EXACT SOLUTIONS AND HEURISTICS FOR MULTI-PRODUCT INVENTORY PRICING PROBLEM

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By

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July, 2014

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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July, 2014

We study the multi-product inventory pricing problem under stochastic and price sensitive demand. We have initial inventory of m resources whose different combinations form n products. Products are perishable and need to be sold by a deadline. Demand for each product is modeled as a non-homogeneous Poisson process whose intensity is a function of the current price of the product itself. The aim is to set the price of each product over the selling period to maximize the expected revenue. This problem is faced in various industries including retail, airlines, automobile, apparel, hotels and car rentals. Our contributions are twofold. First, we provide a closed form solution for the special case of exponential price response where the elasticity parameter of the demand function of all products are equal. Second, we develop two classes of dynamic pricing heuristics: one using the value approximation approach of dynamic programming and the other using the deterministic version of the problem. Our numerical analysis indicates that dynamic pricing yields significantly higher revenues compared to fixed price policies. One of the dynamic pricing heuristics based on the deterministic problem provides around $5-15\%$ additional revenue compared to fixed price policies. Moreover, two value approximation heuristics that we suggest result in at most $\sim 0.5\%$ and $\sim 3.4\%$ gaps in the expected revenue compared to the optimal dynamic pricing policy for general form of exponential price response. These additional revenues can have a profound effect on the profitability of firms, so dynamic pricing should be preferred over fixed price policies in practice.

Keywords: Dynamic pricing, network revenue management, inventory pricing.

ÖZET

ÇOKLU ÜRÜN FİYATLANDIRMA PROBLEMİ İÇİN KESİN VE SEZGİSEL ÇÖZÜM YÖNTEMLERİ

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Tez Yöneticisi: Doç. Dr. Alper Şen
Temmuz, 2014

Bu çalışmada, birden fazla tipte üründen oluşan bir envanterin, rassal talep durumundaki fiyatlandırılması problemi ele alınmıştır. Başlangıçta elimizde m farklı ara üründen belirli miktarlarda bulunmaktadır. Bu ara ürünler farklı kombinasyonlarda birleşerek n farklı ürünü meydana getirmektedir. Ürünler belli bir süre sonunda değerlerini yitirmektedir. Her bir ürün için talep, yoğunluğu zaman içerisindeki anlık fiyata bağlı olarak değişim gösteren bir Poisson süreci olarak modellenmiştir. Amaç, bu satış sürecinden en yüksek geliri elde etmektir. Bu problem ile perakendecilik, havayolları, otomobil, giyim, otelcilik ve kiralık araba işletmeciliği gibi birçok endüstride karşılaşılmaktadır. Bu konuda çalışmamızın katkıları iki kısma ayrılabilir. İlk olarak talebin üstel fonksiyon halinde tanımlandığı ve her bir ürüne ait esneklik parametresinin aynı olduğu özel bir durum için analitik çözüm sunulmuştur. İkinci katkımız ise problemin çözümü için ortaya koyduğumuz iki farklı tipteki sezgisel yöntemlerdir. Birinci tipteki yöntemler dinamik programlamada kullanılan değer fonksiyonunun tahminini kullanmaktadır. İkinci tip sezgisel yöntemler ise problemin deterministik halinden faydalanmaktadır. Sayısal analizimiz dinamik fiyatlandırmanın, sabit fiyat politikalarına göre önemli ölçüde daha yüksek gelir sağladığını göstermektedir. Deterministik problemi kullanan sezgisel yöntemlerden biri, farklı başlangıç envanterleri için $\sim \%5 - \%15$ daha yüksek gelir sağlamıştır. Ayrıca, değer fonksiyonu tahminini kullanan sezgisel yöntemlerden ikisi, mümkün olan en yüksek ortalama gelirden en fazla $\sim \%0.5$ ve $\sim \%3.4$ oranında daha az ortalama gelir sağlamıştır. Dinamik fiyatlandırmanın sağladığı fazladan gelir, firmaların karlılığı açısından önemli rol oynayabilir. Bu sebeple, uygulamada dinamik fiyatlandırma, sabit fiyat politikalarına tercih edilmelidir.

Anahtar sözcükler: Dinamik fiyatlandırma, hasılat yönetimi, envanter fiyatlandırma.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Motivation

In both manufacturing and service industries, companies have the ultimate goal of increasing their profit. They perform various activities to achieve this goal. Increasing sales volume or market share through marketing and advertisement activities or decreasing operating costs through quality control and more efficient logistics can all be employed to this end. Yet another means of increasing profits is efficient pricing, which is basically achieved by setting the price of a good or service in a way that maximizes the profitability of the company subject to certain constraints on supply.

According to a study by Marn et al. [1], a price rise of one percent would generate an eight percent increase in operating profits in the average income statement of an S&P 1500 company. This is in contrast to about a 5.4 percent increase via decreasing variable costs by one percent, or only about a 2.5 percent increase via increasing sales volume by one percent. Hence, it can be argued that pricing has a profound effect on profitability.

Dynamic pricing is a strategy in which the price of a product is flexible and controlled depending on various determinants such as customer valuation for the product, inventory levels, remaining time in the selling season, prices of complementary and substitutable products and competitors' decisions, etc. Dynamic pricing strategy is being used extensively in different industries such as retail, apparel, automobiles, consumer electronics and telecommunications. Sahay [2] reports that EBay Inc. sold $20 billion worth of goods in 2005 and Ford Motor Co. sold more than $50 billion worth of automobiles in 2003 through dynamically pricing their products. These examples indicate how extensively dynamic pricing is used as an important way of increasing profits in many different industries.

In apparel sector, the traditional pricing policy is to set fixed prices during season followed by large markdowns towards the end of the season. Similar pricing policies can be observed in consumer electronics. In such cases, dynamic pricing throughout the selling period can provide higher revenues due to more efficient demand management and customer segmentation based on willingness to pay. An interesting example of successful demand management is provided in Sahay [2]. They consider a hairdresser in London, who turns away customers at weekends due to limited capacity. However, he is idle most of the time in weekdays. They increased the price of a haircut at weekends and decreased it on Tuesdays and Wednesdays. This practice resulted in 10% increase in the revenue of the hairdresser which is a result of successful customer segmentation based on willingness to pay.

As for the capacity constrained service companies such as airlines, hotels, car rentals and cruise-lines, efficient pricing is even more important since variable costs are relatively small in their operating activities and it is very costly or impossible to increase their capacities in the short run. Realization of high demand may lead to lost sales due to the constrained capacity in which case one would prefer increasing prices at the expense of losing some demand. On the other hand, an increase in prices may result in unsold products, which is not desirable especially for perishable items. Besides, there may be some considerations other than revenue. For instance, there may be an attendance target for a concert to be held. A striking example of pricing perishable items was the London 2012 Olympic Games where organizers had to price 8 million tickets while meeting the revenue and attendance targets at the same time according to Bertini and

Gourville [3].

It is worth noting here that pricing is an indispensable part of revenue management practice which has its origins in airline industry. In particular, it can be seen that pricing and capacity allocation problems are interrelated from the perspective of the capacity constrained service companies. What is meant by the capacity allocation problem may vary in different contexts. Determination of the number of seats reserved for different fare classes in a single flight leg can be considered as an example of this problem. Besides, in the case of an airline network, the number of seats in a flight leg reserved for different itineraries is another example from the same industry. The same problem is faced in hotels and cruise-lines while determining the capacity allocated for early bookings with lower prices. As Gallego & van Ryzin [4] point out, pricing and capacity allocation problems are interrelated because pricing has an influence on demand statistics which directly affect capacity allocation decisions. One can close a fare class by setting a sufficiently high price so that the demand rate for that fare class gets close to zero. Hence, capacity allocation problems can be studied in a pure pricing framework, which is the approach of this thesis as well.

Dynamic pricing practice is observed in traditional brick-and-mortar retailers as well as online retailers. Although it is difficult to update the prices of around 50,000 SKUs frequently in traditional retailing with shelf labels, it becomes more applicable with the use of electronic shelf labels. Thompson [5] reports that in 2013, a start-up company which offers dynamic pricing solutions and electronic shelf labels to retailers raised $ 1.7 million in venture capital funding. Retailers employ dynamic pricing in various ways e.g., competitor based or time based pricing. They sometimes reprice depending on the prices of their competitors. In time based pricing, they increase the prices during the time periods with high demand and vice versa. For instance, customers who choose online shopping usually prefer evenings for shopping, so retailers set higher prices during evenings. Retailers also frequently apply bundling strategy in which customers are offered to buy bundles with prices usually lower than the sum of the prices of individual products that constitute the bundle. In this case, one should decide on the number of products reserved for bundles and individual sale which can be seen as a capacity allocation problem.

As mentioned above, there are many different factors having influence on the price of an item, one of which is the inventory level. Under stochastic and price sensitive demand, one may intuitively think that excess inventory should lead to lower prices to increase the demand and avoid unsold products. However, the increase in demand may not compensate the negative effect of price reduction on the revenue in some cases. This strongly depends on how demand responds to price changes. Low inventory levels, on the other hand, would lead one to intuitively expect higher prices in order to sell to those customers with higher reservation prices. Again, the demand response to price changes is binding here because the increase in price may not compensate the negative effect of demand loss on the revenue.

The length of selling period is another factor that affects the pricing decisions. If the selling period is relatively long, one can expect to have higher prices to benefit from the customer surplus as much as possible. On the other hand, prices should be set lower if the selling period is relatively shorter in order to trigger the demand up and to sell all the products on hand in this short period. Again, the response of consumers to price changes must be kept in mind because selling all the products with low prices may not be the right way to maximize the revenue.

Since the 1980s, the network effects in revenue management received considerable attention because the expansion of hub-and-spoke networks led to an increase in passenger itineraries including different flight legs [6]. A new branch of revenue management came out in this regard, namely the network revenue management, in which a set of resources may yield multiple products, and each resource may be demanded by different products [7]. The multiple resource - multiple product structure makes the pricing problem even more complicated. To give an example, there are $4,000$ flights and $350,000$ passenger itineraries per day in United/Lufthansa/SAS ORION System [6]. Due to this large problem size, it is very difficult to find and implement optimal revenue management strategies in practice. Consequently, we see a growing interest in academia for the study of these problems.

In conclusion, efficient pricing is crucial for many industries including capacity constrained service companies and retailers. It is strongly connected to revenue management and considered as an indispensable part of it in both practice and theory. Determining the right price for a product is a complex task which depends on many different factors. Network effects and large problem sizes make this task even more difficult. All of these motivate practitioners and academicians to work on this subject.

1.2 Contribution

We consider the problem of a seller who needs to sell a fixed inventory of multiple items over a finite horizon. Customers arrive following a Poisson process based on their willingness-to-pay and the current price set by the seller. Consequently, the sales rate or demand rate can be represented as a function of the prices set by the retailer. The objective of the seller is to maximize its expected revenue over the horizon by changing the prices as a function of remaining time and inventory levels. This problem is faced in many different settings, so the model and solution approaches are aimed to be generic in the sense that they can be applied in various industries. As the main contribution, a closed form solution when the demand rate, or price response function is an exponential price response function is provided. The closed form solution is then used to obtain approximations for expected revenue under general form of exponential and linear price responses. These approximate revenues are then used in two heuristics. In addition, two dynamic pricing heuristics based on the deterministic version of the problem are also proposed for large scale problem instances. The dynamic pricing heuristics, together with two fixed price heuristics from the literature are compared in terms of the performances on expected revenue through a substantial numerical analysis. As a result, we emphasize the advantage of dynamic over fixed pricing since it yields significantly higher expected revenues.

1.3 Overview

The rest of the thesis is organized as follows:

In Chapter 2, we review the earlier research on revenue management. First, we consider the revenue management literature in general and then consider four subcategories determined by McGill & van Ryzin [6]: Forecasting, overbooking, capacity allocation and pricing. We put more emphasis on capacity allocation and pricing since they are closely related with this study.

In Chapter 3, the problem is formulated. The required assumptions of the model are explained. The deterministic version of the problem is also presented, since it provides an upper bound on the optimal expected revenue of the stochastic problem and it is also used in heuristics.

Chapter 4 is dedicated to present the closed form solution of the problem for a special case of exponential price response where the parameter α_j of the demand function of all products are assumed to be equal. Some structural results arising from the closed form solution are also reported here.

Heuristic methods are explained in Chapter 5. There are two types of heuristics. The first type includes two heuristics both of which use value approximation approach of dynamic programming. The second type of heuristics uses the deterministic version of the problem and four heuristics are included in this type. Two of them are fixed price heuristics from earlier research and the other two are new dynamic pricing heuristics.

Numerical analysis to examine the performance of heuristics are reported in Chapter 6. We conclude in Chapter 7.

Chapter 2

Literature Search

2.1 Revenue Management in General

Revenue management (or yield management) is a broad field of research. Many authors define revenue management in their own words. According to Belobaba [8], *yield* is the revenue per passenger-mile of traffic carried by an airline. Talluri & van Ryzin [9] define revenue management as the collection of strategies and tactics firms use to scientifically manage demand for their products and services. Netessine & Shumsky [10] refer it as the techniques to allocate limited resources, such as airplane seats or hotel rooms, among a variety of customers, such as business or leisure travelers.

In this section, we will briefly review the revenue management literature, classify it into subsections and provide some references. In particular, we are interested in a problem where a seller needs to sell a given stock of items by a deadline, where the demand is stochastic and price sensitive. We will focus on this problem while reviewing the literature.

The origin of revenue management is the overbooking practice which is to sell flight tickets beyond the capacity of the aircraft in order to prevent flights with empty seats which was a huge problem for the airlines in 1960s. Research on overbooking strategy strongly depends on the statistical information on demand, cancellations and no-shows. Thus, overbooking led to a research interest on forecasting. Airlines have the advantage of having considerably useful demand data due to sophisticated software and technology which makes overbooking so successful and popular.

In 1970s, airlines started to offer discounted flight tickets for early booking passengers. This gave birth to the problem of capacity allocation (or seat inventory control) since it is important to determine how much of the capacity should be reserved for those customers who are willing to pay higher but book later. This also generated a broad literature and developed information systems that are capable of handling different classes of customers (i.e., fare classes).

We would like to follow the classification of revenue management literature provided by McGill & van Ryzin [6]. They divide the revenue management research into four broad categories: forecasting, overbooking, seat inventory control and pricing. It is an extensive overview of revenue management literature that refers many studies some of which we will mention here.

The forecasting category includes the publications of approaches to airline forecasting and models for demand distributions and arrival processes. An early example is Beckmann & Bobkowski [11] in which different probability distributions for total number of passengers are tested and Gamma distribution is used to propose an overbooking level. In the seminal work of Littlewood [12], methods of passenger forecasting are described. It also introduces the idea of maximizing revenue instead of number of passengers carried which is accepted and followed in many of the subsequent studies in revenue management literature.

As a relatively more recent work, Mcgill [13] studies data censoring in regression analysis of multiple classes of demand that are subjected to a common resource constraint. Censoring of the data arise from the fact that demand is not recorded after all seats are sold, i.e., historical booking data reveals sales information rather than demand. They achieved to provide maximum likelihood estimates of the parameters of the demand model under censorship.

In overbooking category, early research dealt with limiting the probability of denied boardings through nondynamic approaches which ignore cancellations and reservations after the overbooking decision is made. Beckmann [14] uses Gamma distribution again to model cancellations and no-shows and determines sales limits. Littlewood [12] describes a model to fit a probability distribution of departed loads due to overbooking which can be used to calculate the expected number of passengers carried and off-load. Shlifer & Vardi [15] extends overbooking models for two fare classes and two flight legs. There are also dynamic approaches for overbooking problem one of which is Rothstein's [16] Ph.D. thesis, which introduces a dynamic programming approach for overbooking problem for the first time.

The remaining two categories of the literature, seat inventory control and pricing, will be discussed in more detail in the next section. The problem we studied is closely related with these two categories of the literature.

