EXISTENCE IN AN OVERLAPPING GENERATIONS MODEL WITH PRODUCTION

A Thesis Submitted to the Department of Economics and the Institute of Economics and Social Sciences of Bilkent University in Partial Fulfillment of the Requirements for the Degree of

MASTER OF ARTS IN ECONOMICS

by

Atila Abdulkadiroğlu

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tarafından bağışlanmıştır.

I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Farhad Husseinov

I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Asad Zaman

I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Serdar Sayan

Approved by the Institute of Economics and Social Sciences

Director:

ABSTRACT

EXISTENCE IN AN OVERLAPPING GENERATIONS MODEL WITH PRODUCTION

Atila Abdulkadiroğlu

M.A. in Economics

Supervisor: Prof. Farhad Husseinov

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This thesis proves the existence of competitive equilibrium in an overlapping generations model (OLG) with production. In the proof, existence of equilibrium in the classical Arrow-Debreu Model is essential, and the work is similar in spirit to that presented in Balasko, Cass and Shell [2], except some tricks used in the proof. The assumptions do not deviate from standard assumptions, so the model can be taken as a first step in developing more general models.

KEYWORDS: Overlapping generations model - Kakutani's Fixed Point Theorem - Arrow-Debreu Model - Competitive equilibrium - Compensated equilibrium

ÖZET

BİR ÜRETİMLİ KESİŞEN NESİLLER MODELİNDE REKABETÇİ DENGENİN VARLIĞI

Atila Abdulkadiroğlu
Yüksek Lisans Tezi, İktisat Bölümü
Tez Yöneticisi: Prof.Dr. Farhad Husseinov

44 sayfa
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Bu tez bir üretimli kesişen nesiller modelinde rekabetçi dengenin varlığını ispatlar. İspatta, klasik Arrow-Debreu modelinde denge varlığı esastır, ve çalışma Balasko, Cass ve Shell [2] tarafından yapılan çalışmaya, kullanılan bazı metodlar hariç benzerdir. Varsayımlar standart varsayımlardan farklı değildir, dolayısıyla bu model yeni modeller geliştirmek için ilk adım olarak kullanılabilir.

ANAHTAR KELİMELER: Kesişen nesiller modeli - Kakutani'nin sabit nokta teoremi - Arrow-Debreu modeli - Rekabetçi denge - Teeşvikli denge

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Contents

1 Introduction

The overlapping generations model was firstly introduced by Samuelson [5]. The model has a wide area of applications in the literature, and some important implications, which cannot be observed in other types of macroeconomic models. The existence of equilibrium was established in an overlapping generations model of pure exchange case by Balasko and Shell [1]. In a subsequent paper [2], Balasko,Cass and Shell relaxed some of the restrictive assumptions of the model and proved the existence.

The purpose of this thesis is to incorporate production to the model and then to prove the existence of competitive equilibrium. Our model differs from those papers cited above in that we follow the modern cardinalist approach. Assumptions of the model are fairly general, and can be seen in almost all standard models. Yet, the model is open to further generalization, especially by applying the investigations of Arrow, Debreu and Uzzawa as discussed in Nikaido [4].

The existence in the Arrow-Debreu Model (henceforth the Basic Model) is essential in the proof of existence in our model. We establish the existence in the Basic Model, through a proof that is similar to Nikaido's proof [4]. The way of overcoming the difficulties in that proof, is also essential in establishing the existence in the OLG model. Restricting the competitive equilibrium allocations of the OLG model in some compact sets as in the basic model allows us to find the subsequences of sequences of competitive equilibria, with limits which are established to be the competitive equilibria of the OLG model. This last approach is similar, in spirit, to the method introduced by Balasko, Cass and Shell [2].

The plan of the thesis is as follows. The next section establishes the existence in the basic model, and discusses on the basic model. Subsection 2.1 introduces the model, then Subsection 2.2 establishes the existence.To prove the existence in the OLG model we need to use the limits of sequences of equilibrium allocations. But the discontinuity of the standard competitive equilibrium causes difficulties in the proof. So, in Subsection 2.3 we introduce an alternative definition of equilibrium namely the compensated equilibrium which is continuous in some appropriate sense. And we examine the relations between those two types of equilibria. Continuity of that new type of equilibrium allows us to find limits of the sequences of compensated equilibria. This and the equivalence of competitive and compensated equilibria of the basic model under fair conditions allow us to establish the existence of competitive equilibrium in the OLG model in Section 3.

2 Existence in the Basic Model

2.1 The Basic Model

Here, we introduce the standard Arrow-Debreu Model and call it "the Basic Model". We list the standard assumptions of the model and establish the existence of competitive equilibrium following Nikaido [4].

In the basic model there are n categories of goods denoted by $j = 1, \cdots, n$; l consumers denoted by $i = 1, \cdots, l$; m producers denoted by $k = 1, \cdots, m$. Each consumer i has a preference field $(X_i, \succeq_i)$, where X_i is his consumption set in R^n, $a_i = (a_i^j) \in R^n$, is the initial holding of consumer i, the jth component, a_i^j, stands for the amount of the jth good initially held by him. Each producer k has a production technology set Y_k. Each producer decides to produce y_k from the technology set Y_k. Then the sum

$$y = \sum_{k=1}^{m} y_k \tag{1}$$

is the corresponding aggregate production.Hence, the aggregate technology set of the economy can be written as the vectorial sum of the Y_k's:

$$Y = \sum_{k=1}^{m} Y_k. \tag{2}$$

The aggregate supply vectors available in the economy can be represented by the following set

$$\{a\} + Y \tag{3}$$

where

$$a = \sum_{i=1}^{l} a_i \tag{4}$$

is the aggregate initial holding.

Turning our attention to the demand side of allocation, an allocation of consumption is affected by choosing a menu of consumption x_i from the consumption set X_i for each $i = 1, \cdots, l$. Then the sum

$$x = \sum_{i=1}^{l} x_i \tag{5}$$

is the corresponding aggregate demand vector.

It should be noted that the initial holding may consist not only of finished consumption goods, which are ready for distribution among consumers, but also of semi-finished goods as well as fixed capital stocks, land and labor. And some of the components of an aggregate demand vector may be negative. These components correspond to the positive supplies of such goods as labor in the individual consumers' menus of consumption, which may serve as inputs in production.

An allocation of production and consumption means the choice of an $(l + m)$-tuple

$$(x_1, \cdots, x_i, \cdots, x_l, y_1, \cdots, y_k, \cdots, y_m) \tag{6}$$

of

$$x_i \in X_i \quad (i = 1, \cdots, l), \tag{7}$$

$$y_k \in Y_k \quad (k = 1, \cdots, m). \tag{8}$$

Such an allocation will be called a "feasible allocation" if

$$\sum_{i=1}^{l} x_i \leq a + \sum_{k=1}^{m} y_k \tag{9}$$

The following assumptions complete the basic model:

(C.1)The consumption set X_i is a closed convex set in R^n for each consumer i.

(C.2)Each X_i has a lower bound c_i which satisfies

$$x_i \geq c_i \text{ for all } x_i \in X_i \tag{10}$$

relative to the semi-order based on the component-wise comparison among vectors.

(C.3)Each preference field $(X_i, \succeq_i)$ satisfies the following:

$$\forall x, y \in X_i : x \succ y \Rightarrow \lambda x + (1-\lambda)y \succ y \tag{11}$$

for each $\lambda \in (0, 1]$

(C.4)Each preference relation, $\succeq_i$, is continuous in the sense that the sets $\{x_i \mid x_i \in X_i, x_i \succeq x_0\}$ and $\{x_i \mid x_i \in X_i, x_0 \succeq x_i\}$ are closed subsets of X_i for each $x_0 \in X_i$.

(C.5)Each consumer i has an initial holding a_i, which is a vector of R^n.

(P.1)Each technology set Y_k is a subset of R^n containing the origin of R^n.

(P.2)Y_k is convex and closed in R^n.

(P.3)The aggregate technology set $Y = \sum_{k=1}^{m} Y_k$ satisfies $Y \cap R^n_+ = \{0\}$ (Arrow-Hahn) where R^n_+ is the nonnegative orthant of R^n.

(P.4)$Y \cap (-Y) = \{0\}$

(C-P)There are lm constants $\alpha_{ij} \geq 0 (i = 1, \cdots, l; k = 1, \cdots, m)$ satisfying $\sum_{i=1}^{l} \alpha_{ik} = 1, (k = 1, \cdots, m)$ with α_{ik} standing for the relative share of consumer i in the profit π_k which is $\alpha_{ik}\pi_k$.

