2| < 1$$

$$\Omega_P^I : \quad \max\{|p_1|, |p_2|\} < 0.2411$$

$$\Omega_S^I : \quad (p_1^2 + p_2^2)^{1/2} < 0.296$$

from (3.27) -(3.29);

$$\begin{aligned}\Omega_D^{II}: &\quad 2.8985|p_1| + 3.1732|p_2| < 1\\ \Omega_P^{II}: &\quad \max\{|p_1|, |p_2|\} < 0.2109\\ \Omega_S^{II}: &\quad (p_1^2 + p_2^2)^{1/2} < 0.3019\end{aligned}$$

from (3.46) - (3.48); and

$$\begin{aligned}\Omega_D^{III}: &\quad 2.3219|p_1| + 1.8595|p_2| < 1\\ \Omega_P^{III}: &\quad \max\{|p_1|, |p_2|\} < 0.2725\\ \Omega_S^{III}: &\quad (p_1^2 + p_2^2)^{1/2} < 0.3649\end{aligned}$$

from (3.49) - (3.51).

These stability regions are shown in Figures 3.1, 3.2, 3.3. Note that the stability regions obtained from (3.49) - (3.51) are superior to others.

Finally, by using (3.55) and modifying Φ and p_2 as

$$\Phi' = \Phi + 0.2725E_2 \quad p_2' = p_2 - 0.2725$$

i.e. shifting the origin along the p_2 axis, then the following bounds are obtained from (3.49) - (3.51).

$$2.5443|p_1| + 2.242|p_2'| < 1 \tag{3.55}$$

$$\max\{|p_1| + |p_2'|\} < 0.2377 \tag{3.56}$$

$$(p_1^2 + p_2'^2)^{1/2} < 0.3201 \tag{3.57}$$

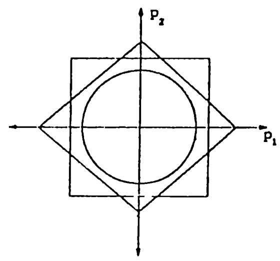

Figure 3.1. Stability regions obtained using (3.27) - (3.29)

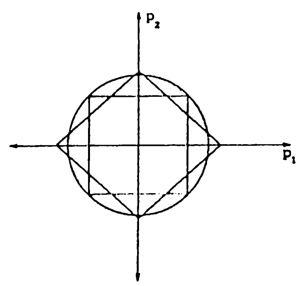

Figure 3.2. Stability regions obtained using (3.46) - (3.48)

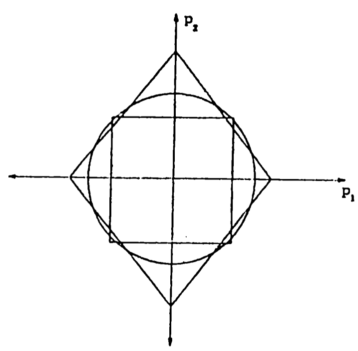

Figure 3.3. Stability regions obtained using (3.49) - (3.51)

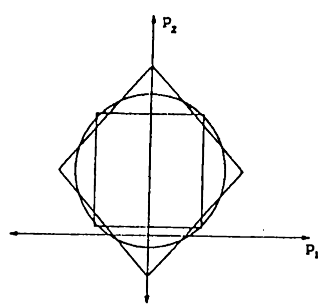

Figure 3.4. Stability regions obtained by shifting the origin

Chapter 4

APPLICATION TO DISCRETE-TIME INTERCONNECTED SYSTEMS

A natural way to describe a complex system is to view it as an interconnection of dynamic parts, or subsystems. In such a description, the essential uncertainty lies in the interconnection parameters, which reflect the strength of coupling, or interaction, among more precisely modeled subsystems. The concept of connective stability, put forward by Šiljak [26], refers to the stability of an interconnected system, where the subsystems are disconnected and connected again during operation. Since overall stability of the system when all the subsystems are decoupled requires the stability of individual subsystems, in connective stability analysis the interconnections are treated as undesired perturbations. This brings into picture the issue of robustness.

In this chapter, we apply the results of the previous chapter to obtain robustness bounds for a discrete-time interconnected system described as

$$\mathcal{D}_p: \; x_i(k+1) = \Phi_i x_i(k) + \sum_{j=1}^{N} p_{ij} \Phi_{ij} x_j(k), \quad i = 1, 2, \ldots, N. \tag{4.1}$$

In (4.1), $x_i(k) \in \mathcal{R}^{n_i}$ is the state of the ith subsystem,

$$\mathcal{D}_i : \; x_i(k+1) = \Phi_i x_i(k) \tag{4.2}$$

which is assumed to be stable; Φ_{ij} are fixed interconnection matrices, and p_{ij} are interconnection gains which are treated as perturbation parameters.