2.2 Pricing and Capacity Allocation

The problem of selling a given stock of items by a deadline where demand is stochastic and price sensitive received considerable attention in the literature since the early 1960s. Kincaid & Darling [17] was first to address this problem. According to their model, potential buyers arrive in accordance with a Poisson process and their reservation prices have a probability distribution which is known by the seller. It is assumed that unsold items at the end of the selling period is disposed with a given salvage value and no backordering is allowed. The objective is to maximize the expected revenue. Although their work requires no background in dynamic programming, they suggest that the problem can be formulated with a dynamic programming approach, which is actually adopted in many of the following studies.

Another seminal work on the same problem is Stadje [18]. They characterize the maximum expected gain and the optimal price path as a system of differential equations. Unfortunately, they state that these differential equations turn out to have no explicit solution but must be solved numerically in many of the examples. On the other hand, they are able to propose closed form solutions for two special cases of the distribution function of the reservation price.

Gallego & van Ryzin [19] also study the same pricing problem. They consider a market with imperfect competition, i.e., the firm is a price maker for the product. The demand is modeled as a non-homogeneous Poisson process whose intensity depends on the remaining time in the selling period and the remaining inventory. They propose a dynamic programming formulation and give an implicit Hamilton-Jacobi optimality condition which yields the maximum expected revenue after solving a system of partial differential equations. As an important structural result, they show that optimal expected revenue is strictly increasing and strictly concave in both the length of the selling period and number of items on hand. The optimal price is strictly increasing in the remaining time and strictly decreasing in the number of items on hand. In addition, two fixed price heuristics based on the solution of the deterministic version of the problem are proposed and proved to be asymptotically optimal if both the number of items and expected demand are large.

It should be noted here that Gallego & van Ryzin's [19] model does not include the notion of reservation price; however, their model does not have any loss of generality. The reason is that they use aggregate demand functions which is explained in more detail by Huang et al. [20]. They show that any demand model with reservation prices following one of uniform, exponential, logistic, Weibull and Pareto distributions can be translated into the aggregate demand functions of type linear, log-linear, logistic, exponential and power, respectively.

Gallego & van Ryzin [19] assume that the demand rate of the arrival process depends only on the current price, i.e., demand is a time-invariant function of the price. Zhao & Zheng [21] relax this assumption and consider the case where the distribution function of the reservation price changes over time. This implies that intensity of the sales process depends on time as well as the current price. In this case, Zhao & Zheng [21] question the previous structural results of Gallego and van Ryzin [19]. They show that the optimal price still decreases in the number of items on hand, yet it may not be increasing in the remaining time. Indeed, the optimal price is increasing in remaining time if the conditional probability that a customer will buy at a higher price, given that she is willing to buy at a lower price, is decreasing over time. This sufficient condition seems to hold for fashion goods but does not hold for travel services. They also question the effectiveness of fixed price heuristics and conclude that dynamic pricing significantly outperforms fixed price policy even if prices are selected from a discrete and finite price set.

Although Gallego & van Ryzin [19] show that fixed price heuristics are asymptotically optimal and that dynamic pricing has only secondary effect on revenues, Şen [22] further investigates the problem and obtains higher revenues via dynamic pricing which may be of importance in practice. He proposes two practical dynamic pricing heuristics that continuously update prices based on the remaining inventory and the time in the selling period. One of their heuristics, namely the revenue approximation heuristic, provides significant improvement leading to at most 0.2% gap compared to optimal dynamic pricing.

All of the above studies consider the pricing of one type of item. On the other hand, the common practice in the revenue management is to offer multiple fare classes for a single type of inventory. This practice can be viewed as a single resource - multiple product model where different fare classes represents the multiple products. As mentioned in Chapter 1, when there are multiple products, capacity allocation problem must be taken into consideration as well as the pricing problem. Lee & Hersh [23] is an example of studies considering the capacity allocation aspect in a single resource - multiple product structure. A similar approach for two demand classes can be found in Gerchak et al. [24]. They develop a discrete time dynamic programming model to find an optimal booking policy. They divide the selling period into small enough intervals such that no more than one request occur during that interval. In each interval, an accept/reject decision is made based on the time at which request is received and the available seats. They conclude that if there are no multiple seat bookings, the optimal booking policy can be reduced to two sets of critical values based on booking capacity and decision periods.

Gallego & van Ryzin [4] study the multiple resource - multiple product version of the problem proposed in Gallego & van Ryzin [19]. In this case, there is a set of products that are formed by using different combinations of a set of resources. There is a given initial stock of each resource at the beginning of the finite selling period and the problem is to find the price path of each product that maximizes the expected revenue. This product-resource structure is generic in the sense that it can be applied in many different application areas of revenue management. The distinguishing feature of their work is that capacity allocation and pricing problems are jointly solved as mentioned in Chapter 1. They formulate the problem and provide an implicit optimality condition similar to that of the single item case. The fixed price heuristics based on the deterministic version of the problem are proved to be asymptotically optimal for the multiple-product case as well. They also run some simulations to test the performance of the heuristics numerically in different scenarios and conclude that fixed price heuristics perform well.

Cooper [25] consider the capacity allocation problem in a multiple resource - multiple product structure. He emphasizes the advantages of the LP-based deterministic allocation policy which ignores the randomness in the demand but provides with a way of overcoming the curse of dimensionality. His primary conclusion is that normalized revenues obtained by implementing allocation policies based on the deterministic problem converges in distribution to a constant upper bound on the optimal value in a stochastic demand environment. He actually investigates the mechanism under the asymptotic optimality of fixed price heuristics introduced in Gallego & van Ryzin [4]. A counterintuitive example from their work is also worth to mention at this point. He questions the effectiveness of resolving the deterministic problem during the selling period and updating the allocations. One may expect that resolving yields better solutions since it includes more information which is the realized demand up to the point of resolving. However, Cooper [25] gives an example where the expected revenue yielded by resolving is strictly worse than solving once.

Maglaras & Meissner [26] suggest a common framework for pricing and capacity allocation problems. The problems defined in Gallego & van Ryzin [4] and

Lee & Hersh [23] can be treated as different instances of this common framework. They verify the structural results obtained in some previous studies by using their formulation. They also propose heuristics based on the deterministic version of the problem and illustrate that dynamic pricing via resolving is asymptotically optimal in their settings contrary to Cooper's [25] example. In their numerical studies, dynamic pricing heuristics tend to outperform static one which highlights the importance of dynamic pricing.

Chapter 3

Model Description

3.1 Stochastic Problem

We will follow the notation and formulation provided in the seminal work of Gallego & van Ryzin [4]. There are m resources and n products, where a unit of product j consumes a_{ij} units of resource i. $\boldsymbol{A} = [a_{ij}]$ is an integer valued matrix with no zero columns. We have x_i units of initial inventory for resource i and a selling period of length T. We will consider the problem of dynamically pricing these n products over the time interval [0,T]. The demand is both stochastic and price sensitive. The arrival of the customers for each product is modeled as a non-homogeneous Poisson process for each of the products. For product j, the intensity of the arrival process is $\lambda_j(p_j)$ if p_j is the current price of product j. It is assumed that the demand for product j is independent of the prices of products other than j. This is certainly a limitation of the model but is required for analytical tractability.

We need to impose some regularity assumptions on the demand function. The demand function for product j, $\lambda_j(p_j)$, is invertible and its inverse is $p_j(\lambda_j)$. The revenue rate for product j is denoted by $r_j(\lambda_j) = \lambda_j \cdot p_j(\lambda_j)$, is assumed to satisfy $\lim_{\lambda_j \to 0} r_j(\lambda_j) = 0$, and is continuous, bounded, concave and has a least maximizer denoted by $\lambda_j^0 = min\left\{\lambda_j : \lambda_j \cdot p_j(\lambda_j) = max_{\lambda_j \geq 0} \lambda_j \cdot p_j(\lambda_j)\right\}$. There exists a *null* price for all products denoted by p_j^∞ for which $\lim_{p_j \to p_j^\infty} p_j \cdot \lambda_j(p_j) = 0$. The price of product j is selected from a set of allowable prices $\mathscr{P}_j = [0, p_j^\infty)$. The corresponding set of allowable rates is denoted by $\Lambda_j = \{\lambda_j(p_j) : p_j \in \mathscr{P}_j\}$.

Let the counting process N_s^j denote the number of product j sold up to time s. A pricing policy sets the price of product j at time s to a certain level which is denoted by p_j^s. The corresponding demand rate for product j at time s is then $\lambda_j^s = \lambda_j(p_j^s)$. Denote by $\mathscr{U}$ the class of pricing policies that satisfy

$$\sum_{j=1}^{n} \int_0^T a_{ij} \, dN_s^j \leq x_i \quad \forall i, \tag{3.1}$$

$$p_j^s \in \mathscr{P}_j \iff \lambda_j^s \in \Lambda_j \quad \forall j, \quad 0 \leq s \leq T. \tag{3.2}$$

Given initial vector of inventory levels $\boldsymbol{x} = (x_1, ..x_m)$ and a deadline T, the problem is to find the optimal pricing policy u^* that maximizes the expected revenue. More formally,

$$u^* = \arg\max_{u \in \mathscr{U}} \left\{ \mathbb{E}_u \left[\sum_{j=1}^{n} \int_0^T p_j^s \, dN_s^j \right] \right\} \tag{3.3}$$

Bremaud [27] show that one can find the Hamilton-Jacobi sufficient conditions for optimal expected revenue to go $J^*(\boldsymbol{x}, s)$ (and the corresponding demand rates and prices), given remaining time s and remaining inventory vector $\boldsymbol{x}$ as

$$\frac{\partial J^*(\boldsymbol{x}, s)}{\partial s} = \sup_{\lambda_1, .., \lambda_n} \left\{ \sum_{j=1}^{n} r_j(\lambda_j) - \sum_{j=1}^{n} \lambda_j \left(J^*(\boldsymbol{x}, s) - J^*(\boldsymbol{x} - \boldsymbol{A}^j, s) \right) \right\} \tag{3.4}$$

where $\boldsymbol{A^j}$ is the j^{th} column of $\boldsymbol{A}$ and J^* satisfies the boundary conditions $J^*(\boldsymbol{x}, s) = 0, \quad \forall s$ and $\boldsymbol{x} : x^i < a_{ij}$ for some i and for all j and $J^*(\boldsymbol{x}, 0) = 0$, $\forall x$.

A formal proof of the above optimality condition can be found in Bremaud [27]; however, we can justify it informally by using simple arguments. In the next small time interval δs, we will observe one unit of demand for product j with probability $\lambda_j \delta s$, no demand with probability $(1 - \lambda_j \delta s)$ and more than one unit of demand with probability $o(\delta s)$. Hence, by the Principle of Optimality, we can write

$$J^*(\boldsymbol{x}, s) = \sup_{\lambda_1,..,\lambda_n} \left\{ \sum_{j=1}^{n} \left(\lambda_j \delta s \Big(p_j(\lambda_j) + J^* \left(\boldsymbol{x} - \boldsymbol{A}^j \right) \Big) \right.\right.$$
$$\left.\left. + (1 - \lambda_j \delta s) J^* \left(\boldsymbol{x}, s - \delta s \right) + o(\delta s) \right) \right\} \tag{3.5}$$

$$\frac{J^*(\boldsymbol{x}, s)}{\delta s} = \sup_{\lambda_1,..,\lambda_n} \left\{ \sum_{j=1}^{n} \left(\lambda_j \Big(p_j(\lambda_j) + J^* \left(\boldsymbol{x} - \boldsymbol{A}^j \right) \Big) \right.\right.$$
$$\left.\left. + \left(\frac{1}{\delta s} - \lambda_j \right) J^* \left(\boldsymbol{x}, s - \delta s \right) + o(\delta s) \right) \right\} \tag{3.6}$$

$$\frac{J^*(\boldsymbol{x}, s) - J^* \left(\boldsymbol{x}, s - \delta s \right)}{\delta s} = \sup_{\lambda_1,..,\lambda_n} \left\{ \sum_{j=1}^{n} \left(r_j - \lambda_j \Big(J^* \left(\boldsymbol{x}, s - \delta s \right) \right.\right.\right.$$
$$\left.\left.\left. - J^* \left(\boldsymbol{x} - \boldsymbol{A}^j \right) + o(\delta s) \Big) \right) \right\} \tag{3.7}$$

By taking the limit as $\delta s \to 0$, we obtain (3.4).

Gallego & van Ryzin [4] state that it is very difficult to find closed form solutions to system of partial differential equations defined in (3.4). Hence, they propose two heuristic pricing policies, namely make-to-stock (MTS) and make-to-order (MTO) policies, which will be discussed after the deterministic version of the problem is presented in the next section.

3.2 Deterministic Problem

In deterministic problem, there are again m resources and n products, where a unit of product j consumes a_{ij} units of resource i. The selling period is T time units. We have the same regularity assumptions on the demand functions as in section 3.1. $\lambda_j(s)$ is the *deterministic* demand rate of product j at time s. The price of product j at time s is a function $p_j(\lambda_j(s))$ of the current demand rate of product j. The revenue rate of each product at time s is $r_j(\lambda_j(s), s) =$

$\lambda_j(s) \cdot p_j(\lambda_j(s))$. We have x_i units of initial inventory for resource i that are continuous quantities. Products can also be sold in continuous amounts. In this setting, the deterministic problem can be formulated as follows:

$$
\begin{aligned}
J_D(\boldsymbol{x}, T) = \max \ & \sum_{j=1}^{n} \int_0^T r_j(\lambda_j(s), s)\, ds \\
\text{s.t.} \ & \sum_{j=1}^{n} \int_0^T a_{ij} \cdot \lambda_j(s)\, ds \leq x_i \quad \forall i, \\
& \lambda_j(s) \in \Lambda_j, \quad \forall j, \quad 0 \leq s \leq T.
\end{aligned} \tag{3.8}
$$

The solution of this problem, if one exists, is a function $\lambda_D(s) : [0, T] \to \mathbb{R}^n$. Gallego & van Ryzin [4] make a simplifying observation for this problem. Since the revenue rate is time invariant, more formally $r_j(\lambda_j(s), s) = r_j(\lambda_j(s))\ \forall s, j$, solutions are always constant intensities (prices) and the problem reduces to the following convex programming problem:

$$
\begin{aligned}
J_D(\boldsymbol{x}, T) = \max_{\lambda_1..\lambda_n} \ & \sum_{j=1}^{n} r_j(\lambda_j) \cdot T \\
\text{s.t.} \ & \sum_{j=1}^{n} a_{ij} \cdot \lambda_j \cdot T \ \leq x_i \quad \forall i, \\
& \lambda_j \geq 0 \quad \forall j.
\end{aligned} \tag{3.9}
$$

Deterministic problem is important because it constitutes an upper bound for the stochastic problem defined in (3.1)-(3.3) and it is stated in Theorem 1.

Theorem 1. *(Gallego and van Ryzin [4], Theorem 1)*
$J^*(\boldsymbol{x}, s) \leq J_D(\boldsymbol{x}, s), \quad \forall \boldsymbol{x} \geq 0.$

This upper bound can be used to test the performance of heuristics. Şen [22] uses it together with a lower bound to obtain a heuristic for the single product case. In particular, it is shown that x times the optimal expected revenue obtained by selling 1 unit of product over a period of length s/x is a lower bound on the optimal expected revenue obtained by selling x units of product over a period of length s. More formally, the following theorem holds:

Theorem 2. *(Şen [22], Theorem 1)*
$x \cdot J^*(1, s/x) \leq J^*(x, s), \quad \forall x \geq 0.$

Unfortunately, the idea behind this lower bound, which is to divide the selling season into x periods, cannot be directly generalized for the multiple product case where initial inventory levels are denoted by the m dimensional vector $\boldsymbol{x}$ rather than the scalar x.

