EXACT AND APPROXIMATE DECOUPLING AND NONINTERACTING CONTROL PROBLEMS

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Nail Akar
September, 1989

EXACT AND APPROXIMATE DECOUPLING AND NONINTERACTING CONTROL PROBLEMS

A THESIS

SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Nail Akar

September, 1989

Nail Akar

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Bülent Özgüler

Assoc. Prof. Dr. A. Bülent Özgüler(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. M. Erol Sezer

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Erol Emre

Approved for the Institute of Engineering and Sciences:

M. Baray

Prof. Dr. Mehmet Baray
Director of Institute of Engineering and Sciences

ABSTRACT

EXACT AND APPROXIMATE DECOUPLING AND NONINTERACTING CONTROL PROBLEMS

Nail Akar
M.S. in Electrical and Electronics Engineering
Supervisor: Assoc. Prof. Dr. A. Bülent Özgüler
September, 1989

In this thesis, we consider "exact" and "approximate" versions of the disturbance decoupling problem and the noninteracting control problem for linear, time-invariant systems. In the exact versions of these problems, we obtain necessary and sufficient conditions for the existence of an internally stabilizing dynamic output feedback controller such that prespecified interactions between certain sets of inputs and certain sets of outputs are annihilated in the closed-loop system. In the approximate version of these problems we require these interactions to be quenched in the $\mathcal{H}_\infty$ sense, up to any degree of accuracy. The solvability of the noninteracting control problems are shown to be equivalent to the existence of a common solution to two linear matrix equations over a principal ideal domain. A common solution to these equations exists if and only if the equations each have a solution and a bilateral matrix equation is solvable. This yields a system theoretical interpretation for the solvability of the original noninteracting control problem.

Keywords. Multivariable systems; control system synthesis; decoupling; almost decoupling; noninteracting control; internal stability; matrix algebra.

ÖZET

TAM VE YAKLAŞIK AYRIŞTIRMA VE ETKİLEŞİMSİZ DENETİM PROBLEMLERİ

Nail Akar
Elektrik ve Elektronik Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Doç. Dr. A. Bülent Özgüler
Eylül, 1989

Bu tezde doğrusal, zamanla değişmeyen bir dizgede, bozucu etkinin sıfırlanması problemi ile etkileşimsiz denetim probleminin "tam" ve "yaklaşık" türleri ele alınmıştır. Bu problemlerin "tam" türlerinde, iç-kararlı bir kapalı döngü dizge elde etmenin yanısıra, bazı belirli giriş ve çıkış kümeleri arasında önceden belirlenmiş etkileşimleri yok eden bir dinamik çıkış geribeslemesinin varlığı için gerekli ve yeterli koşullar elde edilmiştir. Diğer yandan "yaklaşık" problemlerde bu etkileşimlerin $\mathcal{H}_\infty$ anlamında istenen dereceye kadar bastırılması amaçlanmaktadır. Bakılan etkileşimsiz denetim problemlerinin çözülebilirliğinin iki doğrusal matris denkleminin bazı "esas ideal halkaları" (principal ideal domain) içinde ortak çözümlerinin olmasına eşdeğer olduğu gösterilmiştir. Bu ise, denklemlerin kendi aralarında çözülebilirliğine ve ayrıca iki taraflı ve doğrusal bir matris denkleminin çözülebilirliğine denktir. Bu kullanılarak asıl etkileşimsiz denetim probleminin çözülebilirliğini dizgeler teorisi açısından yorumlayabilmemiz sağlanmıştır.

Anahtar kelimeler. Çok girişli, çok çıkışlı dizgeler; denetim dizgesi sentezi; ayrıştırma; yaklaşık ayrıştırma; etkileşimsiz denetim; iç-kararlılık; matris cebiri.

Contents

Chapter 1

INTRODUCTION

This thesis is concerned with the control via dynamic output feedback of linear, time-invariant, finite dimensional, multivariable systems. The control problems we consider are in the general category of "decoupling problems". In particular, we examine "disturbance decoupling" and "noninteracting control" problems. The names disturbance decoupling and noninteracting control are motivated by quite different applications. However, mathematically, both type of problems might be considered under the same heading since their solutions involve zeroing (or making arbitrarily small) a predetermined set of of transfer matrices.

The problems are posed on a linear system having two types of inputs (*control inputs* and *exogenous inputs*) and two types of outputs (*measurement outputs* and *exogenous outputs*). The control inputs represent the control actions that one can employ to influence the behavior of the system. The exogenous inputs represent either unknown influences acting on the plant or inputs that might be used for further control purposes. The measurement outputs are those outputs which are available as inputs to the controller (compensator). Finally, the exogenous outputs represent the response of the system relevant to the outside world. Naturally, a set of inputs (outputs) can be included in both groups of inputs (outputs).

We now comment on the distinction between "exact" and "approximate" decoupling problems. The exact decoupling problems, broadly speaking, consist of finding a dynamic feedback compensator so that, in the closed-loop system, the undesired interactions between certain sets of exogenous inputs and certain sets of exogenous outputs are annihilated (zeroed). In the approximate or (by the now popular usage) "almost" version of these problems, the aim is approximate zeroing of certain transfer matrices instead of exact zeroing. Although there are many alternative ways of quantifying measures of proximity to zero, we shall choose an extreme approach and measure closeness to zero of a transfer matrix by its $\mathcal{H}_\infty$-norm. Moreover, rather than trying to determine a solution which makes this norm as small as possible, we seek conditions under which this norm can be made arbitrarily small. The former problem is one of "optimization" and has occupied a great deal of attention in the recent literature (see, e.g., [1],[2]). The almost decoupling problems we consider, however, turn out to be purely algebraic and has its roots in the works of Willems ([3],[4]).

A fundamental additional requirement in all the decoupling problems we investigate is "internal stability" of the overall system obtained by the interconnection of the plant and the compensator. Internal stability constraint consists of requiring that none of the internal modes of the overall feedback system grow without bound. As is well-known, the constraint of internal stability is essential in all feedback control problems and it forbids any anomaly that might occur when the feedback loop is closed.

The approach we make use of to tackle these control problems is the *stable proper factorization approach* [5]. The central idea of this approach is to represent the transfer matrix of a given system (not necessarily stable) as the ratio of stable proper matrices. One advantage of using this approach is the ease with which the set of internally stabilizing compensators are characterized. Exploiting this, we carry out the following program in obtaining solutions to all the control problems with internal stability. We first parameterize the set of all internally stabilizing compensators in terms of a free parameter and reflect further problem constraints on this free parameter. Most of the further manipulations are directed towards expressing the results in linear matrix equations directly in terms of the problem data so that a system theoretical interpretation becomes transparent.

The following figure will aid the description of the particular decoupling problems we investigate. The figure consists of the feedback configuration which we employ.

Disturbance decoupling problem with internal stability, DDPIS, defined for $N = 2$ in Fig.1 consists of finding an internally stabilizing compensator which decouples

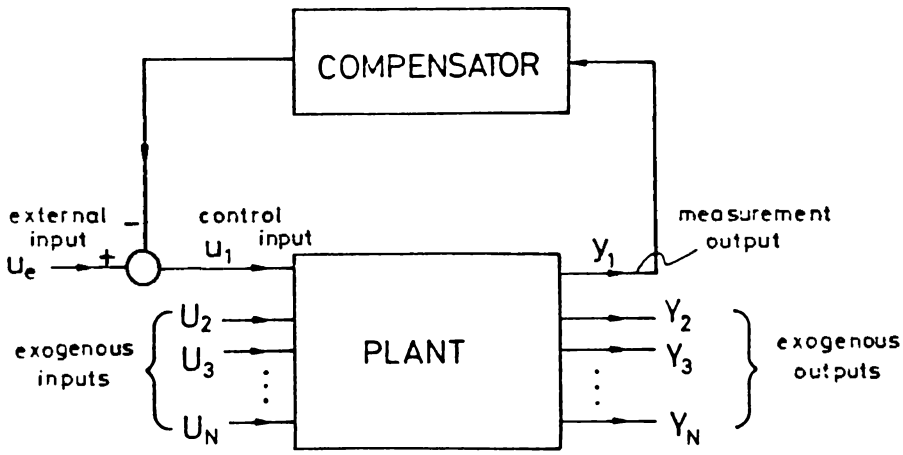

Fig. 1

y_2 (controlled output) from u_2 (disturbance). The *general noninteracting control problem* can be described as follows. Find an internally stabilizing compensator so that the off-diagonal blocks of the closed-loop transfer matrix from the exogenous inputs to the exogenous outputs are annihilated, in other words the closed-loop transfer matrices from u_i to y_j ; $i \neq j$, $1 < i,j \leq N$ are zeroed. We denote this problem for the case $N = 3$ by NICPIS , *noninteracting control problem with internal stability.* In the almost counterparts of DDPIS and NICPIS, abbreviated by ADDPIS and ANICPIS, respectively, we consider almost zeroing of the same transfer matrices. In other words, we seek the conditions under which the $\mathcal{H}_\infty$-norms of these transfer matrices can be made as close to zero as desired by suitable choices of compensation. It should be mentioned that, the descriptions given for the noninteracting control problems above are considerably different from those in the "classical" context of noninteracting control (e.g., [6],[7]). The classical problem of noninteracting control, roughly speaking, can be described as follows: given a plant with a control input and a given number of exogenous outputs, design exogenous input variables, a precompensator having these variables as its inputs, and a compensator from the measured output to the control input so that the closed-loop system is block-diagonal. Some other requirements like output controllability are also imposed on the description of the problems to avoid trivialities. The major distinction between the two set-ups is on the exogenous inputs: in our set-up, they are predetermined and in the classical problem, they are up to the designer's choice.

In the following paragraphs, for each of the problems DDPIS, ADDPIS, NICPIS, and ANICPIS, we describe the relevant results in the literature and the main results of this thesis.

The problem DDPIS has been subject to numerous investigations in the system theory literature. For a full bibliography on this problem, see e.g., [8]. The results of Chapter 4 are mainly restatements of some well-known results on DDPIS in the language of stable proper rational matrices. They are included in this thesis for ease of reference and for being able to contrast with the results obtained for ADDPIS and (A)NICPIS.

We consider ADDPIS for continuous-time systems by taking the stability region as the open left half plane. Different versions of this problem have been solved by geometric techniques in [9] and by frequency domain techniques in [10] and [11]. The constraint of internal stability is with respect to the closed left half plane in [9] and [10], and with respect to the open left half plane in [11]. The basic motivation for our slightly different solution to ADDPIS lies in the fact that, our results are amenable to an easy extension for obtaining a solution to ANICPIS. A relevant remark at this point is that ADDPIS can be viewed as an extreme case of the standard $\mathcal{H}_\infty$-optimization problem. In this optimization problem the purpose is to determine an optimal solution which achieves the infimum cost. On the other hand, ADDPIS can be reformulated as seeking conditions under which the infimum cost is zero. Consequently, a solution to ADDPIS (if it exists) can be obtained by using $\mathcal{H}_\infty$-optimization techniques. Our different approach, however, is still justified since the emphasis here is on determining simple solvability conditions which have interpretations in terms of zeros and poles of the open-loop plant rather than obtaining a solution whenever it exists.

The main motivation for NICPIS is that, if NICPIS is solvable, then we can decompose the overall system into smaller scale subsystems having no interaction among each other. Once this is done by a primary feedback, then this decomposition facilitates the design and implementation of a further feedback law which might be employed for more sophisticated control purposes. The state feedback version of this problem, when full state observation is possible, has been formulated in [12] and has been further developed in [13]. In the measurement feedback version (when the internal stability constraint is absent), this problem has been reduced to the common solvability of a pair of linear matrix equations

$$A_1 = B_1 X C_1 \ , \ A_2 = B_2 X C_2 \tag{1.1}$$

over the ring of proper rational functions, where A_i, B_i, and C_i's $i = 1, 2$ are transfer matrices of various subsystems. Our main result on NICPIS is that, using Theorem 6.1, we reduce the solvability of NICPIS to the solvability of (1.1) over the ring of stable proper rational functions, where A_i, B_i, and C_i's $i = 1, 2$ are now system matrices associated with various subsystems of the system model.

Concerning the almost version ANICPIS, when full state observation is possible, solvability conditions in geometric terms have been obtained in [13]. In the measurement feedback case, when internal stability is not required, the problem (ANICP) has been reduced in [14] to the solvability of (1.1), but this time over the field of rational functions contrary to the exact version of this problem. In our main result on NICPIS, we show that the solvability of the problem is again equivalent to the solvability of equations of the type (1.1) over various relevant rings.

One of the main contributions of this thesis has been the derivation of a set of necessary and sufficient conditions for the solvability of (1.1). Actually, it is well-known that the equations of the type (1.1) can easily be analyzed via the use of Kronecker products and via the theory of the linear vector equation $Ax = b$. This approach, however, leads to an alteration of the given data (i.e., the matrices A_i, B_i, C_i) and makes it difficult to have an intuitive system theoretical interpretation for the solvability for the original problem.

Solvability conditions for these equations, in case all the matrices in (1.1) have elements in a field $\mathcal{F}$ and X is sought over $\mathcal{F}$, have been obtained in [15] and [14]. We show in Theorem 8.1 that, the equations (1.1) have a common solution X over an arbitrary but fixed principal ideal domain $\mathcal{R}$ if and only if they are separately solvable over $\mathcal{R}$ and a matrix equation of the type

$$BX + YC = A \tag{1.2}$$

is solvable over $\mathcal{R}$. The first condition, the separate solvability of these equations, occurs as the solvability condition for DDPIS and the conditions under which an equation $A = BXC$ has a solution is well-known in the literature. The existence of a solution to (1.2) can easily be checked by using the fundamental result of Roth [16]. Such conditions, on the other hand, occur as the solvability conditions for various output regulation and/or tracking problems (see e.g., [17],[18]). The result of Theorem 8.1 constitutes a solution to an open problem posed in [14].

The techniques used in reducing (A)NICPIS to the solvability of a pair of linear matrix equations extend to the general N-channel case. This is part of the objective of Chapter 9. We have, however, not yet been able to derive similar solvability conditions for this general problem to what we have obtained in Theorem 8.1 for the case $N = 3$.

The organization of the material is as follows.

In Chapter 2, we briefly cover the algebraic and analytical background necessary to develop the contents of the subsequent chapters. In Chapter 3, we consider the problem of internal stability and give a parameterization of internally stabilizing compensators. Chapters 4, 5, 6, and 7 are addressed to DDPIS, ADDPIS, NICPIS, and ANICPIS, respectively. The main theme in each chapter is that, we obtain solvability conditions and give synthesis procedures for the solutions of the corresponding problems. Chapter 8 is devoted to an investigation of the equations of the type (1.1). In this chapter, verifiable solvability conditions are stated preserving the structure of matrices occurring in the equation and a procedure for the construction of a solution is given. In the last chapter, we examine the general noninteracting control problem and some special problems relevant to noninteracting control. Our results in Chapters 3 and 4 follow [19] closely and the main results of Chapter 5 is an extension of the main result of [10]. The results of Chapter 8, on the other hand, are in part contained in [20].

Chapter 2

PRELIMINARIES AND NOTATION

The purpose of this chapter is to fix the notation of the thesis and to give some definitions and facts that will be used in the subsequent chapters. Section 2.1 is devoted to algebraic preliminaries concerning matrices over a principal ideal domain (pid). We give certain terminology and facts on some particular matrix norms (Euclidean norm and $\mathcal{H}_\infty$-norm) in Section 2.2.

2.1 Algebraic Preliminaries

In this section, we will mainly consider matrices which have elements from a pid $\mathcal{R}$. We then describe the ring of stable proper rational functions $\mathbf{S}$, which plays a central role in the synthesis problems we investigate. All the facts below, stated without proof, can be found in [21] and [5].

Let $\mathcal{R}$ be a principal ideal domain. If $x \in \mathcal{R}$ has an inverse $y \in \mathcal{R}$ such that $xy = yx = 1$, then x is called a *unit* of $\mathcal{R}$. We say that x divides y if there is an element $z \in \mathcal{R}$ such that $y = xz$ which is denoted by $x|y$. If x and y are elements of $\mathcal{R}$, not both zero, a *greatest common divisor* (gcd) of x and y is any element $d \in \mathcal{R}$ such that (i) $d|x$ and $d|y$ (ii) $c|x, c|y$ implies $c|d$. Now let $\mathcal{R}^{n\times m}$ constitute the set of $n \times m$ matrices whose elements belong to $\mathcal{R}$. A matrix $A \in \mathcal{R}^{n\times m}$ is said to have *rank* l if there is an $l \times l$ nonzero minor of A and every $(l+1) \times (l+1)$ minor of A is zero. If $n = l$ ($m = l$), then A is said to have *full row rank* (*full column rank*). A matrix $U \in \mathcal{R}^{n\times n}$ is called *unimodular* if there exists $U^{-1} \in \mathcal{R}^{n\times n}$ such that $UU^{-1} = U^{-1}U = I$, or equivalently, U is unimodular if $\det(U)$ is a unit in $\mathcal{R}$. A matrix $U \in \mathcal{R}^{n\times m}$ is called *right unimodular* if there is an element $U^\sharp \in \mathcal{R}^{m\times n}$ such that $U^\sharp U = I$. Similarly, U is called *left unimodular* if there exists $U^\sharp \in \mathcal{R}^{m\times n}$ satisfying $UU^\sharp = I$. A matrix $A \in \mathcal{R}^{n\times m}$ is a *left associate* of $B \in \mathcal{R}^{n\times m}$ if there is a unimodular matrix $U \in \mathcal{R}^{n\times n}$ such that $A = UB$. It is a *right associate* of B if there is a unimodular matrix $V \in \mathcal{R}^{m\times m}$ such that $A = BV$. Two matrices $A, B \in \mathcal{R}^{n\times m}$ are called *equivalent* if there exist unimodular matrices U and V such that $A = UBV$.

We now give some facts concerning the standard forms (Hermite and Smith forms) of a matrix $A \in \mathcal{R}^{n\times m}$.

FACT 2.1 (Hermite row form) : The matrix A is a left associate of a matrix of the form

$$\begin{bmatrix} G \\ 0 \end{bmatrix} \text{ if } n > m \ , \ G \text{ if } n = m \ , \ \begin{bmatrix} G & * \end{bmatrix} \text{ if } n < m \ ,$$

where the square matrix G can be chosen to be either upper or lower triangular. The Hermite column form of a matrix can be defined analogously.

FACT 2.2 (Smith form) : If A has rank l, then A is equivalent to a matrix S_A of the form

$$S_A = \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix} ; \hat{A} = diag\{\lambda_1, \lambda_2, \ldots, \lambda_l\} \ ,$$

where λ_i divides λ_{i+1} for $i = 1, 2, \ldots, l-1$. Moreover, $\lambda_1\lambda_2\cdots\lambda_i$ is a greatest common divisor of all $i \times i$ minors of A and the λ_i's are unique up to a multiplication by a unit. We call $\lambda_1, \lambda_2, \ldots, \lambda_l$ as the *invariant factors* of A and in particular we call λ_l as the *largest invariant factor* of A. Using the Smith form of A, one can easily show the existence of unimodular matrices U and V such that

$$UA = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \ , \ AV = \begin{bmatrix} A_2 & 0 \end{bmatrix} \ ,$$

where A_1 is of full row rank and A_2 is of full column rank.