Letting

$$x(k) = [x_1^T(k) \; x_2^T(k) \; \ldots x_N^T(k) \;]^T \tag{4.3}$$

and

$$\Phi = diag\{\Phi_1, \; \Phi_2, \; \ldots, \Phi_N \; \}, \tag{4.4}$$

the collection of decoupled subsystems in (4.2) can be described in a compact way as

$$\mathcal{D} : \; x(k+1) = \Phi x(k) \tag{4.5}$$

Similarly, letting $E_{ij} = (E_{pq}^{ij})_{N \times N}$, where

$$E_{pq}^{ij} = \begin{cases} \Phi_{ij}, & \text{for } p = i, q = j \\ 0, & \text{otherwise} \end{cases} \tag{4.6}$$

the interconnected system in (4.1) can be modelled as

$$\mathcal{D}_p : \; x(k+1) = (\Phi + \sum_{i=1}^{N} \sum_{j=1}^{N} p_{ij} E_{ij}) x(k) \tag{4.7}$$

which has the standard multi-parameter perturbation description.

Choosing $V(x) = x^T \bar{H} x$ as a Liapunov function for $\mathcal{D}$ of (4.5), where

$$\Phi^T \bar{H} \Phi - \bar{H} = -I \tag{4.8}$$

we obtain the following stability regions in the parameter space of $\mathcal{D}_p$.

$$\bar{\Omega}_D \; : \quad \sum_i \sum_j |p_{ij}| \sigma_{max}(\bar{F}_{ij}) < 1 \tag{4.9}$$

$$\bar{\Omega}_P \; : \quad \max_{i,j} |p_{ij}| < \sigma_{max}^{-1}(\sum_i \sum_j \mid \bar{F}_{ij} \mid), \tag{4.10}$$

$$\bar{\Omega}_S \; : \quad (\sum_i \sum_j p_{ij}^2)^{1/2} < \sigma_{max}^{-1/2}(\sum_i \sum_j \bar{F}_{ij}^2) \tag{4.11}$$

where

$$\bar{F}_{ij} = \begin{bmatrix} 0 & \bar{H}^{1/2}E_{ij} \\ E_{ij}^T\bar{H}^{1/2} & -(E_{ij}^T\bar{H}\Phi + \Phi^T\bar{H}E_{ij}) \end{bmatrix} \tag{4.12}$$

We immediately notice from (4.4) and (4.8) that

$$\bar{H} = diag\{\bar{H}_1, \bar{H}_2, \dots, \bar{H}_N\} \tag{4.13}$$

where $V_i(x_i) = x_i^T\bar{H}_i x_i$ are Liapunov functions for the decoupled subsystems $\mathcal{D}_i$ of (4.2), with $\bar{H}_i$ being the solutions of

$$\Phi_i^T\bar{H}_i\Phi_i - \bar{H}_i = -I \tag{4.14}$$

The block diagonal structure of $\bar{H}$, together with the special structures of the perturbation matrices E_{ij} defined in (4.6) allows for obtaining explicit expressions for $\sigma_{max}(\cdot)$ terms in (4.9)-(4.11). As an illustration, for $N = 3$, and $i = 1, j = 2$, $\bar{F}_{ij}$ becomes

$$\bar{F}_{12} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & \bar{H}_1^{1/2}\Phi_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\Phi_1^T\bar{H}_1\Phi_{12} & 0 \\ \Phi_{12}^T\bar{H}_1^{1/2} & 0 & 0 & \Phi_{12}^T\bar{H}_1\Phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{4.15}$$

from which we obtain

$$\begin{aligned} \sigma_{max}(\bar{F}_{12}) &= \lambda_{\max}^{1/2}(\bar{F}_{12}^2) \\ &= \lambda_{\max}^{1/2}(\bar{X}_{12}^T\bar{X}_{12}) \\ &= \sigma_{max}^{1/2}(\Phi_{12}^T\bar{H}_1\Phi_{12} + \Phi_{12}^T\bar{H}_1\Phi_1\Phi_1^T\bar{H}_1\Phi_{12}) \end{aligned} \tag{4.16}$$

where

$$\bar{X}_{12}^T = [\Phi_{12}^T\bar{H}_1^{1/2} \quad -\Phi_{12}^T\bar{H}_1\Phi_1] \tag{4.17}$$

