IMPLEMENTATION IN DOMINANT STRATEGY EQUILIBRIUM

A Thesis Submitted to the Department of Economics and the Institute of Economics and Social Sciences of Bilkent University In Partial Fulfillment of the Requirements for the Degree of

MASTER OF ARTS IN ECONOMICS

by

Özgür KIBRIS

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I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Dr. Semih Koray

I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

IMPLEMENTATION IN DOMINANT STRATEGY EQUILIBRIUM

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A social choice rule is any proposed solution to the problem of collective decision making and it embeds the normative features that can be attached to the mentioned problem. Implementation of social choice rules in dominant strategy equilibrium is the decentralization of the decision power among the agents such that the outcome that is a priori recommended by the social choice rule can be obtained as a dominant strategy equilibrium outcome of the game form which is endowed with the preferences of the individuals. This work has two features. First, it is a survey on the literature on implementation in dominant strategy and its link with the economic theory. Second, it constructs some new relationships among the key terms of the literature. In this framework, it states and proves a slightly generalized version of the Gibbard-Satterthwaite impossibility theorem. Moreover, it states and proves that the cardinality of a single-peaked domain converges to zero as the number of alternatives increase to infinity.

Key Words: Social Choice Rule, Implementation, Game Form, Normal Form Game, Dominant Strategy Equilibrium, Strategy Proofness, Decomposable Preference Domain, Single-Peaked Preference Domain.

ÖZET

BASKIN STRATEJİ DENGELERİ ARACILIĞIYLA UYGULAMA

ÖZGÜR KIBRIS
Yüksek Lisans Tezi, İktisat Bölümü
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Bir grup bireyin ortak karar alma problemine önerilen herhangi bir çözüme bir toplumsal seçim kuralı denir. Toplumsal seçim kurallarının oyun formlarının baskın strateji dengeleri aracılığıyla uygulanması bu kurallarca önerilen sonuçların, karar yetkisinin bireyler arasında dağıtılması sonucu ortaya çıkan ve bireylerin tercihleri ile donanmış olan oyun formlarının baskın strateji denge sonuçları ile elde edilmesi demektir. Bu çalışmanın ikili bir niteliği vardır. Birincisi, bahsi geçen teori ve bunun ekonomi teorisine uygulanımı ile ilgili bir literatür araştırması yapılmaktadır. İkincisi, literatürdeki kimi anahtar terimler arası yeni ilişkiler elde edilmektedir. Bu çerçevede Gibbard–Satterthwaite imkansızlık teoreminin daha genel bir uyarlaması sunulur ve ispatlanır. Bunun dışında, alternatif sayısı sonsuza giderken tek-tepeli tanım kümelerinin kardinalitesinin sıfıra gittiği ispatlanır.

Anahtar Kelimeler: Toplumsal Seçim Kuralı, Uygulama, Oyun Formu, Normal Formlu Oyun, Baskın Strateji Dengesi, Strateji Geçirmezlik, Ayrıştırılabilir Tercih Tanım Kümesi, Tek-tepeli Tercih Tanım Kümesi.

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Contents

1 INTRODUCTION

Collective decision making has been one of the main concerns of political and social sciences for a long time. It refers to a wide range of situations spanning from voting methods to allocation rules. Modelization of collective decision making always involves both normative and positive features. While the normative side includes prescriptive value judgements represented by *social choice rules*, the positive side analyzes the strategic behavior as represented by game theoretic equilibrium concepts.

Social choice theory is concerned with normative decision making: several agents have to decide on some issue of collective interest whereas their preferences about the issue may differ. A social choice rule is any proposed solution to this problem. Its being normative is rooted in its dependency on social norms, ethics, etc..

Given that the society views as desirable certain ethics of collective decision, is it possible, and so how, to decentralize the decision power among individual agents in such a way that by freely exercising this decision power the agents eventually select the very outcome(s) recommended as a priori desirable? This is called as the *implementation problem* and it is central for the link among the normative and the positive properties of collective decision making. Thus, a social choice rule, given the preferences of the individuals, recommends an outcome according to some normative criteria . The process of achieving this outcome (mostly through the decentralization of the decision power) is called as implementation. This task is mainly the obtainment of cooperative goals via noncooperative tools.

This characterization is closely related to the neoclassical definition of the democracy. H. Moulin, in his 1983 book, "The Strategy of Social Choice"[13], defines democracy as follows: "Democracy, in its neoclassical context means that the goals of collective action must rely on the opinion of individuals and these opinions only". Thus, the tools that democracies use to obtain social goals should be identical to those of the mechanisms that are used to implement social choice rules via decentralization of the decision power.

Economics, being a social science itself, has faced the problem of collective decision making in several different ways. One of the most striking fields is the allocation problem. While the normative side of the approach proposes some concepts such as Pareto optimality, the outcome is implemented through pseudo-games which are called as abstract economies. The social equilibrium of an abstract economy is shown to be identical to the competitive equilibrium of a pure exchange economy and is shown to satisfy some socially desirable conditions such as Pareto optimality in the case of private goods. This is nothing but a kind Nash implementation of an allocation rule satisfying some desirable criteria.

It is wide known in economic theory that while in case of private goods a socially desirable allocation rule can be implemented via the social equilibrium of an abstract economy, this is not the case for public goods. The phenomenon is called as the provision of public goods and there is a bunch of literature about this issue which mainly agrees about the occurrence of a "prisoners-dilemma"- like situation in case of public goods. This is a typical case where the individuals benefit through misrepresenting their preferences, and is thus closely related to the strategy proofness concept discussed in this paper.

The concept, strategy proofness (or equivalently nonmanipulability) of a social choice rule, is mainly rooted in the knowledge of the individuals that their preferences about the issue have, up to some degree, an effect on the socially desirable outcome(s) that is (are) chosen by the social choice rule mentioned. The important point is whether an individual has an incentive to misrepresent his/her preferences. If there occurs such a case, the strategical misrepresentation of the individuals may lead to an outcome that is an undesirable one in terms of the criteria defined above.

The penchant that individuals have for strategizing, causes economic theorists trouble because the essence of an individual's strategic choice is to guess correctly the actions of other individuals and then to choose the action that results in the best attainable outcome for himself/herself. But in case of the lack of coordination among individuals, this may lead to undesirable outcomes.

For strategy proof mechanisms, the question of strategy never arises, because no agent has a reason to deviate from the dominant strategy of truth telling. This makes the analysis of strategy proof mechanisms trivial in comparison to the analysis of mechanisms that are not strategy proof, because questions about the information that agents possess about the others can be ignored.

There are two important points about strategy proofness. The first one is that it can only be defined for the social choice rules that are singleton-valued (social choice functions). This is mainly because of the necessity that each individual has to compare the outcome that occurs when he/she tells the truth to the ones that he/she can obtain through misrepresentation for each possible preferences of the other individuals. Since the individuals have preference relations that are on the elements of the alternative set, they can't use these preferences to compare subsets of this alternative set. However, under some circumstances, the concept of strategy proofness can be extended to the concept of implementibility in the dominant strategy equilibria of a mechanism (game form). This is the second important point and is closely related to the implementation problem mentioned above.

A mechanism (game form) is a set of strategy spaces for each individual and a function (the outcome function) that leads a strategy tuple to the outcome space. It lacks a preference profile for the individuals and when attached a preference profile is called as a game. Note that a social choice function can be viewed as a game form where the strategy spaces of each individual is a set of the admissible preferences for that individual and the outcome function is simply the social choice function itself. This kind of mechanisms are called as revelation mechanisms. In such a setting, the sincere revelation of the preferences occurs as a dominant strategy equilibrium of the game that is produced by attaching the (true) preferences of the individuals to the mentioned revelation mechanism in case of strategy proofness.

Dominant strategy implementability of a social choice rule means that there exists a mechanism such that for any (true) preference profile of the individuals, the outcome of the social choice rule one-to-one matches with the outcome(s) generated by the dominant strategy equilibrium(s) of the implementing mechanism. Mostly, the concept of strategy proofness can be used interchangeably with dominant strategy implementability. Though most of the literature about strategy proofness doesn't find it necessary to distinguish between these concepts, there are some conditions that has to be satisfied to use these terns interchangeably.

At this point, there may occur a question of why the concept of dominant strategy equilibria is used instead of other wide known solution concepts such as Nash equilibria. The main reason of this is dominant strategy equilibria being the most noncooperative one among all solution concepts. In dominant strategy equilibria, the individual does not need any information about the others while making his/her strategical choice. That means, if the individuals have dominant strategies that they can utilize, they don't need to make any strategical guess about what the others do. Thus, no information problem occurs for dominant strategy equilibria to be reached in a game.

Since the main aim of the implementation business is the obtainment of cooperative outcomes via noncooperative tools, and since it is hard to obtain cooperation in case of inability to keep the track of deviations from this cooperation, it is a good solution to prepare a playground to the individuals (game form) such that they can act according to their incentives and at last obtain the cooperative outcome. The best way to do this is the dominant strategy implementability of the social choice rule that leads the individuals to cooperative outcomes. Such a situation has two main advantages to the other solution concepts. The first one is the innecessity of information as mentioned above and the second advantage (compared to the Nash concept) is that it is known how the system reaches to the dominant strategy equilibrium. The situation is different for the Nash equilibrium concept. It quarantees that when reached to the Nash equilibrium the individuals have no incentive to deviate from it, but tells nothing about how this equilibrium will be reached.

As a result of the above reasons, the history of the implementation literature starts with dominant strategy implementation. This is followed by the famous Gibbard-Satterthwaite impossibility theorem which tells that under certain conditions it is impossible to find a strategy proof social choice function that is nondictatorial. This result and the restrictiveness of the domains that admit the construction of strategy proof and nondictatorial social choice rules lead the literature to focus on some alternative solution concepts such as Nash equilibrium.

This thesis is a combination of the followings: a survey of the literature on strategy proofness, the task of completing the points which are implicitly assumed by the literature and not formally analyzed up to date, and some new findings that are an addition to the theory of social choice. It mainly aims to be a starting point for who wants to deal with the social choice theory. Thus, all the concepts used are defined and related to each other in an axiomatic approach. Since there is a wide range of different terminology and definitions about some of the concepts in the literature, we found it necessary to combine them under a uniform terminology.

The thesis includes five main parts, excluding the conclusion part. The first chapter, which is called as the "preliminaries", constructs the model, giving the necessary definitions and some relationships among the concepts introduced. In this chapter, the presented relationships are limited to that ones which were proved by other authors. The second chapter is formed of two sections. The first section is a presentation of the Gibbard-Satterthwaite impossibility theorem, its proof and its relationship with Arrow's impossibility theorem. The second section presents three alternative ways to get rid of this impossibility in implementation. In the third chapter we present our main findings about the relationship between strategy proofness and dominant strategy implementability and an extended impossibility theorem together with other findings about the relationships among the other concepts used in the thesis. The fourth chapter relates the strategy proofness concept with economics and presents an introduction of this concept to the allocation problem. The last chapter is about the rareness of the domains that permit the construction of strategy proof mechanisms that are nondictatorial. In this framework, single-peakedness, one of the most well-known examples of this appreciated domains is analyzed and it is shown that the probability of obtaining a single-peaked domain goes to zero as the number of alternatives increases.

2 PRELIMINARIES

Let N={1,...,n} be a society of n individuals who must select a group of alternatives from an alternative set, A={x,y,...,w} which is finite. Each individual $i \in N$ has a complete and transitive (and thus reflexive) binary relation R_i on the set A. The set of all complete and transitive orderings on A is defined as Ω. Moreover the set of all linear orders on A will be called as L(A). The preference domain, $D_i(A)$ for an individual i, will be defined as a subset of Ω, moreover D(A) will be defined as the Cartesian product of $D_i(A)$s of each individual. We will denote the set of nonempty subsets of A as Π. For a binary relation P on A, the set of elements of A that are maximal with respect to this binary relation will be shown as argmaxP; moreover for any $B \in \Pi$, the elements of B that are maximal with respect to P will be shown as $\text{argmax}_B P$.

Definition: (*Social Choice Rule*)

A *Social Choice Rule* (SCR) is a nonempty-valued correspondence from a domain of preference profiles, D(A), which is either a subset of Ω^n or a subset of $L(A)^n$, on A to a range of alternatives, A. That means, F: D(A) $\rightarrow$A is a SCR if it is nonempty valued. From now on we will call F a *social choice function*, SCF, if it is singe valued, and a *social choice correspondence*, SCC, if it is set valued.

Definition: (*Game Form* or *Mechanism*)

A game form, g, is an (N+1) tuple $g=(X_i, i \in N; \pi)$ where

a) For all $i \in N$, X_i is the strategy (message) space of individual i

b) π: X$\rightarrow$A is a function (an outcome function) where $X=\prod_{i \in N} X_i$

From now on the terms *game form* and *mechanism* will be used interchangeably.

Definition: (*Normal Form Game*)

Given a game form, $g=(X_i\ ,\ i\in N;\ \pi\ )$, and a preference profile $R=\prod_{i\in N} R_i$ where $R\in D(A)$, $g[R]=(X_i\ ,\ R_i\ o\pi,\ i\in N)$ is a normal form game (NFG) where agent i's strategy is $x_i \in X_i$ and his/her utility is determined through $R_i\ (\pi(x))$ where $x=(x_1,...,x_n)$.

Definition: (Dominant Strategy)

Given a NFG, $g[R]=(X_i\ ,\ R_i\ o\pi,\ i\in N)$, a strategy x_i of individual i is said to be a *dominant* strategy (DS) of i if for any strategy tuple of the other individuals there doesn't exist another strategy of i which makes him/her strictly better-off. That is, for all $y_{-i}\in X_{-i}$ and for all $z_i\in X_i$, $\pi(x_i\ ,\ y_{-i}\ )\ R_i\ \pi(z_i\ ,\ y_{-i}\ )$.

Definition: (Dominant Strategy Equilibrium of a NFG)

A strategy n-tuple $x=(x_1\ ,...,x_n)$ is said to be a *dominant strategy equilibrium* (DSE) of a NFG if for each individual i, x_i is a dominant strategy of that individual. The set of dominant strategy equilibria of a NFG, $g[R]=(X_i\ ,\ R_i\ o\pi,\ i\in N)$, are shown as $\sigma(g[R])$.

Definition: (Implementability)

A social choice rule (SCR), $F{:}D(A)\rightarrow A$, is said to be *implementable* if there exists a mechanism $g=(X,\pi)$ s.t. for all $R\in D(A)$, $F(R)=\pi(\sigma(g[R]))$. Note that the right hand side is not necessarily the image of a single value, but is used to denote a subset of the range, A, formed of the images of the Dominant strategy equilibria of the normal form game (NFG), g[R], with respect to the outcome function π.

Note that every social choice function can be viewed as a mechanism (game form) where for each individual $i\in N$, $X_i = D_i\ (A)$. This kind of mechanisms are called as *Revelation Mechanisms*. That is, if F is a SCF then $g_F = (D_i(A),\ i\in N;\ F)$ is a *Revelation Mechanism*. These mechanisms have the property that the strategy for each individual is revealing his/her preference ordering on the feasible set, and the outcome function is simply the SCF itself.

For later usage during the construction of the relation between the social choice rules and Arrow's famous impossibility theorem we need to define what a social welfare function is.

Definition: (Social Welfare Function)

A function $f:D(A)\rightarrow B(A)$ is said to be a *social welfare function* where B(A) is a nonempty subset of Ω. Given the preference profile of the society, a social welfare function assigns this profile to a social preference. That is for any $R\in D(A)$, f(R) is a binary relation on $A\times A$.

Now, to be able to construct a social welfare function (SWF) that satisfies the conditions necessary for the presentation of the Arrow's famous impossibility theorem, we need the following properties.

Definition: (Agreeing profiles)

Given a subset B of A, the alternative set, and two admissible preference profiles $P,Q\in D(A)$, P and Q are said to *agree on B* if for each individual $i\in N$, and for each $x,y\in B$, (xP_iy iff xQ_iy) holds.

Definition: (Independence of irrelevant alternatives)

A SWF is said to satisfy the condition of *independence of irrelevant alternatives* (IIA) if for any subset B of A and any two admissible preference profiles $P,Q\in D(A)$ which agree on the set B, the SWF, f, should lead to the same ordering on B for each profile P and Q. That is, for all $x,y\in B$, ($xf(P)y$ iff $xf(Q)y$) should hold.

Definition: (Monotonicity)

Let B and C be subsets of A s.t. $C=B\setminus\{x\}$. Now *monotonicity* is satisfied if whenever

(i) there are profiles P and Q s.t. for all $z,y\in C$ and for all $i\in N$, (zP_iy iff zQ_iy) holds and

(ii) for all $y\in C$, xP_iy implies xQ_iy

then for all $y \in B$, $xf(P)y$ implies $xf(Q)y$.

Then comes the definition of *strategy proofness.*

Definition: (*Strategy Proofness, Nonmanipulability*)

A SCF, $F{:}D(A) \to A$ is said to be *strategy proof (nonmanipulable)* if for any admissible profile $R \in D(A)$, for any individual $i \in N$ and for any preference $Q_i \in D_i(A)$,

$F(R_i,R_{-i})R_iF(Q_i,R_{-i})$.

That is, an individual should not have any incentive to misrepresent his/her sincere preference whatever the others do. It is clear that a SCF is said to be *strategy proof* iff it, as a revelation mechanism, is strategy proof. A revelation mechanism, $g_F=(D(A), F)$, is *strategy proof,* if for each admissible preference profile $R \in D(A)$, $R \in \sigma(g_F[R])$. Strategy proofness is also referred as *nonmanipulability* since in case of strategy proofness no individual has an incentive to manipulate the mechanism via misrepresenting his/her true preference.

This means that for each individual i with the preference ordering R_i , playing anything other than R_i is not strictly preferred to playing R_i whatever the other agents play. The above definition turns into the claim that revealing the true preferences on the outcome should be a dominant strategy for each agent in the society. This is important since the fact that "the individuals can't be forced to report their preferences sincerely" is the crux of the problem considered here.

There is another point worth to mention here. There may be a case where the agents have dominant strategies in revealing their preferences and these dominant strategy revelations are not necessarily the true preferences of the agents. Such mechanisms are called as *dominant strategy revelation mechanisms.* This means that the set of strategy proof revelation mechanisms is a little bit narrower than the set of dominant strategy revelation mechanisms. This does not create a problem because of the 1973 result of Gibbard [9] claiming that every dominant strategy revelation mechanism that is not strategy proof is equivalent to a strategy proof revelation mechanism. Looking at the broader class doesn't add any generality to the analysis. The term *equivalent* is used here to denote that the two mechanisms which are said to be equivalent lead to the same outcomes when the true preferences of the individuals are identical in the two cases.