Next, we extend the definition of gcd to the matrix case. If three matrices over $\mathcal{R}$ are in the relation $A = CG$, then G is called a *right divisor* of A and C is called a *left divisor* of A. Let A be a full column rank matrix over $\mathcal{R}$. A *greatest right divisor* of A is a square nonsingular matrix L over $\mathcal{R}$ such that $A = UL$ for a right unimodular U. A *greatest common right divisor* (gcrd) of two matrices A and B is a greatest right divisor of $\begin{bmatrix} A \\ B \end{bmatrix}$. Every pair of matrices A and B with elements in $\mathcal{R}$ have a gcrd G expressible in the form

$$G = PA + QB ,$$

with P and Q over $\mathcal{R}$. If the composite matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is of full column rank, the matrices A and B have a nonsingular gcrd G and every gcrd of A and B is of the form VG where V is unimodular. Two matrices $A \in \mathcal{R}^{n\times m}, B \in \mathcal{R}^{k\times m}$ are called *right coprime* if a gcrd of A and B is unimodular. Suppose A and B are right coprime and $n + k > m$. Also let U be a unimodular matrix such that

$$U \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix} , \tag{2.1}$$

where $\begin{bmatrix} V \\ 0 \end{bmatrix}$ is the Hermite row form of $\begin{bmatrix} A \\ B \end{bmatrix}$. It follows that $V \in \mathcal{R}^{m\times m}$ is a unimodular gcrd of A and B. Exploiting this, one can show the existence of matrices $K_1, K_2, \tilde{A}, \tilde{B}, \tilde{K}_1$, and $\tilde{K}_2$ over $\mathcal{R}$ of appropriate sizes satisfying

$$\begin{bmatrix} K_1 & K_2 \\ \tilde{A} & \tilde{B} \end{bmatrix} \begin{bmatrix} A & \tilde{K}_1 \\ B & \tilde{K}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} . \tag{2.2}$$

A *greatest left divisor* of a matrix, a *greatest common left divisor* of a pair of matrices, and *left coprimeness* can be defined similarly or via matrix transposition.

Let A and B be two matrices over $\mathcal{R}$ of sizes $p \times q$ and $q \times r$, respectively. The ordered pair (A, B) is called *skew-prime* if there are matrices $X \in \mathcal{R}^{q\times p}$ and $Y \in \mathcal{R}^{r\times q}$ such that $XA + BY = I$. It is shown in [22] that, excluding some trivial cases, (A, B) is skew-prime iff there exist matrices $\tilde{B}$ and $\tilde{A}$ over $\mathcal{R}$ such that $AB = \tilde{B}\tilde{A}$ with A and $\tilde{B}$ left coprime and B and $\tilde{A}$ right coprime. The following fact concerns the equations of the type

$$BX + YC = A . \tag{2.3}$$

The condition (ii) below yields a checkable condition for its solvability [16] and the condition (iii) expresses the solvability of the equation in terms of skew-primeness of certain matrices [23].

FACT 2.4 : Let $A \in \mathcal{R}^{l\times m}, B \in \mathcal{R}^{l\times p}$, and $C \in \mathcal{R}^{q\times m}$. The following statements are equivalent.

(i) The matrix equation $BX + YC = A$ is solvable,

(ii) $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ and $\begin{bmatrix} B & A \\ 0 & C \end{bmatrix}$ are equivalent over $\mathcal{R}$,

(iii) The pair $\left(\begin{bmatrix} I & A \\ 0 & C \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \right)$ is skew-prime.

Let $\mathbf{R}(s)$ denote the set of rational functions with coefficients in $\mathbf{R}$, the field of real numbers, in the indeterminate s. Also let $Z \in \mathbf{R}(s)^{p\times m}$. One can express Z in powers of s^{-1} as

$$Z = \sum_{i=-k}^{\infty} A_i s^{-i} , \tag{2.4}$$

for unique matrices A_i in $\mathbf{R}^{p\times m}$ where $A_{-k} \neq 0$. The highest power of s $(= k)$ in this Laurent series expansion of Z is called the *causality degree* of Z and is denoted by $\deg(Z)$. The rational matrix Z is called *proper* if $\deg(Z) \leq 0$ and *strictly proper* if $\deg(Z) < 0$. If for a square rational matrix Z, $\deg(Z) = 0$ and A_0 in the expansion (2.4) is nonsingular, then Z is called *biproper.*

A *stability region* (stability set) Ω is any conjugate symmetric subset of the set of complex numbers $\mathbf{C}$. A rational matrix Z is called Ω-*stable* if the denominator polynomial of every entry of Z has all its roots in the stability region Ω. When the stability region needs to be emphasized, we denote the set of Ω-stable rational functions by $\mathbf{R}(s)_\Omega$, Ω-stable proper rational functions by $\mathbf{R}(s)_{o\Omega}$, and Ω-stable strictly proper rational functions by $\mathbf{R}(s)_{-\Omega}$. When the stability region is arbitrary but fixed, we denote the set of stable proper rational functions by $\mathbf{S}$, the set of proper rational functions by $\mathbf{P}$, and the set of strictly proper rational functions by $\mathbf{SP}$.

The following fact concerns the existence of a *bicoprime factorization* over $\mathbf{S}$ of a given transfer matrix Z over $\mathbf{P}$.

FACT 2.4 : Given any Z over $\mathbf{P}$, there exists a quadruple $\Sigma = (P, Q, R, W)$ over $\mathbf{S}$ satisfying

(i) $\det(Q) \neq 0$ and $Z = PQ^{-1}R + W$,

(ii) P and Q are right coprime,

(iii) Q and R are left coprime.

If these three conditions are satisfied, the quadruple $\Sigma = (P, Q, R, W)$ is called a bicoprime factorization (representation) of Z over $\mathbf{S}$. Actually, any quadruple $\bar{\Sigma} = (\bar{P}, \bar{Q}, \bar{R}, \bar{W})$ satisfying (i) (but not necessarily (ii) and (iii)) is called a *fractional representation* of Z.

Given a fractional representation $\Sigma = (P, Q, R, W)$ associated of a transfer matrix Z, we can define the decoupling zeros of this representation. Given a stability region ω, a complex number $z \in \mathbf{C}$ is called an *(unstable) input decoupling zero* of Σ if z lies outside ω and given any gcld D of Q and R over $\mathbf{S}$, $\det D(z) = 0$. Similarly, $z \in \mathbf{C}$ is called an *(unstable) output decoupling zero* of Σ if z lies outside ω and given any gcrd C of P and Q over $\mathbf{S}$, $\det C(z) = 0$. The representation Σ is bicoprime iff Σ has no input and output decoupling zeros. We shall call $z_1 \in \mathbf{C}$ as an *(unstable) system zero* iff z_1 lies outside ω and is a zero of the largest invariant factor of

$$\Pi = \begin{bmatrix} Q & R \\ -P & W \end{bmatrix}, \tag{2.5}$$

where Π is called the *system matrix* associated with the quadruple (P, Q, R, W). We note that many other definitions of system zeros exist in the literature.

Finally, we give a brief description of the "Kronecker product" of two matrices based on [24]. The description will be given for matrices over $\mathcal{R}$, but it is valid for arbitrary rings.

If $A \in \mathcal{R}^{m\times n}$ and $B \in \mathcal{R}^{p\times q}$, then the *right Kronecker product* of A and B, denoted by $A \otimes B$, is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

For a matrix $A \in \mathcal{R}^{m\times n}$ write $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$ where $A_i \in \mathcal{R}^m$; $i = 1, 2, \ldots, n$. The vector

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$$

is said to be the *vec-function* of A and we denote it by $\vec{A}$. It is the vector formed by stacking the columns of A into one long vector. The fact below demonstrates the relationship between the Kronecker product and the vec-function.

FACT 2.5 : Let $A \in \mathcal{R}^{m\times n}, B \in \mathcal{R}^{k\times s}, X \in \mathcal{R}^{n\times k}$. Then $\Delta := AXB$ if and only if

$$\vec{\Delta} = (B^T \otimes A)\vec{X} .$$

2.2 Preliminaries on Matrix Norms

Let $\mathbf{C}_+, \mathbf{C}_{j\omega}$, and $\mathbf{C}_-$ denote the open right half plane, the $j\omega$-axis, and the open left half plane, respectively. Also let $\mathbf{S}$ denote the set of proper rational functions which are stable with respect to $\mathbf{C}_-$, throughout this section. The $\mathcal{H}_\infty$-norm of a matrix $A \in \mathbf{S}^{p\times m}$ is defined by

$$\|A\|_\infty = \sup_{Re\, s\geq 0} \bar{\sigma}[A(s)] , \tag{2.6}$$

where $\bar{\sigma}(B)$ denotes the *largest singular value* of a matrix B (i.e., the square root of the largest eigenvalue of the symmetric matrix B^*B where $*$ stands for conjugate transpose). Recall that the Euclidean norm of a vector x in $\mathbf{C}^n$ is defined by

$$\|x\|_2 = (x^*x)^{1/2} .$$

If $B \in \mathbf{C}^{p\times m}$, then its Euclidean induced norm is defined by

$$\|B\|_2 = \sup_{x\in\mathbf{C}^m - 0} \frac{\|Bx\|_2}{\|x\|_2}$$

and equals $\bar{\sigma}(B)$. Thus, the $\mathcal{H}_\infty$-norm of a matrix $A \in \mathbf{S}^{p\times m}$ can also be defined by

$$\|A\|_\infty = \sup_{Re\, s\geq 0} \|A(s)\|_2 . \tag{2.7}$$

A significant fact on the $\mathcal{H}_\infty$-norm of a matrix A over S is that, the norm of A can be computed based on the behavior of $A(\cdot)$ on the $j\omega$-axis only. This is formally shown as follows:

$$\|A\|_\infty = \sup_{Res\geq 0} \bar{\sigma}[A(s)] = \sup_{\omega\in\mathbf{R}} \bar{\sigma}[A(j\omega)]\,. \tag{2.8}$$

Finally, we consider the *inner-outer factorization* of a matrix $A \in \mathbf{S}^{n\times m}$. Let $G^{\sim}(s)$ denote $G(-s)^T$, the transpose of $G(-s)$. A matrix $G \in \mathbf{S}^{n\times m}$ is called *inner* if $G^{\sim}(s)G(s) = I$ and *outer* if $rankG(s) = n, \;\; \forall s \in \mathbf{C}_+$, or equivalently, if G has a right inverse which is analytic in $\mathbf{C}_+$.

FACT 2.6 : Suppose $A \in \mathbf{S}^{n\times m}$ and has rank $\min(n,m)$. Then A has a factorization $A = A_iA_o$ where A_i is inner and A_o is outer. If $n \geq m$, then A_o is square, whereas if $n \leq m$, then A_i is square.

An important property of inner matrices is that left multiplication by an inner matrix preserves $\mathcal{H}_\infty$-norms. That is, given $F \in \mathbf{S}^{m\times k}$ and given an inner matrix $G \in \mathbf{S}^{n\times m}$ there holds

$$\|GF\|_\infty = \|F\|_\infty.$$

Chapter 3

INTERNAL STABILITY PROBLEM

In this chapter, we consider the internal stability problem from a fractional viewpoint. In the first section, we define the internal stability of a feedback loop consisting of a plant and a compensator. In the second and the third sections of the chapter, we obtain a convenient characterization of all internally stabilizing compensators.

3.1 Stability of a Feedback Loop

In this section, we are concerned with the internal stability of a feedback loop consisting of a plant and a compensator. The plant has the transfer matrix representation

$$y = Z_{11}u \ , \tag{3.1}$$

where $Z_{11} \in \mathbf{SP}^{p\times q}$, and the compensator has the transfer matrix representation

$$y_c = Z_c u_c \ , \tag{3.2}$$

where $Z_c \in \mathbf{P}^{q\times p}$. The plant and the compensator are connected in a feedback loop by the laws

$$u = u_e - y_c \ , \ u_c = u_{c\epsilon} + y \ , \tag{3.3}$$

where u_ϵ and u_{ce} are external inputs to the system which may serve as new control inputs in case of additional control applications. The resulting closed-loop system has the transfer matrix representation

$$\begin{bmatrix} y \\ y_c \end{bmatrix} = \begin{bmatrix} Z_{11} - Z_{11}Y_cZ_{11} & -Z_{11}Y_c \\ Y_cZ_{11} & Y_c \end{bmatrix} \begin{bmatrix} u_e \\ u_{ce} \end{bmatrix} , \tag{3.4}$$

where

$$Y_c := Z_c(I + Z_{11}Z_c)^{-1} , \tag{3.5}$$

which is proper by strict properness of Z_{11}. We are now ready to define the internal stability problem as follows.

The pair (Z_{11}, Z_c) *is internally stable if and only if the transfer matrix in* (3.4) *is over* S, *or equivalently, all the four transfer matrices*

$$Y_c, Y_cZ_{11}, Z_{11}Y_c, Z_{11} - Z_{11}Y_cZ_{11} \tag{3.6}$$

are matrices over S.

In order to justify the word "internally" in this definition, we need to examine the implication of this type of stability on the internal modes of the closed-loop system. This can be done by considering the state-space realizations of Z_{11} and Z_c. An alternative way, however, is to examine a suitable fractional representation of the closed-loop system. For this purpose let

$$Z_{11} = P_1Q_{11}^{-1}R_1 + W_{11} , \tag{3.7}$$

$$Z_c = P_cQ_c^{-1}R_c \tag{3.8}$$

be some fractional representations of Z_{11} and Z_c over S. We do not assume at the outset that the fractional representation of Z_{11} is bicoprime.

In fact, the cancellations that occur in the right hand side of (3.7) are of primary importance for the control problems we are going to investigate. We therefore examine this fractional representation closely, identify the possible cancellations, and obtain a natural bicoprime representation for Z_{11} in (3.7). Therefore, let

$$C_1 := gcrd(P_1, Q_{11}) \ , \ D_1 := gcld(Q_{11}, R_1)$$

so that

$$Q_{11} = Q_1C_1 = D_1Q_2 \ , \ P_1 = PC_1 \ , \ R_1 = D_1R_0 \tag{3.9}$$

for a right coprime pair (P, Q_1) and a left coprime pair (Q_2, R_0) of matrices over S. Further, let

$$C := gcrd(P_1, Q_2) \ , \ D := gcld(Q_1, R_1)$$

so that

$$P_1 = P_0C \ , \ Q_2 = Q_0C \ , \ Q_1 = DQ \ , \ R_1 = DR \tag{3.10}$$

for a right coprime pair (P_0, Q_0) and a left coprime pair (Q, R) of matrices over S. By definitions of *gcrd* and *gcld*, it follows that

$$C_1 = C_0C \ , \ D_1 = DD_0$$

for some matrices C_0 and D_0 over S. It follows that

$$QC_0 = D_0Q_0 \ , \tag{3.11}$$

where (Q, D_0) is left coprime and (Q_0, C_0) is right coprime by left coprimeness of the pair (Q, R) and right coprimeness of the pair (P_0, Q_0). Moreover, both of the fractional representations in

$$Z_{11} = P_0Q_0^{-1}R_0 + W = PQ^{-1}R + W \tag{3.12}$$

are bicoprime, where $W := W_{11}$. By (3.11), we have $\det(C_0) = u\det(D_0)$ where u is a unit of S and the unstable zeros of $\det(C_0)$ will be called *the (unstable) input-output decoupling zeros* of (P_1, Q_{11}, R_1, W). Recall that, the unstable zeros of $\det(C_1)$ and $\det(D_1)$ are, respectively, *the output* and *the input decoupling zeros of* (P_1, Q_{11}, R_1, W).

Let

$$PQ^{-1} = Q_l^{-1}R_l \ , \ Q^{-1}R = P_rQ_r^{-1}, \tag{3.13}$$

for some left coprime matrices (Q_l, R_l) and right coprime matrices (P_r, Q_r) over S. Thus, the following two equalities hold.

$$\begin{bmatrix} K & -L \\ R_l & Q_l \end{bmatrix} \begin{bmatrix} Q & N_l \\ -P & M_l \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} , \tag{3.14}$$

and

$$\begin{bmatrix} Q & R \\ -L_r & K_r \end{bmatrix} \begin{bmatrix} M & P_r \\ N & Q_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} , \tag{3.15}$$

for some matrices $K, L, M, N, M_l, N_l, K_r$ and L_r over S. We also have

$$Z_{11} = (PP_r + WQ_r)Q_r^{-1} = Q_l^{-1}(Q_lW + R_lR) \ . \tag{3.16}$$

Moreover, the first representation above is right coprime and the second one is left coprime since we can write

$$Q_l(M_l + PMN_l - WNN_l) + (Q_lW + R_lR)NN_l = I \ , \tag{3.17}$$

$$(K_r + L_r K R - L_r L W)Q_r + L_r L(PP_r + WQ_r) = I\ . \tag{3.18}$$

We shall frequently have to refer to equalities (3.9)-(3.18) in this chapter and in the subsequent chapters.

The previous definition of the internal stability problem is in terms of the compensator Z_c. We are now prepared to give an equivalent definition in terms of the triplet (P_c, Q_c, R_c), in the following lemma.

LEMMA 3.1 : *Given* (3.12), there exists a compensator Z_c such that (Z_{11}, Z_c) *is internally stable if and only if there exists a triplet* (P_c, Q_c, R_c) *with* $P_c \in \mathbf{S}^{q\times s}, Q_c \in \mathbf{S}^{s\times s}, R_c \in \mathbf{S}^{s\times p}$ *and* Q_c *biproper such that*

$$\Phi := \begin{bmatrix} Q & RP_c \\ -R_cP & Q_c + R_cWP_c \end{bmatrix} \tag{3.19}$$

is unimodular. Further, if Z_c *is an internally stabilizing compensator for* Z_{11}, *then the triplet of matrices in any bicoprime fractional representation of* $Z_c = P_cQ_c^{-1}R_c$ *is such that* Φ *is unimodular and , conversely, given any triplet* (P_c, Q_c, R_c) *with* Q_c *biproper and* Φ *is unimodular,* Z_c *defined by* $Z_c := P_cQ_c^{-1}R_c$ *is such that* (Z_{11}, Z_c) *is internally stable.*

Proof : Writing $Z_c = P_cQ_c^{-1}R_c$, it can easily be shown that

$$\begin{bmatrix} Z_{11} - Z_{11}Y_cZ_{11} & Z_{11}Y_c \\ Y_cZ_{11} & -Y_c \end{bmatrix} = \begin{bmatrix} P & WP_c \\ 0 & -P_c \end{bmatrix} \Phi^{-1} \begin{bmatrix} R & 0 \\ R_cW & R_c \end{bmatrix} + \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} . \tag{3.20}$$

Now let (Z_{11}, Z_c) be internally stable so that the left hand side of (3.20) is a matrix over $\mathbf{S}$. Let $P_cQ_c^{-1}R_c$ be any bicoprime fractional representation of Z_c over $\mathbf{S}$. By right coprimeness of the pairs (P, Q) and (P_c, Q_c), if follows that the pair

$$\left(\begin{bmatrix} P & -WP_c \\ 0 & -P_c \end{bmatrix}, \begin{bmatrix} Q & RP_c \\ -R_cP & Q_c + R_cWP_c \end{bmatrix} \right)$$

is right coprime. By left coprimeness of the pairs (Q, R) and (Q_c, R_c), it follows that the pair

$$\left(\begin{bmatrix} Q & RP_c \\ -R_cP & Q_c + R_cWP_c \end{bmatrix}, \begin{bmatrix} R & 0 \\ R_cW & R_c \end{bmatrix} \right)$$

is left coprime. Hence, the representation (3.20) is bicoprime yielding that Φ^{-1} is a matrix over $\mathbf{S}$, or equivalently, Φ is unimodular. Conversely, let a triplet (P_c, Q_c, R_c)

be such that all three matrices are over S and Q_c is biproper. It follows that $Z_c = P_c Q_c^{-1} R_c$ is a matrix over **P**. In case Φ is unimodular, then the right hand side of (3.20) is over S, that is, all the four transfer matrices $Y_c, Z_{11}Y_c, Y_cZ_{11}, Z_{11} - Z_{11}Y_cZ_{11}$ are over S. □

3.2 Solutions to the Internal Stability Problem

In this section, we first construct a solution to the internal stability problem and then we give a characterization of all internally stabilizing compensators, by making use of the factorizations introduced in Section 3.1.