Chapter 4

Exponential Price Response with Identical Elasticity Parameters

In this chapter, we will focus on a special case of exponential price response, where the demand rate for product j is denoted by $\lambda_j(p_j) = a_j\, \mathrm{e}^{-\alpha_j p_j}$ and α_j's are identical. The elasticity of exponential price response decreases linearly with price. Elasticity can be defined as the ratio of the percentage change in demand to the percentage of causative change in price. Hence, by definition, each product has elasticity $\frac{d\lambda_j/\lambda_j}{dp_j/p_j} = -\alpha_j \cdot p_j$. In our special case, α_j's of all products are the same. Closed form solutions to the stochastic problem (3.1)-(3.3) and some structural results will be presented for this special case.

Assume that $\alpha_1 = \alpha_2 = .. = \alpha_n$. In this case, we can take $\alpha_j = 1 \quad \forall j$ by changing the units of prices to $p'_j = \alpha_j p_j$. As a result, the demand rate for product j can be denoted by $\lambda_j(p_j) = a_j\, \mathrm{e}^{-p_j}$. We will now solve (3.4) and obtain closed form representations of $J^*(\boldsymbol{x}, s)$ for any product-resource structure in general.

Theorem 3. *If the demand rate for product j is given by $\lambda_j(p_j) = a_j \mathrm{e}^{-p_j}$, the* *optimal expected revenue $J^*(x,s)$ has the following closed form:*

$$J^*(\boldsymbol{x},s) = \ln\left(\sum_{\substack{\boldsymbol{A}\cdot\boldsymbol{i}\le\boldsymbol{x}\\ \boldsymbol{i}\in Z_+^n}} \left(\frac{s}{\mathrm{e}}\right)^{i_1+..+i_n} \frac{a_1^{i_1}..a_n^{i_n}}{i_1!..i_n!}\right) \tag{4.1}$$

Consequently, optimal price and intensity as a function of $\boldsymbol{x}$ and s for product j can be calculated as

$$p_j^*(\boldsymbol{x},s) = 1 + J^*(\boldsymbol{x},s) - J^*(\boldsymbol{x}-\boldsymbol{A^j},s), \tag{4.2}$$

$$\lambda_j^*(\boldsymbol{x},s) = a_j\mathrm{e}^{-(1+J^*(\boldsymbol{x},s)-J^*(\boldsymbol{x}-\boldsymbol{A^j},s))}. \tag{4.3}$$

Proof. We will prove (4.1) by induction. (4.2) and (4.3) follow immediately.
Base case: The initial amounts of m resources, which is denoted by $\boldsymbol{x}$, allow selling only x unit of a certain type of product, say product j. In other words, $\boldsymbol{x}$ is a positive integer multiple of j^{th} unit vector $\boldsymbol{e_j}$. Note that Gallego & van Ryzin [19] show that the optimal expected revenue $J^*(x,s)$ for the single product case is

$$J^*(x,s) = \ln\left(\sum_{i=0}^{x}\left(\frac{as}{\mathrm{e}}\right)^i \frac{1}{i!}\right) \tag{4.4}$$

Hence, the theorem holds for any $\boldsymbol{x}$ that can be represented as $\boldsymbol{x} = x\cdot\boldsymbol{e_j}$ where x is a positive integer, since (4.1) is equivalent to (4.4).
Inductive hypothesis: Assume w.l.o.g. that $\boldsymbol{A}\cdot\boldsymbol{e_j}\le\boldsymbol{x}$ $\forall j=1..n$, i.e., we assume that there are enough resources to produce any type of product. Then, suppose also that the theorem holds for $\bar{\boldsymbol{x}} = \boldsymbol{x}-\boldsymbol{A^j}$ for all $j=1..n$. This implies that the following holds:

$$J^*(\bar{\boldsymbol{x}},s) = \ln\left(\sum_{\substack{\boldsymbol{A}\cdot\boldsymbol{i}\le\bar{\boldsymbol{x}}\\ \boldsymbol{i}\in Z_+^n}} \left(\frac{s}{\mathrm{e}}\right)^{i_1+..+i_n} \frac{a_1^{i_1}..a_n^{i_n}}{i_1!..i_n!}\right) \tag{4.5}$$

Notice that by subtracting appropriate $\boldsymbol{A^j}$'s from any vector $\boldsymbol{x}$, we can obtain $\bar{\boldsymbol{x}}$ of the form examined in the base case (a positive integer multiple of a unit vector).

Inductive step: We will now argue that (4.5) holds for $\boldsymbol{x}$. We know that the Hamilton-Jacobi sufficient condition for optimal expected revenue is

$$\frac{\partial J^*(\boldsymbol{x},s)}{\partial s} = \sup_{\boldsymbol{\lambda}} \left\{ \sum_{j=1}^{n} r_j(\lambda_j) - \sum_{j=1}^{n} \lambda_j (J^*(\boldsymbol{x},s) - J^*(\boldsymbol{x} - \boldsymbol{A}^j, s)) \right\} \tag{4.6}$$

One can easily show that $\lambda_j^* = \frac{a_j}{\mathrm{e}^{1+J^*(\boldsymbol{x},s)-J^*(\boldsymbol{x}-\boldsymbol{A}^j,s)}}$. By substituting it into (4.6) we get

$$\frac{\partial J^*(\boldsymbol{x},s)}{\partial s} = \sum_{j=1}^{n} \frac{a_j}{\mathrm{e}^{1+J^*(\boldsymbol{x},s)-J^*(\boldsymbol{x}-\boldsymbol{A}^j,s)}} \tag{4.7}$$

By the inductive hypothesis, we have

$$\begin{aligned}
\frac{\partial J^*(\boldsymbol{x},s)}{\partial s} &= \sum_{j=1}^{n} \frac{a_j \displaystyle\sum_{\substack{\boldsymbol{A}\cdot\boldsymbol{i} \le \boldsymbol{x} - \boldsymbol{A}^j \\ \boldsymbol{i}\in Z_+^n}} \left(\frac{s}{\mathrm{e}}\right)^{i_1+..+i_n} \frac{a_1^{i_1}..a_n^{i_n}}{i_1!..i_n!}}{\mathrm{e}^{1+J^*(\boldsymbol{x},s)}} \\
&= \sum_{j=1}^{n} \frac{\displaystyle\sum_{\substack{\boldsymbol{A}\cdot\boldsymbol{i} \le \boldsymbol{x} \\ i_j \ge 1}} \left(\frac{s}{\mathrm{e}}\right)^{i_1+..+i_n-1} \frac{a_1^{i_1}..a_n^{i_n}}{i_1!..(i_j-1)!..i_n!}}{\mathrm{e}^{1+J^*(\boldsymbol{x},s)}} \\
&= \frac{\displaystyle\sum_{\substack{\boldsymbol{A}\cdot\boldsymbol{i} \le \boldsymbol{x} \\ i_1 \ge 1}} i_1 \left(\frac{s}{\mathrm{e}}\right)^{i_1+..+i_n-1} \frac{a_1^{i_1}..a_n^{i_n}}{i_1!..i_n!} + ... + \sum_{\substack{\boldsymbol{A}\cdot\boldsymbol{i} \le \boldsymbol{x} \\ i_n \ge 1}} i_n \left(\frac{s}{\mathrm{e}}\right)^{i_1+..+i_n-1} \frac{a_1^{i_1}..a_n^{i_n}}{i_1!..i_n!}}{\mathrm{e}^{1+J^*(\boldsymbol{x},s)}} \\
&= \frac{\displaystyle\sum_{\substack{\boldsymbol{A}\cdot\boldsymbol{i} \le \boldsymbol{x} \\ ||\boldsymbol{i}|| > 0}} \frac{(i_1+..+i_n)}{\mathrm{e}} \left(\frac{s}{\mathrm{e}}\right)^{i_1+..+i_n-1} \frac{a_1^{i_1}..a_n^{i_n}}{i_1!..i_n!}}{\mathrm{e}^{J^*(\boldsymbol{x},s)}}
\end{aligned}$$

Now, it is easy to see that

$$J^*(\boldsymbol{x},s) = \ln \left(\sum_{\substack{\mathbf{A}\cdot\mathbf{i} \le \boldsymbol{x} \\ \mathbf{i}\in Z_+^n}} \left(\frac{s}{\mathrm{e}}\right)^{i_1+..+i_n} \frac{a_1^{i_1}..a_n^{i_n}}{i_1!..i_n!} \right)$$

$\square$

We can deduce several structural results and comparative statics by using this closed form solution. The following corollary and conjectures are examples of such results that provide intuitive justification and economic interpretation.

Corollary 3.1. *If the demand rate for product j is given by $\lambda_j(p_j) = a_j \mathrm{e}^{-p_j}$, the optimal expected revenue $J^*(\boldsymbol{x}, s)$ is strictly increasing in s and non-decreasing in all x_i's.*

Proof. The first part of Corollary 3.1 follows directly from the fact that $J^*(x, s)$ is the natural logarithm of a polynomial in s. For the second part, define the set $I_{\boldsymbol{x}} = \{i : A \cdot i \leq x\}$. If $\boldsymbol{x} > \hat{\boldsymbol{x}}$, then $I_{\boldsymbol{x}} \supseteq I_{\hat{\boldsymbol{x}}}$ which implies that either $I_{\boldsymbol{x}} = I_{\hat{\boldsymbol{x}}}$ or $I_{\hat{\boldsymbol{x}}}$ is a strict subset of $I_{\boldsymbol{x}}$. If $I_{\boldsymbol{x}} = I_{\hat{\boldsymbol{x}}}$, then $J^*(\boldsymbol{x}, s) = J^*(\hat{\boldsymbol{x}}, s)$ for $0 \leq s \leq T$. However, if $I_{\boldsymbol{x}} \supset I_{\hat{\boldsymbol{x}}}$, then $J^*(\boldsymbol{x}, s) > J^*(\hat{\boldsymbol{x}}, s)$ for $0 \leq s \leq T$. □

Corollary 3.1 is a formal statement of the intuition that more inventory and/ or time yield higher expected revenues.

Conjecture 3.1. *The optimal price of product j, $p_j^*(\boldsymbol{x}, s)$, (resp., the optimal intensity $\lambda_j^*(\boldsymbol{x}, s)$) is strictly increasing (resp., decreasing) in s for all j's.*

Conjecture 3.1 implies that the optimal price of product j rises for given inventory levels if we have a longer selling period. In other words, the optimal price of product j decreases over time between consecutive demand realizations. A statement similar to Conjecture 3.1 for the single item case is proved in Gallego & van Ryzin [19], Theorem 1. To the best of our knowledge, proof for the multi-product case is not provided in the literature; however, the closed form solution can be used to prove it for the special case of exponential price response. Although we give this result as a conjecture here, it is verified and can be observed in all optimal price path examples provided in Chapter 6.

Conjecture 3.2. *The optimal price of product j, $p_j^*(\boldsymbol{x}, s)$, is decreasing in all x_i's with $a_{ij} > 0$ and increasing in all x_i's with $a_{ij} = 0$.*

Gallego & van Ryzin [19] prove for the single item problem that the optimal price $p^*(x, s)$ is strictly decreasing in s. In other words, a demand realization which decreases the inventory level of the single product by one unit leads to an upward jump on the optimal price path. Conjecture 3.2 is a generalization of this argument for the multi-product case. It implies that a demand realization for product j, which decreases the inventory levels of the resources used by product j, causes an upward jump on the price path of products that share a resource with product j. It conversely causes a fall on the optimal price path of the products that do not share any resources with product j. Beyond the single item case, this conjecture reveals the mechanics of network effects in multi-product dynamic pricing problem. In Chapter 6, these network effects on the optimal price paths are examined and verified by examples.

Chapter 5

Heuristic Methods

It is very difficult, if not impossible, to obtain analytical solutions for the system of partial differential equations in (3.4) for general price response functions. It is possible to solve it numerically, but only for simple product-resource structures and limited number of initial resources. Hence, heuristic methods are crucial in such problems. We will present two types of dynamic pricing heuristics. The first type uses the idea of approximating the value function in the Hamilton-Jacobi equation given in (3.4). The second type of heuristics are based on the deterministic problem.

5.1 Heuristics Using Value Approximation

An important observation is that one can write the optimal demand rate $\lambda_j^*(\boldsymbol{x}, s)$ (or optimal price $p_j^*(\boldsymbol{x}, s)$) at time s with remaining inventory $\boldsymbol{x}$ in terms of $J^*(\boldsymbol{x}, s)$ and $J^*(\boldsymbol{x} - \boldsymbol{A^j}, s)$ by using (3.4) which we restate here as:

$$\frac{\partial J^*(\boldsymbol{x}, s)}{\partial s} = \sup_{\lambda_1,..,\lambda_n} \left\{ \sum_{j=1}^{n} r_j(\lambda_j) - \sum_{j=1}^{n} \lambda_j \left(J^*(\boldsymbol{x}, s) - J^*(\boldsymbol{x} - \boldsymbol{A}^j, s) \right) \right\} \tag{5.1}$$

Hence, if J^* can be approximated in some way, then the demand rates, which are the control variables in (5.1), can be found by using the approximated J^*, say $\tilde{J}$. In other words, the procedure we follow is:

1. Approximate the optimal expected revenue function $J^*(\boldsymbol{x}, s)$ with a proper function and call it $\tilde{J}$.

2. Find the demand rates $\tilde{\lambda}$ by using $\tilde{J}$ in the maximizer of (5.1), or more formally:

$$\tilde{\lambda}_j(\boldsymbol{x}, s) = \arg\sup_{\lambda_j} \left\{ \sum_{j=1}^{n} r_j(\lambda_j) - \sum_{j=1}^{n} \lambda_j \left(\tilde{J}(\boldsymbol{x}, s) - \tilde{J}(\boldsymbol{x} - \boldsymbol{A}^j, s) \right) \right\} \tag{5.2}$$

   For example, for the linear price response function $\lambda_j(p_j) = a_j - b_j\, p$ we can obtain

$$\tilde{\lambda}_j(\boldsymbol{x}, s) = \frac{a_j - b_j\big(\tilde{J}(\boldsymbol{x}, s) - \tilde{J}(\boldsymbol{x} - \boldsymbol{A}^j, s)\big)}{2} \tag{5.3}$$

   and for exponential price response $\lambda_j(p_j) = a_j\, \mathrm{e}^{-\alpha_j p_j}$, we have

$$\tilde{\lambda}_j(\boldsymbol{x}, s) = \frac{a_j}{\mathrm{e}^{1+\alpha_j \left(\tilde{J}(\boldsymbol{x}, s) - \tilde{J}(\boldsymbol{x} - \boldsymbol{A}^j, s) \right)}} \tag{5.4}$$

This is similar to the approximate dynamic programming approach used in [28] and [29] in which the value function of Hamilton-Jacobi equation is approximated. One can find $\tilde{J}$ in various ways, two of which we will discuss in the following subsections. In 5.1.1, we will present an expression for $\tilde{J}$ which is indeed a generalization of the closed form solution in (4.1) given for the special case of exponential price response with equal α_j's. In 5.1.2 and 5.1.3, we will use (4.1) together with two different parameter transformations (one for exponential and the other for linear price response parameters) to find $\tilde{J}$. These approaches give us two heuristics using value approximation. A similar heuristic for the single item problem is given by Şen [22].

