BASES IN BANACH SPACES OF SMOOTH FUNCTIONS ON CANTOR-TYPE SETS

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August, 2013

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

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August, 2013

We construct Schauder bases in the spaces of continuous functions $C^p(K)$ and in the Whitney spaces $\mathcal{E}^p(K)$ where K is a Cantor-type set. Here different Cantor-type sets are considered. In the construction, local Taylor expansions of functions are used. Also we show that the Schauder basis which we constructed in the space $C^p(K)$, is conditional.

Keywords: Schauder bases, C^p-spaces, Whitney spaces, Cantor sets.

ÖZET

CANTOR TİPİ KÜMELERDE DÜZGÜN FONKSİYONLARIN BANACH UZAYINDA BAZ BULUNMASI

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Biz, K bir Cantor-tipi küme olmak üzere, sürekli fonksiyonların uzayı $C^p(K)$'de ve Whitney uzayları $\mathcal{E}^p(K)$'de Schauder bazı oluşturduk. Burada farklı Cantor-tipi kümeler göz önünde bulunduruldu. Baz oluşturulurken fonksiyonların lokal Taylor açılımları kullanıldı. Ayrıca biz $C^p(K)$ uzayında oluşturduğumuz Schauder bazının şartlı baz olduğunu gösterdik.

Anahtar sözcükler: Schauder bazları, C^p-uzayları, Whitney uzayları, Cantor kümeleri.

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Introduction

In the study of Banach spaces and topological vector spaces, the concept of basis is a very useful and very important tool. An importance of the concept of basis lies in the fact that it provides a natural method of approximation of vectors and operators in space. On the other hand, elements of bases of concrete function spaces rather often play a special role in different problems of analysis. For example, the Chebyshev polynomials, the Hermite functions, the Faber polynomials and the Franklin sequence have attracted attention of mathematicians for many years. These systems of functions were well-known in analysis when it was proven that they form topological bases in the spaces $C^\infty[-1,1]$, S-rapidly decreasing C^∞-functions on the line, analytic functions, and H^1- Hardy space, correspodingly.

The notion of a topological basis was introduced by Schauder [1]. But interest in the theory of basis in topological vector spaces has grown essentially after the publication of Banach's book on the theory of linear operators. In his book Banach asked the question of whether every separable Banach space possesses a basis or not. This problem was known as the Banach basis problem. Until the 1970's much of the literature on the theory of basis was devoted to this problem. In 1972, Enflo [2] constructed the first example of a separable Banach space which does not have the approximation property and hence does not possess a basis. Afterwards other such examples were presented. In particular, it was shown that for every p, $1 \leq p \leq \infty$, $p \neq 2$ the space l_p contains a subspace without the approximation property.

Furthermore, another famous basis problem by Grothendieck [3] (about existence of basis in each nuclear Fréchet space) was answered in the negative by Zobin and Mityagin in [4].

In spite of the fact that both fundamental basis problems were solved in the middle of the 1970's, many people continue to work in this field. The following questions attract their attention: construction of bases in concrete function spaces, the problem of quasi-equivalence of bases, existence of a nuclear Fréchet function space without basis (all previous examples were given as artificial constructions or non-metrizable function spaces).

In this work we study bases in the spaces of differentiable functions and construct a basis in the space of smooth functions defined on Cantor-type sets.

First we give the definitions of the topological basis and Schauder basis. Then we start the history of basis in the spaces of differentiable functions.

Definition 1.0.1. A sequence of elements $(e_n)_{n=1}^{\infty}$ in an infinite-dimensional normed space X is said to be a *topological basis* of X if for each $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} a_n e_n.$$

This means that the sequence $\left(\sum_{n=1}^{N} a_n e_n\right)_{N=1}^{\infty}$ converges to x in the norm topology of X.

If $(e_n)_{n=1}^{\infty}$ is a basis of a normed space X, then the maps $x \mapsto a_n$ for $n \in \mathbb{N}$ are linear functionals on X. Let us denote these functionals as e_n^*, then $e_n^*(x) = a_n$. If the linear functionals $(e_n^*)_{n=1}^{\infty}$ are continuous, then we call $(e_n)_{n=1}^{\infty}$ a Schauder basis. In particular, by Banach Open Mapping Theorem, any topological basis in a Banach space is a Schauder basis. Thus we have the following definition.

Definition 1.0.2. Let $(e_n)_{n=1}^{\infty}$ be a sequence in a Banach space X. Suppose there is a sequence $(e_n^*)_{n=1}^{\infty}$ in X^* (topological dual of X) such that

(i) $e_k^*(e_j) = 1$ if $k = j$, and $e_k^*(e_j) = 0$ otherwise, for any k and j in $\mathbb{N}$.

(ii) $x = \sum_{n=1}^{\infty} e_n^*(x) e_n$ for each $x \in X$.

Then $(e_n)_{n=1}^{\infty}$ is called a *Schauder basis* for X and the functionals $(e_n^*)_{n=1}^{\infty}$ are called the *biorthogonal functionals* associated with $(e_n)_{n=1}^{\infty}$.

J. Schauder, first introduced the concept of basis in 1927, and the name Schauder was given after his works. However, such bases were discussed earlier by Faber. In 1910, Faber [5] showed that there exists a basis in the space $C[0,1]$ consisting of the primitives of the Haar functions. Faber used the diadic system of points in his construction. In 1927, Schauder [1] rediscovered the more general form of this result. In our construction we follow the main idea by Schauder to interpolate given functions step by step at a dense sequence of points. As in [6], we interpolate functions locally and since we consider smooth functions, we will use Taylor's interpolation. Now, we examine the Schauder system in details:
Recall that $C[0,1]$ is a Banach space with the norm $\|f\| = \sup_{0 \le t \le 1} |f(t)|$. Let $(s_n)_{n=0}^{\infty}$ be the sequence in the space $C[0,1]$ defined in the following way. Let $s_0(t) = 1$ and $s_1(t) = t$. Let m be the least positive integer such that $2^{m-1} < n \le 2^m$. And when $n \ge 2$, we define s_n as

$$s_n(t) = \begin{cases} 2^m \left(t - (\frac{2n-2}{2^m} - 1)\right) & \text{if } \frac{2n-2}{2^m} - 1 \le t < \frac{2n-1}{2^m} - 1; \\ 1 - 2^m \left(t - (\frac{2n-1}{2^m} - 1)\right) & \text{if } \frac{2n-1}{2^m} - 1 \le t < \frac{2n}{2^m} - 1; \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Clearly, s_n is a continuous piecewise linear function. See Figure 1.1 for the first 9 terms of $(s_n)_{n=1}^{\infty}$.

Suppose that $f \in C[0,1]$. Define a new sequence $(p_n)_{n=0}^{\infty}$ in $C[0,1]$ in the following way. Let $(a_n)_{n=0}^{\infty}$ be the sequnce of points where $a_0 = 0$, $a_1 = 1$ and for $n \ge 2$, $a_n = \frac{2n-1}{2^m} - 1$ where m be the least positive integer such that $2^{m-1} < n \le$

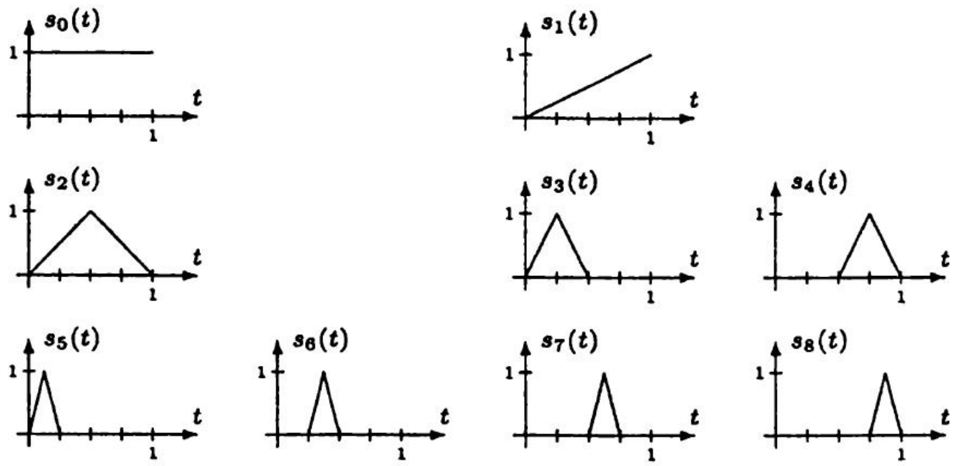


Figure 1.1: The first 9 terms of the classical Schauder basis for $C[0,1]$.

2^m. Then

$$\begin{aligned}
p_0 &= f(0)s_0, \\
p_1 &= p_0 + (f(1) - p_0(1))s_1, \\
p_2 &= p_1 + (f(1/2) - p_1(1/2))s_2, \\
p_3 &= p_2 + (f(1/4) - p_2(1/4))s_3, \\
p_4 &= p_3 + (f(3/4) - p_3(3/4))s_4, \\
p_5 &= p_4 + (f(1/8) - p_4(1/8))s_5, \\
p_6 &= p_5 + (f(3/8) - p_5(3/8))s_6, \\
&\vdots \\
p_n &= p_{n-1} + \left(f\left(a_n\right) - p_{n-1}\left(a_n\right)\right) s_n.
\end{aligned}$$

For each nonnegative integer i, let α_i be the coefficient of s_i in the formula for p_n. Then $p_n = \sum_{i=0}^{n} \alpha_i s_i$ for each n. Here

$$\alpha_0 = f(0), \quad \alpha_1 = f(1) - p_0(1) = f(1) - f(0), \ \cdots \ , \alpha_n = f(a_n) - \sum_{i=0}^{n-1} \alpha_i s_i(a_n).$$

Then since s_i is piecewise linear for all i, p_n is also piecewise linear. Also p_n is interpolating, that is, $p_n(a_n) = 1$ and $p_n(a_j) = 0$ for $j = 0, \ldots, n-1$. From the definition of s_n we have $s_n(a_n) = 1$ and $s_n(a_j) = 0$ for $j = 0, \dots, n-1$. Then

$$\begin{aligned} p_n(a_j) &= \sum_{i=0}^{n} \alpha_i s_i(a_j) \\ &= \sum_{i=0}^{j-1} \alpha_i s_i(a_j) + \alpha_j s_j(a_j) + \sum_{i=j+1}^{n} \alpha_i s_i(a_j) \\ &= \sum_{i=0}^{j-1} \alpha_i s_i(a_j) + \alpha_j \\ &= \sum_{i=0}^{j-1} \alpha_i s_i(a_j) + f(a_j) - \sum_{i=0}^{j-1} \alpha_i s_i(a_j) \\ &= f(a_j). \end{aligned}$$

Thus p_n is the polygonal function interpolating the values of f at the points $a_0, a_1, \dots, a_n$. Then if a_{i_1}, a_{i_2} are any consecutive points of $\{a_i\}_{i=1}^{n-1}$,

$$p_n(\lambda a_{i_1} + (1-\lambda) a_{i_2})) = \lambda f(a_{i_1}) + (1-\lambda) f(a_{i_2}) \quad (0 \le \lambda \le 1). \tag{1.2}$$

Let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous on $[0,1]$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(t_1) - f(t_2)| < \varepsilon$ whenever $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$. Since $\{a_j\}$ is dense in $[0,1]$, there exists a positive integer $N = N(\delta(\varepsilon))$ such that for $n > N$ we have $\max |a_{i_1} - a_{i_2}| < \delta$, where the maximum is taken over all couples of consecutive points of $\{a_i\}_{i=1}^{n-1}$. Now, let $t \in [0,1]$ be arbitrary. Then there exists λ with $0 \le \lambda \le 1$ such that $t = \lambda a_{i_1} + (1-\lambda) a_{i_2}$, where a_{i_1}, a_{i_2} are consecutive points of $\{a_i\}_{i=1}^{n-1}$ satisfying $t \in [a_{i_1}, a_{i_2}]$. Then, by (1.2),

$$\begin{aligned} |f(t) - p_n(t)| &= |f(t) - (\lambda f(a_{i_1}) + (1-\lambda) f(a_{i_2}))| \\ &= |\lambda(f(t) - f(a_{i_1})) + (1-\lambda)(f(t) - f(a_{i_2}))| \\ &\le \max_{t_1, t_2 \in [a_{i_1}, a_{i_2}]} |f(t_1) - f(t_2)| < \varepsilon, \end{aligned} \tag{1.3}$$

where $n > N(\delta(\varepsilon))$. Since $N(\delta(\varepsilon))$ is independent of $t \in [0,1]$ for all $n > N(\delta(\varepsilon))$,

$$\|f - p_n\| < \varepsilon.$$

Therefore, $f = \sum_{n=0}^{\infty} \alpha_n s_n$.

Now we prove the uniqueness. Let $(\beta_n)_{n=0}^{\infty}$ be any sequence of scalars such that $f = \sum_{n=0}^{\infty} \beta_n s_n$. But then $\sum_{n=0}^{\infty} (\alpha_n - \beta_n) s_n = 0$. Also $\sum_{n=0}^{\infty} (\alpha_n - \beta_n) s_n(a_n) = 0$

for all n which implies that $\alpha_n = \beta_n$ for all n. Therefore there is a unique sequence of scalars $(\alpha_n)_{n=0}^{\infty}$ such that $f = \sum_{n=0}^{\infty} \alpha_n s_n$. So the sequence $(s_n)_{n=0}^{\infty}$ is a basis for $C[0,1]$.

Using any basis $(f_n)_1^{\infty}$ of $C[0,1]$ it is not difficult to find a basis in the space $C^p[0,1]$. Recall that the topology in the space $C^p[0,1]$ is given by the norm

$$\|g\|_p = \max_{0\le k\le p} \sup_{0\le x\le 1} |g^{(k)}(x)|.$$

Indeed, let us consider the operator

$$T : C[0,1] \longrightarrow C^p_{\mathcal{F}}[0,1] : f \mapsto \int_0^x \int_0^{x_1} \cdots \int_0^{x_{p-1}} f(x_p)\, dx_p \cdots dx_1$$

where $C^p_{\mathcal{F}}[0,1]$ denotes the subspace of functions that are flat at 0, that is such that $g^{(k)}(0) = 0$ for $0 \le k \le p-1$. Then, by means of the operator T we have an isomorphism $C^p[0,1] \simeq \mathbb{R}^p \oplus C[0,1]$. Let us show this for the case $p = 1$. For all $f \in C[0,1]$ we have $Tf \in C^1_{\mathcal{F}}[0,1]$ since both $Tf(x) = \int_0^x f(t)dt$ and $(Tf)'(x) = f(x)$ are continuous and $Tf(0) = 0$. Also for all $g \in C^1_{\mathcal{F}}[0,1]$, $g \in C[0,1]$, we have $T^{-1}g = g' \in C[0,1]$, so T is a linear bijection. Since

$$\|Tf\|_1 = \max\{\|Tf\|, \|(Tf)'\|\} = \max\left\{ \sup_{0\le x\le 1} \left| \int_0^x f(t)dt \right|, \sup_{0\le x\le 1} |f(x)| \right\} = \|f\|,$$

the operator T is an isometry. Thus the spaces $C[0,1]$ and $C^1_{\mathcal{F}}[0,1]$ are isometric. At the same time, we have trivially $C^1[0,1] \simeq \mathbb{R} \oplus C^1_{\mathcal{F}}[0,1]$, where the corresponding continuous projections are given as

$$P_1 : C^1[0,1] \longrightarrow \mathbb{R} : \; g(x) \mapsto g(0),$$
$$P_2 : C^1[0,1] \longrightarrow C^1_{\mathcal{F}}[0,1] : \; g(x) \mapsto g(x) - g(0).$$

Therefore, $C^1[0,1] \simeq \mathbb{R} \oplus C[0,1]$. By the same method we can show that $C^p[0,1] \simeq \mathbb{R}^p \oplus C[0,1]$. The elements of basis in $C^p[0,1]$ are

$$1,\, x,\, \frac{x^2}{2}\, \ldots,\, \frac{x^p}{p!},\, \int_0^{x_1} \cdots \int_0^{x_{p-1}} f_1(x_p)\, dx_p \cdots dx_1,\, \int_0^{x_1} \cdots \int_0^{x_{p-1}} f_2(x_p)\, dx_p \cdots dx_1, \ldots$$

On the other hand the basis problem for the space $C^p[0,1]^2$ is much more difficult. In 1932 Banach [7] raised this problem in his book: Let $B = C^1[0,1]^2$

be the space of all real-valued continuous functions on the unit square $0 \leq t \leq 1$, $0 \leq s \leq 1$, admitting continuous partial derivatives of order 1, endowed with the norm

$$||x|| = \max_{0 \leq t \leq 1, 0 \leq s \leq 1} |x(t,s)| + \max_{0 \leq t \leq 1, 0 \leq s \leq 1} |x_t(t,s)| + \max_{0 \leq t \leq 1, 0 \leq s \leq 1} |x_s(t,s)|;$$

does B possess a basis? This problem was solved by Ciesielski [8] and Schonefeld [9] independently only 37 years later in 1969. Ciesielski and Schonefeld used the Franklin dyadic functions elements for the basis. Generalization of this system to the case $C^p[0,1]^2$, $p \geq 2$, was rather diffucult and complicated problem. In 1972, Ciesielski and Schonefeld improved this result independently. Ciesielski and Domsta [10] showed the existence of basis for $C^p[0,1]^q$ and Schonefeld [11] constructed a Schauder basis for the space $C^p(\mathbb{T}^q)$ where $\mathbb{T}^q$ is the product of q copies of the one-dimensional torus. This basis is also a basis for $C^1(\mathbb{T}^q), C^2(\mathbb{T}^q), \ldots, C^{p-1}(\mathbb{T}^q)$ and an interpolating basis for $C(\mathbb{T}^q)$. Schonefeld first proved that $C^p(\mathbb{T})$ has a basis.

Let the partition Δ_n be the set of points $\left\{0, \frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}\right\}$ and $(2p+1)$-periodic spline on Δ_n where $p = 1, 2, \ldots$ be an element of $C^{2p}(\mathbb{T})$ whose restriction to each interval $(i/N, (i+1)/N)$, $i = 0, 1, \ldots, N-1$ is a $(2p+1)$-degree polynomial. Next Schonefeld constructed the basis from the following functions: $f_1 \equiv 1$, f_{N+q} is the $(2p+1)$-periodic spline on the partition Δ_{2N} which is zero at every point of the partition Δ_{2N} except $(2q-1)/2N$ where $N = 1, 2, 4, 8, \ldots$ and $q = 1, 2, \ldots, N$ and $f_{N_q}(\frac{2q-1}{2N}) = 1$. Then he defined an operator S_n inductively by the following:

$$a_1 = f(r_1), \quad S_n f = \sum_{i=1}^{n} a_i f_i, \quad a_{n+1} = f(r_{n+1}) - S_n f(r_{n+1}) \quad n = 1, 2, \ldots$$

where $\{r_n; n = 1, 2, \ldots\} = \left\{0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \cdots, \frac{2^{m-1}-1}{2^{m-1}}, \frac{1}{2^m}, \frac{3}{2^m}, \frac{5}{2^m}, \cdots, \frac{2^m-1}{2^m}\right\}$. Therefore, $S_n f$ interpolates f on Δ_N, that is, $S_N f(i/N) = f(i/N)$. He then showed that $\{f_n\}$ is an interpolating basis for $C(T)$, that is, $||f - S_n f|| \leq \varepsilon$. Furthermore, he differentiated $S_n(f) - f$ and by using the properties of divided differences he proved that $(f_n)_1^\infty$ is the desired basis in $C^p(\mathbb{T})$.

At the end of his paper, Schonefeld remarked that the spaces $C^p(\mathbb{T}^q), C^p(I^q)$ (where $I = [0,1]$), $C^p(M)$ (where M is a q-dimensional compact C^p-manifold) and $C^p(D)$ (where D is a domain in $\mathbb{R}^q$ with the boundary such that there exists a linear extension operator $L : C^p(\partial D) \to C^p(D)$ are isomorphic. Thus, with these isomorphisms there also exists a Schauder basis in these spaces. Schonefeld stated this remark according to the theorem of Mitjagin established in [12, Thm 3] that if M_1 and M_2 are n-dimensional smooth manifolds with or without boundary, then the spaces $C^p(M_1)$ and $C^p(M_2)$ are isomorphic. This result essentially enlarges the class of compact sets K with a basis in the space $C^p(K)$, but it cannot be applied to compact sets with infinitely many components, in particular for nontrivial totally disconnected sets.

In 2004, Jonsson [13] used the method of triangulations to construct an interpolating basis for the space $C^p(F)$ where F is a compact subset of $\mathbb{R}$ admitting a sequence of regular triangulations. Jonsson showed in [13, Thm1] that F admits a sequence of regular triangulations if and only if F preserves the Local Markov Inequality. Moreover, a set preserves the Local Markov Inequality if and only if it is uniformly perfect [14, Sec. 2.2]. On the other hand, in [13, p.52] Jonsson defined the space $C^k(F)$ as: "A function f belongs to the space $C^k(F)$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|R_j(x, y)| < \varepsilon |x - y|^{k-j}$ for $0 \le j \le k$ and $|x - y| < \delta$." Here $R_j(x, y)$ denotes the Taylor remainder. This means that, actually Jonsson considered the space of Whitney functions, $\mathcal{E}^p(F)$. However Jonsson considered this space equipped with the norm of the space $C^p(F)$. In general, the space $\mathcal{E}^p(F)$ is not complete in the topology of $C^p(F)$. As a result, Jonsson constructed an interpolating basis in the space $\mathcal{E}^p(F)$ with the norm of $C^p(F)$ where F is a uniformly perfect set on $\mathbb{R}$. For the details see Section 2.4.3.

In this thesis, we construct a Schauder basis in the space $C^p(K)$ of p times differentiable functions and in the Whitney space $\mathcal{E}^p(K)$ on Cantor-type sets K. Now we shortly describe the content of the thesis.

In Chapter 2, we introduce the spaces of differentiable and Whitney functions on any compact set, and give some results concerning these spaces. Then we give some definitions and properties about Taylor polynomials, Cantor-type sets and interpolation methods. Next, we give more detailed information about Jonsson's paper [13] in this chapter, since in Chapter 3 another basis is constructed in the whole space $C^p(F)$ for the case of Cantor-type set F satisfying restrictions from Jonsson's paper.

Chapter 3 contains the result of the master thesis of the author. In this work we construct a Schauder basis in the Banach space of $C^p(K)$ by using the method of local Newton interpolations suggested in [15]. Elements of the basis are polynomials of any preassigned degree and biorthogonal functionals are special linear combinations of the divided differences of functions. Here we construct basis in the Banach space of $C^p(K)$ for uniformly perfect K.

We then construct a Schauder basis in the Banach space $C^p(K)$ of p times differentiable functions and in the Whitney space $\mathcal{E}^p(K)$ on a Cantor-type set K by using the local Taylor expansions of functions. In Chapter 4, we construct a basis in the space $C^p(K_2(\Lambda))$ and $\mathcal{E}^p(K_2(\Lambda))$ on a Cantor type set $K_2(\Lambda)$ which we define in Section 2.3. In Chapter 5, we use the same method and same system of local Taylor expansions of functions to construct a basis in the spaces $C^p(K_\infty(\Lambda))$ and $\mathcal{E}^p(K_\infty(\Lambda))$ where $K_\infty(\Lambda)$ is a generalised Cantor-type set defined in Section 2.3. However our system of local Taylor expansions of functions does not work in the Fréchet spaces $\mathcal{E}(K)$ of Whitney functions of infinite order. We give an explanation for this at the end of Chapter 4. For a basis in the space $\mathcal{E}(K)$, see [6]. It should be noted that in [16] Keşir and Kocatepe used another technique to prove the existence of a basis in the space $\mathcal{E}(K)$ for Cantor-type sets K with the extension property.

In chapter 6, we give the definition and the properties of unconditional basis. Then we show that the basis which we constructed in the space $C^p(K)$ in Chapter 4, is a conditional basis.

Chapter 2

Preliminaries

2.1 Spaces of Differentiable Functions and Whitney Spaces

Let K be a compact set of $\mathbb{R}$, $p \in \mathbb{N}$. We denote by $C^p(K)$ (respectively $C(K)$) the algebra of p times continuously differentiable functions in K, with the topology of uniform convergence of functions and all their partial derivatives on K. This is the topology defined by the norm

$$|f|_p = \sup\{|f^{(k)}(x)| : x \in K,\, k = 0, 1. \dots, p\}$$

For every nonisolated point $x \in K$ we define $f'(x)$ as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If the point x is an isolated point, then $f'(x)$ can be taken arbitrarly. Thus, $C^p(K)$ is a subspace of $\prod_{0 \leq k \leq p} C(K)$.

By Tietze-Uryson Extension Theorem there exists continuous extension of functions from $C^p(K)$.

Theorem 2.1.1 (Tietze-Uryson Extension Theorem)**.** *If X is a normal topological space and $f : K \longrightarrow \mathbb{R}$ is a continuous map from a closed subset of K of X*

into real numbers carrying the standard topology, then there exists a continuous map

$$\tilde{f} : X \longrightarrow \mathbb{R}$$

with $f(x) = \tilde{f}(x)$ for all x in K. The map $\tilde{f}$ is called a continuous extension of f.

The space $\mathcal{E}^p(K)$ of Whitney functions of order p consists of functions from $C^p(K)$ that are extendable to C^p-functions defined on $\mathbb{R}$. The natural topology of $\mathcal{E}^p(K)$ is given by the norm

$$\|f\|_p = |f|_p + \sup\{|(R_y^p f)^{(k)}(x)| \cdot |x-y|^{k-p};\ x, y \in K, x \neq y, k = 0, 1, ...p\},$$

where $T_y^p f(x) = \sum_{0 \leq k \leq p} f^{(k)}(y) \frac{(x-y)^k}{k!}$ is the formal Taylor polynomial and $R_y^p f(x) = f(x) - T_y^p f(x)$ is the Taylor remainder.

Due to Whitney [17], $f = (f^{(k)})_{0 \leq k \leq p} \in \mathcal{E}^p(K)$ if

$$(R_y^p f)^{(k)}(x) = o(|x-y|^{p-k}) \quad \text{for} \quad k \leq p \quad \text{and} \quad x, y \in K \quad \text{as} \quad |x-y| \to 0. \tag{2.1}$$

The Fréchet spaces $C^\infty(K)$ and $\mathcal{E}(K)$ ($\mathcal{E}^\infty(K)$) are obtained as the projective limits of the corresponding sequences of spaces.

Similarly one can define the spaces $C^p(K)$ and $\mathcal{E}^p(K)$ for $K \subset \mathbb{R}^d$.