Those assumption are standard in many models, and the interpretation of those assumptions can be found in Nikaido [4], Arrow-Hahn [3], or in any mathematical economics reference. The definition of $\succ$ is in the following standard manner:

$$x \succ y \text{ iff } x \succeq y \text{ and } y \not\succeq x$$

Definition 1 *An (l+m+1)-tuple*

$$(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p}) \tag{12}$$

of menus of consumption $\hat{x}_i \in X_i (i = 1, \cdots, l)$, production processes $\hat{y}_k \in Y_k, (k = 1, \cdots, m)$ and an n-dimensional price vector $\hat{p} \geq 0$ is called a competitive equilibrium of the basic model and $\hat{p}$ is referred to as an equilibrium price vector if the following conditions (i)-(iii) are fulfilled:
(i) The maximum profit of each producer under $\hat{p}$. That is

$$\pi_k(\hat{p}) = \hat{p}.\hat{y}_k = max(\hat{p}.y) \tag{13}$$

over all $y \in Y_k (k = 1, \cdots, m)$
(ii) The optimum preference of each consumer subject to budget constraint under $\hat{p}$. That is, for each $i = 1, \cdots, l, \hat{x}_i$ is a most preferable menu of consumption among all x in X_i fulfilling the budget constraint

$$\hat{p}.x \leq \hat{p}.a_i + \sum_{k=1}^{m} \alpha_{ik}.\pi_k(\hat{p}) \tag{14}$$

(iii) The balance of aggregate supply and demand. That is

$$\sum_{i=1}^{l} \hat{x}_i \leq \sum_{i=1}^{l} a_i + \sum_{k=1}^{m} \hat{y}_k \tag{15}$$

with equality holding in the jth component relation of the above inequality if the corresponding price $\hat{p}^j$ is positive.

Theorem 1 *(Arrow and Debreu,1954) If each consumer i has a positive initial holding a_i in the sense that there is a commodity bundle b_i in X_i fulfilling*

$$a_i > b_i \tag{16}$$

and if he has no satiation point, then there exists a competitive equilibrium.

2.2 Proof of Theorem 1

2.2.1 Boundedness of Competitive Equilibrium Allocations

The main cause of difficulties in the proof of the theorem is that consumption and technology sets need not be bounded, and this does not ensure the existence of most preferable menus of consumption and most profitable processes. To overcome such difficulties, we will restrict the competitive equilibrium allocations to some bounded subsets of X_i and Y_k for each $i = 1, \cdots, l, j = 1, \cdots, m$. Let $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ be a competitive equilibrium, if there exists any, then it satisfies (iii) of Definition 1 and therefore must satisfy the condition

$$(a + \sum_{k=1}^{m} \hat{y}_k - \sum_{i=1}^{l} \hat{x}_i) \in (a + Y - X) \cap R^n_+, \tag{17}$$

where $X = \sum_{i=1}^{l} X_i, Y = \sum_{i=1}^{m} Y_i, a = \sum_{i=1}^{l} a_i$. Now let us define

$$\tilde{X}_i = \{x_i \mid x_i \in X_i, (a + Y - \sum_{s \neq i} X_s - x_i) \cap R^n_+ \neq \emptyset\} \tag{18}$$

$$(i = 1, \cdots, l)$$

$$\tilde{Y}_k = \{y_k \mid y_k \in Y_k, (A + y_k + \sum_{t \neq k} Y_t - X) \cap R^n_+ \neq \emptyset\} \tag{19}$$

$(k = 1, \cdots, m)$

The following lemma states how we will overcome the difficulty of unboundedness of X_i and $Y_k (i = 1, \cdots, l, k = 1, \cdots, m)$

Lemma 1 *Assume that all the assumptions hold, then*

(i) $\tilde{X}_i \neq \emptyset, tildeY_k \neq \emptyset$ *for any* i, k.

(ii) If $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ *is a competitive equilibrium, then* $\hat{x}_i \in \tilde{X}_i, \hat{y}_k \in \tilde{Y}_k$ *for* $i = 1, \cdots, l, k = 1, \cdots, m$.

(iii) $\tilde{X}_i, \tilde{Y}_k$ *are convex subsets of* X_i, Y_k*, respectively, for any i,k.*

(iv) $\tilde{X}_i, \tilde{Y}_k$ *are bounded for any i,k.*

Proof: (i) Trivially, we have $b_i \in \tilde{X}_i$ and $0 \in \tilde{Y}_k$ for each i and k. So, $\tilde{X}_i \neq \emptyset, \tilde{Y}_k \neq \emptyset$ for any i,k.

(ii) Since $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ satisfies the condition (iii) in Definition 1, inclusion of $\hat{x}_i$ in $\tilde{X}_i$ and of $\hat{y}_i$ in $\tilde{Y}_k$ are clear from the definitions of $\tilde{X}_i$ and $\tilde{Y}_k$ for each i,k.

(iii) They are convex, because they are obtained by the combined operations of taking linear combinations of convex sets and letting them intersect with R^n_+, which is a convex set. By the definitions, they are subsets of X_i, Y_k, respectively.

(iv) To show that $\tilde{X}_i, \tilde{Y}_k$ are bounded, it suffices to show that if $l+m$ sequences $\{x_i^v\}$ in $X_i (i = 1, \cdots, l)$ and $\{y_k^v\}$ in $Y_k (k = 1, \cdots, m)$ satisfy

$$a + \sum_{k=1}^{m} y_k^v - \sum_{i=1}^{l} x_i^v \geq 0, \quad (v = 1, 2, \cdots) \tag{20}$$

then they must be bounded. First, we will prove the boundedness of $\{y_k^v\}$. Suppose the contrary, without loss of generality, assume

$$lim_{v \to +\infty} \mu_v = +\infty \tag{21}$$

where $\mu_v = \| y_k^v \|$ over all $k = 1, \cdots, m$. (20) implies that

$$a + \sum_{k=1}^{m} y_k^v \geq \sum_{i=1}^{l} x_i^v \tag{22}$$

By the assumption of the lower boundedness of X_i, (22) reduces to

$$\sum_{k=1}^{m} y_k^v \geq c - a \tag{23}$$

where $c = \sum_{i=1}^{l} c_i$. Since $lim_{v \to +\infty} \mu_v = +\infty$, we have that, $\mu_v > 0$ for large v and that Y_k is a convex set including 0 by (P.1),(P.2) and including y_k^v, therefore we have that

$$\frac{1}{\mu_v}.y_k^v = \frac{1}{\mu_v}.y_k^v + (1 - \frac{1}{\mu_v}).0 \in Y_k \qquad (k = 1, \cdots, m) \tag{24}$$

Dividing (23) by μ_v, we obtain

$$\sum_{k=1}^{m} \frac{y_k^v}{\mu_v} \geq \frac{(c-a)}{\mu_v} \to 0 \qquad as \quad v \to +\infty \tag{25}$$

On the other hand, the definition of μ_v ensures, for large v,

$$\| \frac{y_k^v}{\mu_v} \| \leq 1 \qquad (k = 1, \cdots, m) \tag{26}$$

which implies the inclusion of $\frac{y_k^v}{\mu_v}$ in the unit ball. Hence, the compactness of the unit ball enables us to assume, without loss of generality, that the m sequences $\{y_k^v\}(k = 1, \cdots, m)$ are convergent, with the corresponding limits

$$lim_{v \to +\infty} y_k^v = y_k^0 \tag{27}$$

Since Y_k is closed for each k, y_k^0 belongs to $Y_k \quad (k = 1, \cdots, m)$, respectively. Now (25) reduces to

$$\sum_{k=1}^{m} y_k^0 \geq 0 \tag{28}$$

Obviously, $\sum_{k=1}^{m} y_k^0 \in Y$, so (28) entails

$$\sum_{k=1}^{m} y_k^0 \in Y \cap R_+^n. \tag{29}$$

By virtue of (P.3), this yields

$$\sum_{k=1}^{m} y_k^0 = 0 \tag{30}$$

On the other hand, we observe that we have, for each t,

$$y_t^0 = \overbrace{0 + \cdots + 0}^{m-1} + y_t^0 \in Y, \tag{31}$$

because $0 \in Y_k (k = 1, \cdots, m)$, while (30) implies

$$-y_t^0 = 0 + \sum_{k \neq t} y_k^0 \in Y \tag{32}$$

and (32) implies

$$y_t^0 \in (-Y) \tag{33}$$

Hence (31),(33) together yield

$$y_t^0 \in Y \cap (-Y) \quad (t = 1, \cdots, m) \tag{34}$$

which, by virtue of (P.4), implies $y_t^0 = 0 (t = 1, \cdots, m)$. Therefore we must have $y_k^0 = 0 (k = 1, \cdots, m)$ in (27), so that

$$lim_{v \to +\infty} max_k \parallel \frac{y_k^v}{\mu_v} \parallel = 0 \tag{35}$$

But (35) contradicts

$$1 = max \mid\mid \frac{y_k^v}{\mu_v} \mid\mid \text{ over } k = 1, \cdots, m \tag{36}$$

which is implied by the definition of μ_v. This completes the proof of the boundedness of $\{y_k^v\}(k = 1, \cdots, m)$. The boundedness of the l sequences $\{x_i^v\}(i = 1, \cdots, l)$ can be seen immediately. Using the assumed lower-boundedness of X_i with lower-bounds $c_i(i = 1, \cdots, l)$ in (C.2) as well as the boundedness of $\{y_k^v\}$, we can establish the boundedness of $\{x_s^v\}$ for each $s = 1, \cdots, l$ by

$$c_s \leq x_s^v \leq a + \sum_{k=1}^{m} y_k^v - \sum_{i \neq s} x_i \leq a + \sum_{k=1}^{m} y_k^v - \sum_{i \neq s} c_i \tag{37}$$

Since the above argument can apply to arbitrary sequences $\{x_i^v\}$ in X_i, $\{y_k^v\}$ in Y_k satisfying (20), the proof is thereby complete. This method of proof is due to Arrow and Debreu (1954), Q.E.D.

2.2.2 Supply and Demand Functions

To overcome the difficulties that occur in the proof, we will search for the competitive equilibrium allocations not in the original sets X_i, Y_k, but instead, in some compact sets by virtue of Lemma 1.