Let us define

$$P_{cr} := NN_l\ ,\ Q_{cr} := M_l + PMN_l - WNN_l\ , \tag{3.21}$$

so that $Q_{cr} = Q_l^{-1} - Z_{11}NN_l$ is biproper by the fact that Q_l is biproper and Z_{11} is strictly proper. Thus, the compensator defined by

$$Z_{cr} := P_{cr}Q_{cr}^{-1} \tag{3.22}$$

is proper and is such that

$$\det(\Phi) = \det \begin{bmatrix} Q & RNN_l \\ -P & M_l + PMN_l \end{bmatrix} = \det \begin{bmatrix} Q & N_l \\ -P & M_l \end{bmatrix}, \tag{3.23}$$

by using suitable column operations. Noting the unimodularity of the last term of (3.23), Z_{cr} defined by (3.21) and (3.22) is an internally stabilizing compensator for Z_{11}. Analogously, it can be shown using (3.15) that the compensator defined by

$$Z_{cl} := Q_{cl}^{-1}R_{cl}\ , \tag{3.24}$$

where

$$Q_{cl} := K_r + L_rKR - L_rLW\ ,\ R_{cl} := L_rL \tag{3.25}$$

is also a stabilizing compensator for Z_{11}.

The method just described above has the advantage of leading us to a characterization of all compensators Z_c such that the pair (Z_{11}, Z_c) is internally stable. We state and prove this result in the following theorem.

THEOREM 3.1 : *The set of all internally stabilizing compensators for Z_{11} is given by one of the following sets :*

$$Z_{cr}(X) := \{(NN_l + Q_rX)(M_l + PMN_l - WNN_l - PP_rX - WQ_rX)^{-1} : X \in \mathbf{S}^{m\times p}\} \tag{3.26}$$

$$Z_{cl}(Y) := \{(K_r + L_rKR - L_rLW + YR_lR + YQ_lW)^{-1}(L_rL - YQ_l) : Y \in \mathbf{S}^{m\times p}\} . \tag{3.27}$$

Proof : Note by Lemma 3.1 that (Z_{11}, Z_{cr}) is internally stable if and only if

$$\Phi_r := \begin{bmatrix} Q_l & (R_lR + Q_lW)P_{cr} \\ -I & Q_{cr} \end{bmatrix} \tag{3.28}$$

is unimodular for any right coprime fractional representation $P_{cr}Q_{cr}^{-1}$ of Z_{cr}. On the other hand, in (3.26), each $Z_{cr}(X)$ is given in a right coprime fraction since

$$Q_l(M_l+PMN_l-WNN_l-WQ_rX-PP_rX)+(Q_lW+R_lR)(NN_l+Q_rX) = I \ , \tag{3.29}$$

by (3.13) and (3.17). Unimodularity of Φ_r can easily be shown by performing suitable elementary operations on Φ_r, in case any element of $Z_{cr}(X)$ is used as the compensator. Consequently, every element in $Z_{cr}(X)$ internally stabilizes Z_{11}. Conversely, given any Z_c which internally stabilizes Z_{11}, let $Z_c = P_{cr}Q_{cr}^{-1}$ be a right coprime fraction for Z_c and note that Φ_r in (3.28) is unimodular by Lemma 3.1. Unimodularity of Φ_r implies that

$$U := Q_lQ_{cr} + (Q_lW + R_lR)P_{cr} \tag{3.30}$$

is also unimodular. Comparing (3.13) and (3.30), we have

$$Q_{cr}U^{-1} - (M_l + PMN_l - WNN_l) = -(PP_r + WQ_r)X \tag{3.31}$$

$$P_{cr}U^{-1} - NN_l = Q_rX \tag{3.32}$$

for some matrix X over $\mathbf{S}$. Now, (3.30) and (3.31) imply that Z_c is in $Z_{cr}(X)$. The fact that $Z_{cl}(Y)$ is an alternative characterization for all internally stabilizing compensators for Z_{11} follows by analogous arguments. □

Chapter 4

DISTURBANCE DECOUPLING PROBLEM

This chapter concerns DDPIS, Disturbance Decoupling Problem with Internal Stability, which is posed for a 2-channel plant. In Section 4.1, the 2-channel plant model is given and DDPIS is defined in terms of the closed-loop system obtained using dynamic compensation by measurement feedback. We state a necessary and sufficient condition for the solvability of DDPIS in terms of the solvability of a linear matrix equation of the type $A = BXC$ in Section 4.2 and we examine the solvability of such matrix equations in Section 4.3.

4.1 System Model and Problem Definition

The basic system model for our two-channel plant is the following input-output model in terms of its transfer matrix Z_p:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Z_p \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{4.1}$$

where $Z_{11} \in \mathbf{P}^{p\times m}$, $Z_{12} \in \mathbf{P}^{p\times n}$, $Z_{21} \in \mathbf{P}^{q\times m}$, and $Z_{22} \in \mathbf{P}^{q\times n}$. We assume that

$$Z_{11} \in \mathbf{SP}^{p\times m}, \tag{4.2}$$

which is a standard simplifying assumption used to avoid complications concerning the well-definedness of the feedback loop when a feedback is applied around the first channel.

This model is widely used for various problems where it is necessary to distinguish between two types of outputs and inputs: the outputs that can be employed for dynamic feedback and those whose behavior need to be changed under feedback, the inputs that can be used for control purposes and those with unwanted influences on the plant. A particular input or output may be included in both of the channels depending on the problem requirements. Motivated by applications, the output vector y_1 is called *the measured output* and y_2 is called *the controlled output*, the input vector u_1 is called *the control input* and u_2 is called *the disturbance input*. Thus, the first channel of the plant is called *the control channel* around which the feedback is applied.

The plant transfer matrix can be represented in matrix fractions over S as

$$Z_p = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} Q_{11}^{-1} \begin{bmatrix} R_1 & R_2 \end{bmatrix} + \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \tag{4.3}$$

where $P_1 \in \mathbf{S}^{p\times r}, P_2 \in \mathbf{S}^{q\times r}, Q_{11} \in \mathbf{S}^{r\times r}, R_1 \in \mathbf{S}^{r\times m}, R_2 \in \mathbf{S}^{r\times n}, W_{11} \in \mathbf{S}^{p\times m}, W_{12} \in \mathbf{S}^{p\times n}, W_{21} \in \mathbf{S}^{q\times m}$ and $W_{22} \in \mathbf{S}^{q\times n}$ with Q_{11} being nonsingular. We assume that this representation is bicoprime, i.e.,

$$([P_1^T \quad P_2^T]^T, Q_{11}) \textit{ is right coprime,}$$
$$(Q_{11}, [R_1 \quad R_2]) \textit{ is left coprime.}$$

In spite of the fact that the overall representation in (4.3) is bicoprime, the representation of the control input-to-measured output subplant

$$Z_{11} = P_1 Q_{11}^{-1} R_1 + W_{11}$$

may not be bicoprime. We now use the same factorizations (3.9) and (3.10) to obtain a bicoprime fractional representation for Z_{11} as in (3.12). Also suppose that (3.13)-(3.18) hold.

Now, define the feedback law

$$u_1 = -Z_c y_1 + u_{e1} , \tag{4.4}$$

where the compensator $Z_c \in \mathbf{P}^{m\times p}$. We then obtain the closed-loop plant

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Z_{11} - Z_{11}Y_cZ_{11} & Z_{12} - Z_{11}Y_cZ_{12} \\ Z_{21} - Z_{21}Y_cZ_{11} & Z_{22} - Z_{21}Y_cZ_{12} \end{bmatrix} \begin{bmatrix} u_{e1} \\ u_2 \end{bmatrix}, \tag{4.5}$$

where the matrix

$$Y_c = (I + Z_c Z_{11})^{-1} Z_c \tag{4.6}$$

is in $\mathbf{P}^{m \times p}$ by (4.2). The solvability of the disturbance decoupling problem will concern the closed-loop transfer matrix

$$Z_{dc} = Z_{22} - Z_{21} Y_c Z_{12} \tag{4.7}$$

between the disturbance input and the controlled output. If the transfer matrix Z_c of the compensator is written in matrix fractions as

$$Z_c = P_c Q_c^{-1} R_c \,, \tag{4.8}$$

where $P_c \in \mathbf{S}^{m \times s}, Q_c \in \mathbf{S}^{s \times s}, R_c \in \mathbf{S}^{s \times p}$, then we can write a natural matrix fractional representation for the closed-loop transfer matrix Z_f of (4.5) as follows:

$$Z_f = \begin{bmatrix} P_1 & -WP_c \\ P_2 & -W_{21}P_c \end{bmatrix} \begin{bmatrix} Q_{11} & R_1 P_c \\ -R_c P_1 & Q_c + R_c W P_c \end{bmatrix}^{-1} \begin{bmatrix} R_1 & R_2 \\ R_c W & R_c W_{12} \end{bmatrix} + \begin{bmatrix} W & W_{12} \\ W_{21} & W_{22} \end{bmatrix} . \tag{4.9}$$

where $W := W_{11}$.

Given the open-loop plant (4.1) *in which* (4.2) *holds,* DDPIS *is determining an internally stabilizing compensator* Z_c *defined by* (4.4) *which decouples the disturbance input from the controlled output. The second condition is expressed by*

$$Z_{dc} = 0 \,, \tag{4.10}$$

where Z_{dc} *is the closed-loop transfer matrix from the disturbance input to the controlled output and is given by* (4.7).

If an internally stabilizing compensator for Z_{11} is applied, then the closed-loop plant Z_f in (4.9) can be expressed as a function of the free parameter X. Employing the right coprime fraction in (3.26) for $Z_{cr}(X)$, the closed-loop transfer matrix between the disturbance input and the controlled output can be written in terms of the free parameter X as

$$Z_{dc} = \begin{bmatrix} P_2 & -W_{21}(NN_l + Q_r X) \end{bmatrix} \begin{bmatrix} Q_{11} & R_1(NN_l + Q_r X) \\ -P_1 & M_l + PMN_l - PP_r X \end{bmatrix}^{-1} \begin{bmatrix} R_2 \\ W_{12} \end{bmatrix} + W_{22} \tag{4.11}$$

Note by (3.9) and (3.10) that

$$Q_{11} = DQC_1 \,, \; R_1 = DR \,, \; P_1 = PC_1 \,. \tag{4.12}$$

By the fact that the representation (4.3) is bicoprime, it follows that (P_2, C_1) is right coprime and (D, R_2) is left coprime. Let us write

$$P_2 C_1^{-1} = \bar{C}_1^{-1} T \ , \ D^{-1} R_2 = S \bar{D}^{-1} \ , \tag{4.13}$$

for left coprime $(\bar{C}_1, T)$ and right coprime $(S, \bar{D})$ over S. Using (3.14) and (3.15), it is easy to verify the following alternative expression for Z_{dc}:

$$Z_{dc} = \bar{C}_1^{-1}(T\Theta_{12} + \Theta_{21}S - \Theta_{21}Q\Theta_{12} + \bar{C}_1 W_{22}\bar{D} - \Omega_{21}X\Omega_{12})\bar{D}^{-1} \ , \tag{4.14}$$

where

$$\Theta_{12} := KS - LW_{12}\bar{D} \ , \ \Theta_{21} := TM - \bar{C}_1 W_{21} N \tag{4.15}$$

and

$$\Omega_{12} := R_l S + Q_l W_{12}\bar{D} \ , \ \Omega_{21} := TP_r + \bar{C}_1 W_{21} Q_r \ . \tag{4.16}$$

The technique of obtaining solutions to DDPIS will be based on reflecting the disturbance decoupling constraint to the free parameter X.

PROPOSITION 4.1 : DDPIS *is solvable if and only if there exists* $X \in \mathrm{S}^{m\times p}$ *satisfying*

$$\Omega_{21}X\Omega_{12} = T\Theta_{12} + \Theta_{21}S - \Theta_{21}Q\Theta_{12} + \bar{C}_1 W_{22}\bar{D} \ , \tag{4.17}$$

where $\Theta_{12}, \Theta_{21}, \Omega_{21}$ and Ω_{12} *are as defined by* (4.15) *and* (4.16).

Proof : If Z_c is a solution to DDPIS, then (Z_{11}, Z_c) is internally stable, in particular. Thus, by Theorem 3.1, there exists $X \in \mathrm{S}^{m\times p}$ such that $Z_c = Z_{cr}(X)$. Now, $Z_{dc}(X)$ is given by (4.11) and by the decoupling property (4.10) of Z_c, X satisfies (4.17). Conversely, given any X satisfying (4.17), let $Z_c := Z_{cr}(X)$ and note, by Theorem 3.1, that (Z_{11}, Z_c) is internally stable. On the other hand, (4.10) immediately follows from (4.14) and (4.17). Therefore, $Z_c = Z_{cr}(X)$ solves DDPIS. □

Although this solvability condition is good enough for all practical purposes, it does not give an idea about the pole-zero structure of the open-loop plant since it is not directly in terms of the problem data. Below we obtain an alternative condition which is devoid of this drawback.

Consider the following system matrices

$$\bar{\Pi}_{12} := \begin{bmatrix} Q & S \\ -P & W_{12}\bar{D} \end{bmatrix}, \bar{\Pi}_{21} := \begin{bmatrix} Q & R \\ -T & \bar{C}_1 W_{21} \end{bmatrix}, \bar{\Pi}_{22} := \begin{bmatrix} Q & S \\ -T & \bar{C}_1 W_{22}\bar{D} \end{bmatrix} \tag{4.18}$$

associated with the fractional representations of transfer matrices $Z_{12}\bar{D}, \bar{C}_1 Z_{21}$, and $\bar{C}_1 Z_{22}\bar{D}$, respectively. By (3.14), (3.15), (4.15), and (4.16), we have

$$\begin{bmatrix} K & -L \\ R_l & Q_l \end{bmatrix} \bar{\Pi}_{12} = \begin{bmatrix} I & \Theta_{12} \\ 0 & \Omega_{12} \end{bmatrix}, \tag{4.19}$$

$$\bar{\Pi}_{21} \begin{bmatrix} M & -P_r \\ N & Q_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\Theta_{21} & \Omega_{21} \end{bmatrix}. \tag{4.20}$$

Note from these that the nontrivial invariant factors of $\bar{\Pi}_{12}$ are the same as those of Ω_{12} and the nontrivial invariant factors of $\bar{\Pi}_{21}$ are the same as those of Ω_{21}. We can now prove the main result of this section.

THEOREM 4.1 : DDPIS *is solvable if and only if there exists* $\hat{X} \in \mathbf{S}^{(r+m)\times(r+p)}$ satisfying

$$\bar{\Pi}_{21}\hat{X}\bar{\Pi}_{12} = \bar{\Pi}_{22}. \tag{4.21}$$

Proof : [Only If] Let DDPIS have a solution so that, by Proposition 4.1, there exists $X \in \mathbf{S}^{m\times p}$ satisfying (4.17). Define

$$\hat{X} := \begin{bmatrix} -P_r \\ Q_r \end{bmatrix} X \begin{bmatrix} R_l & Q_l \end{bmatrix} + \begin{bmatrix} K + M - MQK & MQL - L \\ N - NQK & NQL \end{bmatrix}, \tag{4.22}$$

where M, N, K, L satisfy (3.14) and (3.15). Note that $\hat{X} \in \mathbf{S}^{(r+m)\times(r+p)}$. Moreover,

$$\bar{\Pi}_{21} \begin{bmatrix} -P_r \\ Q_r \end{bmatrix} X \begin{bmatrix} R_l & Q_l \end{bmatrix} \bar{\Pi}_{12} = \begin{bmatrix} 0 & 0 \\ 0 & \Omega_{21} X \Omega_{12} \end{bmatrix}, \tag{4.23}$$

by (4.19) and (4.20), and

$$\bar{\Pi}_{21} \begin{bmatrix} K + M - MQK & MQL - L \\ N - NQK & NQL \end{bmatrix} \bar{\Pi}_{12} = \begin{bmatrix} Q & S \\ -T & \bar{C}_1 W_{22}\bar{D} - \Omega_{21} X \Omega_{12} \end{bmatrix}, \tag{4.24}$$

by (4.17), (4.19), and (4.20). It follows from (4.22), (4.23), and (4.24) that (4.21) holds.

[If] Suppose that (4.21) has a solution $\hat{X}$. Let

$$X := \begin{bmatrix} -L_r & K_r \end{bmatrix} \hat{X} \begin{bmatrix} N_l \\ M_l \end{bmatrix}, \tag{4.25}$$

where the matrices L_r, K_r, N_l, M_l satisfy (3.14) and (3.15). Employing the equalities

$$N_l\Omega_{12} = S - Q\Theta_{12}\ ,\ M_l\Omega_{12} = W_{12}\bar{D} + P\Theta_{12}\ , \tag{4.26}$$

$$\Omega_{21}L_r = T - \Theta_{21}Q\ ,\ \Omega_{21}K_r = \bar{C}_1W_{21} + \Theta_{21}R\ , \tag{4.27}$$

we obtain

$$\begin{aligned}\Omega_{21}X\Omega_{12} &= \begin{bmatrix} \Theta_{21}Q - T & \bar{C}_1W_{21} + \Theta_{21}R \end{bmatrix}\hat{X}\begin{bmatrix} S - Q\Theta_{12} \\ W_{12}\bar{D} + P\Theta_{12} \end{bmatrix} \\ &= \begin{bmatrix} \Theta_{21} & I \end{bmatrix}\bar{\Pi}_{21}\hat{X}\bar{\Pi}_{12}\begin{bmatrix} -\Theta_{12} \\ I \end{bmatrix} \\ &= T\Theta_{12} + \Theta_{21}S - \Theta_{21}Q\Theta_{12} + \bar{C}_1W_{22}\bar{D}\ . \end{aligned} \tag{4.28}$$

Therefore, (4.17) is satisfied by our choice of X in (4.25) implying that DDPIS is solvable, by Lemma 4.1. □

4.2 Two-sided Matrix Equation and Its Solution

We have shown in Section 4.1 that the central solvability condition for DDPIS is the solvability over **S** of a linear matrix equation of the type

$$A = BXC\ . \tag{4.29}$$

Since no special property of the ring $\mathcal{S}$ is required for the development, the following analysis below will be carried out for an arbitrary pid $\mathcal{R}$.

Let $A \in \mathcal{R}^{p\times q}, B \in \mathcal{R}^{p\times r}$ and $C \in \mathcal{R}^{s\times q}$. Also let $M \in \mathcal{R}^{p\times p}$ and $N \in \mathcal{R}^{q\times q}$ be unimodular matrices such that

$$MB = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}\ ,\ CN = \begin{bmatrix} \hat{C} & 0 \end{bmatrix}$$

with $\hat{B}$ of full row rank in $\mathcal{R}^{k\times r}$ and $\hat{C}$ of full column rank in $\mathcal{R}^{s\times l}$, where $k := rank(B)$ and $l := rank(C)$. Set

$$\hat{A} := MAN = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix},$$

partitioned so that $\hat{A}_{11}$ is in $\mathcal{R}^{k\times l}$. Further, let L be a greatest left divisor of $\hat{B}$ and let R be a greatest right divisor of $\hat{C}$ so that

$$\hat{B} = LU \ , \ \hat{C} = VR$$

for a left unimodular U and a right unimodular V.

THEOREM 4.2 : *The equation*

$$A = BXC$$

has a solution X over $\mathcal{R}^{r\times s}$ if and only if

(i) $\hat{A}_{12} = 0 \ , \ \hat{A}_{21} = 0 \ , \ \hat{A}_{22} = 0 \ ,$

(ii) $L^{-1}\hat{A}_{11}R^{-1} \in \mathcal{R}^{k\times l}$.

Proof : Let X be in $\mathcal{R}^{r\times s}$ satisfying $A = BXC$. It follows that

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} X \begin{bmatrix} \hat{C} & 0 \end{bmatrix}$$

which implies (i). Note that $\hat{A}_{11} = \hat{B}X\hat{C}$ which yields $UXV = L^{-1}\hat{A}_{11}R^{-1}$, where the left hand side is over $\mathcal{R}$. Thus, (ii) holds. Conversely, let $U^\sharp \in \mathcal{R}^{r\times k}$ and $V^\sharp \in \mathcal{R}^{l\times s}$ be such that

$$UU^\sharp = I \ , \ V^\sharp V = I \ .$$

On setting

$$X := U^\sharp L^{-1}\hat{A}_{11}R^{-1}V^\sharp$$

and by using (i) and (ii), $A = BXC$ holds with $X \in \mathcal{R}^{r\times s}$. □

Chapter 5

ALMOST DISTURBANCE DECOUPLING PROBLEM

In this section, we will be concerned with ADDPIS, Almost Disturbance Decoupling Problem with Internal Stability, which is a slightly different version of DDPIS examined in Chapter 4. The results of this chapter pertain to continuous-time systems contrary to the results of Chapter 4 where the stability region is arbitrarily chosen. Consequently, we define the particular stability regions ω and Ω as $\omega := \mathbf{C}_- \cup \mathbf{C}_{j\omega}$ and $\Omega := \mathbf{C}_-$ where Ω is the usual stability set for continuous-time systems.