2.1.1 The Spaces $C^p(K)$ and $\mathcal{E}^p(K)$

Let K be a compact subset of $\mathbb{R}^d$. For the space of continuous functions $C(K) = \mathcal{E}^0(K)$. But we can not say this for $p \geq 1$ since in general the spaces $C^p(K)$ and $C^\infty(K)$ contain nonextendable functions and the norms $\| f \|_p$, $|f|_p$ are not equivalent on $\mathcal{E}^p(K)$. Then for which sets K, $C^p(K) = \mathcal{E}^p(K)$? In this section we give a proposition and a corollary about this question.

Definition 2.1.1. Given $r > 0$ a compact set $K \subset \mathbb{R}^d$ is called Whitney r-regular if it is connected by rectifiable arcs, and there exists a constant C such that $\sigma^r(x, y) \leq C\,|x-y|$ for all $x, y \in K$ where σ denotes the intrinsic (or geodesic) distance in K.

The case $r = 1$ gives the property (P) of Whitney [18]. Due to Whitney [18, Thm 1], if K is 1-regular, then $C^p(K) = \mathcal{E}^p(K)$. Also r-regularity of K is a sufficient condition for $C^\infty(K) = \mathcal{E}(K)$ for some r. In this case for an estimation of $\|\cdot\|_p$ by $|\cdot|_p$, see [19, IV,3.11] and [20].

For one-dimensional compact sets we have the following result:

Proposition 2.1.1. *[21, Prop. 1] $C^p(K) = \mathcal{E}^p(K)$ for $2 \leq p \leq \infty$ if and only if $K = \cup_{n=1}^N [a_n, b_n]$ with $a_n \leq b_n$ for $n \leq N$.*

Proof. Assume K is a finite union of closed intervals. Then for any C^p-function there exist an extension of function with the same smoothness. Furthermore, the extension which is analytic outside K, can be choosen.(see e.g. in [22, Cor.2.2.3])

For the other side, suppose K cannot be represented as a finite union of closed segments. Since the complement $\mathbb{R}\backslash K$ contains infinitely many disjoint open intervals , there exists at least one point $c \in K$ which is an accumulation point of these intervals. Let $K \subset [a, b]$ with $a, b \in K$ and assume without loss of generality $[c, b]$ contains a sequence of intervals from $\mathbb{R}\backslash K$. Then $K \subset K_0 := [a, c] \cup \cup_{n=1}^\infty [a_n, b_n]$ with $(a_n)_{n=1}^\infty$, $(b_n)_{n=1}^\infty \subset K$, $b_1 = b$, $a_{n+1} \leq b_{n+1} < a_n$, $(b_{n+1}, a_n) \subset \mathbb{R}\backslash K$ for all n. Given $1 < p < \infty$, let us take $F = 0$ on $[a, c]$ and $F = (a_n - c)^p$ on $[a_n, b_n]$ if $a_n < b_n$. In the case $a_n = b_n$ $F(a_n) = (a_n - c)^p$ and $F^{(k)}(a_n) = 0$ for all $k > 1$. Thus, $F' \equiv 0$. Then $f = F|_K$ belongs to $C^\infty(K)$, but not extendable to C^p-functions on $\mathbb{R}$ because of violation of (2.1) for $y = c$, $x = a_n$, $k = 0$. □

This nonextendable function can be easily approximated in $|\cdot|_p$ by extendable functions. Therefore, by the open mapping theorem, the following is obtained:

Corollary 2.1.1. *[21, Cor. 1] If $1 < p < \infty$ and K is not a finite union of (may be degenerated) segments, then the space $(\mathcal{E}^p(K), |\cdot|_p)$ is not complete. The same result is valid for $(\mathcal{E}(K), (|\cdot|_p)_{p=0}^\infty)$.*

It is interesting that the case $p = 1$ is exceptional here. Now we give two examples about the case $p = 1$. In the first example $C^1(K) = \mathcal{E}^1(K)$ for $K = \{0\} \cup (2^{-n})_{n=1}^\infty$. In the second example $C^1(K) \neq \mathcal{E}^1(K)$ for $K = \{0\} \cup (1/n)_{n=1}^\infty$.

Examples

1. Let $K = \{0\} \cup (2^{-n})_{n=1}^{\infty}$. Then $C^1(K) = \mathcal{E}^1(K)$. Indeed, the function $f \in C^1(K)$ is defined here by two sequences $(f_n)_{n=0}^{\infty}$ and $(f'_n)_{n=0}^{\infty}$ with $\gamma_n := (f_n - f_0) \cdot 2^n - f'_0 \to 0$ and $f'_n \to f'_0$ as $n \to \infty$. The second condition gives (2.1) with $k = 1$. The first condition means (2.1) with $k = 0, y = 0$. For the remaining case $x = 2^{-n}, y = 2^{-m}$, we have

$$f_n - f_m - f'_m(2^{-n} - 2^{-m}) = \gamma_n \cdot 2^{-n} - \gamma_m \cdot 2^{-m} + (2^{-n} - 2^{-m})(f'_0 - f'_m),$$

which is $o(|2^{-n} - 2^{-m}|)$ as $m, n \to \infty$, since $\max\{2^{-n}, 2^{-m}\} \leq 2 \cdot |2^{-n} - 2^{-m}|$. Thus, $f \in \mathcal{E}^1(K)$.

2. Let $K = \{0\} \cup (1/n)_{n=1}^{\infty}$, $f(\frac{1}{2m-1}) = 0, f(\frac{1}{2m}) = \frac{1}{m\sqrt{m}}$ for $m \in \mathbb{N}$, and $f' \equiv 0$ on K. Then $f \in C^1(K)$, but by the mean value theorem, there is no differentiable extension of f to $\mathbb{R}$.

2.2 Taylor Polynomials

The Taylor polynomial is $T_a^n f(x) = \sum_{k=0}^{n} f^{(k)}(a) \frac{(x-a)^k}{k!}$ and the corresponding Taylor remainder is $R_a^n f(x) = f(x) - T_a^n f(x)$. If $m \leq n$ and $a, b \in K$ then we have the following identities:

$$T_a^n \circ T_b^m = T_b^m, \quad R_a^n \circ R_b^m = R_a^n, \quad R_a^n \circ T_b^m = 0. \tag{2.2}$$

First we show $T_a^n \circ T_b^m = T_b^m$:

$$\begin{aligned}
(T_a^n \circ T_b^m) f(x) &= \sum_{k=0}^{n} (T_b^m f)^{(k)}(x)(a) \frac{(x-a)^k}{k!} \\
&= \sum_{k=0}^{m} \frac{(x-a)^k}{k!} \sum_{i=k}^{m} f^{(k)}(b) \frac{(a-b)^{i-k}}{(i-k)!} \\
&= \sum_{k=0}^{m} f^{(k)}(b) \sum_{i=k}^{m} \frac{(x-a)^k}{k!} \frac{(a-b)^{i-k}}{(i-k)!} \\
&= \sum_{i=0}^{m} f^{(i)}(b) \sum_{i=k+j} \frac{(x-a)^k}{k!} \frac{(a-b)^j}{j!} \\
&= \sum_{i=0}^{m} f^{(i)}(b) \frac{(x-b)^i}{i!} = (T_b^m f)(x).
\end{aligned}$$

Now we prove $R_a^n \circ R_b^m = R_a^n$:

$$\begin{aligned}
R_a^n \circ R_b^m &= (1 - T_a^n) \circ (1 - T_b^m) \\
&= 1 - T_a^n - T_b^m + T_a^n \circ T_b^m \\
&= 1 - T_a^n - T_b^m + T_b^m = R_a^n.
\end{aligned}$$

Lastly we prove $R_a^n \circ T_b^m = 0$:

$$\begin{aligned}
R_a^n \circ T_b^m &= (1 - T_a^n) \circ T_b^m \\
&= T_b^m - T_a^n \circ T_b^m \\
&= T_b^m - T_b^m = 0.
\end{aligned}$$

2.3 Cantor-type Sets

In this thesis, we consider the following Cantor-type set. Let $(N_s)_{s=0}^{\infty}$ be a sequence of integers. Let $\Lambda = (l_s)_{s=0}^{\infty}$ be a sequence of positive numbers such that $l_0 = 1$ and $0 < N_{s+1} l_{s+1} \le l_s$ for $s \in \mathbb{N}_0 := \{0, 1, \ldots\}$. Let $K_{N_s}(\Lambda)$ be the Cantor set associated with the sequence Λ that is $K_{N_s}(\Lambda) = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0,1]$, E_s is a union of $\prod_{i=0}^{s} N_i$ closed basic intervals $I_{j,s} = [a_{j,s}, b_{j,s}]$ of length l_s and E_{s+1} is obtained by deleting $N_{s+1} - 1$ open uniformly distributed subintervals of length $h_s := \frac{l_s - N_{s+1} l_{s+1}}{N_{s+1} - 1}$ from each $I_{j,s}$, $j = 1, 2, \ldots, \prod_{i=0}^{s} N_i$.

In Chapter 3 and 4, we consider the Cantor-type set $K_2(\Lambda)$ and for shortness we denote $K_2(\Lambda)$ as $K(\Lambda)$. That is $K(\Lambda)$ is the Cantor set such that $K(\Lambda) = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0,1]$, E_s is a union of 2^s closed *basic* intervals $I_{j,s} = [a_{j,s}, b_{j,s}]$ of length l_s and E_{s+1} is obtained by deleting the open concentric subinterval of length $h_s := l_s - 2l_{s+1}$ from each $I_{j,s}$, $j = 1, 2, \ldots, 2^s$. See Figure 2.1.

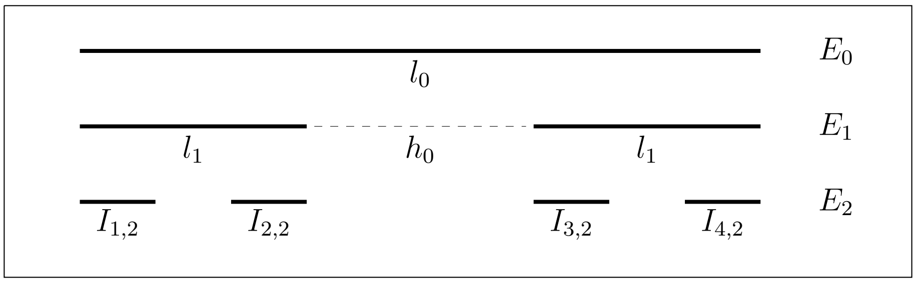


Figure 2.1: First steps of Cantor procedure for $K(\Lambda)$

In Chapter 5, we consider the Cantor-type set $K_\infty(\Lambda)$ where, $(N_s)_{s=0}^{\infty}$ is a increasing sequence such that $N_s \to \infty$ as $s \to \infty$. In the following Figure 2.2, $N_s = s + 1$.

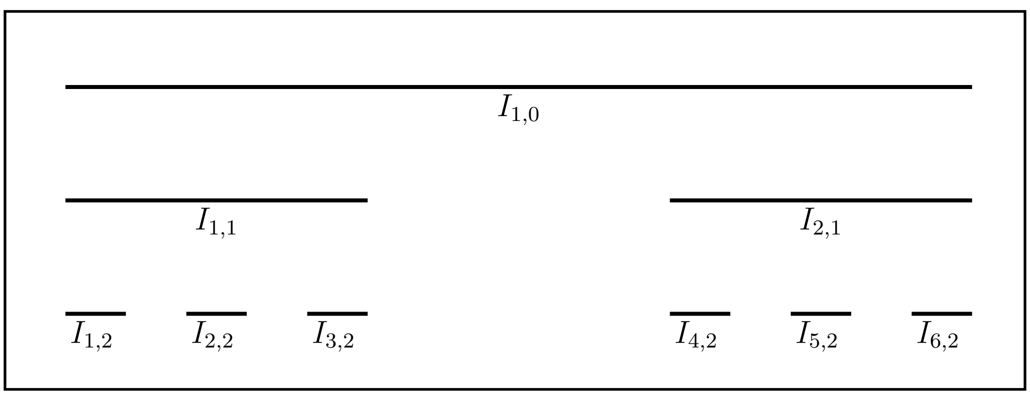


Figure 2.2: The case $N_s = s + 1$

2.4 Interpolation

Let f be a function whose values at two distinct points, say x_0 and x_1 are given. Then we can approximate f by linear function p that satisfies the conditions

$$p(x_0) = f(x_0) \quad \text{and} \quad p(x_1) = f(x_1).$$

Such a polynomial p exists and unique. We call p a linear interpolating polynomial and this process a linear interpolation. We can construct the linear interpolating polynomial directly by using the above two conditions. Then we obtain

$$p(x) = \frac{x_1 f(x_0) - x_0 f(x_1)}{x_1 - x_0} + x\left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right).$$

This can also be expressed in the Lagrange symmetric form

$$p(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0}\right) f(x_1),$$

or in Newton's divided difference form

$$p(x) = f(x_0) + (x - x_0)\left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right).$$

Let us denote the set of all polynomials of degree at most n by P_n. Let f be a function defined on a set of distinct points $x_0, x_1, \ldots, x_n$. Can we find a unique polynomial $p_n \in P_n$ such that $p(x_j) = f(x_j)$ for $j = 0, 1, \ldots, n$? Since

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x_n \quad \text{and} \quad p(x_j) = f(x_j),$$

$$f(x_j) = a_0 + a_1 x_j + \cdots + a_n x_j^n.$$

Then we have $n+1$ unknowns, $a_0, a_1, \ldots, a_n$, and we have $n+1$ linear equations. We can write these equations in the matrix form:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}. \tag{2.3}$$

This system has a unique solution if the matrix

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix},$$

which is known as Vandermonde matrix, is nonsingular. Since

$$\det V = \prod_{i>j}(x_i - x_j)$$

where the product is taken over all i and j such that $0 \leq j < i \leq n$, $\det V \neq 0$, that is, V is nonsingular. So the linear system (2.3) has a unique solution. This polynomial is called the interpolating polynomial.

2.4.1 Lagrange Interpolation

Let f be a function defined on a set of distinct points $x_0, x_1, \ldots, x_n$. Instead of using monomials $1, x, x^2, \ldots, x^n$ as a basis in the polynomial interpolation, let us consider the fundamental polynomials $L_0, L_1, \ldots, L_n$ where

$$L_i(x) = \prod_{j=0,\, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}. \tag{2.4}$$

It follows from the definition of L_i that $L_i(x_i) = 1$ and $L_i(x_j) = 0$. Then the polynomial

$$p_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x), \tag{2.5}$$

is called the Lagrange interpolating polynomial. Since $L_i(x_i) = 1$, $p_n(x_i) = f(x_i)$.

2.4.2 Newton Interpolation

Let f be as above. For a basis in the polynomial interpolation, Newton used the polynomials $\pi_0, \pi_1, \ldots, \pi_n$ where

$$\pi_i(x) = \begin{cases} 1, & i = 0 \\ (x - x_0)(x - x_1)\cdots(x - x_{i-1}), & 1 \leq i \leq n. \end{cases} \tag{2.6}$$

Then we can express the interpolating polynomial as $p_n(x) = \sum_{i=0}^{n} a_i \pi_i$. We can determine the coefficients a_j by using $p_n(x_j) = f(x_j)$ for $0 \leq j \leq n$. Then we have a system of linear equations

$$a_0\pi_0(x_j) + a_1\pi_1(x_j) + \cdots + a_n\pi_n(x_j) = f(x_j)$$

for $0 \leq j \leq n$. Then we obtain

$$a_0 = f(x_0) \quad \text{and} \quad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

We will write

$$a_j = [x_0, x_1, \ldots, x_j]f$$

and we say a_j j-th divided difference. Thus we may write $p_n(x)$ in the form

$$p_n(x) = [x_0]f\,\pi_0(x) + [x_0, x_1]f\,\pi_1(x) + \cdots + [x_0, x_1, \ldots, x_n]f\,\pi_n(x), \tag{2.7}$$

which is known as the Newton's divided difference formula for the interpolating polynomial.

In Chapter 3, we use local Newton interpolation.

Now we give some properties of the divided differences.

Proposition 2.4.1. *[23, Thm. 1.1.1] The divided difference $[x_0, x_1, \ldots, x_n]f$ can be expressed as the following symmetric sum of multiples of $f(x_j)$,*

$$[x_0, x_1, \ldots, x_n]f = \sum_{r=0}^{n} \frac{f(x_r)}{\prod_{j \neq r}(x_r - x_j)}, \tag{2.8}$$

where in the above product of n factors, r remains fixed and j takes all values from 0 to n, excluding r.

Proposition 2.4.2. *[23, Thm. 1.1.2] Let x and the abscissas $x_0, x_1, \ldots, x_n$ be contained in an interval $[a, b]$ on which f and its first n derivatives are continuous, and let $f^{(n+1)}$ exists in the open interval (a, b). Then there exists $\xi_x \in (a, b)$, which depends on x, such that*

$$f(x) - p_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)}(\xi_x)}{(n+1)!}. \tag{2.9}$$

Corollary 2.4.1. *Let $f \in C^n[a,b]$ and let $\{x_i : i = 0, \dots, n\}$ be a set of distinct points in $[a,b]$. Then there exists a point θ, in the smallest interval that contains the points $\{x_i : i = 0, \dots, n\}$ at which the equation*

$$[x_0, x_1, \dots, x_n]f = \frac{f^{(n)}(\theta)}{n!} \tag{2.10}$$

is satisfied.

Proposition 2.4.3. *Let f be defined on a set of distinct points $x_0, x_1, \dots, x_n$. Then for each j, $k \in \mathbb{N}$ with $j + k + 1 \leq n$ we have*

$$[x_j, \dots, x_{j+k+1}]f = \frac{[x_{j+1}, \dots, x_{j+k+1}]f - [x_j, \dots, x_{j+k}]f}{x_{j+k+1} - x_j}. \tag{2.11}$$

The last formula explains the term divided difference. For the proofs of these propositions see [23].

2.4.3 Interpolating Basis

Definition 2.4.1. Let $(f_n)_{n=1}^{\infty}$ be a basis in a function Banach space X with the corresponding biorthogonal functionals $(\xi_n)_{n=1}^{\infty}$. Then $(f_n)_{n=1}^{\infty}$ is called an interpolating basis with nodes $(x_n)_{n=1}^{\infty}$ if for each $f \in X$ and $n \in \mathbb{N}$ we have

$$S_m(x_m) = f(x_m) \quad \text{for} \quad m = 1, 2, \dots, n \tag{2.12}$$

where $S_n = \sum_{k=1}^{n} \xi_k(f) f_k$. Thus, the n-th partial sum S_n interpolates f at n points $x_1, \dots, x_n$.

There are many examples of interpolating bases. The basis of unit vectors $e_1, e_2, \dots$ in c_0 is interpolating. Also Faber-Schauder system is interpolating. (We showed this in Chapter 1). Furthermore, Gurari [24], Bochkarev [25], Grober and Bychkov [26] used interpolating basis in their constructions. Also the basis which we constructed in Chapter 4 and Chapter 5 are also interpolating basis. But it should be noted that not all functional spaces possess interpolating bases [27].

In 2004, Jonsson [13] considered triangulations for subsets of $\mathbb{R}^n$. In particular he constructed an interpolating basis for the space $C^p(F)$ where F is a compact subset of $\mathbb{R}$ preserving a special form of Markov inequality. Here we give more detailed information about this paper, since in Chapter 3 another basis is constructed in $C^p(F)$ for the case of Cantor-type set F satisfying restrictions from Jonsson's paper.

Definition 2.4.2. Let F be a compact subset of $\mathbb{R}^n$. A finite set T of n-dimensional closed, non-degenerated, simplices is called a triangulation of F if the following conditions hold:
A1. For each pair $\Delta_1, \Delta_2 \in T$, the intersection $\Delta_1 \cap \Delta_2$ is empty or a common face of lower dimension.
A2. Every vertex of a simplex $\Delta \in T$ is in F.
A3. $F \subset \bigcup_{\Delta \in T}$.

In his paper, Jonsson considered $\delta = \max_{\Delta \in T} \mathrm{diam}(\Delta)$ as the diameter of the triangulation and denoted the diameter of the sequence of triangulations $\{T_i\}_{i=0}^{\infty}$ as δ_i. The sequence of triangulations $\{T_i\}_{i=0}^{\infty}$, Jonsson defined, satisfied the following conditions:
B1. For each $i \geq 0$, T_{i+1} is a refinement of T_i, i.e., for each $\Delta \in T_{i+1}$ there is $\tilde{\Delta} \in T_i$ such that $\Delta \subset \tilde{\Delta}$.
B2. $\delta_i \to 0$, $i \to \infty$.
B3. If U_i is the set of vertices of T_i, the $U_i \subset U_{i+1}$ for $i \geq 0$.

Then Jonsson defined regular sequence of triangulations.

Definition 2.4.3. [13, Def. 1] Let $F \in \mathbb{R}$, and let $\{T_i\}$ be a sequence of triangulations satisfying B1. Then $\{T_i\}$ is a regular sequence of triangulations if the following conditions hold.
T1. There is a constant $c_2 > 0$, independent of i, such that, for all Δ_1, $\Delta_2 \in T_i$,

$$c_2^{-1}\mathrm{diam}(\Delta_2) \leq \mathrm{diam}(\Delta_1) \leq c_2\mathrm{diam}(\Delta_2).$$

T2. There are constants $0 < c_3 < c_4 < 1$ such that, for all $i \geq 0$,

$$c_3\delta_i \leq \delta_{i+1} \leq c_4\delta_i.$$

T3. There exists a constant $a > 0$, independent of i, such that if $\Delta \in T_i$ and $\Delta' \in T_i$ and the distance between these intervals is less than or equal to $a\delta_i$, then the intervals intersect.

Then Jonsson consider the following version of the Markov's inequality.

Definition 2.4.4. Denote by P_m the set of all polynomials in n variables of total degree less than or equal to m. A closed set $F \subset \mathbb{R}^n$ preserves Markov's inequality if for every fixed positive integer m there exists a constant c, such that for all polynomials $P \in P_m$ and all closed balls $B = B(x_0, r)$, $x_0 \in F$, $0 < r < 1$, holds

$$\max_{F\cap B} |\nabla P| \leq \frac{c}{r} \max_{F\cap B} |P|,$$

where ∇ denotes the gradient.

Some authors call it the Local Markov Inequality whereas the Global Markov Inequality means that

$$\sup_{x\in F} |\nabla P_m(x)| \leq Cm^R \sup_{x\in F} |P_m(x)|$$

where the constants C and R depend only on F.

Then Jonsson stated that [14, Section 2.2] a set preserves Markov's inequality if and only if it is uniformly perfect, that is, there is an $\varepsilon > 0$ such that for any r with $0 < r \leq 1$ and any $x_0 \in F$, the set $F \cap \{x : \varepsilon r \leq |x - x_0| \leq r\}$ is nonempty.

For Cantor type set $K(\Lambda)$ which was defined in Section 2.3, the natural triangulations are given by the sequence $\mathcal{F}_s = \{I_{i,s},\ 1 \leq i \leq 2^s\}$, $s \geq 0$. In our Cantor set we can take $\delta_i = l_i$ where $(l_i)_{i=0}^{\infty}$ is a sequence in the Cantor set such that $l_0 = 1$ and $0 < 2l_{s+1} < l_s$ for $s \in \mathbb{N}$. (For details of Cantor set see Section 2.3) Now we look the regularity conditions. The condition (T1) in the definition 2.4.3 satisfies for all Cantor type sets. If the condition (T2) satisfies, then $c_2 l_i \leq l_{i+1} \leq c_3 l_i$. This means that our Cantor set is uniformly perfect. Also the condition (T3) is satisfies for Cantor type sets. Then Cantor type sets with regular triangulations are uniformly perfect. In Chapter 3, our Cantor-type set is uniformly perfect.

Jonsson showed that F preserves Markov's inequality if and only if there exists a regular triangulation of F. This is the main theorem of Jonsson's article. [13, Theorem1] After this Jonsson constructed a basis in the space $C^k(F)$ of k times differentiable functions on F. But Jonsson defined the space $C^k(F)$ as: "A function f belongs to the space $C^k(F)$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|R_j(x,y)| < \varepsilon|x-y|^{k-j}$ for $0 \leq j \leq k$ and $|x-y| < \delta$." Here $R_j(x,y)$ denotes the Taylor remainder. In this definition the condition $|R_j(x,y)| < \varepsilon|x-y|^{k-j}$ is Whitney's condition (2.1) for the space $\mathcal{E}^p(F)$, that is the Whitney space of functions on K. Therefore Jonsson considered the space $\mathcal{E}^p(F)$ but equipped with the norm of the space $C^p(F)$. Except the case when $\mathcal{E}^p(F) = C^p(F)$ as sets of elements (that is all C^p-functions on F are extendable preserving the class), the space of Whitney functions is not complete in the topology of the space $C^p(F)$. The lack of completeness was remarked by the author in [13] on page 54. In contrast to this we contruct a basis in the whole space $C^p(F)$.

Elements of basis in [13] are restrictions of special Hermite polynomials. Since the construction is rather technical, for details we refer the reader to [13].

The Schauder basis which was given in [28] is another interpolating basis in Banach Besov space on fractals.

2.4.4 Local Interpolation

Let us consider the method of local interpolations suggested in [6] (see also [15]). Suppose we have a chain of compact sets $K_0 \supset K_1 \supset \cdots \supset K_s \supset \cdots$ and finite system of distinct points $(x_k^{(s)})_{k=1}^{N_s} \subset K_s$ for $s \in \mathbb{N}_0$. Some part of the knots on K_{s+1} belongs to the previous set $(x_k^{(s)})_{k=1}^{N_s}$. Let us enumerate these points as $(x_k^{(s+1)})_{k=1}^{M_{s+1}}$. We will interpolate a given function f on K_s up to the degree N_s. After this we continue the interpolation on the set K_{s+1} up to degree N_{s+1}, etc. Since we will take $\operatorname{diam} K_s \to 0$, the approximation properties of the interpolating polynomials will improve. The points of interpolation will be chosen independently on functions. This will allow to construct topological bases in spaces of differentiable functions defined on the set K which is a union of the intersections of all chains (K_j).

Suppose we are given a sequence $(x_n)_{n=1}^{\infty}$ on a compact set $K \subset \mathbb{R}$. Let $e_0 \equiv 1$ and $e_n(x) = \prod_{k=1}^{n}(x - x_k)$ for $n \in \mathbb{N}$. Let $X(K)$ be any Fréchet space of continuous functions on K, containing all polynomials. Then, given $f \in X(K)$ and $n \in \mathbb{N}_0$ we denote by ξ_n the linear functional $\xi_n(f) = [x_1, \ldots, x_{n+1}]f$. Trivially we have

Lemma 2.4.1. *[6, Lem. 1] If a sequence $(x_n)_{n=0}^{\infty}$ of distinct points is dense on a perfect compact set $K \subset \mathbb{R}$; then the system $(e_n; \xi_n)_{n=0}^{\infty}$ is biorthogonal and the sequence of functionals $(\xi_n)_{n=0}^{\infty}$ is total on $X(K)$; that is whenever $\xi_n(f) = 0$ for all n, it follows that $f = 0$.*

Since below we consider different basic systems, the following convolution property of coefficients of basic expansions will be useful.