Definition: (Equivalence in DSE)

Two mechanisms, $g_1=(S_1, \pi_1)$ and $g_2=(S_2, \pi_2)$ are said to be *equivalent in DSE* if for each admissible preference profile $R \in D(A)$, $\pi_1(\sigma(g_1[R]))=\pi_2(\sigma(g_2[R]))$.

Proposition: (Gibbard)

Let $h=(D(A), \pi)$ be a revelation mechanism which implements a SCF, F, in dominant strategy equilibrium but is not strategy proof. Then there exists a strategy proof revelation mechanism $g=(D(A), G)$ which is equivalent to h.

Proof: Assume that $h=(D(A), \pi)$ is a dominant strategy revelation mechanism which implements a SCF, F, in dominant strategy equilibrium but is not strategy proof. Take any $R \in D(A)$, then $R \notin \sigma(h[R])$ since h is not strategy proof. Moreover since h is a dominant strategy mechanism $\sigma(h[R]) \neq \emptyset$, and since h implements F for all $s \in \sigma(h[R])$, $\pi(s)=F(R)$.

Now for each individual i, define the function d_i: $D_i(A) \rightarrow D_i(A)$ such that $d_i(R_i)$ gives a dominant strategy of individual i with the preference R_i. Define d as an n-tuple of these functions, i.e. $d=(d_1,\ldots,d_n)$. Now let $G=\pi o d$. To show that g is strategy proof suppose the contrary, i.e. there exists a profile $R \in D(A)$ and an ordering $s_i \in D_i(A)$ s.t. in the normal form game, g[R],

$$G(R) \sim R_i\ G(s_i, R_{-i})$$

$$\text{i.e. } \pi(d_{-i}(R_{-i}), d_i(R_i)) \sim R_i\ \pi(d_{-i}(R_{-i}), s_i)$$

which contradicts with the assumption that $d_i(R_i)$ is a dominant strategy of the i'th individual. So g is strategy proof. Moreover, for all $R \in D(A)$,

$$F(R)=\pi(\sigma(h[R]))=G(\sigma(g[R]))=G(R).$$

Thus g is a strategy proof mechanism.

QED

Now since the set of dominant strategy mechanisms is broader than the set of strategy proof mechanisms, one can easily find an example where the mechanism is strategy proof but doesn't implement the SCR it is associated with. Such an example and a characterization of the equivalence between strategy proofness and dominant strategy implementibility in SCF's will be presented as a result in the following chapters.

Having defined strategy proofness both in terms of social choice functions and direct revelation mechanisms, we will now deal with the question of whether one can build strategy proof mechanisms satisfying certain other criteria. To illustrate what one can expect to have additional to strategy proofness in a mechanism, we will give certain examples.

Example 1: (Imposed Mechanisms)

The first example is a mechanism which leads to a certain outcome independent of the strategical choice of the agents, i.e. $g=(X, \pi)$ where π: $X \rightarrow A$ is s.t. for all $x \in X$, $\pi(x)=a$ where a is defined to be a unique element of A. This mechanism is a dominant strategy mechanism because of the fact that the strategical choice of an individual doesn't affect the outcome of the mechanism makes any strategy a dominant strategy(The mechanism, also, is strategy proof if it is a revelation mechanism). Though this mechanism satisfies the appreciated property of being a dominant strategy mechanism, one has to accept the fact that this kind of a mechanism will not be approved by the individuals in any situation of social choice. Here the distribution of power doesn't create a major problem solely for the reason that the power is not distributed. There may be another case where the power is distributed among the individuals but unjustly.

Definition: (Dictatorial SCR) .

Given a SCR F: $D(A) \rightarrow A$, an individual $d \in N$ is said to be a *dictator* of F if for all $R \in D(A)$, $F(R) \subseteq \mathrm{argmax}(R_d)$. A SCR F: $D(A) \rightarrow A$ is said to be dictatorial if there exists an individual d who is a dictator in F.

Definition: (Dictatorial Mechanism)

A mechanism, $g=(X,\pi)$ is said to be dictatorial if for any admissible preference profile $R \in D(A)$, $\pi(\sigma(g[R])) \subseteq \mathrm{argmax}(R_d)$ where d is defined to be a dictator in the society, N.

Example 2: (Dictatorial Mechanism)

Let $g=(X,\pi)$ be s.t. for each individual i, $X_i=A$ and π: $X \rightarrow A$ is s.t. for any $x \in X$, $\pi(x)=x_d$. Thus the dictator, d, tells which outcome he/she wants to obtain and it occurs as the outcome of the mechanism. Now this is a dominant strategy mechanism since for each individual other than the dictator any strategy is a dominant strategy and the dominant strategy of the dictator is one of his/her topmost choices. Though this mechanism has dominant strategy equilibria for any admissible preference profile, it is unacceptable (of course from the view point of the tenants) since the distribution of power is unjust.

Both of these examples are about the cases where a big number of individuals have no decision power at all. The distribution of power among the agents in the society will be one of our main concerns and we will try to obtain mechanisms which give sufficient scope for individual preferences to affect the social choice. In the first step we will try to obtain this through two properties called as the Pareto Criterion and Nondictatorship.

Definition: (Pareto Criterion, Quasi Pareto Criterion)

A mechanism $g=(X, \pi)$ {a SCF F:$D(A) \rightarrow A$} is said to satisfy the *Quasi Pareto Criterion* if for any $R \in D(A)$ and any $x,y \in A$, xR_iy and $y \sim R_ix$ for each individual, then $\pi(\sigma(g[R])) \neq y$

$\{F(R)\neq y\}$[1]. A mechanism $g=(X, \pi)$ { a SCF $F:D(A)\rightarrow A$} is said to satisfy the *Pareto Criterion* if for any $R\in D(A)$ and any $x,y\in A$, xR_iy for each individual, and $y\sim R_jx$ for some individual j, then $\pi(\sigma(g[R]))\neq y$ { $F(R)\neq y$ }.

This means that an alternative to which another alternative is preferred by all of the society can't be chosen as socially optimal. When the set of admissible preferences is restricted to be a subset of the linear orders on A, strategy proofness automatically implies the Pareto Criterion. This will be shown later, in a more general framework. Nondictatorship is simply the case of unexistence of a dictator in a mechanism.

Additional to these requirements we require the alternative space not be limited and the set of admissible preferences as broad as possible. At this point there occurs the question of whether one can obtain a SCF which satisfies all of these requirements. Unfortunately the answer of this question is no. This is because of the famous Gibbard-Satterthwaite Impossibility Theorem which was independently proved by Mark A. Satterthwaite [16] and Alan Gibbard [9] in 1973.

For a more detailed analysis of the concepts mentioned above we have to introduce some new definitions and some propositions about the relationships between these new definitions and the ones above. First of all we have to extend the definition of a SCR to apply it in the analysis of the relation between social choice rules and social welfare functions.

Definition: (Generalized Social Choice Rule)

A nonempty valued correspondence $F:D(A)\times\Pi\rightarrow A$ is said to be a *generalized social choice rule* (GSCR) if for all $B\in\Pi$ and for all $R\in D(A)$, $F(R,B)\subseteq B$. From now on F will be called as a *generalized social choice function* (GSCF) if it is singleton valued and a *generalized social choice correspondence* (GSCC) otherwise.

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[1] This definition is usually referred as Pareto Criterion in many texts, but formally is weaker than the original definition of Pareto Criterion. Thus from now on a distinction will be made between these two definitions.

Definition: (Pseudo-game form)

Given a game form $h=(S,\pi)$ and the outcome space A, a *pseudo-game form* $h_B=(S_B,\pi_B)$ for a $B\in\Pi$ is s.t. $S_B=\{s\in S \,/\, \pi(s)\in B\}$ and $\pi_B:S_B\rightarrow B$ is the outcome function.

Definition: (Dominant strategy equilibrium of a Pseudo-game)

Given a pseudo-game $h_B[R]$, a member $s^*\in S_B$ is a *dominant strategy equilibrium* of $h_B[R]$ if for all $i\in N$, s_i^* is a *dominant strategy of individual i relative to B.*

For any strategy $s_i\in S_i$ of i , define $S_{B,-i}(s_i)=\{s_{-i}\in S_{B,-i} \,/\, (s_i,s_{-i})\in S_B\}$ and for any $s_{-i}\in S_{B,-i}$ define $S_{B,i}(s_{-i})=\{s_i\in S_{B,i} \,/\, (s_i,s_{-i})\in S_B\}$.

A strategy s_i^* of individual i is said to be a *dominant strategy of i relative to B* if $S_{B,-i}(s_i^*)\neq\emptyset$, for all $s_{-i}\in S_{B,-i}(s_i^*)$ and for all $s_i'\in S_{B,i}(s_{-i})$, $\pi_B(s_i^*,s_{-i})R_i\pi_B(s_i',s_{-i})$.

Definition: (Pseudo Implementation in DSE)

We say that a mechanism $h=(S,\pi)$ *pseudo implements* a GSCR $F:D(A)\times\Pi\rightarrow A$ *in Dominant Strategy Equilibrium* if for any $B\in\Pi$, $h_B=(S_B,\pi_B)$ implements F(. , B) in DSE relative to B. That is, for all $R\in D(A)$, $F(R,B)=\pi_B(\sigma(h_B[R]))$.

Definition: (Implementation in DSE)

We say that a mechanism class $h=\{\ h_B=(S_B\ ,\pi_B)\ /\ B\in\Pi\ \}$ (where any two mechanisms implementing F for different subset of A need not be related to each other) *implements* a GSCR, $F:D(A)\times\Pi\rightarrow A$, *in Dominant Strategy Equilibrium* if for any $B\in\Pi$, $h_B=(S_B,\pi_B)$ implements F(. , B) in DSE. That is, for all $R\in D(A)$, $F(R,B)=\pi_B(\sigma(h_B[R]))$.

The main difference between pseudo implementibility and implementibility is that while the rectangularness of the strategy space is sacrificed for the sake of creating the mechanism class from a single mechanism that implements F(. ,A) in case of pseudo implementibility, the other extreme is held in implementibility. That is, for the sake of having rectangular strategy spaces (which need not be related) for each subset of A, the relationship among the mechanisms in the implementing class is left aside.

At this point whether one can find a kind of implementibility that is endowed with both of these appreciated properties arises as an interesting question. This kind of an implementibility, which we will call as total implementibility, can be defined as follows.

Definition: (Total Implementation in DSE)

We say that a mechanism class $h=\{ h_B=(S_B, \pi_B) / B\in\Pi \}$ ***totally implements*** a GSCR, $F:D(A)\times\Pi\rightarrow A$, *in Dominant Strategy Equilibrium* if for any $B\in\Pi$, $h_B=(S_B,\pi_B)$,which is itself a game form and is a restriction of $h_A=(S_A, \pi_A)$ implements $F(. , B)$ in DSE. That is, for all $R\in D(A)$, $F(R,B)=\pi_B(\sigma(h_B[R]))$.

In total implementation the essential point is that while the strategy space remains rectangular for each subset of A, moreover, it is obtained through the restriction of the mechanism h_A which implements $F(. ,A)$. That is, for each $i\in N$, $S_{i,B}$ is obtained through the restriction of $S_{i,A}$ and $\pi_B:S_B\rightarrow B$ is a function. One example of this kind of a mechanism class is the one that is obtained through the restrictions of the revelation mechanism $h_A=(D(A), F_A)$ where $F_A=F$. For any $B\in\Pi$, the mechanism that implements $F(. ,B)$ is $h_B=(D(B), F_B)$ where for each $i\in N$, $D_i(B)$ is the restriction of the preference domain of i, $D_i(A)$ on B, and $F_B:D(B)\rightarrow B$ is s.t. for any $R_B\in D(B)$, $F_B(R_B)=F(R,B)$ (where R_B is the restriction of R on B).

To associate the generalized social choice rules with social welfare functions, we need the following property on a generalized social choice rule.

Definition: (Rationality)

Let $F:D(A)\times\Pi\rightarrow A$ be a GSCR. F is said to be a *rational* SCR if there exists a SWF, f, s.t. $F=F_f$ where F_f is s.t. for all $R\in D(A)$ and for all $B\in\Pi$, $F(R,B)=\text{argmax}_B f(R)$.

Rationality implies a SCR be consistent among different alternative sets. That means, if F chooses an alternative x from the grand set A, it should also choose x in another set $B \subseteq A$ which also includes x.

Definition: (Choice function)

A function $c: \Pi \rightarrow \Pi$ is called a choice function for A if for all $B \in \Pi$, $c(B) \subseteq B$.

Definition: Let R be a relation on A. We define $c(. ,R): \Pi \rightarrow 2^A$ by

$c(B,R)=\{x \in B \ / \text{ for all } y \in B,\ y \sim Rx\}$ for any $B \in \Pi$.

Definition: (Hauthakker's axiom)

Let $c: \Pi \rightarrow \Pi$ be a choice function. We say that c satisfies *Hauthakker's axiom* (HA) if for all $B, C \in \Pi$ and for all $x, y \in B \cap C$, $[x \in c(A)$ and $y \in c(B)$ implies $x \in c(B)]$.

3 STATE OF THE ART

Now, we will present the main paths followed in the dominant strategy implementation theory up to date. We will first present the Gibbard-Satterthwaite impossibility theorem, which is a cornerstone in the dominant strategy implementation theory, together with a formal proof of it which is essential for the link between this theorem and the famous impossibility theorem of Arrow. We will also present a proof for the two individuals and three alternatives case to help the reader to gain a better understanding of the Gibbard-Satterthwaite theorem. Then, we will present the main paths that were followed in history to overcome the impossibility problem in implementation via altering the framework.

3.1 Impossibility Theorems

Theorem: (Gibbard- Satterthwaite)
If $|A| \geq 3$ and preferences are unrestricted ($D_i(A)= \Omega$ or $L(A)$ for all $i \in N$) then a SCF, F, can not simultaneously be strategy proof and satisfy both the Quasi Pareto Criterion and nondictatorship.

This impossibility theorem is closely related to Arrow's 1963 result which states the impossibility of the existence of a social welfare function which satisfies certain conditions.

A social welfare function (SWF) for the alternative set A is a single valued function, f, that maps the set of admissible preference profiles to a subset of Ω. With each SWF, f, one can associate a SCF, F_f, s.t. for all feasible profile P,$F_f(P)$=argmaxf(P).

To understand the relation between Gibbard-Satterthwaite theorem and Arrow theorem, firstly we have to gain a deeper insight about Arrow's Impossibility Theorem. Arrow's theorem investigates the social welfare functions, f, which satisfy the conditions of Pareto Criterion, nondictatorship and two additional conditions: independence of irrelevant alternatives (IIA) and monotonicity.

Theorem: (Arrow)
If the cardinality of the alternative set, A, is greater than or equal to three and preferences are restricted to be either the full domain of L(A) or Ω, then a SWF can't simultaneously satisfy the Pareto Criterion, nondictatorship, IIA and monotonicity.

Now having given the Arrow impossibility theorem we can start to the discussion of the Gibbard-Satterthwaite theorem. There exists two independent proofs to Gibbard-Satterthwaite theorem. Satterthwaite's proof, which was at the same time a part of his Ph.D. thesis, is mainly based on a counting procedure[16]. Though it has the advantage of not using Arrow's impossibility theorem, I prefer to mention here Gibbard's proof [9]which is more instructing about strategy proofness and the relation between Gibbard-Satterthwaite theorem and Arrow's theorem. Additional to Gibbard's proof, I will give Feldman's 1979 proof [8] for the two individuals, three alternatives case which, I hope, will help to gain an intuition about strategy proofness.

In his 1973 paper, Gibbard proves that any nondictatorial voting scheme with at least three possible outcomes is subject to individual manipulation. The term 'voting scheme" is used for any scheme which makes a community's choice depend entirely on individuals' professed preferences among the alternatives.

Gibbard's proof is based on Arrow's impossibility theorem. Showing that a SWF derived from a strategy proof mechanism satisfies all Arrow's conditions except nondictatorship, he claims that every strategy proof mechanism which has a number of alternatives greater than 2 is dictatorial. In his paper he uses the term *chain ordering* to denote a linear order.

Definition: (Chain Ordering)
A *chain ordering* is an ordering in which no distinct items are indifferent, i.e. $P \in \Omega$ is a chain ordering on A if for all $x,y \in A$, xPy and yPx implies x=y.

Definition: (Strict preference relation and indifference relation)

For any $B \subseteq A$, $R_i \in \Omega$ and $x,y \in A$ define P_i and I_i as follows:

(i) xP_iy iff $y \sim R_ix$

(ii) xI_iy iff yR_ix and xR_iy

Thus, P_i is defined to be a strict preference relation while I_i is an indifference relation.

Definition:

Let Q be a chain ordering of A. For any $B \subseteq A$, $R_i \in \Omega$ define P_i*B as

(i) If $x,y, \in B$ then $x(P_i*B)y$ iff xP_iy or (xI_iy and xQy)

(ii) If $x \in B, y \notin B$ then $x(P_i*B)y$

(iii) If $x,y \notin B$ then $x(P_i*B)y$ iff xQy

This new ordering will create an strict preference relation on the alternative set A which orders the elements of B automatically above those of A\B.

Proposition: (Gibbard)

From the above definitions are derived

(i) For all $i \in N$, P_i*B is a chain ordering.

(ii) If $C \subseteq B$, then for any $i \in N$, $(P_i*B)*C = P_i*C$.

(iii) Suppose for all $i \in N$, (xP_iy iff xQ_iy) and (yP_ix iff yQ_ix) for all $x,y \in B$. Then $P*B=Q*B$ where $P=(P_1,\ldots,P_n)$ and $Q=(Q_1,\ldots,Q_n)$

(iv) If $xf(P)y$ then $y \sim f(P)x$

(v) (Independence of Irrelevant Alternatives, IIA)

Take any $x,y \in A$. Suppose for all $i \in N$, (xP_iy iff xQ_iy) and (yP_ix iff yQ_ix). Then $xf(P)y$ iff $xf(Q)y$.

The proof of derivation (v) is a simple consequence of derivation (iii) of the proposition. From (iii), $P*\{x,y\}=Q*\{x,y\}$ and hence $x \in F(P*\{x,y\})$ iff $x \in F(Q*\{x,y\})$. But this, by definition, implies that $xf(P)y$ iff $xf(Q)y$.