Given the bicoprime fractional representation (4.3) *of* Z_p *defined in* (4.1) *over* $\mathbf{R}(s)_{o\Omega}$, ADDPIS *can be described as follows : Determine the conditions under which for every real number* $\varepsilon > 0$, *there exists a compensator* $Z_c(\varepsilon)$ *which internally* Ω*-stabilizes the plant and for which* $\|Z_{dc}(\varepsilon)\|_\infty \leq \varepsilon$. *Further, give a synthesis procedure for such a compensator* $Z_c(\varepsilon)$ *for a given* $\varepsilon > 0$, *when the problem is solvable.*

To avoid too much technicality, we will have the assumption that $\bar{C}_1$ and $\bar{D}$ defined through (4.13) are unimodular over $\mathbf{R}(s)_{o\Omega}$ which means that the fractional representation of Z_{11} is free of input and output decoupling zeros. Under this assumption, we immediately have the following proposition.

PROPOSITION 5.1 : ADDPIS *is solvable if and only if for any given* $\varepsilon > 0$, *there exists* $X(\varepsilon) \in \mathbf{R}_{o\Omega}^{m \times p}$ *satisfying*

$$\|\Omega_{21}X(\varepsilon)\Omega_{12} - T\Theta_{12} - \Theta_{21}S + \Theta_{21}Q\Theta_{12} - W_{22}\|_\infty \leq \varepsilon \ , \tag{5.1}$$

where $\Theta_{12}, \Theta_{21}, \Omega_{12}$, *and* Ω_{21} *are as defined by* (4.15) *and* (4.16).

Proof : Noting that $\bar{C}_1$ and $\bar{D}$ can be taken as identity matrices of suitable sizes, the proof of the proposition is an immediate consequence of the problem definition.

Instead of using this post-introduced data in Proposition 5.1 in the synthesis of ADDPIS, we can rather deal with system matrices as in DDPIS. This will be carried out by the following lemma.

LEMMA 5.1 : ADDPIS *is solvable if and only if for any given* $\varepsilon > 0$, *there exists* $\hat{X} \in \mathbf{R}(s)_{o\Omega}^{(r+m)\times(r+p)}$ *such that*

$$\|\bar{\Pi}_{21}\hat{X}(\varepsilon)\bar{\Pi}_{12} - \bar{\Pi}_{22}\| \leq \varepsilon \ . \tag{5.2}$$

Proof : [Only If] Let $\varepsilon > 0$ be given and also let ADDPIS have a solution so that, by Proposition 5.1, there exists $X(\varepsilon) \in \mathbf{R}(s)_{o\Omega}^{m\times p}$ satisfying (5.1). Set

$$\hat{X}(\varepsilon) := \begin{bmatrix} -P_r \\ Q_r \end{bmatrix} X(\varepsilon) \begin{bmatrix} R_l & Q_l \end{bmatrix} + \begin{bmatrix} K + M - MQK & MQL - L \\ N - NQK & NQL \end{bmatrix} ,$$

where all the above matrices are defined as in Chapter 3 over $\mathbf{R}(s)_{o\Omega}$. Note that, $\hat{X}(\varepsilon)$ is Ω-stable proper. Moreover,

$$\bar{\Pi}_{21}\hat{X}(\varepsilon)\bar{\Pi}_{12} - \bar{\Pi}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & \Omega_{21}X(\varepsilon)\Omega_{12} + \Theta_{21}Q\Theta_{12} - \Theta_{21}S - T\Theta_{12} - W_{22} \end{bmatrix} ,$$

by (4.8), (4.10), and (4.11). (5.2) immediately follows from here.

[If] Let $\varepsilon > 0$ be given. Also let

$$\tilde{\varepsilon} := \frac{\varepsilon}{\left\| \begin{bmatrix} \Theta_{21} & I \end{bmatrix} \right\|_\infty \left\| \begin{bmatrix} -\Theta_{12} \\ I \end{bmatrix} \right\|_\infty} .$$

By (5.2), there exists $\hat{X}(\tilde{\varepsilon}) \in \mathbf{R}(s)_{o\Omega}^{(r+m)\times(r+p)}$ such that

$$\|\bar{\Pi}_{21}\hat{X}(\tilde{\varepsilon})\bar{\Pi}_{12} - \bar{\Pi}_{22}\|_\infty \leq \tilde{\varepsilon} \ .$$

Define

$$X(\varepsilon) := \begin{bmatrix} -L_r & K_r \end{bmatrix} \hat{X}(\bar{\varepsilon}) \begin{bmatrix} N_l \\ M_l \end{bmatrix},$$

where the matrices L_r, K_r, N_l, and M_l satisfy (3.10) and (3.11). Using (4.17) and (4.18) we obtain

$$\begin{aligned} &\| \ \Omega_{21}X(\varepsilon)\Omega_{12} - T\Theta_{12} - \Theta_{21}S + \Theta_{21}Q\Theta_{12} + W_{22}\|_\infty \\ = &\left\| \begin{bmatrix} \Theta_{21} & I \end{bmatrix} (\bar{\Pi}_{21}\hat{X}(\bar{\varepsilon})\bar{\Pi}_{12} - \bar{\Pi}_{22}) \begin{bmatrix} -\Theta_{12} \\ I \end{bmatrix} \right\|_\infty \\ \leq &\ \varepsilon \ . \end{aligned}$$

This completes the proof of Lemma 5.1.

The lemma below easily follows from Lemma 5.1 and the unimodularity of $\bar{C}_1$ and $\bar{D}$.

LEMMA 5.2 : ADDPIS *is solvable if and only if for any given* $\varepsilon > 0$, *there exists* $\hat{X}(\varepsilon) \in \mathbf{R}_{o\Omega}^{(r+m)\times(r+p)}$ *satisfying*

$$\|\Pi_{21}\hat{X}(\varepsilon)\Pi_{12} - \Pi_{22}\|_\infty \leq \varepsilon \ . \tag{5.3}$$

Before giving solvability conditions for ADDPIS, let us introduce the notation

$$T(\infty) := \lim_{s\to\infty} T(s) \ ,$$

for any proper rational matrix T. Also, let $jw_1, jw_2, \ldots, jw_M$ be distinct and finite zeros of the largest invariant factor of either Π_{21} or Π_{12} on the nonnegative jw-axis. The following theorem is the main result of this section.

THEOREM 5.1 : ADDPIS *is solvable if and only if the following three conditions hold.*

(C1) *There exists a matrix* $X_0 \in \mathbf{R}^{(r+m)\times(r+p)}$ *satisfying*

$$\Pi_{21}(\infty)X_0\Pi_{12}(\infty) = \Pi_{22}(\infty) \ , \tag{5.4}$$

(C2) *For each* w_i, $i = 1, 2, \ldots, M$ *there exists a matrix* $X_{w_i} \in \mathbf{C}^{(r+m)\times(r+p)}$ *satisfying*

$$\Pi_{21}(jw_i)X_{w_i}\Pi_{12}(jw_i) = \Pi_{22}(jw_i) \ , \tag{5.5}$$

(C3) *There exists* $X \in \mathbf{R}(s)_{\omega}^{(r+m)\times(r+p)}$ *such that*

$$\Pi_{21} X \Pi_{12} = \Pi_{22} \ . \tag{5.6}$$

Proof : [Only If] Let ADDPIS be solvable and let $\varepsilon > 0$ be given. By (5.3), there exists a Ω-stable proper matrix $X(\varepsilon)$ such that

$$\|\Pi_{21} X(\varepsilon) \Pi_{12} - \Pi_{22}\|_{\infty} \leq \varepsilon \ ,$$

which yields

$$\|\Pi_{21}(jw) X(\varepsilon) \Pi_{12}(jw) - \Pi_{22}(jw)\|_2 \leq \varepsilon \ , \forall w \in \mathbf{R} \ .$$

Therefore, in particular

$$\|\Pi_{21}(\infty) X(\varepsilon, \infty) \Pi_{12}(\infty) - \Pi_{22}(\infty)\|_2 \leq \varepsilon \ ,$$

where $X(\varepsilon, \infty) := \lim_{s\to\infty} X(\varepsilon)$.

Let M_∞ and N_∞ be real nonsingular matrices with unity $\|\cdot\|_2$ norms so that

$$M_\infty \Pi_{21}(\infty) = \begin{bmatrix} \hat{\Pi}_{21}(\infty) \\ 0 \end{bmatrix} , \ \Pi_{12}(\infty) N_\infty = \begin{bmatrix} \hat{\Pi}_{12}(\infty) & 0 \end{bmatrix} , \tag{5.7}$$

with $\hat{\Pi}_{21}(\infty)$ of full row rank and $\hat{\Pi}_{12}(\infty)$ of full column rank. Set

$$M_\infty \Pi_{22}(\infty) N_\infty = \begin{bmatrix} \hat{\Pi}_1(\infty) & \hat{\Pi}_2(\infty) \\ \hat{\Pi}_3(\infty) & \hat{\Pi}_4(\infty) \end{bmatrix} . \tag{5.8}$$

Using (5.7) and (5.8), one can show that

$$\left\| \begin{bmatrix} \hat{\Pi}_1(\infty) - \hat{\Pi}_{21}(\infty) X(\varepsilon, \infty) \hat{\Pi}_{12}(\infty) & \hat{\Pi}_2(\infty) \\ \hat{\Pi}_3(\infty) & \hat{\Pi}_4(\infty) \end{bmatrix} \right\|_2 \leq \varepsilon \ .$$

Since the above statement is valid for all $\varepsilon > 0$ and $\hat{\Pi}_2(\infty), \hat{\Pi}_3(\infty)$, and $\hat{\Pi}_4(\infty)$ are independent of ε, it is clear that

$$\hat{\Pi}_i(\infty) = 0 \quad i = 2, 3, 4. \tag{5.9}$$

On referring to the solvability condition (i) of Theorem 4.2, (5.8) directly implies (C1). Note that, the solvability condition (ii) of Theorem 4.2 is automatically satisfied since $\mathbf{R}$ is a field rather than a principal ideal domain. The necessity of (C2) can be shown similarly, by following the same steps.

In order to show (C3), let M and N be unimodular matrices over $\mathbf{R}(s)_{o\Omega}$ with unity $\|\cdot\|_\infty$ norms such that

$$M\Pi_{21} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ \Pi_{12}N = \begin{bmatrix} C & 0 \end{bmatrix}, \tag{5.10}$$

with B of full row rank and C of full column rank. Write

$$M\Pi_{22}N := \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \tag{5.11}$$

partitioned so that A_1 has as many rows as B and has as many columns as C. Using (5.10) and (5.11), it is clear that

$$\left\| \begin{bmatrix} A_1 - BX(\varepsilon)C & A_2 \\ A_3 & A_4 \end{bmatrix} \right\|_\infty \leq \varepsilon .$$

Since this is valid for all $\varepsilon > 0$ and A_2, A_3, and A_4 are independent of ε,

$$A_i = 0\,;\ i = 2,3,4. \tag{5.12}$$

Further, consider the inner-outer factorization of B so that we can write

$$B = B_i B_o,\ B_i \text{ inner},\ B_o \text{ outer}. \tag{5.13}$$

Similarly, inner-outer factorization of C' yields

$$C = C_o C_i,\ C_i' \text{ inner},\ C_o' \text{ outer}. \tag{5.14}$$

Using (5.12), (5.13), and (5.14), we have

$$\sup_{w\in\mathbf{R}} \|T_1(jw) - T_2(jw)\|_2 \leq \varepsilon , \tag{5.15}$$

where

$$T_1 := B_i^{\sim} A_1 C_i^{\sim},\ T_2 := B_o X(\varepsilon) C_o . \tag{5.16}$$

We will now show that T_1 may not have any $\mathbf{C}_+$ pole. In order to show this, let the least common multiple of all the denominators of T_1 have $\mathbf{C}_+$ zeros $\sigma_1, \sigma_2, \ldots, \sigma_N$ with multiplicities $m_1, m_2, \ldots, m_N$, respectively. Define

$$g(s) := \frac{\prod_{i=1}^{N}(s-\sigma_i)^{m_i}}{\prod_{i=1}^{N}(s+\sigma_i)^{m_i}}, \tag{5.17}$$

Noting that T_2 is Ω-stable rational, $g(s)[T_1(s) - T_2(s)]$ is analytic in the closed right half plane. Moreover, for any $i \in \{1, 2, \ldots, N\}$ there holds

$$g(s)T_1(s)\,|_{s=\sigma_i} \neq 0 ,$$

$$g(s)T_2(s)\mid_{s=\sigma_i}=0\ .$$

Thus, $g(s)[T_1(s)-T_2(s)]\mid_{s=\sigma_i}$ is nonzero and independent of ε. By the maximum modulus principle it follows that

$$\begin{aligned}\sup_{w\in\mathbf{R}}\|T_1(jw)-T_2(jw)\|_2 &= \sup_{w\in\mathbf{R}}\|g(jw)[T_1(jw)-T_2(jw)\|_2\\ &\geq \|g(s)T_1(s)\|_2\mid_{s=\sigma_i}\ .\end{aligned}$$

This contradicts (5.15), therefore T_1 is free of $\mathbf{C}_+$ poles which implies that T_1 is ω-stable rational. Recalling the left unimodularity of outer matrices over $\mathbf{R}(s)_{o\omega}$ we have

$$B_oB_o^\sharp=I\ ,\ C_o^\sharp C_o=I\ , \tag{5.18}$$

where $B_o^\sharp$ and $C_o^\sharp$ are ω-stable matrices. On letting $X:=B_o^\sharp T_1C_o^\sharp$ which is ω-stable rational and using (5.12), $\Pi_{21}X\Pi_{12}=\Pi_{22}$ holds. This implies (C3) and thus the necessity part of Theorem 5.1 is established.

[If] Before giving a synthesis procedure for the solution of ADDPIS, we need the following lemma.

LEMMA 5.3 : *Let $A\in\mathbf{R}_{-\Omega}^{p\times q}(s), B\in\mathbf{R}_{o\Omega}^{p\times r}(s)$, and $C\in\mathbf{R}_{o\Omega}^{s\times q}(s)$. Also let $\eta:=\|A\|_\infty$. If there exists $X\in\mathbf{R}_\omega^{r\times s}(s)$ such that $\|A-BXC\|_\infty\leq\varepsilon$, then there exists $X'\in\mathbf{R}_\omega^{r\times s}(s)$ with $\deg(X')=\deg(X)-1$ such that*

$$\|A-BX'C\|_\infty\leq 2\varepsilon\ .$$

Proof : Since A is strictly proper, there exists a positive real number R such that

$$\sup_{|w|>R}\bar\sigma[A(jw)]<\varepsilon\ .$$

Now, let λ be a real number satisfying $0<\lambda<\frac{\varepsilon}{R\sqrt{\eta^2-\varepsilon^2}}$ and define

$$f(s):=\frac{1}{1+\lambda s}\ .$$

Note that, $\|f\|_\infty=1$. Setting $X':=fX$ which is ω-stable with $\deg(X')=\deg(X)-1$, it is clear that

$$A-BX'C=f(A-BXC)+(1-f)A$$

and

$$\|A-BX'C\|_\infty\leq\|A-BXC\|_\infty+\|(1-f)A\|_\infty\ . \tag{5.19}$$

Since $(1-f)A$ is an w-stable strictly proper matrix, its $\mathcal{H}_\infty$-norm exists and is determined as follows:

$$\begin{aligned}
\|(1-f)A\|_\infty &= \sup_{w\in\mathbf{R}} |1-f(jw)|\bar{\sigma}[A(jw)]\,, \\
&= \max\left(\sup_{|w|\le R} \sqrt{\tfrac{\lambda^2 w^2}{\lambda^2 w^2+1}}\,\eta \quad , \quad \sup_{|w|>R} \sqrt{\tfrac{\lambda^2 w^2}{\lambda^2 w^2+1}}\,\varepsilon \right)\,, \\
&= \max\left(\frac{\varepsilon}{\eta}\eta\,,\ 1\cdot\varepsilon\right)\,, \\
&= \varepsilon\,.
\end{aligned}$$

Consequently, using (5.19), $\|A-BX'C\|_\infty \le 2\varepsilon$. □

Now, given $\varepsilon > 0$, the first objective is to construct a ω-stable proper rational matrix $\tilde{X}(\varepsilon)$ such that

$$\|\Pi_{21}\tilde{X}(\varepsilon)\Pi_{12} - \Pi_{22}\|_\infty \le \varepsilon_1 = \varepsilon/2\,. \tag{5.20}$$

Suppose that (C1) and (C3) hold. Let $\tau := \deg(X)$. If $\tau \le 0$, then X is ω-stable proper, set $\tilde{X}(\varepsilon) := X$ and we are done. Therefore, assume $\tau > 0$. By (C1), the matrix W defined by

$$W := \Pi_{22} - \Pi_{21}X_0\Pi_{12} \tag{5.21}$$

is Ω-stable and strictly proper. Let $\eta := \|W\|_\infty$. If $\eta \le \varepsilon_1$, then set $\tilde{X}(\varepsilon) := X_0$ satisfying (5.20). Therefore, assume $\eta > \varepsilon_1$ and define $\hat{X} := X - X_0$, $\hat{\varepsilon} := 2^{-\tau}\varepsilon_1$. It is then clear that $\|W - \Pi_{21}\hat{X}\Pi_{12}\|_\infty = 0 \le \hat{\varepsilon}$. Then, applying Lemma 5.3 τ times, we show the existence of $X'(\varepsilon) \in \mathbf{R}(s)_{ow}^{(r+m)\times(r+p)}$ such that

$$\|W - \Pi_{21}X'(\varepsilon)\Pi_{12}\|_\infty = \|\Pi_{22} - \Pi_{21}(X_0 + X'(\varepsilon))\Pi_{12}\|_\infty \le 2^\tau\hat{\varepsilon} = \varepsilon_1\,.$$

By defining $\tilde{X}(\varepsilon) := X_0 + \hat{X}(\varepsilon)$, which is clearly proper and ω-stable, (5.20) is satisfied. Now, given that (5.20) holds, our aim is to construct a Ω-stable proper rational matrix $X(\varepsilon)$ such that

$$\|\Pi_{21}X(\varepsilon)\Pi_{12} - \Pi_{22}\|_\infty \le \varepsilon\,. \tag{5.22}$$

Since $\tilde{X}(\varepsilon)$ defined above may have jw-axis poles, let us define the polynomial n as the least common multiple of all the denominators of $\tilde{X}(\varepsilon)$ and factorize it as

$$n := n_{jw}n_o\,, \tag{5.23}$$

where n_{jw} is monic and has zeros on the jw-axis and n_o has zeros on the open left or right half plane. Without loss of generality, we can assume that n_{jw} is in the form

$$n_{jw} = (s^2+w_1^2)^{\mu_1}(s^2+w_2^2)^{\mu_2}\cdots(s^2+w_M^2)^{\mu_M}\,, \tag{5.24}$$

because, if $\tilde{X}(\varepsilon)$ has a pole different from any zero of the largest invariant factor of Π_{21} or Π_{12}, then we can still find a ω-stable $X'(\varepsilon)$ satisfying (5.20) with corresponding n'_{jw} in the form (5.24). Also, choose a monic polynomial n_δ whose zeros are in $\mathbf{C}_-$ and which converge to those of n_{jw}, as $\delta \to 0$. Finally, let

$$f_\delta := \frac{n_{jw}}{n_\delta} . \tag{5.25}$$

We now need the following lemma in the construction of $X(\varepsilon)$ satisfying (5.22).

