FOUR ESSAYS ON OVERLAPPING GENERATIONS RESOURCE ECONOMIES: OPTIMALITY, SUSTAINABILITY AND DYNAMICS

A Ph.D. Dissertation

by
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Ankara
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To Happy and Our Future Generations…

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Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

BURCU FAZLIOĞLU

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of
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in

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ANKARA

August 2012

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Economics.

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ABSTRACT

FOUR ESSAYS ON OVERLAPPING GENERATIONS RESOURCE ECONOMIES: OPTIMALITY, SUSTAINABILITY AND DYNAMICS

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This dissertation is made up of four essays on overlapping generations resource economies. The first essay studies the effects of energy saving technological progress and substitution of renewable energy resources with non-renewable resources on natural resource depletion and long run growth. A growth model in two-period overlapping generations framework incorporating the presence of both resources and resource augmenting technological progress is developed. The effect of an increase in the intensity of the renewable resources in producing energy on long run growth is found to be positive. Although exhaustible resources are essential in production the economy can be sustained and the balanced growth path is optimal.

In the second essay, the implications of assuming different energy intensities for physical capital accumulation and the final good production is studied in an overlapping generations resource economy where energy is obtained from the extraction of the natural resources. Apart from the standard literature, physical capital accumulation is assumed to be relatively more energy-intensive

than consumption. Multiple steady states, indeterminacy and bifurcations are obtained, without taking non-linearizing assumptions evident in the literature. For the non-renewable resources if the share of energy resources is low enough, local indeterminacy and hopf bifurcations may arise in the model.

The aim of third essay is to analyze can costly resource extraction and differentiating energy intensities induce dynamics other than saddles in an overlapping generations resource economy. The capital accumulation sector is assumed to be more energy intensive. The energy input is extracted from the natural resources with some extraction costs. The main finding of the essay is that both naturally evident assumptions contribute to the richness of the dynamics. Depending on the share of resources in capital accumulation dynamics other than saddle –indeterminacy, flip and hopf bifurcations– can arise in the model for the non-zero steady state.

In the fourth essay, a feedback mechanism between population and natural resource to a standard model of renewable resource based OLG economy is incorporated to check the stability of the dynamics. Multiple steady states and indeterminacy have been obtained even in the absence of logistic regeneration and independent of intertemporal elasticity of substitution. In particular, transcritical bifurcations may arise in the model varying the rate of constant regeneration with respect to population growth rate.

Keywords: Overlapping Generations Model, Natural Resources, Endogenous population growth, Harvest Costs, Optimality, Sustainability, Dynamics, Bifurcations, Indeterminacy.

ÖZET

ARDIŞIK NESİLLER KAYNAK EKONOMİLERİ ÜZERİNE DÖRT MAKALE: OPTİMALİTE, SÜRDÜRÜLEBİLİRLİK VE DİNAMİKLER

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Bu çalışma, ardışık nesiller kaynak ekonomileri üzerine dört makaleden oluşmaktadır. İlk makalede, enerji tasarrufu sağlayan teknolojik gelişmenin ve yenilenebilir enerji kaynaklarının yenilenemeyen kaynaklarla ikamesinin, kaynakların tükenmesine ve uzun dönem büyümeye olan etkisi incelenmektedir. Bu kapsamda, enerji kaynaklarını ve kaynaklardan tasarruf eden teknolojik ilerlemeyi içeren 2 periyotluk ardışık nesiller büyüme modeli geliştirilmiştir. Enerji üretiminde, yenilenebilir enerji kaynaklarının yoğunluğunun artmasının büyüme üzerinde olumlu bir etkisinin olduğu bulunmuştur. Modelde her ne kadar yenilemeyen enerji kaynakları üretim için gerekli olsa da; ekonominin sürdürülebilir olduğu ve dengeli büyüme patikasının pareto optimal olduğu sonuçlarına ulaşılmıştır.