Assertion 1: Define $\sigma=(\sigma_1,..,\sigma_n)$ where for each $i\in N$, $\sigma_i:D_i(A)\rightarrow S_i$ be the dominant strategy function for the mechanism $h=(S,\pi)$. Let $s=\sigma(P)$. Take any $x,y\in A$ and $s'\in S$ satisfying,

(a) For all $i\in N$, yP_ix implies $s_i'=s_i$

(b) For all $i\in N$, $x\sim I_iy$

(c) $yf(P)x$, i.e. $x\neq F(P*\{x,y\})=\pi(s)$

Then $x\neq\pi(s')$

Proof of Assertion 1:

Suppose $x=\pi(s')$. Let $P^*=P*\{x,y\}$ and let $t=\sigma(P^*)$. Then

$$yRx \text{ implies } x=y \text{ or } x\neq F(P^*)$$

$$\text{which means } x\neq F(P^*)=\pi(\sigma(P^*)=\pi(t)$$

$$\text{which implies that } x\neq\pi(t)$$

Define the sequence

$s^0=(s_1',s_2',\ldots,s_k',s_{k+1}',\ldots,s_n'),\ldots,s^k=(t_1,\ldots,t_k,s_{k+1}',\ldots,s_n'),\ldots,s^n=(t_1,\ldots,t_k,t_{k+1},\ldots,t_n)$

where $\pi(s^0)=x$ and $\pi(s^n)\neq x$.

Let k be the least indexed individual s.t. $\pi(s^k)\neq x$, $\pi(s^{k-1})=x$.

Case 1: $\pi(s^k)=y$ and yP_kx

$\pi(s^k)=y$, $\pi(s^{k-1})=x$ and yP_kx implies that $\pi(s^k)P_k\pi(s^{k-1})$ which in turn implies that $\pi(s^{k-1})\sim R_k\pi(s^k)$ and this means that s_k' is not a dominant strategy for k. But since yP_kx, by (a), $s_k'=s_k$ and s_k is by definition a dominant strategy for k. But this leads to a contradiction.

Thus $x\neq\pi(s')$ in this case.

Case 2: $\pi(s^k)\neq y$ or xP_ky

If $\pi(s^k)=y$ then xP_ky. But this implies that xP_k^*y which in turn implies $xP_k^*\pi(s^k)$. If $\pi(s^k)\neq y$ then since $\pi(s^k)\neq x$ we have $\pi(s^k)\notin\{x,y\}$ and by (b), $xP_k^*\pi(s^k)$. So $xP_k^*\pi(s^k)$. Now $x=\pi(s^{k-1})$. Hence $\pi(s^{k-1})P_k^*\pi(s^k)$. Thus t_k is not a dominant strategy for individual k. But since $t_k=\sigma_k(P_k^*)$, t_k is a dominant strategy for individual k. But this leads to a contradiction. Thus $x\neq\pi(s')$. QED.

Assertion 2: The dictator for a SWF ,f, which is derived from a SCR, F, is also a dictator for the mechanism $g=(D(A),F)$.

Proof of Assertion 2: Assume that individual d is a dictator for the SWF f. Then d is a dictator for g if for all $y \in A$, there is a $s_d(y) \in D_d(A)$ s.t. for all $s \in D(A)$ s.t. $s_d=s_d(y)$, $F(s)=y$. Let $P_d{}^*$ be the SPR s.t. for all $x \in A$ s.t. $x \neq y$, $yP_d{}^*x$ and let $s_d(y)=\sigma_d(P_d{}^*)$. Let $s \in D(A)$ be s.t. $s_d=s_d(y)$ and take any $x \in A$ s.t. $x \neq y$. We shall show that $F(s) \neq x$. Let $P \in D(A)$ be any preference profile s.t. $P_d=P_d{}^*$ and for all $i \in N \backslash \{d\}$, xP_iy. Let $s'=\sigma(P)$. Then $s_d'=s_d(y)$. Now under this preference profile, P, assumptions (a) and (b) of Assertion 1 are satisfied. Moreover since d is a dictator of f, $yf(P)x$ and thus $x \sim f(P)y$, i.e. (c) is also satisfied. Therefore, $x \neq F(s)$. Since this holds for all $x \in A \backslash \{y\}$, $y=F(s)$. Thus for any $P \in D(A)$ s.t. the dictator d strictly prefers y to x, y has to be chosen by g. Thus d is a dictator for g.

QED.

Proof of Gibbard-Satterthwaite Theorem: According to the Arrow's Impossibility Theorem, every SWF violates at least one of the following conditions.

(Scope) (i) A has at least three elements

(Unanimity) (ii) If for all $i \in N$, xP_iy then $xf(P)y$

(IIA) (iii) If for all $i \in N$, (xP_iy iff xQ_iy) and (yP_ix iff yQ_ix) then $xf(P)y$ iff $xf(Q)y$

(Nondictatorship)(iv) There is no dictator for f where a dictator $k \in N$ is s.t. for all $R \in D(A)$, $x,y \in A$: if xP_ky then $xf(P)y$

Now, (i) is satisfied by assumption and (iii) is shown to be satisfied by the above proposition. So if one shows that (ii) is satisfied, (iv) can't be satisfied by f.

Now, (ii) will be obtained as a corollary to Assertion 1:

Assume that there is a profile P and alternatives x and y ($x \neq y$) s.t. for each individual $i \in N$, xP_iy. Now we want to show that $xf(P)y$. Now since x is an outcome there is a strategy s' s.t. $x=\pi(s')$. Take $s=\sigma(P)$. Now we are going to use Assertion 1 in opposite direction. That is, we took $s=\sigma(P)$ and took any $x,y \in A$. Moreover $s' \in S$ is s.t. $x=\pi(s')$. Now assumptions (a) and (b) of Assertion 1 still hold in this setting. Thus Assumption (c) doesn't hold, which means that x=F(P*{x,y}). But this holds iff xf(P)y. Thus condition (ii) of the theorem holds, but by Arrow's theorem means that f violates the fourth condition of nondictatorship. Moreover if f is dictatorial then F is also dictatorial by Assertion 2 above. Thus every strategy proof game form with at least three outcomes is dictatorial.

QED.

Though Gibbard's proof is illustrating in terms of the relation between a SCF and a SWF, one can gain a better insight to the Gibbard-Satterthwaite theorem by examining the Feldman's proof additionally. Feldman in 1979 has devised a proof for the simple case of two individuals and three alternatives: The SCF, F, is defined on the domain $L(A)^2$, and for the set of alternatives A={a,b,c}. Initially we will assume that F satisfies the strategy proofness and the Pareto criterion conditions. Table 1 gives the set of possible outcomes for each preference profile of the society. The table is formed according to the Pareto Criterion, thus while in some cases (as cases of unanimity) it surely determines the outcome, in some other cases it determines what can't be the outcome (according to the violation of the Pareto Criterion). The question marks in the table represent the cases where no alternative can be eliminated. Shortly speaking, Table 1 gives the restrictions imposed on F by the Pareto Criterion.

Table 1.

Agent 1	Agent 2: 1 (xyz)	2 (xzy)	3 (yxz)	4 (yzx)	5 (zxy)	6 (zyx)
1 (xyz)	x	x	$\neq z$ [6]	$\neq z$ [5]	$\neq y$ [2]	? [4]
2 (xzy)	x	x	$\neq z$ [7]	? [8]	$\neq y$ [1]	$\neq y$ [3]
3 (yxz)	$\neq z$	$\neq z$	y	y	? [12]	$\neq x$ [9]
4 (yzx)	$\neq z$	?	y	y	$\neq x$ [11]	$\neq x$ [10]
5 (zxy)	$\neq y$	$\neq y$	?	$\neq x$	z	z
6 (zyx)	?	$\neq y$	x	$\neq x$	z	z

Since the SCF is single valued, a single alternative must be assigned to each cell. We will begin with the assumption that element x is assigned to the cell labeled 1, and show that this will lead to the dictatoriality of individual 1. In the alternative assumption of assigning z to cell 1, the same proof will lead to the dictatoriality of individual 2.

Now, assigning x to cell 1 implies that x must be assigned to cell 2. This is by strategy proofness of F. Now y can't be assigned to cell 2 by Pareto Criterion, suppose z is assigned to cell 2. Then at profile (1,5) (i.e. {(xyz),(zxy)}), individual 1 can manipulate F by declaring (xzy) instead of his/her sincere preference of (xyz). Thus, assigning z to cell 2 violates the strategy proofness assumption of F. The same logic leads to the following results which are indicated in Table 2.

Table 2.

Cell	Assigned outcome	Alternative outcome	Manip. situation	Manip. agent	Manip. strategy	Manip. outcome
2	x	z	F(1,5)=z	one	F(2,5)	x
3	x	z	F(2,5)=x	two	F(2,6)	z
4	x	y or z	F(1,6)=y or z	one	F(2,6)	x
5	x	y	F(1,6)=x	two	F(1,4)	y
6	x	y	F(1,6)=x	two	F(1,3)	y
7	x	y	F(2,3)=y	one	F(1,3)	x
8	x	y or z	F(2,4)=y or z	one	F(1,4)	x
9	y	z	F(3,6)=z	one	F(2,6)	x
10	y	z	F(4,6)=z	one	F(3,6)	y
11	y	z	F(4,6)=y	two	F(4,5)	z
12	y	x or z	F(3,5)=x or z	one	F(4,5)	y

Filling in each indeterminate cell in this manner, both above and below the diagonal results in individual 1 being a dictator, i.e. individual 1's topmost choice is chosen independent of what individual 2 declares.

QED.

Having defined and proved the Gibbard-Satterthwaite impossibility theorem, now we will deal with the ways to get rid of his impossibility in implementation.

3.2 Ways to Bypass the Impossibility Problem Via Altering the Framework

The impossibility result of Gibbard and Satterthwaite, despite being a very important result in the implementation of group decision making, is really a discouraging one. The result simply says that it is impossible to distribute the decision power among the individuals fairly if one expects to have fulfill some essential criteria. To break the pessimism about the future of group decision making, three main paths are followed. The first one is imposing some restrictions on the preference domain such that the social choice rules defined on this restricted domain can overcome the problem of impossibility in implementation in the dominant strategy equilibria of a nondictatorial mechanism. The second path is changing the equilibrium concept, that is leaving the dominant strategy implementibility aside and trying something more accessible such as the Nash implementation. The third and the last path is giving up the social choice functions and dealing with social choice correspondences instead. Now we will analyze each of these paths in detail.

3.2.1 Domain Restrictions

In the task of restricting the domain of admissible preferences three approaches have been followed. The first approach is taking a specific social choice function and looking at the domain restrictions that are sufficient to make it strategy proof. This is closely related to the work of Sen and Pattanaik (1969) [19] and will not be mentioned here. The second approach begins with a domain with economic restrictions on preferences such as convexity, continuity, etc. and then looks for strategy proof mechanisms which are not dictatorial. This approach will be discussed in detail in the next section and will be mainly based on a recent work of Salvador Barbera and Matthew Jackson (1992). The third approach looks for necessary and sufficient conditions on the preferences such that the resulting domain permits the construction of a strategy proof social choice function that satisfies some additional restrictions on the power distribution as the Pareto criterion and nondictatorship.

In this section we will mainly concentrate on this last approach. Since we know that when the domain of preferences is left to be a full domain of complete preorders or linear orders, it is impossible to find a nondictatorial social choice function which is strategy proof and Pareto optimal, we have to impose some restrictions on the preference domain to obtain a subset appropriate for building acceptable social choice functions. One very famous example of such a limitation is the single-peaked domains characterization. They will be analyzed in detail in one of the following chapters and it will be shown that single-peakedness limitation becomes extraordinarily binding when the cardinality of the alternative set goes to infinity.

What we want in this section is to obtain necessary and sufficient conditions on the preference domain such that the resulting social choice functions will be nondictatorial and will also satisfy strategy proofness and the Pareto criterion. One limitation of the result is its only holding for the generalized social choice functions which are rational. The main reason for this is that the characterization is based on social welfare functions and thus doesn't hold for nonrational generalized social choice functions (i.e. the ones which can't be paired with a social welfare function). Before introducing the theorem we must formalize the structure necessary for this characterization. For this, we need the below definitions.

Definition: (Ordered pairs)

Let A be the alternative set which is finite and let D(A) be s.t for all $i \in N$, $D_i(A) \subseteq L(A)$ and for all $i,j \in N$, $D_i(A)=D_j(A)$. Now the set of *ordered pairs* within A is defined as $T=\{(x,y) \in A \times A \ / \ x \neq y \}$. Moreover, the set of *trivial ordered pairs* within D(A) is $TR(D(A))=\{(x,y) \in T$ / there exists a $P \in D(A)$ s.t. xPy and there doesn't exist a $Q \in D(A)$ s.t. $yQx\}$.

Definition: (Being closed under decisiveness implications)

A subset R of T is said is said to be *closed under decisiveness implications* (closed DI) if for all (x,y), $(x,z) \in T \setminus TR(D(A))$ the following conditions hold:

DI1: If there are $P^1, P^2 \in D(A)$ with xP^1yP^1z and yP^2zP^2x then

DI1a: $(x,y) \in R$ implies $(x,z) \in R$

DI1b: $(z,x) \in R$ implies $(y,x) \in R$

DI2: If there is a $P \in D(A)$ with xPyPz then

DI2a: $(x,y)\in R$ and $(y,z)\in R$ implies $(x,z)\in R$

DI2b: $(z,x)\in R$ implies either $(y,x)\in R$ or $(x,y)\in R$

Definition: (Decomposable domain)

A domain of preferences, D(A), is said to be a ***decomposable domain*** if there exists an $R\subset T$ s.t. $TR(D(A))\subset R\subset T$ and R is closed under decisiveness implications.

Definition: (Nondictatorial domain)

A domain of preference profiles, D(A), is said to be a ***nondictatorial domain*** if there exists a nondictatorial n-person SWF on D(A) which satisfies monotonicity, IIA and Pareto optimality conditions.

A nondictatorial domain as can be seen above, if can be obtained, is sufficient for the characterization of social welfare functions that we appreciate. That simultaneously leads to a characterization of rational generalized social choice rules that are strategy proof and Pareto optimal and also nondictatorial. There are three main theorems built by Kalai and Muller (1977) [11] that construct the needed characterization. The first one is used in the proof of the others and claims that one can find an n-person nondictatorial social welfare function if, and only if one can find a 2-person nondictatorial social welfare function:

Theorem 1: (Kalai-Muller)

For $n\geq 2$, there exists a nondictatorial n-person SWF on D(A) which satisfies monotonicity, IIA and Pareto optimality iff there exists a nondictatorial 2-person SWF on D(A) which satisfies monotonicity, IIA and Pareto optimality.

The following two theorems, using this theorem as an input, characterize nondictatorial domains and the relation between these domains, social welfare functions and rational generalized social choice functions, respectively:

Theorem 2: (Kalai-Muller)

A preference domain, D(A), is nondictatorial iff it is decomposable.

This theorem simply says that if one can construct a decomposable domain, D(A), then whatever social welfare function defined on this domain satisfies nondictatoriality, monotonicity, IIA and the Pareto criterion. Now it is time to construct the relationship of these appreciated social welfare functions with the rational generalized social choice rules:

Theorem 3: (Kalai-Muller)

Let n be any integer s.t. $n \geq 2$. The following three statements are equivalent for every $D(A) \subseteq L(A)^n$:

I. D(A) admits an n-person, nondictatorial, strategy proof, rational generalized SCF which satisfies the Pareto criterion.

II. D(A) admits an n-person nondictatorial SWF which satisfies monotonicity, IIA and the Pareto criterion.

III. D(A) is decomposable.

This theorem is a full characterization of the domain restrictions that admit the formation of nondictatorial generalized social choice functions which are strategy proof and Pareto optimal. That is, if one constructs a decomposable domain, he/she can obtain a rational generalized social choice function defined on this domain satisfying nondictatoriality, strategy proofness and the Pareto criterion, moreover any rational generalized social choice function that satisfies nondictatoriality, strategy proofness and Pareto criterion turns out to have a decomposable domain. As one can guess single-peaked domains are decomposable. This will be shown in our analysis of the single-peaked domains. Now having completed the characterization of the domain restrictions we will analyze the alternative ways to get rid of the impossibility problem in dominant strategy implementation.

There are some additional comments about decomposibility. The first one is its being designed for the preference domains that are subsets of linear orders. That is, the whole characterization depends on the preferences satisfying the properties of linear orders. One can build a social choice function that is formed on a domain that is not a subset of linear orders (and thus is not decomposable) and is both strategy proof and nondictatorial. The following is an example of this.

Example 1:

Let $F: D(A) \times \Pi \rightarrow A$ be a GSCF where $A=\{a,b,c\}$, $N=\{1,2\}$, $D(A)=D_1(A) \times D_2(A)$.

Let $D_i(A)=\{R,Q\}$ for $i \in N$. Define R as aIbPc (i.e R is indifferent between a and b and both are strictly preferred to c) and Q as cPaIb (i.e Q strictly prefers c to both a and b and is indifferent between a and b).

Define F as follows:

$F(R,R;A)=a$	$F(R,R;\{a,b\})=a$	$F(R,R;\{a,c\})=a$	$F(R,R;\{b,c\})=a$
$F(R,Q;A)=a$	$F(R,Q;\{a,b\})=a$	$F(R,Q;\{a,c\})=a$	$F(R,Q;\{b,c\})=b$
$F(Q,R;A)=a$	$F(Q,R;\{a,b\})=a$	$F(Q,R;\{a,c\})=a$	$F(Q,R;\{b,c\})=b$
$F(Q,Q;A)=c$	$F(Q,Q;\{a,b\})=b$	$F(Q,Q;\{a,c\})=c$	$F(Q,Q;\{b,c\})=c$

Now F is strategy-proof and nondictatorial.

In the above example though the domain is not decomposable (since it is not a subset of the linear orders), the social choice function formed on it satisfies both strategy proofness and nondictatoriality.

Another weakness of the decomposable domains is that: though it drives out the dictatorial social choice functions, most of the social choice functions formed on decomposable domains distribute the decision power among two agents, the other agents have no decision power at all. This kind of a social choice function, though nondictatorial, is not so much satisfactory. Thus, additional properties, such as essentiality and symmetry (essentiality is the case where for each agent there exists a profile where he/she can affect the outcome by changing his/her declaration; symmetry is simply anonymity, i.e. any permutation of a preference profile should lead to the same outcome), are imposed on decomposable domains. More can be found about this subject in the 1983 paper of Blair and Muller[3].

3.2.2 Nash Implementation

The second path followed in the task of getting rid of the impossibility problem in implementation is changing the equilibrium concept. The historical development of the literature shows that after the limitations necessary and sufficient to characterize domains that admit the construction of a strategy proof social choice function were characterized, the scientists that found this too limiting for the preference domain tried another equilibrium concept. The main path followed was the trial of Nash implementation, leaving aside dominant strategy implementation. Much of the work done on this subject is very recent, the main developments were obtained in the end of eighties and the beginning of nineties. To gain a better understanding of the literature and the theorems that will be presented, we must construct the additional framework necessary for this task.