Lemma 1 is a very important step in the proof of existence. That is, this lemma enables us to lock allocations $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m)$ associated with competitive equilibria in the bounded sets $\tilde{X}_i, \tilde{Y}_k$, although the existence of competitive equilibria themselves have not yet been established. To define the supply and demand functions, first we choose a sufficiently large cube

$$E = \{x \mid \xi_j \leq x_j \leq \eta_j(j = 1, \cdots, n)\} \tag{38}$$

such that $0, b_i \in E, \tilde{X}_i, \tilde{Y}_k \subset E^\circ (i = 1, \cdots, l; k = 1, \cdots, m)$ where E° denotes the interior of E. Now we obtain $l+m$ nonempty convex sets $X_i \cap E, Y_k \cap E (i = 1, \cdots, l; k = 1, \cdots, m)$. By letting the consumers and producers pursue their best satisfaction on the narrowed ranges of choice $X_i \cap E, Y_k \cap E$, we define individual demand and supply functions on the set of all semi-positive price vectors.

Individual Supply Functions: The supply function $\Psi_k(p)$ of producer k and his profit function $\pi_k(p)$ are defined for all $p \geq 0$ by

$$\Psi_k(p) = \{y_k \mid p.y_k = max p.y \text{ over all } y \in Y_k \cap E\}, \tag{39}$$

$$\pi_k(p) = max(p.y) \text{ over all } y \in Y_k \cap E \quad (k = 1, \cdots, m) \tag{40}$$

Remember that $Y_k \cap E$ is a nonempty compact set, and since the continuous function p.y is defined on these compact sets, the definitions of $\Psi_k(p)$ and $\pi_k(p)$ make sense.

Lemma 2 *Ψ_k is a set valued mapping with $\Psi_k(p)$ being nonempty compact convex subset of $Y_k \cap E$.*

Proof: Since $Y_k \cap E$ is a nonempty compact set, and $p.y$ is a continuous function on that set, $p.y$ attains its maximum for some $y_k \in Y_k \cap E$. So, $\Psi_k(p)$ is nonempty for any $p \geq 0$. Since $Y_k \cap E$ is compact, it is bounded, so is $\Psi_k(p) \subset Y_k \cap E$. Now take any sequence $\{y^v\}$ in $\Psi_k(p)$ such that $y^v \to y^0$. Then for each $v \geq 1$, we have $p.y^v = max(p.y)$ over all $y \in Y_k \cap E$. Since $p.y$ is a continuous function and $y^v \to y^0$, we obtain that $p.y^0 = max(p.y)$ over all $y \in Y_k \cap E$. So $y^0 \in \Psi_k(p)$, that is, $\Psi_k(p)$ is also closed. Hence, $\Psi_k(p)$ is compact.

Now take any $y^1, y^2 \in \Psi_k(p)$, and any $\lambda \in [0,1]$. Define $y^0 = \lambda.y^1 + (1-\lambda).y^2$. We have that $p.y^1 = p.y^2 = max_{y \in Y_k \cap E}(p.y)$. Then, trivially, we have

$$\begin{aligned} p.y^0 = p.(\lambda.y^1 + (1-\lambda).y^2) &= \lambda p.y^1 + (1-\lambda)p.y^2 \\ &= \lambda.max_{y \in Y_k \cap E}(p.y) + (1-\lambda).max_{y \in Y_k \cap E}(p.y) \\ &= max_{y \in Y_k \cap E}(p.y) \end{aligned}$$

So, $y^0 \in \Psi_k(p)$. Hence, $\Psi_k(p)$ is convex for any $p \geq 0$, QED.

On the other hand, $\pi_k(p)$ is a numerical-valued function satisfying $\pi_k(p) \geq 0$ for all $p \geq 0$. This is clear from the inclusion of 0 in $Y_k \cap E$ by (P.1) and (40).

Individual Demand Functions: The demand function $\Phi_i(p)$ of consumer i is defined for all $p \geq 0$ by

$$\begin{aligned} \Phi_i(p) = \{x_i \mid \quad & x_i \in X_i \cap E, x_i \succeq_i x \text{ for all } x \in X_i \cap E \\ & \text{subject to } p.x \leq p.a^i + \textstyle\sum_{k=1}^m \alpha_{ik}\pi_k(p)\} \end{aligned} \tag{41}$$

$$(1 = 1, \cdots, l)$$

Lemma 3 *Φ_i is a set valued mapping with $\Phi_i(p)$ being nonempty compact convex subset of $X_i \cap E$.*

Proof: From Lemmas 1,2 in Appendix, non-emptiness, compactness and convexity of $\Phi_i(p)$ is obvious. QED.

The Aggregate Excess Supply Function: The aggregate supply function $\Psi(p)$, the aggregate demand function $\Phi(p)$, and the aggregate excess supply function $\chi(p)$ are defined by

$$\Psi(p) = a + \sum_{k=1}^{m} \Psi_k(p) \tag{42}$$

$$\Phi(p) = \sum_{i=1}^{l} \Phi_i(p) \tag{43}$$

$$\chi(p) = \Psi(p) - \Phi(p) \tag{44}$$

Walras Law: In the following lines of the proof, we will refer to a very important identity, called the Walras Law, which connects $\Phi(p)$ to $\Psi(p)$. Let $x \in \Phi(p)$ and $y \in \Psi(p)$, and let

$$x = \sum_{i=1}^{l} x_i, \quad x_i \in \Phi_i(p) \qquad (i = 1, \cdots, l) \tag{45}$$

$$y = a + \sum_{k=1}^{m} y_k, \quad y_k \in \Psi_k(p) \qquad (k = 1, \cdots, m) \tag{46}$$

be their decompositions. Then, since x_i satisfies the budget constraint, we have

$$p.x_i \leq p.a_i + \sum_{k=1}^{m} \alpha_{ik}\pi_k(p) \qquad (i = 1, \cdots, l) \tag{47}$$

And, since $y_k \in \Psi_k(p)$, we have that

$$\pi_k(p) = p.y_k \qquad (k = 1, \cdots, m) \tag{48}$$

So, summing up (47) and (48) over all i, k and remembering that $\sum_{i=1}^{l} \alpha_{ik} = 1$ for all $k = 1, \cdots, m$ we obtain

$$\begin{aligned}
p.(\sum_{i=1}^{l} x^i) &= \sum_{i=1}^{l} p.x^i \\
&\leq \sum_{i=1}^{l} p.a^i + \sum_{i=1}^{l}\sum_{k=1}^{m} \alpha_{ik}\pi_k(p) \\
&= p.(\sum_{i=1}^{l} a^i) + \sum_{k=1}^{m}\sum_{i=1}^{l} \alpha_{ik}\pi_k(p) \\
&= p.a + \sum_{k=1}^{m} \pi_k(p) \\
&= p.a + \sum_{k=1}^{m} p.y^k \\
&= p.(a + \sum_{k=1}^{m} y^k)
\end{aligned}$$

Hence $p.x \leq p.y$ for all $x \in \Phi(p), y \in \Psi(p)$, or expressed in terms of the excess supply function $\chi(p)$

$$p.u \geq 0 \quad \forall u \in \chi(p) \tag{49}$$

From the economic point of view, the Walras Law states that the sole origin of effective demand is the spending of income.These laws play a central part in letting the Walrasian system admit a competitive equilibrium. In the following we will define another type of equilibrium, and establish the connection between this new type of equilibrium and the competitive equilibrium.

2.2.3 Equilibrium Solution of $\chi(p)$

It is in order to introduce another type of equilibrium solution and its relation to the competitive equilibrium of Definition 1.

Definition 2 *A triplet $(\hat{x}, \hat{y}, \hat{p})$ of an aggregate demand vector $\hat{x}$, an aggregate supply vector $\hat{y}$, and a semi-positive price vector $\hat{p}$ is said to be an equilibrium solution of the aggregate supply and demand functions $\Psi(p), \Phi(p)$ if,*

$$\hat{x} \in \Phi(\hat{p}), \hat{y} \in \Psi(\hat{p}), \tag{50}$$

$$\hat{y} \geq \hat{x}. \tag{51}$$

The price vector $\hat{p}$ is termed an equilibrium price vector of the aggregate supply and demand functions $\Psi(p), \Phi(p)$.

It is obvious that an equilibrium price vector of $\Psi(p), \Phi(p)$ is characterized in terms of the aggregate excess supply function $\chi(p)$ as such a price vector $\hat{p}$ that

$$\chi(\hat{p}) \cap R^n_+ \neq \emptyset \tag{52}$$

The following theorem establishes the link between the two types of equilibrium solutions.