İkinci makale, fiziki sermaye birikimi ile nihai ürün üretiminin enerji yoğunluklarının farklılaştırılmasını, enerjinin doğal kaynaklardan çıkartıldığı bir ardışık nesiller modeli kapsamında incelemektedir. Literatürden farklı olarak, sermaye

birikiminin nihai ürün üretimine kıyasla daha enerji yoğun olduğu varsayılmıştır. Literatürdeki doğrusal olmayan dinamiklere yol açacak varsayımlar yapılmadan modelde birden çok durağan noktanın, belirsizliğin ve dallanmaların olduğu bulunmuştur. Yenilenemeyen kaynaklara odaklanıldığında ise kaynaklarının sermaye üretimindeki payının yeterince düşük olduğu durumlarda yerel belirsizliğin ve hopf dallanmalarının ortaya çıkabileceği gösterilmiştir.

Üçüncü makale, doğal kaynakları çıkartmanın maliyetlerini ve fiziki sermaye birikimi ile nihai ürün üretiminin enerji yoğunluklarının farklı olduğunu dikkate alan ardışık nesiller kaynak ekonomilerinde, eyer noktası karalılığı dışında dinamiklere ulaşılıp ulaşılamayacağını analiz etmektedir. Sermaye birikiminin nihai ürün üretimine kıyasla daha enerji yoğun olduğu ve kaynakları çıkartmanın maliyetli olduğu varsayılmıştır. Söz konusu varsayımların dinamikleri zenginleştirdiği tespit edilmiştir. Kaynakların tükenmemiş olduğu durağan noktada yerel belirsizliğin transkritik ve hopf dallanmalarına yol açabileceği gösterilmiştir.

Son makalede, ardışık nesiller ekonomilerinde nüfus büyümesi ve kaynaklar arasında bir geribildirim mekanizması kurulmuş ve söz konusu mekanizmanın dinamiklerin durağanlığını nasıl etkilediği incelenmiştir. Lojistik yenilenme oranı alınmadan ve dönemler arası ikame elastikiyeti üzerinde varsayımlar yapılmadan, durağan noktalarda çoğulluk ve lineer olarak alınan yenilenme oranının nüfus büyümesiyle ilişkisine göre belirsizlik ve transkritik dallanmalar elde edilmiştir.

Anahtar Kelimeler: Ardışık Nesiller Modeli, Doğal Kaynaklar, İçsel Nüfus Artış Hızı, Hasat Maliyetleri, Pareto Optimalite, Sürdürülebilirlik, Dinamikler, Dallanmalar, Belirsizlik.

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CHAPTER 1

INTRODUCTION

How does scarcity of resources limits economic growth? To what extent substitution of renewable energy resources with non-renewables, developing energy saving technologies or physical capital accumulation relax this constraint on growth? What is the role of population growth in this scenario? Can growth under the presence of scarce resources be sustainable? These research questions have been the subject of several scholarly papers in the literature of resource economics dating as back as to Dasgupta and Heal (1974, 1979), Solow (1974), and Stiglitz (1974).

The first essay of the thesis (Chapter 2) studies the effects of the first two solutions "energy saving technological progress" and "substitution of renewable energy resources with non-renewable resources" on natural resource depletion and long run growth. Although there are numerous papers addressing these issues with non-renewables (Guruswamy Babu and Kavi Kumar, 1997; John and Peccheccino, 1994) and renewables (Gerlagh and Zwan, 2001; Koskela et al., 2008) separately, there are limited studies within OLG framework considering these resources as alternative sources of energy and analyzing their effect on dynamics of growth. Besides, there are several papers studying the trade-off between energy saving technological progress, energy consumption and growth (see Van Zon and Yetkiner, 2003; Boucekkine and Pommeret, 2004; Azomahou et al., 2004; Perez-Barahona and Zou, 2006; Yuan et al., 2009). Yet none of them gives particular attention to the intergenerational aspects (such as sustainability) or focuses on the presence of the natural resources in an OLG economy.