Lemma 2.4.2. *[6, Lem. 2] Let $(x^{(s)})_k^{\infty}$, $s = 1, 2, 3$, be three sequences such that for a fixed superscript s all points in the sequence $(x^{(s)})_k^{\infty}$ are different. Let $e_{ns} = \prod_{k=1}^{n}(x - x_k^{(s)})$ and $\xi_{ns}(f) = [x_1^{(s)}, x_2^{(s)}, \ldots, x_n^{(s)}]f$ for $n \in \mathbb{N}_0$. Then*

$$\sum_{p=q}^{r} \xi_{p3}(e_{q2})\xi_{q2}(e_{r1}) = \xi_{p3}(e_{r1}) \quad \textit{for} \quad p \leq r.$$

Suppose K_0 and K_1 are infinite compact sets such that $K_1 \subset K_0$ and $K_0 \backslash K_1$ is closed. Let natural numbers N_0, N_1, M_1 be given with $N_0 \geq 2$, $M_1 \leq N_0$, $M_1 \leq N_1$, $N_0 - M_1 \leq N_1$. Let for $s \in \{0, 1\}$ we have a finite system of points $(x_k^{(s)})_{k=1}^{N_s} \subset K_s$. Here we suppose that $x_k^{(s)} \neq x_l^{(s)}$ for $k \neq l$ and $(x_k^{(0)})_{k=1}^{N_0 - M_1} \subset K_0 \backslash K_1$, $x_{N_0 - M_1 + r}^{(0)} = x_r^{(1)}$ for $r = 1, \ldots, M_1$. Let us set $\tilde{e}_{ns}(x) = \prod_{k=1}^{n}(x - x_k^{(s)})$ for $s \in \{0, 1\}$, $0 \leq n \leq N_s$ and let e_{ns} be the restriction of $\tilde{e}_{ns}$ to K_s, otherwise $e_{ns}(x) = 0$. Also for any function defined on K_s let $\xi_{ns}(f) = [x_1^{(s)}, x_2^{(s)}, \ldots, x_{n+1}^{(s)}]f$, where $x_{N_0+1}^{(0)} := x_{M_1+1}^{(1)}$ and $x_{N_1+1}^{(1)} \in K_1$ is any point differing from $x_k^{(1)}$, $k = 1, \ldots, N_1$.

By means of Lemma 2.4.2 we can construct new biorthogonal systems corresponding the local interpolation of functions. For fixed level of s, the system $(e_{ns}, \xi_{ns})_{n=0}^{N_s}$ is biorthogonal, that is $\xi_{ns}(e_{ms}) = \delta_{mn}$ For $n = M_1 + 1, \ldots, N_1$, $e_{n1} =$

$\prod_{k=1}^{n}(x-x_k^{(1)})$ for $x \in K_1$ and $e_{n1}=0$ for $x \in K_0\backslash K_1$. Since $K_0\backslash K_1$ is closed e_{n1} is continuous on K_0. Then $\xi_{n0}(e_{m,1})=0$, because the number $\xi_{n0}(f)$ is defined by the values of f at some points on $K_0\backslash K_1$ and at some points from $(x_k^{(1)})_{k=1}^{M_1}$ where the function $e_{m,1}$ is zero. Clearly, $\xi_{n,1}(e_{m0})=0$ for $n>m$. But for $n<m$, the functional $\xi_{n,1}$, in general, is not biorthogonal to e_{m0}. For this reason we use the functionals

$$\eta_{n,1}=\xi_{n,1}-\sum_{k=n}^{N_0}\xi_{n,1}(e_{k0})\xi_{k0}.$$

Now, the functional $\eta_{n,1}$ is biorthogonal to e_{m0} by means of Lemma 2.4.2.

Given f on K_0 let us denote by $Q_n(f,(x_k)_{k=1}^{n+1},\cdot)$ the Newton interpolation polynomial of degree n for f with nodes at $x_1,\ldots,x_{n+1}$. Let us denote $\Pi_N(A)$ the set of functions coinciding on the set A with some polynomial of degree not greater than N and also $\Pi_N:=\Pi_N(\mathbb{R})$.

Let us consider the function $S_n(f,x)=Q_n(f,(x_k^{(0)})_{k=1}^{n+1},x)$ for $n=0,\ldots,N_0$ and

$$S_{N_0+r}(f,x)=Q_{N_0}(f,(x_k^{(0)})_{k=1}^{N_0+1},x)+\sum_{k=M_1+1}^{M_1+r}\eta_{k,1}(f)e_{k,1}(x) \tag{2.13}$$

for $r=1,\ldots,N_1-M_1$. Then $S_{N_0+r}\in\Pi_{N_0}(K_0\backslash K_1)$ and $S_{N_0+r}\in\Pi_{\max\{N_0,M_1+r\}}(K_1)$.

Lemma 2.4.3. *[15, Lem. 1] Given function f defined on K_0 and $n=0,1,\ldots,N_0+N_1-M_1$, the function $S_n(f,\cdot)$ interpolates f at the first $n+1$ points from the set*

$$\{x_1^{(0)},\ldots,x_{N_0}^{(0)},x_{M_1+1}^{(1)},\ldots,x_{N_1+1}^{(1)}\}.$$

In Chapter 3 all our subsequent considerations are related to Cantor-type sets. Let $\Lambda=(l_s)_{s=0}^{\infty}$ be a sequence such that $l_0=1$ and $0<2l_{s+1}\le l_s$ for $s\in\mathbb{N}_0$. Let $K(\Lambda)$ be the Cantor set associated with the sequence Λ that is $K(\Lambda)=\cap_{s=0}^{\infty}E_s$, where $E_0=I_{1,0}=[0,1]$, E_s is a union of 2^s closed basic intervals $I_{j,s}$ of length l_s and E_{s+1} is obtained by deleting the open concentric subinterval of length $h_s:=l_s-2l_{s+1}$ from each $I_{j,s}$, $j=1,2,\ldots,2^s$.

Given a nondecreasing sequence of natural numbers $(n_s)_0^{\infty}$, let $N_s=2^{n_s}$, $M_s^{(l)}=N_{s-1}/2+1$, $M_r^{(l)}=N_{s-1}/2$ for $s\ge 1$ and $M_0=1$. Here, (l) and

(r) mean left and right respectively. For any basic interval $I_{j,s} = [a_{j,s}, b_{j,s}]$ we choose the sequence of points $(x_{n,j,s})_{n=1}^{\infty}$ using the rule of increase of the type. We take $e_{N,1,0} = \prod_{n=1}^{N}(x - x_{n,1,0}) = \prod_{n=1}^{N}(x - x_n)$ for $x \in K(\Lambda)$, $N = 0, 1, \ldots, N_0$. For $s \geq 1$, $j \leq 2^s$, let $e_{N,j,s} = \prod_{n=1}^{N}(x - x_{n,j,s})$ if $x \in K(\Lambda) \cap I_{j,s}$, and $e_{N,j,s} = 0$ on $K(\Lambda)$ otherwise. Here, $N = M_s^a, M_s^a + 1, \ldots, N_s$ with $a = l$ if j is odd and $a = r$ if j is even. Given function f on $K(\Lambda)$ let the functional $\xi_{N,j,s}(f) = [x_{1,j,s}, \ldots, x_{N+1,j,s}]f$ for $s = 0, 1, \ldots$; $j = 1, 2, \ldots, 2^s$ and $N = 0, 1, \ldots$. Let us set $\eta_{N,1,0} = \xi_{N,1,0}$ for $N \leq N_0$. Every basic interval $I_{j,s}$, $s \geq 1$, is a subinterval of a certain $I_{i,s-1}$ with $j = 2i - 1$ or $j = 2i$. Let

$$\eta_{N,j,s}(f) = \xi_{N,j,s}(f) - \sum_{k=N}^{N_{s-1}} \xi_{N,j,s}(e_{k,i,s-1})\xi_{k,i,s-1}(f)$$

for $N = M_s^a, M_s^a + 1, \ldots, N_s$. As before $a = l$ if $j = 2i - 1$ and $a = r$ if $j = 2i$.

Now we give an example to local interpolation for $N_0 = N_1 = N_2 = 4$ and $f \in C[0, 1]$. Then our points are $x_1 = 0$, $x_2 = 1$, $x_3 = l_1$, and $x_4 = 1 - l_1$. Then the interpolating polynomial

$$\begin{aligned} Q_4 &= f(x_1) + (x - x_1)\,[x_1, x_2]f + \cdots + (x - x_1)\cdots(x - x_4)\,[x_1, x_2, \ldots, x_5]f \\ &= f(0) + (f(1) - f(0))x + \cdots \\ &\quad + x(x-1)(x-l_1)(x-1+l_1)\,[0, 1, l_1, 1-l_1, l_2]f. \end{aligned}$$

As seen in the equation we add one more point $x_5 = l_2$. For $s = 0$,

$$\begin{aligned} e_{0,1,0} &= 1, \\ e_{1,1,0} &= x, \\ e_{2,1,0} &= x(x-1), \\ e_{3,1,0} &= x(x-1)(x-l_1), \\ e_{4,1,0} &= x(x-1)(x-l_1)(x-1+l_1), \end{aligned}$$

and

$$\begin{aligned}\xi_{0,1,0} &= f(0),\\ \xi_{1,1,0} &= [0,1]f,\\ \xi_{2,1,0} &= [0,1,l_1]f,\\ \xi_{3,1,0} &= [0,1,l_1,1-l_1]f,\\ \xi_{4,1,0} &= [0,1,l_1,1-l_1,l_2]f.\end{aligned}$$

Now we look this for the intervals $I_{1,1}$, $I_{2,1} \in I_{1,0} = [0,1]$. $I_{1,1}$ is the left part and $I_{2,1}$ is the right part. Then on $I_{1,1}$

$$\begin{aligned}e_{1,1,1} &= x,\\ e_{2,1,1} &= x(x-1),\end{aligned}$$

and on $I_{1,2}$

$$\begin{aligned}e_{1,2,1} &= x-1,\\ e_{2,2,1} &= (x-1)(x-1+l_1).\end{aligned}$$

Also

$$\begin{aligned}\xi_{0,1,1} &= f(0),\\ \xi_{1,1,1} &= [0,l_1]f,\\ \xi_{2,1,1} &= [0,l_1,l_2]f,\end{aligned}$$

and

$$\begin{aligned}\xi_{0,2,1} &= f(1-l_1),\\ \xi_{1,2,1} &= [1-l_1,1]f,\\ \xi_{1,2,1} &= [1-l_1,1,1-l_1+l_2]f.\end{aligned}$$

So we add a new point $x_6 = 1 - l_1 + l_2$ to $I_{2,1}$. In this way we add new points to left and right intervals and interpolate f. Let us look this. We know that $S_5(f,x) = Q_4$ by Lemma 2.4.3. Then

$$\begin{aligned}S_6 &= Q_4 + \eta_{2,2,1}(f)e_{2,2,1},\\ S_7 &= Q_4 + \eta_{2,2,1}(f)e_{2,2,1} + \eta_{3,1,1}(f)e_{3,1,1},\\ S_8 &= Q_4 + \eta_{2,2,1}(f)e_{2,2,1} + \eta_{3,2,1}(f)e_{3,2,1},\\ &\vdots\end{aligned}$$

Here

$$\begin{aligned}\eta_{2,2,1}(f) &= \xi_{2,2,1} - \sum_{k=2}^{4} \xi_{2,2,1}(e_{k,1,0})\xi_{k,1,0}(f), \\ \eta_{3,1,1}(f) &= \xi_{3,1,1} - \sum_{k=3}^{4} \xi_{3,1,1}(e_{k,1,0})\xi_{k,1,0}(f).\end{aligned}$$

Then

$$\|f(x) - S_7(f,x)\| \leq |\eta_{2,2,1}(f)e_{2,2,1} + \eta_{3,1,1}(f)e_{3,1,1}|$$

since Q_4 is the interpolating polynomial of $f(x)$. In Chapter 3, we find bounds for $\eta(f)$ and $\xi(f)$ and we show that $\|f(x) - S_n(f,x)\|_p \leq \varepsilon$ as $n \to \infty$.

Chapter 3

Schauder Bases in the Space $C^p(K(\Lambda))$ Where $K(\Lambda)$ is Uniformly Perfect

In this chapter we construct basis in the space $C^p(K(\Lambda))$ where $K(\Lambda)$ is a uniformly perfect Cantor-type set. In the construction we use the method of local Newton interpolations (see Section 2.4.4). Elements of basis are polynomials of preassigned degree and biorthogonal functionals are special linear combinations of the divided differences of functions. First we give some estimations, then we give the main theorem.

3.1 Estimations

Given function $f \in C(K)$ on a compact set $K \subset \mathbb{R}$, let $w(f, \cdot)$ be the modulus of continuity of f, that is, $w(f,t) = \sup\{|f(x) - f(y)| : x, y \in K, |x-y| \le t\}$, $t > 0$. Let $N \ge 1$ and $(x_k)_{k=1}^{N+1} \in K$ be such that $x_1 < x_2 < \cdots < x_{N+1}$. Let $e_{N+1}(x) = \prod_{k=1}^{N+1}(x - x_k)$, $\xi_N(f) = [x_1, x_2, \dots, x_{N+1}]f$ and $t = \max_{k \le N} |x_{k+1} - x_k|$. Then ([15, p 26, (3)])

$$|\xi_N(f)| \le N^2 w(f,t)(\min_{k \le N} |e'_{N_1}(x_k)|)^{-1}. \tag{3.1}$$

Let us generalize this inequality to the case $f \in C^p(K)$ for $p \in \mathbb{N}_0$.

Lemma 3.1.1. *For $N \geq 1$ and $x_1, \ldots, x_{N+1} \in K$ with $x_1 < x_2 < \cdots < x_{N+1}$ let $\xi_N(f) = [x_1, x_2, \ldots, x_{N+1}]f$ and $t = \max_{k \leq N} |x_{k+1} - x_k|$. Then*

$$|\xi_N(f)| \leq \frac{N^2 w(f^{(q)}, t)}{q!} \left(\min_{k \leq N-q,\, j \leq N} \left| \prod_{s=1,\, s \neq j}^{k} (x_j - x_s) \prod_{s=k+q+1,\, s \neq j}^{N+1} (x_j - x_s) \right| \right)^{-1}, \tag{3.2}$$

for all q with $0 \leq q \leq p$.

Proof. First we show this for fix p, then we show this for all $0 \leq q \leq p$. Let $e_{N+1}(x) = \prod_{j=1}^{N+1}(x - x_j)$. Then $e'_{N+1}(x_k) = \prod_{j=1,\, j \neq k}^{N+1}(x_k - x_j)$. From Proposition 2.4.1 $\xi_N(f) = \sum_{k=1}^{N+1} \frac{f(x_k)}{e'_{N+1}(x_k)}$. We write $\xi_N(f)$ in terms of divided differences of p-th order.

$$
\begin{aligned}
|\xi_N(f)| &= \left| \sum_{k=1}^{N+1} \frac{f(x_k)}{e'_{N+1}(x_k)} \right| \\
&\leq \left| (x_1 - x_{p+2})[x_1, \ldots, x_{p+2}]f \frac{\prod_{j=2}^{p+1}(x_1 - x_j)}{e'_{N+1}(x_1)} \right| + \\
&\quad \left| (x_2 - x_{p+3})[x_2, \ldots, x_{p+3}]f \left[\frac{\prod_{j=3}^{p+2}(x_1 - x_j)}{e'_{N+1}(x_1)} + \frac{\prod_{j=3}^{p+2}(x_2 - x_j)}{e'_{N+1}(x_2)} \right] \right| + \\
&\quad . \\
&\quad . \\
&\quad . \\
&\quad \left| (x_N - x_{N+p+1})[x_N, \ldots, x_{N+p+1}]f \left[\frac{\prod_{j=N+1}^{N+p}(x_1 - x_j)}{e'_{N+1}(x_1)} + \ldots + \frac{\prod_{j=N+1}^{N+p}(x_N - x_j)}{e'_{N+1}(x_{N+p+1})} \right] \right| \\
&\leq \sum_{k=1}^{N} \left| (x_k - x_{k+p+1})[x_k, \ldots, x_{k+p+1}]f \sum_{j=1}^{k} \frac{\prod_{s=k+1}^{k+p}(x_j - x_s)}{e'_{N+1}(x_j)} \right| \qquad (3.3) \\
&\leq \sum_{k=1}^{N} N \left| (x_k - x_{k+p+1})[x_k, \ldots, x_{k+p+1}]f \right| \left(\min_{j \leq N} \left| \prod_{s=1,\, s \neq j}^{k} (x_j - x_s) \prod_{s=k+p+1,\, s \neq j}^{N+1} (x_j - x_s) \right| \right)^{-1}
\end{aligned}
$$

From Proposition 2.4.3 we can write $[x_k, \ldots, x_{k+p+1}]f$ in such a way that

$$[x_k, \ldots, x_{k+p+1}]f = \frac{[x_{k+1}, \ldots, x_{k+p+1}]f - [x_k, \ldots, x_{k+p+1}]f}{x_{k+p} - x_k}.$$

By Corollary 2.4.1

$$[x_k,\, \ldots,\, x_{k+p+1}]f = \frac{f^{(p)}(\theta)}{p!}$$

where $\theta \in [x_{k+1},\, x_{k+p+1}]$. Then

$$(x_k - x_{k+p+1})[x_k,\, \ldots,\, x_{k+p+1}]f = \frac{f^{(p)}(\theta_1) - f^{(p)}(\theta_2)}{p!} < \frac{w(f^{(p)}, t)}{p!} \tag{3.4}$$

where $t \in [x_k,\, x_{k+p+1}]$. By (3.4)

$$\begin{aligned} \xi_N(f) \leq &\quad \sum_{k=1}^{N} N \frac{w(f^{(p)}, t)}{p!} \left(\min_{j \leq N} \left| \prod_{s=1,\, s\neq j}^{k} (x_j - x_s) \prod_{s=k+p+1,\, s\neq j}^{N+1} (x_j - x_s) \right| \right)^{-1} \\ \leq &\quad \frac{N^2 w(f^{(p)}, t)}{p!} \left(\min_{j \leq N} \left| \prod_{s=1,\, s\neq j}^{k} (x_j - x_s) \prod_{s=k+p+1,\, s\neq j}^{N+1} (x_j - x_s) \right| \right)^{-1} \end{aligned}$$

where $t \in [x_k,\, x_{k+p+1}]$, which is the desired result for fixed p.

Let us prove this for all $q = 0, 1,\, \ldots, p$. To prove this we use induction on q. For $q = 0$, (3.1) satisfies.
Assume 3.3 is true for $q = p - 1$, that is,

$$|\xi_N(f)| \leq \frac{N^2 w(f^{(p-1)}, t)}{(p-1)!} \left(\min_{k \leq N-p+1,\, j \leq N} \left| \prod_{s=1,\, s\neq j}^{k} (x_j - x_s) \prod_{s=k+p,\, s\neq j}^{N+1} (x_j - x_s) \right| \right)^{-1}.$$

Now we show that it is true for $q = p$. By (3.3) we have

$$|\xi_N(f)| \leq \sum_{k=1}^{N} (x_k - x_{k+p+1})[x_k,\, \ldots,\, x_{k+p+1}]f \sum_{j=1}^{k} \frac{\prod_{s=k+1}^{k+p} (x_j - x_s)}{e'_{N+1}(x_j)}$$

By Proposition 2.4.3,

$$[x_k, \ldots, x_{k+p+1}]f = \frac{[x_{k+1}, \ldots, x_{k+p+1}]f - [x_k, \ldots, x_{k+p}]f}{x_{k+p+1} - x_k}.$$

Then

$$\begin{aligned}
|\xi_N(f)| &\leq \sum_{k=1}^{N}([x_k,\dots,x_{k+p}]f-[x_{k+1},\dots,x_{k+p+1}]f)\sum_{j=1}^{k}\frac{\prod_{s=k+1}^{k+p}(x_j-x_s)}{e'_{N+1}(x_j)} \\
&\leq ([x_1,\dots,x_{p+1}]f-[x_2,\dots,x_{p+2}]f)\,\frac{\prod_{s=2}^{p+1}(x_1-x_s)}{e'_{N+1}(x_1)} \\
&\quad +([x_2,\dots,x_{p+2}]f-[x_3,\dots,x_{p+3}]f)\left(\frac{\prod_{s=3}^{p+2}(x_1-x_s)}{e'_{N+1}(x_1)}+\frac{\prod_{s=3}^{p+2}(x_2-x_s)}{e'_{N+1}(x_2)}\right) \\
&\qquad \vdots \\
&\quad +([x_N,\dots,x_{N+p}]f-[x_{N+1},\dots,x_{N+p+1}]f)\sum_{j=1}^{N}\frac{\prod_{s=N+1}^{N+p}(x_j-x_s)}{e'_{N+1}(x_j)} \\
&\leq (x_1-x_{1+p})[x_1,\dots,x_{1+p}]f\,\frac{\prod_{s=2}^{p+1}(x_1-x_s)}{e'_{N+1}(x_1)} \\
&\quad +(x_2-x_{2+p})[x_2,\dots,x_{2+p}]f\left(\frac{\prod_{s=3}^{p+2}(x_1-x_s)}{e'_{N+1}(x_1)}+\frac{\prod_{s=3}^{p+2}(x_2-x_s)}{e'_{N+1}(x_2)}\right) \\
&\qquad \vdots \\
&\quad +(x_N-x_{N+p})[x_N,\dots,x_{N+p}]f\,\sum_{j=1}^{N}\frac{\prod_{s=N+1}^{N+p}(x_j-x_s)}{e'_{N+1}(x_j)} \\
&\quad -(x_{N+1}-x_{N+p+1})[x_{N+1},\dots,x_{N+p+1}]f\,\sum_{j=1}^{N+1}\frac{\prod_{s=N+2}^{N+p+1}(x_j-x_s)}{e'_{N+1}(x_j)} \\
&\leq \sum_{k=1}^{N}(x_k-x_{k+p})[x_k,\dots,x_{k+p}]f\,\sum_{j=1}^{k}\frac{\prod_{s=k+1}^{k+p}(x_j-x_s)}{e'_{N+1}(x_j)} \\
&\quad -(x_{N+1}-x_{N+p+1})[x_{N+1},\dots,x_{N+p+1}]f\,\sum_{j=1}^{N+1}\frac{\prod_{s=N+2}^{N+p+1}(x_j-x_s)}{e'_{N+1}(x_j)}
\end{aligned}$$

and

$$
\begin{aligned}
|\xi_N(f)| &\leq \left|\sum_{k=1}^{N}(x_k - x_{k+p})[x_k,\ldots,x_{k+p}]f \sum_{j=1}^{k} \frac{\prod_{s=k+1}^{k+p}(x_j - x_s)}{e'_{N+1}(x_j)}\right| \\
&\quad + \left|(x_{N+1} - x_{N+p+1})[x_{N+1},\ldots,x_{N+p+1}]f \sum_{j=1}^{N+1} \frac{\prod_{s=N+2}^{N+p+1}(x_j - x_s)}{e'_{N+1}(x_j)}\right| \\
&\leq \frac{N^2 w(f^{(p-1)},t)}{(p-1)!}\left(\min_{k\leq N-p+1,\, j\leq N}\left|\prod_{s=1,\, s\neq j}^{k}(x_j - x_s)\prod_{s=k+p,\, s\neq j}^{N+1}(x_j - x_s)\right|\right)^{-1} \\
&\quad + \frac{N w(f^{(p-1)},t)}{(p-1)!}\left(\min_{k\leq N-p+1,\, j\leq N}\left|\prod_{s=1,\, s\neq j}^{k}(x_j - x_s)\prod_{s=k+p,\, s\neq j}^{N+1}(x_j - x_s)\right|\right)^{-1} \\
&\leq \frac{(N+1)^2 w(f^{(p-1)},t)}{(p-1)!}\left(\min_{k\leq N-p+1,\, j\leq N}\left|\prod_{s=1,\, s\neq j}^{k}(x_j - x_s)\prod_{s=k+p,\, s\neq j}^{N+1}(x_j - x_s)\right|\right)^{-1}
\end{aligned}
$$

So it satisfies for $q = p$ and the proof complete. □

Lemma 3.1.2. *Let* $e_{N+1}(x) = \prod_{k=1}^{N+1}(x - x_k)$ *and* $e_{N+1}^{(p)}$ *be the p-th derivative of* e_{N+1}. *Then for* $p \geq 1$

$$|e_{N+1}^{(p)}| \leq N^p \max_{k\leq N}\left|\prod_{s=1}^{k}(x - x_s)\prod_{s=k+p+1}^{N+1}(x - x_s)\right|. \tag{3.5}$$

In our work all our subsequent considerations are related to Cantor-type sets. Let $\Lambda = (l_s)_{s=0}^{\infty}$ be a sequence such that $l_0 = 1$ and $0 < 2l_{s+1} \leq l_s$ for $s \in \mathbb{N}_0$. Let $K(\Lambda)$ be the Cantor set associated with the sequence Λ that is $K(\Lambda) = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0,1]$, E_s is a union of 2^s closed basic intervals $I_{j,s}$ of length l_s and E_{s+1} is obtained by deleting the open concentric subinterval of length $h_s := l_s - 2l_{s+1}$ from each $I_{j,s}$, $j = 1, 2, ..., 2^s$.

We will consider Cantor-type sets with the restriction

$$\exists\, A \,:\, l_k \leq A\, h_k,\ \forall\, k. \tag{3.6}$$

Without loss of generality we suppose $A \geq 2$.

Let x be an endpoint of some basic interval. Then there exists the minimal number s such that x is the endpoint of some $I_{j,m}$ for every $m \geq s$.

By K_s we denote $K(\Lambda)\cap l_s$. Given K_s with $s\in\mathbb{N}_0$, let us choose the sequence $(x_n)_1^\infty$ by including all endpoints of basic intervals, using the rule of increase of the type. For the points of the same type we first take the endpoints of the largest gaps between the points of this type; here the intervals $(\infty, x), (x, \infty)$ are considered as gaps. From points adjacent to the equal gaps, we choose the left one x and then $l_s - x$. Thus, $x_1 = 0$, $x_2 = l_s$, $x_3 = l_{s+1}, \ldots,$ $x_7 = l_{s+1} - l_{s+2}, \ldots,$ $x_{2k+1} = l_{s+k}, \ldots$

Let

$$\mu_{s,N} := \frac{\max_{x\in K_s}|e_N(x)|}{\min_{j\le N}|e'_{N+1}(x_j)|}, \qquad L_{N,j} = \prod_{k=1,\,k\ne j}^{N}\frac{x-x_j}{x_k-x_j},$$

that is, $L_{N,j}$ denotes the fundamental Lagrange polynomial.