Theorem 2 *(i) If an $(l+m+1)$-tuple $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ of $\hat{x}_i \in X_i, \hat{y}_k \in Y_k, \hat{p} \geq 0$ is a competitive equilibrium of the basic model then*

$$\hat{x}_i \in \Phi_i(\hat{p})(i = 1, \cdots, l) \tag{53}$$

$$\hat{y}_k \in \Psi_k(\hat{p})(k = 1, \cdots, m) \tag{54}$$

and the triplet $(\hat{x}, \hat{y}, \hat{p})$*, where*

$$\hat{x} = \sum_{i=1}^{l} \hat{x}_i \tag{55}$$

$$\hat{y} = a + \sum_{k=1}^{m} \hat{y}_k \tag{56}$$

is an equilibrium of the aggregate supply and demand functions $\Psi(p), \Phi(p)$.
(ii) If a triplet $(\hat{x}, \hat{y}, \hat{p})$ *is an equilibrium of* $\Psi(p), \Phi(p)$*, then any* $(l+m+1)$*-tuple* $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ *obtained by performing the decomposition (53)-(56) is a competitive equilibrium of the basic model.*

Proof: (i)Since $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ is a competitive equilibrium, condition (51) follows from (iii) in Definition 1. From (ii) in Definition 1, $\hat{x}_i$ is one of the most preferable menus of X_i subject to the budget constraint of consumer i, and $\hat{y}_k$ is one of the most profitable processes in Y_k for producer k. Also, by Lemma 1(ii) and by (38), $\hat{x}_i \in X_i \cap E$, $\hat{y}_k \in Y_k \cap E$. Then $\hat{y}_k$ maximizes $p.y$ over $Y_k \cap E$, because it does even over Y_k by (i) in Definition 1. Similarly, $\hat{x}_i$ is a most preferable element of $X_i \cap E$ subject to the budget constraint, because it is even a most preferable element of X_i subject to the same budget constraint by (ii) in Definition 1. Hence, $\hat{x}_i \in \Phi_i(\hat{p})$, $\hat{y}_k \in \Psi_k(\hat{p})$, which proves condition (50). So, $(\hat{x}, \hat{y}, \hat{p})$ is an equilibrium of the aggregate supply and demand functions $\Psi(p), \Phi(p)$.
(ii)Assume that $(\hat{x}, \hat{y}, \hat{p})$ is an equilibrium solution of the corresponding aggregate supply and demand functions. Moreover, assume that the $(l+m+1)$-tuple $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ is obtained by performing the decomposition

(53)-(56). (51) implies that

$$\hat{x}_i \in \tilde{X}_i \qquad (i = 1, \cdots, l), \tag{57}$$

$$\hat{y}_k \in \tilde{Y}_k \qquad (k = 1, \cdots, m), \tag{58}$$

by the definitions of $\tilde{X}_i, \tilde{Y}_k$. Now we have to establish (i)-(iii) of Definition 1.
Proof of (i): Suppose $\hat{p}.y_k > \hat{p}.\hat{y}_k$ for some $y_k \in Y_k$. This y_k may be such that $y_k \notin Y_k \cap E$, but for any z_k on the segment with end points $y_k, \hat{y}_k$, we have $\hat{p}.z_k > \hat{p}.\hat{y}_k$ as long as $z_k \neq \hat{y}_k$. Clearly, $z_k \in Y_k$ by the convexity of Y_k. On the other hand, $\hat{y}_k$ must be an interior point of E by (38) and (58). Hence we can take z_k so close to $\hat{y}_k$ that $z_k \in Y_k \cap E$. This contradicts $\hat{y}_k \in \Psi_k(\hat{p})$. This proves (i).
Proof of (ii): Suppose

$$x_i \succ_i \hat{x}_i, \quad \hat{p}.x_i \leq \hat{p}.a_i + \sum_{k=1}^{m} \alpha_{ik}\pi_k(\hat{p}),$$

for some $x_i \in X_i$. Then, (C.3) ensures

$$w_i \succ_i \hat{x}_i$$

for any w_i on the segment with and points $x_i, \hat{x}_i$ so long as $w_i \neq \hat{x}_i$. Moreover, w_i belongs to X_i by the convexity of X_i and fulfills the budget constraint. Since $\hat{x}_i$ must be an interior point of E by (38) and (57), we can take w_i so close to $\hat{x}_i$ that $w_i \in X_i \cap E$. This contradicts $\hat{x}_i \in \Phi_i(\hat{p})$. This completes the proof of (ii).
Proof of (iii): Condition (51) ensures that

$$\sum_{i=1}^{l} \hat{x}_i \leq a + \sum_{k=1}^{m} \hat{y}_k.$$

It remains to be shown that

$$\hat{x}_j = \hat{y}_j \quad if \quad \hat{p}_j > 0. \tag{59}$$

It can be seen that, under (51), (59) is equivalent to

$$\hat{p}.\hat{x} = \hat{p}.\hat{y}. \tag{60}$$

Thus it suffices to prove (60). To prove (60), we have only to show that equality holds in the budget constraint in (ii) for any consumer i. We first observe that for each consumer i, there is some $v_i \in X_i$ satisfying

$$v_i \succ_i \hat{x}_i \tag{61}$$

because of the insatiability of consumer i, which is assumed in Theorem 1 explicitly. Hence, (61) and (C.3) entail

$$x_i \succ_i \hat{x}_i \tag{62}$$

for any x_i on the segment with end points $v_i, \hat{x}_i$ whenever $x_i \neq \hat{x}_i$. Now suppose that strict inequality holds in the budget constraint of some consumer i. Then we can take the above x_i so close to $\hat{x}_i$ that x_i can satisfy the budget constraint. In such a situation (62) contradicts (ii), since $\hat{x}_i$ is a most preferable element of X_i subject to the budget constraint, QED.

Now we have transformed the competitive equilibria of the basic model to equilibrium solutions of the corresponding aggregate supply and demand functions. In the following lines, existence of equilibrium solutions of the aggregate supply and demand functions will be established, so it is in order to formulate a powerful existence theorem in a rather simple way by means of an excess supply function.

2.2.4 A Fixed Point Theorem

Theorem 3 *(Gale,1955; Nikaido,1956; Debreu,1959). Put* $P_n = \{p \mid p \geq 0, \sum_{j=1}^{n} p_j = 1\}$ *and let* Γ *be a compact convex subset of* R^n. *Suppose that there is a given set-valued mapping* $\chi : p_n \rightarrow 2^{\Gamma}$ *(which will be called an excess supply function). It is assumed that the mapping satisfies the following conditions (i),(ii):*
(i)χ : $P_n \rightarrow 2^{\Gamma}$ *is a closed mapping that carries each point of* P_n *to a nonempty convex subset of* Γ.
(ii) The Walras Law holds, i.e.,

$$p.u \geq 0 \quad \mathit{for}\; u \in \chi(p). \tag{63}$$

Then there is some $\hat{p}$ *in* P_n *such that*

$$\chi(\hat{p}) \cap R_+^N \neq \emptyset \tag{64}$$

Proof: The proof will be worked out by constructing a suitable mapping, and applying Kakutani's fixed point theorem to it. One can refer to Nikaido [4] to sketch an extensive proof of Kakutani's famous fixed point theorem. To this end, first define a single-valued mapping

$$\theta : \Gamma \times P_n \rightarrow P_n$$

by the formulas

$$\theta(u,p) = (\theta_i(u,p)), \quad u = (u_i) \in \Gamma, \quad p = (p_i) \in P_n, \tag{65}$$

$$\theta_i(u,p) = \frac{p_i + max(-u_i, 0)}{1 + \sum_{j=1}^{n} max(-u_j, 0)} \quad (i = 1, \cdots, n) \tag{66}$$

Then, by constructing the Cartesian product of θ with the mapping χ, which can be viewed as a mapping

$$f = \chi \times \theta : \Gamma \times P_n \to 2^{\Gamma \times P_n} \tag{67}$$

$$f(u,p) = \chi(p) \times \{\theta(u,p)\}. \tag{68}$$

We will show that all the conditions of Kakutani's fixed point theorem are fulfilled by f. This will be done in steps (i)-(iii):

(i) Both Γ and P_n are compact convex sets in R^n. Hence $\Gamma \times P_n$ is a compact convex set in $R^n \times R^n$ by Theorem 1 in the Appendix.

(ii) $\chi(p)$ is by assumption a convex subset of Γ for each p, while $\{\theta(u,p)\}$ is a special convex subset, consisting of one element, of P_n for each $(u,p) \in \Gamma \times P_n$. Whence, for each $(u,p) \in \Gamma \times P_n$ the image $f(u,p)$ defined by (68) is a convex subset of $\Gamma \times P_n$.

(iii) To prove the closedness of f, it suffices to show, in the light of Theorem 4 in Appendix, that the component mappings χ, θ are closed. The closedness of χ, viewed as mapping on $\Gamma \times P_n$ is an immediate consequence of the assumed closedness of χ, as a mapping on P_n. On the other hand, θ is obviously closed, since it is a single-valued continuous mapping from $\Gamma \times P_n$ to P_n.

Therefore, by virtue of Kakutani's fixed point theorem, f has a fixed point $(\hat{u}, \hat{p})$ so that

$$(\hat{u}, \hat{p}) \in f(\hat{u}, \hat{p}). \tag{69}$$

Relation (69) can be reduced, by using (66),(68), to the relations for the component mappings

$$\hat{u} \in \chi(\hat{p}), \tag{70}$$

$$\hat{p} = \theta(\hat{u}, \hat{p}). \tag{71}$$

The theorem will have been proved if it is seen that $\hat{u} \geq 0$, using (66), we rewrite (71) to obtain

$$\hat{p}_i = \frac{\hat{p}_i + max(-\hat{u}_i, 0)}{1 + \sum_{j=1}^{n} max(-\hat{(u)}_j, 0)} (i = 1, \cdots, n). \tag{72}$$

Multiplying the ith equation by $\hat{u}_i$ in (72), and summing over all i, we have,

$$(\hat{p}.\hat{u}) \sum_{j=1}^{n} max(-\hat{u}_j, 0) = \sum_{i=1}^{n} \hat{u}_i max(-\hat{u}_i, 0) \tag{73}$$

Note that $t max(t, 0) = (max(t, 0))^2$, so we can reduce (73) to

$$-(\hat{p}.\hat{u}) \sum_{j=1}^{n} max(-\hat{u}_j, 0) = \sum_{i=1}^{n} (max(-\hat{u}_i, 0))^2. \tag{74}$$

We observe that $\hat{p}.\hat{u} \geq 0$ in (74), and $\sum_{j=1}^{n} max(-\hat{u}_j, 0) \geq 0$. Hence the right-hand side of (74) must be non-positive, so that

$$max(-\hat{u}_i) = 0 \quad (i = 1, \cdots, n). \tag{75}$$

Equation (75) implies $-\hat{u}_i \leq max(-\hat{u}_i, 0) = 0$ for all i, whence $\hat{u}_i \geq 0$ for all i. This establishes that $\hat{u} \geq 0$, and the proof is complete, QED.