The main contribution of this essay is that it is the first study that analyzes the presence of both renewable and non-renewable energy resources and resource augmenting technological progress in an analytically solvable overlapping generations model. In this essay, an analytical characterization of the balanced growth path is provided and the conditions for the economy to exhibit positive long run growth is analyzed. Then, the effects of discount factor, resource augmenting technological progress and intensity of resources in energy production on the depletion rate is investigated. In addition, whether the long run growth is sustainable or optimal is examined.

In parallel with the OLG literature (Galor and Ryder, 1989; Agnani et al., 2005) the model necessitates sufficiently high labor share for the economy to exhibit positive growth. However, compared with previous models the condition required is less binding. In fact, the share of renewables in energy production, the technological progress in producing energy and the regeneration factor are found among the key variables affecting the required labor share and hence the possibility of long run growth. In terms of policy implications, the first essay shows that increasing the intensity of the renewable resources in producing energy and developing technologies improving the regeneration rate of renewables promotes long run growth. Moreover, promoting energy saving technologies will support the sustainability of the resources for future generations.

As a third solution, greater physical capital accumulation is suggested to overcome the constraint that the natural resources put on growth. However, the vast majority of the literature, assumes the same technology for the consumption and capital accumulation sector which is contradictory with the evidence on energy intensities of these sectors. The data suggests that physical capital production is relatively more energy-intensive than consumption. Differing energy intensities has not been considered within the overlapping generations (OLG) framework.

Inspired by this idea, the second essay (Chapter 3) and third essay (Chapter 4) analyzes the implications of assuming different energy intensities for physical capital accumulation and the final good production in an overlapping generations resource economy. In the second essay, harvest is assumed to be costless. How do the standard results on stability of the dynamics, growth and optimality are modified under differing energy intensities are evaluated. The analytical characterization of the balanced growth path is presented, optimality of the balanced growth path is discussed and the dynamics are studied. It is shown in the essay that dynamics are drastically changing taking above mentioned assumption into account. In fact, for the non-renewable resources local indeterminacy and hopf bifurcations are found if the share of energy resources is low enough.

In the field of macroeconomics, as economic oscillations / cyclical fluctuations has been observed for a long time, many theories have been built to explain these cyclic behavior. There is a line of literature explaining the above mentioned cycles by non-linear dynamics. Observing Hopf bifurcations is important as they give rise to the existence of limit cycles[1] varying a parameter. In addition, limit cycles are quite important as they resemble business cycles. In the second essay, Hopf bifurcations are obtained varying the share of energy resources. Therefore, this model claims that the intensity of energy resources in the equipment good sector can serve as an additional channel for explaining the cyclical fluctuations in the economies. Finding local indeterminacy varying the share of energy resources helps to explain the cross-country income differences. In this context, this model shows that even though the countries will eventually converge to the same steady state they may follow different paths which can be welfare improving or decreasing. The path that the countries choose are vulnerable to speculative attacks. Finding local indeterminacy can put a light on how bubbles occur in economies.

The aim of third essay is to study the effects of costly resource extraction in addition to differentiating energy intensities on dynamics. Within the overlapping generations framework, Bednar–Friedl and Farmer (2010, 2011) are the only studies focusing on harvest costs. As mentioned above, to my knowledge there is no pa-

[1] Limit cycles are unforced and self-excited period oscillations.

per within OLG framework considering differing energy intensities of consumption good and capital accumulation sector. This essay attempts to analyze dynamics that could arise by integrating costly extraction, different technologies for equipment good and final good sector in an overlapping generations resource economy. The uniqueness of the steady state as well as the dynamics around the steady state is analyzed. The net effect of modelling harvest costs and as well as differing technologies are revealed independently.