In general, we have

$$\sigma_{max}(\bar{F}_{ij}) = \sigma_{max}^{1/2}(\Phi_{ij}\bar{H}_i\Phi_{ij} + \Phi_{ij}\bar{H}_i\Phi_i\Phi_i^T\bar{H}_i\Phi_{ij}), \tag{4.18}$$

which provides an explicit expression for $\sigma_{max}(\bar{F}_{ij})$ in terms of system matrices. Majorizing (4.18) further, we obtain

$$\sigma_{max}(\bar{F}_{ij}) \leq \sigma_{max}^{1/2}(\bar{H}_i)\sigma_{max}^{1/2}(\Phi_{ij}^T\bar{H}_i\Phi_{ij}) \tag{4.19}$$

resulting a stability region

$$\bar{\Omega}_D' : \qquad \sum_i \sum_j |p_{ij}|\sigma_{max}^{1/2}(\bar{H}_i)\sigma_{max}^{1/2}(\Phi_{ij}^T\bar{H}_i\Phi_{ij}) < 1 \tag{4.20}$$

which is smaller than Ω_D of (4.9), but easier to compute.

We now turn our attention to stability analysis of $\mathcal{D}_P$ via composite Liapunov functions (Sezer and Šiljak [18]). Let $V_i(x_i) = x_i^T H_i x_i$ be the subsystem Liapunov functions, where

$$\Phi_i^T H_i \Phi_i - H_i = -G_i, \quad i = 1, 2, \ldots, N \tag{4.21}$$

for positive definite matrices G_i. Computing the increment of V_i along the solutions of the interconnected system $\mathcal{D}_P$ of (4.1), we obtain

$$\begin{aligned}
\Delta V_i(x_i) &= (x_i^T\Phi_i^T + \sum_j p_{ij}x_j^T\Phi_{ij}^T)H_i(\Phi_i x_i + \sum_j p_{ij}\Phi_{ij}x_j) - x_i^T H_i x_i \\
&= -x_i^T G_i x_i + 2x_i^T\Phi_i^T H_i^{1/2}\sum_j p_{ij}H_i^{1/2}\Phi_{ij}x_j \\
&\quad +(\sum_j p_{ij}H_i^{1/2}\Phi_{ij}x_j)^T(\sum_j p_{ij}H_i^{1/2}\Phi_{ij}x_j) \\
&\leq -\sigma_{min}(G_i)\|x_i\|^2 + 2\sigma_{max}^{1/2}(\Phi_i^T H_i \Phi_i)\,\|x_i\|\,(\sum_j |p_{ij}|\,\xi_{ij}\,\|x_j\|) \\
&\quad +(\sum_j |p_{ij}|\,\xi_{ij}\,\|x_j\|)^2 \\
&= -[\sigma_{min}(G_i) + \sigma_{max}(H_i - G_i)]\|x_i\|^2 \\
&\quad +[\sigma_{max}^{1/2}(H_i - G_i)\|x_i\| + \sum_j |p_{ij}|\,\xi_{ij}\,\|x_j\|]^2 \qquad (4.22) \\
&= -[\alpha_i^2\|x_i\|^2 - (\beta_i\|x_i\| + \sum_j |p_{ij}|\,\xi_{ij}\,\|x_j\|)^2] \qquad (4.23)
\end{aligned}$$

where

$$\begin{aligned} \alpha_i^2 &= \sigma_{min}(G_i) + \sigma_{max}(H_i - G_i) \\ \beta_i &= \sigma_{max}^{1/2}(H_i - G_i) \\ \xi_{ij} &= \sigma_{max}^{1/2}(\Phi_{ij}^T H_i \Phi_{ij}) \end{aligned} \tag{4.24}$$

We now choose

$$V(x) = \sum_{i=1}^{N} d_i V_i(x_i) \tag{4.25}$$

as a candidate for a Liapunov function for $\mathcal{D}_P$, where $d_i > 0$ are to be determined. Using (4.23), we get

$$\begin{aligned} \Delta V(x) &\leq -\sum_{i=1}^{N} d_i[\alpha_i^2 \|x_i\| - (\beta_i \|x_i\| + \sum_j |p_{ij}|\ \xi_{ij}\ \|x_j\|)^2] \\ &= -U^T(\|x\|)(C^T DC - B^T DB)U(\|x\|) \end{aligned} \tag{4.26}$$

where

$$U^T(\|x\|) = [\|x_1\|,\ \|x_2\|,\ \ldots,\ \|x_n\|\ ] \tag{4.27}$$

and

$$C = diag\{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_N\}$$

$$D = diag\{d_1,\ d_2,\ \ldots,\ d_N\} \tag{4.28}$$

$$B = (b_{ij})_{N\times N} \tag{4.29}$$

with

$$b_{ij} = \begin{cases} \beta_i + |p_{ii}|\xi_{ii} & ,\ j = i \\ |p_{ij}|\xi_{ij} & ,\ j \neq i \end{cases} \tag{4.30}$$