Definition: (Nash equilibrium of a normal form game)
Let $R \in D(A)$, S be the Cartesian product of strategy spaces of the individuals, and let $\pi: S \rightarrow A$ be the outcome function of the game. Then, given a normal form game $h[R]=(S,\pi)$, $s^* \in S$ is said to be a *Nash equilibrium* of $h[R]$ (and is formally written as $s^* \in \sigma_0(h[R])$) if for all $i \in N$ and for all $s_i \in S_i$, $\pi(s_i^*, s_{-i}^*) R_i \pi(s_i, s_{-i}^*)$. $\sigma_0(h[R])$ is defined as the set of all Nash equilibria of the normal form game $h[R]$.

That means, given that the other agents play their Nash strategies, individual i can't be better of by deviating from his/her Nash strategy.

Definition: (Implementation in Nash equilibrium)
Let $F: D(A) \rightarrow A$ be a SCR and let $h=(S,\pi)$ be a mechanism where $\pi: S \rightarrow A$. We say that h *fully {weakly} implements F in Nash Equilibrium* if for all $R \in D(A)$, $F(R)=\pi(\sigma_0(h[R]))$

$\{\pi(\sigma_0(h[R]))\subseteq F(R)\}$. F is said to be *Nash implementable* if there exists a mechanism h which implements F fully in Nash equilibrium.

While all the main theorems in the literature are on full implementation, weak implementation is also an important concept. The main point in weak implementation is obtaining a subset of the socially desirable outcomes (which are collected in the set F(R)) and to leave aside socially undesirable ones. This is the reason for the mechanism to obtain a subset of F(R). A\F(R) is simply the socially undesirable alternatives. Moreover, one can always obtain a mechanism that implements a social choice rule in a way that for all $R\in D(A)$, $F(R)\subseteq\pi(\sigma_0(h[R]))$ though this is of no value in terms of implementing the socially desirable outcomes. Such a mechanism is illustrated in the following example.

Example 2:

Take any $F:D(A)\to A$ with $\#N\geq 3$. Take the mechanism $h=(S,\pi)$ as the following:

For any $i\in N$, $S_i=\{(R^i,a^i)\in D(A)\times A \,/\, a^i\in F(R^i)\}$ and for any $s\in S$,

$\pi(s) = a$ if there is an $R\in D(A)$ s.t. $a\in F(R)$ and there is an $I\subset N$ with $\#I=n-1$ s.t. for all $i\in I$, $s_i=(R,a)$

$= a^1$ otherwise

Then $s^*\in S$ is a Nash equilibrium if for all $i\in N$, $s_i^*=(R,a)$

Then $F(R)\subseteq\pi(\sigma_0(h[R]))$

Definition: (Lower contour set, upper contour set)

Given a preference $R\in\Omega$ and alternative $a\in A$, the *lower contour set of a w.r.t. R* is simply $L(a,R)=\{x\in A \,/\, aRx\}$. Moreover, the *upper contour set of a w.r.t. R* is $U(a,R)=\{y\in A \,/\, yRa\}$.

Additional to these, for a normal form game, $h[R]=(S,\pi)$, and for a strategy tuple $s\in S$, define $\pi(S_i, s_{-i})=\{x\in A \,/\, \text{there is a } s_i'\in S_i \text{ s.t. } \pi(s_i',s_{-i})=x\}$

Definition: (Monotonic SCR)

A SCR $F:D(A)\rightarrow A$ is said to be *monotonic* if for all R, $R'\in D(A)$ and for all $a\in A$, $\{a\in F(R)$ and $L(a,R_i)\subseteq L(a,R_i')$ for all $i\in N\}$ implies $a\in F(R')$.

This simply means that an alternative that is chosen as the social optimal under a profile R should also be chosen as the social optimal under the profile R' if its rank in the preference ordering of the individuals doesn't worsen while passing from the profile R to the profile R'.

Definition: (No veto power)

A social choice rule F is said to satisfy *no veto power* if for all $R\in D(A)$, for all $a\in A$ and for all $i\in N$, [$a\in \text{argmax}R_j$ for each $j\in N\backslash\{i\}$] implies $a\in F(R)$.

The first result that will be presented belongs to E. Maskin (1977)[12]. It constructs a relationship between Nash implementibility and monotonicity.

Theorem 1: (Maskin)

If a SCR $F:D(A)\rightarrow A$ is Nash implementable then F is monotonic.

Given this theorem, one thinks whether the converse is true. This question is answered by Maskin's well-known example given below.

Example 3: (A monotonic SCR which is not Nash implementable)

Take $N=\{1,2,3\}$, $A=\{a,b,c\}$ and $D_i(A)=L(A)$ for each $i\in N$.

$F:D(A)\rightarrow A$ is s.t.

$a\in F(R)$ iff a is top ranked by 1

$b\in F(R)$ iff b is top ranked by 1

$c\in F(R)$ iff c is Pareto optimal w.r.t. R and c is not bottom ranked by 1

Now, F is monotonic. Take the following profiles, R*, R**, R*** as R*=[(bca), (cab), (cab)], R**=[(abc), (cba), (cab)], R***=[(bac), (abc), (abc)] . Now F(R*)={b,c}, F(R**)={a} and F(R***)={b} according to the above rules.

Suppose $h=(S,\pi)$ is a mechanism Nash implementing F. Then, there is a $s^*\in\sigma_0(h[R^*])$ s.t. $\pi(s^*)=c$.

Now $b\notin\pi(S_1, s_{-1}^*)$ since s^* is a Nash equilibrium of $h[R^*]$ and individual 1 mustn't have any incentive to deviate from his/her Nash strategy s_i^*.

Suppose that $a\in\pi(S_1, s_{-1}^*)$. Then there exists an $s_1'\in S_1$ s.t. $\pi(s_1',s_{-1}^*)=a$. But then

$$a=\pi(s_1',s_{-1}^*)\in\pi(\sigma_0(h[R^{***}]))=F(R^{***})$$

which contradicts with F(R***)={b}. So, $a\notin\pi(S_1, s_{-1}^*)$.

Suppose that $c\in\pi(S_1, s_{-1}^*)$. Then for all $s_1\in S_1$, $\pi(s_1, s_{-1}^*)=c$. But then

$$c=\pi(s_1,s_{-1}^*)\in\pi(\sigma_0(h[R^{**}]))=F(R^{**})$$

which contradicts with F(R**)={a}. So $c\notin\pi(S_1, s_{-1}^*)$.

Thus $\pi(S_1, s_{-1}^*)=\emptyset$. This is a contradiction, so F is not Nash implementable.

So we now know that there is a one-sided relationship between Nash implementability and monotonicity. Monotonicity itself is not a sufficient condition to satisfy Nash implementibility. However, monotonicity and no veto power together implies Nash implementibility.

Theorem 2: (Maskin)

Let $F:D(A)\rightarrow A$ be a SCR where $\#N\geq 3$. If F is monotonic and satisfies no veto power, then F is Nash implementable.

Moreover, one can give examples showing that Nash implementibility doesn't imply No veto power and No veto power doesn't imply Nash implementibility.

At this point in history there was not a full characterization of Nash implementibility. Monotonicity and No veto power were known to be sufficient, but only monotonicity was a necessary condition. Moreover these implications were true only for the case of $\#N\geq3$. In case of two individuals and Pareto optimality dictatoriality occurred as a necessary and sufficient condition for Nash implementability. The following theorem shows this impossibility.

Theorem 3:

Let $F:D(A)\rightarrow A$ be a SCR with $\#N=2$ and for all $i\in N$, $D_i(A)=\Omega$. Moreover, let F be Pareto optimal. Then, F is Nash implementable iff F is dictatorial.

This impossibility theorem showed that if one requires Pareto optimality and if there are only two individuals in the society, then one surely gets a dictatorial social choice rule. Then, in 1990, Moore and Repullo [14] presented a full characterization of Nash implementibility for the case of $\#N\geq3$. This was really a path-breaking invention in the task of Nash implementibility.

Theorem 4: (Moore and Repullo)

If $\#N\geq3$, then a SCR F can be implemented in Nash equilibrium iff it satisfies condition m given below:

Condition m: A SCR is said to satisfy condition m iff there exists a $B\subseteq A$ and, for all $R\in D(A)$, $a\in F(R)$ and $i\in N$, there is a nonempty set $C_i(a,R)=\pi(S_i, s_{-i}*)$ (where $s*\in S$ is a Nash equilibrium of the normal form game h[R]) with $a\in C_i(a,R)\subseteq(L(a,R_i)\cap B)$ satisfying:

(mi) If $R,R'\in D(A)$ and $a\in F(R)$ are s.t. for all $i\in N$, $C_i(a,R)\subseteq L(a,R_i')$ then $a\in F(R')$

(mii) If $R,R'\in D(A)$ and $a\in F(R)$ are s.t. there is a $b\in A$ with $b\in C_i(a,R)\subseteq L(b,R_i')$ and for all $j\neq i$, $B\subseteq L(b,R_j')$, then $b\in F(R')$

(miii) If $R'\in D(A)$ and $c\in B$ are s.t. for all $i\in N$, $B\subseteq L(c,R_i')$ then $c\in F(R')$

This is a full characterization of Nash implementibility which came about very recently, in 1990. The only limitation on the characterization is its being for a society with a population of three or more. However, there exists another characterization which deals with the case of a population of two. This second characterization is at the same time with the one above and is shown independently with two different teams, namely Moore and Repullo [14] and Dutta and Sen [7]. The theorem is as follows.

Theorem 5: (Moore and Repullo , Dutta and Sen)

If #N=2, then a SCR can be implemented in Nash equilibrium iff it satisfies condition m2 given below:

Condition m2: A SCR is said to satisfy condition m2 iff there exists a $B \subseteq A$ and, for all $R \in D(A)$, $a \in F(R)$ and $i \in N$, there is a nonempty set $C_i(a,R)=\pi(S_i , s_{-i}^*)$ (where $s^* \in S$ is a Nash equilibrium of the normal form game h[R]) with $a \in C_i(a,R) \subseteq (L(a,R_i) \cap B)$ satisfying condition m, and additionally the below condition:

(miv) For every $R,R' \in D(A)$, $a \in F(R)$ and $a' \in F(R')$, there exists some $c \in C_1(a,R) \cap C_2(a,R)$ s.t. if $R^* \in D(A)$ is s.t. $C_1(a,R) \subseteq L(c,R_1^*)$ and $C_2(a',R') \subseteq L(c,R_2^*)$ then $c \in F(R^*)$

These two theorems bring about a full characterization of Nash implementibility. Thus and SCR that satisfies condition m or m2 (depending on the number of individuals in the society) surely is implementable in the Nash equilibrium of a mechanism and any SCR that is Nash implementable must satisfy one of these conditions (depending on the number of individuals in the society). This very recent works show that without a restriction on the admissible set of preferences, one can check whether a SCR is Nash implementable and also can construct Nash implementable social choice rules.

Though there is a nice characterization of Nash implementable social choice rules, Nash implementation is limiting in terms of the information needed to obtain a Nash Equilibrium; that means, one has to know about the Nash strategies of the others in order to decide on whether his/her Nash strategy is the best he/she can obtain. This implies a full knowledge about the preferences and the strategy spaces of the other individuals. The second point about the Nash equilibrium is the problem of reaching the equilibrium. Nash equilibrium is self-enforcing, that means, no one deviates from his/her Nash strategy once they fall into the Nash equilibrium but there is no proposal embedded in the Nash concept about how this equilibrium will be achieved. Another point about the Nash equilibrium occurs whenever there are more than one equilibrium in such a case there may occur a coordination problem as in the famous story of O'Henry. Both players may play one of their Nash strategies that lead to different Nash equilibria and the result of such an uncoordinated play may be the worst outcome that may occur. These are the main reasons that led the scientists first to try dominant strategy implementation of social choice rules. However, if one finds the domain restrictions that characterize dominant strategy implementable social choice rules to be very restrictive, Nash implementation always occurs as an alternative way of implementation.

3.2.3 Social Choice Correspondences

Since the implementation task in social choice functions (if the domain is not restricted to be a decomposable one) hits the walls of impossibility, one can wonder whether using social choice correspondences instead is a solution to this problem. Another reason is the curiosity about the reason of why one can extend the analysis to social choice correspondences in case of Nash implementation while for strategy proofness (dominant strategy implementation) he/she is limited to social choice functions. There is not much literature about this business since in the strategy proofness concept the outcome is what matters and to display the manipulation of a social choice rule one needs functions whose values are definite outcomes. Thus a social choice function differs from a social welfare function which it resembles in most other respects, since a social welfare function doesn't have to have a single maximum.

In case of social choice correspondences strategy proofness looses its meaning. Since the preferences of the individuals are defined on the alternatives rather then groups of alternatives, the question of evaluation of an outcome relative to another remains open.

Though one can propose ways to derive new preferences (that are binary relations on the power set of the alternative set) from the preferences on single alternatives, the question of in which degree these reflect the true preferences of the agents remain open. Moreover one can define a social choice correspondence as a function where the individuals directly have preferences over the subsets of the alternative set and the social choice function, $F':D(\Pi)\rightarrow\Pi$ derived from the social choice correspondence $F:D(A)\rightarrow A$ is used instead of the social choice correspondence F.

Another important point is that a dominant strategy implementable social choice correspondence turns out to be singleton-valued when the preferences are restricted to be a subset of linear orders on A. This result will be demonstrated as a proposition below.

These are the main points that lead the literature to deal only with social choice functions instead of taking social choice correspondences into account. Thus, this section will demonstrate an analysis of social choice correspondences instead of proposing a solution to the impossibility problem in implementation.

While dealing with social choice correspondences there occur many questions in mind which we will try to answer here. The first one is about the effect of the preference domain in determination of whether a dominant strategy implementable social choice correspondence is singleton-valued or not. The below proposition proves that if the preference domain is restricted to be a subset of linear orders on A, then a dominant strategy implementable social choice correspondence turns out to be singleton-valued.

Proposition:

Let $F:D(A)\rightarrow A$ be a SCC where for all $i\in N$, $D_i(A)\subseteq L(A)$. Now, if F is dominant strategy implementable, then F is singleton-valued.

proof: Assume that F is dominant strategy implementable and let $h=(S,\pi)$ be the mechanism implementing F in DSE.

Suppose F is not singleton-valued. Then there is an $R\in D(A)$ s.t. $F(R)$ is not a singleton. Now $F(R)=\pi(\sigma(h[R]))$.

Take any $a_0,a_n\in F(R)$ s.t. $a_0\neq a_n$.

Now there are $s,q\in\sigma(h[R])$ s.t. $\pi(s)=a_0$ and $\pi(q)=a_n$.

Take the sequence $\pi(s_1,s_2,\ldots,s_n)=a_0$, $\pi(q_1,s_2,\ldots,s_n)=a_1$, ... , $\pi(q_1,\ldots,q_n)=a_n$.
Now for any $k\in N$ both s_k and q_k are dominant strategies. Thus k must be indifferent among the outcomes a_{k-1} and a_k because of the antisymmetry property of linear orders.
Since this holds for all $k\in N$, $a_0=a_n$. but this contradicts with $a_0\neq a_n$. Thus the supposition is wrong and F is singleton-valued.

QED.

Another question that arises is whether the Gibbard-Satterthwaite theorem is still valid with social choice correspondences. Now since the preference domain can be taken as a full domain of complete preorders in the impossibility theorem we do not need to have singleton-valued social choice correspondences. Moreover if one checks Gibbard's proof of the impossibility theorem, he/she can see that it is based on social welfare functions and implementation in dominant strategy equilibrium. This brings the intuition that the Gibbard-Satterthwaite theorem can be extended to hold for dominant strategy implementibility of social choice correspondences. This is achieved in the following chapter, in page fifty. This means that using social choice correspondences doesn't let us to get rid of impossibility in implementation.

Another question in mind is whether the individuals are indifferent among the elements of the outcome set, F(R). That means, if a social choice correspondence is dominant strategy implementable is it utilitywise singleton? Unfortunately one can find examples in which there is an individual who strictly prefers an element of the outcome set to another. The following example illustrates this.

Example 4:

Let F be a SCC as follows. $F:D(A)\rightarrow A$ is s.t. $A=\{a,b\}$, $N=\{1,2\}$ and for all $i\in N$, $D_i(A)\subseteq\Omega$. $F(R)=\text{argmax}R_1$.

Now, F is implemented in DSE by the mechanism below.

$m=(S,\pi)$ where $S_1=\{a,b\}$ and $S_2=\{a,b\}$ and $\pi(s_1,s_2)=s_1$.

Now for any $R\in D(A)$, $F(R)=\pi(\sigma(m[R]))$, and thus m implements F in DSE. Moreover $F(R)=\{a,b\}$ where aR_1b and bR_1a (i.e. R_1 is an indifference relation) and $b{\sim}R_2a$ (i.e.

individual 2 strictly prefers a to b). Here the second individual is not indifferent between a and b and F(R)={a,b}.

As can be seen in the above example, in case of dictatoriality there may exist individuals who are not indifferent among the outcomes generated by the social choice correspondence. What if dictatoriality is dropped? Then, for the two agents two goods case, we see that the individuals should be indifferent among the outcomes to achieve implementation in dominant strategy equilibrium.

Proposition:

Let $F:D(A)\rightarrow A$ where $A=\{a,b\}$, $N=\{1,2\}$ and for all $i\in N$, $D_i(A)$ be a nondictatorial SCC which is implementable by a mechanism $h=(S,\pi)$ in DSE. Then for each $R\in D(A)$, for all $i\in N$, individual i is indifferent among the elements of F(R).

proof: Suppose there is an $R\in D(A)$ s.t. $F(R)=\{a,b\}$ and $b\sim R_1 a$ (i.e. individual 1 strictly prefers a to b). Now since h implements F in DSE, there are $s^a,s^b\in\sigma(h[R])$ s.t. $\pi(s^a)=a$ and $\pi(s^b)=b$. Now $\pi(s_1^a,s_2^b)=b$ since otherwise s_1^b would not be a dominant strategy of individual 1. (He/she would gain by deviating to s_1^a from s_1^b). Similarly $\pi(s_1^b,s_2^a)=a$. Thus, the outcome is independent of the first individual's strategy. But this means that the second individual is a dictator of the mechanism, thus also of the SCC it implements in DSE. This contradicting with the assumption of nondictatoriality implies that the supposition is wrong.

QED.

As can be seen in the above analysis, there is not a gain offered by the usage of social choice correspondences instead of functions. The only, above the surface, change is the definition of strategy proofness not holding for correspondences. However there occurs the same problems with dominant strategy implementability instead of strategy proofness.

4 MAIN RESULTS

At this chapter it is aimed to clarify some points about the relationship among strategy proofness and other concepts. While some of these relationships are intrinsically assumed to hold, there doesn't exist a formal construction of these relationships in the literature. Moreover, this chapter introduces some new results about the characterization of the relationships among the concepts given, together with a new impossibility theorem that is extended to generalized social choice rules.