The main finding of the chapter is that both naturally evident assumptions contribute to the richness of the dynamics. Multiple steady states exists in the model. Depending on the share of resources in capital accumulation dynamics other than saddle –indeterminacy, transcritical and hopf bifurcations– can arise in the model for the non-zero steady state. As multiple steady states are evident in this essay, this chapter contributes to the explanation of long term patterns of income across countries. The model explains the occurrence of converge clubs across countries. The essay shows that depending on the initial conditions of the economies some countries may converge to the good steady state –with higher income levels– or vice versa. Besides, if the energy intensity parameter can be considered as a choice for countries, depending on the choice of energy intensities countries with different initial conditions may converge to the same steady state indicating conditional convergence. As conveyed in the second essay, through Hopf bifurcations the intensity of energy resources in the equipment good sector becomes an additional channel for explaining the cyclical fluctuations in the economies. Finding local indeterminacy varying the share of energy resources helps to explain the cross-country income differences.

Vast of the standard economic growth literature assumes labor force grows at a constant rate, following exponential growth. Allowing population to grow in an exponential manner is not realistic, as scarce environmental resources will put a constraint on growth. Smith (1974), describes such a constraint on population growth by defining a feedback mechanism between population growth and carrying capacity of the environment.

As the carrying capacity of the environment is directly linked with the availability of natural resources, the final essay of the thesis evaluates whether a feedback mechanism between the population growth rate and per capita resource extraction and resource availability modifies the standard results in the area. Specifically, the possibilities of non-linearities in an OLG growth model where the natural resource is essential in production is investigated. The main contribution of this essay is to show that multiple steady states and complex dynamics have been obtained even in the absence of logistic regeneration and independent of intertemporal elasticity of substitution.

Overall, the aim of this theses is to understand and present mechanics of resource use and its implications for macroeconomic dynamics in an overlapping generations framework with a special focus on sustainability (Chapter 2), optimality (Chapter 2,3) and especially dynamics (Chapter 3,4,5).

As mentioned above OLG framework is preferred to infinitely lived agents framework in this thesis. In OLG framework agents have finite life time and are not perfectly altruistically linked. Infinitely lived agents framework can be seen as a special case of OLG models in which agents are perfectly altruistic –care about their descendants– and have an infinite horizon. OLG framework offers a better explanatory power for the discussion of resource problems due to three main reasons:

(i) First of all, besides being an input to energy production, resources are store of values between generations (see Koskela et al., 2002; Valente, 2008; Birgit Bednar–Friedl and Farmer; 2011) and are not held by one representative generation forever as infinitely lived representative agent framework assumes.

(ii) Secondly, current decisions on resource extraction taken by short-lived and selfish individuals have consequences not only on current but on future generations as well. Thus, both intratemporal and intertemporal effects should be considered. Solow (1974), Padilla (2002), Agnani (2005) note that these intergenerational aspects should be taken into account when analyzing environmental issues and/or natural resource economies.

(iii) Finally, contrary to what the infinitely-lived representative agent models claim, there exists some empirical evidence that agents are not perfectly altruistically linked (Altonji et al., 1992; Balestra, 2003).

CHAPTER 2

ENERGY SAVING TECHNOLOGICAL PROGRESS IN OVERLAPPING GENERATIONS ECONOMIES WITH RENEWABLE AND NON-RENEWABLE RESOURCES

As the worldwide energy demand has continuously been increasing, the question of whether the scarcity of natural energy resources limits economic growth receives special attention. The recent fluctuations in the oil prices along with the threat of climate change have further stimulated the interest in the issue of sustainability as well. Among others, substitution of renewable energy resources with non-renewables and developing energy saving technologies are the most prominent suggestions to overcome the problem. While substitution to renewable resources is accepted to contribute to more sustainable economic development paths (Daly, 1990; Andre and Cerda, 2005); energy efficiency programs are also offered as a policy response by several policy makers and environmental groups (Cabinet Office, 2001; DEFRA, 2005; Allan et. al, 2007).