5.1.1 A Generalization for Exponential Price Response With Unequal α_j's

We can modify the closed form solution given in (4.1) for the case of general form of exponential price-response i.e. $\lambda_j(p_j) = a_j\, \mathrm{e}^{-\alpha_j p_j}$. This approach gives a good approximation of the optimal expected revenue under general form of exponential price-response where α_j's are not identical. This approximation can be used as $\tilde{J}$ and is expressed as follows:

$$\tilde{J}(\boldsymbol{x}, s) = \ln\left[\sum_{\substack{\boldsymbol{A}*\boldsymbol{i}=\boldsymbol{x}\\ \boldsymbol{i}\in\mathbb{Z}_+^n}} \left(\left(\sum_{k_1=0}^{i_1} \frac{\left(\frac{s\,a_1}{\mathrm{e}}\right)^{k_1}}{k_1!}\right)^{\frac{1}{\alpha_1}} \cdots \quad \left(\sum_{k_n=0}^{i_n} \frac{\left(\frac{s\,a_n}{\mathrm{e}}\right)^{k_n}}{k_n!}\right)^{\frac{1}{\alpha_n}}\right)\right] \tag{5.5}$$

This expression is basically a more general form of (4.1). When $\alpha_j = 1\ \forall j$, (5.5) is equivalent to (4.1). If we plug this expression into (5.1) in order to check whether it is the solution of the system of differential equations, we unfortunately see that it is not. However, it gives very close results for the optimal expected revenue when compared with the numerical solutions.

After finding $\tilde{J}$, corresponding $\tilde{\lambda}_j$ values can be found as explained in the second step of the procedure above. The expected revenue obtained by this heuristic is denoted with J_{RA1} in the numerical analysis provided in Chapter 6. J_{RA1} can be calculated by plugging $\tilde{\lambda}_j$'s in (5.1), or more formally

$$\frac{\partial J_{RA1}(\boldsymbol{x}, s)}{\partial s} = \sum_{j=1}^{n} r_j(\tilde{\lambda}_j) - \sum_{j=1}^{n} \tilde{\lambda}_j \left(J_{RA1}(\boldsymbol{x}, s) - J_{RA1}(\boldsymbol{x} - \boldsymbol{A}^j, s)\right) \tag{5.6}$$

5.1.2 Using Closed Form Solution in General Form of Exponential Price Response

In the exponential price response with equal α_j's, we write the demand function as $\lambda_j(p) = a_j \mathrm{e}^{-p}$, so we have only one parameter for each product, namely a_j. On the other hand, the demand function is $\lambda_j(p) = a_j \mathrm{e}^{-\alpha_j p}$ in the general form of exponential price response with unequal α_j's. So, we have two parameters for each product in this case, namely a_j and α_j. In order to use (4.1) as $\tilde{J}$ when the actual demand is exponential with unequal α_j's, we need to find the single parameter a_j of the special case of exponential price response in some way. Hence, our aim is to find a correspondence between the single parameter of the special case, a_j (a'_j hereafter to avoid confusion), and two parameters of the general case, a_j and α_j.

In Table 5.1, maximizers λ^0 and p^0 of the revenue rate $r(\lambda)$ for exponential and linear price responses are shown . We will build the correspondence between a'_j and $(a_j,\, \alpha_j)$ by equalizing the maximum instantaneous revenue rates $r_j(\lambda_j^0)$ of special and general forms of exponential price response. Hence, we find a'_j by the following transformation:

$$r_j(\lambda_j^0) = \frac{a_j}{\mathrm{e}}\frac{1}{\alpha_j} = \frac{a'_j}{\mathrm{e}} \Rightarrow a'_j = \frac{a_j}{\alpha_j} \tag{5.7}$$

Then, $\tilde{J}$ is calculated as

$$\tilde{J}(\boldsymbol{x}, s) = \ln\left(\sum_{\substack{\boldsymbol{A}\cdot\boldsymbol{i}\leq\boldsymbol{x} \\ \boldsymbol{i}\in Z_+^n}} \left(\frac{s}{\mathrm{e}}\right)^{i_1+..+i_n} \frac{(a'_1)^{i_1}..(a'_n)^{i_n}}{i_1!..i_n!}\right) \tag{5.8}$$

Table 5.1: Maximizers λ^0 and p^0 of revenue rate function $r(\lambda)$

	$\lambda(p)$	$p(\lambda)$	λ^0	p^0
Exponential	$a\mathrm{e}^{-\alpha p}$	$\frac{ln(a)-ln(\lambda)}{\alpha}$	$\frac{a}{\mathrm{e}}$	$\frac{1}{\alpha}$
Linear	$a-bp$	$\frac{a-\lambda}{b}$	$\frac{a}{2}$	$\frac{a}{2b}$

Again, $\tilde{\lambda}_j$'s can be calculated by using $\tilde{J}$. The performance of this heuristic is further discussed in Chapter 6 with some numerical results. We denote the expected revenue obtained by this heuristic as J_{RA2} which can be calculated similar to J_{RA1}.

5.1.3 Using Closed Form Solution in General Price Response

The same idea of using the closed form solution of the special case to approximate the value function J^* can be applied for any price response in general. The only difference is in the conversion of the parameters. Here, we will explain how it is done for linear price response. We will again equalize maximum revenue rates $r_j(\lambda_j^0)$ but we have different parameters in this case. If we denote the parameters of the linear price response with a_j^L's and b_j's and the single parameter of the special case of exponential price response with a_j^E's, then the conversion is as follows:

$$r(\lambda_j^*) = \frac{a_j^L}{2}\frac{a_j^L}{2b} = \frac{a_j^E}{\mathrm{e}} \Rightarrow a_j^E = \frac{(a_j^L)^2\mathrm{e}}{2b} \tag{5.9}$$

Now, $\tilde{J}$ can be calculated as

$$\tilde{J}(\mathbf{x}, s) = \ln\left(\sum_{\substack{\mathbf{A}\cdot\mathbf{i}\leq\mathbf{x}\\ \mathbf{i}\in Z_+^n}} \left(\frac{s}{\mathrm{e}}\right)^{i_1+..+i_n} \frac{(a_1^E)^{i_1}..(a_n^E)^{i_n}}{i_1!..i_n!}\right) \tag{5.10}$$

We denote the the revenue obtained by this heuristic as J_{RA2} in the numerical results presented in Chapter 6.

5.2 Heuristics Based on Deterministic Problem

In this section, heuristic approaches based on the deterministic problem will be proposed. First, fixed price heuristics proposed by Gallego & van Ryzin [4] will be described in Section 5.2.1. Then, we will present two dynamic pricing heuristics in Sections 5.2.2 and 5.2.3. In Chapter 6, the performance comparison for all heuristics will be done with numerical examples.

5.2.1 Fixed Price Heuristics

Gallego & van Ryzin [4] proposes two heuristics that are asymptotically optimal as the initial inventories and the length of selling period tend to infinity. These heuristics, namely *make to stock* (MTS) and *make to order* (MTO) heuristics, determine fixed prices over the entire selling period based on the solution of the deterministic problem. In MTS heuristic, resources are initially allocated to products that are to be sold until their allocated resource capacity is exhausted or the selling period ends. No resource transfers can be made among products after initial allocation. In MTO heuristic, no initial allocation is done, the products are sold in a first-come-first-served order. The deterministic problem is easy to solve and the fixed price is easy to implement; however, it is reasonable to question whether it is possible to acquire more revenue via dynamic pricing. This additional revenue may have a profound effect on profitability in industries such as airlines and retail.

Suppose the solution of the deterministic problem is λ_j^D and the corresponding fixed price is p_j^D for products $j = 1..n$. Then, denote the number of product j to be sold (according to the deterministic problem) during the selling period with $y_j = \lfloor \lambda_j^D \cdot T \rfloor$. Remember from Chapter 3 that the counting process N_T^j represents the number of product j sold up to time T. Expected revenue obtained from MTS heuristic is then

$$J_{MTS} = \sum_{j=1}^{n} p_j^D \cdot \mathbb{E}\Big[\min \left\{ y_j, N_T^j \right\} \Big] \tag{5.11}$$

We find the expected revenue obtained from MTO heuristic, J_{MTO} by plugging λ_j^D into the system of partial differential equations in (3.4). Hence, we can write

$$\frac{\partial J_{MTO}(\boldsymbol{x}, s)}{\partial s} = \sum_{j=1}^{n} r_j(\lambda_j^D) - \sum_{j=1}^{n} \lambda_j^D \left(J_{MTO}(\boldsymbol{x}, s) - J_{MTO}(\boldsymbol{x} - \boldsymbol{A}^j, s) \right) \tag{5.12}$$

5.2.2 Resolving the Deterministic Problem Continuously

In order to reveal the advantage of dynamic pricing, we propose two heuristics. The first one solves the deterministic problem continuously over the selling period and sets the prices of each product dynamically. The main idea here is to take advantage of the information of demand realization up to current time in the selling period. This is basically a resolving approach which received considerable attention in the literature as mentioned in Chapter 1. (See Cooper [25], Maglaras & Meissner [26]) This heuristic is similar to the *run-out rate heuristic* proposed by Şen [22] for single product case. Hence, we will call this heuristic as run-out rate (RR) heuristic.

5.2.3 Dynamic Pricing after Resource Allocation

The second heuristic decomposes the multiple product problem into simpler single product problems. As in MTS heuristic, resources are allocated to products based on the solution of the deterministic problem, then optimal dynamic pricing is implemented for each product. Hence, this is a mixture of the MTS heuristic and the optimal pricing of single products. This heuristic is abbreviated by ATD which stands for *allocate-then-dynamic*. The expected revenue obtained from this heuristic, J_{ATD}, can be written as

$$J_{ATD} = \sum_{j=1}^{n} J_j^*(y_j, T) \tag{5.13}$$

where $J_j^*(y_j, T)$ is the optimal expected revenue function for the single item j for a given inventory level y_j and selling period T, $y_j = \lambda_j^D \cdot T$ and λ_j^D is the solution of the deterministic problem.

Chapter 6

Numerical Analysis

In this chapter, numerical results are reported to compare the performance of heuristics. All operations in numeric analysis including solution of the deterministic problem and system of partial differential equations are performed in Maple 15 with default methods and error tolerances. There are two product-resource structures in particular that are covered by our numerical analysis. The first one is an example from retail industry where bundling is a common practice. This example is related with the dynamic pricing of bundle and individual products. The second example is from airline industry where there is a network of flight legs which constitute many different itineraries. Dynamic pricing of these itineraries is of great importance to this industry as mentioned in Chapter 1. The airline network we used in the numerical analysis is the same network proposed by Gallego & van Ryzin [4] which enables fair comparisons.

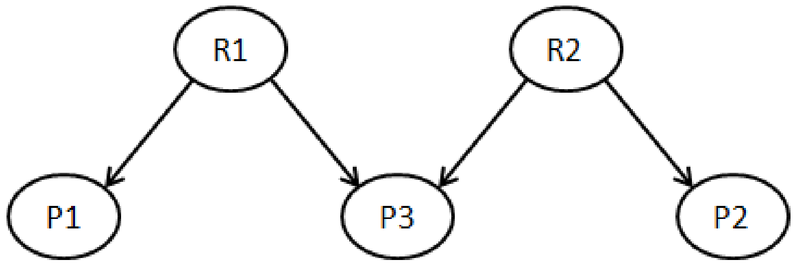


Figure 6.1: Product resource structure of retail example

6.1 Performance of Heuristics Based on Deterministic Problem

6.1.1 Example from Retail Industry

In this section, we will consider the product resource structure shown in Figure 6.1. There are two resources R1 and R2 and three products P1, P2 and P3. In words, P1 and P2 are individual products and P3 is the bundle.

6.1.1.1 Linear Price Response

In linear price response, the demand rate for product j is denoted by $\lambda_j(p_j) = a_j - b_j p_j$. We used three parameter sets in the following numerical results. In all sets, $a_1 = a_2 = a_3 = 2$ and $b_1 = b_2 = 1$, but b_3 changes in each parameter set and takes one of the values from the set $\{2/3, 4/7, 1/2\}$. So, the demand function of the third product changes as shown in Figure 6.2.

Expected revenues of each heuristic and their percentages to optimal expected revenue J^* for all parameter sets are tabulated in Table 6.1, 6.2 and 6.3. There are two levels of T, one represents short ($T = 10$) and the other represents long selling period ($T = 40$). The column x_j shows the initial inventory for each resource type $j = 1, 2$. In fact, $x_1 = x_2$ in all cases.

The values in the J^* column are obtained by solving the system of partial

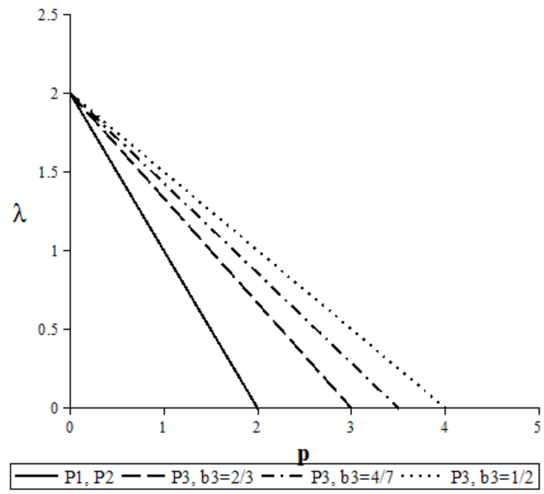


Figure 6.2: Linear demand vs. price

differential equations in (3.4) numerically. The values in J_{MTS}, J_{MTO} and J_{ATD} columns are calculated as shown in (5.11), (5.12) and (5.13), respectively. The deterministic problem given in (3.9), which we solve continuously over the selling period in RR heuristic, is a linearly constrained quadratic programming problem in linear price response case. We were able to write the optimal solution of the deterministic problem for linear price response as a piecewise function of the problem parameters by using KKT conditions. In other words, we constructed a piecewise function which has 16 branches with conditions on remaining time s, remaining inventory vector $\boldsymbol{x}$, a_j's and b_j's and gives the optimal demand rate. When we plug this function in (3.4) as the demand rate, we obtain the expected revenue of the RR heuristic, J_{RR}, which solves the deterministic problem continuously.