Lemma 3.1.3. *[15, Lem. 2] Suppose the Cantor-type set $K(\Lambda)$ satisfies* (3.6) *and for $N\ge 1$ the points $(x_k)_1^{N+1}\subset K_s$ are chosen by the rule of increase of the type. Then*

$$\mu_{s,N}\le A^N \qquad \textit{and} \qquad \max_{j\le N,\,x\in K_s}|L_{N,j}(x)|\le A^{N-1}.$$

Let us set

$$\varphi_{s,N} := \frac{\max_{k\le N-p}\left|\prod_{s=1}^{k}(x-x_s)\prod_{s=k+p+1}^{N+1}(x-x_s)\right|}{\min_{k\le n-p,\,j\le N}\left|\prod_{s=1,\,s\ne j}^{k}(x_j-x_s)\prod_{s=k+p+1,\,s\ne j}^{N+1}(x_j-x_s)\right|}.$$

Suppose the Cantor-type set $K(\Lambda)$ where $K(\Lambda)$ is uniformly perfect and satisfies (3.6). Since $K(\Lambda)$ is uniformly perfect, there exists $B\in\mathbb{R}$ such that $l_s\le Bl_{s+1}$.

Lemma 3.1.4. *For $N\ge p+1$ the points $(x_k)_1^{N+1}\subset K_s$ are chosen by the rule of increase of the type. Then*

$$\varphi_{s,N}\le A^{N-p}B^{p\,\log_2(N)}.$$

Proof. Let $N = 2n+\nu$ with $0\le\nu<2^n$.Then $(x_k)_1^{N+1}$ consists of all endpoints basic intervals of the type $s+n-1$ and $\nu+1$ points of the type $s+n$. Fix any $x\in K_s$ and $x_j, j\le N+1$.

By $(y_k)_1^N$ we denote the points $(x_k)_1^N$ arranged in the order of distances $|x-x_k|$, that is, $|x-y_k| = |x-x_{\sigma_k}|\uparrow$. Then $Y = (y_k)_1^N = \bigcup_{m=0}^n Y_{s+m}$ where $Y_r = \{y_k : h_r\le|x-y_k|\le l_r\}$, $r = s,\ldots,s+n$.

Similarly $Z = (z_k)_1^N$ consist of all points $(x_k)_{k=1,\, k\neq j}^N$ arranged in the order of distances $|x_j - x_k|$, that is, $|x_j - z_k| = |x_j - x_{\tau_k}| \uparrow$. As before, $Z = \bigcup_{m=0}^n Z_{s+m}$ where $Z_r = \{y_k : h_r \le |x_j - z_k| \le l_r\}$, $r = s, \dots, s+n$. Let $a_p = |Y_p|$, $b_p = |Z_p|$ be the cardinalities of the corresponding sets. Since $(x_k)_1^{N+1}$ are uniformly distributed on K_s, it follows that the numbers of points x_k in two basic intervals $I_{i,r}$, $I_{j,r}$ of equal length are the same of differ by 1. But the point x_j is not included into the computation of b_r. Hence for $r = s, \dots, s+n$ we have the following inequality

$$a_{s+n} + \cdots + a_r \ge b_{s+n} + \cdots + b_r.$$

Next to find the maximum of the product $\left|\prod_{s=1}^k (x - x_s) \prod_{s=k+p+1}^{N+1} (x - x_s)\right|$ we choose p points which are very close to x. So the distance between x and the other Np points, is maximum. We know that $a_s + \cdots + a_{s+n} = N$. Then $a_s + \cdots + a_{s+n} - p = N - p$. Let we choose $v_p, c_{1p} \in \mathbb{N}$ such that

$$a_s + \cdots + a_{s+v_p} + c_{1p} = N - p, \qquad c_{1p} \le a_{s+v_p+1}, \qquad v_p \le n. \tag{3.7}$$

Hence

$$\max_{k\le N} \left| \prod_{s=1}^k (x - x_s) \prod_{s=k+p+1}^{N+1} (x - x_s) \right| \le l_s^{a_s} l_{s+1}^{a_{s+1}} \cdots l_{s+v_p}^{a_{s+v_p}} l_{s+v_p+1}^{c_{1p}}. \tag{3.8}$$

Also to find the minimum of the product $\left|\prod_{s=1,\, s\neq j}^k (x_j - x_s) \prod_{s=k+p+1,\, s\neq j}^{N+1} (x_j - x_s)\right|$, first we fix $j = \tilde{j}$ such that

$$\min_{k\le N,\, j\le N} \left| \prod_{s=1,\, s\neq j}^k (x_j - x_s) \prod_{s=k+p+1,\, s\neq j}^{N+1} (x_j - x_s) \right| = \min_{k\le N} \left| \prod_{s=1,\, s\neq \tilde{j}}^k (x_{\tilde{j}} - x_s) \prod_{s=k+p+1,\, s\neq \tilde{j}}^{N+1} (x_{\tilde{j}} - x_s) \right|.$$

where $\tilde{j} \in \mathbb{N}$ and $\tilde{j} \le N$. Then we choose p points which are far away from $x_{\tilde{j}}$. So the distance between other points and $x_{\tilde{j}}$ is minimum. We know that $b_s + \cdots + b_{s+n} = N$. Let we choose u_p, $c_{2p} \in \mathbb{N}$ such that

$$b_s + \cdots + b_{s+u_p} + c_{2p} = N - p, \qquad c_{2p} \le b_{s+u_p-1}, \qquad u_p \le n \tag{3.9}$$

Then

$$\min_{k\leq N}\left|\prod_{s=1,\, s\neq\tilde{j}}^{k}(x_{\tilde{j}}-x_s)\prod_{s=k+p+1,\, s\neq\tilde{j}}^{N+1}(x_{\tilde{j}}-x_s)\right|\geq l_{s+n}^{b_{s+n}}h_{s+n-1}^{b_{s+n-1}}\cdots h_{s+u_p}^{b_{s+u_p}}h_{s+u_p-1}^{c_2p}. \tag{3.10}$$

Then by (3.8) and (3.10)

$$\begin{aligned}|\varphi_{s,N}| &\leq \frac{l_s^{a_s}l_{s+1}^{a_{s+1}}\cdots l_{s+v_p}^{a_{s+v_p}}l_{s+v_p+1}^{c_1p}}{l_{s+n}^{b_{s+n}}h_{s+n-1}^{b_{s+n-1}}\cdots h_{s+u_p}^{b_{s+u_p}}h_{s+u_p-1}^{c_2p}}\\ &\leq \prod_{k=s}^{s+N}l_k^{a_k-b_k}\prod_{k=s}^{s+n-1}(l_k/h_k)^{b_k}\frac{h_s^{b_s}+\cdots+h_{s+u_p-1}^{b_{s+u_p-1}-c_2p}}{l_{s+v_p+1}^{a_s+v_p+1-c_1p}+\cdots+l_{s+n}^{a_{s+n}}}.\end{aligned}$$

By [15, Lem. 2] we know that

$$\prod_{k=s}^{s+N}l_k^{a_k-b_k}\prod_{k=s}^{s+n-1}(l_k/h_k)^{b_k}\leq A^N.$$

Then

$$|\varphi_{s,N}|\leq A^N\prod_{k=s}^{s+u_p-1}(h_k/l_k)^{b_k}\left(\frac{h_{s+u_p-1}}{l_{s+u_p-1}}\right)^{-c_{2p}}\frac{l_s^{b_s}+\cdots+l_{s+u_p-1}^{b_{s+u_p-1}-c_2p}}{l_{s+v_p+1}^{a_s+v_p+1-c_1p}+\cdots+l_{s+n}^{a_{s+n}}}.$$

By (3.6), $h_k/l_k\geq 1/A$ and by (3.9) $b_s+\cdots+b_{s+u_p-1}-c_{2p}=p$. So

$$\prod_{k=s}^{s+u_p-1}(h_k/l_k)^{b_k}\left(\frac{h_{s+u_p-1}}{l_{s+u_p-1}}\right)^{-c_{2p}}\leq\frac{1}{A^p}.$$

Since $l_{s+n}<l_{s+n-1}$ and by (3.7) $a_{s+v_p+1}+\cdots+a_{s+n}-c_{1p}=p$,

$$|\varphi_{s,N}|\leq A^{N-p}\frac{l_s^p}{l_{s+n}^p}.$$

Since $K(\Lambda)$ is uniformly perfect, there exists $B\in\mathbb{R}$ such that $l_s\leq Bl_{s+1}$. So $l_s\leq B^n l_{s+n}$. Since $N\geq 2^n$, $\log_2 N\geq n$. Then

$$|\varphi_{s,N}|\leq A^{N-p}B^{p\log_2 N},$$

which is the desired result. □

3.2 Interpolating Bases

Fix $s \in \mathbb{N}$. Let natural numbers n_{s-1}, n_s be given with $n_{s-1} \leq n_s$. Set $N_s = 2^{n_s}$ and $N_{s-1} = 2^{n_{s-1}}$. Given N with $1 \leq N \leq N_{s-1}$ we choose the points $(x_k^{(s-1)})_{k=1}^{N_{s-1}+1}$ on K_{s-1} and $(x_k)_{k=1}^{N}$ on K_s by the rule of increase of the type. As above, $\xi_{k,s-1}(f) = [x_1^{(s-1)}, \dots, x_{k+1}^{(s-1)}]f$, $e_{k,s-1}(x) = \prod_{j=1}^{k}(x - x_j^{(s-1)})|_{K_{s-1}}$ for $k = 1, 2, \dots, N_{s-1}$. Also let $e_N(y) = \prod_{j=1}^{N}(y - x_j)|_{K_s}$.

Lemma 3.2.1. *[15, Lem. 3] For fixed $f \in C(K(\Lambda))$, $x \in K_s$ let*

$$\begin{aligned} \tilde{\xi}(f) &= [x_1, \dots, x_N, x]f, \\ \tilde{\eta}(f) &= \tilde{\xi}(f) - \sum_{k=N}^{N_{s-1}} \tilde{\xi}(e_{k,s-1})\xi_{k,s-1}(f). \end{aligned}$$

Then

$$|\tilde{\eta}(f)e_N(x)| \leq N_{s-1}^4\, A^{2N_{s-1}}\, w(f, l_{s-1}).$$

In the case $K(\Lambda) = K^{(\alpha)}$ we have

$$|\tilde{\eta}(f)e_N(x)| \leq e^6\, N_{s-1}^4\, w(f, l_{s-1}),$$

provided the condition $N_s l_s^{\alpha-1} \leq 1$ is fulfilled.

Proof. By $\tilde{e}$ we denote the function $\tilde{e}(y) = (y-x)e_N(y)$. Then by (3.1), $|\xi_N(f)| \leq N^2 w(f,t)(\min_{k\leq N} |e'_{N_1}(x_k)|)^{-1}$. Since $e_N(x)/\tilde{e}'(x_j) = -L_{N,j}(x)$, Lemma 3.1.3 implies

$$|\tilde{\xi}(f)e_N(x)| \leq N^2\, A^{N-1}\, w(f, l_s). \tag{3.11}$$

The representation $\tilde{\xi}(e_{k,s-1}) = \dfrac{-e_{k,s-1}(x)}{e_N(x)} + \displaystyle\sum_{j=1}^{N} \dfrac{e_{k,s-1}(x_j)}{\tilde{e}'(x_j)}$ gives

$$\begin{aligned} |\tilde{\xi}(e_{k,s-1})\xi_{k,s-1}(f)e_N(x)| &\leq |\xi_{k,s-1}(f)e_{k,s-1}| \\ &\quad + \sum_{j=1}^{N} \frac{|e_{k,s-1}(x_j)|}{|\tilde{e}'(x_j)|} \cdot \frac{|e_N(x)|}{\min_{i\leq k} |e'_{k+1,s-1}(x_i)|}\, k^2 w(f, l_{s-1}). \end{aligned}$$

By (3.1) and Lemma 3.1.3, $|\xi_{k,s-1}(f)e_N(x)| \leq k^2 A^k w(f, l_{s-1})$. By Lemma 3.1.4, $\dfrac{|e_{k,s-1}(x_j)|}{|\tilde{e}'(x_j)|} \leq A^k$ and $\dfrac{|e_N(x)|}{\min_{i\leq k} |e'_{k+1,s-1}(x_i)|} \leq A^{N-1}$. Then we get

$$|\tilde{\xi}(e_{k,s-1})\xi_{k,s-1}(f)e_N(x)| \leq (1 + NA^{N-1})k^2 A^k w(f, l_{s-1}),$$

and

$$
\begin{aligned}
|\tilde{\eta}(f)e_N(x)| &= \left|\tilde{\xi}(f)e_N(x) - \sum_{k=N}^{N_{s-1}} \tilde{\xi}(e_{k,s-1})\xi_{k,s-1}(f)e_N(x)\right| \\
&\leq |\tilde{\xi}(f)e_N(x)| + \sum_{k=N}^{N_{s-1}} |\tilde{\xi}(e_{k,s-1})\xi_{k,s-1}(f)e_N(x)| \\
&\leq N^2\, A^{N-1}\, w(f, l_s) + \sum_{k=N}^{N_{s-1}} (1 + NA^{N-1})k^2 A^k w(f, l_{s-1}) \\
&\leq N^2\, A^{N-1}\, w(f, l_s) + (1 + NA^{N-1}) w(f, l_{s-1}) \sum_{k=N}^{N_{s-1}} k^2 A^k \\
&\leq N^2\, A^{N-1}\, w(f, l_s) + (1 + NA^{N-1}) w(f, l_{s-1}) A^{N_{s-1}} {N_{s-1}}^3 \\
&\leq N_{s-1}^4\, A^{2N_{s-1}}\, w(f, l_{s-1})
\end{aligned}
$$

which is the desired result. □

In the same manner we obtain the desired bound in the case $K(\Lambda) = K^{(\alpha)}$.

Lemma 3.2.2. *For fixed $f \in C^p(K(\Lambda))$, $x \in K_s$ let $\tilde{\xi}(f) = [x_1, \ldots, x_N, x]f$, $\tilde{\eta}(f) = \tilde{\xi}(f) - \sum_{k=N}^{N_{s-1}} \tilde{\xi}(e_{k,s-1})\xi_{k,s-1}(f)$. Then*

$$
|\tilde{\eta}(f)e_N^{(p)}(x)| \leq \frac{N_{s-1}^{2p+3}\, A^{2N_{s-1}-2p}\, B^{2p \log_2 N_{s-1}}\, w(f^{(p)}, l_{s-1})}{p!}.
$$

Proof. By Lemma 3.1.1,

$$
|\tilde{\xi}_N(f)| \leq \frac{N^2 w(f^{(p)}, l_{s-1})}{p!} \left(\min_{k \leq N-p,\, j \leq N} \left| \prod_{s=1,\, s \neq j}^{k} (x_j - x_s) \prod_{s=k+p+1,\, s \neq j}^{N+1} (x_j - x_s) \right| \right)^{-1}.
$$

By Lemma 3.1.2,

$$
|e_{N+1}^{(p)}| \leq N^p \max_{k \leq N} \left| \prod_{s=1}^{k} (x - x_s) \prod_{s=k+p+1}^{N+1} (x - x_s) \right|
$$

Then by Lemma 3.1.4, we get

$$
\begin{aligned}
|\tilde{\xi}_N(f)e_{N+1}^{(p)}| &\leq \frac{N^{p+2} w(f^{(p)}, l_{s-1})}{p!} \cdot \frac{|\varphi_{s,N}|}{\min_{j \leq N} |x - x_j|} \\
&\leq \frac{A^{N-p}\, B^{p \log_2 N}\, N^{p+2}\, w(f^{(p)}, l_{s-1})}{p!}. \qquad (3.12)
\end{aligned}
$$

The representation $\tilde{\xi}_{k,s-1} = -\frac{e_{k,s-1}(x)}{e_N(x)} + \sum_{j=1}^{N} \frac{e_{k,s-1}(x_j)}{\tilde{e}'(x_j)}$ gives

$$|\tilde{\xi}_{k,s-1}\xi_{k,s-1}(f)e_N(x)| \le |\xi_{k,s-1}(f)e_{k,s-1}(x)| + \sum_{j=1}^{N} \frac{|e_{k,s-1}(x_j)|}{|\tilde{e}'(x_j)|} \cdot \frac{e_N(x)}{\min_{i\le k}|e'_{k+1,s-1}(x_i)|} k^2 w(f^{(p)}, l_{s-1}). \quad (3.13)$$

Then the first term

$$|\xi_{k,s-1}(f)e_{k,s-1}(x)| \le \frac{A^{k-p}B^{p\log_2 k}k^{p+2}w(f^{(p)}, l_{s-1})}{p!}$$

by Lemma 3.1.1, Lemma 3.1.2 and Lemma 3.1.4. The parts of the two fractions in the second sum will be considered cross-wise. Applying Lemma 3.1.4 twice we get

$$|\tilde{\xi}_{k,s-1}\xi_{k,s-1}(f)e_N(x)| \le (1 + N^p A^{N-p} B^{p\log_2 N}) k^{p+2} A^{k-p} B^{p\log_2 N} w(\tilde{f}^{(p)}, l_{s-1}).$$

Clearly, $\sum k = 1^n k^{p+2} A^{k-p} \le n^{2p+3} A^{n-p}$ for $n \ge 2$. Summing over k we get the general estimation of $|\tilde{\eta} e_N(x)|$. □

The task is now to show that the biorthogonal system suggested in [15] as a basis for the space $C(K(\Lambda))$ forms a topological basis in the space $C^p(K(\Lambda))$ as well, provided a suitable choice of degrees of polynomials.

Given a nondecreasing sequence of natural numbers $(n_s)_0^\infty$, let $N_s = 2^{n_s}$, $M_s^{(l)} = N_{s-1}/2 + 1$, $M_s^{(r)} = N_{s-1}/2$ for $s \ge 1$ and $M_0 = 1$. Here, (l) and (r) mean left and right respectively. For any basic interval $I_{j,s} = [a_{j,s}, b_{j,s}]$ we choose the sequence of points $(x_{n,j,s})_{n=1}^\infty$ using the rule of increase of the type.

Let $e_{N,1,0} = \prod_{n=1}^N (x - x_{n,1,0}) = \prod_{n=1}^N (x - x_n)$ for $x \in K(\Lambda)$, $N = 0, 1, \ldots, N_0$. For $s \ge 1$, $j \le 2^s$ let $e_{n,j,s} = \prod_{n=1}^N (x - x_{n,j,s})$ if $x \in K(\Lambda) \cap I_{j,s}$ and $e_{n,j,s} = 0$ on $K(\Lambda)$ otherwise. Here, $N = M_s^{(a)}, M_s^{(a)}+1, \ldots, N_s$ with $a = l$ for odd j and $a = r$ if j is even. The functionals are given as follows: for $s = 0, 1, \ldots; j = 1, 2, \ldots, 2^s$ and $N = 0, 1, \ldots,$ let $\xi_{N,j,s}(f) = [x_{1,j,s}, \ldots, x_{N+1,j,s}]f$. Set $\eta_{N,1,0} = \xi_{N,1,0}$ for

$N \le N_0$. Every basic interval $I_{j,s},\ s \ge 1$, is a subinterval of a certain $I_{i,s-1}$ with $j = 2i-1$ or $j = 2i$. Let

$$\eta_{N,j,s}(f) = \xi_{N,j,s}(f) - \sum_{k=N}^{N_{s-1}} \xi_{N,j,s}(e_{k,i,s-1})\xi_{k,i,s-1}(f)$$

for $N = M_s^{(a)},\ M_s^{(a)}+1, \ldots, N_s$. As before $a = l$ if $j = 2i-1$, and $a = r$ if $j = 2i$.

In the space $C(K(\Lambda))$ there is no unconditional basis. Thus we have to enumerate the elements $(e_{N,j,s})_{s=0,\, j=1,\, N=M_s}^{\infty,\quad 2^s,\quad N_s}$ in a reasonable way. We arrange them by increasing the level s. Elements of the same level are ordered by increasing the degree, that is with respect to N. For fixed s and N the elements $e_{N,j,s}$ are ordered by increasing j, that is from left to right. In this way we introduce an injective function $\sigma : (N, j, s) \mapsto M \in \mathbb{N}$. At the beginning we have for zero level: $\sigma(0,1,0) = 1, \ldots, \sigma(N_0,1,0) = N_0+1$. Since the degree of the first element on $I_{1,1}$ is greater that on $I_{2,1}$, we start the first level from $e_{N_0/2,2,1} : \sigma(N_0/2,2,1) = N_0+2$, $\sigma(N_0/2+1,1,1) = N_0+3$, $\sigma(N_0/2+1,2,1) = N_0+4, \cdots, \sigma(N_1,2,1) = N_0+1+2(N_1-N_0/2)+1 = 2(N_1+1)$ and we finish all elements of the first level. For $s = 2$ we have two elements $e_{N_1/2,2,2}$, $e_{N_1/2,4,2}$ of the smaller degree, so they have a priority: $\sigma(N_1/2,2,2) = 2(N_1+1)+1$, $\sigma(N_1/2,4,2) = 2(N_1+1)+2$ Then $\sigma(N_1/2+1,1,2) = 2(N_1+1)+3$, $\sigma(N_1/2+1,2,2) = 2(N_1+1)+4, \cdots, \sigma(N_2,4,2) = 2(N_1+1)+4(N_2-N_1/2)+2 = 4(N_2+1)$. Continuing in this manner after completing of the s-th level we get the value $\sigma(N_s, 2^s, s) = 2^s(N_s+1)$.

By injectivity of the function σ there exists the inverse function σ^{-1}. Let $f_m = e_{\sigma^{-1}(m)},\ m \in \mathbb{N}$.

Theorem 3.2.3. *[15, Thm. 1] Let a Cantor-type set $K(\Lambda)$ satisfy (3.6). Then for any bounded sequence $(N_s)_0^\infty$ the system $(f_m)_1^\infty$ forms a Schauder basis in the space $C(K(\Lambda))$.*

Theorem 3.2.4. *Let a Cantor-type set $K(\Lambda)$ be a uniformly perfect set which satisfies (3.6). Then for any bounded sequence $(N_s)_0^\infty$ the system $(f_m)_1^\infty$ forms a Schauder basis in the space $C^p(K(\Lambda))$.*

Proof. Let $S_M(f,)$ be the M-th partial sum of the expansion of f with respect to the system $(f_m)_1^\infty$ for given $f \in C(K(\Lambda))$. Then $S_M(f,x) = \sum \eta_{N,j,s}(f)e_{N,j,s}(x)$, where the sum is taken over all N, j, s with $\sigma(N,j,s)\le M$. By Lemma 2.4.3, $S_M(f,x)=Q_{M-1}(f,(x_{n,1,0})_{n=1}^{M},x)$ if $1\le M\le N_0+1$. But the next function S_{N_0+2} is not a polynomial on $I_{1,0}$. The restriction of S_{N_0+2} to the interval $I_{1,1}$ is Q_{N_0}, whereas $S_{N_0+2}|_{I_{2,1}}=Q_{N_0}+\eta_{N_0/2,2,1}(f)e_{N_0/2,2,1}$. In both cases we get the polynomials of degree N_0 that interpolate f at $N_0/2+1$ points each. And always the subscript M gives the total number of points where S_M interpolates f.

If we continue this process, we see that the restriction of the function $S_{2^s(N_s+1)}$ to any interval $I_{j,s}$, $j=1,\dots,2^s$, coincides with $Q_{N_s}(f,(x_{n,j,s})_{n=1}^{N_s+1},\cdot)$. Then we add the next terms $\eta_\cdot(f)e_\cdot$ to $S_{2^s(N_s+1)}$ and we get certain polynomials of degree N_s that interpolate f at some points on the intervals $I_{j,s+1}$, $j=1,\dots,2^{s+1}$. Continuing in this way we get $S_{2^{s+1}N_s+1}$ that has a degree N_s on $I_{1,s+1}$ and interpolates f on this interval at N_s+1 points; so here it is the usual interpolating polynomial. Then the restriction of $S_{2^{s+1}(N_s+1)}$ to the interval $I_{j,s+1}$ gives $Q_{N_s}(f,(x_{n,j,s+1})_{n=1}^{N_s+1},\cdot)$ and $S_{2^{s+1}(N+1)}|_{I_{j,s+1}}$ produces $Q_N(f,(x_{n,j,s+1})_{n=1}^{N+1},x)$ for $N\ge N_s$. It will continue up to the value $N=N_{s+1}$, after which we do the next splitting.

Now we show that the expansion $f=\sum\eta_{N,j,s}(f)e_{N,j,s}(x)$ is unique. Suppose $\eta_{N,j,s}(f)=0$ for all N,j,s. Then, by considering step by step all triples $\sigma^{-1}(m)$, $m\in\mathbb{N}$, we get $\xi_{N,j,s}(f)=0$ for all N,j,s. Since the set of nodes of the corresponding divided differences is dense in $K(\Lambda)$, $f=0$ which means that the expansion $f=\sum\eta_{N,j,s}(f)e_{N,j,s}(x)$ is unique. Then, it is enough to check only the convergence of $S_M(f,\cdot)$ to f in the norm of the space $C^p(K(\Lambda))$.

Let $N_s\le D$ for $s\in\mathbb{N}_0$. Fix $f\in C^p(K(\Lambda))$, $\varepsilon>0$ and s_ε such that $w(\tilde f,l_{s_\varepsilon})\le D^{-2p-3}A^{-2D+2p}B^{-2\log_2 D}\varepsilon$. For any $M\ge M_\varepsilon$ we get $2^{s-1}(N_{s-1}+1)\le M\le 2^s(N_s+1)$ with $s\ge s_\varepsilon+1$.

Fix $x\in K(\Lambda)$. Without loss of generality let $x\in K(\Lambda)\cap[0,l_s]$. We have two cases:

For the first case if $2^{s-1}(N_{s-1}+1)\le M\le 2^sN_{s-1}$, then

$$S_M(f,x)=Q_{N_{s-1}}(f,(x_{n,1,s-1})_{n=1}^{N_{s-1}+1},x)+\sum\eta_{N,1,s}(f)e_{N,1,s}(x),\tag{3.14}$$

where the sum is taken over all $N,\ j,\ s$ with $2^{s-1}(N_{s-1}+1) < \sigma(N,j,s) \leq M$. Since the degree N_{s-1} will appear for the first time when $\sigma^{-1}(2^s N_{s-1}+1) = (N_{s-1}, 1, s)$, for the values N with (3.14) we have $N \leq N_{s-1} - 1$.

For the second case $2^s N_{s-1} + 1 \leq M \leq 2^s(N_s + 1)$ we get

$$S_M(f,x) = Q_N(f, (x_{n,1,s})_{n=1}^{N+1}, x)$$

with some $N,\ N_{s-1} \leq N \leq N_s$.