This paper aims to answer whether substitution of non-renewable energy resources with renewables and progress in energy saving technologies will bring growth in the long run. A two-period overlapping generations model in which the energy is an essential input in production and exogenous resource augmenting technical change drives long-run growth is developed. To analyze how scarcity limits can be alleviated by technological progress or substitution of non-renewable resources with renewables, the following questions will be addressed:

(i) Under which circumstances will the economy prevail long run growth?

(ii) What determines the rate of depletion?

(iii) How will the intensity of renewables in energy production affect growth?

(iv) What will be the effect of the energy saving technological progress on the long run growth?

(v) How will the patience of the generations and the population growth rate affect these results? (vi) Will the long run growth be optimal and sustainable?

Although there is a vast literature analyzing the sustainability of growth in the presence of non-renewable or renewable resources, most of these papers focus on just one type of resources. There are endogenous growth models with infinitely lived agents (ILA) dealing with sustainability of long run growth under exhaustible resources (Stollery, 1998; Schou, 2000, 2002; Grimaud and Rouge, 2005, 2008; Groth and Schou, 2007). They conclude that under technological progress no matter it is taken as exogenous or endogenous growth is sustainable in the long-run despite the finite resource stock. Although there are numerous papers addressing these issues with non-renewables (Guruswamy Babu and Kavi Kumar, 1997; John and Peccheccino, 1994) and renewables (Gerlagh and Zwan, 2001; Koskela et al., 2008) separately, there are limited studies within OLG framework considering these resources as alternative sources of energy and analyzing their effect on dynamics of growth.

Few exceptions in the literature are Tahvonen and Salo (2001), Andre and Cerda (2005), Di Vita (2006), Nguyen and Nguyen-Van (2008), Maltsoglou (2009) and Hung and Quyen (2008). Tahvonen and Salo (2001) considers the problem of substitutability between exhaustible and renewable resources in terms of their costs but not in terms of relative scarcity. Although Andre and Cerda (2005) takes natural growth and technological substitution possibilities into account, they focus on the optimal combination of these resources in case of no technological progress and other inputs (such as capital, labor). Di Vita (2006), Nguyen-Van (2008) and Maltsoglou (2009) consider labor, physical capital and both types of energy resources as inputs to production. Yet, these studies analyze the behavior of the economies with infinitely lived agents. They do not also consider the effects of energy saving technological progress[2]. Hung and Quyen (2008) is the only study within the OLG framework while considering both inputs. However, they focus on the effects of endogenous fertility without any technological progress and renewable resource is solar energy which is produced from backstop capital.

It is well known that although improvements in technology lowers the energy consumption, through economic growth, it will in turn create further energy demand. In fact, there are several papers studying the trade-off between energy saving technological progress, energy consumption and growth (see Van Zon and Yetkiner, 2003; Boucekkine and Pommeret, 2004; Azomahou et al., 2004; Perez-Barahona and Zou, 2006; Yuan et al., 2009). Yet none of them gives particular attention to the intergenerational aspects (such as sustainability) or focuses on the presence of the natural resources. Perez-Barahona (2011) investigates the effect of energy saving technological progress on growth under the presence of an exhaustible resource but does not consider an OLG framework and alternative sources of energy. Valente (2005) accounts for a renewable resource in an OLG economy under resource augmenting technology leaving alternative resource aside.

This paper tries to fulfill the above mentioned gaps in the literature through studying the presence of both renewable and non-renewable energy resources and resource augmenting technological progress in an analytically solvable exogenous growth overlapping generations model. To analyze the presence of both resources, they are differentiated according to their relative scarcity. Non-renewable resources are scarce whereas renewables are not due to their regeneration property . The reason behind assuming resource-saving technological progress stems from the evidence

[2]Nguyen and Nguyen-Van (2008) mentions that if they assume a Cobb-Douglas production function then a parameter could capture the resource saving technological progress yet in the rest of the paper they do not focus on the effects of this parameter.

that the energy-saving technological progress has proved to be significant in the last two decades. Newell et al. (1999) reveals that increasing energy prices result in energy saving innovations in the USA. Through investigating the sectors of Dutch economy, Kuper and Soest (2006) shows that energy saving technological progress occurs after periods of high and rising energy prices.