Thus $\Delta V(x)$ is negative definite if the matrix $CDC - B^T DB$ is positive definite for some suitable choice of the diagonal matrix D. However, the latter is equivalent to the aggregate matrix

$$W = C - B \tag{4.31}$$

being an M-matrix. Letting $W = (w_{ij})_{N\times N}$, we observe that

$$\begin{aligned} w_{ii} &= \alpha_i - \beta_i \\ &= [\sigma_{min}(G_i) + \sigma_{max}(H_i - G_i)]^{1/2} - \sigma_{max}^{1/2}(H_i - G_i) \end{aligned} \tag{4.32}$$

and

$$w_{ij} = -|p_{ij}|\xi_{ij} = -|p_{ij}|\sigma_{max}^{1/2}(\Phi_{ij}^T H_i \Phi_{ij}), \;\; j \neq i \tag{4.33}$$

That is, each interconnection gain $|p_{ij}|$ appears in an off-diagonal element of W. The M-matrix conditions in terms of the leading principal minors of W provide a set of inequalities in $|p_{ij}|$'s, which define a stability region in the parameter space.

To obtain explicit expressions for the robustness regions defined through W, let us choose $G_i = \bar{G}_i = I$, so that (4.32) and (4.33) become

$$\bar{w}_{ii} = \sigma_{max}^{1/2}(\bar{H}_i) - [\sigma_{max}(\bar{H}_i) - 1]^{1/2}, \tag{4.34}$$

$$\bar{w}_{ij} = -|p_{ij}|\sigma_{max}^{1/2}(\Phi_{ij}^T \bar{H}_i \Phi_{ij}), \tag{4.35}$$

We also note that $\bar{W} = (\bar{w}_{ij})_{N\times N}$ is an M-matrix if and only if $\tilde{W} = (\tilde{w}_{ij})_{N\times N}$ is an M-matrix, where

$$\tilde{w}_{ij} = \begin{cases} 1, & j = i \\ \bar{w}_{ij}/\bar{w}_{ii}, & j \neq i \end{cases} \tag{4.36}$$

From (4.34) - (4.36), we can write

$$\tilde{W} = I - \sum_i \sum_{j\neq i} |p_{ij}|\tilde{F}_{ij}, \tag{4.37}$$

where $\tilde{F}_{ij}$ has a single nonzero element in the (i,j)-th position given by

$$\frac{\sigma_{max}^{1/2}(\Phi_{ij}^T \bar{H}_i \Phi_{ij})}{\sigma_{max}^{1/2}(\bar{H}_i) - [\sigma_{max}(\bar{H}_i) - 1]^{1/2}}. \tag{4.38}$$

Obviously, $\tilde{W}$ in (4.37) is an M-matrix if

$$\sigma_{max}(\sum_i \sum_{j\neq i} |p_{ij}|\tilde{F}_{ij}) < 1 \tag{4.39}$$

from which we obtain the stability regions

$$\Omega_D^W \quad \sum_i \sum_{j\neq i} |p_{ij}|\sigma_{max}(\tilde{F}_{ij}) < 1 \tag{4.40}$$

$$\Omega_P^W \; : \; \max_{i\neq j} |p_{ij}| < \sigma_{max}^{-1}(\sum_i \sum_{j\neq i} \tilde{F}_{ij}) \tag{4.41}$$

$$\Omega_S^W \; : \; (\sum_i \sum_{j\neq i} p_{ij}^2)^{1/2} < \lambda_{max}^{-1/2}(\sum_i \sum_{j\neq i} \tilde{F}_{ij}\tilde{F}_{ij}^T) \tag{4.42}$$

In other words, the role of $\bar{F}_{ij}$ in (4.9)-(4.11) are taken by $\tilde{F}_{ij}$ in (4.40) - (4.42). However, it is much easier to evaluate the expressions in (4.40) - (4.42). From the structure of $\tilde{F}_{ij}$, it follows that

$$\sigma_{max}(\tilde{F}_{ij}) = \frac{\sigma_{max}^{1/2}(\Phi_{ij}^T \bar{H}_i \Phi_{ij})}{\sigma_{max}^{1/2}(\bar{H}_i) - [\sigma_{max}(\bar{H}_i) - 1]^{1/2}} \tag{4.43}$$

$$\sigma_{max}(\sum_i \sum_j \tilde{F}_{ij}) = \sigma_{max}(\tilde{F}) \tag{4.44}$$

where $\tilde{F} = [\sigma_{max}(\tilde{F}_{ij})]_{N \times N}$, and

$$\lambda_{\max}^{1/2}(\sum_i \sum_{j \neq i} \tilde{F}_{ij} \tilde{F}_{ij}^T) = \max_{i, j \neq i}\{\sigma_{max}(\tilde{F}_{ij})\}, \tag{4.45}$$

which provide very simple characterization of the stability regions in (4.40) - (4.42).