We prove the convergence of $S_M(f,\cdot)$ to f for these two cases and we start with the more simple second case. With the notation $\tilde{\xi}(f) = [x_{1,1,s}, \ldots, x_{N+1,1,s}, x]f$, we have the polynomial $\tilde{Q}_{N+1}(\cdot) = Q_N(\cdot) + \tilde{\xi}(f)e_{N+1,1,s}(\cdot)$ that interpolates f also at the point x. Therefore here

$$f(x) - S_M(f,x) = \tilde{\xi}(f)e_{N+1,1,s}(x). \tag{3.15}$$

Then for $k = 0, 1, \ldots, p$

$$|f(x) - S_M(f,x)|_k = \sup\{|(\tilde{\xi}(f)e_{N+1,1,s}(x))^{(i)}| : \ x \in K(\Lambda),\ i = 0, 1, \ldots, k\}. \tag{3.16}$$

By (3.12)

$$\begin{aligned} |\tilde{\xi}_N(f)e_{N+1}^{(i)}| &\leq \frac{A^{N_s - i}\, B^{k \log_2 N_s}\, N_s^{i+2}\, w(f^{(i)}, l_{s-1})}{i!} \\ &\leq \frac{A^{D-i}\, B^{i \log_2 D}\, D^{i+2}\, w(f^{(i)}, l_{s-1})}{i!} \\ &\leq \varepsilon. \end{aligned}$$

Here $\tilde{\xi}(f) = [x_{1,1,s}, \ldots, x_{N+1,1,s}, x]f$ also depends on x. We know that

$$\tilde{\xi}(f) = \frac{-f(x)}{e_N(x)} + \sum_{j=1}^{N} \frac{f(x_j)}{\tilde{e}'(x_j)}.$$

Then

$$\tilde{\xi}(f)e_{N+1,1,s}(x) = \frac{-f(x)e_{N+1,1,s}(x)}{e_N(x)} + \sum_{j=1}^{N} \frac{f(x_j)e_{N+1,1,s}(x)}{\tilde{e}'(x_j)}.$$

Since we show that the second part, it is enough to show that $\left(\frac{-f(x)e_{N+1,1,s}(x)}{e_N(x)}\right)^{(i)} \leq \varepsilon$. But

$$\left|\frac{f(x)e_{N+1,1,s}(x)}{e_N(x)}\right|^{(i)} \leq \varepsilon,$$

since we approximate in terms of the terms l_s.

If $2^{s-1}(N_{s-1}+1) \leq M \leq 2^s N_{s-1}$, then

$$|f(x) - S_M(f,x)| = |\tilde{\eta}(f) e_{R+1,1,s}(x)|. \tag{3.17}$$

Then by Lemma 3.2.1

$$\begin{aligned} |\tilde{\eta}(f) e^{(k)}_{R+1,1,s}(x)| &\leq \frac{N_{s-1}^{2k+3}\, A^{2N_{s-1}-2k}\, B^{2k \log_2 N_{s-1}}\, w(f^{(k)}, l_{s-1})}{k!} \\ &\leq \frac{D^{2p+3}\, A^{2D-2k}\, B^{2k \log_2 D}\, w(f^{(k)}, l_{s-1})}{k!} \\ &\leq \varepsilon. \end{aligned}$$

In the same way we can find a bound for the differentiation of $\tilde{\eta}(f)$. Therefore $|f(x) - S_M(f,x)|_k \leq \varepsilon$ for all $k = 1, 2, \ldots, p$ which is the desired conclusion. □

Chapter 4

Schauder Bases in the Spaces $C^p(K(\Lambda))$ and $\mathcal{E}^p(K(\Lambda))$

In this chapter, we construct a basis in the space of continuous functions and in Whitney spaces on Cantor type sets. Let $K \subset \mathbb{R}$ be a compact set, and let $f = (f^{(k)})_{0 \le k \le n} \in C^p(K)$. Let $K(\Lambda)$ be the Cantor set associated with the sequence Λ which we define in Section 2.3. In this chapter, by using the local Taylor expansions of functions, we construct a biorthogonal system. Then we show that this biorthogonal system is a Schauder basis both in the spaces $C^p(K(\Lambda))$ and $\mathcal{E}^p(K(\Lambda))$.

4.1 Local Taylor Expansions on Cantor-type Set $K(\Lambda)$

We consider the set of all left endpoints of basic intervals of our Cantor set $K(\Lambda)$. As we defined in Section 2.3, $K(\Lambda)$ is the Cantor set such that $K(\Lambda) = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0,1]$, E_s is a union of 2^s closed *basic* intervals $I_{j,\,s} = [a_{j,s}, b_{j,s}]$ of length l_s and E_{s+1} is obtained by deleting the open concentric subinterval of length $h_s := l_s - 2l_{s+1}$ from each $I_{j,\,s}$, $j = 1, 2, ...2^s$. Then $a_{j,s} = a_{2j-1,s+1}$ for

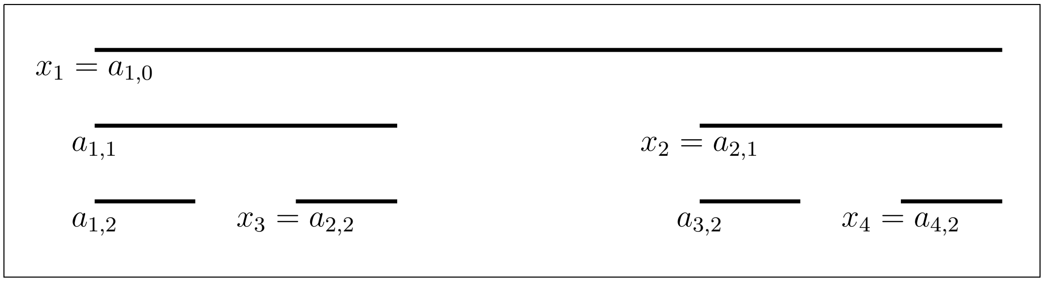


Figure 4.1: First steps of Cantor procedure and the points $a_{j,s}$

$j \le 2^s$. So any such point has infinitely many representations in the form $a_{j,s}$. We select the representation with the minimal second subscript and call it the *minimal representation.* If j is even, then the representation $a_{j,s}$ is minimal for the corresponding point. If j is odd, for $j = 2^q(2m+1)+1 > 1$, $a_{j,s} = a_{2m+2,s-q}$ where $m \in \mathbb{N}$ and for $j = 1$, $a_{1,s} = a_{1,0}$ for all s. As seen in the Figure 4.1 $a_{1,0} = a_{1,1} = a_{1,2}$ and $a_{2,1} = a_{3,2}$. Therefore we have a bijection between the set of all left endpoints of basic intervals and the set $A = a_{1,0} \cup (a_{2j,s})_{j=1,s=1}^{2^{s-1},\,\infty}$.

Let us enumerate the set A by first increasing s, then j. Then $x_1 = a_{1,0} = 0$, $x_2 = a_{2,1} = 1 - l_1$, $x_3 = a_{2,2} = l_1 - l_2$, $x_4 = a_{4,2} = 1 - l_2, \ldots$ and, in general, $x_{2^s+k} = a_{2k,s+1}$ for $k = 1, 2, \ldots, 2^s$.

Let us fix $p \in \mathbb{N}$. For $s \in \mathbb{N}_0$, $j \le 2^s$ and $0 \le k \le p$ let $e_{k,j,s}(x) = (x - a_{j,s})^k/k!$ if $x \in K(\Lambda) \cap I_{j,s}$ and $e_{k,j,\,s} = 0$ on $K(\Lambda)$ otherwise. Given $f = (f^{(k)})_{0\le k\le p} \in \prod_{0\le k\le p} C(K(\Lambda))$, let $\xi_{k,j,\,s}(f) = f^{(k)}(a_{j,s})$ stand for the same values of s, j, and k as above. For the fixed level s, the system $(e_{k,j,\,s}, \xi_{k,j,\,s})$ is biorthogonal, that is $\xi_{k,j,\,s}(e_{n,i,\,s}) = \delta_{kn}\cdot\delta_{ij}$. For example, for fixed s, $\xi_{2,4,\,s}(f) = f^{(2)}(a_{4,s})$ and $e_{3,4,s}(x) = (x - a_{4,s})^3/3!$ if $x \in K(\Lambda) \cap I_{4,s}$ and $e_{3,4,\,s} = 0$ on $K(\Lambda)$ otherwise. Then clearly $\xi_{2,4,\,s}(e_{3,4,\,s}) = 0$. Also $\xi_{2,4,\,s}(f) = f^{(2)}(a_{4,s})$ and $e_{2,2,s}(x) = (x - a_{2,s})^2/2!$ if $x \in K(\Lambda) \cap I_{2,s}$ and $e_{2,2,\,s} = 0$ on $K(\Lambda)$ otherwise, then $\xi_{2,4,\,s}(e_{2,2,\,s}) = 0$. Furthermore, $\xi_{k,j,\,s}$ takes zero value at all elements $(e_{k,i,\,n})_{k=0}^p$ with $n \ge s$, except $e_{k,j,\,s}$, where it equals 1. But when $n < s$, the system $(e_{k,j,\,n}, \xi_{k,j,\,s})$ is not biorthogonal. For example, $\xi_{2,4,\,3}(f) = f^{(2)}(a_{4,3})$ and $e_{2,2,2}(x) = (x - a_{2,2})^2/2!$ if $x \in K(\Lambda) \cap I_{2,2}$ and $e_{2,2,\,2} = 0$ on $K(\Lambda)$ otherwise, then $\xi_{2,4,\,3}(e_{2,2,\,2}) = 1$.

In order to obtain biorthogonality as well with regard to s, we will use the following convolution property of the values of functionals on the basis elements (see Section 2.4.4). Let $I_{i,n} \supset I_{j,s-1}$. Then

$$\sum_{m=k}^{p} \xi_{k,2j,s}(e_{m,j,s-1}) \cdot \xi_{m,j,s-1}(e_{q,i,n}) = \xi_{k,2j,s}(e_{q,i,n}) \quad \text{for all} \quad q \leq p.$$

Indeed, $(e_{k,i,n})_{k=0}^{p}, (e_{k,j,s-1})_{k=0}^{p}, (e_{k,2j,s})_{k=0}^{p}$ are three bases in the space $\mathcal{P}_p(I_{2j,s})$ of polynomials of degree not greater than p on the interval $I_{2j,s}$. If $M_{r\leftarrow t}$ denotes the transition matrix from the $t-$th basis to the $r-$th basis, then the identity above means $M_{3\leftarrow 2}M_{2\leftarrow 1} = M_{3\leftarrow 1}$.

On the other hand, in our case, this identity is the corresponding binomial expansion:

$$\sum_{m=k}^{q} \frac{(a_{2j,s} - a_{j,s-1})^{m-k}}{(m-k)!} \cdot \frac{(a_{j,s-1} - a_{i,n})^{q-m}}{(q-m)!} = \frac{(a_{2j,s} - a_{i,n})^{q-k}}{(q-k)!}. \tag{4.1}$$

Here we consider summation until q since for $q < m \leq p$, the terms $\xi_{m,j,s-1}(e_{q,i,n})$ vanish. We obtain (4.1) by the binomial expression

$$(x+y)^n = \sum_{m=0}^{n} \frac{n!}{(n-m)!m!} x^m y^{n-m}$$

where we take $x = a_{2j,s} - a_{j,s-1}$, $y = a_{j,s-1} - a_{i,n}$, $n = q-k$ and we change the index.

We restrict our attention only to the functions $(e_{k,1,0})_{k=0}^{p}$ and $(e_{k,2j,s})_{k=0,j=1,s=1}^{p,\ 2^{s-1},\ \infty}$ corresponding to the set A. Let us enumerate this family in the lexicographical order with respect to the triple (s, j, k) : $f_n = e_{n-1,1,0} = \frac{1}{(n-1)!}(x - x_1)^{n-1} \cdot \chi_{1,0}$ for $n = 1, 2, \cdots, p+1$. Here and in what follows, $\chi_{j,s}$ denotes the characteristic function of the interval $I_{j,s}$. After this, $f_n = e_{n-p-2,2,1} = \frac{1}{(n-p-2)!}(x-x_2)^{n-p-2} \cdot \chi_{2,1}$ for $n = p+2, p+3, \cdots, 2(p+1)$ and in general, if $(m-1)(p+1)+1 \leq n \leq m(p+1)$, then $f_n = \frac{1}{k!}(x - x_m)^k \cdot \chi_{2i,s+1} = e_{k,2i,s+1}$. Here $m = 2^s + i$ with $1 \leq i \leq 2^s$ and $k = n-(m-1)(p+1)-1$. We see that all functions of the type $\frac{1}{k!}(x-x_m)^k \cdot \chi_{2i,s+1}$ with $0 \leq k \leq p$ and $m = 2^s + i \in \mathbb{N}$ are included into the sequence $(f_n)_{n=1}^{\infty}$.

For the same values of parameters as above, we define the functionals $\eta_{k,1,0} =$

$\xi_{k,1,0}$ for $k = 0, 1, \cdots, p$ and

$$\eta_{k,2j,\,s} = \xi_{k,2j,\,s} - \sum_{m=k}^{p} \xi_{k,2j,\,s}(e_{m,j,\,s-1}) \cdot \xi_{m,j,\,s-1}$$

for $s \in \mathbb{N}$, $j = 1, 2, \cdots, 2^{s-1}$, and $k = 0, 1, \cdots, p$. In what follows, we will use the minimal representations of the points $a_{j,s}$ and the corresponding functionals $\xi_{m,j,\,s}$. For example, $\eta_{k,2,\,s} = \xi_{k,2,\,s} - \sum_{m=k}^{p} \xi_{k,2,\,s}(e_{m,1,0}) \cdot \xi_{m,1,0}$. This agreement is justified by the fact that the value $\xi_{m,j,\,s}(f) = f^{(m)}(a_{j,s})$ does not depend on the representation of the point $a_{j,s}$ and the functions $e_{m,j,\,s-1}$, $e_{m,r,\,s-q}$ coincide on the interval $I_{2j,s}$ if $a_{j,s-1} = a_{r,s-q}$.

The crucial point of the construction is that the functionals $\eta_{k,2j,\,s}$ are biorthogonal, not only to all elements $(e_{k,2j,\,s-1})_{k=0}^{p}$, but also, by the convolution property, to all $(e_{k,2i,\,n})_{k=0}^{p}$ with $n = 0, 1, \cdots, s-2$ and $i = 1, 2, \cdots, 2^{n-1}$. For example,

$$\eta_{2,4,\,3} = \xi_{2,4,\,3} - \sum_{m=2}^{p} \xi_{2,4,\,3}(e_{m,2,\,2}) \cdot \xi_{m,2,\,2} \quad \text{and} \quad e_{2,2,\,2} = \frac{(x - a_{2,2})^2 \cdot \chi_{2,2}}{2!}.$$

Then

$$\begin{aligned}
\eta_{2,4,\,3}(e_{2,2,\,2}) &= \xi_{2,4,\,3}(e_{2,2,\,2}) - \sum_{m=2}^{p} \xi_{2,4,\,3}(e_{m,2,\,2}) \cdot \xi_{m,2,\,2}(e_{2,2,\,2}) \\
&= 1 - \sum_{m=2}^{p} \frac{d}{dx^2}\left(\frac{(x - a_{2,2})^m}{m!}\right)_{x=a_{4,3}} \frac{d}{dx^m}\left(\frac{(x - a_{2,2})^2}{2!}\right)_{x=a_{4,3}} \\
&= 1 - \frac{d}{dx^2}\left(\frac{(x - a_{2,2})^2}{2!}\right)_{x=a_{4,3}} \frac{d}{dx^2}\left(\frac{(x - a_{2,2})^2}{2!}\right)_{x=a_{4,3}} = 1 - 1 = 0
\end{aligned}$$

Also

$$\begin{aligned}
\eta_{2,4,\,3}(e_{3,1,\,0}) &= \xi_{2,4,\,3}(e_{3,1,\,0}) - \sum_{m=2}^{p} \xi_{2,4,\,3}(e_{m,2,\,2}) \cdot \xi_{m,2,\,2}(e_{3,1,\,0}) \\
&= \xi_{2,4,\,3}(e_{3,1,\,0}) - \xi_{2,4,\,3}(e_{2,2,\,2}) \cdot \xi_{2,2,\,2}(e_{3,1,\,0}) - \xi_{2,4,\,3}(e_{3,2,\,2}) \cdot \xi_{3,2,\,2}(e_{3,1,\,0}) \\
&= \xi_{2,4,\,3}(e_{3,1,\,0}) - \xi_{2,2,\,2}(e_{3,1,\,0}) - \xi_{2,4,\,3}(e_{3,2,\,2}) \\
&= 6(a_{4,3} - a_{1,0}) - 6(a_{2,2} - a_{1,0}) - 6(a_{4,3} - a_{2,2}) = 0.
\end{aligned}$$

In addition, the functional $\eta_{k,2j,\,s}$ takes zero value at all elements $(e_{k,2i,\,n})_{k=0}^{p}$ with $n \geq s$, except $e_{k,2j,\,s}$, where it equals 1.

In the same lexicographical order as above, we arrange all functionals $(\eta_{k,1,0})_{k=0}^{p}$ and $(\eta_{k,2j,s})_{k=0,j=1,s=1}^{p,\ 2^{s-1},\ \infty}$ into the sequence $(\eta_n)_{n=1}^{\infty}$. That is, $\eta_n(f) = \xi_{n-1,1,0}$ for $n = 1, 2, \cdots, p+1$ and in general, $\eta_n(f) = \eta_{k,2i,\,s+1}$ where $k = n-(m-1)(p+1)-1$ and $m = 2^s + i$ with $1 \le i \le 2^s$.

Our next goal is to express the sum $S_N(f) := \sum_{n=1}^{N} \eta_n(f) \cdot f_n$ in terms of the Taylor polynomials of the function f. For $1 \le N \le p+1$,

$$S_N(f) = \sum_{n=1}^{N} \eta_n(f) \cdot f_n = \sum_{n=1}^{N} f^{(n-1)}(x_1) \cdot \frac{(x-x_1)^{n-1}}{(n-1)!} = T_0^{N-1} f.$$

Suppose $p+2 \le N \le 2(p+1)$. Then $S_N(f) = T_0^p f$ on $I_{1,1}$. On the interval $I_{2,1}$, we obtain

$$\begin{aligned} S_N(f) &= T_0^p f + \sum_{n=p+2}^{N} \eta_{n-p-2,2,1}(f) \cdot e_{n-p-2,2,1} \\ &= T_0^p f + \sum_{k=0}^{N-p-2} \left[\xi_{k,2,1}(f) - \sum_{m=k}^{p} \xi_{k,2,1}(e_{m,1,0}) \cdot \xi_{m,1,0}(f) \right] \frac{1}{k!} (x-a_{2,1})^k \\ &= T_0^p f + \sum_{k=0}^{N-p-2} \left[f^{(k)}(a_{2,1}) - \sum_{m=k}^{p} \frac{1}{(m-k)!} a_{2,1}^{m-k} \cdot f^{(m)}(0) \right] \frac{1}{k!} (x-a_{2,1})^k \\ &= T_0^p f + \sum_{k=0}^{N-p-2} (R_0^p f)^{(k)}(a_{2,1}) \frac{1}{k!} (x-a_{2,1})^k = T_0^p f + T_{a_{2,1}}^{N-p-2}(R_0^p f). \end{aligned}$$

Therefore, $S_N(f) = T_0^p f$ on $I_{1,1}$ and $S_N(f) = T_0^p f + T_{a_{2,1}}^{N-p-2}(R_0^p f)$ on $I_{2,1}$. Particularly, $S_{2p+2}(f) = T_0^p f + T_{a_{2,1}}^p(R_0^p f) = T_0^p f + T_{a_{2,1}}^p(f - T_0^p f) = T_0^p f + T_{a_{2,1}}^p(f) - T_0^p f = T_{a_{2,1}}^p f$, by (2.2). In addition, $S_N^{(k)}(f, a_{2,1}) = f^{(k)}(a_{2,1})$ for $0 \le k \le N-p-2$, as is easy to check.

Continuing in this way, the values $2p+3 \le N \le 3(p+1)$ correspond to the passage on the interval $I_{2,2}$. On $I_{1,2}$, $S_N(f) = T_0^p f$. On $I_{2,2}$, $S_N(f) = T_0^p f + T_{a_{2,2}}^{N-2p-3}(R_0^p f)$. On $I_{3,2}$ and $I_{4,2}$, $S_N(f) = T_{a_{2,1}}^p f$.

As seen in the Figure 4.2, we have three different sets.

For example, the values $5(p+1)+1 \le N \le 6(p+1)$ (since $a_{4,3} = x_6$) correspond the passage on the interval $I_{4,3}$. On the interval $I_{4,3}$, $S_N(f) = T_{a_{2,2}}^p f +$

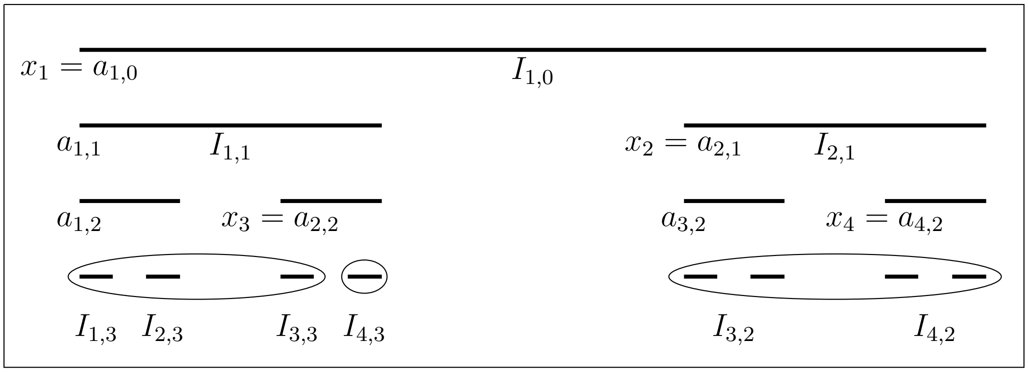


Figure 4.2: Decomposition of $K(\Lambda)$ into three different sets

$T_{a_{4,3}}^{N-5p-6}(R_{a_{2,2}}^p f)$. On the intervals $I_{k,3}$ where $k = 1, 2, 3$, $S_N(f) = T_{a_{k,3}}^p f$, that is, on $I_{1,3}$, $S_N(f) = T_{a_{1,3}}^p f = T_{a_{1,0}}^p f$; on $I_{2,3}$, $S_N(f) = T_{a_{2,3}}^p f$ and on $I_{3,3}$, $S_N(f) = T_{a_{3,3}}^p f = T_{a_{2,2}}^p f$. On the interval $I_{3,2}$, $S_N(f) = T_{a_{3,2}}^p f = T_{a_{2,1}}^p f$ and on $I_{4,2}$, $S_N(f) = T_{a_{4,2}}^p f$.

Combining all considerations of this section yields the following result:

Lemma 4.1.1. *[21, Lem. 1] The system $(f_n, \eta_n)_{n=1}^{\infty}$ is biorthogonal. Given $f = (f^{(k)})_{0 \le k \le p} \in \prod_{0 \le k \le p} C(K(\Lambda))$ and $N = 2^s(p+1) + j(p+1) + m + 1$ with $s \in \mathbb{N}_0, 0 \le j < 2^s$, and $0 \le m \le p$ we have $S_N(f) = T_{a_{k,s+1}}^p f$ on $I_{k,s+1}$ with $k = 1, 2, \cdots, 2j+1$, $S_N(f) = T_{a_{k,s}}^p f$ on $I_{k,s}$ with $k = j+2, j+3, \cdots, 2^s$, and $S_N(f) = T_{a_{j+1,s}}^p f + T_{a_{2j+2,s+1}}^m (R_{a_{j+1,s}}^p f)$ on $I_{2j+2,s+1}$.*

4.2 Schauder Bases in the Spaces $C^p(K(\Lambda))$ and $\mathcal{E}^p(K(\Lambda))$

In this section we show that the biorthogonal system suggested in Section 4.1 is a Schauder basis in both spaces $C^p(K(\Lambda))$ and $\mathcal{E}^p(K(\Lambda))$. Here, as before, $p \in \mathbb{N}$. Given g on $K(\Lambda)$, let $\omega(g, \cdot)$ be the modulus of continuity of g, that is

$$\omega(g,t) = \sup\{\, |\, g(x) - g(y)\, | : x, y \in K(\Lambda),\ |x-y| \le t\},\ t > 0.$$

If $x \in I = [a, a + l_s]$, then for any $i \leq p$ we have

$$|(R_a^p f)^{(i)}(x)| = \left| f^{(i)}(x) - f^{(i)}(a) - \sum_{k=1}^{p-i} f^{(i+k)}(a) \frac{(x-a)^k}{k!} \right| < \omega(f^{(i)}, l_s) + l_s \cdot 2\,|f|_p \tag{4.2}$$

and

$$|(R_a^p f)^{(i)}(x)| < 4\,|f|_p. \tag{4.3}$$

Lemma 4.2.1. *[21, Lem. 2] The system $(f_n, \eta_n)_{n=1}^{\infty}$ is a Schauder basis in the space $C^p(K(\Lambda))$.*

Proof. Given $f \in C^p(K(\Lambda))$ and $\varepsilon > 0$, we want to find N_ε with $|f - S_N(f)|_p \leq \varepsilon$ for $N \geq N_\varepsilon$. Let us take S such that for all $i \leq p$ we have

$$3 \cdot \omega(f^{(i)}, l_S) + 14 \cdot l_S \cdot |f|_p < \varepsilon. \tag{4.4}$$

Set $N_\varepsilon = 2^S(p+1)$. Then any $N \geq N_\varepsilon$ has a representation in the form $N = 2^s(p+1) + j(p+1) + m + 1$ with $s \geq S$, $0 \leq j < 2^s$, and $0 \leq m \leq p$. Let us fix $i \leq p$ and apply Lemma 4.1.1 to $R := (f - S_N(f))^{(i)}(x)$ for $x \in K(\Lambda)$.

If $x \in I_{k,s+1}$ with $k = 1, \cdots, 2j+1$, then $|R| = |(f - T^p_{a_{k,s+1}} f)^{(i)}(x)| = |(R^p_{a_{k,s+1}} f)^{(i)}(x)| < \varepsilon$, by (4.2) and (4.4).

If $x \in I_{k,s}$ with $k = j+2, j+3, \cdots, 2^s$, then $|R| = |(f - T^p_{a_{k,s}} f)^{(i)}(x)| = |(R^p_{a_{k,s}} f)^{(i)}(x)| < \varepsilon$, by (4.2) and (4.4).