LEMMA 5.4 : *Given a real number $w_0 > 0$ and a complex number $\rho := Re^{j\theta}$, $\theta \in [0, 2\pi)$, there exists an Ω-stable proper rational function $q(s)$ such that*

$$q(jw_0) = \rho . \tag{5.26}$$

Proof : The proof is by construction. Choose q as follows:

$$q(s) := \begin{cases} R & \text{if } \theta = 0 \\ R\frac{s-\alpha}{s+\alpha} \ , \ \alpha^2 = \frac{1-\cos\theta}{1+\cos\theta} w_0^2 & \text{if } 0 < \theta < \pi \\ -R & \text{if } \theta = \pi \\ -R\frac{s-\alpha}{s+\alpha} \ , \ \alpha^2 = \frac{1+\cos\theta}{1-\cos\theta} w_0^2 & \text{if } \pi < \theta < 2\pi \end{cases}$$

It is immediate to see that $q(s)$ is Ω-stable proper rational and satisfies (5.26). In case $w_0 = 0$ and ρ is real, simply choose $q(s) := \rho$. The matrix generalization of this result is that, given a real number $w_0 > 0$ and a matrix $R \in \mathbf{C}^{m \times n}$, there exists a Ω-stable proper rational matrix Q such that $Q(jw_0) = R$.

Using Lemma 5.4, it is easy to show that there exist Ω-stable proper rational matrices Q_i, $i = 1, 2, \ldots, M$ such that

$$Q_i(jw_i) = X_{wi}(jw_i + 1)^{2(M-1)} . \tag{5.27}$$

Now, define

$$Q' := \frac{1}{(s+1)^{2(M-1)}} \sum_{i=1}^{M} Q_i \prod_{\substack{j=1 \\ j \neq i}}^{M} \frac{s^2 + w_j^2}{w_j^2 - w_i^2} \tag{5.28}$$

and note that Q' is Ω-stable proper rational and satisfies

$$Q'(jw_i) = X_{w_i} \ , \ Q'(-jw_i) = X_{wi}^* \ , \ i = 1, \ldots, M \tag{5.29}$$

where X_{wi}^* denotes the complex conjugate of X_{wi} without transposition. Now, we are ready to define $X(\varepsilon)$ explicitly so that (5.22) holds. Set

$$X(\varepsilon, \delta) := f_\delta \tilde{X}(\varepsilon) + (1 - f_\delta)Q' , \tag{5.30}$$

which is clearly Ω-stable and proper. Observe that

$$\Pi_{22} - \Pi_{21}X(\varepsilon,\delta)\Pi_{12} = f_\delta(\Pi_{22} - \Pi_{21}\tilde{X}(\varepsilon)\Pi_{12}) + (1 - f_\delta)(\Pi_{22} - \Pi_{21}Q'\Pi_{12}) \ . \quad (5.31)$$

Also note that there exist $K_1 > 0, K_2 > 0$, and $\delta_1 > 0$ such that

$$|1 - f_\delta(jw)| \leq K_1 \ , \ \|\Pi_{22}(jw) - \Pi_{21}(jw)Q'(jw)\Pi_{12}(jw)\|_2 \leq K_2 \ ,$$

$\forall w \in \mathbf{R}$ and $0 < \delta < \delta_1$. Moreover, for each zero w_k of n_{jw}, there exists an open neighborhood Ω_k around w_k on the jw-axis such that within this neighborhood

$$\|\Pi_{22}(jw) - \Pi_{21}(jw)Q'(jw)\Pi_{12}(jw)\|_2 \leq \varepsilon_1/K_1 \ .$$

This is clear by (C3) and (5.29). Now, let Ω be the union of all the sets Ω_k and $\Omega^c = \mathbf{R} - \Omega$. Since Ω^c is compact and does not contain any zeros of f_δ, $|1 - f_\delta(jw)|$ converges to zero uniformly in Ω^c. Hence, there exists $\delta_2 > 0$ such that

$$|1 - f_\delta(jw)| \leq \varepsilon_1/K_2 \ , \ \forall w \in \Omega^c \text{ and } 0 < \delta < \delta_2 \ .$$

On letting $\delta_0 := \min(\delta_1, \delta_2)$ and choosing $X(\varepsilon) := X(\varepsilon, \delta_0)$ in the definition (5.30), we obtain

$$\|\Pi_{22} - \Pi_{21}X(\varepsilon)\Pi_{12}\|_\infty \leq \varepsilon_1 + \varepsilon_1 = \varepsilon \ ,$$

by employing (5.20) and (5.31). This completes the proof of the sufficiency part of Theorem 5.1. □

Chapter 6

NONINTERACTING CONTROL PROBLEM

We now consider what might be considered as the core problem of noninteracting control ; the simplest case of a problem which can be posed for N-channel plants. In this part, we will, particularly be dealing with three channel systems which have two exogenous inputs and two exogenous outputs in addition to a control input and a control output. If a system of this kind is controlled by a dynamic feedback compensator which processes the measurement output, we obtain a closed loop system with two exogenous inputs and two exogenous outputs. The Noninteracting Control Problem with Internal Stability (NICPIS) can be described as follows : *Find an internally stabilizing compensator such that the off-diagonal blocks of the transfer matrix of the closed loop system from the exogenous inputs to the exogenous outputs are identically equal to zero.*

There are many different versions of noninteracting control problems in the literature. It is more or less agreed that, such problems are among the more difficult algebraic control problems in the sense that the solvability conditions are rarely obtained in compact form and even when they are obtained, their implications on the open loop plant structure are not usually clear. However, in the three channel case considered here, the solvability conditions we obtain are conceptually simple yielding intuitive system theoretical interpretations.

In this chapter exact problem definition of NICPIS will be stated and the solvability of NICPIS will be reduced to the simultaneous solvability of a pair of linear matrix equations of the type $A_1 = B_1XC_1$, $A_2 = B_2XC_2$ over $\mathbf{S}$.

The basic system model for our three channel plant is the following input-output model in terms of its transfer matrix Z_p :

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = Z_p \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} , \tag{6.1}$$

where $Z_{11} \in \mathbf{P}^{p_1 \times q_1}, Z_{12} \in \mathbf{P}^{p_1 \times q_2}, Z_{13} \in \mathbf{P}^{p_1 \times q_3}, Z_{21} \in \mathbf{P}^{p_2 \times q_1}, Z_{22} \in \mathbf{P}^{p_2 \times q_2}, Z_{23} \in \mathbf{P}^{p_2 \times q_3}, Z_{31} \in \mathbf{P}^{p_3 \times q_1}, Z_{32} \in \mathbf{P}^{p_3 \times q_2}$ and $Z_{33} \in \mathbf{P}^{p_3 \times q_3}$. We also assume Z_{11} to be strictly proper to avoid complications concerning the well-definedness of the feedback loop when a feedback is applied around the first channel. The output vector y_1 is called the measured output and y_2 and y_3 are called the exogenous outputs. The input vector u_1 is called the control input and u_2 and u_3 are called the exogenous inputs.

Let $Z_c \in \mathbf{P}^{q_1 \times p_1}$ and consider the feedback law

$$u_1 = -Z_c y_1 \tag{6.2}$$

resulting in the closed loop plant

$$\begin{bmatrix} y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \hat{Z}_{22} & \hat{Z}_{23} \\ \hat{Z}_{32} & \hat{Z}_{33} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} Z_{22} - Z_{21}Y_cZ_{12} & Z_{23} - Z_{21}Y_cZ_{13} \\ Z_{32} - Z_{31}Y_cZ_{12} & Z_{33} - Z_{31}Y_cZ_{13} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}, \tag{6.3}$$

where the matrix

$$Y_c := (I + Z_cZ_{11})^{-1}Z_c = Z_c(I + Z_{11}Z_c)^{-1} \tag{6.4}$$

is in $\mathbf{P}^{q_1 \times p_1}$, because Z_{11} is strictly proper. We now give the following definition for NICPIS.

NICPIS *is solvable if and only if there exists an internally stabilizing compensator* $Z_c \in \mathbf{P}^{q_1 \times p_1}$ *such that*

$$(i) \ \hat{Z}_{23} = 0 \, , \tag{6.5}$$

$$(ii) \ \hat{Z}_{32} = 0 \, . \tag{6.6}$$

Let the plant transfer matrix be represented in matrix fractions over S as

$$Z_p = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} Q_{11}^{-1} \begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} + \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix},$$

with $Q_{11} \in \mathbf{S}^{r \times r}$. We assume that the representation is bicoprime, i.e.,

$$([P_1^T \quad P_2^T \quad P_3^T]^T,\ Q_{11}) \text{ is right coprime,}$$

$$(Q_{11},\ [R_1 \quad R_2 \quad R_3]) \text{ is left coprime.}$$

If the transfer matrix Z_c of the compensator is written in matrix fractions over S as $Z_c = P_c Q_c^{-1} R_c$, the closed loop transfer matrix $\hat{Z}$ from the exogenous inputs to the exogenous outputs is written in matrix fractions as

$$\hat{Z} = \begin{bmatrix} P_2 & -W_{21}P_c \\ P_3 & -W_{31}P_c \end{bmatrix} \begin{bmatrix} Q_{11} & R_1 P_c \\ -R_c P_1 & Q_c + R_c W_{11} P_c \end{bmatrix}^{-1} \begin{bmatrix} R_2 & R_3 \\ R_c W_{12} & R_c W_{13} \end{bmatrix} + \begin{bmatrix} W_{22} & W_{23} \\ W_{32} & W_{33} \end{bmatrix}. \tag{6.7}$$

The off-diagonal blocks $\hat{Z}_{23}$ and $\hat{Z}_{32}$ concerning the noninteracting control problem can be written by using (6.7) as

$$\hat{Z}_{23} = \begin{bmatrix} P_2 & -W_{21}P_c \end{bmatrix} \Phi^{-1} \begin{bmatrix} R_3 \\ R_c W_{13} \end{bmatrix} + W_{23}\,, \tag{6.8}$$

$$\hat{Z}_{32} = \begin{bmatrix} P_3 & -W_{31}P_c \end{bmatrix} \Phi^{-1} \begin{bmatrix} R_2 \\ R_c W_{12} \end{bmatrix} + W_{32}\,, \tag{6.9}$$

where

$$\Phi := \begin{bmatrix} Q_{11} & R_1 P_c \\ -R_c P_1 & Q_c + R_c W_{11} P_c \end{bmatrix}. \tag{6.10}$$

Since the internal stability of the pair (Z_{11}, Z_c) is the fundamental requirement in NICPIS, we should rather deal with internally stabilizing compensators for Z_{11}, not with arbitrary compensators. Note that, the set of all internally stabilizing compensators is given by either one of the sets $Z_{cr}(X)$ or $Z_{cl}(Y)$ in (3.26) and (3.27), respectively. Using the characterization of all internally stabilizing compensators as

$$Z_{cr}(X) = (NN_l + Q_r X)(M_l + PMN_l - WNN_l - PP_r X + WQ_r X)^{-1}\,,$$

the closed loop transfer matrices $\hat{Z}_{23}$ and $\hat{Z}_{32}$ can be written in terms of the free parameter X as follows :

$$\hat{Z}_{23} = \begin{bmatrix} P_2 & -W_{21}(NN_l + Q_r X) \end{bmatrix} \begin{bmatrix} Q_{11} & R_1(NN_l + Q_r X) \\ -P_1 & M_l + PMN_l - PP_r X \end{bmatrix}^{-1} \begin{bmatrix} R_3 \\ W_{13} \end{bmatrix} + W_{23}, \tag{6.11}$$

$$\hat{Z}_{32} = \left[\begin{array}{cc} P_3 & -W_{31}(NN_l + Q_rX) \end{array} \right] \left[\begin{array}{cc} Q_{11} & R_1(NN_l + Q_rX) \\ -P_1 & M_l + PMN_l - PP_rX \end{array} \right]^{-1} \left[\begin{array}{c} R_2 \\ W_{12} \end{array} \right] + W_{32}. \tag{6.12}$$

Note by (3.9) and (3.10) that

$$Q_{11} = DQC_1, R_1 = DR, P_1 = PC_1.$$

By the fact that the representation (6.7) is bicoprime, it follows that $\left(\left[\begin{array}{c} P_2 \\ P_3 \end{array} \right], C_1 \right)$ is right coprime and $(D, [R_2 \ \ R_3])$ is left coprime. Let us write

$$\left[\begin{array}{c} P_2 \\ P_3 \end{array} \right] C_1^{-1} = \bar{C}_1^{-1} \left[\begin{array}{c} T_2 \\ T_3 \end{array} \right] , \ D^{-1} \left[\begin{array}{cc} R_2 & R_3 \end{array} \right] = \left[\begin{array}{cc} S_2 & S_3 \end{array} \right] \bar{D}^{-1} , \tag{6.13}$$

for left coprime $\left(\bar{C}_1, \left[\begin{array}{c} T_2 \\ T_3 \end{array} \right] \right)$ and right coprime $([S_2 \ \ S_3], \bar{D})$ over S. Employing (3.14), (3.15), and (5.13) , it is possible to come up with the alternative expression below for $\hat{Z}_{23}$:

$$\hat{Z}_{23} = \bar{C}_1^{-1}(T_2\Theta_{13} + \Theta_{21}S_3 - \Theta_{21}Q\Theta_{13} + \bar{C}_1W_{23}\bar{D} - \Omega_{21}X\Omega_{13})\bar{D}^{-1} \tag{6.14}$$

where

$$\Theta_{13} := KS_3 - LW_{13}\bar{D} \ , \ \Theta_{21} := T_2M - \bar{C}_1W_{21}N \tag{6.15}$$

and

$$\Omega_{13} := R_lS_3 + Q_lW_{13}\bar{D} \ , \ \Omega_{21} := T_2P_r + \bar{C}_1W_{21}Q_r. \tag{6.16}$$

Similarly, one can show that

$$\hat{Z}_{32} = \bar{C}_1^{-1}(T_3\Theta_{12} + \Theta_{31}S_2 - \Theta_{31}Q\Theta_{12} + \bar{C}_1W_{32}\bar{D} - \Omega_{31}X\Omega_{12})\bar{D}^{-1}, \tag{6.17}$$

where

$$\Theta_{12} := KS_2 - LW_{12}\bar{D} \ , \ \Theta_{31} := T_3M - \bar{C}_1W_{31}N \tag{6.18}$$

and

$$\Omega_{12} := R_lS_2 + Q_lW_{12}\bar{D} \ , \ \Omega_{31} := T_3P_r + \bar{C}_1W_{31}Q_r. \tag{6.19}$$

We can now state a first set of solvability conditions for NICPIS. These are in terms of the existence of a common solution to a pair of linear matrix equations of the type $A_i = B_iXC_i$, $i = 1,2$ over S.

LEMMA 6.1 : NICPIS *is solvable if and only if there exists* $X \in \mathbf{S}^{q_1 \times p_1}$ *such that the following two equalities hold:*

$$\Omega_{21} X \Omega_{13} = T_2 \Theta_{13} + \Theta_{21} S_3 - \Theta_{21} Q \Theta_{13} + \bar{C}_1 W_{23} \bar{D},$$
$$\Omega_{31} X \Omega_{12} = T_3 \Theta_{12} + \Theta_{31} S_2 - \Theta_{31} Q \Theta_{12} + \bar{C}_1 W_{32} \bar{D}. \tag{6.20}$$

Proof : If Z_c is a solution to NICPIS, then in particular, the pair (Z_{11}, Z_c) is internally stable. Thus, by Theorem 3.1, there exists X over S such that $Z = Z_{cr}(X)$. Now, $\hat{Z}_{23}(X)$ and $\hat{Z}_{32}(X)$ are given as in (6.14) and (6.17) and by definition of NICPIS, X satisfies (6.20). Conversely, given any X satisfying (6.20), let $Z_c := Z_{cr}(X)$ and note that Z_c internally stabilizes the system. On the other hand, $\hat{Z}_{23} = 0$ and $\hat{Z}_{32} = 0$ immediately follows from (6.14) and (6.17). Therefore, the choice of $Z_c = Z_{cr}(X)$ solves NICPIS.□

In order to eliminate the subsidiary matrices employed in (6.20), we introduce system matrices associated with particular transfer matrices, similar to what we have done while solving DDPIS. Thus, consider the system matrices

$$\bar{\Pi}_{21} = \begin{bmatrix} Q & R \\ -T_2 & \bar{C}_1 W_{21} \end{bmatrix}, \bar{\Pi}_{13} = \begin{bmatrix} Q & S_3 \\ -P & W_{13}\bar{D} \end{bmatrix}, \bar{\Pi}_{23} = \begin{bmatrix} Q & S_3 \\ -T_2 & \bar{C}_1 W_{23}\bar{D} \end{bmatrix} \tag{6.21}$$

and

$$\bar{\Pi}_{31} = \begin{bmatrix} Q & R \\ -T_3 & \bar{C}_1 W_{31} \end{bmatrix}, \bar{\Pi}_{12} = \begin{bmatrix} Q & S_2 \\ -P & W_{12}\bar{D} \end{bmatrix}, \bar{\Pi}_{32} = \begin{bmatrix} Q & S_2 \\ -T_3 & \bar{C}_1 W_{23}\bar{D} \end{bmatrix} \tag{6.22}$$

associated with the suitable fractional representations of the transfer matrices $\bar{C}_1 Z_{21}$, $Z_{13}\bar{D}$, $\bar{C}_1 Z_{23}\bar{D}$ and $\bar{C}_1 Z_{31}$, $Z_{12}\bar{D}$, $\bar{C}_1 Z_{32}\bar{D}$, respectively. We also have

$$\begin{bmatrix} K & -L \\ R_l & Q_l \end{bmatrix} \bar{\Pi}_{13} = \begin{bmatrix} I & \Theta_{13} \\ 0 & \Omega_{13} \end{bmatrix}, \tag{6.23}$$

$$\bar{\Pi}_{21} \begin{bmatrix} M & -P_r \\ N & Q_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\Theta_{21} & \Omega_{21} \end{bmatrix}, \tag{6.24}$$

$$\begin{bmatrix} K & -L \\ R_l & Q_l \end{bmatrix} \bar{\Pi}_{12} = \begin{bmatrix} I & \Theta_{12} \\ 0 & \Omega_{12} \end{bmatrix}, \tag{6.25}$$

and

$$\bar{\Pi}_{31} \begin{bmatrix} M & -P_r \\ N & Q_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\Theta_{31} & \Omega_{31} \end{bmatrix}. \tag{6.26}$$

It follows that the nontrivial invariant factors of $\bar{\Pi}_{13}$, $\bar{\Pi}_{21}$, $\bar{\Pi}_{12}$, and $\bar{\Pi}_{31}$ are the same as those of Ω_{13}, Ω_{21}, Ω_{12}, and Ω_{31} respectively.

We are now ready to prove the main result of this section.

THEOREM 6.1 : NICPIS *is solvable if and only if there exists* $\hat{X} \in \mathbf{S}^{(r+q_1)\times(r+p_1)}$ *such that*

$$\bar{\Pi}_{21}\hat{X}\bar{\Pi}_{13} = \bar{\Pi}_{23}\ ,\ \bar{\Pi}_{31}\hat{X}\bar{\Pi}_{12} = \bar{\Pi}_{32}. \tag{6.27}$$

Proof : Using (6.23), (6.24),(6.25), and (6.26), it is now a matter of straightforward calculation, to show that if X satisfies (6.20), then the matrix

$$\hat{X} := \begin{bmatrix} -P_r \\ Q_r \end{bmatrix} X \begin{bmatrix} R_l & Q_l \end{bmatrix} + \begin{bmatrix} K + M - MQK & MQL - L \\ N - NQK & NQL \end{bmatrix} \tag{6.28}$$

satisfies (6.27), and conversely, if $\hat{X}$ satisfies (6.27), then

$$X := \begin{bmatrix} -L_r & K_r \end{bmatrix} \hat{X} \begin{bmatrix} N_l \\ M_l \end{bmatrix} \tag{6.29}$$

satisfies (6.20). □

Chapter 7

ALMOST NONINTERACTING CONTROL PROBLEM

We now consider the almost version of NICPIS. This problem is called ANICPIS, Almost Noninteracting Control with Internal Stability. For the synthesis of this problem, we will concentrate on the three-channel plant (6.1) and its corresponding matrix fractional representation over $\mathbf{R}(s)_{o\Omega}$, where $\Omega := \mathbf{C}_-$. The stability region $\omega := \mathbf{C}_- \cup \mathbf{C}_{jw}$ employed in the synthesis of ADDPIS, will also be used in determining solvability conditions for ANICPIS.