Finally, it is interesting to compare the stability regions obtained by the two approaches. Since,

$$\sigma_{max}^{1/2}(\bar{H}_i) \leq \frac{1}{\sigma_{max}^{1/2}(\bar{H}_i) - [\sigma_{max}(\bar{H}_i) - 1]^{1/2}} \tag{4.46}$$

from (4.19) and (4.43), we get

$$\sigma_{max}(\bar{F}_{ij}) \leq \sigma_{max}(\tilde{F}_{ij}) \tag{4.47}$$

Thus,

$$\Omega_D^W \subset \bar{\Omega}_D' \subset \bar{\Omega}_D \tag{4.48}$$

where $\bar{\Omega}_D$, $\bar{\Omega}_D'$ and Ω_D^W are defined in (4.9), (4.20) and (4.40) respectively. This shows that, for the choice $\bar{G}_i = I$, the composite Liapunov function approach does not provide any improvement over ordinary Liapunov functions as far as the diamond-shaped stability region is considered. Since explicit expressions for other stability regions are not available, it is not possible to compare $\bar{\Omega}_P$ and $\bar{\Omega}_S$ of (4.10) and (4.11) with Ω_P^W and Ω_S^W of (4.41) and (4.42). However, Ω_P^W and Ω_S^W are so easy to characterize compared to $\bar{\Omega}_P$ and $\bar{\Omega}_S$ that, any loss in the estimate of stability regions should be outweighed by the enormous reduction in computational effort.

Example 4.1 Consider an interconnection of three subsystems

$$\mathcal{D}_p : x_i(k+1) = \Phi_i x_i(k) + \sum_{j=1}^{3} p_{ij}\Phi_{ij}x_j(k) \quad i = 1,2,3 \tag{4.49}$$

where

$$\Phi_1 = \begin{bmatrix} 0.25 & -0.5 \\ 0.5 & 0.75 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} -0.9 & -0.6 \\ 0.3 & 0 \end{bmatrix}, \quad \Phi_3 = \begin{bmatrix} 0.5 & 1 \\ 0 & -0.5 \end{bmatrix},$$

$$\Phi_{12} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Phi_{23} = \begin{bmatrix} 0.1 & 0.2 \\ 0 & -0.1 \end{bmatrix}, \quad \Phi_{32} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix},$$

and all other Φ_{ij}'s are zero.

Using $V_i(x_i) = x_i^T \bar{H}_i x_i$ as subsystem Liapunov functions, where $\bar{H}_i$ satisfy (4.14), the stability regions in (4.9) - (4.11) are evaluated as

$$\bar{\Omega}_D \ : \ 0.1626|p_{12}| + 0.6003|p_{23}| + 0.3907|p_{32}| < 1 \tag{4.50}$$

$$\bar{\Omega}_P \ : \ \max_{i,j} |p_{ij}| < 1.6288 \tag{4.51}$$

$$\bar{\Omega}_S \ : \ (p_{12}^2 + p_{23}^2 + p_{32}^2)^{1/2} < 0.8791 \tag{4.52}$$

On the other hand, computing $\tilde{W}$ of (4.37) as

$$\tilde{W} = \begin{bmatrix} 1 & 0.402|p_{12}| & 0 \\ 0 & 1 & 1.1201|p_{23}| \\ 0 & 1.0641|p_{32}| & 1 \end{bmatrix},$$

we observe that , $\tilde{W}$ is an M-matrix, if

$$\tilde{W} : \quad |p_{23}|\,|p_{32}| < 0.8389 \tag{4.53}$$

It is interesting to compare the stability regions $\tilde{\Omega}$ in (4.53)with the ones in (4.50) - (4.52)

1. While each of $\bar{\Omega}_D, \bar{\Omega}_P$ and $\bar{\Omega}_S$ are bounded in the p_{12}- direction, $\tilde{\Omega}$ is not. This is because $\tilde{\Omega}$ carries more information about the structure of $\mathcal{D}_p$, in which the subsystems do not form a loop through $\mathcal{D}_1$.

2. $\tilde{\Omega}$ includes $\bar{\Omega}_D$ if $|p_{12}| \geq 0.6941$, $\bar{\Omega}_P$ if $|p_{12}| \geq 1.6288$ and $\bar{\Omega}_S$ for all $|p_{12}|$.