Suppose $x \in I_{2j+2,s+1}$. Then

$$\begin{aligned} |R| &= |(f - T^p_{a_{j+1,s}} f - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f))^{(i)}(x)| \\ &\leq |(R^p_{a_{j+1,s}} f)^{(i)}(x)| + |(T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f))^{(i)}(x)|. \end{aligned}$$

For the first term we use (4.2). Second term vanishes if $m < i$. Otherwise, it is

$$\left| (R^p_{a_{j+1,s}} f)^{(i)}(x) - (R^p_{a_{j+1,s}} f)^{(i)}(a_{2j+2,s+1}) - \sum_{k=i+1}^{m} (R^p_{a_{j+1,s}} f)^{(k)}(a_{2j+2,s+1}) \frac{(x - a_{2j+2,s+1})^{k-i}}{(k-i)!} \right|.$$

Here, we estimate the terms $(R^p_{a_{j+1,s}} f)^{(i)}(x)$, $(R^p_{a_{j+1,s}} f)^{(i)}(a_{2j+2,s+1})$ by means of (4.2). For the remaining sum, we use (4.3):

$$\left| \sum_{k=i+1}^{m} (R^p_{a_{j+1,s}} f)^{(k)}(a_{2j+2,s+1}) \frac{(x - a_{2j+2,s+1})^{k-i}}{(k-i)!} \right| \leq 4\,|f|_p \sum_{k=i+1}^{m} \frac{l_{s+1}^{k-i}}{(k-i)!} < l_{s+1} \cdot 8\,|f|_p.$$

Combining these we conclude that $|R| \leq 3(\omega(f^{(i)}, l_s) + l_s \cdot 2\,|f|_p) + l_{s+1} \cdot 8\,|f|_p$. This does not exceed ε due to the choice of S. Therefore, $|f - S_N(f)|_p \leq \varepsilon$ for $N \geq N_\varepsilon$. □

The main result is given for Cantor-type sets under mild restriction:

$$\exists\ C_0 :\ \ l_s \leq C_0 \cdot h_s, \qquad \text{for} \quad s \in \mathbb{N}_0. \tag{4.5}$$

Theorem 4.2.2. *[21, Thm 3] Let $K(\Lambda)$ satisfy (4.5). Then the system $(f_n, \eta_n)_{n=1}^{\infty}$ is a Schauder basis in the space $\mathcal{E}^p(K(\Lambda))$.*

Proof. Given $f \in \mathcal{E}^p(K(\Lambda))$, we show that the sequence $(S_N(f))$ converges to f as well in the norm

$$\|\, f \,\|_p = |f|_p + \sup\left\{|(R_y^p f)^{(k)}(x)| \cdot |x - y|^{k-p};\ x, y \in K, x \neq y, k = 0, 1, ...p\right\}.$$

Because of Lemma 4.2.1, we only have to check that the second term of the norm $|(R_y^p(f - S_N(f)))^{(i)}(x)| \cdot |x - y|^{i-p}$ is uniformly small (with respect to $x, y \in K$ with $x \neq y$ and $i \leq p$) for large enough N. Fix $\varepsilon > 0$. Due to Whitney Theorem (condition (2.1)), we can take S such that

$$|(R_y^p f)^{(k)}(x)| < \varepsilon\ |x - y|^{p-k} \quad \text{for } k \leq p \quad \text{and} \quad x, y \in K(\Lambda) \ \text{ with } \ |x - y| \leq l_S. \tag{4.6}$$

As above, let $N_\varepsilon = 2^S(p + 1)$ and $N = 2^s(p + 1) + j(p + 1) + m + 1$ with $s \geq S, 0 \leq j < 2^s$, and $0 \leq m \leq p$.

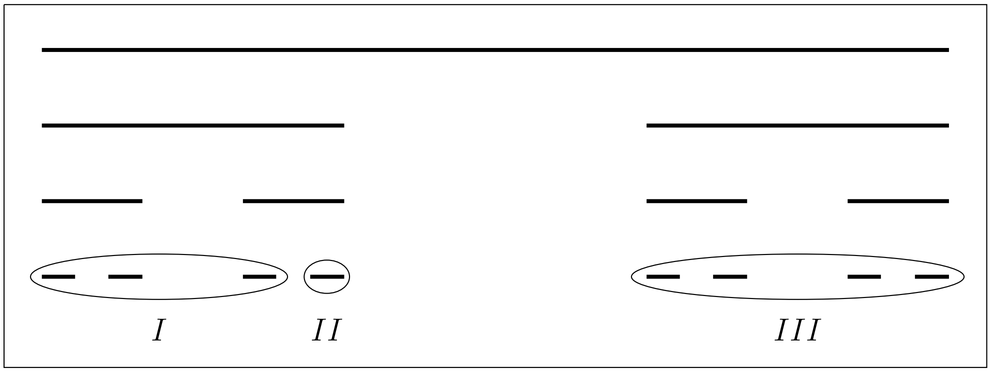
For simplicity, we take the value $i = 0$ since the general case can be analyzed in the same manner. We will consider different positions of x and y on $K(\Lambda)$ in order to show

$$|R_y^p(f - S_N(f))(x)| < C\ \varepsilon\ |x - y|^p,$$

where the constant C does not depend on x and y. As seen in the following figure we have three different sets on $K(\Lambda)$ and we will consider the positions of x and y according to these sets. In all cases, we use the representation of

$$S_N(f) = \begin{cases} T^p_{a_{k,s+1}} f & x \in I_{k,s+1} \ \text{ and } \ k = 1, 2, \cdots, 2j + 1 \\ T^p_{a_{k,s}} f & x \in I_{k,s} \ \text{ and } \ k = j + 2, \cdots, 2^s \\ T^p_{a_{j+1,s}} f + T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f) & x \in I_{2j+2,s+1} \end{cases}$$

given in Lemma 4.1.1.



Suppose first that x, y belong to the same interval $I_{k,s+1}$ with some $k = 1, \ldots, 2j+1$. Then $(f - S_N(f))(x) = R^p_{a_{k,s+1}} f(x)$. From (2.2) it follows that $R^p_y(f - S_N(f))(x) = R^p_y(R^p_{a_{k,s+1}} f(x))(x) = R^p_y f(x)$. Then since $|x - y| \leq l_{s+1}$, by (4.6)

$$|R^p_y(f - S_N(f))(x)| = |R^p_y f(x)| \leq \varepsilon \ |x - y|^p$$

which is the desired bound.

Also for the case $x, y \in I_{k,s}$ with $k = j+2, j+3, \cdots, 2^s$, $(f - S_N(f))(x) = R^p_{a_{k,s}} f(x)$ and $R^p_y(f - S_N(f))(x) = R^p_y(R^p_{a_{k,s}} f(x))(x) = R^p_y f(x)$. Here, $|x - y| \leq l_s$, so as before (4.6) can be applied.

If $x, y \in I_{2j+2,s+1}$, then $(f - S_N(f))(x) = (R^p_{a_{j+1,s}} f)(x) - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)(x)$ for $m < p$ and $(f - S_N(f))(x) = (R^p_{a_{2j+2,s+1}} f)(x)$ for $m = p$. Since $R^p(T^m) = 0$ for $m < p$, in both cases we get $R^p_y(f - S_N(f))(x) = R^p_y f(x)$ with $|x - y| \leq l_s$ and (4.6) can be applied once again.

We now turn to the cases when x and y lie on different intervals. Let $x \in$

$I_{k,s+1}, y \in I_{m,s+1}$ with distinct $k, m = 1, \cdots, 2j+1$. Then

$$\begin{aligned}
R^p_y(f - S_N(f))(x) &= R^p_y(f - T^p_{a_{k,s+1}} f)(x) \\
&= R^p_y(R^p_{a_{k,s+1}})(x) \\
&= R^p_{a_{k,s+1}} f(x) - \sum_{i=0}^{p} (R^p_{a_{m,s+1}} f)^{(i)}(y) \frac{(x-y)^i}{i!} \\
&\le \varepsilon \cdot l^p_{s+1} + \varepsilon \cdot \sum_{i=0}^{p} l^{p-i}_{s+1} \frac{|x-y|^i}{i!} \\
&\le \varepsilon \cdot (C_o|x-y|)^p + \varepsilon \cdot \sum_{i=0}^{p} \frac{(C_o|x-y|)^{p-i}\ |x-y|^i}{i!} \\
&< C_0^p(e+1) \cdot \varepsilon \cdot |x-y|^p.
\end{aligned}$$

Here, $|x - a_{k,s+1}| \le l_{s+1}$, $|y - a_{m,s+1}| \le l_{s+1}$ and by hypothesis $|x-y| \ge h_s \ge C_0^{-1} l_s$. Also by (4.6),

$$|R^p_{a_{k,s+1}} f(x)| < \varepsilon\, |x - a_{k,s+1}|^p < \varepsilon\, l^p_{s+1}$$

and

$$|(R^p_{a_{m,s+1}} f)^{(i)}(y)| < \varepsilon\, |y - a_{m,s+1}|^{p-i} < \varepsilon\, l^{p-i}_{s+1}.$$

As a result, $|R^p_y(f - S_N(f))(x)| < C_0^p(e+1) \cdot \varepsilon \cdot |x-y|^p$, which establishes the desired result. Clearly, the same conclusion can be drawn for $x \in I_{k,s}, y \in I_{m,s}$ with distinct $k, m = j+2, \cdots, 2^s$, as well for the case when one of the points x, y belongs to $I_{k,s+1}$ with $k \le 2j+1$ whereas another lies on $I_{m,s}$ with $m = j+2, \cdots, 2^s$.

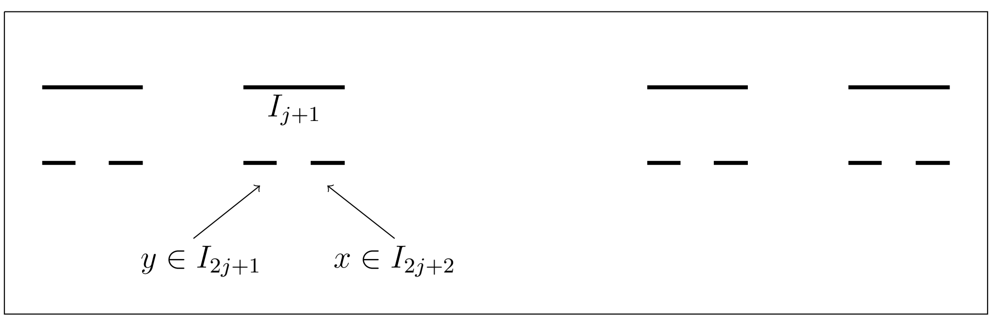


Figure 4.3: The case $x \in I_{2j+2,s+1}$ and $y \in I_{2j+1,s+1}$.

It remains to consider the most difficult cases: just one of the points x, y belongs to $I_{2j+2,s+1}$. Suppose $x \in I_{2j+2,s+1}$. We can assume that $y \in I_{2j+1,s+1}$

since other positions of y only enlarge $|x-y|$. Let $g = f - S_N(f)$. Then

$$g(x) = (f - T^p_{a_{j+1,s}} f - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f))(x) = (R^p_{a_{j+1,s}} f - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f))(x),$$

since $x \in I_{2j+2,s+1}$. Also since $y \in I_{2j+1,s+1}$ and $a_{j+1} = a_{2j+1}$,

$$g(y) = (f - T^p_{a_{j+1,s}})(y) = (R^p_{a_{2j+1,s}} f))(y).$$

Then,

$$\begin{aligned} R^p_y(f - S_N(f))(x) &= R^p_y\, g(x) \\ &= g(x) - \sum_{i=0}^{p} \frac{g^{(i)}(y)(x-y)^i}{i!} \\ &= R^p_{a_{j+1,s}} f(x) - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)(x) \\ &\quad - \sum_{i=0}^{p} \frac{(R^p_{a_{2j+1,s+1}} f)^{(i)}(y)(x-y)^i}{i!} \end{aligned}$$

We only need to estimate the intermediate T^m since other terms can be handled in the same way as above. Now,

$$\begin{aligned} |T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)(x)| &\leq \sum_{i=0}^{m} \frac{|(R^p_{a_{j+1,s}} f)^{(i)}(a_{2j+2,s+1})|\ |x - a_{2j+2,s+1}|^i}{i!} \\ &\leq \sum_{i=0}^{m} \frac{\varepsilon\, |a_{2j+2,s+1} - a_{j+1,s}|^{p-i}\ |x - a_{2j+2,s+1}|^i}{i!} \\ &\leq \sum_{i=0}^{m} \frac{\varepsilon\, C_0^{p-i} |x-y|^{p-i}\ C_0^i |x-y|^i}{i!} \\ &\leq C_0^p\, e\, \varepsilon\ |x-y|^p. \end{aligned}$$

Since by (4.6), $|(R^p_{a_{j+1,s}} f)^{(i)}(a_{2j+2,s+1})| \leq \varepsilon\, |a_{2j+2,s+1} - a_{j+1,s}|^{p-i}$. In addition, $|a_{2j+2,s+1} - a_{j+1,s}|$ and $|x - a_{2j+2,s+1}|$ do not exceed $C_0\, |x-y|$. As a result we obtain $|T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)(x)| \leq C_0^p\, e\, \varepsilon\ |x-y|^p$ which is the desired result.

In the last case $x \in I_{2j+1,s+1},\ y \in I_{2j+2,s+1}$. Then since $x \in I_{2j+1,s+1}$,
$g(x) = (f - T^p_{a_{2j+1,s}} f)(x) = R^p_{a_{2j+1,s}} f(x)$ and since $y \in I_{2j+2,s+1}$,
$g(y) = (f - T^p_{a_{j+1,s}} - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f))(y) = (R^p_{a_{2j+1,s}} f - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f))(y)$.

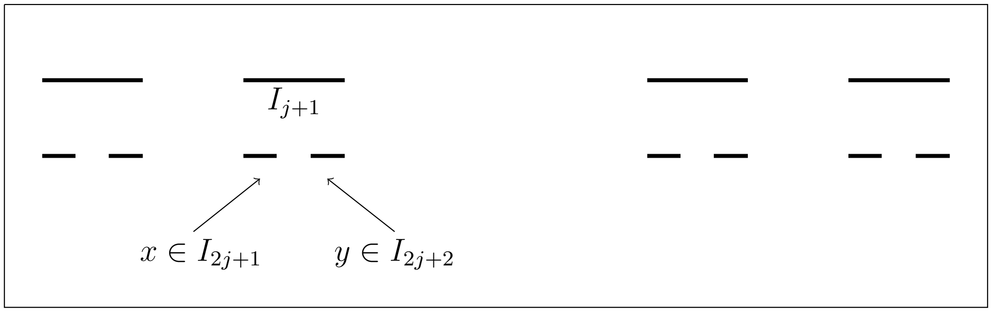


Figure 4.4: The case $y \in I_{2j+2,s+1}$ and $x \in I_{2j+1,s+1}$.

Then we have,

$$\begin{aligned} R_y^p(f - S_N(f))(x) &= R_y^p\, g(x) \\ &= g(x) - \sum_{i=0}^{p} \frac{g^{(i)}(y)(x-y)^i}{i!} \\ &= R^p_{a_{j+1,s}} f(x) - \sum_{i=0}^{p} \frac{(R^p_{a_{j+1,s}} f - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f))^{(i)}(y)(x-y)^i}{i!} \end{aligned}$$

As above, it is sufficient to consider only $\sum_{i=0}^{p} [T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)]^{(i)}(y)(x-y)^i/i!$ since for other terms we have the desired bound. Since T^m term vanishes when $i > m$, the genuine summation here is until $i = m$. Then

$$\begin{aligned} &\sum_{i=0}^{m} \frac{[T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)]^{(i)}(y)(x-y)^i}{i!} \\ &\quad \leq \sum_{i=0}^{m} \frac{|x-y|^i}{i!} \cdot \sum_{k=i}^{m} \frac{|R^p_{a_{j+1,s}} f(y)|^{(k)} \, |y - a_{2j+2,s+1}|^{k-i}}{(k-i)!} \\ &\quad \leq \sum_{i=0}^{m} \frac{|x-y|^i}{i!} \cdot \sum_{k=i}^{m} \frac{\varepsilon \, |y - a_{j+1,s}|^{p-k} \, |l_{s+1}|^{k-i}}{(k-i)!} \\ &\quad \leq \sum_{i=0}^{m} \frac{|x-y|^i}{i!} \cdot \sum_{k=i}^{m} \frac{\varepsilon \, l_s^{p-k} \, l_{s+1}^{k-i}}{(k-i)!} \\ &\quad \leq \sum_{i=0}^{m} \frac{|x-y|^i \, \varepsilon \, l_s^{p-i} \, e}{i!} \\ &\quad \leq C_0^p \, e^2 \, \varepsilon \, |x-y|^p. \end{aligned}$$

Since by (4.6), $|(R^p_{a_{j+1,s}} f)^{(k)}(y)| \leq \varepsilon \, |y - a_{j+1,s}|^{p-k}$. In addition, $|y - a_{2j+2,s+1}| \leq l_{s+1}$ and l_s do not exceed $C_0 \, |x-y|$. As a result we obtain the desired result. Then the proof is complete. □

Now we give two remarks. One is about whether our construction works in the space $\mathcal{E}(K(\Lambda))$. The other remark about the restriction in our theorem.

Remark 1. This biorthogonal system will not have the basis property in the space $\mathcal{E}(K(\Lambda))$. Let us enumerate all functions from $(e_{k,1,0})_{k=0}^{\infty} \cup (e_{k,2j,s})_{k=0,j=1,s=1}^{\infty,\ 2^{s-1},\ \infty}$ and the corresponding functionals η into a biorthogonal sequence $(f_n, \eta_n)_{n=1}^{\infty}$ in such way that for some increasing sequences $(N_p)_{p=0}^{\infty}, (q_p)_{p=0}^{\infty}$ the sum $S_{N_p}(f) = \sum_{n=1}^{N_p} \eta_n(f) \cdot f_n$ coincides with $T_{a_{j,p}}^{q_p} f$ on $I_{j,p}$ for $1 \leq j \leq 2^p$. The sequence $(f_n, \eta_n)_{n=1}^{\infty}$ will not have the basis property in the space $\mathcal{E}(K(\Lambda))$. Let $F \in C^{\infty}[0,1]$ solve the Borel problem for the sequence $(n!\, l_n^{-n})_{n=0}^{\infty}$, that is $F^{(n)}(0) = n!\, l_n^{-n}$ for $n \in \mathbb{N}_0$. Let $f = F|_{K(\Lambda)}$. Since $S_{N_p}(f) = \sum_{n=1}^{N_p} \eta_n(f) \cdot f_n$ coincides with $T_{a_{j,p}}^{q_p} f$ on $I_{j,p}$ for $1 \leq j \leq 2^p$, $(f - S_{N_p}(f))(x) = (f - T_{a_{j,p}}^{q_p} f)(x) = R_{a_{j,p}}^{q_p} f(x)$. Then

$$
\begin{aligned}
|f - S_{N_p}(f)|_0 = |R_{a_{j,p}}^{q_p} f|_0 \geq |R_0^{q_p} f(l_p)| &\geq \left| f(l_p) - \sum_{k=0}^{q_p} f^{(k)}(0) \frac{l_p^k}{k!} \right| \\
&\geq \left| f(l_p) - f(0) - \sum_{k=1}^{q_p} k! l_k^{-k} \frac{l_p^k}{k!} \right| \quad \text{since} \quad f^{(k)}(0) = k! l_k^{-k} \\
&> 1 - |f(l_p) - f(0)|
\end{aligned}
$$

The last expression has a limit 1 as $p \to \infty$, so $S_N(f)$ does not converge to f in $|\cdot|_0$.

For a basis in the space $\mathcal{E}(K(\Lambda))$, see [6].

Remark 2. As we notice in Section 2.4.3, the natural triangulations in Jonsson's paper [13] are given by the sequence $\mathcal{F}_s = \{I_{i,s},\ 1 \leq i \leq 2^s\}$, $s \geq 0$ for Cantor set $K(\Lambda)$. The regularity conditions in Jonsson's paper [13] satisfied when $c_2 l_i \leq l_{i+1} \leq c_3 l_i$, that is,

$$
\liminf_{s \to \infty} \frac{l_{s+1}}{l_s} > 0. \tag{4.7}
$$

Thus, [13, Prop. 2], for the Cantor set $K(\Lambda)$ provided these conditions, the expansion of $f \in \mathcal{E}^p(K(\Lambda))$ with respect to Jonsson's interpolating system converges, at least in $|\cdot|_p$, to f. It is interesting to check the corresponding convergence in topology given by the norm $\|\cdot\|_p$. Our construction can be applied to any small Cantor set with arbitrary fast decrease of the sequence $(l_s)_{s=0}^{\infty}$. The basis problem for the space $\mathcal{E}^p(K(\Lambda))$ in the case of large Cantor set with $l_s/h_s \to \infty$ is open.

Chapter 5

Schauder Bases in the Spaces $C^p(K_\infty(\Lambda))$ and $\mathcal{E}^p(K_\infty(\Lambda))$

Let $K \subset \mathbb{R}$ be a compact set, and let $f = (f^{(k)})_{0\leq k\leq n} \in C^p(K)$ be defined as in Section 2.1. In this chapter, we construct a basis in the space of continuous functions and in the Whitney space and also on $K_\infty(\Lambda)$, the Cantor set associated with the sequence Λ which we defined in Section 2.3. We use the same method as in Chapter 4. In the same way, by using the local Taylor expansions of functions, we construct a biorthogonal system. Then we show that this biorthogonal system is a Schauder basis both in the space $C^p(K_\infty(\Lambda))$ and $\mathcal{E}^p(K_\infty(\Lambda))$.

5.1 Local Taylor Expansions on $K_\infty(\Lambda)$

Let $(N_s)_{s=0}^\infty$ be an increasing sequence such that $N_s \to \infty$ as $s \to \infty$. Let $\Lambda = (l_s)_{s=0}^\infty$ be a sequence such that $l_0 = 1$ and $0 < N_{s+1}l_{s+1} \leq l_s$ for $s \in \mathbb{N}_0 := \{0, 1, \cdots\}$. Let $K_\infty(\Lambda)$ be the Cantor set associated with the sequence Λ that is $K_\infty(\Lambda) = \bigcap_{s=0}^\infty E_s$, where $E_0 = I_{1,0} = [0, 1]$, E_s is a union of $\prod_{i=0}^s N_i$ closed basic intervals $I_{j,s} = [a_{j,s}, b_{j,s}]$ of length l_s and E_{s+1} is obtained by deleting $N_{s+1} - 1$ open concentric subintervals of length $h_s := \frac{l_s - N_{s+1}l_{s+1}}{N_{s+1}-1}$ from each $I_{j,s}$, $j = 1, 2, \ldots, \prod_{i=0}^s N_i$.

Let $\phi_s = \prod_{i=0}^{s} N_i$ and $\tau_s(x) = [\![\frac{x}{N_s}]\!]$ where $[\![\cdot]\!]$ denote the greatest integer function. We use these expressions many times in our work.

Let us consider the set of all left endpoints of basic intervals. Since $a_{j,s} = a_{N_{s+1}(j-1)+1,s+1}$ for $j \leq \phi_s$, any such point has infinitely many representations in the form $a_{j,s}$. We select the representation with the minimal second subscript and call it the *minimal representation*. Let $B_s = \{j : 1 \leq j \leq \phi_s,\ j \neq N_s t+1,\ 0 \leq t < \phi_{s-1},\ j,t \in \mathbb{N}\}$. For $s \geq 1$ the representation $a_{j,s}$ where $j \in B_s$, is minimal for the corresponding point. For example, as seen in the following figure for $N_s = s+1$ the points $a_{1,0}$, $a_{2,1}$, $a_{2,2}$, $a_{3,2}$, $a_{5,2}$, $a_{6,2}$, $a_{2,3}$, $a_{3,3}, \cdots$ are minimal. Therefore we have a bijection between the set of all left endpoints of basic intervals and the set $A = a_{1,0} \bigcup (a_{j,s})_{s=1, j=1, j\in B_s}^{\infty,\ \ \phi_s}$.

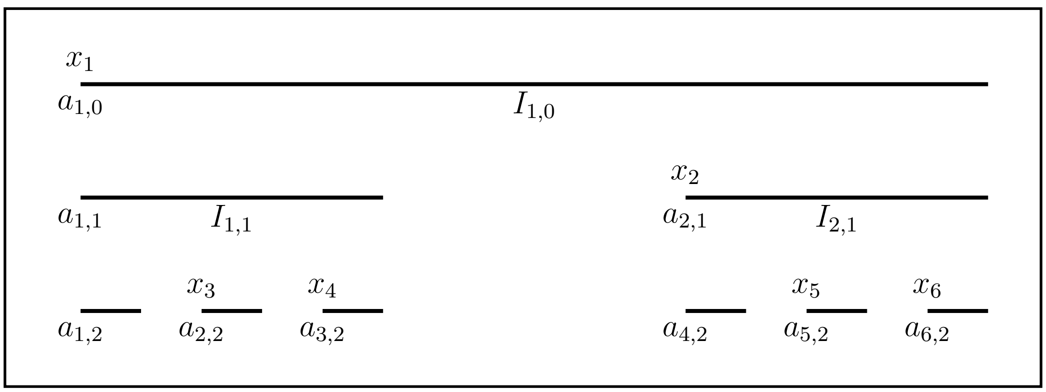


Figure 5.1: First two steps of Cantor procedure with $N_s = s+1$

Let us enumerate the set A by first increasing s, then j: $x_1 = a_{1,0} = 0$, $x_2 = a_{2,1}$, and, in general, $x_{\phi_{s-1}+k} = a_{\tau_s(k+\tau_s(k))+k+1,s}$ for $k = 1, 2, \cdots, \phi_s - \phi_{s-1}$. As seen in the figure above, for $N_s = s+1$, $x_1 = a_{1,0} = 0$, $x_2 = a_{2,1}$, $x_3 = a_{2,2}$, $x_4 = a_{3,2}$, $x_5 = a_{5,2}$, $x_6 = a_{6,2}, \ldots$.