2.2.5 Proof of the Existence

Now, the general idea of the method of proof is to apply Theorem 3 to the excess supply function (64).

Lemma 4 *(i) $\pi_k(p)$ is a continuous function on P_n.*
(ii) $\Psi_k : P_n \to 2^E$ is a closed mapping whose image set $\Psi_k(p)$ is convex for any $p \in P_n$.

Proof: We will prove (i) and (ii) together. Let $p, p^v \in P_n, y_k, y_k^v \in E$, $y_k^v \in \Psi_k(p^v)$ $(v = 1, 2, \cdots)$, and $p^v \to p$, $y_k^v \to y_k$ as $v \to +\infty$. Then we have, by definition, for $v = 1, 2, \cdots$,

$$y_k^v \in Y_k \cap E, \tag{76}$$

$$\pi_k(p^v) = p^v . y_k^v \geq p^v . y \text{ for each } y \in Y_k \cap E, \tag{77}$$

so by virtue the closedness of $Y_k \cap E$ in E and the continuity of the inner product, in the limit (76) and (77) become

$$y_k \in Y_k \cap E \tag{78}$$

$$p.y_k \geq p.y \text{ for each } y \in Y_k \cap E. \tag{79}$$

Relations (78) and (79) mean that y_k maximizes $p.y$ over $y \in Y_k \cap E$, with $p.y_k$ being the corresponding maximum $\pi_k(p)$. This establishes that $\lim_{v \to +\infty} \pi_k(p^v) = \pi_k(p)$, $y_k \in \Psi_k(p)$, proving the continuity of $\pi_k(p)$ and the closedness of Ψ_k. The convexity of $\Psi_k(p)$ has already been established in the Lemma (2), QED.

Lemma 5 $\Phi_i : P_n \to 2^E$ *is a closed mapping whose image* $\Phi_i(p)$ *is a convex set for each* $p \in P_n$.

Proof: Let $p, p^v \in P_n, x_i, x_i^v \in E, x_i^v \in \Phi_i(p^v)$ $(v = 1, 2, \cdots)$, and $p^v \to p$, $x_i^v \to x_i$ as $v \to +\infty$. To prove the closedness, we have to show that $x_i \in \Phi_i(p)$.

Since $X_i \cap E$ is closed, first note that x_i belongs to $X_i \cap E$, and satisfies the budget constraint under p, because of the continuity of the inner product. Now, it remains to be shown that $x \in X_i \cap E$, with x satisfying the budget constraint, implies that

$$x_i \succeq_i x. \tag{80}$$

Now define

$$I_i(p) = p.(a_i - b_i) + \sum_{k=1}^{m} \alpha_{ik}\pi_k(p) \qquad (i = 1, \cdots, l). \tag{81}$$

Notice that, since $a_i > b_i$ by the assumption in Theorem 1, we have $p.(a_i - b_i) > 0$, so $I_i(p) > 0$ for any semi-positive p. Now, we define

$$\lambda_v = \frac{I_i(p^v)}{max(I_i(p^v), p^v.(x - b_i))} \qquad (v = 1, 2, \cdots) \tag{82}$$

Since x satisfies the budget constraint, we have that $p.(x - b_i) \leq I - i(p)$ for any semi-positive p. So, we have

$$1 \geq \lambda_v > 0, \tag{83}$$

$$\lim_{v \to +\infty} \lambda_v = \frac{I_i(p)}{max(I_i(p), p.(x - b_i))} = 1, \tag{84}$$

If we let

$$x^v = (1 - \lambda_v)b_i + \lambda_v x \qquad (v = 1, 2, \cdots), \tag{85}$$

by the convexity of $X_i \cap E$ containing b_i, x, and by (82),(84) we have

$$x^v \in X_i \cap E \qquad (v = 1, 2, \cdots), \tag{86}$$

$$p^v.(x^v - b_i) \leq I_i(p^v) \qquad (v = 1, 2, \cdots), \tag{87}$$

$$\lim_{v \to +\infty} x^v = x. \tag{88}$$

Since $x_i^v \in \Phi_i(p^v)$ implies that x_i^v is a most preferable element of $X_i \cap E$ subject to the budget constraint under p^v, from (86) and (87) we have

$$x_i^v \succeq_i x^v \qquad (v = 1, 2, \cdots). \tag{89}$$

Then, by the continuity of $\succeq_i$, (89) becomes

$$x_i \succeq_i x. \tag{90}$$

This proves the closedness of Φ_i. Convexity of $\Phi_i(p)$ has already been established in the Lemma 3, QED.

From Lemma 4 and Lemma 5, we have that $\Psi_k : P_n \to 2^E$ is a closed mapping whose image set $\Psi_k(p)$ is convex for any $p \in P_n$; $\Phi_i : P_n \to 2^E$ is a closed mapping whose image set $\Phi_i(p)$ is a convex set. Now, in the light of Theorem 2, it suffices to see that the corresponding excess supply function $\chi(p)$ has an equilibrium price vector $\hat{p}$ for which $X(\hat{p}) \cap R^n_+ \neq \emptyset$. To this end, it will be shown that this $\chi(p)$ satisfies all the conditions of Theorem 3. Φ_i, Ψ_k are set-valued mappings whose image sets are subsets of the cube E chosen by (38). Define

$$\Gamma = \{a\} + (\overbrace{E + \cdots + E}^{m}) - (\overbrace{E + \cdots + E}^{l}) \tag{91}$$

Now, the excess supply function $\chi(p)$ can be regarded as a mapping whose image sets are subsets of Γ ,which is a compact convex subset of R^n by Appendix, Theorem 2.10.

We shall show that the mapping $\chi : P_n \to 2^\Gamma$ satisfies the conditions (i) and (ii) of Theorem 3:

(i)Note that all of $\Phi_i, \Psi_k : P_n \to E$ are closed mappings by Lemma 4 and Lemma 5. In addition, a constant mapping $\Phi_\circ : P_n \to \{a\}$ defined by $\Phi_\circ(p) = a$ for any $p \in P_n$ is naturally closed. The mapping χ can therefore be thought of as the composite of two mappings, i.e., the Cartesian product

$$\Phi_\circ \times \prod_{i=1}^{l} \Phi_i \times \prod_{k=1}^{m} \Psi_k \tag{92}$$

of the $l + m + 1$ closed mappings $\Phi_\circ, \Phi_i, \Psi_k$ followed by the single-valued mapping

$$(\hat{x}_0, \hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m) \to \hat{x}_0 + \sum_{k=1}^{m} y_k - \sum_{i=1}^{l} x_i \tag{93}$$

from the Cartesian product $\{a\} \times E \times \cdots \times E$ to Γ. Hence χ is closed by virtue of Appendix, Theorem 4.5(i) and Theorem 4.6(i). On the other hand, from the convexity of the image sets of $\Phi_\circ, \Phi_i, \Psi_k$ ensured in Lemmas 4 and 5 follows the corresponding property of χ by Appendix Theorem 2.10. This verifies (i).

(ii)Previously it has been verified that the Walras Law holds for $\chi(p)$. Thus, applying Theorem 3 to $\chi(p)$, we can guarantee the existence of equilibrium price vector of $\Psi(p), \Phi(p)$. In turn, by virtue of Theorem 2, this implies the existence of competitive equilibrium of the basic model.Q.E.D.