Finally, the stability regions in (4.40) - (4.42) are found as

$$\Omega_D^W \; : \; 0.402|p_{12}| + 1.1201|p_{23}| + 1.064|p_{32}| < 1 \tag{4.54}$$

$$\Omega_P^W \; : \; \max_{i,j} |p_{ij}| < 1.6364 \tag{4.55}$$

$$\Omega_S^W \; : \; (p_{12}^2 + p_{23}^2 + p_{32}^2)^{1/2} < 0.8791 \tag{4.56}$$

We observe that although $\Omega_D^W \subset \bar{\Omega}_D$ as expected, $\Omega_P^W \supset \bar{\Omega}_P$ and $\Omega_S^W = \bar{\Omega}_S$. Thus, the regions Ω^W are comparable to $\bar{\Omega}$ although they are much easier to compute.

Chapter 5

FURTHER RESEARCH AREAS

Usually, the more general the problem is, the harder the solution is. This has been the case of robust stability problem in state-space. Although, necessary and sufficient bounds are available for some special cases, in general we have only sufficient bounds.

The robust stability problem in state-space can be handled in two ways; Liapunov and Non-Liapunov approaches. Liapunov approach uses Liapunov stability theory to determine to what extend the nominal system is still stable when it is perturbed. This approach usually yields bounds which are conservative, but easier to compute. It is possible to reduce the conservatism of the bounds if some information about the structure of the perturbations exists. But, this brings out the question of choosing a Liapunov function for the nominal system which has the best decaying rate, i.e. choosing the best G in the Liapunov equation, $A^T H + HA = -G$ or $\Phi^T H \Phi - H = -G$. We know that the best choice of G depends on both the structures of the system matrix and the perturbations, but the answer is not straightforward.

Non-Liapunov approaches, including frequency-domain techniques, yield bounds which are better for some cases, but harder to compute. Use of operators like Kronecker sums or products causes an increase in the matrix dimensions. Introducing new operators which have smaller dimensions and easier to compute, would be for the benefit of the analysis of robust stability problem.

One of the open questions is modelling the perturbations. Usually, a perturbation model is assumed and the problem is solved in either frequency domain or in state-space. As in all control problems, the question of whether the perturbation model is physically meaningful or not is important. In addition, since robust stability problem can be analyzed in frequency domain or in state-space, the relations between the existing bounds in each domain should be revealed.

If the perturbations are due to the nonlinear functions of some parameters, this case should also be studied extensively. Up to now, only results about linear parametric perturbations are at hand, which can be extended to polynomial case. But, nonlinear parametric perturbation case is an open question.

Another open question is the stability robustness analysis of sampled-data systems. Since perturbations in the continuous-time system seems highly nonlinear when the system is sampled, a special care must be taken for the analysis of this case.

Searching a norm type bound on the perturbation matrix is equivalent to finding the distance of the system matrix to the unstable real matrices. However, we still don't know the distance of a stable matrix to the unstable real matrices whereas the distance to the unstable complex matrices is known.

Consequently, robust stability problem is still an active, promising research area.

Appendix A

BACKGROUND MATERIAL

In this appendix, we briefly review stability of linear systems and summarize some results from matrix algebra.

A.1 Lyapunov Theory for Linear Systems

Consider the linear, continuous time system

$$\mathcal{S}: \quad \dot{x}(t) = Ax(t) \tag{A.1}$$

which has an equilibrium at $x_e = 0$. It is well-known [21] that due to linearity of $\mathcal{S}$, stability, asymptotic stability in the large, and exponential stability of the equilibrium $x_e = 0$ imply each other. They also imply that origin is the unique equilibrium of $\mathcal{S}$. In this thesis, we use the phrase "stability of $\mathcal{S}$" to mean these equivalent concepts of stability.

Let $V(x) = x^T Hx$, where H is a symmetric, positive definite matrix. The derivative of $V(x)$ along the solutions of $\mathcal{S}$ is given by

$$\dot{V}(x)|_{\mathcal{S}} = x^T(A^T H + HA)x \tag{A.2}$$

Then, a symmetric matrix G can be defined as

$$A^T H + HA = -G \tag{A.3}$$

We now state a basic result about the stability of $\mathcal{S}$:

Theorem A.1 *(Vidyasagar [21]) The following are equivalent :*

1. *$\mathcal{S}$ is stable.*

2. *All eigenvalues of A have negative real parts.*

3. *For every symmetric, positive-definite matrix G, equation (A.3) has a unique, symmetric, positive-definite solution H.*

Consider a stable system $\mathcal{S}$, and let

$$\sigma_0 = \min_i \{ \ |Re[\lambda_i(A)]| \ \}$$

where $Re[\lambda_i(A)]$ denote real part of the eigenvalues of A. From the solution properties of $\mathcal{S}$, it can easily be shown that there exists $M > 0$ such that

$$\|x(t)\| \leq M e^{-\sigma_0 t} \|x(0)\|, \quad \forall t \in \mathcal{R}_+$$

for all initial states $x(0)$. In other words, σ_0 is the degree of exponential stability of $\mathcal{S}$.