Let us fix $p \in \mathbb{N}$. For $s \in \mathbb{N}_0$, $j \leq \phi_s$ and $0 \leq k \leq p$ let $e_{k,j,s}(x) = (x - a_{j,s})^k / k!$ if $x \in K(\Lambda) \cap I_{j,s}$ and $e_{k,j,s} = 0$ on $K(\Lambda)$ otherwise. Given $f = (f^{(k)})_{0 \leq k \leq p} \in C^p(K(\Lambda))$, let $\xi_{k,j,s}(f) = f^{(k)}(a_{j,s})$ stand for the same values of s, j, and k as above. For the fixed level s, the system $(e_{k,j,s}, \xi_{k,j,s})$ is biorthogonal, that is $\xi_{k,j,s}(e_{n,i,s}) = \delta_{kn} \cdot \delta_{ij}$. For example, for $N_s = s+1$, $\xi_{3,5,2}(f) = f^{(3)}(a_{5,2})$ and $e_{3,3,2}(x) = (x - a_{3,2})^3/3!$ if $x \in K(\Lambda) \cap I_{3,2}$ and $e_{3,2,2} = 0$ on $K(\Lambda)$ otherwise, then clearly $\xi_{3,5,2}(e_{3,3,2}) = 0$. Also $\xi_{3,5,2}(f) = f^{(3)}(a_{5,2})$ and $e_{4,5,2}(x) = (x -$

$a_{5,2})^4/4!$ if $x \in K(\Lambda) \cap I_{5,2}$ and $e_{4,5,2} = 0$ on $K(\Lambda)$ otherwise, then $\xi_{3,5,2}(e_{4,5,2}) = 0$. Furthermore, $\xi_{k,j,s}$ takes zero value at all elements $(e_{k,i,n})_{k=0}^{p}$ with $n \geq s$, except $e_{k,j,s}$, where it equals 1. But when $n < s$, the system $(e_{k,j,n}, \xi_{k,j,s})$ is not biorthogonal. For example, for $N_s = s+1$ $\xi_{3,5,2}(f) = f^{(3)}(a_{5,2})$ and $e_{3,2,1}(x) = (x - a_{2,1})^3/3!$ if $x \in K(\Lambda) \cap I_{2,1}$ and $e_{3,2,1} = 0$ on $K(\Lambda)$ otherwise, then $\xi_{3,5,2}(e_{3,2,1}) = 1$.

In order to obtain biorthogonality as well with regard to s, we will use the convolution property of the values of functionals on the basis elements as in Section 4.1. Let $I_{i,n} \supset I_{j,s-1}$. Then

$$\sum_{m=k}^{p} \xi_{k,j,s}(e_{m,\tau_s(j)+1,s-1}) \cdot \xi_{m,\tau_s(j)+1,s-1}(e_{q,i,n}) = \xi_{k,j,s}(e_{q,i,n}) \quad \text{for all} \quad q \leq p.$$

We define the functionals

$$\eta_{k,j,s} = \xi_{k,j,s} - \sum_{m=k}^{p} \xi_{k,j,s}(e_{m,\tau_s(j)+1,s-1}) \cdot \xi_{m,\tau_s(j)+1,s-1}$$

for $s \in \mathbb{N}$, $j = 1, 2, \cdots, \phi_s$, $j \in B_s$, and $k = 0, 1, \cdots, p$. Now the functionals $\eta_{k,j,s}$ are biorthogonal, not only to all elements $(e_{k,j,s-1})_{k=0}^{p}$ but also, by the convolution property (see Section 2.4.4), to all $(e_{k,j,n})_{k=0}^{p}$ with $n = 0, 1, \cdots, s-2$ and $j = 1, 2, \cdots, \phi_n$, $j \in B_s$. In addition, the functionals $\eta_{k,j,s}$ takes zero value at all elements $(e_{k,j,n})_{k=0}^{p}$ with $n \geq s$, except $e_{k,j,s}$ where it equals 1. For example, for $N_s = s+1$,

$$\begin{aligned}
\eta_{3,5,2}(e_{3,2,1}) &= \xi_{3,5,2}(e_{3,2,1}) - \sum_{m=3}^{p} \xi_{3,5,2}(e_{m,2,1}) \cdot \xi_{m,2,1}(e_{3,2,1}) \\
&= 1 - \xi_{3,5,2}(e_{3,2,1}) \cdot \xi_{3,2,1}(e_{3,2,1}) = 1 - 1 = 0.
\end{aligned}$$

Also

$$\begin{aligned}
\eta_{3,5,2}(e_{5,1,0}) &= \xi_{3,5,2}(e_{5,1,0}) - \sum_{m=3}^{p} \xi_{3,5,2}(e_{m,2,1}) \cdot \xi_{m,2,1}(e_{5,1,0}) \\
&= \frac{(a_{5,2} - a_{1,0})^2}{2} - \xi_{3,5,2}(e_{3,2,1}) \cdot \xi_{3,2,1}(e_{5,1,0}) \\
&\quad - \xi_{3,5,2}(e_{4,2,1}) \cdot \xi_{4,2,1}(e_{5,1,0}) - \xi_{3,5,2}(e_{5,2,1}) \cdot \xi_{5,2,1}(e_{5,1,0}) \\
&= \frac{(a_{5,2} - a_{1,0})^2}{2} - \frac{(a_{2,1} - a_{1,0})^2}{2} - (a_{5,2} - a_{2,1})(a_{2,1} - a_{1,0}) \\
&\quad - \frac{(a_{5,2} - a_{2,1})^2}{2} \\
&= 0
\end{aligned}$$

Here, we will use the minimal representations of the points $a_{j,s}$ and the corresponding functionals $\xi_{m,j,\,s}$.

We restrict our attention only to the functions $(e_{k,1,0})_{k=0}^{p}$ and $(e_{k,2j,s})_{k=0,j=1,s=1}^{p,\ 2^{s-1},\ \infty}$ corresponding to the set A. As in Chapter 4 we enumerate this family in the lexicographical order with respect to the triple (s,j,k) : $f_n = e_{n-1,1,0} = \frac{1}{(n-1)!}(x-x_1)^{n-1}\cdot\chi_{1,0}$ for $n=1,2,\cdots,p+1$. Here and in what follows, $\chi_{j,s}$ denotes the characteristic function of the interval $I_{j,s}$. After this, $f_n = e_{n-p-2,2,1} = \frac{1}{(n-p-2)!}(x-x_2)^{n-p-2}\cdot\chi_{2,1}$ for $n=p+2,p+3,\cdots,2(p+1)$ and in general, if $(m-1)(p+1)+1\le n\le m(p+1)$, then $f_n = \frac{1}{k!}(x-x_m)^k\cdot\chi_{\tau_s(i+\tau_s(i))+i+1,s} = e_{k,\tau_s(i+\tau_s(i))+i+1,\,s}$. Here $m=\phi_{s-1}+i$ with $1\le i\le\phi_s$ and $k=n-(m-1)(p+1)-1$. We see that all functions of the type $\frac{1}{k!}(x-x_m)^k\cdot\chi_{\tau_s(i+\tau_s(i))+i+1,s}$ with $0\le k\le p$ and $m=\phi_{s-1}+i\in\mathbb{N}$ are included into the sequence $(f_n)_{n=1}^{\infty}$.

In the same lexicographical order as above, we arrange all functionals $(\eta_{k,j,s})_{k=0,s=1,j=1,j\in B_s}^{p,\ \ \infty,\ \ \phi_s}$ into the sequence $(\eta_n)_{n=1}^{\infty}$.

Our next goal is to express the sum $S_N(f) := \sum_{n=1}^{N}\eta_n(f)\cdot f_n$ in terms of the Taylor polynomials of the function f. Clearly, $S_N(f)=T_0^{N-1}f$ for $1\le N\le p+1$. Suppose $p+2\le N\le 2(p+1)$. Then $S_N(f)=T_0^pf$ on $I_{1,1}$. On the interval $I_{2,1}$, we obtain

$$
\begin{aligned}
S_N(f) &= T_0^pf + \sum_{n=p+2}^{N}\eta_{n-p-2,2,1}(f)\cdot e_{n-p-2,2,1}\\
&= T_0^pf + \sum_{k=0}^{N-p-2}\left[\xi_{k,2,1}(f) - \sum_{m=k}^{p}\xi_{k,2,1}(e_{m,1,0})\cdot\xi_{m,1,0}(f)\right]\frac{1}{k!}(x-a_{2,1})^k\\
&= T_0^pf + \sum_{k=0}^{N-p-2}\left[f^{(k)}(a_{2,1}) - \sum_{m=k}^{p}\frac{1}{(m-k)!}\,a_{2,1}^{m-k}\cdot f^{(m)}(0)\right]\frac{1}{k!}(x-a_{2,1})^k\\
&= T_0^pf + \sum_{k=0}^{N-p-2}(R_0^pf)^{(k)}(a_{2,1})\,\frac{1}{k!}(x-a_{2,1})^k = T_0^pf + T_{a_{2,1}}^{N-p-2}(R_0^pf).
\end{aligned}
$$

Therefore, $S_N(f)=T_0^pf$ on $I_{1,1}$ and $S_N(f)=T_0^pf+T_{a_{2,1}}^{N-p-2}(R_0^pf)$ on $I_{2,1}$. Particularly, $S_{2p+2}(f) = T_0^pf + T_{a_{2,1}}^p(R_0^pf) = T_0^pf + T_{a_{2,1}}^p(f-T_0^pf) = T_0^pf + T_{a_{2,1}}^p(f) - T_0^pf = T_{a_{2,1}}^pf$, by (2.2). In addition, $S_N^{(k)}(f,a_{2,1}) = f^{(k)}(a_{2,1})$ for $0\le k\le N-p-2$, as is easy to check.

Continuing in this way, the values $2p+3 \leq N \leq 3(p+1)$ correspond to the passage on the interval $I_{3,1}$ or $I_{2,2}$ from the polynomial $T_0^p f$ to the polynomial $T^p_{a_{3,1}} f$ or $T^p_{a_{2,2}} f$. As seen in the following figure, here we have four different sets. We explain this by an example. Let $N_s = s+1$. The values $12(p+1)+1 \leq N \leq 13(p+1)$ correspond to the passage on the interval $I_{10,3}$. On the interval $I_{10,3}$ (set 2 in the Figure 5.2) $S_N(f) = T^p_{a_{3,2}} f + T^{N-12p-13}_{a_{10,3}}(R^p_{a_{3,2}} f)$. On the intervals $I_{k,3}$ with $k = 1, ..., 9$ (set 1 in the Figure 5.2), we have $S_N(f) = T^p_{a_{k,3}} f$. On the intervals $I_{11,3}$ and $I_{12,3}$ (set 3 in the Figure 5.2), we have $S_N(f) = T^p_{a_{3,2}} f$. On the intervals $I_{k,2}$ with $k = 4, 5, 6$ (set 4 in the Figure 5.2), we have $S_N(f) = T^p_{a_{k,2}} f$.

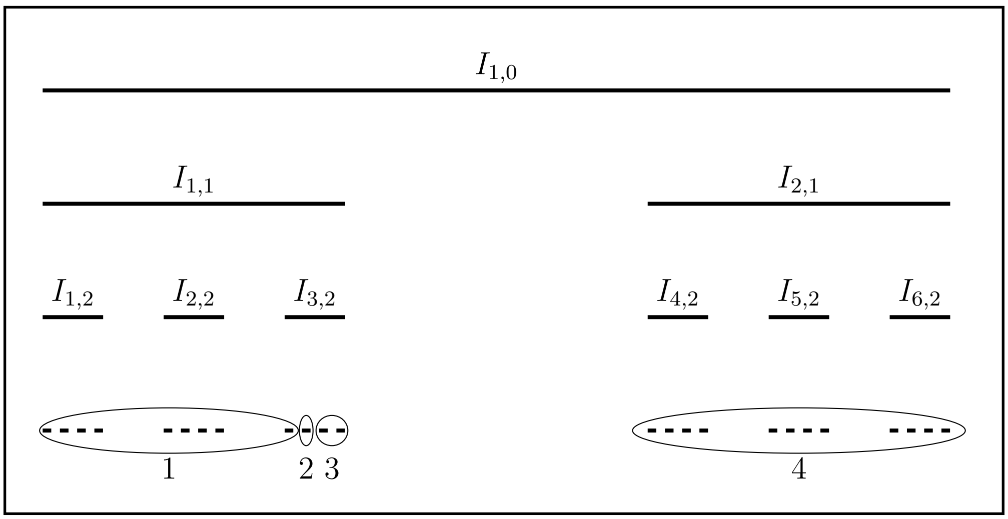


Figure 5.2: Decomposition of Cantor procedure into four different sets with $N_s = s+1$

Combining all considerations of this section yields the following result:

Lemma 5.1.1. *[29] The system $(f_n, \eta_n)_{n=1}^{\infty}$ is biorthogonal. Let $t = \tau_{s+1}(j)+1$. Given $f = (f^{(k)})_{0\leq k\leq p} \in \prod_{0\leq k\leq p} C(K_\infty(\Lambda))$ and $N = \phi_s(p+1) + j(p+1) + m + 1$ with $s \in \mathbb{N}_0, 0 \leq j < \phi_s$, and $0 \leq m \leq p$ we have*

$$S_N(f) = \begin{cases} T^p_{a_{k,s+1}} f & x \in I_{k,s+1} \ \ and \ \ k = 1, 2, \cdots, j-1 \\ T^p_{a_{k,s}} f & x \in I_{k,s} \ \ and \ \ k = t+1, \cdots, \phi_s \\ T^p_{a_{t,s}} f + T^m_{a_{j,s+1}}(R^p_{a_{t,s}} f) & x \in I_{j,s+1} \ \ and \ \ j \in B_s \\ T^p_{a_{t,s}} f & x \in I_{k,s+1} \ \ and \ \ k = j+1, \cdots, tN_{s+1} \end{cases} \tag{5.1}$$

5.2 Schauder Bases in the Spaces $C^p(K_\infty(\Lambda))$ and $\mathcal{E}^p(K_\infty(\Lambda))$

In this section we show that the biorthogonal system suggested in Section 5.1 is a Schauder basis in both spaces $C^p(K_\infty(\Lambda))$ and $\mathcal{E}^p(K_\infty(\Lambda))$. Here, as before, $p \in \mathbb{N}$. Given g on $K(\Lambda)$, let $\omega(g, \cdot)$ be the modulus of continuity of g, that is

$$\omega(g,t) = \sup\{\, |\, g(x) - g(y)\,| : x, y \in K(\Lambda),\ |x-y| \le t\},\ t > 0.$$

If $x \in I = [a, a + l_s]$, then for any $i \le p$ we have

$$|(R^p_a f)^{(i)}(x)| < \omega(f^{(i)}, l_s) + l_s\, 2\, |f|_p \tag{5.2}$$

and

$$|(R^p_a f)^{(i)}(x)| < 4\, |f|_p. \tag{5.3}$$

Lemma 5.2.1. *[29] The system $(f_n, \eta_n)_{n=1}^\infty$ is a Schauder basis in the space $C^p(K(\Lambda))$.*

Proof. Given $f \in C^p(K(\Lambda))$ and $\varepsilon > 0$, we want to find N_ε with $|f - S_N(f)|_p \le \varepsilon$ for $N \ge N_\varepsilon$. Let us take S such that for all $i \le p$ we have

$$3\,\omega(f^{(i)}, l_S) + 14\, l_S\, |f|_p < \varepsilon. \tag{5.4}$$

Set $N_\varepsilon = \phi_S(p+1)$. Then any $N \ge N_\varepsilon$ has a representation in the form $N = \phi_s(p+1) + j(p+1) + m + 1$ with $s \ge S$, $0 \le j < \phi_s$, and $0 \le m \le p$. Let us fix $i \le p$ and apply Lemma 5.1.1 to $R := (f - S_N(f))^{(i)}(x)$ for $x \in K(\Lambda)$.

If $x \in I_{k,s+1}$ with $k = 1, \cdots, j-1$, then $|R| = |(f - T^p_{a_{k,s+1}} f)^{(i)}(x)| = |(R^p_{a_{k,s+1}} f)^{(i)}(x)| < \varepsilon$, by (5.2) and (5.4).

If $x \in I_{k,s}$ with $k = t+1, t+2, \cdots, \phi_s$, then $|R| = |(f - T^p_{a_{k,s}} f)^{(i)}(x)| = |(R^p_{a_{k,s}} f)^{(i)}(x)|$ and the same arguments can be used.

If $x \in I_{k,s+1}$ with $k = j+1, j+2, \cdots, tN_{s+1}$, then $|R| = |(f - T^p_{a_{k,s+1}} f)^{(i)}(x)| = |(R^p_{a_{k,s+1}} f)^{(i)}(x)|$ and the same arguments can be used.

Suppose $x \in I_{j,s+1}$, $j \in B_s$. Then

$$\begin{aligned}|R| &= |(f - T^p_{a_{t,s}} f - T^m_{a_{j,s+1}}(R^p_{a_{t,s}} f))^{(i)}(x)| \\ &\le |(R^p_{a_{t,s}} f)^{(i)}(x)| + |(T^m_{a_{j,s+1}}(R^p_{a_{t,s}} f))^{(i)}(x)|.\end{aligned}$$

For the first term we use (5.2). The second term vanishes if $m < i$. Otherwise, it is

$$\left|(R^p_{a_{t,s}} f)^{(i)}(x) - (R^p_{a_{t,s}} f)^{(i)}(a_{2j+2,s+1}) - \sum_{k=i+1}^{m} (R^p_{a_{t,s}} f)^{(k)}(a_{j,s+1}) \frac{(x - a_{j,s+1})^{k-i}}{(k-i)!}\right|.$$

Here, we estimate the first and the second terms by means of (5.2). For the remaining sum, we use (5.3):

$$\left|\sum_{k=i+1}^{m} (R^p_{a_{t,s}} f)^{(k)}(a_{j,s+1}) \frac{(x - a_{j,s+1})^{k-i}}{(k-i)!}\right| \le 4\,|f|_p \sum_{k=i+1}^{m} \frac{l_{s+1}^{k-i}}{(k-i)!} < l_{s+1} \cdot 8\,|f|_p.$$

Combining these we conclude that $|R| \le 3(\omega(f^{(i)}, l_s) + l_s \cdot 2\,|f|_p) + l_{s+1} \cdot 8\,|f|_p$. This does not exceed ε due to the choice of S. Therefore, $|f - S_N(f)|_p \le \varepsilon$ for $N \ge N_\varepsilon$. □

Also for this problem the main result is given for Cantor-type sets under restriction:

$$\exists\, C_0 : \ \ l_s \le C_0 \cdot h_s, \qquad \text{for} \quad s \in \mathbb{N}_0. \tag{5.5}$$

Theorem 5.2.2. *[29] Let $K(\Lambda)$ satisfy (5.5). Then the system $(f_n, \eta_n)_{n=1}^{\infty}$ is a Schauder basis in the space $\mathcal{E}^p(K_\infty(\Lambda))$.*

Proof. Given $f \in \mathcal{E}^p(K(\Lambda))$, we show that the sequence $(S_N(f))$ converges to f as well in the norm

$$\| f \|_p = |f|_p + \sup\left\{|(R^p_y f)^{(k)}(x)| \cdot |x - y|^{k-p};\ x, y \in K, x \ne y, k = 0, 1, ...p\right\}.$$

Because of Lemma 5.2.1, we only have to check that the second term of the norm $|(R^p_y(f - S_N(f)))^{(i)}(x)| \cdot |x - y|^{i-p}$ is uniformly small (with respect to $x, y \in K$ with $x \ne y$ and $i \le p$) for large enough N. Fix $\varepsilon > 0$. Due to Whitney Theorem (condition (2.1)), we can take S such that

$$|(R^p_y f)^{(k)}(x)| < \varepsilon\ |x - y|^{p-k} \quad \text{for } k \le p \quad \text{and} \quad x, y \in K_\infty(\Lambda) \ \text{with} \ \ |x - y| \le l_S. \tag{5.6}$$

As above, let $N_\varepsilon = \phi_S(p+1)$ and $N = \phi_s(p+1) + j(p+1) + m + 1$ with $s \geq S, 0 \leq j < \phi_s$, and $0 \leq m \leq p$.

For simplicity, we take the value $i = 0$ since the general case can be analyzed in the same manner. We will consider different positions of x and y on $K_\infty(\Lambda)$ in order to show

$$|R_y^p(f - S_N(f))(x)| < C\ \varepsilon\ |x-y|^p,$$

where the constant C does not depend on x and y. As seen in the following figure we have four different sets on $K_\infty(\Lambda)$ and we will consider the positions of x and y according to these sets. In all cases, we use the representation of

$$S_N(f) = \begin{cases} T^p_{a_{k,s+1}} f & x \in I_{k,s+1} \text{ and } k = 1, 2, \cdots, j-1 \\ T^p_{a_{k,s}} f & x \in I_{k,s} \text{ and } k = t+1, \cdots, \phi_s \\ T^p_{a_{t,s}} f + T^m_{a_{j,s+1}}(R^p_{a_{t,s}} f) & x \in I_{j,s+1} \text{ and } j \in B_s \\ T^p_{a_{t,s}} f & x \in I_{k,s+1} \text{ and } k = j+1, \cdots, tN_{s+1} \end{cases}$$

given in Lemma 5.1.1.

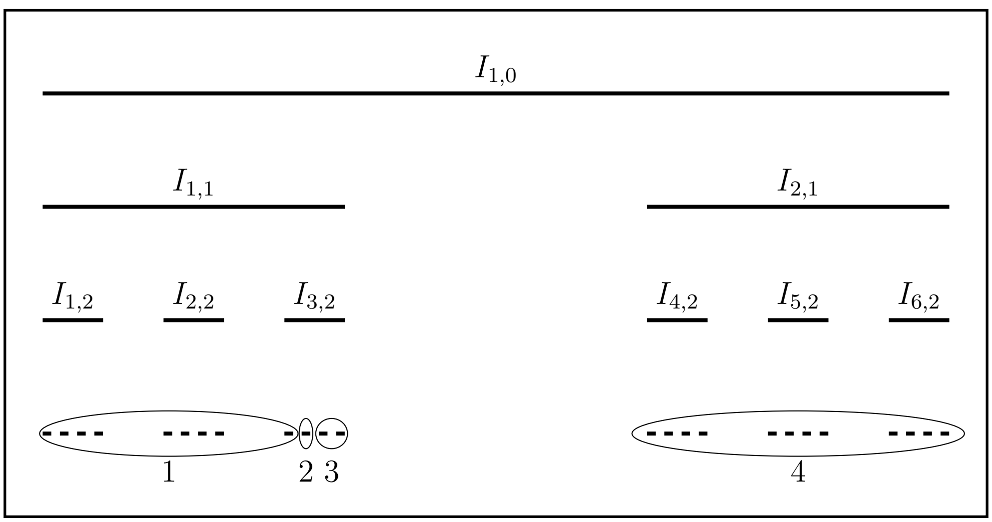


Figure 5.3: Decomposition of Cantor procedure into four different sets with $N_s = s+1$

Suppose first that x, y belong to the same interval $I_{k,s+1}$ with some $k = 1, \cdots, j-1$. Then $(f - S_N(f))(x) = R^p_{a_{k,s+1}} f(x)$. From (2.2) it follows that $R^p_y(f - S_N(f))(x) = R^p_y(R^p_{a_{k,s+1}} f(x))(x) = R^p_y f(x)$. Here, $|x-y| \leq l_{s+1}$, so we have the desired bound by (5.6).

Also for the case $x, y \in I_{k,s}$ with $k = t+1, j+3, \cdots, \phi_s$, $(f - S_N(f))(x) = R^p_{a_{k,s}} f(x)$ and $R^p_y(f - S_N(f))(x) = R^p_y(R^p_{a_{k,s}} f(x))(x) = R^p_y f(x)$. Here, $|x-y| \leq l_s$, so (5.6) can be applied.

For the case $x, y \in I_{k,s+1}$ with $k = j+1, j+3, \cdots, tN_{s+1}$, $(f - S_N(f))(x) = R^p_{a_{k,s+1}} f(x)$ and $R^p_y(f - S_N(f))(x) = R^p_y(R^p_{a_{k,s+1}} f(x))(x) = R^p_y f(x)$. Here, $|x-y| \leq l_{s+1}$, so (5.6) can be applied.

If $x, y \in I_{j,s+1}$, then

$$(f - S_N(f))(x) = \begin{cases} (R^p_{a_{j+1,s}} f)(x) - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)(x) & \text{for } m < p \\ (R^p_{a_{2j+2,s+1}} f)(x) & \text{for } m = p \end{cases}$$

Since $R^p(T^m) = 0$ for $m < p$, in both cases we get $R^p_y(f - S_N(f))(x) = R^p_y f(x)$ with $|x-y| \leq l_s$ and (5.6) can be applied once again.

We now turn to the cases when x and y lie on different intervals. Let $x \in I_{k,s+1}, y \in I_{m,s+1}$ with distinct $k, m = 1, \cdots, j-1$. Then

$$\begin{aligned} R^p_y(f - S_N(f))(x) &= R^p_y(f - T^p_{a_{k,s+1}} f)(x) \\ &= R^p_y(R^p_{a_{k,s+1}})(x) \\ &= R^p_{a_{k,s+1}} f(x) - \sum_{i=0}^{p} (R^p_{a_{m,s+1}} f)^{(i)}(y) \frac{(x-y)^i}{i!} \\ &\leq \varepsilon\, l^p_{s+1} + \varepsilon \sum_{i=0}^{p} l^{p-i}_{s+1} \frac{|x-y|^i}{i!} \\ &\leq \varepsilon\, (C_o|x-y|)^p + \varepsilon \sum_{i=0}^{p} \frac{(C_o|x-y|)^{p-i}\ |x-y|^i}{i!} \\ &< C^p_0(e+1)\, \varepsilon\, |x-y|^p. \end{aligned}$$

Here, $|x - a_{k,s+1}| \leq l_{s+1}$, $|y - a_{m,s+1}| \leq l_{s+1}$ and by hypothesis $|x-y| \geq h_s \geq C_0^{-1} l_s$. Also by (5.6),

$$|R^p_{a_{k,s+1}} f(x)| < \varepsilon\, |x - a_{k,s+1}|^p < \varepsilon\, l^p_{s+1}$$

and

$$|(R^p_{a_{m,s+1}} f)^{(i)}(y)| < \varepsilon\, |y - a_{m,s+1}|^{p-i} < \varepsilon\, l^{p-i}_{s+1}.$$

As a result, $|R^p_y(f - S_N(f))(x)| < C^p_0(e+1) \cdot \varepsilon \cdot |x-y|^p$, which establishes the desired result.

Also, for the cases $x \in I_{k,s}, y \in I_{m,s}$ with distinct $k, m = t+1, \cdots, \phi_s$ and $x \in I_{k,s+1}, y \in I_{m,s+1}$ with distinct $k, m = j+1, \cdots, tN_{s+1}$, same conclusion can be drawn. Since for the cases $R_y^p(f - S_N(f))(x) = R_y^p(R_{a_{k,s}}^p)(x)$ and $R_y^p(f - S_N(f))(x) = R_y^p(R_{a_{k,s+1}}^p)(x)$, respectively and we can apply same procedure.

It remains to consider the most difficult cases. Let $g = f - S_N(f)$.

Case 1: Suppose $x \in I_{j,s+1}$ where $j \in B_s$ and $y \in I_{j-1,s+1}$.