The OLG framework is preferred to infinitely lived agents (ILA) since the latter ignore 'generation overlap and treat society in each period as a single generation caring about (and also discounting) the welfare of its immediate descendants, which has complete control over the rate of resource use and the saving rate' (Mourmouras, 1991, p. 585). In addition, as Agnani et al. (2005) indicates, the OLG models can be preferred to ILA in analyzing the sustainability of long-run growth with exhaustible resources since the natural resources may act as stores of values between different generations.

An analytical characterization of the balanced growth path (BGP) is provided and the conditions for a positive long run growth is investigated. The model is building upon Agnani et al. (2005), which studies the BGP of an OLG economy with exogenous technical progress where exhaustible resources is an essential input to the production. They show that a sufficiently high labor share is necessary for the economy to exhibit a positive steady state growth rate. The results also reveal that the share of labor in production has to be sufficiently high in order to yield positive growth along the BGP. However, the constraint on labor share is less binding, compared to Agnani et al. (2005). In this essay, it is shown that the share of renewables in energy production, the resource saving technological progress and the regeneration factor are among the key variables having an effect on the required labor share and hence the possibility of long run growth.

What determines the rate of depletion and how it is determined, is quite important as it paves the way for understanding the limits to growth. To answer this question, the effects of discount factor, resource augmenting technological progress and intensity of resources in energy production on the depletion rate are investigated.

As Smulders (2005) emphasizes, the increase in the discount factor and hence the patience of the households, is expected to decrease the depletion rate –which is also confirmed in the results– whereas the effect of the exogenous resource saving technological progress on the rate of depletion is accepted to be ambiguous due to opposing income and substitution effects. Under the productivity gains, households would attach a greater value to energy resources in future periods since these resources will be more productive. Households, thus, save more on these resources which demonstrates the substitution channel. On the other hand, more output would be obtained given a resource stock when the productivity increases. As a result, the households would know that they will have more income in the future. The income effect works through consumption smoothing and the households will consume more. It is found that along the BGP, the substitution effect dominates the income effect as long as the depletion rate is slightly higher than its lower bound. Thus, the higher the resource saving technological progress the economy deplete its energy resources less.

As regards the circumstances, increasing/decreasing the resource intensity of energy production promote growth, the main finding of the chapter is that the effects of an increase in the intensity of the renewable resources in producing energy has positive long run growth effects. It is shown that the patience of generations has important long run implications in this context. For more patient economies, an increase in resource saving technological progress will result in higher growth. With the presence of renewables, it is shown that the constraint on the labor share which is required to guarantee the long run growth is relaxed. It is also revealed that the sustainability of the economy depends on the energy saving effect of the technological progress and the depletion rate of the resources which in turn depend on the rest of the parameters in the economy. Finally, it is found that the BGP can turn out to be optimal.

The paper is structured as follows: Section 1 presents the model and Section 2 defines the equilibrium conditions for a decentralized economy. Section 3 analyzes the existence and the uniqueness of the BGP. The optimality and the sustainability conditions for BGP is analyzed in Section4. Section 5 performs the comparative statics analysis and Section 6 concludes.

2.1. The Model

A two-period overlapping generations model in discrete time with an infinite horizon is considered. At each period t, a generation of agents appears and lives for two periods, young and old. The population in period t consists of N_t young and N_{t-1} old individuals. The growth rate of population is assumed to be constant so that $N_{t+1} = (1+n)N_t$.