The first task is the clarification of the relationship between strategy proofness and dominant strategy implementibility. Now since the set of dominant strategy mechanisms is broader than the set of strategy proof mechanisms, one can easily find an example where the mechanism is strategy proof but doesn't implement the SCR it is associated with.

Example 1: (A strategy proof mechanism which doesn't implement the SCF (with which it is associated) in DSE.)

Let $h=(\Omega^n,F)$ be a strategy proof mechanism. Now for any $R\in\Omega^n$, $R\in\sigma(h[R])$.

For h to implement F in DSE, for all $R\in\Omega^n$ and for all $B\in\Pi$, $F(R,B)=F(\sigma(h[R]))$ must hold. Take R^* to be the preference profile where for each individual i, R_i^* is an indifference relation. Now $\sigma(h[R^*])= \Omega^n$ i.e. any preference profile is a dominant strategy equilibrium of the game $h[R^*]$. Now $F(\sigma(h[R^*]))=A$ but $F(R^*)$ is a singleton and thus $F(R^*)\neq F(\sigma(h[R^*]))$. So h (though it satisfies strategy proofness condition) doesn't implement F in DSE.

We saw in the above example that there doesn't have to be an equivalence relation between strategy proofness and dominant strategy implementation. This is mainly because of taking the preference domain as a subset of the complete preorders on the alternative set, A. Now we will claim that restricting the preference domain to the linear orders is a sufficient condition for this equivalence.

Proposition:

Let $D_i(A) \subseteq L(A)$ be the preference domain for each individual $i \in N$. Let F be a social choice function. Now, F is strategy proof iff it is dominant strategy implementable.

proof: Assume that F is strategy proof. Now, $F: D(A) \rightarrow A$ is said to be strategy proof if the associated revelation mechanism $h=(D(A),F)$ is strategy proof. That means for all $R \in D(A)$, $R \in \sigma(h[R])$.

Now suppose that F is not dominant strategy implementable. Take the mechanism $h=(D(A),F)$ which also doesn't implement F in DSE by the supposition. That means, there exists a profile $P \in D(A)$ s.t. $P \in \sigma(h[P])$ and $F(P)=c_0$ since F is strategy proof, and moreover there exists a $Q \in D(A)$ s.t. $Q \neq P$, $Q \in \sigma(h[P])$ and $F(Q)=c_n \neq c_0$.

Now take the following sequence,

$F(P_1,\ldots,P_n)=c_0$, $F(Q_1,P_2,\ldots,P_n)=c_1$, ... , $F(Q_1,\ldots,Q_n)=c_n$.

Now since both P_i and Q_i are dominant strategies of individual i, he/she must be indifferent between the outcomes c_{i-1} and c_i. But this, by antisymmetry property of linear orders, implies that $c_{i-1}=c_i$. Moreover this holds for all $i \in N$. Thus $c_0=c_1=\ldots=c_n$ which means that $c_0=c_n$. This contradicts with $c_0 \neq c_n$, thus the supposition is wrong, F is dominant strategy implementable.

For the converse case, assume that F is dominant strategy implementable.

Let $h=(X,\pi)$ be the mechanism implementing F in DSE. Then for all $R \in D(A)$, $F(R)=\pi(\sigma(h[R]))$.

Now since h is a dominant strategy mechanism, each individual have a dominant strategy in the game h[R]. Call s* as a dominant strategy equilibrium of h[R].Take an individual $i \in N$.

Then for all $s_i' \in X_i$, and any $s_{-i} \in X_{-i}$, $\pi(s_i^*,s_{-i})R_i\pi(s_i'.s_{-i})$.

That is , for all $P_i \in D_i(A)$, $\pi(\sigma(h[R]))R_i\pi(\sigma(h[P_i,R_{-i}]))$.

But that means, for all $P_i \in D_i(A)$, $F(R_{-i},R_i)R_iF(R_{-i},P_i)$.

Since this holds for all $R \in D(A)$ and for all $i \in N$, F is strategy proof.

QED.

Here, though restricting the preference domain is needed for strategy proofness to imply dominant strategy implementibility, this kind of a restriction is not necessary for dominant strategy implementibility to imply strategy proofness. That is, any social choice function that is dominant strategy implementable is strategy proof. To extend the results on social choice rules to the generalized social choice rules, we first have to clarify some points about the concepts that will be used.

The first point is about the relationship between strategy proofness and rationality of a generalized social choice function. The following two examples illustrate that one of these concepts doesn't imply the other to exist.

Example 2: (A strategy proof GSCF which is not rational)

Take the imposed GSCR $F:D(A)\times\Pi\rightarrow A$, where $A=\{a,b,c\}$. construct F as

$F(.,A)=a$

$F(.,\{a,b\})=b$

$F(.,\{b,c\})=b$

$F(.,\{a,c\})=a$

$F(.,\{x\})=x$ for all $x\in A$.

Now, since F is an imposed GSCF, it is strategy proof (i.e. no individual has any incentive to manipulate F) but it is not rational.

Example 3: (A rational GSCF which is not strategy proof)

Let $F:D(A)\times A\rightarrow A$ be a rational GSCF. Let $f:D(A)\rightarrow L(A)$ be the SWF associated with F. For each $R\in D(A)$ and $B\in\Pi$, define F as $F(R,B)=\text{argmin}_B f(R)$. Now, each individual has an incentive to change the social order via misrepresenting his/her preferences (namely declaring the inverse of his/her preference ordering). Thus F is not strategy proof.

The second point of clarification is about the relationship of strategy proofness, rationality and the Pareto optimality of a generalized social choice function. While analyzing the relationship between strategy proofness and the quasi Pareto criterion we see that rationality is a sufficient condition for strategy proofness to imply the quasi Pareto criterion. We will first demonstrate an example where a social choice function, though it is strategy proof, does not satisfy the quasi Pareto criterion. Then we will claim that any rational generalized social choice function which is strategy proof is automatically quasi Pareto optimal.

Example 4: (A SCF which is strategy proof but not quasi Pareto optimal)

Let $F:D(A)\to A$ be a social choice function where $A=\{a,b,c,d\}$, $D(A)=D_1(A)\times D_2(A)$. $D_1(A)=\{R_1,Q_1\}$ where $aR_1bR_1cR_1d$ and $bQ_1aQ_1dQ_1c$ and $D_2(A)=\{R_2,Q_2\}$ where $dR_2bR_2cR_2a$ and $bQ_2dQ_2aQ_2c$. That is, both individuals have two alternative linear orders on the set A. Moreover F is as below:

$F(R_1,R_2)=c$

$F(R_1,Q_2)=a$

$F(Q_1,R_2)=d$

$F(Q_1,Q_2)=b$

Now F is easily checked to be strategy proof. Moreover F doesn't satisfy the quasi Pareto criterion since there exists a profile, (R_1,R_2), where each individual strictly prefers the alternative, b, to the outcome of the SCF at that profile, c. Thus F, though it is strategy proof, doesn't satisfy the quasi Pareto criterion. Now we will claim that the rationality is a sufficient condition for this implication.

Proposition:

Let $F:D(A)\times\Pi\to A$ be a GSCF. If F is strategy proof and rational then F satisfies the Quasi Pareto Criterion.

proof: Let $F:D(A)\times\Pi\to A$ be a strategy proof and rational GSCF. Suppose that F doesn't satisfy the Quasi Pareto criterion. Then there exists a profile $R\in D(A)$ and $x,y\in B\in\Pi$ s.t. for all $i\in N$, xR_iy and $y\sim R_ix$ is satisfied and $F(R,B)=y$. Define $B'=\{x,y\}$.

Now since F is rational $F(R,B')=y$. Moreover since $x\in B'$, there exists an $R'\in D(A)$ s.t. $F(R',B')=x$. Define the sequence $F_0,\ldots,F_n$ as follows

F_0=F((R_1,...,R_n), B')=y , F_1=F((R_1',R_2,...,R_n), B') , ... , F_k=F((R_1',...,R_k',R_{k+1},...,R_n), B'), and F_n=F((R_1',...,R_n'), B')=x

Now there exists an individual k∈N s.t. F_k=x and F_{k-1}≠x which means F_{k-1}=y.

Thus $F_kR_kF_{k-1}$ i.e. F((R_1',...,R_k',R_{k-1},...,R_n), B')R_kF((R_1',...,R_k,R_{k+1},...,R_n), B')

which implies that R_k is not a dominant strategy for k. But this in turn means that F is not a strategy proof GSCF. This leads to a contradiction, so F satisfies the quasi Pareto criterion.

QED.

The last point of clarification about strategy proof generalized social choice functions is on the relationship between strategy proofness and some of the Arrow's conditions. This relationship, though proved independently by us, was shown to be true by Blin and Satterthwaite [5] and Blair and Muller [3]. The idea is summarized in the following proposition.

Proposition:

Let F:D(A)×Π→A be a rational GSCF and let f be the SWF associated with F. Then F is strategy proof iff f satisfies monotonicity and IIA.

proof: Assume that f satisfies monotonicity and IIA conditions. Suppose that F is not strategy proof. Then there exists an i∈N s.t. i can manipulate F at profile P∈D(A) and feasible set B by playing Q_i. This means that there are x,y∈B s.t. x=F(P,B), y=F(P_{-i},Q_i, B) and yP_ix. This, by rationality, implies that xf(P)y and yf(P_{-i},Q_i)x.

If yQ_ix then since initially there was yP_ix, the assumption of IIA is violated.

If xQ_iy then the assumption of monotonicity is violated since i's changing from yP_ix to xQ_iy (i.e. increasing the rank of x in his/her preference ordering) leads x not to be chosen.

For the converse part assume that F is strategy proof.

Case 1: Suppose f is not monotonic.

Then there are R,R'∈D(A) , B∈Π and x∈B s.t. for C=B\{x}

(i) R and R' agree on C

(ii) For all $i \in N$, and for all $y \in C$, xR_iy implies that $xR_i'y$

(iii) $x = \text{argmax}_B f(R)$

and $x \neq \text{argmax}_B f(R') = z$ where $z \in C$.

Now $\text{argmax}_B f(R) = x$,by rationality, implies that $\text{argmax}_{\{x,z\}} f(R) = x$.

So $xf(R)z$ but $zf(R')x$ i.e.

$f_0 = \text{argmax}_{\{x,z\}} f(R_1,\ldots,R_n) = x$, $f_1 = \text{argmax}_{\{x,z\}} f(R_1',R_2,\ldots,R_n)$,..., $f_n = \text{argmax}_{\{x,z\}} f(R_1',\ldots,R_n') = z$

Then there is a $k \in N$ s.t. $f_k = z$ and $f_{k-1} = x$ i.e. if k plays R_k x is chosen and if k plays R_k' z is chosen.

Suppose xR_kz is the case. But this implies $xR_k'z$ and while all the preferences are constant the outcome changes then. (This will be shown to be implied by strategy proofness in one of the following propositions)

So $x \sim R_kz$ is the case. Now it may be that $x \sim R_k'z$, but the same contradiction above occurs in this case. So $xR_k'z$. But then player k can manipulate F at profile R and feasible set {x,z} by playing R_k'. But this contradicts with the assumption that F is strategy proof. So f is monotonic.

Case 2: Suppose f doesn't satisfy IIA .

That is, there are $P,Q \in D(A)$ which agree on a $B \in \Pi$, but $x = F(P,B) \neq F(Q,B) = y$,

i.e. $x = \text{argmax}_B f(P) \neq \text{argmax}_B f(Q) = y$

Now form the sequence

$f_0 = \text{argmax}_B f(P) = x$, $f_1 = \text{argmax}_B f(Q_1,P_2,\ldots,P_n)$, ... , $f_n = \text{argmax} f(Q_1,\ldots,Q_n) = y$

Now there is a $k \in N$ s.t. $F_k = y$ and $F_{k-1} \neq y$.

Since P and Q agree on B, for all $x,y \in B$, (xP_ky iff xQ_ky) and (yP_kx iff yQ_kx) must hold. But then monotonicity which was shown to be implied by strategy proofness and rationality is violated. (This is because the social choice changes from x to y though no preference changes from f_k to f_{k-1}). This means that the above supposition leads to a contradiction. Thus f satisfies. IIA.

QED.

Having clarified relationships among the main concepts that are related with strategy proof social choice functions we now can extend the characterization of the relationship between strategy proofness and dominant strategy implementibility to the generalized social choice rules. While doing this task, to maintain a relationship among the mechanisms, each of which implements the generalized social choice rule for a certain subset of A, is an important criteria. Though it is possible to show that there exists a class of mechanisms (the elements of which are not necessarily related with each other) which implements a strategy proof generalized social choice function (and vice versa), this kind of a characterization does not tell much about how to obtain that specific class of mechanisms. Thus, the concepts of "total implementation" and "pseudo implementation" are introduced. This, specifically, makes the characterization wider and more powerful.

As a result of the above reasons, we will first introduce a characterization about the relationship between strategy proofness and pseudo implementation. Then we will generalize it to total implementability and to a characterization of the equivalence among all these related concepts.

Lemma:

Let $F:D(A)\times\Pi\rightarrow A$ be a GSCF, where for all $i\in N$, $D_i(A)\subseteq L(A)$. Now F is strategy proof iff F is pseudo implementable in DSE.

proof: Firstly assume that F is strategy proof. Take the class of mechanisms $m=\{m_B=(D(A),F_B) \,/\, B\in\Pi\}$ where for all $R\in D(A)$, $F_B(R)=F(R,B)\in B$.

Suppose F is not pseudo implementable in DSE. Then, specifically, the class m doesn't pseudo implement F. This means that there is a $B\in\Pi$ s.t. $m_B=(D(A), F_B)$ doesn't pseudo implement $F(\,.\,,B)$ in DSE relative to B.

Now $R\in\sigma(m_B[R])$ since for all $Q_{-i}\in D_{-i}(A)$ and $P_i\in D_i(A)$, $F(R_i,Q_{-i},B)R_iF(P_i,Q_{-i},B)$ by strategy proofness of F. But this means that $F_B(R_i,Q_{-i})R_iF_B(P_i,Q_{-i})$.

Call $F_B(R)=c_0$. Then $F_B(\sigma(m_B[R]))\supseteq F_B(R)=\{c_0\}$.

Now since F is not implementable in DSE by supposition, there is a $P\in D(A)$ s.t. $P\in\sigma(m_B[R])$ and $F_B(P)=c_n\neq c_0$.

Now take the following sequence,

$F_B(R_1,\ldots,R_n)=c_0$, ... , $F_B(P_1,\ldots,P_k,R_{k+1},\ldots,R_n)=c_k$, ... , $F_B(P_1,\ldots,P_n)=c_n$

Now take any $k \in N$. Since the only difference between the strategy profiles leading to c_{k-1} and c_k is individual k's strategy (R_k for c_{k-1} and P_k for c_k) and since both are his/her dominant strategies he/she must be indifferent among the outcomes c_{k-1} and c_k. But since his/her preference relation is a linear order, by antisymmetry of linear orders $c_{k-1}=c_k$.

Since this holds for all $k \in N$, $c_0=c_1=\ldots=c_n$ which means $c_0=c_n$.

But this contradicts with $c_0 \neq c_n$.

So F is pseudo implemented by the class m in DSE.

For the converse assume that F is pseudo implementable in DSE. Then, there's a class of mechanisms $h=\{h_B=(S_B,\pi_B) \,/\, B \in \Pi\}$ s.t. for all $B \in \Pi$, h_B pseudo implements F(. ,B) in DSE relative to B. That is, for all $R \in D(A)$, $F(R,B)=\pi_B(\sigma(h_B[R]))$.

Now, we want to show that for all $B \in \Pi$, for all $R \in D(A)$, and for all $P_i \in D_i(A)$, $F(R_{-i},R_i ,B) R_i F(R_{-i},P_i ,B)$.

Take any $B \in \Pi$, $R \in D(A)$, $P_i \in D_i(A)$. Take any $s^* \in \sigma(h_B[R])$ and any $i \in N$.

Now since s_i^* is a dominant strategy of i $\pi(s_i^*,s_{-i}^*) R_i \pi(s_i,s_{-i}^*)$ for any $s_i \in S_{B,i}(s_i^*)$.

Take the two pseudo games $h_B[R]$ and $h_B[R_{-i},P_i]$.

Now for all $j \in N \backslash \{i\}$, s_j^* which was a dominant strategy of j in $h_B[R]$ is still a dominant strategy in $h_B[R_{-i},P_i]$. Since $\pi_B(\sigma(h_B[R_{-i},P_i]))=F(R_{-i},P_i,B) \neq \emptyset$, there still exists a dominant strategy for i in $h_B[R_{-i},P_i]$. Take $s' \in \sigma(h_B[R_{-i},P_i])$ s.t. for all $j \in N \backslash \{i\}$, $s_j'=s_j^*$. Then there exists an $s_i' \in S_{B,i}(s_{-i}')$ s.t. $(s_i',s_{-i}')=(s_i',s_{-i}^*) \in \sigma(h_B[R_{-i},P_i])$.

Now since $\pi(s_i^*,s_{-i}^*) R_i \pi(s_i,s_{-i}^*)$ for all $s_i \in S_{B,i}(s_{-i}^*)$,

$\pi(s_i^*,s_{-i}^*) R_i \pi(s_i',s_{-i}^*)$ since $s_i' \in S_{B,i}(s_{-i}^*)$.

But this means that $\pi(\sigma(h_B[R])) R_i \pi(\sigma(h_B[R_{-i},P_i]))$, and is equivalent to saying $F(R,B) R_i F(R_{-i},P_i ,B)$. Thus F is strategy proof.

QED.

Though this lemma brings a full characterization of the relationship between strategy proofness and pseudo implementability in dominant strategies, pseudo implementability is a new concept and in some terms weaker than the property of implementability. Because of this reason, a full characterization of the relationships between strategy proofness and other concepts can't be considered to be completed. Thus, with the help of the following proposition, a full characterization of these relationships will be given in an equivalence theorem which will relate strategy proofness, pseudo implementability, implementability and total implementability all together.

Proposition:

Let $F:D(A)\times\Pi\rightarrow A$ be a strategy proof GSCF where $D_i(A)\subseteq L(A)$ for each $i\in N$. Let $R,Q\in D(A)$ be s.t. $R_B=Q_B$ for some $B\in\Pi$ (where R_B is the restriction of the profile R on the set B). Then $F(R,B)=F(Q,B)$.

proof: Suppose that $F(R,B)\neq F(Q,B)$.

Take $F(R,B)=a_0\in B$ and $F(Q,B)=a_n\in B$ s.t. $a_0\neq a_n$.

Now form the following sequence,

$F(R,B)=a_0$ ..... $F(Q_1,...,Q_k,R_{k-1},...,R_n)=a_k$,..., $F(Q,B)=a_n$ where $\{a_0,a_1,...,a_n\}\subseteq B$.

Now, for any individual, k, at profile $(Q_1,...,Q_{k-1},R_k,...,R_n)$, there must be $a_{k-1}R_ka_k$ by strategy proofness.