Table 6.1: Expected Revenues, Linear Demand, $a = [2, 2, 2], b = [1, 1, 2/3]$

T	x_j	J^*	J_{MTS}	J_{MTS}/J^*	J_{MTO}	J_{MTO}/J^*	J_{RR}	J_{RR}/J^*	J_{ATD}	J_{ATD}/J^*
	1	3.340	2.402	0.719	2.402	0.719	3.278	0.982	3.333	0.998
	2	6.324	5.251	0.830	5.251	0.830	6.246	0.988	6.265	0.991
	3	9.071	7.915	0.873	7.915	0.873	8.969	0.989	8.833	0.974
	4	11.634	9.716	0.835	10.303	0.886	11.515	0.990	11.333	0.974
10	5	14.028	12.101	0.863	12.714	0.906	13.902	0.991	13.561	0.967
	10	23.708	21.826	0.921	22.684	0.957	23.555	0.994	22.879	0.965
	20	33.305	30.621	0.919	31.481	0.945	32.532	0.977	32.412	0.973
	30	34.957	33.785	0.966	34.924	0.999	34.941	1.000	34.769	0.995
	1	3.810	2.497	0.655	2.497	0.655	3.748	0.984	3.810	1.000
	2	7.502	5.689	0.758	5.689	0.758	7.401	0.986	7.502	1.000
	3	11.085	8.962	0.808	8.962	0.808	10.958	0.989	11.084	1.000
	4	14.565	12.230	0.840	12.230	0.840	14.422	0.990	14.561	1.000
40	5	17.943	15.460	0.862	15.460	0.862	17.794	0.992	17.936	1.000
	10	33.491	30.621	0.914	30.621	0.914	33.346	0.996	33.304	0.994
	20	60.420	55.279	0.915	56.718	0.939	60.212	0.997	59.750	0.989
	30	83.060	77.734	0.936	79.389	0.956	82.853	0.998	81.956	0.987

Table 6.2: Expected Revenues, Linear Demand, $a = [2, 2, 2], b = [1, 1, 4/7]$

T	x_j	J^*	J_{MTS}	J_{MTS}/J^*	J_{MTO}	J_{MTO}/J^*	J_{RR}	J_{RR}/J^*	J_{ATD}	J_{ATD}/J^*
	1	3.375	2.402	0.712	2.402	0.712	3.300	0.978	3.333	0.988
	2	6.504	5.251	0.807	5.251	0.807	6.381	0.981	6.265	0.963
	3	9.441	7.353	0.779	7.910	0.838	9.297	0.985	9.182	0.973
	4	12.198	10.017	0.821	10.636	0.872	12.048	0.988	11.750	0.963
10	5	14.783	12.510	0.846	13.226	0.895	14.633	0.990	14.315	0.968
	10	25.266	23.190	0.918	24.076	0.953	25.104	0.994	24.321	0.963
	20	35.666	32.808	0.920	33.655	0.944	34.750	0.974	34.727	0.974
	30	37.454	36.198	0.966	37.417	0.999	37.436	1.000	37.252	0.995
	1	3.810	2.497	0.655	2.497	0.655	3.749	0.984	3.810	1.000
	2	7.504	5.689	0.758	5.689	0.758	7.407	0.987	7.502	1.000
	3	11.096	8.962	0.808	8.962	0.808	10.976	0.989	11.084	0.999
	4	14.599	12.230	0.838	12.230	0.838	14.462	0.991	14.561	0.997
40	5	18.032	15.460	0.857	15.460	0.857	17.874	0.991	17.936	0.995
	10	34.400	29.585	0.860	30.717	0.893	34.158	0.993	34.079	0.991
	20	63.563	57.816	0.910	59.322	0.933	63.303	0.996	62.787	0.988
	30	88.162	82.311	0.934	84.003	0.953	87.933	0.997	86.936	0.986

Table 6.3: Expected Revenues, Linear Demand, $a = [2, 2, 2], b = [1, 1, 1/2]$

T	x_j	J^*	J_{MTS}	J_{MTS}/J^*	J_{MTO}	J_{MTO}/J^*	J_{RR}	J_{RR}/J^*	J_{ATD}	J_{ATD}/J^*
	1	3.516	2.402	0.683	2.402	0.683	3.419	0.973	3.333	0.948
	2	6.843	4.804	0.702	5.238	0.765	6.703	0.980	6.667	0.974
	3	9.978	7.653	0.767	8.227	0.825	9.822	0.984	9.598	0.962
	4	12.926	10.502	0.812	11.136	0.861	12.767	0.988	12.530	0.969
10	5	15.692	13.166	0.839	13.892	0.885	15.535	0.990	15.098	0.962
	10	26.914	24.736	0.919	25.610	0.952	26.741	0.994	25.943	0.964
	20	38.041	34.996	0.920	35.829	0.942	36.976	0.972	37.042	0.974
	30	39.951	38.611	0.966	39.910	0.999	39.931	0.999	39.736	0.995
	1	3.862	2.497	0.647	2.497	0.647	3.791	0.982	3.810	0.986
	2	7.671	4.994	0.651	5.444	0.710	7.549	0.984	7.619	0.993
	3	11.426	8.186	0.716	8.802	0.770	11.265	0.986	11.311	0.990
	4	15.127	11.378	0.752	12.064	0.797	14.936	0.987	15.003	0.992
40	5	18.776	14.651	0.780	15.469	0.824	18.561	0.989	18.586	0.990
	10	36.254	30.920	0.853	32.013	0.883	35.976	0.992	35.871	0.989
	20	67.474	61.242	0.908	62.701	0.929	67.197	0.996	66.607	0.987
	30	93.823	87.513	0.933	89.175	0.950	93.584	0.997	92.482	0.986

One important observation from Table 6.1,6.2 and 6.3 is that the optimal expected revenue is inreasing in both $\boldsymbol{x}$ and s, which is parallel to Corollary 3.1 for a different price response. It can be seen graphically from Figure 6.3. Besides, expected revenues are increasing in both $\boldsymbol{x}$ and s in all heuristics, which is intuitive. MTO heuristic outperforms MTS heuristic in all cases. A similar observation is done by Gallego & van Ryin [4]. Their interpretation is that "protecting" resources for certain products, which is done in MTS heuristic, does not perform well. The first-come-first-served order, which is followed in MTO heuristic, yields better results. In other words, the inventory flexibility provided by MTO heuristic has a value in terms of the expected revenues. However, it is possible in principle for MTS heuristic to outperform MTO heuristic.

Another observation is that RR and ATD heuristics consistently outperform fixed price heuristics, which indicates the advantage of dynamic pricing. (The only exception is when $x_1 = x_2 = 30$ and when $T = 10$. For this instance, ATD

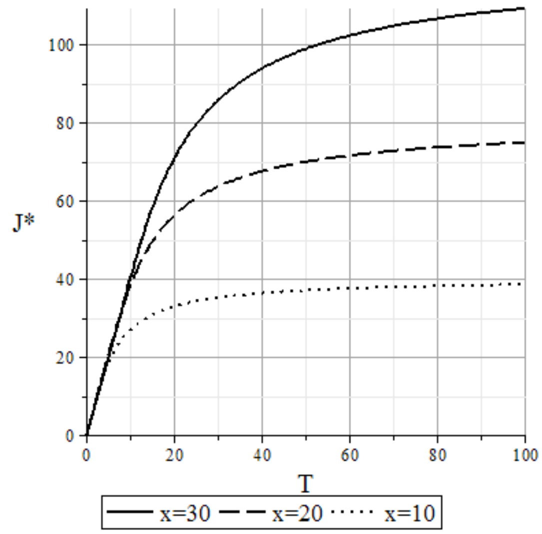


Figure 6.3: Optimal expected revenue is increasing in $\boldsymbol{x}$ and s

performs slightly worse than MTO) RR heuristic provides $\sim 15\%$ additional revenue compared to MTO heuristic for $x_j \leq 5$ and $\sim 5\%$ additional revenue for $x_j = 10, 20, 30$. It is not surprising that ATD heuristic always outperforms MTS heuristic, since prices are set according to optimal dynamic pricing in ATD heuristic whereas fixed prices of the deterministic solution are used in MTS heuristic, after the same initial resource allocation is done in both. A more insightful observation is the comparison of ATD and MTO heuristics. This comparison gives an idea about whether price flexibility of ATD heuristic or inventory flexibility of MTO heuristic is more favorable. It seems that price flexibility is more important in almost all cases with the only exceptional instance with short selling period ($T = 10$) and high initial inventory levels ($x_1 = x_2 = 30$).

It is observed that revenues increase as b_3 gets larger. This is intuitive since the same demand level can be achieved with higher prices for product 3 as b_3 gets larger, which can be seen in Figure 6.2.

It is also interesting to see how the price of each product behaves under different policies. To this end, we will consider a certain demand realization which is shown in Figure 6.4. We will observe one unit of demand for product 1,2 and 3 at time $s = 3, 6, 9$, respectively. Suppose that we have 5 units of both resources at the beginning of the selling period which is of length $T = 10$. Then, the price paths of each product under different pricing policies are shown in Figure 6.5, 6.6, 6.7 and 6.8.

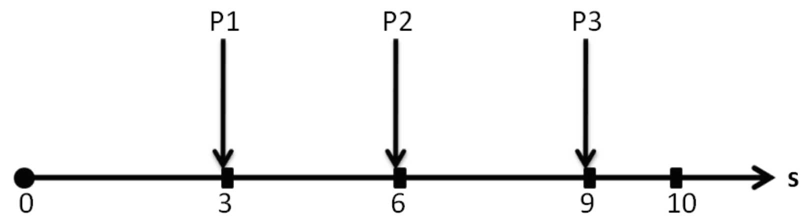


Figure 6.4: Demand realization

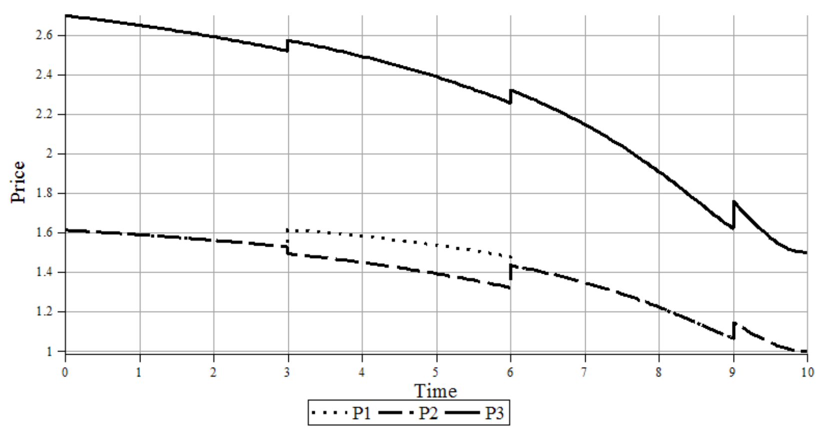


Figure 6.5: Price paths of 3 products under optimal dynamic pricing policy

MTS and MTO heuristics are fixed price heuristics, so the price level does not change during the selling period. The fixed demand rates (and corresponding price levels) are determined by the deterministic problem given in (3.9).

Notice that a realization of demand for a product causes an upward jump on the optimal price path of that particular product. Besides, a realization of demand for a product also causes an upward jump on the price paths of the other products that shares common resources. As an example, at $s = 3$, we observe a demand for product 1 which created an upward jump on the price path of product 1. It also caused an upward jump on the price path of product 3 since product 1 and 3 shares the common resource 1. Although products 1 and 2 do not share any resource, the price of product 2 is also affected by the sales of a unit of product 1

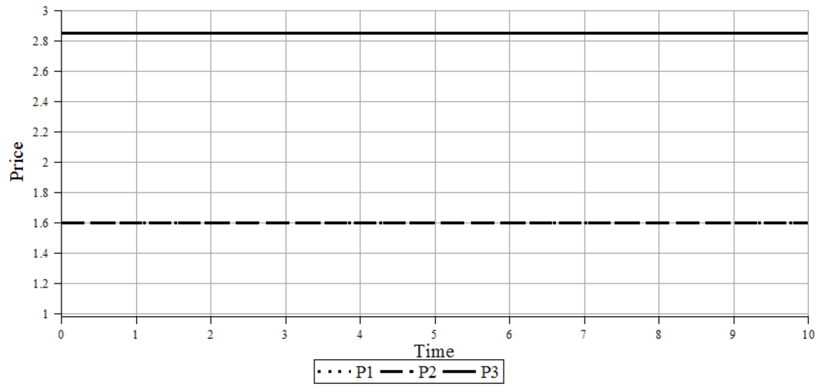


Figure 6.6: Price paths of 3 products under MTS and MTO heuristics

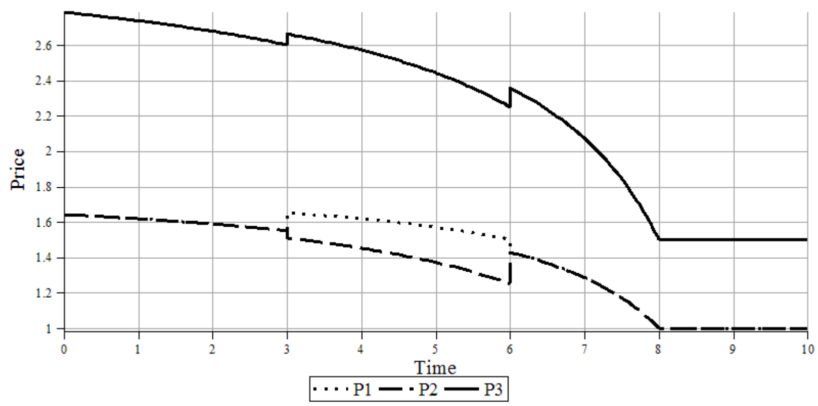


Figure 6.7: Price paths of 3 products under RR heuristics

at $s = 3$. This is the network effect which makes the problem interesting. Notice also that prices reduce between two consecutive demand realizations for each product. Figure 6.9 indicates how close the price path of product 3 is determined by different pricing policies compared to the optimal dynamic pricing policy. Note that the same demand realization given in Figure 6.4 is assumed for all pricing policies just for demonstration; however, the actual demand realization depends on the prices determined and hence differs under each pricing policy.

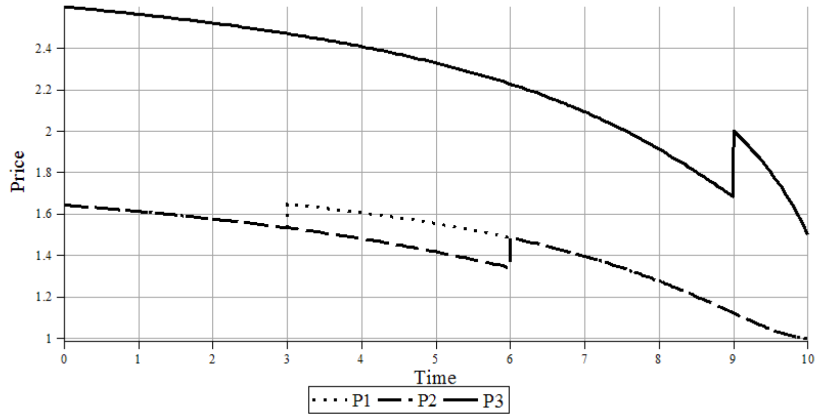


Figure 6.8: Price paths of 3 products under ATD heuristics

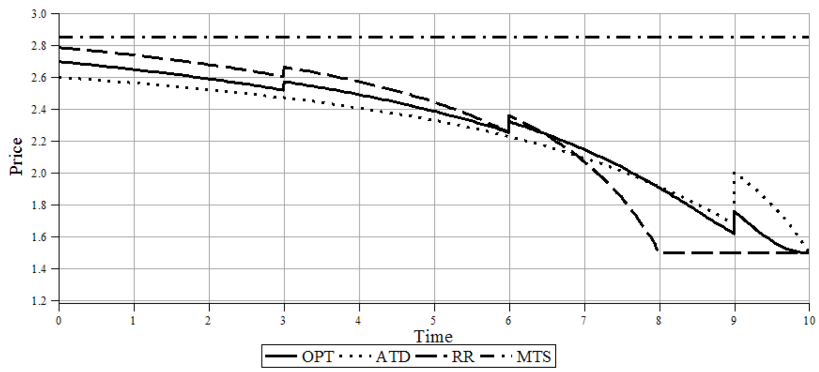


Figure 6.9: Price paths of product 3 in all pricing policies

6.1.1.2 Exponential Price Response

In exponential price response, the demand rate for product j is denoted by $\lambda_j(p_j) = a_j\, e^{-\alpha_j p_j}$. We again used three parameter sets in the following numerical results. In all sets, $a_1 = a_2 = a_3 = \mathrm{e}$ and $\alpha_1 = \alpha_2 = 1$, but α_3 changes in each parameter set and takes one of the values from the set $\{2/3, 4/7, 1/2\}$. Thus, the demand function of the third product changes as shown in Figure 6.10. Expected revenues of each heuristic and their percentages to optimal expected revenue J^* for all parameter sets are tabulated in Table 6.4, 6.5 and 6.6.

The values in the columns of J^*, J_{MTS}, J_{MTO} and J_{ATD} are obtained as they are calculated in linear price response case. However, J_{RR} values are calculated with a different technique. In exponential price response, we cannot construct

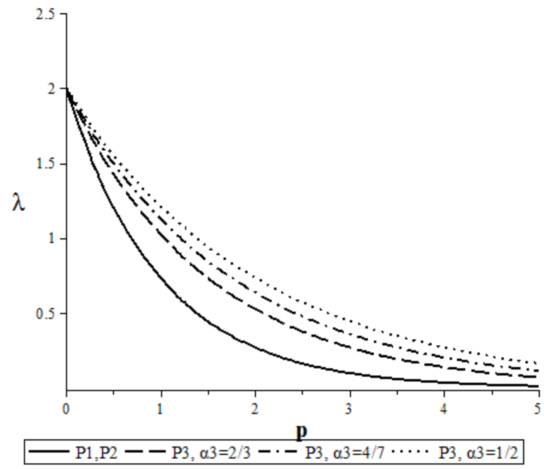


Figure 6.10: Exponential Demand vs. Price

the piecewise demand function which we used in calculating J_{RR} values of linear price response case, since objective function of the deterministic problem has a more complicated structure with natural logarithms in this case. This makes it difficult to handle KKT conditions parametrically. Hence, we use simulation where we discretize the selling period by dividing it into 100 and 400 intervals for $T = 10$ and $T = 40$, respectively. In each interval, the following procedure is followed:

1. Solve the deterministic problem with remaining inventory $\boldsymbol{x}$ and remaining time s to obtain optimal demand rates λ_j for each product.