Now let H be the positive-definite solution of (A.3) for some given positive-definite matrix G. Then $V(x) = x^T H x$ is a Liapunov function for $\mathcal{S}$, having a negative-definite time derivative.

$$\dot{V}(x)|_{\mathcal{S}} = -x^T G x \tag{A.4}$$

Using the inequalities

$$\sigma_{min}(M) \, \|x\|^2 \leq x^T M x \leq \sigma_{max}(M) \, \|x\|^2 \tag{A.5}$$

where $\sigma_{min}(M)$ and $\sigma_{max}(M)$ denote the minimum and maximum singular values of the symmetric matrix M, it follows from (A.4) that

$$V(x(t)) \leq \exp[-\sigma_{min}(G)/\sigma_{max}(H)] V(x(0)) \tag{A.6}$$

and that

$$\|x(t)\| \leq M_v e^{-\sigma_v t}\|x(0)\| \tag{A.7}$$

where

$$M_v = \sigma_{max}^{1/2}(H)/\sigma_{min}^{1/2}(H) \tag{A.8}$$

$$\sigma_v = \sigma_{min}(G)/2\sigma_{max}(H) \tag{A.9}$$

Thus, if $V(x) = x^T H x$ is a Liapunov function for $\mathcal{S}$, then σ_v provides an estimate of the degree of stability σ_0 such that

$$\sigma_v \leq \sigma_0$$

It is also well-known that [21], the solution H (A.3) is given by

$$H = -\int_0^\infty e^{A^T t} G e^{At} dt \tag{A.10}$$

Using this expression it can be shown that the estimate σ_v of the degree of stability given by (A.9) is maximized for the choice of $G = I$. That is the main reason for choosing the corresponding $V(x)$ as a Liapunov function in most of studies on robustness analysis.

All the stability concepts and the results mentioned so far also apply to discrete-time linear systems described as

$$\mathcal{D}: \quad x_{k+1} = \Phi x_k, \quad k \in \mathcal{Z}_+ \tag{A.11}$$

In this case, Theorem A.1 becomes :

Theorem A.2 *The following are equivalent :*

1. *$\mathcal{D}$ is stable.*

2. *All eigenvalues of Φ have moduli less than unity.*

3. For every symmetric, positive-definite G, the equation

$$\Phi^T H \Phi - H = -G \tag{A.12}$$

has a unique, symmetric, positive-definite solution H.

When $\mathcal{D}$ is stable,

$$\rho_0 = \max_i \{|\lambda_i(\Phi)|\}$$

is the degree of exponential stability in the sense that there exists an $M > 0$ such that

$$\|x_k\| \leq M \rho_0^k \|x_0\|, \quad \forall k \in \mathcal{Z}_+, \tag{A.13}$$

for all initial states x_0.

As in continuous-time systems, with H being the solution of (A.12) for some positive-definite G, $V(x) = x^T H x$ is a Liapunov function for $\mathcal{D}$ with a negative-definite increment.

$$\Delta V(x)|_{\mathcal{D}} = -x^T G x \tag{A.14}$$

From (A.14) it follows that

$$V(x_k) \leq [1 - \frac{\sigma_{min}(G)}{\sigma_{max}(H)}]^k V(x_0), \tag{A.15}$$

and hence,

$$\|x_k\| \leq M_v \rho_v^k \|x_0\|, \tag{A.16}$$

where M_v is given in (A.8) and

$$\rho_v = [1 - \frac{\sigma_{min}(G)}{\sigma_{max}(H)}]^{1/2} \tag{A.17}$$

is an estimate of the degree of stability of $\mathcal{D}$ such that

$$0 < \rho_v \leq \rho_0 < 1 \tag{A.18}$$

Using the fact that the solution of (A.12) is given by

$$H = \sum_{i=0}^{\infty} (\Phi^T)^i G \Phi^i \tag{A.19}$$

It can be shown (Sezer and Šiljak [18]) that the best estimate of the degree of stability is obtained by choosing $G = I$.