Moreover, at profile $(Q_1,...,Q_k,R_{k-1},...,R_n)$, $a_kQ_ka_{k-1}$ for the same reason.

But since both a_{k-1} and a_k are in B and since R and Q completely agree on B,

$a_{k-1}R_ka_k$ implies $a_{k-1}Q_ka_k$ and $a_kQ_ka_{k-1}$ implies $a_kR_ka_{k-1}$.

This means that, in both profiles R and Q, individual k is indifferent between a_{k-1} and a_k.

Thus, by the antisymmetry property of the linear orders, $a_k=a_{k-1}$.

Since this holds for any $k\in N$, $a_0=a_1=...=a_n$ which implies $a_0=a_n$.

But this contradicts with the supposition, thus the supposition is wrong. Thus $F(R,B)=F(Q,B)$.

QED.

This theorem is a full characterization of the relationships among strategy proofness and all the introduced implementation concepts.

Theorem:

Let $F:D(A)\times\Pi\to A$ be a GSCF where $D_i(A)\subseteq L(A)$ for each $i\in N$. Now, the followings are equivalent:

(i) F is strategy proof.

(ii) F is pseudo implementable in dominant strategies.

(iii) F is implementable in dominant strategies.

(iv) F is totally implementable in dominant strategies.

proof:

(i) iff (ii):

This was shown to be true by the above lemma.

(ii) if (iv) and (iii) if (iv):

These simply hold by definition of total implementation.

(iv) if (i):

Assume that F is a strategy proof GSCF. Define the class $m=\{m_B=(D(B),F_B \,/\, B\in\Pi\}$ as $D(B)=D_1(B)\times...\times D_n(B)$ where $D_i(B)$ is the restriction of $D_i(A)$ on B and $F_B:D(B)\to B$ is s.t. $F_B(R_B)=F(R,B)$ for any $R_B\in D(B)$ (which is the restriction of $R\in D(A)$ on B).

Now F_B is a well-defined function as a result of the preceding proposition. If the proposition was not true, there could be a case where F(R,B) would not be equal to F(Q,B) while $R_B=Q_B$ This would leave to a contradiction as follows: $F_B(R_B)=F(R,B)\neq F(Q,B)=F_B(Q_B)=F_B(R_B)$. Thus, the above proposition is essential for m to be a well-defined mechanism class.

Suppose that F is not totally implementable. This specifically means that the mechanism class m doesn't implement F. Then there exists a subset B of A s.t m_B doesn't implement F(. ,B) in dominant strategy equilibrium.

Now since F is strategy proof, for all $R\in D(A)$, for all $i\in N$ and for all $Q_i\in D(A)$, $F(R_i,R_{-i},B)R_iF(Q_i,R_{-i},B)$. This is equivalent to $F_B(R_{i,B},R_{-i,B})R_iF_B(Q_{i,B},R_{-i,B})$.

This means that R_B is a dominant strategy equilibrium of $m_B[R]$, i.e. $R_B\in\sigma(m_B[R])$. Call $F_B(R_B)=c_0$. Now, $c_0=F_B(R_B)\subset F_B(\sigma(m_B[R]))$. That is, since m_B doesn't implement F(. ,B) by supposition, there exists an $P_B \in \sigma(m_B[R])$ s.t. $P_B \neq R_B$ and $F_B(P_B)=c_n \neq c_0$ .Form the following sequence,

$F_B(R_{1,B},\ldots,R_{n,B})=c_0,\ldots,F_B(P_{1,B},\ldots,P_{k,B},R_{k-1,B},\ldots,R_{n,B})=c_k,\ldots,F_B(P_{1,B},\ldots,P_{n,B})=c_n$.

Then since for all $i \in N$, $R_{i,B}$ and $P_{i,B}$ are dominant strategies of $i \in N$, and both R_i and P_i are in $L(A)$, $c_{i-1}=c_i$ by the antisymmetry property of linear orders. That implies $c_0=c_1=\ldots=c_n$ which in turn means that $c_0=c_n$ contradicting with $c_0 \neq c_n$. Thus the supposition is wrong and F is totally implementable (specifically by the class m).

(i) if (iii):

Assume that F is implementable in dominant strategies. Then there exists a class of mechanisms $h=\{h_B=(S_B,\pi_B)\ /\ B \in \Pi\}$ that implements F in dominant strategies. That means, for each $B \in \Pi$, a mechanism, h_B, implements $F(\ .\ ,B)$ in dominant strategies.

That is, for all $R \in D(A)$, $F(R,B)=\pi_B(\sigma(h_B[R]))$.

Now, we want to show that for all $B \in \Pi$, for all $R \in D(A)$, and for all $P_i \in D_i(A)$,

$F(R_{-i},R_i,B)R_iF(R_{-i},P_i,B)$.

Take any $B \in \Pi$, $R \in D(A)$, $P_i \in D_i(A)$. Take any $s^* \in \sigma(h_B[R])$ and any $i \in N$.

Now since s_i^* is a dominant strategy of i $\pi(s_i^*,s_{-i}^*)R_i\pi(s_i,s_{-i}^*)$ for any $s_i \in S_{B,i}(s_i^*)$.

Take the two games $h_B[R]$ and $h_B[R_{-i},P_i]$.

Now for all $j \in N \backslash \{i\}$, s_j^* which was a dominant strategy of j in $h_B[R]$ is still a dominant strategy in $h_B[R_{-i},P_i]$. Since $\pi_B(\sigma(h_B[R_{-i},P_i]))=F(R_{-i},P_i,B) \neq \emptyset$, there still exists a dominant strategy for i in $h_B[R_{-i},P_i]$. Take $s' \in \sigma(h_B[R_{-i},P_i])$ s.t. for all $j \in N \backslash \{i\}$, $s_j'=s_j^*$. Then there exists an $s_i' \in S_{B,i}(s_{-i}')$ s.t. $(s_i',s_{-i}')=(s_i',s_{-i}^*) \in \sigma(h_B[R_{-i},P_i])$.

Now since $\pi(s_i^*,s_{-i}^*)R_i\pi(s_i,s_{-i}^*)$ for all $s_i \in S_{B,i}(s_{-i}^*)$,

$\pi(s_i^*,s_{-i}^*)R_i\pi(s_i',s_{-i}^*)$ since $s_i' \in S_{B,i}(s_{-i}^*)$.

But this means that $\pi(\sigma(h_B[R]))R_i\pi(\sigma(h_B[R_{-i},P_i]))$, and is equivalent to saying $F(R,B)R_iF(R_{-i},P_i,B)$. Thus F is strategy proof.

Now, via the above implications which are proven to be true, any of the four concepts imply any other. Thus the equivalence holds.

QED.

One problem with this characterization is its being designed for the domains that are subsets of linear orders. However, since it is shown that in case of the lack of this assumption strategy proofness doesn't necessarily imply implementability, this is not a main problem. One other result of this restriction is , as it was shown in the section for social choice correspondences, that a dominant strategy implementable social choice correspondence turns out to be singleton-valued when the preference domain is restricted to be a subset of the linear orders. Moreover, when the preference domain is not restricted to be a subset of linear orders, the individuals need not be indifferent among the outcomes of a dominant strategy implementable social choice correspondence (This was also shown in the section about social choice correspondences).

Leaving the preference domain unrestricted (to be a subset of linear orders) creates an additional problem. In such a situation, two profiles that agree on a subset of the alternative set does not necessarily lead to the same outcome. The following is an example of such a situation.

Example 5: (A strategy proof and rational GSCF which doesn't satisfy the equivalent outcomes property of the agreeing profiles)

Let $F: D(A)\times\Pi \rightarrow A$ be a GSCF where $A=\{a,b,c\}$, $N=\{1,2\}$, $D(A)=D_1(A)\times D_2(A)$.

Let $D_i(A)=\{R,Q\}$ for $i\in N$. Define R as aIbPc (i.e R is indifferent between a and b and both are strictly preferred to c) and Q as cPaIb (i.e Q strictly prefers c to both a and b and is indifferent between a and b).

Define F as follows:

$F(R,R;A)=a$	$F(R,R;\{a,b\})=a$	$F(R,R;\{a,c\})=a$	$F(R,R;\{b,c\})=a$
$F(R,Q;A)=a$	$F(R,Q;\{a,b\})=a$	$F(R,Q;\{a,c\})=a$	$F(R,Q;\{b,c\})=b$
$F(Q,R;A)=a$	$F(Q,R;\{a,b\})=a$	$F(Q,R;\{a,c\})=a$	$F(Q,R;\{b,c\})=b$
$F(Q,Q;A)=c$	$F(Q,Q;\{a,b\})=b$	$F(Q,Q;\{a,c\})=c$	$F(Q,Q;\{b,c\})=c$

Now, F is strategy proof, rational, nondictatorial. Moreover, though $(Q,R)_{\{a,b\}}=(Q,Q)_{\{a,b\}}$ $F(Q,R,\{a,b\})\neq F(Q,Q,\{a,b\})$.

The existence of such a situation leads to the necessity of an additional assumption about the case in the following impossibility theorem which is a generalization of the Gibbard-Satterthwaite theorem for generalized social choice rules. The example also shows that the characterization of decomposable domain is of no use when the preference domain is not restricted to be a subset of linear orders.

The following theorem is a generalization of the Gibbard-Satterthwaite theorem for generalized social choice rules. It requires the additional assumptions of rationality and that the outcomes generated by the two agreeing profiles should be the same on that specific subset of the alternative set.

Theorem:

Let $F:D(A)\times\Pi\to A$ be a rational GSCC where the cardinality of A is greater than or equal to three, $D(A)=L(A)^n$ or $D(A)=\Omega^n$. Assume that for any $R,Q\in D(A)$ s.t. $R_B=Q_B$ for some $B\in\Pi$, $F(R,B)=F(Q,B)$. (Note that this assumption is not necessary when $D(A)=L(A)^n$). Then F is pseudo implementable in dominant strategies iff F is dictatorial.

proof: First assume that F is pseudo implementable. We will show that the associated SWF, f, of F satisfies all Arrow's conditions except nondictatoriality. Then using Arrow's theorem, we will claim that f is dictatorial. Finally we will show that this implies F to be dictatorial.

Claim 1: f satisfies IIA.

proof: Suppose f doesn't satisfy IIA. Then there exist $x,y\in A$ and $P,Q\in D(A)$ s.t. P and Q agree on $\{x,y\}$ and f(P) and f(Q) doesn't agree on $\{x,y\}$.

W.l.o.g. assume that $xf(P)y$ and $x\sim f(Q)y$. That means, $\text{argmax}_{\{x,y\}}f(P)\neq \text{argmax}_{\{x,y\}}f(Q)$ which implies $F(P,\{x,y\})\neq F(Q,\{x,y\})$. But this contradicts with the above assumption. Thus the supposition is wrong, f satisfies IIA.

Claim 2: f satisfies monotonicity.

proof: Suppose f is not monotonic. Then, there are $R,R'\in D(A)$, $B\in\Pi$, $x\in B$ s.t. for $C=B\backslash\{x\}$,

(i) $R_C=R'_C$

(ii) For all $i \in N$, for all $y \in C$, xR_iy implies xR'_i

(iii) $x \in \text{argmax}_B f(R)$

and $x \notin \text{argmax}_B f(R')$.

Take any $z \in \text{argmax}_B f(R')$. Now, by rationality, $x \in \text{argmax}_{\{x,z\}} f(R)$ and $z = \text{argmax}_{\{x,z\}} f(R')$. For any $k \in \{0,\dots,n\}$, define the profile $R^k = (R'_1,\dots,R'_k,R_{k+1},\dots R_n)$. Now form the following sequence,

$f_0 = \text{argmax}_{\{x,z\}} f(R^0), \dots, f_k = \text{argmax}_{\{x,z\}} f(R^k), \dots, f_n = \text{argmax}_{\{x,z\}} f(R^n)$ where $x \in f_0$ and $z = f_n$.

Now, there exists a $k \in N$ s.t. $x \in f_{k-1}$ and $z = f_k$. That is,

$$x \in \text{argmax}_{\{x,z\}} f(R^{k-1}) = F(R^{k-1},\{x,z\})$$

$$z = \text{argmax}_{\{x,z\}} f(R^k) = F(R^k,\{x,z\})$$

Since F is pseudo implementable, there is a pseudo game form $h=(S,\pi)$ which implements F relative to $\{x,z\}$. Moreover, $x \in \pi(\sigma(h[R^{k-1}]))$ and $z = \pi(\sigma(h[R^k]))$.

From $h[R^{k-1}]$ to $h[R^k]$ the only thing that changes is agent k's preference relation on $\{x,z\}$. That means, there exists an $s_{-k}^* \in S_{-k}$ where for all $i \in N \setminus \{k\}$, s_i^* is a dominant strategy of i; moreover, there exists an $s_k^x \in S_k$ s.t. $\pi(s_k^x, s_{-k}^*) = x$ and there exists an $s_k^z \in S_k$ s.t. $\pi(s_k^z, s_{-k}^*) = z$. Since $x \notin \pi(\sigma(h[R^k]))$, s_k^x is not a dominant strategy of k with the preference relation R'_k, i.e. zP'_kx (where P'_k is the strict preference relation derived from R'_k). But this, by assumption (ii) implies that zP_kx (where P_k is the strict preference relation derived from R_k). Then, $R_{\{x,z\}} = R'_{\{x,z\}}$ which implies that $F(R,\{x,z\}) = F(R',\{x,z\})$. That means $\text{argmax}_{\{x,z\}} f(R) = \text{argmax}_{\{x,z\}} f(R')$ contradicting with the supposition. Thus f is monotonic.

Now f satisfies the following conditions:

(i) Cardinality of A is greater than or equal to three.

(ii) f is monotonic.

(iii) f satisfies IIA.

Thus, by Arrow's impossibility theorem, f is dictatorial.

To show that F is dictatorial, let $d \in N$ be the dictator for f. Now since d is a dictator for f, for all $R \in D(A)$, $f(R) = R_d$. That implies, for all $R \in D(A)$, for all $B \in \Pi$,

$argmax_B f(R) = argmax_B R_d$ which is equivalent to saying $F(R,B) = argmax_B R_d$. Thus $d \in N$ is a dictator for F: thus, F is dictatorial.

For the reverse implication assume that F is a dictatorial GSCC. Let $d \in N$ be a dictator for F. Now, for all $R \in D(A)$, for all $B \in \Pi$, $F(R,B) = argmax_B R_d$.

Take the following class $h = \{h_B = (S_B, \pi_B) \ / \ B \in \Pi\}$ s.t. for all $i \in N$, $S_{i,B} = B$ and $\pi_B(s) = s_d$ for any $s \in S_B$. Take any $R \in D(A)$ and $B \in \Pi$. Now, $s \in \sigma(h_B[R])$ iff $s_d \in argmax_B R_d$. Thus $\pi_B(\sigma(h_B[R])) = argmax_B R_d = F(R,B)$. Since this holds for all $R \in D(A)$ and $B \in \Pi$, the class h pseudo implements F in dominant strategies.

QED.

The last theorem of this chapter is a characterization of rationality in terms of Hauthakker's Axiom. This is mainly done to gain a deeper understanding of the rationality condition and to be able to judge what it adds to a generalized social choice rule. It will be seen that the rationality condition is a restriction that imposes the necessity of consistency among different subsets of the alternative set for a generalized social choice rule. My personal judgment about this concept is that rationality is a support to the concept of generalized social choice rules than being a restriction on them. The following theorem relates rationality with the famous axiom of Hauthakker.

Theorem:

Let $F: D(A) \times \Pi \to A$ be a GSCR. Now F is rational iff for all $P \in D(A)$, $F(P, .) : \Pi \to A$ satisfies Hauthakker's axiom.

proof: Firstly assume that F is a rational SCC. Now by definition of rationality there is a SWF, $f: D(A) \to B(A)$, s.t. for all $P \in D(A)$ and for all $B \in \Pi$: $F(P,B) = argmax_B f(P)$. Now since $f(P) \in B(A)$ it is a complete preorder (i.e. satisfies completeness, transitivity and reflexivity). Define a binary relation R_P on A as for all $x,y \in A$, xR_Py is satisfied iff $xf(P)y$ and $y{\sim}f(P)x$ are satisfied. Now R_P is a strict preference relation (SPR) on A. Define $c(. , R_P): \Pi \to 2^A$ as for any $B \in \Pi$, $c(B,R_P) = \{x \in B \ / \text{ for all } y \in B, \ y{\sim}R_Px\}$. Now for any $B \in \Pi$,

$c(B,R_P)=\{x\in B$ / for all $y\in B$, $y\sim R_P x\}$

$=\{x\in B$ / for all $y\in B$, $xf(P)y\}$

$=\text{argmax}_B f(P)$

$=F(P,B)$

To complete the proof, we will use the following lemma:

Lemma:

Let $c':\Pi\rightarrow\Pi$ be a choice function. Then the followings are equivalent.

(i) There is a SPR, R_P on A s.t. for all $B\in\Pi$, $c'(B)=c(B,R_P)$

(ii) c' satisfies HA

Now $c(. , R_P)$ is equivalent to a choice function since it is nonempty valued as shown above. Thus the first condition of the above lemma is satisfied. But this implies that the choice function c' which is nothing but F(P, .) here satisfies HA. Since this holds for all admissible profiles $P\in D(A)$ the only if part of the proof is completed.

For the if part of the proof assume that for all $P\in D(A)$, $F(P, .):\Pi\rightarrow\Pi$ satisfies HA. Then, by the above lemma there is an SPR, S, on A s.t. for all $B\in\Pi$, $F(P,B)=c(B,S)$ where $c(B,S)=\{x\in B)=\{x\in B$ / for all $y\in B$, $y\sim Sx\}$. Then

$F(P,B)=c(B,S)=\{x\in B$ / for all $y\in B$, $y\sim Sx\}$

$=\{x\in B$ / for all $y\in B$, $xf(P)y\}$

$=\text{argmax}_B f(P)$

Since this holds for all $P\in D(A)$, $f:D(A)\rightarrow B(A)$ can be defined to be a SWF. Thus F is rational.

QED.

5 AN EXAMPLE OF DOMAIN RESTRICTION IN ECONOMIC ENVIRONMENTS

One of the most striking applications of the social choice theory in economic theory is allocation rules. These rules, given the preferences and the endowments of the individuals, re-allocate these endowments among the individuals in such a way to maximize a given objective. Moreover, Walrasian equilibrium is one of the most famous types of equilibria for pure exchange economies (PEE) in economic theory. But in the achievement of the Walrasian equilibria it's presumed that an invisible hand knows the true utilities of the agents and sets the optimal price for exchange to occur in the economy. This can also be thought as the case where each agent knows all about the others and the agents (who are thought to be wise enough to achieve this) together determine a price to achieve a final allocation which satisfies certain optimality conditions.