2.3 On The Basic Model

In the proof of the existence in our main model, the essential tool is the existence in the basic model. However, the uncompensated (competitive) equilibrium defined in Definition 1 sometimes fails to be continuous. To overcome this difficulty, we introduce another type of equilibrium, namely the compensated equilibrium, and the interrelations between the two types of equilibrium. An uncompensated (competitive) equilibrium has the usual meaning in the economic literature: a set of prices and production and consumption allocation such that each firm maximizes profits at the given prices, each household chooses one of the most preferable menus from the consumption set at the given prices and with the income implied by those prices and initial holdings of assets and profit shares, aggregate consumption is feasible in the sense of not exceeding the sum of aggregate production and initial endowments. It is convenient to introduce another type of equilibrium, which we call a compensated equilibrium. This differs from the competitive equilibrium in the assumptions about the consumer behavior. Formally,

Definition 3 *An* $(l+m+1)$*-tuple* $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ *of menus of consumption* $\hat{x}_i \in X_i (i = 1, \cdots, l)$*, production processes* $\hat{y}_k \in Y_k (k = 1, \cdots, m)$*, and an n-dimensional price vector* $\hat{p} \geq 0$ *is called a compensated equilibrium of the basic model and the* $\hat{p}$ *is referred to as a compensated equilibrium price vector if the following conditions (i)-(iii) are fulfilled:*

(i)The maximum profit of each producer under $\hat{p}$*. That is,* $\pi_k(\hat{p}) = \hat{p}.\hat{y}_k = max(\hat{p}.y) overall y \in Y_k (k = 1, \cdots, m)$

(ii)Minimum cost of consumption for each consumer under $\hat{p}$*. That is,* $\hat{x}_i$

minimizes $\hat{p}.x_i$ *subject to* $x_i \succeq_i \hat{x}_i (i = 1, \cdots, l)$

(iii) $\hat{p}.\hat{x}_i = \hat{p}.a_i + \sum_{k=1}^m \alpha_{ik}\pi_k(\hat{p}), i = 1, \cdots, l$

(iv) Market clearing condition, i.e. $\sum_{i=1}^l \hat{x}_i \leq a + \sum_{k=1}^m \hat{y}_k$

Let us define

$$M_i(p) = p.a_i + \sum_{k=1}^m \alpha_{ik}\pi_k(p). \tag{94}$$

Lemma 6 *If* $\hat{x}_i$ *is a preferred vector subject to a budget constraint,* $p.x_i \leq M_i$, $x_i \in X_i$, *then* $\hat{x}_i$ *minimizes* $p.x_i$ *subject to the constraint* $x_i \succeq_i \hat{x}_i$.

<u>Proof</u>: Suppose the conclusion is false. Then there exists $x_i^1 \succeq_i \hat{x}_i$, for which $p.x_i^1 < p.\hat{x}_i$. By local nonsatiation, there exists x_i^2 arbitrarily close to x_i^1, for which $x_i^2 \succeq_i x_i^1$, and therefore $x_i^2 \succeq_i \hat{x}_i$. By choosing x_i^2 close enough to x_i^1 we can guarantee $p.x_i^2 \leq p.\hat{x}_i \leq M_i(p)$, which contradicts the hypothesis that $\hat{x}_i$ is preferred in the budget constraint.

Lemma 7 *If* $\hat{x}_i$ *minimizes* $p.x_i$ *subject to the constraint* $x_i \succeq_i x_i^0$ *and if* $p.\hat{x}_i > p.x_i^1$ *for some* $x_i^1 \in X_i$, *then* $\hat{x}_i$ *is preferred in the budget constraint,* $p.x_i \leq p.\hat{x}_i$.

<u>Proof</u>: Consider any $x_i' \in X_i$ for which $p.x_i' \leq \hat{x}_i$. Let

$$x_i(\alpha) = (1-\alpha)x_i' + \alpha x_i^1; \qquad \text{then}$$

$$px_i(\alpha) = (1-\alpha)px_i' + \alpha p\hat{x}_i \text{ for } 0 < \alpha \leq 1.$$

If $x_i(\alpha) \succeq_i x_i^0$, then by hypothesis, $p\hat{x}_i \leq px_i(\alpha)$. Hence, $x_i^0 \succ_i x_i(\alpha)$ and, therefore, $x_i(\alpha) \in \{x_i \mid x_i^0 \succeq_i x_i\}$. Since this last set is closed, it contains $\lim_{\alpha \to 0} x_i(\alpha) = x_i'$. Since $\hat{x}_i \succeq_i x_i^0$, $\hat{x}_i \succeq_i x_i'$ for any x_i' for which $px_i' \leq p\hat{x}_i$. This completes the proof, QED.

It is evident from those Lemmas 6 and 7 that there is a close relationship between the two kinds of equilibria. In one direction the relation is very simple.

Theorem 4 *If* $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ *is a competitive equilibrium, then* $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ *is a compensated equilibrium.*

Proof: Definition 1(i),(iii) are identical with Definition 3(i),(iv). Lemma 6 asserts that a consumption vector that is a most preferable menu subject to a budget constraint also minimizes the cost of consumption among the vectors those are preferred to or indifferent with that consumption vector, so that Definition 1(ii) implies Definition 3(ii). Finally, suppose that Definition 3(iii) does not hold. From Definition 1(ii), $\hat{p}\hat{x}_i < M_i(\hat{p})$. But then, by the local nonsatiation assumption of the consumer i we can choose x_i' arbitrarily close to $\hat{x}_i$ such that $x_i' \succ_i \hat{x}_i$ and $px_i' \leq M_i(\hat{p}$, which is a contradiction to the assumption that $\hat{x}_i$ is a most preferable element in the budget constraint. This completes the proof, QED.

To obtain a partial converse of Theorem 4, first note that, for any $i \in \{1, \cdots, l\}$, if $M_i(p) > 0$ then, $M_i(p) > pb_i$ obviously, since $a_i > b_i$ by the assumption in Theorem 1. The existence of such a vector, like b_i, in the consumption set is important in obtaining the partial converse of Theorem 4 by using Lemma 7. Arrow-Hahn establishes the existence of such a vector in more general situation, but with some different framework in [3] pp.108-109.

Theorem 5 *If* $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ *is a compensated equilibrium and* $M_i(\hat{p}) > 0$*, all* i*, then* $(\hat{x}_1, \cdots, \hat{x}_l, \hat{y}_1, \cdots, \hat{y}_m, \hat{p})$ *is a competitive equilibrium.*

Proof: Definition 3(i),(iv) are identical with Definition 1(i),(iii). If $M_i(\hat{p}) > O$, then by Definition 3(iii), $\hat{p}\hat{x}_i = M_i(\hat{p}) > pb_i$, as just remarked. By Lemma 7, it follows from Definition 3(ii) that $\hat{x}_i$ is a most preferable element subject to the budget constraint $\hat{p}x_i \leq M_i(\hat{p})$. This completes the proof, QED.

In particular, there is a possibility that the uncompensated demand functions are discontinuous. Consider the following example: Suppose there are two goods in the economy, and an individual has an initial stock of good 1, but none of good2 and no share of any firm. Hold the price of good 2 constant, and let p_1 approach zero. The budget constraint requires that

$$p_1.x^1 \leq p_1.x^1 + p_2.x^2 \leq p_1.\bar{x}^1$$

So that for $p_1 > 0, x^1 \leq \bar{x}^1$. On the other hand, when $p_1 = 0$, the set of commodity vectors compatible with the budget constraint is the set of all pairs $(x^1, 0)$, and the chosen value of x^1 may be considerably larger than $\bar{x}^1$. This discussion lightens the difficulty that arises in the proof. The following lemma provides a powerful tool in overcoming this difficulty.

Definition 4 *The compensated demand correspondence of consumer i, $\Phi_i(p, x_i^0)$ is defined as*

$$\Phi_i(p, x_i^0) = \{x_i' \mid x_i' \textit{ minimizes } p.x_i \textit{ subject to } x_i \in X_i, x_i \succeq x_i^0\} \tag{95}$$

Lemma 8 *$\Phi_i(p, x_i^0)$ is upper semi-continuous in p for fixed x_i^0.*

Proof: Let $\{p^v\}$ be a sequence with $p^v \to p$; suppose $x_i^v \in \Phi_i(p^v, x_i^0)$ and $x_i^v \to x_i$. Then $x_i^v \succeq_i x_i^0$, all v, and by the continuity of preferences, $x_i \succeq_i x_i^0$. Take any $x_i' \succeq_i x_i^0$. Then, by definition of $\Phi_i(p^v, x_i^0)$, $p^v x_i^v \leq p^v x_i'$. In the limit, $px_i \leq px_i'$, QED.

3 Overlapping Generations Model

3.1 The OLG Model with Production

Our model is an infinite horizon overlapping generations model. In each period t, $(t = 1, 2, \cdots)$ there is a finite number, n, of completely perishable, nonstorable, physical commodities. Each consumer h, $(h = 0, 1, 2, \cdots)$ is born at period $t = h$ and lives only two periods during $t = h$ and $t = h + 1$. However, in each period t, there are two generations of consumers, old generation who was born at the beginning of period $(t - 1)$ and young generation who is born at the beginning of period t.

There is a firm in the economy which is owned by the old generation in each period t. At the end of period t, the ownership of the firm is transferred to the young consumer of that period without cost.

Each consumer h has his consumption set, formally as:

$$x_h = x_h^{h+1} = (x_h^{h+1,1}, \cdots, x_h^{h+1,n}) \in X_h^{h+1} \text{ for } h = 0 \tag{96}$$

$$x_h = (x_h^h, x_h^{h+1}) = (x_h^{h,1}, \cdots, x_h^{h,n}, x_h^{h+1,1}, \cdots, x_h^{h+1,n}) \in X_h^h \times X_h^{h+1} \tag{97}$$

for $h \geq 1$ where X_h^h, X_h^{h+1} are closed convex sets in R^n for each $h \geq 0$, and there exist lower bounds c_h^h, c_h^{h+1} of X_h^h, X_h^{h+1} , respectively, such that

$$x_h^s \geq c_h^s \text{ for all } x_h^s \in X_h^s, \quad s = h, h+1 \tag{98}$$

relative to the semi-order based on the component-wise comparison among vectors. Define $X_0 = X_0^1$ for h=0 and $X_h = X_h^h \times X_h^{h+1}$ for $h \geq 1$. Each consumer h has a preference field $(X_h, \succeq_h)$ such that

$$\forall x_h, y_h \in X_h : x_h \succ_h y_h \Rightarrow \lambda.x_h + (1 - \lambda)y_h \succ_h y \text{ for each } \lambda \in (0, 1] \tag{99}$$

Each preference relation $\succeq_h$ is continuous. Each consumer h has an initial holding w_h such that

$$w_h = w_h^{h+1} = (w_h^{h+1,1}, \cdots, w_h^{h+1,n}) \in R_{++}^n \text{ for } h = 0 \tag{100}$$

and

$$w_h = (w_h^h, w_h^{h+1}) = (w_h^{h,1}, \cdots, w_h^{h,n}, w_h^{h+1,1}, \cdots, w_h^{h+1,n}) \in R_+^n \times R_{++}^N$$

for $h \geq 1$.For each consumer h, there is a commodity bundle b_h in X_h fulfilling $w_h > b_h$ and he has no satiation point.