A.2 Kronecker Products and Sums of Matrices

In this part of the Appendix, we present some results on Kronecker products and sums of matrices, which are borrowed mainly from Lancaster and Tismenetsky [22] and Fuller [23]

The Kronecker product of the $p \times q$ matrix $A = (a_{ij})$ and the $m \times n$ matrix $B = (b_{ij})$ is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1q}B \\ a_{21}B & a_{22}B & & \vdots \\ \vdots & & & \vdots \\ a_{p1}B & & \dots & a_{pq}B \end{bmatrix} \tag{A.20}$$

The Kronecker sum of the $n \times n$ matrix N and the $m \times m$ matrix M is the $nm \times nm$ matrix defined as

$$N \oplus M = N \otimes I_m + I_n \otimes M \tag{A.21}$$

The following identities involving Kronecker products can easily be shown using the definition.

$$(\mu A) \otimes B = A \otimes (\mu B) = \mu(AB)$$

$$(A + B) \otimes C = (A \otimes C) + (B \otimes C)$$

$$A \otimes (B + C) = (A \otimes B) + (A \otimes C)$$

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C$$

$$(A \otimes B)^T = (A^T \otimes B^T)$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

$$A \otimes B = (A \otimes I)(I \otimes B)$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \text{ if } A^{-1} \text{ and } B^{-1}\text{exists.}$$

$$det(A \otimes B) = (det\, A)^n (det\, B)^n$$

$$tr(A \otimes B) = (tr\, A)(\, tr\, B)$$

$$\text{rank}(A \otimes B) = (\text{rank}A)(\text{rank}B)$$

The following theorem states a basic result about the eigenvalues of the Kronecker product of matrices.

Theorem A.3 *(Stéphanos [24])*

Let $\{A_p\}$ and $\{B_q\}$ be finite sets of $n \times n$ and $m \times m$ matrices having eigenvalues λ_i^p, $i = 1, \ldots, n$ and μ_j^q, $j = 1, 2, \ldots, m$. Then, the eigenvalues of the matrix

$$\sum_{p,q} h_{pq} A_p \otimes B_q \tag{A.22}$$

are the nm values $\sum_{p,q} h_{pq} \lambda_i^p \mu_j^q \qquad i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$

Corollary A.1 *Let A and B be square matrices of dimensions n and m, and having eigenvalues λ_i, $i = 1, 2, \ldots, n$ and μ_j, $j = 1, 2, \ldots, m$. Then the matrices $A \otimes B$ and $A \oplus B$ have the eigenvalues $\lambda_i \mu_j$ and $\lambda_i + \mu_j$, $i = 1, 2, \ldots, n$ $j = 1, 2, \ldots, m$, respectively.*

As an illustration of Corollary A.1, consider the 2×2 matrix

$$A = \begin{bmatrix} 0 & 1 \\ -\Pi & \Sigma \end{bmatrix}, \qquad \Sigma = \mu_1 + \mu_2,\ \Pi = \mu_1\mu_2$$

which has the eigenvalues $\lambda_i = \mu_i\ i = 1, 2$. The matrix

$$A \oplus A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -\Pi & \Sigma & 0 & 1 \\ -\Pi & 0 & \Sigma & 1 \\ 0 & -\Pi & -\Pi & 2\Sigma \end{bmatrix}$$

has the characteristic polynomial

$$(s - \Sigma)^2(s^2 - 2\Sigma s + 4\pi),$$

and the eigenvalues

$$\begin{aligned} \lambda_1 &= \Pi + \sqrt{\Pi^2 - 4\Pi} = 2\mu_1 \\ \lambda_2 &= \Sigma = \mu_1 + \mu_2 \\ \lambda_3 &= \Sigma = \mu_1 + \mu_2 \\ \lambda_4 &= \Sigma - \sqrt{\Sigma^2 - 4\Pi} = 2\mu_2, \end{aligned}$$

verifying Corollary A.1.

A.3 M-matrices

A class of matrices, which play an important role in dynamical modelling of economic systems as well as in stability analysis of large-scale systems via composite Liapunov functions, is M-matrices characterized by the following theorem.

Theorem A.4 *(Araki [25])*

Let A be a real square matrix with non-negative off-diagonal elements. Then, the following statements are equivalent.

1. *The principal minors of A are all positive.*

2. *There is a vector x (or y) whose elements are all positive such that the elements of Ax (or $A^T y$) are all positive.*

3. *The leading principal minors of A are all positive.*

4. **A** *is nonsingular and the elements of A^{-1} are all nonnegative.*

5. *(Liapunov-type condition) There is a diagonal matrix D=diag($d_1, \ldots, d_n$) with $d_j > 0$, such that $A^T D + DA$ is a positive definite matrix.*

A matrix A satisfying the above conditions is called an **M**-matrix.

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