But the question of what can be achieved in case of incomplete information (this can be thought of as the unexistence of an invisible hand or simply the case of individuals being unaware of the preferences of the others) there occur problems in the achievement of the Walrasian equilibria. Hurwicz in 1972 showed that the Walrasian equilibria are manipulable via one agent affecting the prices through misrepresenting his/her true preferences when the information is not complete and the number of agents in the economy is finite [10]. Though Postlewaite and Roberts in 1976 showed that the gains from manipulation can be limited to a given amount for a big enough number of replications of the economy, there still exists gains from manipulation unless there are an infinite number of agents in the economy.

Since the Walrasian equilibria can't be obtained through decentralization of the system, one searches for alternative procedures through which the decentralization of the system doesn't lead to individual manipulation. This can be achieved via a mechanism which implements a strategy proof allocation rule in DSE. Barbera and Jackson in 1993 obtained such a mechanism which they call as a fixed-price trading(for the case of two individuals) or a fixed-proportions trading (a more general terminology for the case of more than two individuals) mechanism [1]. Moreover they showed that the only way to implement strategy proof allocation rules (social choice rules) in a pure exchange economy is to devise a fixed-proportions trading mechanism. What these mechanisms do is simply to force the individuals to trade according to some pre-specified proportions. One main problem of these mechanisms is efficiency which means that the pre-specified trade proportions doesn't necessarily imply the individuals to maximize utility through trade with respect to these proportions. Now we will start to an analysis of the Barbera, Jackson 1992 paper called *Strategy Proof Exchange*.

The economy taken is a classical exchange economy (a pure exchange economy). There are n agent and l goods where both n and l are finite numbers. An element $e \in \mathbf{R}_+^{nl}$ (where R denotes the set of real numbers) is said to be an allocation of the endowments of the goods among n agents. The set $A=\{x \in R_+^{nl} / \sum_{i=1}^{n} x^i = \sum_{i=1}^{n} e^i\}$ shows the set of balanced allocations and implies the restriction that no good can be thrown away. An element $x^i \in \mathbf{R}_+^l$ shows the l-dimensional allocation of goods to agent i. Moreover, $x_k^i \in \mathbf{R}_+$ shows the allocation of the k'th good to agent i. It is assumed that $\sum_{i=1}^{n} e_k^i > 0$ for each good k. That means that initially there exists a positive amount of each good in the economy.

The function $u^i : R^l \rightarrow R$ is said to be a utility function of agent i. The set U denotes the set of all utility functions which are continuous, strictly-quasiconcave, and increasing. Here the usage of utility functions instead of preferences doesn't create an important problem. The main limitation is the assumption of continuity. This is mainly because we know that on a compact set, A, there exists a continuous strict preference relation, P, iff there exists a continuous utility function, u, with $u = u_P$.

Moreover the above restrictions on U lead it to be a subset of single-peaked preferences set since for each $u^i \in U$, u^i is continuous, increasing and strictly-quasiconcave, u^i is a single-peaked utility function. (That means it has only one global maximum at the point where the agent i takes all the endowments in the economy, and no local maximum at all). Moreover the vector $u=(u^1,\dots,u^n)$ and $(u^{-i},u'^i)=(u^1,\dots,u^{i-1},u'^i,u^{i+1},\dots,u^n)$.

A function (which is nothing more than a social choice rule) $f:U^n \rightarrow A$ is said to be an allocation rule. A_f is the range of f and $f^i(u)$ denotes the allocation given to agent i at the profile u.

Definition: (strategy proofness)

An allocation rule f is strategy proof if for all $i \in N$, for all $u \in U^n$ and for all $u'^i \in U$,

$$u^i(f^i(u)) \geq u^i(f^i(u^{-i}, u'^i))$$

Definition: (individual rationality)

An allocation rule, f is said to be individually rational (w.r.t. the endowment e) if for all $i \in N$ and for all $u \in U^n$,

$$u^i(f^i(u)) \geq u^i(e^i)$$

Definition: (anonymity)

An allocation rule, f is said to be anonymous if for all $u \in U^n$ and for all $i,j \in N$ s.t. $i \neq j$,

$$u'^i = u'^j \text{ and } u^i = u^j \text{ implies } f^i(u^{-(i,j)}, u^i, u'^j) = f^j(u^{-(i,j)}, u'^i, u^j)$$

Definition: (diagonality)

A set $B \subseteq A$ is said to be *diagonal* if for all $i \in N$ and for all $x,y \in B$ s.t. $x \neq y$, $x^i \sim\geq y^i$ and $y^i \sim\geq y^i$.

Definition:

Given three points $a,b,c \in A$, $ab = \{x \in A \ / \text{ there is a } t \in [0,1] \text{ s.t. } x = ta + (1-t)b\}$. Moreover, given a preference ordering $R_i \in D_i(A)$, cR_iab if there is a $t \in [0,1]$ s.t. $c^i R_i\, ta + (1-t)b$.

At the beginning we will take the case of two agents and two goods, and construct a mechanism called a fixed-price trading mechanism. Later we will give a theorem saying that an allocation rule f is strategy proof iff it's implementable by a fixed-price trading mechanism.

The mechanism that will be mentioned though is not a revelation mechanism, it is a dominant strategy mechanism. The mechanism simply uses the following procedure:

Given an endowment point $e \in A$

(i) Choose an agent and call him/her *agent 1*.

(ii) Choose two relative prices (i.e. number of good 2 for one unit of good 1)

P_s: the price at which agent 1 can offer to sell the first good

P_b: the price at which agent 2 can offer to sell the first good

according to the limitation $P_s \leq P_b$.

The limitation on the prices implies that for any possible allocations $x, y \in A_f$, exactly one of the following holds: $x \in ey$ or $y \in ex$ or $eR_i xy$ for some individual $i \in N$. When the preferences are realized agent one declares whether he/she wants to buy or sell the first good and up to which amount. At the same time agent two declares the amount he/she wants to buy the first good and the amount he/she wants to sell it. The reason for agent two's declaring two amounts is his/her being unaware of the direction of the trade (which will be determined solely by agent one) at that moment. This of course causes an asymmetry between the agents. After the amounts are declared, the trade occurs in the direction declared by the first agent and the volume of trade is determined as the minimum of the declarations of the two agents.

If one wants to write the mechanism it is as follows:

Definition: (A Fixed-price mechanism for the two agents, two goods case)

Given an endowment point $e \in R_+^{nl}$ and two prices P_s and P_b (P_s , $P_b \in \mathbf{R}_+$) s.t. $P_s \leq P_b$, a fixed-price trading mechanism, $m(e, P_s, P_b)$ is defined as follows:

$m(e, P_s, P_b) = (X, \pi)$ where $X = X_1 \times X_2$ and $\pi: X \rightarrow A$ is an outcome function.

$X_1 = \{n \in R \ / \ -e_1^1 \leq n \leq e_2^1/P_b\}$ and $X_2 = \{(w,q) \in R_+^2 \ / \ w \leq e_2^2/P_s \text{ and } q \leq e_1^2\}$

$$\pi(n,(w,q)) = (x^1, x^2) \text{ where } x^1 = [e_1^1 + \min\{n,q\},\ e_1^2 - P_b \min\{n,q\}] \quad \text{if } n \geq 0$$

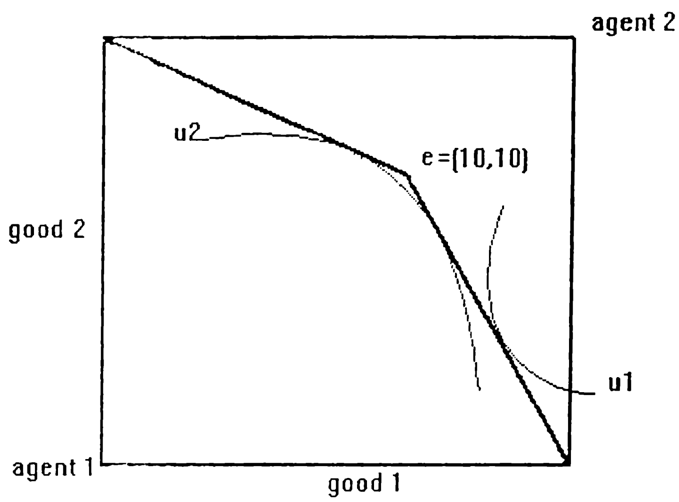
$$= [e_1^1 - \min\{-n,w\},\ e_2^1 + P_s \min\{-n,w\}] \quad \text{if } n < 0$$

$$\text{and} \quad x^2 = [e_1^2 - \min\{n,q\},\ e_2^2 + P_b \min\{n,q\}] \quad \text{if } n \geq 0$$

$=[e_1^2-\min\{-n,w\},\ e_2^2-P_s\min\{-n,w\}]$ if $n<0$

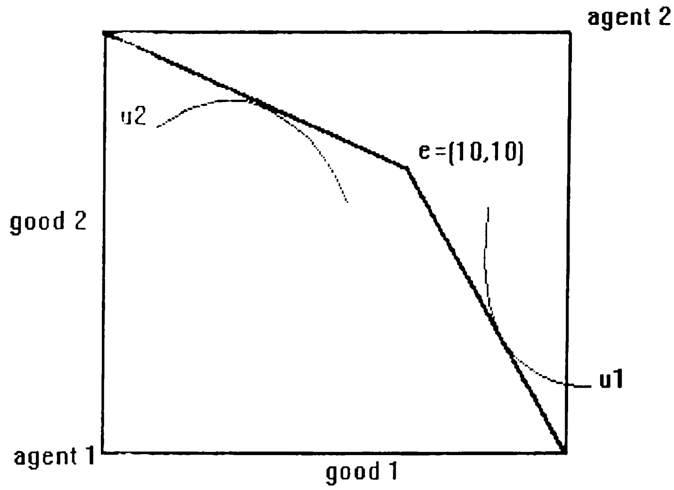
Now for any $u\in U^n$, m[u] is a normal form game with the strategy spaces as given above. Given the quasi-concavity of the preferences and the condition that $P_s \leq P_b$ agent one will be willing to either to buy or to sell the first good but not both. Moreover agent two can choose one or two directions and amounts of trade or can refuse to trade by declaring $(0,0)\in R_-^2$. Moreover the limitations on the preferences imply the dominant strategy of individuals to be that element(s) of A_f that maximizes their utilities (this will be a single point for the first individual though there is no such limitation that is imposed by the mechanism for the second individual). To gain a deeper understanding of the mechanism we will analyze an example.

Example 1: (A fixed-price trading rule for a two agents and two goods economy)
Let the endowments of the agents be $e^1=(10,10)$ and $e^2=(5,5)$. Moreover let the prices be given as $P_s=1$ and $P_b=2$. That is, agent one pays two units of the second good per unit of the first good if he/she wants to buy the first good, and gets one unit of the second good per unit of the first good if he/she wants to sell the first good. Let the preferences of the agents be as in the figure below:



If the preferences are of this type, then the trade will occur in the direction of agent one's buying the first good and agent two's selling the first good. Here, because of the kinkedness of the budget line while agent one has a unique maximizing point, agent two has two maximizing points, first of which can be reached by buying the first good in exchange for the second good and second by selling the first good in exchange for the second good. The direction of trade is determined through agent one's declaration and the amount of trade is determined through satisfying the short side of the market which is nothing but agent one's trade proposal.

There may be an alternative case where agent two's preferences are only through one direction of trade. In such a case if the direction of trade declared by the first agent matches with that of the second agent, trade is achieved. Otherwise, no trade occurs.



In the figure above, agent two's utility is only maximized through buying the first good and thus refuses any trade in the other direction by simply signaling a zero amount for selling the first good. As a result no trade occurs. As also can be seen in the above example, the individuals have no incentive to manipulate the mechanism given the restrictions imposed on their utilities. They can't gain from declaring an amount of trade more than they really want to, since if the other declares a lesser amount the outcome doesn't change, else if the other agent declares a greater amount than the manipulating agent is worse-off since he/she can't obtain his/her maximizing outcome and obtains his/her manipulated outcome. The case is similar for a manipulation through the declaration of a lesser amount of trade. the nonmanipulability of the mechanism is because of the utilities of the agents being chosen from a set of continuous, strictly-quasiconcave and increasing functions. Essentially, in fixed- price trading agents indicate how much they are willing to buy or sell according to two fixed prices, and then the short side of the market is rationed. Thus declaring the trade volume and direction that maximizes his/her utility is a dominant strategy of an agent independent of whose declaration is accepted for trade in the mechanism. The following theorem is a special case for the two agents two goods case and claims that the only allocation rules which are strategy proof are the ones that are implementable by a fixed-price trading mechanism in DSE (the reverse implication is also claimed to hold). This is nothing but a characterization of all strategy proof allocation rules in a pure exchange economy with two agents and two goods.

Theorem 1:

A social choice function defined on a two person, two goods exchange economy is strategy proof and individually rational w.r.t. an endowment point iff it's implementable by a fixed-price trading mechanism in DSE.

The proof of this theorem is a simple extension of the general case that will be mentioned later. One can extend this theorem to a two agents and more than two goods case. The only thing that changes is the number of prices given in the beginning and the name changing from a fixed-price mechanism to a fixed-proportions mechanism. The definition of the fixed-proportions mechanism is quite similar to the prior case. Agent one declares a direction and volume of trade and agent two declares his/her demand for each direction. The mechanism starts with k≥l proportions for trade (which can be thought of as prices) and the agents are limited to those proportions (i.e. they can't obtain a convex combination of them). The only limitation on these proportions is that for any $x,y \in A_f$, $x \in ey$ or $y \in ex$ or $e^i R_i xy$ for some $i \in N$. Thus, essentially the trading procedure is a simple extension of the fixed-price trading mechanism. If wants to write it formally:

$X_1=\{n \in R_-^k$ / for each $i \in \{1,\ldots k\}$, $n_i>0$ implies $n_j=0$ for $j \in \{1,\ldots,l\} \setminus \{i\}$ and $n_i \leq \min\{e_j^i/p_{i,j}$ / $j \in \{1,\ldots l\} \setminus \{i\}\}$

$X_2=\{ m \in R_-^k$ / for each $i \in \{1,\ldots k\}$, $m_i \leq \min\{e_j^i/p_{i,j} / j \in \{1,\ldots,l\} \setminus \{i\}\}$

Here $p_{i,j}$ denotes the j'th component of the i'th price vector $p_i=(p_{i,1},\ldots,p_{i,l})$ where $i \in \{1,\ldots k\}$ since there are k different proportions (price vectors). Given the strategy spaces above. agent one chooses a direction and an amount, and agent two declares the amounts of trade for each direction. Then the direction is determined through agent one's strategy and the amount is determined as the minimum of the two amounts. The conditions, as before, implies that the first agent has a unique maximal choice on the outcome space, A_f. Moreover, the second agent, depending on the number of his/her maximizing points on A_f, declares some positive or zero amounts. This is a zero vector in $\mathbf{R}_+^1$ where the zero coordinates imply that the second agent doesn't have any maximizing allocation for this direction of trade. For this case Theorem 1 can be generalized to Theorem 2 below.

Theorem. 2: A two person social choice function is strategy proof and individually rational w.r.t. an endowment point iff it is the result of fixed-proportion trading.

One problem associated with these theorems is that the fixed-proportion trading rules are asymmetric. That is, changing the roles of the individuals alters the outcome. In other words, the fixed-proportion trading rules are not anonymous. But, in fact, the set of fixed-proportions trading rules include a subset of anonymous fixed-proportions trading rules. This subset is characterized through the specification of the selling and the buying prices. this leads to the below corollary which is a characterization of the anonymous social choice rules which are strategy proof.

Corollary 1: A two person social choice function is strategy proof and anonymous iff it is the result of fixed-proportion trading along a line segment centered at the equal split of the total endowment.

What the above corollary says is that if the budget curve is not kinked and passes through the equal split of the total endowment point, then there doesn't arise any need for the asymmetry. The main reason for this is the cancellation of the chance that agent two can have more than one maximizing allocation. In this case $P_s=P_b$ and both agents declare their maximizing allocations as trade proposals. This brings a complete symmetry to the system. That is, the choice of the first agent doesn't alter the outcome. Moreover, the strategy spaces can be identical in such a case since in case of a linear budget constraint (and given the restrictions on the utilities) each agent will have a single maximizing allocation point on A_f.

Now, having gained a deeper understanding of the situation, we can generalize the above result to a case of n agents and l goods. To simplify the analysis we will impose a certain condition on f which is nothing but the non-bossiness property defined by Satterthwaite and Sonnenschein (1981) [18].

Definition: (Non-bossiness)

A social choice function f is *non-bossy* if for any $i \in N$, $u \in U^n$ and $u'^i \in U$

$$f^i(u)=f^i(u^{-i},u'^i) \text{ implies } f(u)=f(u^{-i},u'^i)$$

This condition simply claims that if an agent changes his/her preference and his/her outcome is not changed, then the outcome of the other agents should not also change. It rules out a series of social choice functions which are not dictatorial but are degenerate in other ways. An example ruled out by this condition is an allocation rule where the entire endowment is given to the second or to the third agent depending on agent one's preferences [18].

The interesting thing for the general case is that it is possible to incorporate a number of proportions (prices). Consider the following example for an n agents two goods case: There are trade proposals for the integers declining from n-1 to n/2 each of which is a set of proportions. For each trade proposal, P(k), the agents declare their demands for each proportion. If there is a proportion in P(n-1) s.t. at that proportion exactly n-1 agents are willing to buy or exactly n-1 agents are willing to sell , then the trade occurs at that proportion. Else, the n-2 trade proposal is checked to see whether there are exactly n-2 buyers or n-2 sellers at that proposal. This goes on until the last proposal. If that can't also be matched, no trade occurs. Now, we will give a definition of the fixed-proportions trading rule for n agents and l goods in three parts:

1. Trade can occur in one proportion which is selected from an a priori fixed set of proportions which satisfy some additional restrictions. The proportions are grouped

into subsets called trade proposals. So there are trade proposals P(n-1),...,P(t) where t is the smallest integer greater than or equal to n/2.

2. The proportion according to which the trade occurs is selected by examining the demands of the agents. Each trade proposal is assigned a number as above and a proportion in that proposal is *matched* if exactly that number of agents demand trades in the same direction.
3. Agents are given trades in the direction of their demanded trade in the selected proportion. No one's trade is larger than his/her demanded trade; but one or more parts of the market may be rationed. The rationing is done uniformly so that the rationed agents are rationed equally.

Definition: (Trade proposal)

A trade proposal $P \subseteq R^l$ is a set of feasible trade proportions which satisfies the following:

(i) if $a \in P$, then $a \not\geq 0$ and $a \not\leq 0$ and

(ii) if $b \in P$ and $b \neq a$, then there exists $\alpha \in (0,1)$ s.t. $\alpha a+(1-\alpha)b \leq 0$

The collection of trade proposals $\{P(k) / n>k \geq n/2\}$ should be chosen to satisfy the following property for the strategy proof social choice function, f, to be implemented by this mechanism.