2. Generate a random number that is uniformly distributed over $[0, 1]$. If it is less than $(\lambda_j \cdot 0.1)$, then we assume that a demand for product j is realized. In other words, a demand for product j in the current time interval is observed with probability $(\lambda_j \cdot 0.1)$. Here, we assumed that the selling period is divided into small enough time intervals so that the Poisson process is approximated by the sequence of Bernoulli trials with success probability $(\lambda_j \cdot 0.1)$. Note that λ_j is recalculated in each time interval based on the remaining time and resource amounts to reflect the effect of resolving in RR heuristic. This step is repeated for all products in each interval.

3. Check the resource constraints for the products with realized demand. If

they are satisfied, then add the current price $p_j(\lambda_j)$ to the revenue and decrease the current inventory by $\boldsymbol{A^j}$. If resource constraint is not satisfied for a product, then skip that product.

4. Move to the next time interval.

We have simulated 50 replications for each entry in the J_{RR} column of Table 6.4, 6.5 and 6.6. The values in the tables are the average revenues of these 50 replications as estimates for expected revenues with relative error of $\pm 5\%$ with 95% confidence.

As for the ATD heuristic, it is possible to calculate the exact values of expected revenue since the optimal expected revenue in exponential price response for single item can be calculated by (4.4). After allocation of resources, single product revenues must be calculated for each product separately and summed up to obtain exact values of expected revenues.

Table 6.4: Expected Revenues, Exponential Demand, $a = [\mathrm{e}, \mathrm{e}, \mathrm{e}], \alpha = [1, 1, 2/3]$

T	x_j	J^*	J_{MTS}	J_{MTS}/J^*	J_{MTO}	J_{MTO}/J^*	J_{RR}	J_{RR}/J^*	J_{ATD}	J_{ATD}/J^*
	1	5.172	4.175	0.807	4.175	0.807	5.166	0.999	4.796	0.927
	2	9.232	7.307	0.791	8.070	0.874	8.954	0.970	8.393	0.909
	3	12.611	10.744	0.852	11.557	0.916	12.182	0.966	11.819	0.937
	4	15.502	13.322	0.859	14.259	0.920	15.307	0.987	14.388	0.928
10	5	18.016	15.971	0.886	16.895	0.938	17.853	0.991	17.022	0.945
	10	26.774	24.431	0.912	25.445	0.950	26.550	0.992	25.544	0.954
	20	33.849	30.621	0.905	31.481	0.930	33.676	0.995	33.112	0.978
	30	34.969	33.785	0.966	34.924	0.999	34.939	0.999	34.825	0.996
	1	7.681	5.928	0.772	5.928	0.772	7.643	0.995	7.427	0.967
	2	14.181	10.374	0.732	11.457	0.808	13.920	0.982	12.998	0.917
	3	19.969	16.103	0.806	17.327	0.868	19.943	0.999	19.040	0.953
	4	25.248	21.161	0.838	22.471	0.890	25.225	0.999	24.272	0.961
40	5	30.131	25.458	0.845	26.955	0.895	29.733	0.987	28.803	0.956
	10	50.530	45.126	0.893	46.901	0.928	50.360	0.997	48.943	0.969
	20	79.705	73.788	0.926	75.733	0.950	79.418	0.996	77.928	0.978
	30	100.001	93.881	0.939	95.872	0.959	98.392	0.984	98.133	0.981

Table 6.5: Expected Revenues, Exponential Demand, $a = [\mathrm{e}, \mathrm{e}, \mathrm{e}], \alpha = [1, 1, 4/7]$

T	x_j	J^*	J_{MTS}	J_{MTS}/J^*	J_{MTO}	J_{MTO}/J^*	J_{RR}	J_{RR}/J^*	J_{ATD}	J_{ATD}/J^*
	1	5.420	4.175	0.770	4.175	0.770	5.417	0.999	4.796	0.885
	2	9.736	7.829	0.804	8.587	0.882	8.690	0.893	8.992	0.924
	3	13.349	11.266	0.844	12.108	0.907	13.162	0.986	12.418	0.930
	4	16.450	14.274	0.868	15.201	0.924	16.378	0.996	15.416	0.937
10	5	19.151	16.922	0.884	17.861	0.933	18.859	0.985	18.050	0.942
	10	28.598	26.176	0.915	27.177	0.950	28.592	1.000	27.368	0.957
	20	36.255	32.808	0.905	33.655	0.928	36.104	0.996	35.477	0.979
	30	37.467	36.198	0.966	37.417	0.999	37.387	0.998	37.313	0.996
	1	7.962	5.928	0.745	5.928	0.745	7.944	0.998	7.427	0.933
	2	14.799	11.115	0.751	12.191	0.824	14.777	0.999	13.926	0.941
	3	20.920	16.844	0.805	18.110	0.866	20.698	0.989	19.968	0.954
	4	26.521	21.857	0.824	23.277	0.878	26.290	0.991	25.255	0.952
40	5	31.714	26.915	0.849	28.435	0.897	31.642	0.998	30.487	0.961
	10	53.495	47.783	0.893	49.583	0.927	52.795	0.987	51.863	0.970
	20	84.822	78.458	0.925	80.444	0.948	84.693	0.998	82.898	0.977
	30	106.708	100.135	0.938	102.157	0.957	106.697	1.000	104.688	0.981

Table 6.6: Expected Revenues, Exponential Demand, $a = [\mathrm{e}, \mathrm{e}, \mathrm{e}], \alpha = [1, 1, 1/2]$

T	x_j	J^*	J_{MTS}	J_{MTS}/J^*	J_{MTO}	J_{MTO}/J^*	J_{RR}	J_{RR}/J^*	J_{ATD}	J_{ATD}/J^*
10	1	5.733	4.175	0.728	4.175	0.728	5.464	0.953	4.796	0.836
	2	10.321	8.351	0.809	9.104	0.882	10.278	0.996	9.592	0.929
	3	14.173	11.788	0.832	12.659	0.893	14.141	0.998	13.018	0.918
	4	17.483	15.225	0.871	16.143	0.923	16.790	0.960	16.443	0.941
	5	20.370	17.874	0.877	18.828	0.924	19.169	0.941	19.078	0.937
	10	30.474	27.921	0.916	28.908	0.949	30.472	1.000	29.193	0.958
	20	38.669	34.996	0.905	35.829	0.927	37.869	0.979	37.842	0.979
	30	39.964	38.611	0.966	39.910	0.999	38.240	0.957	39.800	0.996
40	1	8.386	5.928	0.707	5.928	0.707	8.373	0.999	7.427	0.886
	2	15.626	11.856	0.759	12.926	0.827	15.547	0.995	14.854	0.951
	3	22.129	17.585	0.795	18.892	0.854	22.117	0.999	20.896	0.944
	4	28.088	23.314	0.830	24.720	0.880	27.639	0.984	26.938	0.959
	5	33.617	28.372	0.844	29.915	0.890	32.838	0.977	32.171	0.957
	10	56.840	50.782	0.893	52.577	0.925	56.761	0.999	55.146	0.970
	20	90.295	83.510	0.925	85.499	0.947	90.115	0.998	88.247	0.977
	30	113.693	106.675	0.938	108.702	0.956	113.470	0.998	111.531	0.981

Observations similar to linear demand case can be made from the numerical results of exponential case. Optimal expected revenue is increasing in both $\boldsymbol{x}$ and s. All heuristics give higher expected revenues as $\boldsymbol{x}$ and s increases. Dynamic pricing heuristics (RR and ATD) again outperform fixed price heuristics.(The only exception is when $x_1 = x_2 = 30$ and when $T = 10$. For this instance, RR and ATD performs slightly worse than MTO) The advantage of dynamic pricing is, not surprisingly, more profound when $T = 40$. The effect of the increase in α_3 is similar to the effect of b_3 in linear price response. The same demand level can be achieved with higher prices for product 3 as α_3 gets larger, which can be seen in Figure 6.10.

6.1.2 Example from Airline Industry

The product resource structure in the example from retail industry was relatively a simple structure. What is really expected from the heuristics is that they should be tested and implemented in more complicated product resource structures with larger number of initial inventories. Hence, heuristics are also tested in a product resource structure which is constructed from the airline network shown in Figure 6.11 which we get from Gallego & van Ryzin [4]. The number next to an arc shows the capacity of that flight leg. The products are the origin-destination pairs and the resources are flight legs in this example. All products have exponential price response in general form. Products, resources, problem parameters and the solution of the deterministic problem are shown in Table 6.7. The number of seats allocated to passengers for each origin-destination pair is the vector $\boldsymbol{y} = (y_1, .., y_n)$ where $y_j = \lambda_j^D \cdot T$ as we defined in Chapter 5 (Indeed, we solved a slightly different version of (3.9) where λ_j's are not continuous but discrete, that is why number of seats are always integer values). Number of seats are equal to λ_j^D values for each product since the length of the selling period is $T = 1$ time unit in this example. The optimal deterministic revenue obtained from a single O-D pair can be found by multiplying the number of seats and prices. Total deterministic revenue is the sum of revenues obtained from all O-D pairs.

We have implemented 5 different scales of this problem to compare performances of MTO, MTS, ATD and RR heuristics. For instance, if the scale factor is 2, then the flight leg capacities in Figure 6.11, the parameter a and the number of seats in Table 6.7 are doubled. The optimal deterministic prices do not change with scaling. The aim here is to indicate how heuristics' performance change as the problem size gets larger.

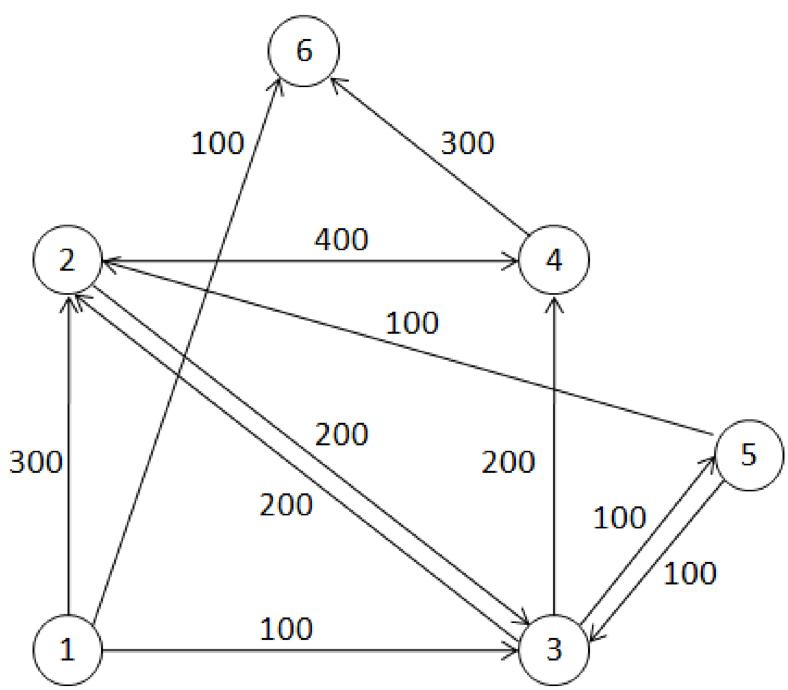


Figure 6.11: An airline network with 6 cities and 11 flight legs

Expected revenues for MTO heuristic are determined by simulation. To do so,

Table 6.7: Origin-destination pairs, exponential demand function parameters and solution of deterministic problem

O	D	α	a	Path	# Seats	Price
1	2	0.0045	815.4845	1-2	135	$396.11
1	3	0.0055	996.0351	1-3	62	$507.71
1	4	0.0050	2,216.7168	1-2-4	165	$519.24
1	5	0.0040	815.4845	1-3-5	38	$769.57
1	6	0.0040	667.6623	1-6	100	$474.65
2	3	0.0043	815.4845	2-3	169	$361.93
2	4	0.0045	737.8809	2-4	143	$365.34
2	5	0.0100	2,216.7168	2-3-5	31	$427.14
2	6	0.0050	815.4845	2-4-6	92	$436.02
3	2	0.0050	815.4845	3-2	200	$281.09
3	4	0.0087	2,216.7168	3-4	132	$324.09
3	5	0.0167	2,216.7168	3-5	32	$255.21
3	6	0.0133	2,216.7168	3-4-6	15	$377.01
4	6	0.0067	815.4845	4-6	161	$242.91
5	2	0.0050	815.4845	5-2	100	$419.72
5	3	0.0133	2,216.7168	5-3	47	$289.17
5	4	0.0063	815.4845	5-3-4	21	$583.27
5	6	0.0043	815.4845	5-3-4-6	32	$746.19

the deterministic problem is solved to find optimal deterministic demand rates (and prices) for each product. According to these fixed demand rates, $10,000$ different realization of Poisson arrival processes are generated. In each replication, products are sold in first come first served order and revenues are calculated based on the optimal deterministic prices. The values in the J_{MTO} column of Table 6.8 are the average of revenues as an estimate for expected revenues with relative error less than $\pm 0.01\%$ with 95% confidence.

It is easier to calculate the expected revenues obtained by MTS heuristic since the problem reduces to single product case once the resource allocation is done. Expected number of arrivals for each product can be calculated easily by using the probability mass function of Poisson distribution. Then, expected revenue obtained via MTS heuristic can be calculated by (5.11).

ATD heuristic is similar to MTS heuristic in the sense that resource are allocated to products at the beginning of the selling period. Then, optimal dynamic

Table 6.8: Expected revenues obtained by heuristics in airline example

Scale	Upper Bound	MTO	MTS	ATD	RR
0.1	65,748	53,922	55,937	59,267	62,034
0.5	328,738	315,871	308,346	319,813	326,028
1	657,477	640,454	630,215	648,487	650,991
2	1,314,953	1,293,374	1,276,061	1,304,355	n/a
10	6,574,767	6,550,660	6,484,796	6,557,380	n/a

Table 6.9: Ratio of the expected revenues to upper bound

Scale	MTO	MTS	ATD	RR
0.1	0.820	0.851	0.901	0.944
0.5	0.961	0.938	0.973	0.992
1	0.974	0.959	0.986	0.990
2	0.984	0.970	0.992	n/a
10	0.996	0.986	0.997	n/a

pricing is applied to single products separately and expected revenues are calculated as it is done in the exponential price response case of the retail example.

Results of RR heuristic are obtained by simulation similar to the exponential price response case in the retail example. The results for scale 0.1, 0.5 and 1 are given in Table 6.8. The $[0,1]$ time interval is divided into 100, 500 and 1000 in scales 0.1, 0.5 and 1, respectively. We have simulated 30 replications for each entry in the J_{RR} column of Table 6.8. These values are the average revenues of 30 replications as estimates for expected revenues with relative error of $\pm 0.1\%$ with 95% confidence.

What we observe from Table 6.8 is that the expected revenues increase as the problem size gets larger. This is intuitive since leg capacities increases with the problem size and more tickets are sold. The demand increases with the problem size as well since the parameter a is multiplied with the scale factor. Dynamic pricing heuristics (ATD, RR) significantly outperform fixed price heuristics especially for smaller scales. As the scale increases, the performance of fixed price heuristics gets better, which can be seen from the ratios presented in Table 6.9. This can be explained by the fact that fixed price heuristics are asymptotically optimal as the volume of sales tends to infinity. RR heuristic yields the best results among four heuristics; however, we could not not calculate the expected revenue of RR heuristic for the scales 2 and 10 since it takes a prohibitive amount of time to generate demand arrivals and simulate it. Although it is difficult to measure the performance of it, implementation is not difficult since it is just solving the deterministic problem frequently enough in practice.