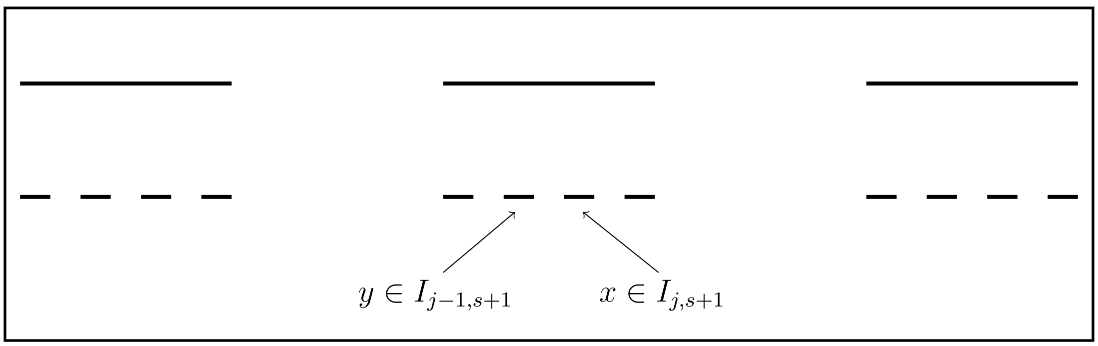


Figure 5.4: The case $x \in I_{j,s+1}$, $y \in I_{j-1,s+1}$

Then, since $x \in I_{j,s+1}$,

$$g(x) = (f - T_{a_{t,s}}^p f - T_{a_{j,s+1}}^m (R_{a_{t,s}}^p f))(x) = (R_{a_{t,s}}^p f - T_{a_{j,s+1}}^m (R_{a_{t,s}}^p f))(x)$$

and since $y \in I_{j-1,s+1}$,

$$g(y) = (f - T_{a_{j-1,s}}^p)(y) = (R_{a_{j-1,s}}^p f))(y).$$

Then we have

$$\begin{aligned} R_y^p(f - S_N(f))(x) &= R_y^p\, g(x) \\ &= g(x) - \sum_{i=0}^{p} \frac{g^{(i)}(y)(x-y)^i}{i!} \\ &= R_{a_{t,s}}^p f(x) - T_{a_{j,s+1}}^m (R_{a_{t,s}}^p f)(x) \\ &\quad - \sum_{i=0}^{p} \frac{(R_{a_{j-1,s+1}}^p f)^{(i)}(y)(x-y)^i}{i!} \end{aligned}$$

We only need to estimate the intermediate T^m since other terms can be handled in the same way as above. Now,

$$\begin{aligned}
|T^m_{a_{j,s+1}}(R^p_{a_{t,s}}f)(x)| &\leq \sum_{i=0}^{m} \frac{|(R^p_{a_{t,s}}f)^{(i)}(a_{j,s+1})|\ |x-a_{j,s+1}|^i}{i!} \\
&\leq \sum_{i=0}^{m} \frac{\varepsilon\,|a_{j,s+1}-a_{t,s}|^{p-i}\ |x-a_{j,s+1}|^i}{i!} \\
&\leq \sum_{i=0}^{m} \frac{\varepsilon\, C_0^{p-i}|x-y|^{p-i}\ C_0^i|x-y|^i}{i!} \\
&\leq C_0^p\, e\ \varepsilon\ |x-y|^p.
\end{aligned}$$

Since by (5.6), $|(R^p_{a_{t,s}}f)^{(i)}(a_{j,s+1})| \leq \varepsilon\,|a_{j,s+1}-a_{t,s}|^{p-i}$. In addition, $|a_{j,s+1}-a_{t,s}|$ and $|x-a_{j,s+1}|$ do not exceed $C_0\,|x-y|$. As a result we obtain $|T^m_{a_{j,s+1}}(R^p_{a_{t,s}}f)(x)| \leq C_0^p\, e\ \varepsilon\ |x-y|^p$ which is the desired result.

Case 2: Suppose $y \in I_{j,s+1}$ where $j \in B_s$ and $x \in I_{j-1,s+1}$.

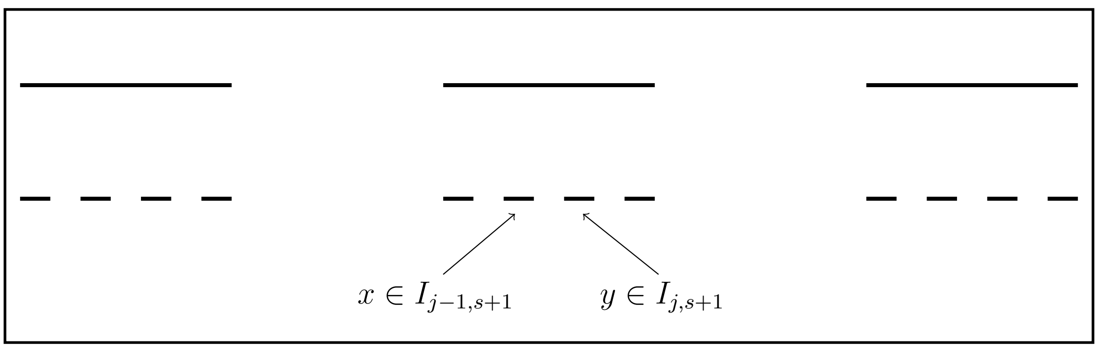


Figure 5.5: The case $x \in I_{j-1,s+1}$, $y \in I_{j,s+1}$

Then, since $y \in I_{j,s+1}$,

$$g(y) = (f - T^p_{a_{t,s}}f - T^m_{a_{j,s+1}}(R^p_{a_{t,s}}f))(y) = (R^p_{a_{t,s}}f - T^m_{a_{j,s+1}}(R^p_{a_{t,s}}f))(y)$$

and since $x \in I_{j-1,s+1}$,

$$g(x) = (f - T^p_{a_{j-1,s}})(x) = (R^p_{a_{j-1,s}}f))(x).$$

Then we have

$$\begin{aligned} R^p_y(f - S_N(f))(x) &= R^p_y\, g(x) \\ &= g(x) - \sum_{i=0}^{p} \frac{g^{(i)}(y)(x-y)^i}{i!} \\ &= R^p_{a_{j-1,s}} f(x) - \sum_{i=0}^{p} \frac{(R^p_{a_{t,s}} f - T^m_{a_{j,s+1}}(R^p_{a_{t,s}} f))^{(i)}(y)(x-y)^i}{i!} \end{aligned}$$

As above, it is sufficient to consider only $\sum_{i=0}^{p}[T^m_{a_{j,s+1}}(R^p_{a_{t,s}}f)]^{(i)}(y)(x-y)^i/i!$ since for other terms we have the desired bound. Since T^m term vanishes when $i > m$, the genuine summation here is until $i = m$. Then

$$\begin{aligned} \sum_{i=0}^{m} \frac{[T^m_{a_{j,s+1}}(R^p_{a_{t,s}}f)]^{(i)}(y)(x-y)^i}{i!} &\leq \sum_{i=0}^{m} \frac{|x-y|^i}{i!} \sum_{k=i}^{m} \frac{|R^p_{a_{t,s}} f(y)|^{(k)}\, |y - a_{j,s+1}|^{k-i}}{(k-i)!} \\ &\leq \sum_{i=0}^{m} \frac{|x-y|^i}{i!} \sum_{k=i}^{m} \frac{\varepsilon\, |y - a_{t,s}|^{p-k}\, |l_{s+1}|^{k-i}}{(k-i)!} \\ &\leq \sum_{i=0}^{m} \frac{|x-y|^i}{i!} \sum_{k=i}^{m} \frac{\varepsilon\, l_s^{p-k}\, l_{s+1}^{k-i}}{(k-i)!} \\ &\leq \sum_{i=0}^{m} \frac{|x-y|^i\, \varepsilon\, l_s^{p-i}\, e}{i!} \\ &\leq C_0^p\, e^2\, \varepsilon\, |x-y|^p. \end{aligned}$$

Since by (5.6), $|(R^p_{a_{t,s}} f)^{(k)}(y)| \leq \varepsilon\, |y - a_{t,s}|^{p-k}$. In addition, $|y - a_{j,s+1}| \leq l_{s+1}$ and l_s do not exceed $C_0\, |x-y|$. As a result we obtain the desired result.

Case 3: Suppose $x \in I_{j,s+1}$ where $j \in B_s$ and $y \in I_{j+1,s+1}$.

Then, since $x \in I_{j,s+1}$,

$$g(x) = (f - T^p_{a_{t,s}} f - T^m_{a_{j,s+1}}(R^p_{a_{t,s}} f))(x) = (R^p_{a_{t,s}} f - T^m_{a_{j,s+1}}(R^p_{a_{t,s}} f))(x)$$

and since $y \in I_{j+1,s+1}$,

$$g(y) = (f - T^p_{a_{t,s}})(y) = (R^p_{a_{t,s}} f))(y).$$

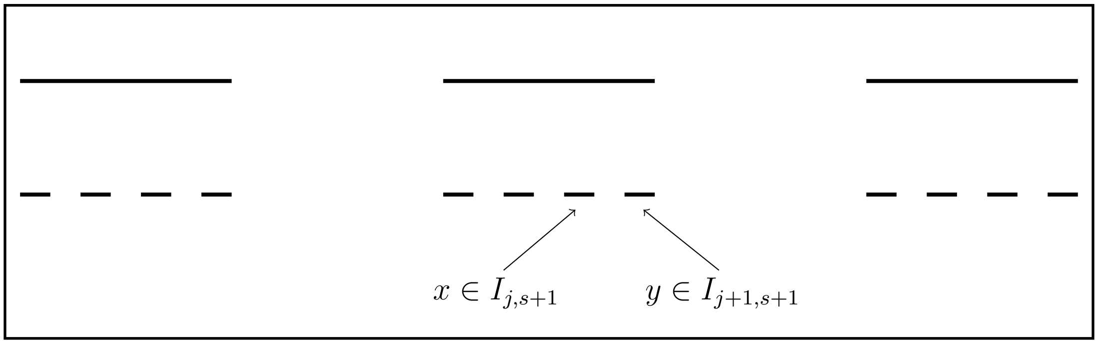


Figure 5.6: The case $x \in I_{j,s+1}$, $y \in I_{j+1,s+1}$

Then we have

$$\begin{aligned} R_y^p(f - S_N(f))(x) &= R_y^p\, g(x) \\ &= g(x) - \sum_{i=0}^{p} \frac{g^{(i)}(y)(x-y)^i}{i!} \\ &= R_{a_{t,s}}^p f(x) - T_{a_{j,s+1}}^m (R_{a_{t,s}}^p f)(x) - \sum_{i=0}^{p} \frac{(R_{a_{t,s+1}}^p f)^{(i)}(y)(x-y)^i}{i!} \end{aligned}$$

We only need to estimate the intermediate T^m since other terms can be handled in the same way as above. Now,

$$\begin{aligned} |T_{a_{j,s+1}}^m (R_{a_{t,s}}^p f)(x)| &\le \sum_{i=0}^{m} \frac{|(R_{a_{t,s}}^p f)^{(i)}(a_{j,s+1})|\ |x - a_{j,s+1}|^i}{i!} \\ &\le \sum_{i=0}^{m} \frac{\varepsilon\, |a_{j,s+1} - a_{t,s}|^{p-i}\ |x - a_{j,s+1}|^i}{i!} \\ &\le \sum_{i=0}^{m} \frac{\varepsilon\, C_0^{p-i} |x-y|^{p-i}\ C_0^i |x-y|^i}{i!} \\ &\le C_0^p\, e\ \varepsilon\ |x-y|^p. \end{aligned}$$

Since by (5.6), $|(R_{a_{t,s}}^p f)^{(i)}(a_{j,s+1})| \le \varepsilon\, |a_{j,s+1} - a_{t,s}|^{p-i}$. In addition, $|a_{j,s+1} - a_{t,s}|$ and $|x - a_{j,s+1}|$ do not exceed $C_0\, |x-y|$. As a result we obtain $|T_{a_{j,s+1}}^m (R_{a_{t,s}}^p f)(x)| \le C_0^p\, e\ \varepsilon\ |x-y|^p$ which is the desired result.

Case 4: Suppose $y \in I_{j,s+1}$ where $j \in B_s$ and $x \in I_{j+1,s+1}$.

Then, since $y \in I_{j,s+1}$,

$$g(y) = (f - T_{a_{t,s}}^p f - T_{a_{j,s+1}}^m (R_{a_{t,s}}^p f))(y) = (R_{a_{t,s}}^p f - T_{a_{j,s+1}}^m (R_{a_{t,s}}^p f))(y)$$

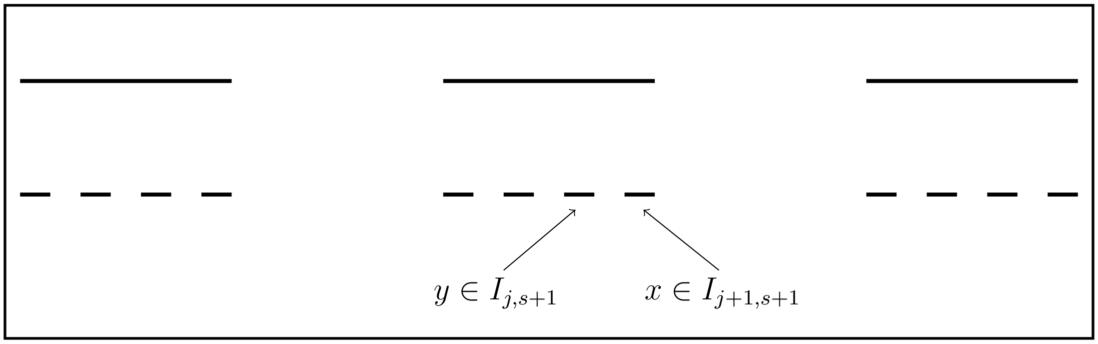


Figure 5.7: The case $x \in I_{j+1,s+1}$, $y \in I_{j,s+1}$

and since $x \in I_{j+1,s+1}$,

$$g(x) = (f - T^p_{a_{t,s}})(x) = (R^p_{a_{t,s}} f)(x).$$

Then we have

$$\begin{aligned} R^p_y(f - S_N(f))(x) &= R^p_y\, g(x) \\ &= g(x) - \sum_{i=0}^{p} \frac{g^{(i)}(y)(x-y)^i}{i!} \\ &= R^p_{a_{t,s}} f(x) - \sum_{i=0}^{p} \frac{(R^p_{a_{t,s}} f - T^m_{a_{j,s+1}}(R^p_{a_{t,s}} f))^{(i)}(y)(x-y)^i}{i!}. \end{aligned}$$

As above, it is sufficient to consider only $\sum_{i=0}^p [T^m_{a_{j,s+1}}(R^p_{a_{t,s}} f)]^{(i)}(y)(x-y)^i/i!$ since for other terms we have the desired bound. Since T^m term vanishes when $i > m$, the genuine summation here is until $i = m$. Then

$$\begin{aligned} \sum_{i=0}^{m} \frac{[T^m_{a_{j,s+1}}(R^p_{a_{t,s}} f)]^{(i)}(y)(x-y)^i}{i!} &\le \sum_{i=0}^{m} \frac{|x-y|^i}{i!} \sum_{k=i}^{m} \frac{|R^p_{a_{t,s}} f(y)|^{(k)}\, |y - a_{j,s+1}|^{k-i}}{(k-i)!} \\ &\le \sum_{i=0}^{m} \frac{|x-y|^i}{i!} \sum_{k=i}^{m} \frac{\varepsilon\, |y - a_{t,s}|^{p-k}\, |l_{s+1}|^{k-i}}{(k-i)!} \\ &\le \sum_{i=0}^{m} \frac{|x-y|^i}{i!} \sum_{k=i}^{m} \frac{\varepsilon\, l_s^{p-k}\, l_{s+1}^{k-i}}{(k-i)!} \\ &\le \sum_{i=0}^{m} \frac{|x-y|^i\, \varepsilon\, l_s^{p-i}\, e}{i!} \\ &\le C_0^p\, e^2\, \varepsilon\, |x-y|^p. \end{aligned}$$

Since by (5.6), $|(R^p_{a_{t,s}}f)^{(k)}(y)| \leq \varepsilon\, |y - a_{t,s}|^{p-k}$. In addition, $|y - a_{j,s+1}| \leq l_{s+1}$ and l_s do not exceed $C_0\, |x - y|$. As a result we obtain the desired result.

Since other positions of y when $x \in I_{j,s+1}$, $j \in B_s$, only enlarge $|x - y|$ this cases are enough. So the proof is complete. $\square$

Chapter 6

Some Properties of Bases

6.1 Unconditional Bases

Definition 6.1.1. Let $(x_n)_{n=1}^{\infty}$ be a sequence in a Banach space X. A (formal) series $\sum_{n=1}^{\infty} x_n$ in X is said to be unconditionally convergent if $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of $\mathbb{N}$.

Lemma 6.1.1. *[30, Lem. 2.4.2] Given a series $\sum_{n=1}^{\infty} x_n$ in a Banach space X, the following are equivalent:*

(a) $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent;

(b) The series $\sum_{n=1}^{\infty} x_{n_k}$ converges for every increasing sequence of integers $(n_k)_{k=1}^{\infty}$;

(c) The series $\sum_{n=1}^{\infty} \epsilon_n x_n$ converges for every choice of signs (ϵ_n);

(d) For every $\epsilon > 0$ there exists an n so that if F is any finite subset of $\{n+1, n+2, \ldots\}$ then

$$\Big\| \sum_{j \in F} x_j \Big\| < \epsilon.$$

Proposition 6.1.1. *[30, Prop. 2.4.9] A series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is unconditionally convergent if and only if $\sum_{n=1}^{\infty} t_n x_n$ converges (unconditionally) for all $(t_n) \in l_\infty$.*

Definition 6.1.2. A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is called unconditional if for each $x \in X$ the series $\sum_{n=1}^{\infty} e_n^*(x) e_n$ converges uncontionally. A basis for a Banach space is conditional if it is not unconditional.

We can easily say by this definition that; $(e_n)_{n=1}^{\infty}$ is an unconditional basis of X if and only if $(e_{\pi(n)})_{n=1}^{\infty}$ is a basis of X for all permutations $\pi : \mathbb{N} \to \mathbb{N}$.

The term 'unconditional basis' due to R.C. James [31]. But such bases was studied before James and was named different names. For example, Karlin [32] called such bases 'absolute' in his study before James. It should be noted that the term 'absolute basis' is used now for a more strong condition.

An example of an unconditional basis is the standard unit vector basis for the spaces c_0 and l_p. The summing basis of c_0, defined as

$$f_n = e_1 + \cdots + e_n, \quad n \in \mathbb{N},$$

is a conditional basis.

Proposition 6.1.2. *[30, Prop. 3.1.3] A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is unconditional if and only if there is a constant $K \geq 1$ such that for all $N \in \mathbb{N}$, whenever $a_1, \ldots, a_N, b_1, \ldots, b_N$ are scalars satisfying $|a_n| \leq |b_n|$ for $n = 1, \ldots, N$, then the following inequality holds:*

$$\Big\| \sum_{n=1}^{N} a_n e_n \Big\| \leq K \Big\| \sum_{n=1}^{N} b_n e_n \Big\|. \tag{6.1}$$

Definition 6.1.3. Let $(e_n)_{n=1}^{\infty}$ be an unconditional basis of a Banach space X. The unconditional basis constant, K_u, of (e_n) is the least constant K so that the inequality (6.1) holds. We then say that (e_n) is K-unconditional whenever $K \geq K_u$.

Theorem 6.1.2. *[33, Thm. 4.2.35] The classical Schauder basis for $C[0,1]$ is a conditional basis.*

Let us consider the proof of this theorem, since its arguments will be used below.

Proof. Let $(s_n)_{n=0}^{\infty}$ be the classical Schauder basis for $C[0,1]$ defined in (1.1). (Also you can look Figure 1.1) Let we define a subsequence (t_n) of (s_n) as follows. Let

$t_1 = s_2$, the member of the basis that is nonzero precisely on $(0,1)$,

$t_2 = s_3$, the member of the basis that is nonzero precisely on $(0,1/2)$,

$t_3 = s_6$, the member of the basis that is nonzero precisely on $(1/4,1/2)$,

$t_4 = s_{11}$, the member of the basis that is nonzero precisely on $(1/4,3/8)$,

and t_n be nonzero on an interval half as long as is the case for t_{n-1} and the corresponding interval for t_{n-1} to share left endpoints if n is even and right endpoints if n is odd.

For each positive integer n, let v_n be the midpoint of the interval on which t_n is nonzero, and let $a_n = (\sum_{j=1}^{n} t_j)(v_n)$. Then $(v_1, a_1) = (1/2, 1)$, $(v_2, a_2) = (1/4, 3/2)$. Then $a_1 = 1$, $a_2 = 3/2$ and for $n \geq 3$

$$a_n = 1 + \frac{a_{n-1} + a_{n-2}}{2}.$$

Since

$$a_n - a_{n-1} = 1 - \frac{a_{n-1} - a_{n-2}}{2}$$

when $n \geq 3$, by induction $1/2 \leq a_n - a_{n-1} \leq 3/4$ when $n \geq 2$. This shows that (a_n) is strictly increasing and unbounded. Also $\| \sum_{j=1}^{n} t_j \|_{\infty} = a_n$ for each n.

Let $b_n = (\sum_{j=1}^{n} (-1)^{j+1} t_j)(v_n)$. Then $(v_1, b_1) = (1/2, 1)$, $(v_2, b_2) = (1/4, -1/2)$. Then $b_1 = 1$, $b_2 = -1/2$ and for $n \geq 3$

$$b_n = \begin{cases} \frac{1}{2}(b_{n-1} + b_{n-2}) + 1, & \text{if } n \text{ is odd} \\ \frac{1}{2}(b_{n-1} + b_{n-2}) - 1, & \text{if } n \text{ is even} \end{cases}$$

Then by induction $1 \leq b_n \leq 2$ when n is odd and $-1 \leq b_n \leq 0$ when n is even. So

$$\left\| \sum_{j=1}^{n} (-1)^{j+1} t_j \right\|_{\infty} = \max\{|b_j| : 1 \leq j \leq n\} \leq 2$$

for each n.

Assume (s_n) is unconditional. Then K_u is its unconditional constant. Then it have to be true that

$$a_n = \left\| \sum_{j=1}^{n} t_j \right\|_\infty \leq K_u \left\| \sum_{j=1}^{n} (-1)^{j+1} t_j \right\|_\infty \leq 2K_u$$

for each n. But this contradicts the fact that (a_n) is an unbounded sequence of positive numbers. So the basis (s_n) is conditional. $\square$

Later, it was shown by Karlin [32] (by using another technique) that there is no unconditional basis in the space $C[0, 1]$.

Theorem 6.1.3. *The system $(f_n, \eta_n)_{n=1}^{\infty}$ which is Schauder basis in the space $C^p(K(\Lambda))$ given in Lemma 4.2.1, is a conditional basis.*

Proof. Let (f_n) be the Schauder basis in the space $C^p(K(\Lambda))$ where we defined in Section 4.1. Let we define a subsequence of (f_n) and call it (g_n). Let $g_1 = f_1 = \chi_{1,0}$, $g_2 = f_{3p+4} = \chi_{2,2}$, $g_3 = f_{6p+7} = \chi_{4,3}$ and in general $g_n = f_{(2^{n-1}+2^{n-2})(p+1)+1} = \chi_{2^{n-1},n}$. In general we take the function f_n which is equal to the characteristic function of the interval whose right end point is $b_{1,1}$. See the following figure.

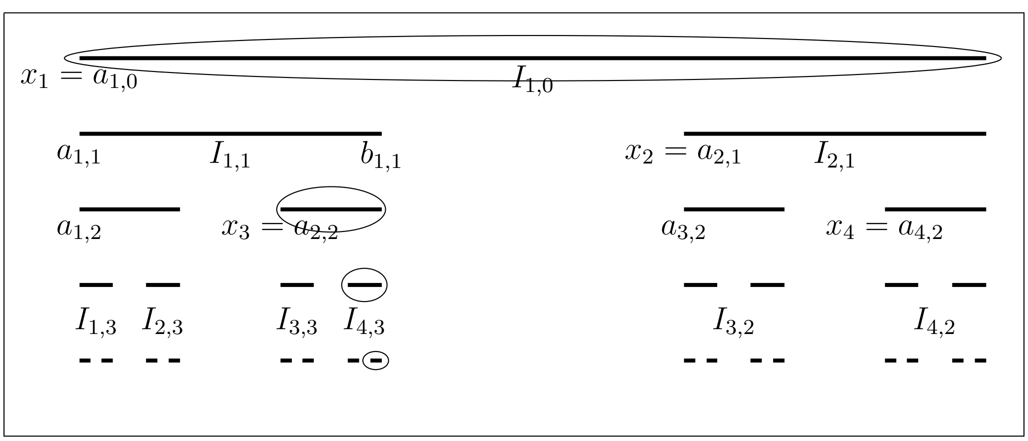


Figure 6.1: Cantor procedure with new subsequence g_n

Then

$$\sum_{j=1}^{n} g_j = \sum_{j=1}^{n} \chi_{2^{j-1},j} = \begin{cases} 1, & \text{on } I_{1,0}/I_{2,2}, \\ 2, & \text{on } I_{2,2}/I_{4,3}, \\ 3, & \text{on } I_{4,3}/I_{8,4}, \\ \vdots & \\ n, & \text{on } I_{2^{n-1},n}/I_{2^{n-2},n-1}. \end{cases} \tag{6.2}$$

Then the sequence $a_n = |\sum_{j=1}^{n} g_j|_k$ where $k = 0, 1, \ldots, p$ is an unbounded sequence.

Also

$$\sum_{j=1}^{n} (-1)^{j+1} g_j = \sum_{j=1}^{n} (-1)^{j+1} \chi_{2^{j-1},j} = \begin{cases} 1, & \text{on } I_{1,0}/I_{2,2}, \\ 0, & \text{on } I_{2,2}/I_{4,3}, \\ 1, & \text{on } I_{4,3}/I_{8,4}, \\ \vdots & \\ 1, & \text{on } I_{2^{n-1},n}/I_{2^{n-2},n-1} \text{ if } n \text{ is odd}, \\ 0, & \text{on } I_{2^{n-1},n}/I_{2^{n-2},n-1} \text{ if } n \text{ is even}. \end{cases} \tag{6.3}$$

Assume (f_n) were unconditional. Then K_u were its unconditional constant. Then it would have to be true that

$$a_n = \left| \sum_{j=1}^{n} g_j \right|_k \leq K_u \left| \sum_{j=1}^{n} (-1)^{j+1} g_j \right|_k \leq K_u,$$

where $k = 0, 1, ..., p$, for each n. But this contradicts the fact that (a_n) is an unbounded sequence of positive numbers. So the basis (f_n) is conditional. $\square$

This method does not work for the space $\mathcal{E}^p(K(\Lambda))$ since $\| \sum_{j=1}^{n} (-1)^{j+1} g_j \|_k$ is not bounded in the space $\mathcal{E}^p(K(\Lambda))$. Therefore, the question regarding the unconditional property of the basis remains open. Of course, any basis in $\mathcal{E}(K)$ is unconditional, since this space is nuclear and, by the Dynin-Mityagin theorem, the basis is absolute.

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