Definition: (Nestedness of a set of trade proposals)

A collection of trade proposals $\{P(k) / n>k \geq n/2\}$ are *nested* if for each k'<k and $a \in P(k')$ and $b \in P(k)$, either there exists $\alpha > 0$ s.t. $\alpha b \leq a$ or there exists an $\alpha \in (0,1)$ s.t. $\alpha a+(1-\alpha)b \leq 0$.

For any $k \in \{n-1,\ldots,t\}$ and $a \in P(k)$ let $o(u^i,a,e^i)$ be the $o \in \mathbf{R}$ which maximizes $u^i(e^i+oa)$ subject to $e^i+oa \in R_+^l$. Now we will define what does a utility profile matching a proportion $a \in P(k)$:

Definition: (Matching)

Given a nested set of trade proposals $\{P(k) \,/\, n>k \geq n/2\}$, we say that a utility profile $u \in U^n$ *matches* an $a \in P(k)$ if there exists a $C \subseteq N$ with $\#C=k$ s.t. $o(u^i,a,e^i) > 0$ for each $i \in C$ and $o(u^i,a,e^i) \leq 0$ for each $i \notin C$.

Given the structure above, the below lemma claims that only one proportion matches for a given utility profile of the agents.

Lemma:

If $b \in P(k)$ is matched and $a \neq b$ is in $P(k')$ for some $k' \neq k$, then a is not matched.

Since the proportion along which the trade occurs is selected by the signs of the demands rather than their sizes, it will often be necessary to ration. The rationing will be done uniformly to preserve the dominant strategy implementability of f by the given mechanism.

Definition: (Uniform rationing)

Consider a $u \in U^n$ which matches $a \in P(k)$ and such that for each $i \in N$ there exists and $r^i \in [0,1]$ s.t. $f^i(u)=e^i+r^i\, o(u^i,a,e^i)\, a$. f satisfies *uniform rationing* at u if

(a) $\text{sign}[o(u^i,a,e^i)]=\text{sign}[o(u^j,a,e^j)]$, and $|o(u^j,a,e^j)| \geq r^i |o(u^i,a,e^i)|$ and $r^i<1$, imply that $f^i(u)=f^j(u)$.

(b) $\text{sign}[o(u'^i,a,e^i)]=\text{sign}[o(u^i,a,e^i)]$, and either $|o(u'^i,a,e^i)| \geq r^i |o(u^i,a,e^i)|$ and $r^i<1$, or $|o(u'^i,a,e^i)| = |o(u^i,a,e^i)|$ imply that $f(u^{-i},u'^i)=f(u)$.

If trade occurs, then according to the uniform rationing all those who are rationed on a given side are rationed to the same trade. If some individual is rationed when announcing a given utility, then the outcome is the same when that individual announces any utility which requests a trade as large or larger along that same proportion. Having formed the necessary background, we can now give the main characterization theorem of Barbera and Jackson.

Theorem:

A social choice function f is strategy proof, anonymous, and non-bossy iff it is the result of fixed-proportion trading away from the equal split point.

As was mentioned above, this is a full characterization of the strategy proof allocation rules that satisfy some additional conditions in an economic environment. The fixed-proportions mechanism defined above implements all these allocation rules and no other allocation rules. Thus, when a fixed-proportions trading rule is used as the exchange rule during the decentralization of the decision power, the individuals have no incentives to manipulate the mechanism as it was the case for the Walrasian equilibria in case of incomplete information and a finite number of individuals. It is worthwhile to mention that though the outcome of the mechanism can turn out to be a Walrasian equilibrium of the economy accidentally, there is no guarantee for the designer to find and use the optimal prices in the mechanism since it is being presumed that the information is incomplete. The critical point in the achievement of strategy proofness in this mechanism is that the designer puts the prices, or so to say the proportions, totally independently of the utilities of the agents. It would be interesting to analyze a case where the designer would bargain with an agent to set some specific prices (that makes that individual strictly better-off)in turn of some bribe paid to him/her.

6 ON THE CARDINALITY OF SINGLE PEAKED DOMAINS

The task of limiting the preference domain so as to overcome the impossibility in the obtainment of strategy proof social choice rules which are nondictatorial is one of the most fruitful paths followed in the literature. However decomposability, being an abstract concept, can rarely be related to everyday applications of social choice theory. One of these rare examples is single-peakedness. Single-peaked domains, additional to being decomposable, can be applied to many cases in which they seem reasonable. In this chapter we will first define single-peakedness, relate it to the concepts mentioned before (as decomposability) and will finally prove a proposition about the limitations that are embedded in the single-peakedness definition.

A single-peaked preference relation is simply a one that has a unique global maximum and no local maxima. That means, given a linear ordering (permutation), σ, of the alternative set, A, a single-peaked utility function increases up to some point, reaches its maximum at this point and starts to decrease (w.r.t. the linear order mentioned). A standard example of a single peaked preference relation is about drinking beer. You feel better with every additional glass up to the n'th glass (where n depends on how much you like drinking beer) and then start to feel worse with every additional glass. Here the linear order on the alternative set is (1 glass, 2 glasses,..., n glasses,...,K glasses).

In the definition of a single-peaked preference, the ordering of the alternative set gains importance. For this purpose we will use a permutation function as follows.

Definition: (Permutation function)

Let $A=\{a_1,\ldots,a_n\}$ be the alternative set. A function $\sigma:A\rightarrow A$ is said to be a *permutation function* if it is a 1-1 and onto mapping . The value of σ is shown as $\sigma(A)=(a_\sigma^1,\ldots,a_\sigma^n)$. The identity permutation σ_I takes A to its initial ordering, i.e. $\sigma_I(A)=(a_1,\ldots,a_n)$.

Given a certain ordering of the alternative set, A, a permutation function simply changes this ordering or leaves it unchanged. If σ doesn't change the ordering of A, then it is called an identity permutation and is denoted with σ_I. Given an alternative set with cardinality n, there exists n! different permutations on A.

For the sake of simplicity we will use utility functions instead of preferences. This doesn't impose a limitation to our analysis since the alternative set is finite. (Note that this implies that any preference relation on A can be paired with a utility function and any utility function on A can be paired with a preference relation on A). A utility function on A is defined as follows.

Definition: (Utility function)

Given an alternative set, A, with cardinality n, a 1-1 onto function, $u:A\rightarrow\{1,...,n\}$ is said to be a *utility function* on A.

Given the above definitions, we now can define a single-peaked preference relation or so to say a single-peaked utility function.

Definition: (Single-peakedness w.r.t. a permutation)

Given a permutation, $\sigma(A)=(a_\sigma^1,...,a_\sigma^n)$. on A, a utility function, u, is said to be *single-peaked w.r.t.* σ if there exists an $a_\sigma^k \in A$ s.t. $u(a_\sigma^1),...,u(a_\sigma^k)$ is increasing and $u(a_\sigma^k),...,u(a_\sigma^n)$ is decreasing. Here a_σ^k denotes that element of A which is the k'th one w.r.t the permutation, σ. The following example illustrates the concept.

Example 1:

Let $A=\{a_1, a_2, a_3\}$. Now a utility function, u, s.t. $u(a_1)=1$, $u(a_2)=3$, $u(a_3)=2$ is single-peaked w.r.t. the identity permutation, σ_I. But a utility function u' s.t. $u'(a_1)=3$, $u'(a_2)=1$, $u'(a_3)=2$ is not single-peaked w.r.t. σ_I. Moreover, given a permutation σ s.t. $\sigma(A)=(a_2, a_1, a_3)$, u' is single-peaked w.r.t. σ since $u'(a_\sigma^1)=u'(a_2)=1$, $u'(a_\sigma^2)=u'(a_1)=3$, $u'(a_\sigma^3)=u'(a_3)=2$.

Single-peaked preferences were first introduced by D. J. Black in 1948 [2]. Then M. Dummet and R. Farquharson [6] showed that in case of a majority rule with Borda completion as the mechanism, if the true preference profiles of the group's members are single-peaked and the ballots which the members are permitted to cast are restricted so as to allow expression only of single-peaked orderings, then every member's dominant strategy is to cast that ballot which faithfully represents his/her true preferences. In other words, they stated that a majority rule with Borda completion is strategy proof whenever both sincere preferences and ballots are a priori restricted to be single-peaked. Later J. M. Blin and M. A. Satterthwaite [4] showed that single-peakedness of preferences alone without a restriction on admissible ballots is insufficient to guarantee strategy proofness.

The single-peaked domains are shown to be decomposable by Kalai and Muller in their 1977 paper [11]. This is simply as follows:

Proposition: (Kalai and Muller)

A domain of single-peaked preferences is decomposable

proof: Take any permutation, σ, of the alternative set, A, and define the set of single-peaked preferences w.r.t σ by $S_\sigma=\{P\in L(A)$: for every $x,y,z\in A$ s.t. $x\neq y\neq z$ if $x\sigma y\sigma z$ (which means that x precedes y and y precedes z in the permutation σ) then it is not the case that xPy and zPy $\}$.

Let $R_1=\{(x,y)\in T \,/\, x\sigma y\}$. Clearly $\emptyset=TR\not\subset R_1\not\subset T$. So, all that is left to show is that R_1 is closed under decisiveness implications.

DI1a. We suppose that $(x,y)\in R_1$ and for some P^1, $P^2\in S_\sigma$ xP^1yP^1z and yP^2zP^2x. These relations imply that in σ, x can not be between y and z, and z can not be between x and y. Thus y must be the middle one and since $x\sigma y$ we must have $x\sigma y\sigma z$. Thus $(x,z)\in R_1$.

DI1b. $(z,x)\in R_1$, xP^1yP^1z and yP^2zP^2x Again y must be the middle one so we must have $z\sigma y\sigma x$. Thus, $(y,x)\in R_1$.

DI2a. $(x,y)\in R_1$, $(y,z)\in R_1$ and for some $P\in S_\sigma$, $xPyPz$. This shows that $x\sigma y\sigma z$. Thus $(x,z)\in R_1$.

DI2b. $(z,x) \in R_1$ and for some $P \in S_\sigma$, $xPyPz$. This shows that $x\sigma y$ or equivalently $(z,y) \in R_1$.

QED.

The above result states that a single-peaked domain, being decomposable, admits the construction of a strategy proof, Pareto optimal and nondictatorial social choice function, which is known to be implemented by the majority rule with Borda completion.

Definition: (A majority rule with Borda completion)

Given a ballot profile, $P \in L(A)^n$, a majority rule with Borda completion, $\pi(P)$, is defined as follows: $\pi(P)=a$ if there is a Condorcet winner $a \in A$.(i.e. for all $(a,b) \subseteq A$ where $b \neq a$, a beats b in majority voting). If there is not a Condorcet winner, each $x \in A$ is assigned a total point as the sum of each individuals given points, which is for each individual $i \in N$, $\#(A)-k_i-1$, where k_i is the rank of x in P_i. The element with the highest points wins. If there occurs a tie, then the ballot of the first individual is used to break the tie.

Up to now, we saw that single-peaked domains are sufficient for majority rules with Borda completion to satisfy strategy proofness, Pareto optimality and nondictatoriality. But how much limitation does the single-peakedness assumption impose to the system? One essential point regarding this question is that the individuals have to have single-peaked preferences w.r.t. the same permutation. If two different individuals have single-peaked preferences w.r.t different permutations then that profile can't be called a single-peaked profile. Given the structure above we can easily claim that for a certain permutation, σ, there exists exactly 2^{n-1} distinct preferences (utility functions) which are single peaked w.r.t σ.

Proposition:

Given a permutation σ on A, there are exactly 2^{n-1} distinct utility functions which are single-peaked w.r.t. σ.

proof: Now a permutation, σ, gives a linear orders of the alternatives as $(a_\sigma^1,\dots,a_\sigma^n)$. We will simply count the admissible utility profiles.

Take any $a_\sigma^k \in (a_\sigma^1,\dots,a_\sigma^n)$. Assume that u is maximized at a_σ^k. Then $u(a_\sigma^k)=n$. Now for the first k-1 parts on the left of a_σ^k there are C(n-1,k-1) different ways to assign utility values without damaging single-peakedness. Determination of the left side automatically determines the right side. So, for each $k\in\{1,\dots,n\}$ there are C(n-1,k-1) different utility functions to place which are single-peaked w.r.t the given σ.

Since this holds for all $k\in\{1,\dots,n\}$, there are $C(n-1,0)+C(n-1,1)+\dots+C(n-1,n-1)=2^{n-1}$ distinct utility functions which are single-peaked w.r.t σ.

QED.

Using the same method, one can also prove that there exists 2^{n-1} distinct permutations w.r.t which a given utility function is single-peaked. Given this limitation imposed by a permutation, σ, on the number of distinct utility functions we can ask the question of what is the probability of the existence of a society in which all the individuals have single-peaked preferences w.r.t the same permutation. For this analysis let us create the following scenario. There is a group of k individuals who have to make a social choice among n alternatives. Each individual randomly selects a utility function for himself/herself from a bowl of utility functions. There are n! distinct utility functions. Note that any utility function is single-peaked w.r.t 2^{n-1} distinct permutations. Now there are $(n!)^k$ different profiles that can be obtained via this process. Moreover one can count the admissible profiles (i.e. single-peaked w.r.t. the same σ), #Ad, as follows:

$$\#Ad=\#Ad1+\dots+\#Ad(n!)-\sum_{i\neq j}\#(S_{\sigma_i}\cap S_{\sigma_j})+\sum_{i\neq j\neq k}\#(S_{\sigma_i}\cap S_{\sigma_j}\cap S_{\sigma_k})-\dots\mp\#(S_{\sigma_1}\cap\dots\cap S_{\sigma_{n!}})$$

Now , #Adk denotes the number of admissible profiles w.r.t σ_k for each $k\in\{1,\dots,(n!)\}$. But since a utility profile can be single-peaked w.r.t. more than one permutation there occurs a double counting problem. Thus the number of twofold intersections are subtracted, the threefold intersections are added and so on. Here for any $i\in\{1,\dots,(n!)\}$, S_{σ_i} denotes the set of utility profiles which are single-peaked w.r.t. σ_i . Moreover

$\#Ad' = \#Ad1 + \ldots + \#Ad(n!) \geq \#Ad$ since there is a double counting problem. What we claim is the following proposition.

Proposition:

The probability of obtaining a profile in which each preference relation is single-peaked w.r.t. the same permutation goes to zero as the cardinality of the alternative set, A, goes to infinity.

proof: Now, this probability can be defined as P(n)=$\frac{\#Ad}{(n!)^k} \leq \frac{\#Ad'}{(n!)^k}$

Moreover, as n goes to infinity P(n) is bounded from both directions. That is

$$0 \leq \lim_{n\to\infty} \frac{\#Ad}{(n!)^k} \leq \lim_{n\to\infty} \frac{\#Ad'}{(n!)^k}$$

Now we will use the Sandwich theorem to prove the proposition:

$$\lim_{n\to\infty} \frac{\#Ad'}{(n!)^k} = \lim_{n\to\infty} \frac{n!(2^{n-1})^k}{(n!)^k} = \lim_{n\to\infty} \frac{(2^{n-1})^k}{(n!)^{k-1}} = \lim_{n\to\infty} \frac{(2^{\frac{n-1}{k-1}} 2^{n-1})^{k-1}}{(n!)^{k-1}}$$

$$\leq \lim_{n\to\infty} \left(\frac{2^{2(n-1)}}{n!}\right)^{k-1} = \lim_{n\to\infty} \left(\frac{4^{n-1}}{n!}\right)^{k-1} = 0$$

$$0 \leq \lim_{n\to\infty} \frac{\#Ad}{(n!)^k} \leq 0 \Rightarrow \lim_{n\to\infty} \frac{\#Ad}{(n!)^k} = 0$$

Thus the probability of obtaining a single-peaked profile (w.r.t. the same permutation) goes to zero as the cardinality of A goes to infinity.

QED.

This proposition means that if the individuals have preferences on the amount of beer they drink instead of the number of glasses, then it is impossible for them to have single-peaked preferences w.r.t. the same permutation of the alternative set.

7 CONCLUSIONS

In this study we aimed at presenting a survey of the literature on strategy proofness, combining it with the relevant concepts. While doing this we tried to combine a wide variety of different terminology in a single framework so as to create a unity in the presentation for ease of understanding the relationships among these concepts. Some relations were already constructed while we had started to study, we presented them with the relevant references. Moreover, we tried to close the gap in terms of the construction of the relationships among these concepts, and thus to present a full picture of the implementation business.

There are six main concepts that we analyzed and tried to embed into a unity in this study. These are strategy proofness, implementability, impossibility in implementation, use of these concepts in economics literature, ways to get rid of this impossibility problem and the limitations that are brought about while utilizing these ways.

During this work, it is seen that there is not a one-to-one relation between strategy proofness and dominant strategy implementibility. However, when the preference domain is restricted to be a subset of linear orders this can be achieved. Moreover this one-to-oneness can be generalized for the generalized social choice functions and it is seen that in this case strategy proofness turns and all kinds of dominant strategy implementibility turns out to be equivalent.

Another finding is that in case of limiting the preference domain to be a subset of the linear orders, all dominant strategy implementable social choice rules turn out to be singleton-valued. Because of this and the above findings, the limitation of the preference domain to be a subset of the linear orders becomes and important limitation. Most of the implications are shown not to hold when this limitation is not utilized.

There are also some additions to the literature such as pseudo implementation, total implementation and a full characterization of the relationship among these and strategy proofness. There is also a new impossibility theorem which is an extension of the

Gibbard-Satterthwaite impossibility theorem for generalized social choice rules. The characterization of the domains that permit the existence of dominant strategy implementable social choice rules that are nondictatorial, is still open. However, it is seen that the decomposibility characterization is not sufficient to deal with generalized social choice rule.

A well known example of decomposable domains is the single-peakedness characterization. As a result of an analysis of the single-peakedness condition it is found out that single-peaked domains are very rare and the probability of obtaining a single-peaked profile goes to zero as the number of alternatives goes to infinity.

As a result of the negative results that are obtained about the existence of a nondictatorial and dominant strategy implementable social choice rule, Nash implementibility is analyzed briefly. The main theorems about this concept are presented to give an idea of what has been done in this area.

There are two paths that can be followed to improve this study on implementation. The first one is that there is an open question about the generalization of the impossibility result in Nash implementibility for social choice functions. The result is for two individuals; however, we have a feeling that this can be generalized for the case of more than two individuals. The second path is the improvement of the literature on dominant strategy implementation by trying to construct a full characterization of the dominant strategy implementable social choice rules that are nondictatorial.

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