6.2 Performance of Heuristics Using Value Approximation

In this section, we present the numerical results for the heuristics using value approximations discussed in Section 5.1. We use the product resource structure of the retail example shown in Figure 6.1. In Table 6.10, performance of the value approximations of RA1 heuristic is shown. In Table 6.11 and 6.12, performance of the value approximations of RA2 heuristic for exponential and linear price responses is shown, respectively. Note that in some cases $\tilde{J} > J^*$, because $\tilde{J}$ values are just approximations and do not have to be less than the optimal expected revenue J^*. The ratio $\tilde{J}/J^*$ is persistently increasing in initial inventory levels x_j in the value approximation of RA1 heuristic. A similar observation cannot be done for RA2 heuristic. Unfortunately, we cannot find an upper bound on the performance of revenue approximations.

In Table 6.13, 6.14 and 6.15 revenues obtained from both approximation heuristics for three different parameter sets are shown. The first approximation heuristic (J_{RA1}) is for exponential price response, so it is compared with the optimal expected revenues of exponential price response (J^*). However, the second approximation heuristic (J_{RA2}) is applicable to any price response in general. Thus, it is compared with the optimal expected revenue of both exponential and linear price responses (J^*).

What we observe is that the first two approximation heuristics give very close results to optimal expected revenues. The first and second approximation heuristics yield at least ~ 0.995 and ~ 0.967 of the optimal expected revenue for all

Table 6.10: Performance of value approximations for RA1 heuristic for $a_E = [\mathrm{e}, \mathrm{e}, \mathrm{e}], \alpha = [1, 1, \alpha_3]$

		$\alpha_3 = 2/3$			$\alpha_3 = 4/7$			$\alpha_3 = 1/2$		
T	x_j	J^*	$\tilde{J}$	$\tilde{J}/J^*$	J^*	$\tilde{J}$	$\tilde{J}/J^*$	J^*	$\tilde{J}$	$\tilde{J}/J^*$
10	1	5.172	5.059	0.9783	5.420	5.233	0.9656	5.733	5.489	0.9574
	2	9.232	9.061	0.9814	9.736	9.480	0.9736	10.321	10.003	0.9691
	3	12.611	12.424	0.9851	13.349	13.068	0.9789	14.173	13.820	0.9751
	4	15.502	15.322	0.9884	16.450	16.168	0.9828	17.483	17.119	0.9792
	5	18.016	17.856	0.9911	19.151	18.883	0.9860	20.370	20.011	0.9824
	10	26.774	26.838	1.0024	28.598	28.537	0.9978	30.474	30.302	0.9943
	20	33.849	34.801	1.0281	36.255	37.109	1.0236	38.669	39.436	1.0198
	30	34.969	37.106	1.0611	37.467	39.559	1.0558	39.964	42.018	1.0514
40	1	7.681	7.572	0.9859	7.962	7.760	0.9747	8.386	8.120	0.9684
	2	14.181	13.975	0.9855	14.799	14.486	0.9789	15.626	15.260	0.9766
	3	19.969	19.703	0.9867	20.920	20.544	0.9820	22.129	21.695	0.9804
	4	25.248	24.944	0.9880	26.521	26.104	0.9843	28.088	27.604	0.9828
	5	30.131	29.804	0.9891	31.714	31.267	0.9859	33.617	33.096	0.9845
	10	50.530	50.163	0.9927	53.495	52.983	0.9904	56.840	56.216	0.9890
	20	79.705	79.389	0.9960	84.822	84.329	0.9942	90.295	89.648	0.9928
	30	100.001	99.807	0.9981	106.708	106.322	0.9964	113.693	113.133	0.9951

Table 6.11: Performance of value approximations for RA2 heuristic for $a_E = [\mathrm{e}, \mathrm{e}, \mathrm{e}], \alpha = [1, 1, \alpha_3]$

		$\alpha_3 = 2/3$			$\alpha_3 = 4/7$			$\alpha_3 = 1/2$		
T	x_j	J^*	$\tilde{J}$	$\tilde{J}/J^*$	J^*	$\tilde{J}$	$\tilde{J}/J^*$	J^*	$\tilde{J}$	$\tilde{J}/J^*$
10	1	5.172	4.913	0.9499	5.420	4.931	0.9098	5.733	4.949	0.8631
	2	9.232	8.639	0.9358	9.736	8.698	0.8934	10.321	8.755	0.8482
	3	12.611	11.710	0.9286	13.349	11.823	0.8857	14.173	11.930	0.8418
	4	15.502	14.335	0.9247	16.450	14.511	0.8821	17.483	14.676	0.8394
	5	18.016	16.625	0.9228	19.151	16.869	0.8809	20.370	17.098	0.8394
	10	26.774	24.829	0.9273	28.598	25.475	0.8908	30.474	26.070	0.8555
	20	33.849	32.579	0.9625	36.255	34.107	0.9408	38.669	35.513	0.9184
	30	34.969	34.761	0.9940	37.467	37.004	0.9876	39.964	39.124	0.9790
40	1	7.681	7.462	0.9716	7.962	7.468	0.9380	8.386	7.474	0.8912
	2	14.181	13.605	0.9594	14.799	13.626	0.9208	15.626	13.647	0.8733
	3	19.969	18.997	0.9513	20.920	19.042	0.9102	22.129	19.086	0.8625
	4	25.248	23.869	0.9454	26.521	23.945	0.9029	28.088	24.018	0.8551
	5	30.131	28.347	0.9408	31.714	28.459	0.8974	33.617	28.566	0.8498
	10	50.530	46.864	0.9275	53.495	47.221	0.8827	56.840	47.558	0.8367
	20	79.705	73.280	0.9194	84.822	74.322	0.8762	90.295	75.290	0.8338
	30	100.001	92.050	0.9205	106.708	93.918	0.8801	113.693	95.638	0.8412

parameter sets of exponential price response. However, second approximation heuristic's performance is worse for linear price response. Its performance gets consistently better as the initial inventory increases. For instance, it yields at least ~ 0.951 of the optimal expected revenue among all parameter sets when $x_j = 30$ for $j = 1, 2$. In Figure 6.12-6.14, the percentage gap of all approximation heuristics for $x_j = 5, 10$ and 20 for $j = 1, 2$ and for the given parameter set is shown. Percentage gap is calculated as follows:

$$Percentage\ Gap = \frac{J^* - J_{RA1}}{J^*} \times 100 \tag{6.1}$$

We can also show how close the price path of a product is to the optimal price path under different approximation heuristics. We will again consider the demand realization shown in Figure 6.4. For instance, the price path of the

Table 6.12: Performance of value approximations for RA2 heuristic for $a_L = [2, 2, 2], b = [1, 1, b_3]$

		$b_3 = 2/3$			$b_3 = 4/7$			$b_3 = 1/2$		
T	x_j	J^*	$\tilde{J}$	$\tilde{J}/J^*$	J^*	$\tilde{J}$	$\tilde{J}/J^*$	J^*	$\tilde{J}$	$\tilde{J}/J^*$
10	1	3.340	6.155	1.8430	3.375	6.165	1.8270	3.516	6.176	1.7567
	2	6.324	11.043	1.7463	6.504	11.080	1.7036	6.843	11.117	1.6246
	3	9.071	15.223	1.6782	9.441	15.298	1.6204	9.978	15.370	1.5404
	4	11.634	18.916	1.6259	12.198	19.037	1.5607	12.926	19.154	1.4818
	5	14.028	22.242	1.5855	14.783	22.417	1.5164	15.692	22.583	1.4392
	10	23.708	35.305	1.4891	25.266	35.814	1.4175	26.914	36.288	1.3483
	20	33.305	51.725	1.5531	35.666	53.077	1.4882	38.041	54.318	1.4279
	30	34.957	61.225	1.7514	37.454	63.516	1.6958	39.951	65.607	1.6422
40	1	3.810	8.807	2.3118	3.810	8.810	2.3124	3.862	8.813	2.2819
	2	7.502	16.263	2.1679	7.504	16.275	2.1687	7.671	16.286	2.1229
	3	11.085	22.942	2.0695	11.096	22.967	2.0698	11.426	22.991	2.0122
	4	14.565	29.077	1.9964	14.599	29.120	1.9947	15.127	29.162	1.9277
	5	17.943	34.796	1.9392	18.032	34.861	1.9333	18.776	34.924	1.8600
	10	33.491	59.269	1.7697	34.400	59.494	1.7295	36.254	59.711	1.6470
	20	60.420	96.761	1.6015	63.563	97.481	1.5336	67.474	98.163	1.4548
	30	83.060	125.917	1.5160	88.162	127.282	1.4437	93.823	128.559	1.3702

Table 6.13: Expected revenues obtained by approximation heuristics for $a_E = [\mathrm{e}, \mathrm{e}, \mathrm{e}], \alpha = [1, 1, 2/3], a_L = [2, 2, 2], b = [1, 1, 2/3]$

		Exponential					Linear		
T	x_j	J^*	J_{RA1}	J_{RA1}/J^*	J_{RA2}	J_{RA2}/J^*	J^*	J_{RA2}	J_{RA2}/J^*
10	1	5.172	5.166	0.9989	5.146	0.9950	3.340	1.968	0.5893
	2	9.232	9.224	0.9991	9.182	0.9945	6.324	4.483	0.7089
	3	12.611	12.600	0.9991	12.548	0.9950	9.071	7.177	0.7912
	4	15.502	15.486	0.9989	15.436	0.9957	11.634	9.901	0.8511
	5	18.016	17.991	0.9986	17.953	0.9965	14.028	12.419	0.8853
	10	26.774	26.694	0.9970	26.756	0.9993	23.708	22.597	0.9531
	20	33.849	33.671	0.9947	33.809	0.9988	33.305	32.067	0.9628
	30	34.969	34.881	0.9975	34.960	0.9997	34.957	34.389	0.9837
40	1	7.681	7.675	0.9993	7.656	0.9968	3.810	1.968	0.5166
	2	14.181	14.172	0.9994	14.114	0.9953	7.502	4.483	0.5975
	3	19.969	19.959	0.9995	19.856	0.9943	11.085	7.177	0.6474
	4	25.248	25.238	0.9996	25.091	0.9938	14.565	9.965	0.6842
	5	30.131	30.122	0.9997	29.935	0.9935	17.943	12.809	0.7139
	10	50.530	50.518	0.9998	50.209	0.9937	33.491	27.424	0.8188
	20	79.705	79.675	0.9996	79.412	0.9963	60.420	55.574	0.9198
	30	100.001	99.939	0.9994	99.864	0.9986	83.060	79.025	0.9514

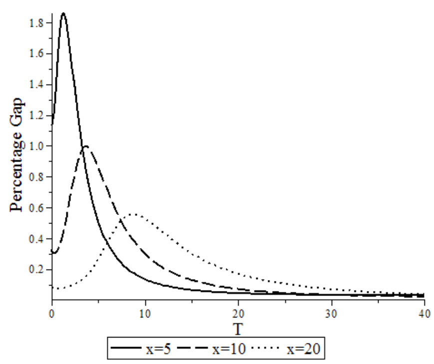
third product under first and second approximation heuristics together with the optimal price path are shown in Figure 6.15. Price path of the third product under the second approximation heuristic for linear price response is shown in Figure 6.16. Note that the price of the third product does not exceed the value 3 since the price response is linear with parameters $a_3 = 2$, $b_3 = 2/3$ and $p_3^\infty = 3$. If the price exceeds 3, then the corresponding demand rate becomes negative. Price paths of the first and the second products under both approximation heuristics for exponential and linear price responses are shown in Figure 6.17 - 6.20. In all price paths, demand realization for a certain product causes an upward jump on the price path of that product and the other products having common resources with it. Price of product 1 (2) drops when a unit of product 2 (1) is sold. Besides, prices decrease over time between consecutive demand realizations.

Table 6.14: Expected revenues obtained by approximation heuristics $a_E = [\mathrm{e},\mathrm{e},\mathrm{e}], \alpha = [1, 1, 4/7], a_L = [2, 2, 2], b = [1, 1, 4/7]$

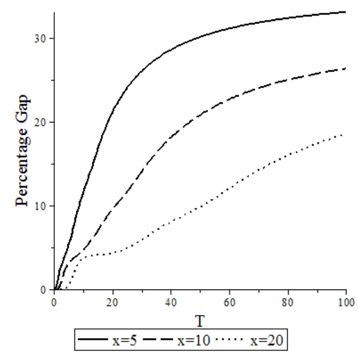
		Exponential					Linear		
T	x_j	J^*	J_{RA1}	J_{RA1}/J^*	J_{RA2}	J_{RA2}/J^*	J^*	J_{RA2}	J_{RA2}/J^*
10	1	5.420	5.408	0.9979	5.343	0.9858	3.375	2.121	0.6284
	2	9.736	9.725	0.9988	9.603	0.9863	6.504	4.817	0.7406
	3	13.349	13.336	0.9990	13.189	0.9880	9.441	7.696	0.8152
	4	16.450	16.434	0.9990	16.288	0.9901	12.198	10.635	0.8718
	5	19.151	19.129	0.9989	19.001	0.9922	14.783	13.437	0.9090
	10	28.598	28.526	0.9975	28.554	0.9985	25.266	24.353	0.9639
	20	36.255	36.080	0.9952	36.170	0.9976	35.666	34.316	0.9622
	30	37.467	37.380	0.9977	37.441	0.9993	37.454	36.749	0.9812
40	1	7.962	7.946	0.9980	7.866	0.9880	3.810	2.121	0.5566
	2	14.799	14.780	0.9987	14.577	0.9850	7.504	4.817	0.6419
	3	20.920	20.902	0.9991	20.579	0.9837	11.096	7.696	0.6936
	4	26.521	26.503	0.9993	26.073	0.9831	14.599	10.668	0.7308
	5	31.714	31.697	0.9995	31.173	0.9829	18.032	13.697	0.7596
	10	53.495	53.480	0.9997	52.691	0.9850	34.400	29.245	0.8501
	20	84.822	84.796	0.9997	84.124	0.9918	63.563	59.885	0.9421
	30	106.708	106.653	0.9995	106.380	0.9969	88.162	85.214	0.9666

Table 6.15: Expected revenues obtained by approximation heuristics for $a_E = [\mathrm{e},\mathrm{e},\mathrm{e}], \alpha = [1, 1, 1/2], a_L = [2, 2, 2], b = [1, 1, 1/2]$

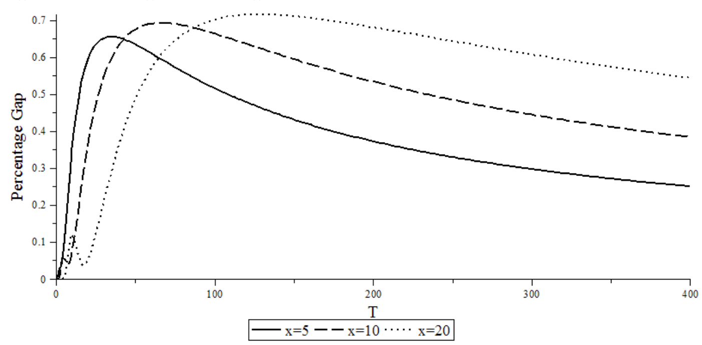
		Exponential					Linear		
T	x_j	J^*	J_{RA1}	J_{RA1}/J^*	J_{RA2}	J_{RA2}/J^*	J^*	J_{RA2}	J_{RA2}/J^*
10	1	5.733	5.716	0.9970	5.566	0.9708	3.516	2.390	0.6799
	2	10.321	10.305	0.9985	10.056	0.9743	6.843	5.355	0.7826
	3	14.173	14.157	0.9989	13.867	0.9784	9.978	8.506	0.8525
	4	17.483	17.466	0.9990	17.178	0.9825	12.926	11.708	0.9057
	5	20.370	20.348	0.9990	20.090	0.9863	15.692	14.681	0.9356
	10	30.474	30.408	0.9978	30.393	0.9973	26.914	26.171	0.9724
	20	38.669	38.496	0.9955	38.528	0.9963	38.041	36.581	0.9616
	30	39.964	39.878	0.9978	39.913	0.9987	39.951	39.109	0.9789
40	1	8.386	8.361	0.9971	8.131	0.9696	3.862	2.390	0.6189
	2	15.626	15.601	0.9984	15.114	0.9672	7.671	5.355	0.6981
	3	22.129	22.105	0.9989	21.387	0.9665	11.426	8.506	0.7444
	4	28.088	28.064	0.9992	27.150	0.9666	15.127	11.749	0.7767
	5	33.617	33.595	0.9993	32.515	0.9672	18.776	15.047	0.8014
	10	56.840	56.821	0.9997	55.304	0.9730	36.254	31.932	0.8808
	20	90.295	90.270	0.9997	89.000	0.9857	67.474	64.894	0.9618
	30	113.693	113.643	0.9996	113.082	0.9946	93.823	91.712	0.9775