The basic assumption of Chapter 5 that $\bar{C}_1 = I$ and $\bar{D} = I$ is still valid throughout this chapter. The details of some of the proofs in this section will be omitted since their contents are more or less the same as in the previous sections.

We define ANICPIS as follows: *Determine the conditions under which for every real number $\varepsilon > 0$, there exists a compensator $Z_c(\varepsilon)$ which internally Ω-stabilizes the plant and for which $\|\hat{Z}_{23}(\varepsilon)\|_\infty \leq \varepsilon$, $\|\hat{Z}_{32}(\varepsilon)\|_\infty \leq \varepsilon$, where $\hat{Z}_{23}$ and $\hat{Z}_{32}$ are as defined in* (6.8) *and* (6.9), *respectively. Further give a synthesis procedure for such a compensator $Z_c(\varepsilon)$ for a given $\varepsilon > 0$, when the problem is solvable.*

We now consider (6.14) and (6.17) involving the expressions of $\hat{Z}_{23}$ and $\hat{Z}_{32}$ in terms of the free parameter X. These expressions constitute the set of all transfer matrices obtained when an internally stabilizing compensator is employed. We now have the following proposition which we state without proof.

PROPOSITION 7.1 : ANICPIS *is solvable if and only if for any given* $\varepsilon > 0$, *there exists* $X(\varepsilon) \in \mathbf{R}(s)_{o\Omega}^{q_1 \times p_1}$ *such that the following hold.*

$$\|\Omega_{21}X(\varepsilon)\Omega_{13} - T_2\Theta_{13} - \Theta_{21}S_3 + \Theta_{21}Q\Theta_{13} - W_{23}\|_\infty \leq \varepsilon \ , \tag{7.1}$$

$$\|\Omega_{31}X(\varepsilon)\Omega_{12} - T_3\Theta_{12} - \Theta_{31}S_2 + \Theta_{31}Q\Theta_{12} - W_{32}\|_\infty \leq \varepsilon \ , \tag{7.2}$$

where all the matrices above are defined in (6.13)-(6.19) *over* $\mathbf{R}(s)_{o\Omega}$.

The proof of the proposition is omitted since it is just the same as the proof of Lemma 6.1. We make use of the manipulations in the proofs of Lemma 5.1 and Theorem 6.1 so that the following proposition follows.

PROPOSITION 7.2 : ANICPIS *is solvable if and only if for any given* $\varepsilon > 0$, *there exists* $\tilde{X}(\varepsilon)$ *over* $\mathbf{R}(s)_{o\Omega}$ *such that both of the following hold.*

$$\|\Pi_{21}\tilde{X}(\varepsilon)\Pi_{13} - \Pi_{23}\|_\infty \leq \varepsilon \ , \ \|\Pi_{31}\tilde{X}(\varepsilon)\Pi_{12} - \Pi_{32}\|_\infty \leq \varepsilon \ . \tag{7.3}$$

where $\Pi_{21}, \Pi_{13}, \Pi_{23}, \Pi_{31}, \Pi_{12}$, *and* Π_{32} *are the system matrices associated with the transfer matrices* $Z_{21}, Z_{13}, Z_{23}, Z_{31}, Z_{12}$, *and* Z_{32}, *respectively.*

Before giving our main result on ANICPIS, we introduce some preliminary information which we will require in giving solvability conditions for this problem.

Let $S_1 := \{jw_1, jw_2, \ldots, jw_{m1}\}$ be the set of all distinct and finite zeros of the largest invariant factor of either Π_{21} or Π_{13}, on the nonnegative jw-axis. Also, let $S_2 := \{j\sigma_1, j\sigma_2, \ldots, j\sigma_{m2}\}$ be the set of all such zeros of either Π_{31} or Π_{12} on the same interval. We define $S := S_1 \cap S_2$ and we assume that $S = \{jw_1, jw_2, \ldots, jw_m\}$ where $m \leq \min(m1, m2)$. We are now ready to state the main result of this section.

THEOREM 7.1 : ANICPIS *is solvable if and only if the following three conditions hold.*

(C1) *There exists a matrix* $X_0 \in \mathbf{R}^{(r+q_1)\times(r+p_1)}$ *satisfying*

$$\Pi_{21}(\infty)X_0\Pi_{13}(\infty) = \Pi_{23}(\infty) \ , \ \Pi_{31}(\infty)X_0\Pi_{12}(\infty) = \Pi_{32}(\infty), \tag{7.4}$$

where $[\cdot](\infty) := \lim_{s\to\infty}[\cdot]$.

(C2) *For each* w_i $i = 1,2,\ldots,m$ *there exists a matrix* $X_{w_i} \in \mathbf{C}^{(r+q_1)\times(r+p_1)}$ *such that*

$$\Pi_{21}(jw_i)X_{w_i}\Pi_{13}(jw_i) = \Pi_{23}(jw_i)\ ,\ \Pi_{31}(jw_i)X_{w_i}\Pi_{12}(jw_i) = \Pi_{32}(jw_i), \quad (7.5)$$

(C3) *There exists* $X \in \mathbf{R}(s)_{omega}^{(r+q_1)\times(r+p_1)}$ *such that*

$$\Pi_{21}X\Pi_{13} = \Pi_{23}\ ,\ \Pi_{31}X\Pi_{12} = \Pi_{32}\ . \quad (7.6)$$

Proof : [Only If] Let ANICPIS be solvable and let $\varepsilon > 0$ be given so that there exists a Ω-stable rational matrix $\tilde{X}(\varepsilon)$ such that (7.3) holds.

Let $A \otimes B$ denote the *right Kronecker product* of the matrices A and B, and $\vec{A}$ denote the *vec*-function of the matrix A. Note that if $A \in \mathbf{R}(s)_{o\Omega}^{m\times n}$ with $\|A\|_\infty \le \varepsilon$, then $\|\vec{A}\|_\infty \le mn\varepsilon$. It then follows that

$$\|(\Pi_{13}^T \otimes \Pi_{21})\vec{X}(\varepsilon) - \vec{\Pi}_{23}\|_\infty \le \varepsilon(r+p_2)(r+q_3)\ , \quad (7.7)$$

$$\|(\Pi_{12}^T \otimes \Pi_{31}\vec{X}(\varepsilon) - \vec{\Pi}_{32}\|_\infty \le \varepsilon(r+p_3)(r+q_2)\ . \quad (7.8)$$

Using the triangle inequality in matrix norms, it is clear that

$$\left\| \begin{bmatrix} \Pi_{13}^T \otimes \Pi_{21} \\ \Pi_{12}^T \otimes \Pi_{31} \end{bmatrix} \vec{X}(\varepsilon) - \begin{bmatrix} \vec{\Pi}_{23} \\ \vec{\Pi}_{32} \end{bmatrix} \right\|_\infty \le \varepsilon L\ , \quad (7.9)$$

where $L = 2[(r+p_2)(r+q_3) + (r+p_3)(r+q_2)]$. Since this is true for all $\varepsilon > 0$, we can apply our result on ADDPIS in Theorem 5.1 to have the existence of a ω-stable rational matrix X_1 such that

$$\begin{bmatrix} \Pi_{13}^T \otimes \Pi_{21} \\ \Pi_{12}^T \otimes \Pi_{31} \end{bmatrix} X_1 = \begin{bmatrix} \vec{\Pi}_{23} \\ \vec{\Pi}_{32} \end{bmatrix}\ . \quad (7.10)$$

It is clear that (7.10) implies (C3) on choosing $X \in \mathbf{R}(s)_\omega^{(r+q_1)(r+p_1)}$ such that $\vec{X} = X_1$. The necessity of (C1) and (C2) can be shown by following the the same steps. It is in fact true that, a condition of the type (C1) and (C2) is to be satisfied for all points on the jw-axis, but in order to make the solvability conditions checkable, we reduce them to a finite number of points as in (C1) and (C2).

[If] Before giving a synthesis procedure for the construction of a solution for ANICPIS, we first need the following lemma.

LEMMA 7.1 : Let $A_1 \in \mathbf{R}(s)_{-\Omega}^{p_1 \times q_1}, A_2 \in \mathbf{R}(s)_{-\Omega}^{p_2 \times q_2}, B_1 \in \mathbf{R}(s)_{o\Omega}^{p_1 \times r}, B_2 \in \mathbf{R}(s)_{o\Omega}^{p_2 \times r}, C_1 \in \mathbf{R}(s)_{o\Omega}^{s \times q_1}$, and $C_2 \in \mathbf{R}(s)_{o\Omega}^{s \times q_2}$. Also let $\eta_i := \|A_i\|_\infty$ $i = 1,2$. If there exists $X \in \mathbf{R}(s)_{\omega}^{r \times s}$ such that $\|A_1 - B_1 X C_1\| \leq \varepsilon$ and $\|A_2 - B_2 X C_2\|_\infty \leq \varepsilon$, then there exists ω-stable rational X' with $\deg(X') = \deg(X) - 1$ such that

$$\|A_1 - B_1 X' C_1\|_\infty \leq 2\varepsilon \ , \ \|A_2 - B_2 X' C_2\|_\infty \leq 2\varepsilon \ . \tag{7.11}$$

Proof : Since A_1 and A_2 are strictly proper matrices, there exist positive real numbers ρ_1 and ρ_2 such that

$$\sup_{|w| > \rho_1} \bar{\sigma}[A_1(jw)] < \varepsilon \ , \ \sup_{|w| > \rho_2} \bar{\sigma}[A_2(jw)] < \varepsilon \ .$$

On letting $\rho := \max(\rho_1, \rho_2)$ and $\eta := \max(\eta_1, \eta_2)$, the choice of X' expressed in terms of X, η, ρ, and ε in Lemma 5.3 will satisfy (7.11). Since this verification is quite similar to that of Lemma 5.3, it will be omitted.

Now let $\varepsilon > 0$ be given and suppose that (C1), (C2), and (C3) hold. Our primary aim is to construct a ω-stable matrix $\tilde{X}(\varepsilon)$ such that

$$\|\Pi_{21} \tilde{X}(\varepsilon) \Pi_{13} - \Pi_{23}\|_\infty \leq \varepsilon_1 \ , \ \|\Pi_{31} \tilde{X}(\varepsilon) \Pi_{12} - \Pi_{32}\|_\infty \leq \varepsilon_1 \ . \tag{7.12}$$

where $\varepsilon_1 := \varepsilon/2$. Let $\tau := \deg(X)$, where X satisfies (C3). If $\tau \leq 0$, then X is ω-stable proper, set $\tilde{X}(\varepsilon) := X$ which will clearly yield (7.12). In case $\tau > 0$, the matrices defined by

$$W_1 := \Pi_{23} - \Pi_{21} X_0 \Pi_{13} \ , \ W_2 := \Pi_{32} - \Pi_{31} X_0 \Pi_{12}$$

are Ω-stable and strictly proper. Let $\eta_i := \|W_i\|_\infty$ $i = 1,2$. If $\eta := \max(\eta_1, \eta_2) \leq \varepsilon_1$, then set $\tilde{X}(\varepsilon) := X_0$ satisfying (7.12). Therefore assume $\eta > \varepsilon_1$ and define $Y := X - X_0$, $\hat{\varepsilon} := 2^{-\tau} \varepsilon_1$. It follows that

$$\|W_1 - \Pi_{21} Y \Pi_{13}\|_\infty = \|W_2 - \Pi_{31} Y \Pi_{12}\|_\infty = 0 \leq \hat{\varepsilon}.$$

We can apply Lemma 8.1 τ times to show the existence of a ω-stable proper rational matrix $\tilde{X}(\varepsilon)$ such that (7.12) holds. Now, given that (7.12) holds, our objective is to construct a Ω-stable proper rational matrix $X(\varepsilon)$ satisfying

$$\|\Pi_{21} X(\varepsilon) \Pi_{13} - \Pi_{23}\|_\infty \leq \varepsilon \ , \ \|\Pi_{31} X(\varepsilon) \Pi_{12} - \Pi_{32}\|_\infty \leq \varepsilon \ . \tag{7.13}$$

Considering the construction of $\tilde{X}(\varepsilon)$ out of X, we note that the jw-axis poles of $\tilde{X}(\varepsilon)$ are precisely those of X. We can also assume without loss of generality that the jw-axis poles of X are contained in the set S which has been defined prior to Theorem 7.1. Otherwise, we can still find an ω-stable rational matrix X' satisfying (7.6) and having all its jw-axis poles in the set S. Therefore, let the polynomial p denote the least common multiple of all the denominators of $\tilde{X}(\varepsilon)$ and factorize p as

$$p = p_{jw} p_0$$

where p_{jw} is monic and has all of its zeros in the set S and p_0 has zeros on the open left or the open right half plane. Also choose a monic polynomial p_δ whose zeros are in $\mathbf{C}_-$ and which converge to those of p_{jw}, as $\delta \to 0$.

Using Lemma 5.4, it is easy to show that there exist Ω-stable proper rational matrices M_i, $i = 1, 2, \ldots, m$ such that

$$M_i(jw_i) = X_{wi}(jw_i + 1)^{2(m-1)} . \tag{7.14}$$

Now, define M' in a similar way to the definition of Q' in (5.28) as follows:

$$M' := \frac{1}{(s+1)^{2(m-1)}} \sum_{i=1}^{m} M_i \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{s^2 + w_j^2}{w_j^2 - w_i^2} \tag{7.15}$$

and note that M' is Ω-stable proper rational and satisfies

$$M'(jw_i) = X_{w_i} \ , \ M'(-jw_i) = X_{wi}^* \ , \ i = 1, \ldots, m \tag{7.16}$$

where X_{wi}^* denotes the complex conjugate of X_{wi} without transposition.

We now claim that the choice of

$$X(\varepsilon, \delta) := \frac{p_{jw}}{p_\delta} \tilde{X}(\varepsilon) + (1 - \frac{p_{jw}}{p_\delta}) M'$$

will satisfy (7.13), provided δ is chosen small enough so that

$$\left\| (1 - \frac{p_{jw}}{p_\delta}) M'(jw) \right\|_2 \leq \varepsilon_1 \ , \ \ \forall w \in \mathbf{R} \ .$$

The verification part of this proof will be omitted, since the technical details are the same as those in the sufficiency part of Theorem 5.1.□

Chapter 8

A COMMON SOLUTION TO TWO MATRIX EQUATIONS OVER A PID

In Chapters 6-7, we have encountered the solvability of a pair of linear matrix equations

$$A_1 = B_1XC_1 \ , \ A_2 = B_2XC_2 \ , \tag{8.1}$$

over S for the NICPIS case and over the ring of ω-stable rational functions for the ANICPIS case where the stability set $\omega = \mathbf{C}_- \cup \mathbf{C}_{j\omega}$. Actually, one can use the Kronecker products for reducing the equations (8.1) into one linear matrix equation of the form $Ax = b$ whose solvability is well-known in the literature. However, this approach alters the form of the given matrices in (8.1) causing difficulties in giving system theoretical interpretations for the original control problem. Therefore, we give an alternative solvability condition in order to avoid this difficulty. Note that the main rings of interest of Chapters 6-7 are all principal ideal domains. We therefore focus our attention to the solvability of (8.1) over an arbitrary pid which we denote by $\mathcal{R}$.

Let $A_1 \in \mathcal{R}^{p_1 \times q_1}$, $A_2 \in \mathcal{R}^{p_2 \times q_2}, B_1 \in \mathcal{R}^{p_1 \times r}, B_2 \in \mathcal{R}^{p_2 \times r}, C_1 \in \mathcal{R}^{s \times q_1}$, and $C_2 \in \mathcal{R}^{s \times q_2}$. Also let $M_1 \in \mathcal{R}^{p_1 \times p_1}, M_2 \in \mathcal{R}^{p_2 \times p_2}, N_1 \in \mathcal{R}^{q_1 \times q_1}$ and $N_2 \in \mathcal{R}^{q_2 \times q_2}$ be unimodular matrices such that

$$M_1B_1 = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}, M_2B_2 = \begin{bmatrix} \hat{B}_2 \\ 0 \end{bmatrix}, C_1N_1 = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}, C_2N_2 = \begin{bmatrix} \hat{C}_2 & 0 \end{bmatrix}, \quad (8.2)$$

where $\hat{B}_1 \in \mathcal{R}^{k_1 \times r}, \hat{B}_2 \in \mathcal{R}^{k_2 \times r}$ are of full row rank and $\hat{C}_1 \in \mathcal{R}^{s \times l_1}, \hat{C}_2 \in \mathcal{R}^{s \times l_2}$ are of full column rank. Now set

$$\hat{A}_1 := M_1A_1N_1 = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{13} & \hat{A}_{14} \end{bmatrix}, \; \hat{A}_2 := M_2A_2N_2 = \begin{bmatrix} \hat{A}_{21} & \hat{A}_{22} \\ \hat{A}_{23} & \hat{A}_{24} \end{bmatrix}, \quad (8.3)$$

partitioned so that $\hat{A}_{11} \in \mathcal{R}^{k_1 \times l_1}$ and $\hat{A}_{21} \in \mathcal{R}^{k_2 \times l_2}$. Further, let L_1, L_2 be greatest left divisors of $\hat{B}_1, \hat{B}_2$ and R_1, R_2 be greatest right divisors of $\hat{C}_1, \hat{C}_2$, respectively, such that

$$\hat{B}_1 = L_1U_1 \; , \; \hat{B}_2 = L_2U_2 \; , \; \hat{C}_1 = V_1R_1 \; , \; \hat{C}_2 = V_2R_2 \quad (8.4)$$

for some left unimodular U_1, U_2 and right unimodular V_1, V_2. Define

$$W_1 := L_1^{-1}\hat{A}_{11}R_1^{-1} \; , \; W_2 := L_2^{-1}\hat{A}_{21}R_2^{-1} \; . \quad (8.5)$$

Now, we are ready to state the main result of this chapter.

THEOREM 8.1 : *The linear matrix equations*

$$A_1 = B_1XC_1 \; , \; A_2 = B_2XC_2$$

have a common solution X over $\mathcal{R}$ if and only if the following conditions hold.

(C1) $\hat{A}_{i2} = 0 \; , \; \hat{A}_{i3} = 0 \; , \; \hat{A}_{i4} = 0 \; ; \; i = 1,2.$

(C2) $W_i \in \mathcal{R}^{k_i \times l_i} \; ; \; i = 1,2.$

(C3) *There exist $X_1 \in \mathcal{R}^{r \times l_1}, X_2 \in \mathcal{R}^{r \times l_2}, Y_1 \in \mathcal{R}^{k_1 \times s}$ and $Y_2 \in \mathcal{R}^{k_2 \times s}$ such that*

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} + \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} W_1 & 0 \\ 0 & -W_2 \end{bmatrix} . \quad (8.6)$$

Proof: **[Only if]** Suppose $X \in \mathcal{R}^{r \times s}$ is such that (8.1) holds. By Theorem 4.2, this immediately implies (C1) and (C2). It is easy to check with

$$X_1 := XV_1 \; , \; X_2 := 0 \; , \; Y_1 := 0 \; , \; Y_2 := -U_2X$$

equality (8.6) also holds.