At each period t, the firm has a technology set $Y_t(t = 1, 2, \cdots)$, which is a subset of R^n containing the origin of R^n. Y_t is convex and closed in R^n. We also have that

$$Y_t \cap R_+^n = \{0\} \text{ and } Y_t \cap (-Y_t) = \{0\}. \tag{101}$$

Since the firm belongs to the old consumer and ownership of the firm transfers to the other consumer at the end of that period without any cost, as stated the profit of the firm is transferred to the old consumer in each period. There exists a futures market in the model such that each consumer h can bargain for the commodities of future periods.

Consumption sequences are denoted by $x = (x_0, x_1, \cdots, x_h, \cdots)$, price sequences are denoted by $p = (p^1, p^2, \cdots, p^t, \cdots)$, and production sequences are denoted by $y = (y^1, y^2, \cdots, y^t, \cdots)$ such that $x_h \in X_h$ for each $h \geq 0$ and $p^t \geq 0$ for each $t \geq 1$, and $y^t \in Y_t$ for each $t \geq 1$. $\pi^t(p)$ is the maximum profit that the firm attains under the given price sequence p in period t, that is,

$$\pi^t(p) = max(p^t.y) \text{ over all } y \in Y_t, \tag{102}$$

and $Y_t(p)$ is the supply function of the firm in period t, given the price sequence p,

$$Y_t(p) = \{y^t \mid \pi^t(p) = p^t.y^t, \quad y^t \in Y_t\}. \tag{103}$$

$X_h(p)$ represents the ordinary lifetime consumption profiles for consumer h in his budget constraint, given the price sequence p, that is,

$$X_h(p) = \{x'_h \mid x'_h \in X_h \text{ and } x'_h \succeq x_h \text{ subject to } x_h \in B_h(p)\} \tag{104}$$

where

$$B_0(p) = \{x_0 \mid x_0 \in X_0, p^1.x_0^1 \leq p^1.w_0^1 + \pi^1(p^1)\}, \text{ for } h = 0 \tag{105}$$

and

$$B_h(p) = \{x_h \mid x_h \in X_h, \; p_h^h.x_h^h + p_h^{h+1}.x_h^{h+1} \leq p^h.W_h^H + p^{h+1}w_h^{h+1} + \pi^{h+1}(p^{h+1})\} \tag{106}$$

for $h \geq 1$.

$X_h(p, \bar{x}_h)$ represents the compensated lifetime consumption profiles for consumer h given the price sequence p. That is

$$X_h(p, \bar{x}_h) = argmin_{x_h \succeq_h \bar{x}_h} p.x_h \tag{107}$$

It is now time to define the competitive equilibrium for the overlapping generations model.

Definition 5 *The triplet of sequences $(\hat{x}, \hat{y}, \hat{p})$ of consumption, production and price vectors, respectively, is called a competitive equilibrium of the OLG* *model if the following conditions (i)-(iii) are fulfilled:*

(i)The maximum profit in each period under $\hat{p}^t$. That is,

$$\pi(\hat{p}^t) = \hat{p}^t.\hat{y}^t = max(\hat{p}^t.y) \text{ over all } y \in Y_t(t = 1, 2, \cdots), \tag{108}$$

or in other words $\hat{y}^t \in Y_t(p)$. (ii)The optimum preference of each consumer subject to budget constraint under $\hat{p}^t$. That is, $\hat{x}_h \in X_h(\hat{p})$ for given price sequence $\hat{p}$.

(iii)The balance of aggregate supply and demand in each period t. That is,

$$\hat{x}^t_{t-1} + \hat{x}^t_t \leq w^t_{t-1} + w^t_t + \hat{y}^t \text{ for } t \geq 1 \tag{109}$$

with equality holding in the jth component relation of the above inequality if the corresponding component of $\hat{p}^t$, namely $\hat{p}^{tj}$, is positive.

Theorem 6 *Under the assumptions stated above, there exists a competitive equilibrium of the OLG model.*

3.2 Proof of the Theorem

The proof of this Existence Theorem is similar in spirit to that presented in Balasko, Cass, and Shell [2]. Firstly, existence of the suitably defined competitive equilibria in the suitably truncated economies are established. Then, it is shown that the limit of those competitive equilibria yields the competitive equilibria of the OLG model.

Definition 6 *The initial holdings and the tastes of the consumers born up through period t and the technology sets of the firm up through period t is called the t-truncated economy, and it is denoted by ξ^t. In addition, the triplet of sequences (x,y,p) is a t-equilibrium if the following conditions are satisfied:*

(i) $\pi^s(p) = p^s.y^s = max(p^s.y)$ *over all* $y \in Y_s$ $s = 1, 2, \cdots, t$

(ii) $x_h \in x_h(p), h = 0, 1, 2, \cdots, t$

(iii) $x^h_{h-1} + x^h_h \leq w^h_{h-1} + w^h_h + y^h, h = 1, 2, \cdots, t$ *holding equality in the jth component if the corresponding price* $p^h_j > 0$.

Notice that if (x,y,p) $= ((x_0, \cdots, x_t, \cdots), (y^1, \cdots, y^t, \cdots), (p^1, \cdots, p^t, p^{t+1}))$ is a t-equilibrium then any (x',y',p') satisfying

$$x'_h = x_h \text{ for } h = 0, \cdots, t \tag{110}$$

$$y'^s = y^s \text{ for } s = 1, \cdots, t \tag{111}$$

$$p'^s = p^s \text{ for } s = 1, \cdots, t+1 \tag{112}$$

is also a t-equilibrium.That is, the components $(x_{t+1}, x_{t+2}, \cdots)$,$(y^{t+1}, y^{t+2}, \cdots)$, $(p^{t+2}, p^{t+3}, \cdots)$ are completely indeterminate at a t-equilibrium.

Lemma 9 *There exists a t-equilibrium* $(x(t), y(t), p(t))$ *of the t-truncated economy* ξ^t*, for each* $t \geq 1$.

Proof: Completing the system of the t-truncated economy ξ^t by $x^{t+1}_t \leq w^{t+1}_t + y_{t+1}$ where $x_t = (x^t_t, x^{t+1}_t) \in x_t(p)$ and $y_{t+1} \in Y^t(p^t)$ and holding equality at the jth component if $p^{t+1j} \geq 0$, ξ^t exhibits all the standard assumptions of the basic model. So there exists a competitive equilibrium of the economy, then by completing the rest of sequences by suitable members this competitive equilibrium obviously yields a t-equilibrium of ξ^t, QED.

Lemma 10 *If $t' > t$ and $(x(t'), y(t'), p(t'))$ is a t'-equilibrium of $\xi^{t'}$ then $(x(t'), y(t'), p(t'))$ is a t-equilibrium of ξ^t , too.*

Proof: This is obvious from the definition of t-equilibrium of ξ^t, QED.

If $(x(t), y(t), p(t))$ is a t-equilibrium of ξ^t, then in fact, the first $(t+1)$ elements of the sequence $\{x(t)\}$, i.e., $\{x(t)_0, x(t)_1, \cdots, x(t)_t\}$, and the first t elements of the sequence $\{y(t)\}$, i.e., $\{y(t)^1, \cdots, y(t)^t\}$ can be restricted to some compact sets as it was done in the proof of the existence in the basic model. In addition, it is easy to see that, those compact sets are independent of truncation t.