(a) RA1 for exponential price response

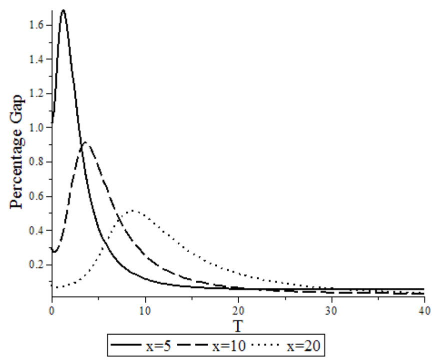


(b) RA2 for linear price response

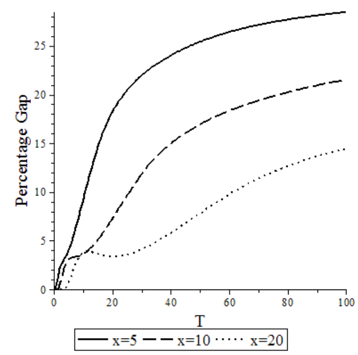


(c) RA2 for exponential price response

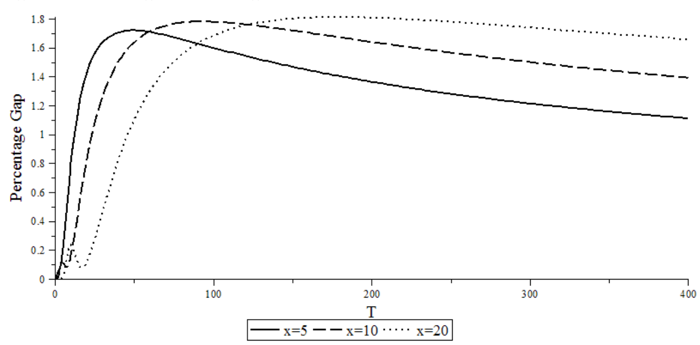
Figure 6.12: Percentage gap of approximation heuristics for $a_E = [\mathrm{e}, \mathrm{e}, \mathrm{e}], \alpha = [1, 1, 2/3], a_L = [2, 2, 2], b = [1, 1, 2/3]$



(a) RA1 for exponential price response

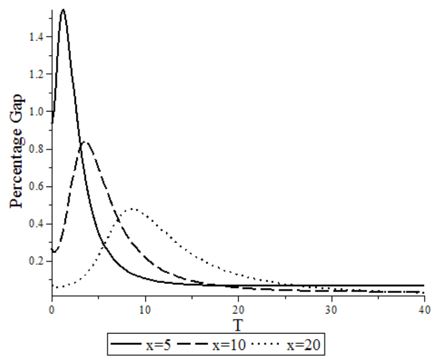


(b) RA2 for linear price response

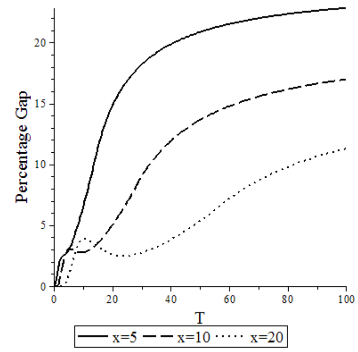


(c) RA2 for exponential price response

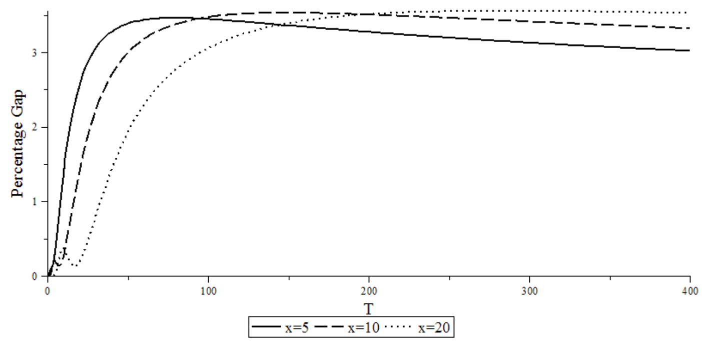
Figure 6.13: Percentage gap of approximation heuristics for $a_E = [\mathrm{e}, \mathrm{e}, \mathrm{e}], \alpha = [1, 1, 4/7], a_L = [2, 2, 2], b = [1, 1, 4/7]$



(a) RA1 for exponential price response



(b) RA2 for linear price response



(c) RA2 for exponential price response

Figure 6.14: Percentage gap of approximation heuristics for $a_E = [\mathrm{e}, \mathrm{e}, \mathrm{e}], \alpha = [1, 1, 1/2], a_L = [2, 2, 2], b = [1, 1, 1/2]$

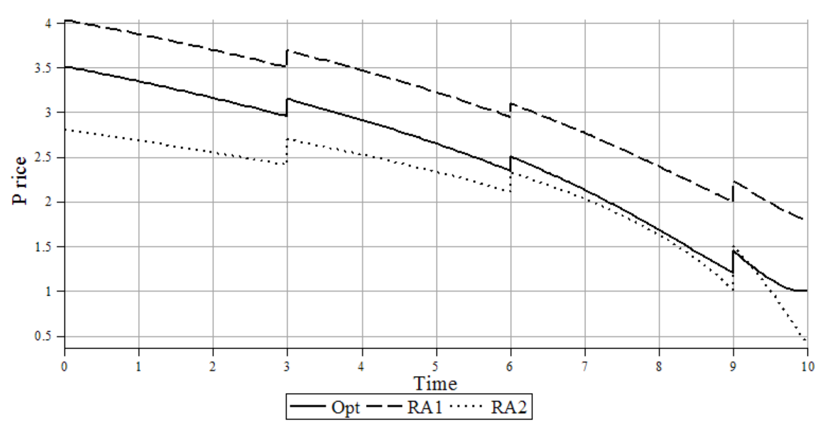


Figure 6.15: Price path of the third product under the RA1 and RA2 heuristics

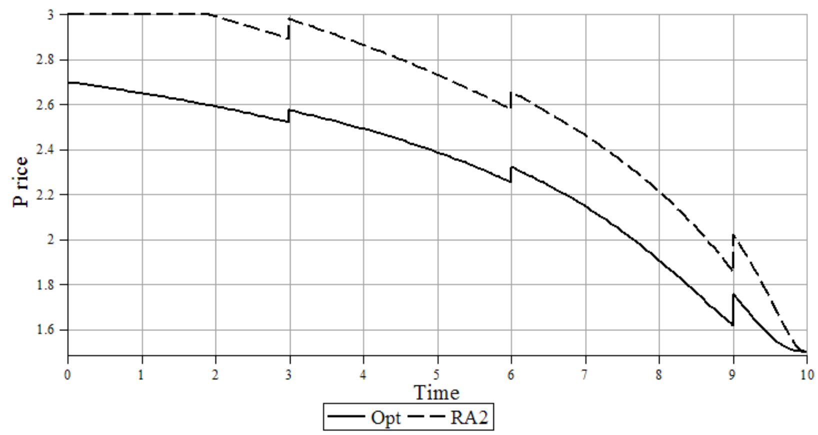


Figure 6.16: Price path of the third product under RA2 heuristic for linear price response

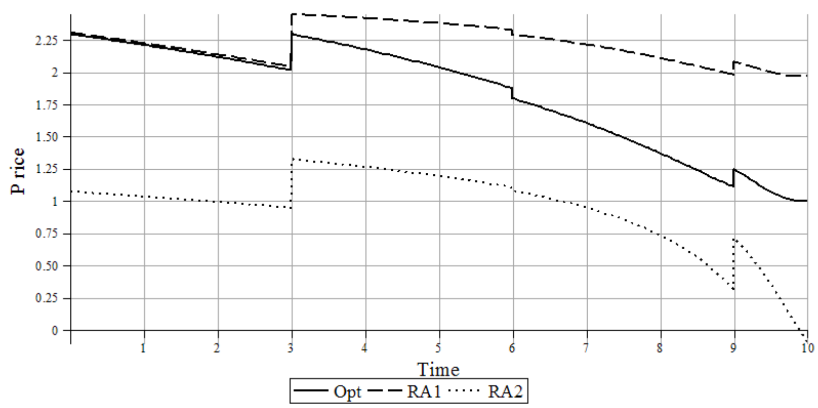


Figure 6.17: Price path of the first product under RA1 and RA2 heuristics

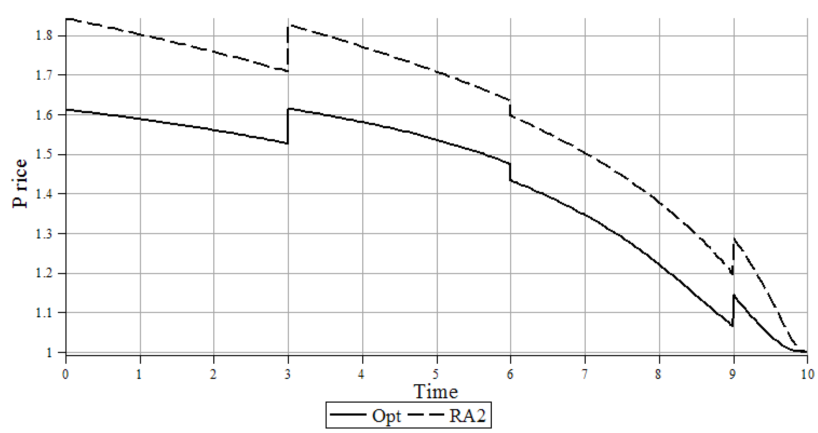


Figure 6.18: Price path of the first product under RA2 heuristic for linear price response

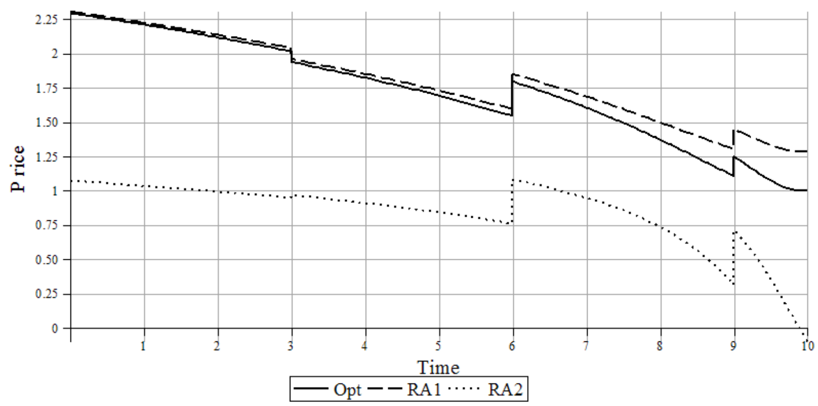


Figure 6.19: Price path of the second product under RA1 and RA2 heuristics

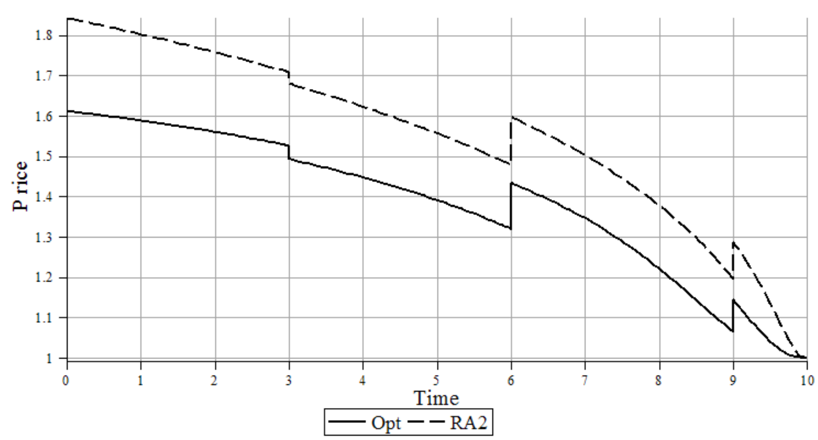


Figure 6.20: Price path of the second product under RA2 heuristic for linear price response

Chapter 7

Conclusion

We study the problem of multi-product inventory pricing under stochastic and price sensitive demand. Demand for each product is modeled as a non-homogeneous Poisson process whose intensity is a function of the current price of the product itself. We have initial inventory of m resources whose different combinations form n products. Products are perishable and the selling period is of length T. The aim is to set the price of each product over the time interval $[0, T]$ to maximize the expected revenue.

This problem takes place in the pricing category of the revenue management literature, but it is also related with capacity allocation, since pricing and capacity allocation problems are interrelated as mentioned in the introduction. There are similar studies in the literature which considers selling a given inventory of items over a finite selling period. Some examples are Kincaid & Darling [17], Stadje [18], Gallego & van Ryzin [4] and Sen [22]. These studies consider either a single type of product or fixed price policies for multiple types of products. Our work differs from those by considering dynamic pricing for a network. The aim of this study is to emphasize the advantage of dynamic pricing.

Dynamic pricing of multiple products is a quite challenging problem. Even for single item case, analytical solution of the Hamilton-Jacobi equation (3.4) is difficult. Gallego & van Ryzin [19] provide a closed form solution of the single product problem for exponential price response. To the best of our knowledge, no closed form solution for the multi-product case is given in the literature. In multi-product case though, it is even difficult to solve the system of partial differential equations in (3.4) numerically. The number of partial differential equations in the system gets larger as the initial inventories and number of resource types increases. For instance, there are $30 \times 30 = 900$ partial differential equations in the system for the product resource structure shown in Figure 6.1 with initial inventories $x_1 = x_2 = 30$. Hence, even obtaining a numerical solution is a computational challenge.

Our contributions are twofold. First, we provided a closed form solution for the multi-product pricing problem for the special case of exponential price response. In this special case, the parameter α_j of the demand function of each product is assumed to be identical. Considering the similarity of products and their customer targets, this is not a very strong assumption. Second, we provided two types of dynamic pricing heuristics: one using the value approximation approach and the other using the deterministic version of the problem. First two heuristics of the value approximation type is applicable to exponential price response, whereas the third heuristic using value approximation is for general price response. Heuristics based on the deterministic problem (RR, ATD) are conceptually for general price response, but are tested for only linear and exponential price responses. We have provided a substantial numerical analysis regarding the performance of all heuristics.

As for the future research, there are many opportunities. The foremost is to solve (3.4) analytically for different price responses. The assumption that the demand of a certain product is a function of only the current price of that product itself can be relaxed. The demand can be modeled in such a way that the demand of a product may depend on the price of other products, which may either be complements or substitutes. In this thesis, we assumed that the price response does not change over time, i.e., the past and future prices do not have an effect on the current demand. This assumption can also be relaxed to model the strategic behaviour of customers which makes the problem more interesting and challenging at the same time. Finally, structural results can be derived analytically.

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