[If] Suppose (C1), (C2), and (C3) hold. Employing Theorem 4.2 (C1) and (C2) immediately imply that there exist Z_1 and Z_2 over $\mathcal{R}$ such that

$$U_1 Z_1 V_1 = W_1 \ , \ U_2 Z_2 V_2 = W_2 \ . \tag{8.7}$$

Let $M \in \mathcal{R}^{r\times r}$ and $N \in \mathcal{R}^{s\times s}$ be unimodular matrices such that

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} M = \begin{bmatrix} \tilde{U}_1 & 0 \\ \tilde{U}_2 & 0 \end{bmatrix} \ , \ N \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \\ 0 & 0 \end{bmatrix} \ , \tag{8.8}$$

where $\tilde{U}_1 \in \mathcal{R}^{k_1\times t}, \tilde{U}_2 \in \mathcal{R}^{k_2\times t}, \tilde{V}_1 \in \mathcal{R}^{d\times l_1}, \tilde{V}_2 \in \mathcal{R}^{d\times l_2}$, $\begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{bmatrix}$ is of full column rank and $\begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \end{bmatrix}$ is of full row rank. It is clear that, $\tilde{U}_1, \tilde{U}_2$ are left unimodular and $\tilde{V}_1, \tilde{V}_2$ are right unimodular. Now let

$$M^{-1} Z_1 N^{-1} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{13} & Z_{14} \end{bmatrix} \ , \ M^{-1} Z_2 N^{-1} = \begin{bmatrix} Z_{21} & Z_{22} \\ Z_{23} & Z_{24} \end{bmatrix} \ ,$$

partitioned so that $Z_{11} \in \mathcal{R}^{t\times d}$ and $Z_{21} \in \mathcal{R}^{t\times d}$. By (8.7), they satisfy

$$\tilde{U}_1 Z_{11} \tilde{V}_1 = W_1 \ , \ \tilde{U}_2 Z_{21} \tilde{V}_2 = W_2 \ . \tag{8.9}$$

Defining

$$M^{-1} \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 \\ \tilde{X}_3 & \tilde{X}_4 \end{bmatrix} \ , \ \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} N^{-1} = \begin{bmatrix} \tilde{Y}_1 & \tilde{Y}_3 \\ \tilde{Y}_2 & \tilde{Y}_4 \end{bmatrix} \ , \tag{8.10}$$

partitioned so that $\tilde{X}_1 \in \mathcal{R}^{t\times l_1}, \tilde{X}_2 \in \mathcal{R}^{t\times l_2}, \tilde{Y}_1 \in \mathcal{R}^{k_1\times d}$ and $\tilde{Y}_2 \in \mathcal{R}^{k_2\times d}$, (8.6) and (8.10) yield the equality

$$\begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{bmatrix} \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 \end{bmatrix} + \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{bmatrix} \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} W_1 & 0 \\ 0 & -W_2 \end{bmatrix} . \tag{8.11}$$

Note that, if we can find a common solution $X_s \in \mathcal{R}^{t\times d}$ to

$$\tilde{U}_1 X_s \tilde{V}_1 = W_1 \ , \ \tilde{U}_2 X_s \tilde{V}_2 = W_2$$

by using (8.9) and (8.11), then the matrix $M \begin{bmatrix} X_s & 0 \\ 0 & 0 \end{bmatrix} N$ will be a common solution to the equations (8.1). This is clear by (8.2), (8.3), (8.4), (8.5), and (C1).

Let G be a greatest common right divisor of $\tilde{U}_1$ and $\tilde{U}_2$ so that

$$\tilde{U}_1 = \Theta_1 G \ , \ \tilde{U}_2 = \Theta_2 G \ , \tag{8.12}$$

for some right coprime Θ_1 and Θ_2 over $\mathcal{R}$. Since, $\tilde{U}_1$ and $\tilde{U}_2$ are left unimodular, there exist $\tilde{U}_1^\sharp \in \mathcal{R}^{t\times k_1}$ and $\tilde{U}_2^\sharp \in \mathcal{R}^{t\times k_2}$ such that

$$\tilde{U}_1\tilde{U}_1^\sharp = I \ , \ \tilde{U}_2\tilde{U}_2^\sharp = I \tag{8.13}$$

Setting $\Theta_1^\sharp := G\tilde{U}_1^\sharp$ and $\Theta_2^\sharp := G\tilde{U}_2^\sharp$ will immediately yield

$$\Theta_1\Theta_1^\sharp = I \ , \ \Theta_2\Theta_2^\sharp = I \tag{8.14}$$

It is clear by (8.8) that $k_1 + k_2 \geq t$. In case strict inequality holds, since Θ_1 and Θ_2 are right coprime there exist matrices $K_1 \in \mathcal{R}^{t\times k_1}, K_2 \in \mathcal{R}^{t\times k_2}, \tilde{K}_1 \in \mathcal{R}^{k_1\times k_1+k_2-t}, \tilde{K}_2 \in \mathcal{R}^{k_2\times k_1+k_2-t}, \tilde{\Theta}_1 \in \mathcal{R}^{k_1+k_2-t\times k_1}$ and $\tilde{\Theta}_2 \in \mathcal{R}^{k_1+k_2-t\times k_2}$ such that the following identity holds

$$\begin{bmatrix} K_1 & K_2 \\ \tilde{\Theta}_1 & \tilde{\Theta}_2 \end{bmatrix} \begin{bmatrix} \Theta_1 & \tilde{K}_1 \\ \Theta_2 & \tilde{K}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} . \tag{8.15}$$

Similarly, let F be a greatest common left divisor of $\tilde{V}_1$ and $\tilde{V}_2$ so that

$$\tilde{V}_1 = F\Psi_1 \ , \ \tilde{V}_2 = F\Psi_2 \ , \tag{8.16}$$

for some left coprime Ψ_1 and Ψ_2 over $\mathcal{R}$. Since $\tilde{V}_1$ and $\tilde{V}_2$ are left unimodular, there exist $\tilde{V}_1^\sharp \in \mathcal{R}^{l_1\times d}$ and $\tilde{V}_2^\sharp \in \mathcal{R}^{l_2\times d}$ such that

$$\tilde{V}_1^\sharp\tilde{V}_1 = I \ , \ \tilde{V}_2^\sharp\tilde{V}_2 = I \ . \tag{8.17}$$

Setting $\Psi_1^\sharp := \tilde{V}_1^\sharp F$ and $\Psi_2^\sharp := \tilde{V}_2^\sharp F$, we obtain

$$\Psi_1^\sharp\Psi_1 = I \ , \ \Psi_2^\sharp\Psi_2 = I \ . \tag{8.18}$$

Note that by (8.8), $l_1 + l_2 \geq d$. If strict inequality holds, then since Ψ_1 and Ψ_2 are left coprime there exist matrices $L_1 \in \mathcal{R}^{l_1\times d}, L_2 \in \mathcal{R}^{l_2\times d}, \tilde{L}_1 \in \mathcal{R}^{l_1+l_2-d\times l_1}, \tilde{L}_2 \in \mathcal{R}^{l_1+l_2-d\times l_2}, \tilde{\Psi}_1 \in \mathcal{R}^{l_1\times l_1+l_2-d}$ and $\tilde{\Psi}_2 \in \mathcal{R}^{l_2\times l_1+l_2-d}$ such that the following identity holds

$$\begin{bmatrix} \Psi_1 & \Psi_2 \\ \tilde{L}_1 & \tilde{L}_2 \end{bmatrix} \begin{bmatrix} L_1 & \tilde{\Psi}_1 \\ L_2 & \tilde{\Psi}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} . \tag{8.19}$$

The problem can be investigated in two cases as follows :

Case 1: $l_1 + l_2 > d$, $k_1 + k_2 > t$

For this case, we define the following matrices which we will soon make use of :

$$\tilde{X} := \tilde{X}_1 L_1 + \tilde{X}_2 L_2 \,, \tag{8.20}$$

$$\tilde{Y} := K_1 \tilde{Y}_1 + K_2 \tilde{Y}_2 \,, \tag{8.21}$$

$$\hat{X} := \tilde{X}_1 \tilde{\Psi}_1 + \tilde{X}_2 \tilde{\Psi}_2 \,, \tag{8.22}$$

$$\hat{Y} := \tilde{\Theta}_1 \tilde{Y}_1 + \tilde{\Theta}_2 \tilde{Y}_2 \,, \tag{8.23}$$

$$Z := Z_{11} - Z_{21} \,. \tag{8.24}$$

On premultiplying equality (8.11) by $\begin{bmatrix} K_1 & K_2 \\ \tilde{\Theta}_1 & \tilde{\Theta}_2 \end{bmatrix}$, postmultiplying it by $\begin{bmatrix} L_1 & \tilde{\Psi}_1 \\ L_2 & \tilde{\Psi}_2 \end{bmatrix}$ and using (8.20), (8.21), (8.22), (8.23), and (8.24) we obtain the following equalities :

$$G\tilde{X} + \tilde{Y}F = K_1 \Theta_1 G Z_{11} F \Psi_1 L_1 - K_2 \Theta_2 G Z_{21} F \Psi_2 L_2 \,, \tag{8.25}$$

$$G\hat{X} = K_1 \Theta_1 G Z_{11} F \Psi_1 \tilde{\Psi}_1 - K_2 \Theta_2 G Z_{21} F \Psi_2 \tilde{\Psi}_2 \,, \tag{8.26}$$

$$\hat{Y}F = \tilde{\Theta}_1 \Theta_1 G Z_{11} F \Psi_1 L_1 - \tilde{\Theta}_2 \Theta_2 G Z_{21} F \Psi_2 L_2 \,, \tag{8.27}$$

$$0 = \tilde{\Theta}_1 \Theta_1 G Z F \Psi_1 \tilde{\Psi}_1 = \tilde{\Theta}_2 \Theta_2 G Z F \Psi_2 \tilde{\Psi}_2 \,. \tag{8.28}$$

We now show that the matrix

$$X := M \begin{bmatrix} X_s & 0 \\ 0 & 0 \end{bmatrix} N$$

extended so that $X \in \mathcal{R}^{r \times s}$ and where

$$\begin{aligned} X_s \;=\; & Z_{11} + (Z_{11} F \Psi_1 \tilde{\Psi}_1 - \hat{X})(\tilde{L}_2 - \tilde{L}_1 \Psi_1^\sharp \Psi_2)\tilde{V}_2^\sharp \\ & + \tilde{U}_2^\sharp (\tilde{K}_2 - \Theta_2 \Theta_1^\sharp \tilde{K}_1)(\tilde{\Theta}_1 \Theta_1 G Z_{11} - \hat{Y}) \\ & + (\tilde{U}_1^\sharp \tilde{U}_1 - I)(\tilde{X} - Z_{11} F \Psi_1 L_1)(I - \Psi_1 \Psi_1^\sharp)\Psi_2 \tilde{V}_2^\sharp \\ & + \tilde{U}_2^\sharp \Theta_2 (\Theta_1^\sharp \Theta_1 - I)(\tilde{Y} + K_2 \Theta_2 G Z_{11})(I - \tilde{V}_1 \tilde{V}_1^\sharp) \end{aligned} \tag{8.29}$$

is a common solution to the equations (8.1). We first claim that, X_s is a common solution to the following equations

$$\tilde{U}_1 X_s \tilde{V}_1 = W_1 \,, \tag{8.30}$$

$$\tilde{U}_2 X_s \tilde{V}_2 = W_2 \, . \tag{8.31}$$

In order for (8.30) to be satisfied, the last four terms of X_s, when premultiplied by $\tilde{U}_1$ and postmultiplied by $\tilde{V}_1$, should vanish. By (8.12) and (8.16), $\tilde{U}_1 X_s \tilde{V}_1 = \Theta_1 G X_s F \Psi_1$. Note that

$$\begin{aligned} \Theta_1 G(Z_{11} F \Psi_1 \tilde{\Psi}_1 - \hat{X}) &= \Theta_1 K_2 \Theta_2 G Z F \Psi_1 \tilde{\Psi}_1 \, , \\ &= \tilde{K}_1 \tilde{\Theta}_2 \Theta_2 G Z F \Psi_2 \tilde{\Psi}_2 \, , \\ &= 0 \, . \end{aligned}$$

by (8.26), (8.15), and (8.28) respectively. Also

$$\begin{aligned} (\tilde{\Theta}_1 \Theta_1 G Z_{11} - \hat{Y}) F \Psi_1 &= \tilde{\Theta}_1 \Theta_1 G Z F \Psi_2 L_2 \Psi_1 \, , \\ &= \tilde{\Theta}_1 \Theta_1 G Z F \Psi_1 \tilde{\Psi}_1 \tilde{L}_1 \, , \\ &= 0 \, . \end{aligned}$$

by (8.27), (8.19), and (8.28) respectively.

Finally, $\tilde{U}_1(\tilde{U}_1^\sharp \tilde{U}_1 - I) = 0$ by (8.13) and $(I - \tilde{V}_1 \tilde{V}_1^\sharp)\tilde{V}_1 = 0$ by (8.17) implying that $\tilde{U}_1 X_s \tilde{V}_1 = \tilde{U}_1 Z_{11} \tilde{V}_1 = W_1$ by (8.9). Therefore, X_s defined as in (8.29) satisfies (8.30).

Now, consider (8.31). Note that, in order for (8.31) to be satisfied, by referring to (8.9) and (8.24), the last four terms when premultiplied by $\tilde{U}_2$ and postmultiplied by $\tilde{V}_2$, should be equal to $-\tilde{U}_2 Z \tilde{V}_2$. By (8.12) and (8.16), $\tilde{U}_2 X_s \tilde{V}_2 = \Theta_2 G X_s F \Psi_2$. Also note that

$$\begin{aligned} \tilde{U}_2(Z_{11} F \Psi_1 \tilde{\Psi}_1 - \hat{X})(\tilde{L}_2 - \tilde{L}_1 \Psi_1^\sharp \Psi_2) \tilde{V}_2^\sharp \tilde{V}_2 &= \Theta_2 (K_2 \Theta_2 G Z F \Psi_1 \tilde{\Psi}_1)(\tilde{L}_2 - \tilde{L}_1 \Psi_1^\sharp \Psi_2) \, , \\ &= \Theta_2 (-K_2 \Theta_2 G Z F \Psi_1 L_1 - K_2 \Theta_2 G Z F \Psi_1 \Psi_1^\sharp \\ &\quad + K_2 \Theta_2 G Z F \Psi_1 L_1 \Psi_1 \Psi_1^\sharp) \Psi_2 \end{aligned}$$

by (8.12), (8.17), (8.26), and (8.19) respectively. Moreover

$$\begin{aligned} \tilde{U}_2 \tilde{U}_2^\sharp (\tilde{K}_2 - \Theta_2 \Theta_1^\sharp \tilde{K}_1)(\tilde{\Theta}_1 \Theta_1 G Z_{11} - \hat{Y}) \tilde{V}_2 &= (\tilde{K}_2 - \Theta_2 \Theta_1^\sharp \tilde{K}_1)(\tilde{\Theta}_1 \Theta_1 G Z_{11} - \hat{Y}) F \Psi_2 \, , \\ &= (\tilde{K}_2 - \Theta_2 \Theta_1^\sharp \tilde{K}_1)(\tilde{\Theta}_1 \Theta_1 G Z F \Psi_2 L_2) \Psi_2 \, , \\ &= \Theta_2 (-K_1 \Theta_1 G Z F \Psi_2 L_2 - \Theta_1^\sharp \Theta_1 G Z F \Psi_2 L_2 \\ &\quad + \Theta_1^\sharp \Theta_1 K_1 \Theta_1 G Z F \Psi_2 L_2) \Psi_2 \end{aligned}$$

by (8.13), (8.16), (8.27), and (8.15). Further

$$\begin{aligned} & \tilde{U}_2(\tilde{U}_1^\sharp \tilde{U}_1 - I)(\tilde{X} - Z_{11}F\Psi_1 L_1)(I - \Psi_1\Psi_1^\sharp)\Psi_2\tilde{V}_2^\sharp \tilde{V}_2 \\ & +\tilde{U}_2\tilde{U}_2^\sharp\Theta_2(\Theta_1^\sharp\Theta_1 - I)(\tilde{Y} + K_2\Theta_2 G Z_{11})(I - \tilde{V}_1\tilde{V}_1^\sharp)\tilde{V}_2 \\ = \; & \Theta_2\{(\Theta_1^\sharp\Theta_1 - I)[G(\tilde{X} - Z_{11}F\Psi_1 L_1) + (\tilde{Y} + K_2\Theta_2 G Z_{11})F](I - \Psi_1\Psi_1^\sharp)\}\Psi_2 \\ = \; & \Theta_2[(\Theta_1^\sharp\Theta_1 - I)(K_2\Theta_2 G Z F \Psi_2 L_2)(I - \Psi_1\Psi_1^\sharp)]\Psi_2 \end{aligned}$$

by (8.13), (8.14), (8.17), (8.18), (8.25), (8.15), and (8.19) respectively. Finally, by employing (8.15),(8.19), and (8.28) several times, we obtain $\tilde{U}_2 X_s \tilde{V}_2 = \tilde{U}_2(Z_{11} - Z)\tilde{V}_2 = \tilde{U}_2 Z_{21}\tilde{V}_2$ which also equals W_2 by (8.9). Therefore, (8.31) is also satisfied.

Considering the matrix equalities in (8.8), it is clear that, X is a common solution to the following matrix equations

$$U_1 X V_1 = W_1 \, , \; U_2 X V_2 = W_2 \, . \tag{8.32}$$

By using (8.2), (8.3), (8.4), and (8.5) together with (C1), (8.32) immediately implies that X is a common solution to the equations (8.1). □

Case 2: (i) $l_1 + l_2 = d$ and/or (ii) $k_1 + k_2 = t$

In this case, the same X defined as in (8.29) works, provided that in the definition of X_s, second term is dropped if (i) holds, third term is dropped if (ii) holds and both the second and the third terms are dropped if (i) and (ii) together hold. The verification in these cases are even simpler than Case 1 and the details of the proof are omitted.

Remark 8.1 : We note by Theorem 7.1 that the solvability conditions of ANICPIS also consist of the solvability of the equations (8.1) over various fields. We can immediately show that (C1) and (C3) in Theorem 8.1 are necessary and sufficient for the solvability of (8.1) over a field $\mathcal{F}$, where the unknown matrices in (8.6) are sought over $\mathcal{F}$.

Remark 8.2 : Considering the result of Woude [14] for the case $\mathcal{R}$ is a field, it may be natural to expect (for pid case) that (C3) may be replaced by (C4) below.

(C4) *There exist X_1, X_2, Y_1 and Y_2 over $\mathcal{R}$ such that*

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} + \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & -A_2 \end{bmatrix} .$$

Evidently, (C3) implies (C4), which together with (C1) and (C2) is necessary for the solvability of the problem. To show that (C1), (C2) and (C4) are not in general sufficient for the solvability of the problem, let $\mathcal{R} = \mathbf{R}[z]$, the ring of polynomials in the indeterminate z. Let $A_1 := z$, $A_2 := z$, $B_1 := \begin{bmatrix} 1 & z \end{bmatrix}$, $B_2 := \begin{bmatrix} 0.5 & 0 \end{bmatrix}$, $C_1 := \begin{bmatrix} z \\ 0 \end{bmatrix}$ and $C_2 := \begin{bmatrix} z \\ 0 \end{bmatrix}$. Note that, with

$$Z_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } Z_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$A_1 = B_1 Z_1 C_1$ and $A_2 = B_2 Z_2 C_2$. Therefore, (C1) and (C2) are satisfied. Also, by letting $X_1 := \begin{bmatrix} 2z \\ -1 \end{bmatrix}$, $X_2 := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $Y_1 := \begin{bmatrix} 0 & 0 \end{bmatrix}$ and $Y_2 := \begin{bmatrix} -1 & 0 \end{bmatrix}$, (C4) is also satisfied. Now, suppose that there exists a common solution $X \in \mathbf{R}[z]^{2\times 2}$ to the matrix equations (8.1), in the form

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} .$$

By simple manipulations, the unique X_{21} is found to be $\frac{-1}{z}$ which is, indeed not a polynomial in z. This contradicts $X \in \mathbf{R}[z]^{2\times 2}$. □

Remark 8.3 : One of the special cases that would eliminate the complexity of the common solution X stated in (8.29) is the unimodularity of G and F over $\mathcal{R}$. In particular, take $G = I$ and $F = I$. We claim that the matrix

$$X := M \begin{bmatrix} X_s & 0 \\ 0 & 0 \end{bmatrix} N$$

extended so that $X \in \mathcal{R}^{r\times s}$ and where

$$X_s = Z_{11} - Z\Psi_2 L_2 - K_2\Theta_2 Z\Psi_1 L_1 \Psi_2 \Psi_2^\sharp + K_1\Theta_1 Z\Psi_2 L_2 \Psi_1 \Psi_1^\sharp$$

is a common solution to the equations (8.1). To show this

$$\begin{aligned} \Theta_1 X_s \Psi_1 &= W_1 - \Theta_1 Z\Psi_2 L_2 \Psi_1 + \Theta_1 K_1 \Theta_1 Z \Psi_2 L_2 \Psi_1 , \\ &= W_1 - \Theta_1 K_2 \Theta_2 Z \Psi_2 L_2 \Psi_1 , \\ &= W_1 , \end{aligned}$$

by (8.9), (8.15), (8.17), (8.19), and (8.28). Similarly, it is not difficult to find out that $\Theta_2 X_s \Psi_2 = W_2$. It follows trivially using the above equalities that, X is a common solution to the equations (8.1).