Now, since for all $t \geq 1$, there exists a t-equilibrium, and for any $t' > t$, any t'-equilibrium is also a t-equilibrium, we can conclude that, for any given τ there exists a subsequence $\{t_v\} \subset \{t\}$ such that

$$lim_{v \to +\infty}((x(t_v)_1, \cdots, x(t_v)_\tau, \quad y(t_v)^1, \cdots, y(t_v)^\tau) \tag{113}$$
$$= (x_1^*, \cdots, x_\tau^*, y^{1*}, \cdots, y^{\tau *})$$
$$= (x^*(\tau), y^*(\tau)). \tag{114}$$

Since this is true for any $\tau \geq 1$, letting $\tau \to \infty$, and using Cantor's diagonalization procedure, we can find a subsequence $\{t_\eta\} \subset \{t\}$ so that

$$\lim_{\eta \to \infty} (x(t_\eta), y(t_\eta)) = (x^*, y^*). \tag{115}$$

Lemma 11 *For every $\tau \geq 1$, there exists $p^*(\tau)$ such that $(x^*, y^*, p^*(\tau))$ is a τ-equilibrium of ξ^τ, where x^* and y^* are as in (114).*

Proof: Given arbitrary $\tau \geq 1$, consider a subsequence $\{t_\eta\} \subset \{t\}$ with corresponding t_η-equilibria $\{(x(t_\eta), y(t_\eta), p(t_\eta))\}$ such that $\lim_{\eta \to \infty}(x(t_\eta), y(t_\eta)) = (x^*, y^*)$. Because every t_η-equilibrium has the property that $x_h \in X_h(p(t_\eta))$ for $h = 0$, thus $p(t_\eta)^1 > 0$, it follows that we can normalize $p(t_\eta)$ so that

$$\| \, (p(t_\eta)^1, p(t_\eta)^2, \cdots, p(t_\eta)^{\tau+1} \, \| = 1. \tag{116}$$

So, without loss of generality, we can assume that the first $(\tau + 1)$ elements of $p(t_\eta)$ converge to $(p^{1*}, p^{2*}, \cdots, p^{\tau+1*}) > 0$ as $\eta \to \infty$. Now, by filling out the rest of the sequence by appropriate members, let us set

$$p^*(\tau) = (p^{1*}, p^{2*}, \cdots, p^{\tau+1*}, p^{\tau+2}, \cdots). \tag{117}$$

In order to verify that $(x^*, y^*, p^*(\tau))$ is a τ-equilibrium, we will employ a standard maneuver, which basically involves switching back and forth between the budget-constrained utility maximization problem and its preference-constrained budget minimization where the preference-constraint is $x_h \succeq_h x_h^*$.

We know that, for every $t_\eta \geq \tau$, the t_η-equilibrium $(x(t_\eta), y(t_\eta), p(t_\eta))$ is also a τ-equilibrium, so is a competitive equilibrium of ξ^τ. From the previous discussions, we know that, this competitive equilibrium of ξ^τ yields a compensated equilibrium of ξ^τ. Moreover, we know that compensated life-time consumption profiles are upper semi-continuous. Thus, going to limit as $\eta \to \infty$, $(x^*, y^*, p^*(\tau))$ is also a compensated equilibrium of ξ^τ. Now invoking the assumption that $w_h > 0$, it can be easily seen that, at the given price sequence $p^*(\tau)$, every consumer h born up through period τ must have positive income, so by Theorem 5, $(x^*, y^*, p^*(\tau))$ yields a competitive equilibrium of ξ^τ, which, in fact, yields a τ-equilibrium, QED.

Lemma 12 *There exist nonnegative, finite constants $0 \leq K^t < \infty$ for $t \geq 1$ with the property that if $(x^*, y^*, p^*(\tau))$ is a τ-equilibrium of ξ^τ, then $p^*(\tau)$ can be normalized so that $\| p^*(\tau)^1 \| = 1$ and $\| p^*(\tau)^{t+1} \| \leq K^t$ for $1 \leq t \leq \tau$.*

Proof: For any $\tau \geq 1$, we can obtain a τ-equilibrium of the specific form $(x^*, y^*, p^*(\tau))$, as stated above. Since $x_o^* \in X_0(p^*(\tau))$, we know that $p^*(\tau)^1 > 0$, or $\| p^*(\tau)^1 \| > 0$. So, in order to prove the lemma, it suffices to show that there exists bounds K^t such that, for every $1 \leq t \leq \tau$, the price sequence $p^*(\tau)$ satisfies

$$\| p^*(\tau)^{t+1} \| \,/\, \| p^*(\tau)^1 \| \leq K^t. \tag{118}$$

Suppose, in contrary that, (119) is not true, that is, for some $1 \leq t \leq \tau$, there are sequences of τ-equilibria $\{(x^*, y^*, p^\eta(\tau))\}$ such that,

$$\lim_{\eta \to \infty} \| p^\eta(\tau)^{t+1} \| \,/\, \| p^\eta(\tau)^1 \| = \infty. \tag{119}$$

Since $p^\eta(\tau) > 0$, we can again normalize $p^\eta(\tau)$ so that $\| (p^\eta(\tau)^1, \cdots, p^\eta(\tau)^{\tau+1})) \| = 1$, and, without loss of generality, assume that the first $(\tau + 1)$ elements of

$p^{\eta}(\tau)$ converge to $(p^{1\infty}, p^{2\infty}, \cdots, p^{\tau+1\infty}) > 0$ as η approaches to ∞. But this leads to the following inconsistency: On the one hand, we know that every price sequence $p^{\infty} = (p^{1\infty}, \cdots, p^{\tau+1\infty}, p^{\tau+2}, \cdots)$ yields a τ-equilibrium (x^*, y^*, p^{∞}), so that $p^{1\infty} > 0$. On the other hand, given the sequence $\{p^{\eta}(\tau)\}$, we know that, for every $0 < K < \infty$, there exists $\eta_K < \infty$ such that

$$\| p^{\eta}(\tau)^1 \| > 0, \quad \| p^{\eta}(\tau)^{t+1} \| \leq 1 \quad \text{and} \tag{120}$$

$$\| p^{\eta}(\tau)^{t+1} \| \, / \, \| p^{\eta}(\tau)^1 \| > K, \tag{121}$$

or $1 > K \| p^{\eta}(\tau)^1 \| > 0$, for $\eta > \eta_K$, so that $p^{1\infty} = 0$. Hence, we conclude that the requisite bounds must in fact exist, QED.

The remainder of the argument is now almost routine. Above lemmas 11,12 guarantee that for every $\tau \geq 1$ there exists a τ-equilibrium of the specific form $(x^*, y^*, p^*(\tau))$, and $p^*(\tau)$ is restricted to the bounded subset of conceivable price sequences

$$\beta^* = \{p \mid p \geq 0, \| \hat{p} \| = 1 and \| p^{t+1} \| \leq K^t\} \tag{122}$$

Hence we can choose a subsequence of periods $\{t_\kappa\} \subset \{t_\eta\}$ with the corresponding t_κ-equilibria $\{(x^*, y^*, p^*(t_\kappa))$ such that

$$lim_{\kappa \to \infty}(p^*(t_\kappa)) = p^* \in \beta^* \tag{123}$$

Finally, employing standard maneuver, it follows that for every $v \geq 1, (x^*, y^*, p^*)$ is a t-equilibrium and, therefore that, by definition (x^*, y^*, p^*) is also a competitive equilibrium of the OLG model, Q.E.D.

4 Conclusion

In the proof of the basic model, restricting the competitive allocations to some compact sets, the tricks used by Balasko, Cass and Shell [2] are essential. The assumptions are fairly general, and can be easily interpreted by any reader who is familiar with mathematical economics.

This model can be taken as a first step in developing the OLG models with production. Its setup facilitates easy understanding and removes the profound complexities in discussion.

The transfer of the ownership of the firm in the model can be understood as bequest. Adding more firms to the model carries no more complication either into the model or into the proof. A model with more than one consumer brings the question of transfers of the ownerships of firms in the model. A given sequence of coefficients, those stand for bequest, is a way of developing the model without further work in the proof. The same proof is valid for this model, too. But the more interesting view is adding a stocks market to the model, dropping the futures market assumption, and examining the existence of equilibrium and its efficiency. For this line of further research, our model may serve as a suitable first step. Existence of futures market improves the efficiency comparing to Samuelson's OLG model [5]. But stocks market may play a similar role as futures market or money in improving the efficiency of the equilibrium. Efficiency of equilibrium, in our model, is another issue requiring further research.

There are many examples of modifications to the basic model in the literature. Using the same arguments, it is possible to generalize our model further.

Appendix

Theorem 1 *If X_i is a convex (closed) (open) (compact) subset of R^{n_i} $(i = 1, \cdots, s)$, then $\prod_{i=1}^{s} X_i$ is a convex (closed) (open) (compact) subset of $\prod_{i=1}^{s} R^{n_i}$ (respectively).*

Theorem 2 *If X_i are convex (compact) subsets of R^n $(i = 1, \cdots, s)$, their linear combination $\sum_{i=1}^{s} X - i$ is also convex (compact).*

Theorem 3 *Let X be a nonempty compact convex set in R^n, and $f : X \rightarrow 2^X$ be a set valued mapping which satisfies*
(a)for each $x \in X$, the image set $f(x)$ is a nonempty convex subset of X; and
(b)f is a closed mapping.
then f has a fixed point.

Theorem 4 *Let f^i $(i = 1, \cdots, s)$ be the set valued mapping.*
(i) If f^i are closed then their Cartesian product is also closed.
(ii) Suppose that the image sets of f^i are compact. Then if the f^i are u.s.c, their Cartesian product is also u.s.c.

Theorem 5 *Let $f : X \rightarrow 2^Y$, and $g : Y \rightarrow 2^Z$ be the set valued mappings, then*
(i) Suppose that a compact subset M of Y exists such that $\cup_{x \in X} f(x) = f(X) \subset M$. Then if f and g are closed, their composite mapping is also closed.
(ii) If f and g are u.s.c., their composite mapping is also u.s.c.

Lemma 1 *Let $(X, \succeq)$ be a preference field, and $M \subset X$. If $\succeq$ continuous and M is compact, M contains at a most(least) preferable element and the totality of all most(least) preferable elements is a compact set.*

Lemma 2 *Let $(X, \succeq)$ be a convex preference field and M be a convex subset of X. Then most preferable elements in M, if any, form a convex subset of M.*

Proofs of those theorems and lemmas above are available in Nikaido [4] as well as in many mathematical economics references.

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