Chapter 9

SPECIAL CASES AND EXTENSIONS

In this chapter, we examine certain special cases of NICPIS and the general noninteracting control problem. In Chapters 6-8, we have considered the case $N = 3$ and we have been able to state solvability conditions on the problem data for NICPIS and for its almost version ANICPIS. As a special case, we can define NICPIS for a 2-channel plant. This problem is going to be our main concern in Section 9.1. Moreover, it is easy to see that most of the results of Chapter 6 can be generalized to the case $N > 3$. In Section 9.2, we briefly discuss this general problem.

9.1 Noninteracting Control for a Two Channel Plant

Let our system model for the 2-channel plant be in terms of its transfer matrix Z_p, defined in (4.1). Also, assume that (3.9)-(3.18) hold. Employing the feedback law (4.4), we can write a matrix fractional representation for the two-by-two closed-loop transfer matrix Z_f in (4.9), in terms of the parameters of the compensator. NICPIS *is the problem of determining a compensator Z_c such that the two off-diagonal block matrices are identically equal to zero with the additional requirement of internal stability.* Using the same kind of manipulations as we made in synthesizing NICPIS in Chapter 6, it is not difficult to prove the following proposition.

PROPOSITION 9.1 : NICPIS *is solvable if and only if there exists* $X \in \mathbf{S}^{(r+q_1)\times(r+p_1)}$ *such that*

$$\bar{\Pi}_{12} = \Pi_{11} X \bar{\Pi}_{12} \ , \ \bar{\Pi}_{21} = \bar{\Pi}_{21} X \Pi_{11} \ , \tag{9.1}$$

where

$$\bar{\Pi}_{12} = \begin{bmatrix} Q & S \\ -P & W_{12}\bar{D} \end{bmatrix}, \bar{\Pi}_{21} = \begin{bmatrix} Q & R \\ -T & \bar{C}_1 W_{21} \end{bmatrix}, \Pi_{11} = \begin{bmatrix} Q & R \\ -P & W_{11} \end{bmatrix},$$

and $S, T, \bar{C}_1$, and $\bar{D}$ are defined through the factorizations (4.13). Note that, in Theorem 6.1, we have encountered similar type of equations. However, in (9.1), the two equations are coupled via Π_{11} and the two sides of each equation are dependent on each other. This difference allows us to obtain simpler solvability conditions for this problem in the sense that separate solvability of the two equations will be enough to determine a common solution. This is established in the following theorem.

THEOREM 9.1 : NICPIS *is solvable if and only if there exist matrices* Y_1, Y_2 *over* $\mathbf{S}$ *such that*

$$\bar{\Pi}_{12} = \Pi_{11} Y_1 \bar{\Pi}_{12} \ , \tag{9.2}$$

$$\bar{\Pi}_{21} = \bar{\Pi}_{21} Y_2 \Pi_{11} \ . \tag{9.3}$$

Proof : The necessity part of the proof is obvious.

[If] Assume that (9.2) and (9.3) hold. On premultiplying the equality (9.2) by $\bar{\Pi}_{21} Y_2$, we have

$$\bar{\Pi}_{21}(Y_2 - Y_1)\bar{\Pi}_{12} = 0 \ . \tag{9.4}$$

At this stage, we need the following lemma.

LEMMA 9.1 : Let $A \in \mathbf{S}^{m\times n}, Y \in \mathbf{S}^{n\times r}, B \in \mathbf{S}^{r\times s}$. If $AYB = 0$, then there exist matrices A_k, B_k, Z_1 and Z_2 over $\mathbf{S}$ such that

$$Y = A_k Z_1 + Z_2 B_k \ , \tag{9.5}$$

where $AA_k = 0$ and $B_k B = 0$.

Proof : Let U be a unimodular matrix over $\mathbf{S}$ such that

$$UA = \begin{bmatrix} \bar{A} \\ 0 \end{bmatrix},$$

where $\bar{A}$ is of full row rank. Furthermore, let L be a greatest left divisor of $\bar{A}$ such that $\bar{A} = L\tilde{A}$, with $\tilde{A}$ being left unimodular. Therefore, let $\tilde{A}^\sharp$ denote the right inverse of $\tilde{A}$ such that

$$\tilde{A}\tilde{A}^\sharp = I \, .$$

It is clear that, if $AYB = 0$, then $\tilde{A}YB = 0$. It follows that, there exist matrices Z_0 and B_k satisfying

$$\tilde{A}Y = Z_0 B_k$$

with $B_k B = 0$, which yields $\tilde{A}(Y - \tilde{A}^\sharp Z_0 B_k) = 0$. Therefore, there exist matrices Z_1 and A_k such that

$$Y = A_k Z_1 + \tilde{A}^\sharp Z_0 B_k$$

with $\tilde{A}A_k = 0$. On choosing $Z_2 := \tilde{A}^\sharp Z_0$ and noting that $AA_k = 0$, (9.5) directly follows. □

Using (9.4) and Lemma 9.1, $Y_2 - Y_1$ can be expressed as

$$Y_2 - Y_1 = \bar{\Pi}_{21k} Z_1 + Z_2 \bar{\Pi}_{12k} \, , \tag{9.6}$$

for some matrices $\bar{\Pi}_{21k}, \bar{\Pi}_{12k}, Z_1$ and Z_2 where

$$\bar{\Pi}_{21}\bar{\Pi}_{21k} = 0 \, , \; \bar{\Pi}_{12k}\bar{\Pi}_{12} = 0 \, .$$

Now, we define

$$X := Y_2 - \bar{\Pi}_{21k} Z_1 = Y_1 + Z_2 \bar{\Pi}_{12k} \tag{9.7}$$

so that (9.1) holds. □

9.2 The General Noninteracting Control Problem

In this section, we consider the general noninteracting control problem with internal stability for N-channel plants with $N > 3$. Let the bicoprime fractional representation of the open-loop plant Z_p over S be as follows:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = Z_p \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}$$

and

$$Z_p = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix} Q_{11}^{-1} \begin{bmatrix} R_1 & R_2 & \cdots & R_N \end{bmatrix} + \begin{bmatrix} W_{11} & W_{12} & & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{N1} & W_{N2} & \cdots & W_{NN} \end{bmatrix}, \quad (9.8)$$

where $P_i \in \mathbf{S}^{p_i \times r}$, $i = 1, \ldots, N$, $Q_{11} \in \mathbf{S}^{r \times r}$, $R_i \in \mathbf{S}^{r \times q_i}$, $i = 1, \ldots, N$, $W_{ij} \in \mathbf{S}^{p_i \times q_j}$, $i, j = 1, \ldots, N$. Also assume that (3.9)-(3.18) hold. Now, if the feedback law (4.4) is applied to the system, then we obtain a natural matrix fractional description of the closed-loop transfer matrix Z_f in terms of P_c, Q_c, and R_c as follows:

$$Z_f = \begin{bmatrix} P_2 & -W_{21}P_c \\ P_3 & -W_{31}P_c \\ \vdots & \vdots \\ P_N & -W_{N1}P_c \end{bmatrix} \Phi^{-1} \begin{bmatrix} R_2 & R_3 & \cdots & R_N \\ R_cW_{21} & R_cW_{31} & \cdots & R_cW_{1N} \end{bmatrix} + \begin{bmatrix} W_{21} & W_{22} & \cdots & W_{2N} \\ W_{31} & W_{32} & \cdots & W_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ W_{N1} & W_{N2} & \cdots & W_{NN} \end{bmatrix} \quad (9.9)$$

where

$$\Phi = \begin{bmatrix} Q_{11} & R_1P_c \\ -R_cP_1 & Q_c + R_cW_{11}P_c \end{bmatrix}.$$

Under this set-up, NICPIS can be defined as follows: *Determine the solvability conditions under which there exists an internally stabilizing compensator* $Z_c = P_cQ_c^{-1}R_c$ *such that the off-diagonal blocks of* Z_f *defined in* (9.9) *are identically equal to zero.*

For this purpose, let us define

$$\begin{bmatrix} P_2 \\ P_3 \\ \vdots \\ P_N \end{bmatrix} C_1^{-1} = \hat{C}_1^{-1} \begin{bmatrix} T_2 \\ T_3 \\ \vdots \\ T_N \end{bmatrix},$$

$$D^{-1} \begin{bmatrix} R_2 & R_3 & \cdots & R_N \end{bmatrix} = \begin{bmatrix} S_2 & S_3 & \cdots & S_N \end{bmatrix} \hat{D}^{-1}.$$

Also let us define

$$\bar{\Pi}_{i1} = \begin{bmatrix} Q & R \\ -T_i & \hat{C}_1W_{i1} \end{bmatrix}, \quad i = 2, 3, \ldots, N.$$

$$\bar{\Pi}_{1j} = \begin{bmatrix} Q & S_j \\ -P & W_{1j}\hat{D} \end{bmatrix}, \quad j = 2, 3, \ldots, N.$$

and

$$\bar{\Pi}_{ij} = \begin{bmatrix} Q & S_j \\ -T_i & \hat{C}_1 W_{ij} \hat{D} \end{bmatrix}, \quad i,j = 2,3,\ldots,N.$$

We are now ready to state the main result of this section.

THEOREM 9.2 : NICPIS *is solvable if and only if there exists* $X \in \mathbf{S}^{(r+q_1)\times(r+p_1)}$ *satisfying*

$$\bar{\Pi}_{ij} = \bar{\Pi}_{i1} X \bar{\Pi}_{1j}\ , \quad i,j = 2,3,\ldots,N\ , \quad i \neq j\ . \tag{9.10}$$

Proof : We do not go into the details of the proof, but, instead we give the outline. Since internal stability is the fundamental constraint, first replace P_c by $P_{cr}(X)$, Q_c by $Q_{cr}(X)$, and R_c by I_{p_1} where $Z_{cr}(X) = P_{cr}(X)Q_{cr}(X)^{-1}$ is the set of all internally stabilizing compensators and is defined via (3.26). Then, as an intermediate step, obtain $N^2 - 3N + 2$ equations similar to what we have obtained in (6.20). Here, the necessary manipulations would be exactly the same as we did in proving Lemma 6.1. The final step is to use the same transformations (6.28) and (6.29) to conclude that the solvability of (9.10) is equivalent to the solvability of NICPIS in the general N-channel case. □

Another interesting problem is NICPISDP, Noninteracting Control Problem with Internal Stability and Diagonal Preservation [14]. In this problem, we not only require the diagonal blocks of the open-loop plant Z_p to be preserved after closing the loop, but also we require that NICPIS is solvable. This matching problem can be stated in terms of a matrix equation of the type $A = BXC$. We present this fact by the following proposition which we state without proof.

PROPOSITION 9.2 : NICPISDP *is solvable if and only if there exists* $X \in \mathbf{S}^{(r+q_1)\times(r+p_1)}$ *such that*

$$\begin{bmatrix} \bar{\Pi}_{21} \\ \bar{\Pi}_{31} \\ \vdots \\ \bar{\Pi}_{N1} \end{bmatrix} X \begin{bmatrix} \bar{\Pi}_{12} & \bar{\Pi}_{13} & \cdots & \bar{\Pi}_{1N} \end{bmatrix} = \begin{bmatrix} \bar{\Pi}_{22} & \bar{\Pi}_{23} & \cdots & \bar{\Pi}_{2N} \\ \bar{\Pi}_{32} & \bar{\Pi}_{33} & \cdots & \bar{\Pi}_{3N} \\ \vdots & \vdots & & \vdots \\ \bar{\Pi}_{N2} & \bar{\Pi}_{N3} & \cdots & \bar{\Pi}_{NN} \end{bmatrix}. \tag{9.11}$$

The significance of this result is that, the solvability of NICPISDP is reduced to the solvability of an equation of the type $A = BXC$ for which we have given verifiable and easily interpretable solvability conditions in Theorem 4.2.

Chapter 10

CONCLUSIONS

The main contributions of this thesis are Theorems 5.1, 6.1, 7.1, and 8.1. Theorems 5.1, 6.1, and 7.1 yield new results on the problems ADDPIS, NICPIS, and ANICPIS, respectively. The conditions are in terms of the solvability of linear matrix equations involving system matrices associated with certain natural subplants of the original plant. The conditions of Theorem 5.1 can be restated in the language of geometric (state-space) theory (see [25],[26]). This condition amounts to a " zero-cancellation" condition in the frequency domain terminology and to an "invariant space inclusion" condition in geometric terminology.

Theorem 8.1 is crucial for obtaining a geometric counterpart to the conditions of Theorems 6.1 and 7.1 which are essentially the common solvability of two linear matrix equations of the type encountered in Theorems 4.2 or 5.1. By the result of Theorem 8.1, a common solution to these equations exists if and only if they are separately solvable (zero-cancellation occurs) and a bilateral matrix equation is solvable. A geometric counterpart for the solvability of this bilateral matrix equation can also be obtained using the results of [27] and the geometric interpretation of skew-primeness equations. Thus, the condition (C3) of Theorem 8.1 amounts to a "decomposition condition for certain invariant subspaces," a condition encountered in [28] for the solvability of the regulator problem. In frequency domain, the condition (C3) amounts to a disjointness condition among appropriate zeros associated with certain matrices. Therefore, when reflected to Theorems 6.1 and 7.1, the condition (C3) will yield a disjointness condition among the system zeros of suitable subsystems, or equivalently, it is expected to yield a "decomposition condition" [28, Corollary 7.3] for certain invariant subspaces of the state-space of the original system.

It is clear that, there is work to be done towards explicitly obtaining the geometric counterparts of the solvability conditions we have given in Theorems 6.1 and 7.1. This will in principle be easily accomplished via a geometric counterpart of the condition (C3) of Theorem 8.1 for which a readily available result is Theorem 5.12 of [27]. This line of research has been deliberately avoided in this thesis since its development requires a rather different algebraic background.

A problem which is left open in this thesis is stating an analogue of Theorem 8.1 for the common solvability of N linear matrix equations when $N \geq 3$. As far as checking the solvability and obtaining a solution (when it exists) of the general NICP, Theorem 9.2 serves well since it may be restated in terms of an equation of the type $Ax = b$ over a principal ideal domain. However, exactly what type of constraints this imposes on the structure of the open-loop plant is not clear at this stage.

Bibliography

[1] B. A. Francis, J. W. Helton, and G. Zames, "$\mathcal{H}^{\infty}$-optimal feedback controllers for linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 888-900, 1984.

[2] J. Doyle, K. Glover, P. Khargonekar, and B. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *Proc. American Control Conference*, Atlanta, 1691-1696, 1988.

[3] J. C. Willems, "Almost invariant subspaces: an approach to high gain feedback design - part I : almost controlled invariant subspaces," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 235-252, 1981.

[4] J. C. Willems, "Almost invariant subspaces: an approach to high gain feedback design - part II : almost conditionally invariant subspaces," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 1071-1085, 1982.

[5] M. Vidyasagar, *Control System Synthesis : A Factorization Approach*, The MIT Press, Cambridge, Massachusetts, London, 1985.

[6] A. S. Morse and W. M. Wonham, "Status of noninteracting control," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 568-581, 1971.

[7] M. L. J. Hautus and M. Heymann, "New results on linear feedback decoupling," *Proc. 4th Intern. Conf. Anal. and Opt. Systems*, Versailles. Lecture notes in *Control and Inf. Sciences*, vol. 28, pp. 562-575, Springer Verlag, Berlin, 1980.

[8] V. Eldem and A. B. Özgüler, "Disturbance decoupling problems by measurement feedback : a characterization of all solutions and fixed modes," *SIAM J. Contr. Optimiz.*, vol. 26, pp. 168-185, 1988.

[9] S. Weiland and J. C. Willems, "Almost disturbance decoupling with internal stability," *IEEE Trans. Automat. Contr.*, vol. AC-34, pp. 277-286, 1989.

[10] A. B. Özgüler, "Almost disturbance decoupling with internal stability : frequency domain conditions," *IEEE. Trans. Automat. Contr.*, to appear.

[11] A. Linnemann, I. Postlethwaite, and B. D. O. Anderson, "Almost disturbance decoupling with stabilization by measurement feedback," *preprint*, Australian National University, Canberra, ACT 2601, Australia.

[12] J. C. Willems, "Almost noninteracting control design using dynamic state feedback," *Proc. 4th Inter. Conf. Anal. and Opt. Systems*, Versailles. Lecture notes in *Control and Inf. Sciences*, vol. 28, pp. 555-561, Springer Verlag, Berlin, 1980.

[13] H. L. Trentelman and J. W. van der Woude, "Almost invariance and noninteracting control : a frequency domain analysis," *MEMORANDUM COSOR 86-09, Eindhoven University of Technology*, 1986.

[14] J. W. van der Woude, "Feedback decoupling and stabilization for linear systems with exogenous variables," *Technische Universiteit Eindhoven, Ph. D. Thesis*, Eindhoven, The Netherlands, 1987.

[15] S. K. Mitra, "Common solutions to a pair of linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$," *Proc. Cambridge. Philos. Soc.*, pp. 213-216, 1974.

[16] E. W. Roth, "The equation $AX - YB = C$ and $AX - XB = C$ in matrices," *Proc. Amer. Math. Soc.*, 3, pp. 392-396, 1952.

[17] W. A. Wolovich and P. Ferreira, "Output regulation and tracking in linear multivariable systems," *Division of engineering, Brown University, Providence, RI, Tech. Rep.* Eng MH 78-1, Mar. 1978.

[18] P. Khargonekar and A. B. Özgüler, "Regulator problem with internal stability : a frequency domain solution," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 331-343, 1984.

[19] A. B. Özgüler, "Lecture notes on control of two-channel systems via a matrix fractional approach," *Bilkent University Reports*, P.O. Box 8, Maltepe 06572, Ankara, Turkey.

[20] A. B. Özgüler and N. Akar, "A common solution to a pair of linear matrix equations over a principal ideal domain," *Linear Algebra Appl.*, submitted for publication.

[21] C. C. Mac Duffee, *Theory of Matrices*, Chelsea, New York, 1946.

[22] W. A. Wolovich, "Skew-prime polynomial matrices," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 880-887, 1978.

[23] P. Fuhrmann, *unpublished notes.*

[24] P. Lancester and M. Tismenetsky, *The Theory of Matrices*, Academic Press, Orlando, Florida, 1985.

[25] A. B. Özgüler, "Polynomial characterizations of (H, F)-invariant subspaces with applications," *Linear Algebra Appl.*, vol. 73, pp. 1-31, 1986.

[26] A. B. Özgüler, "Geometric solvability conditions for the almost disturbance decoupling problem," *Bilkent University Reports*, P.O. Box 8, Maltepe 06572, Ankara, Turkey.

[27] P. P. Khargonekar, T. T. Georgiou, and A. B. Özgüler, "Skew-prime polynomial matrices : The polynomial-model approach," *Linear Algebra Appl.*, vol. 50, pp. 403-435, 1983.

[28] W. M. Wonham, *Linear Multivariable Control : A Geometric Approach*, 2nd ed., Springer, New York, 1979.