In comparison with a standard OLG model[3], the novel feature of this analysis is to consider energy as an essential input to production and take into account that it is built upon both renewable and non-renewable resources[4]. At each period, a single final good is produced in the economy by means of physical capital K, labor N, and energy Ξ. This physical good is either consumed or invested to build future capital. The energy input is obtained from the stock of renewable and non-renewable energy resources denoted by R and E, respectively. The renewable resource is assumed to regenerate itself with $g(R)$ where $g'(R) > 0$ at every period. These resources can act as both stores of value and inputs to the production process.

All agents have rational expectations and each generation consists of a single representative agent. Moreover, all agents in this economy are price-takers and all the markets are competitive.

At a given date, young households work, consume and invest a part of their income in physical capital which is rented and used by the firms in the next period. They invest another part of their income to purchase ownership rights for the renewable and the non-renewable energy resources. When old, they consume their entire income generated from the returns on their savings, and from selling their stock of energy resources to the firms.

[3]See de la Croix and Michel (2002) for a comprehensive treatment of the OLG models.

[4]See, for the discussion of energy being an essential input to production, Ayres et al. (2003; 2005; 2007) and Warr et al. (2006; 2008).

Following Koskela et al. (2008), at the beginning of each period t, the old agents (generation $t-1$) are assumed to own the stock of all energy resources and sell them to the firms. As in Dasgupta and Heal (1974), it is assumed that there are no extraction costs. Firms decide on the amount of renewable and non-renewable energy resources that will be used in the production process, Z_t and X_t, respectively. Before the end of the each period t, firms sell the remaining stock of renewable resources R_{t+1} and the non-renewable resources E_{t+1} to the young agents (generation t)[5]. Accordingly, the evolution dynamics of the energy resources can be formalized as follows:

$$R_{t+1} = R_t + g(R_t) - Z_t,$$

$$E_{t+1} = E_t - X_t.$$

In his first period of life (when young at period t), the representative individual is endowed with one unit of labor that he supplies inelastically to firms. His income is equal to the real wage w_t. He allocates this income among current consumption c_t, savings s_t invested in firms and the purchase of the ownership rights for the renewable r_{t+1} and the non-renewable resources e_{t+1}. The budget constraint of period t is

$$w_t = c_t + s_t + P_t^r r_{t+1} + P_t^e e_{t+1},$$

where P_t^r and P_t^e denote the prices of renewable and non-renewable energy resources, respectively. Note that $R_{t+1} = N_{t+1} r_{t+1}$, and $E_{t+1} = N_{t+1} e_{t+1}$.

In the second period of his life, the agent is retired and he consumes his entire income generated from the returns on his savings $Q_{t+1} s_t = (1 + q_{t+1}) s_t$, and the revenue from selling his stock of energy resources to the firms. Accordingly, his

[5] In Olson and Knapp (1997), although old agents own the resource stock they do not sell all of the resource to the firms. Instead, they choose how much of their stock will be sold to the production sector. Then through the asset market the unextracted resource stock is transferred from the old generation to the young generation. The resource accumulation equations does not differ by this specification.

consumption is

$$d_{t+1} = Q_{t+1}s_t \ + P^r_{t+1}(r_{t+1} + g(r_{t+1})) + P^e_{t+1}e_{t+1}.$$

where $z_{t+1} = \frac{Z_{t+1}}{N_{t+1}}$, $x_{t+1} = \frac{X_{t+1}}{N_{t+1}}$.

The preferences of the representative agent is defined over his consumption bundle (c_t, d_{t+1}). The preferences are represented by an additively separable life-cycle utility function $U(c,d) = u(c) + \beta u(d)$, where $\beta \in (0,1)$ is the subjective discount factor. In particular, a logarithmic instantaneous utility function u is adopted since the main concern of the paper is the existence of the balanced growth path and its qualitative properties[6].

Taking the prices of the energy resources and wages as given, the representative agent maximizes his life-time utility by choosing the young and the old periods' consumption and the ownership of the energy resources. The optimization problem of the representative agent born at time t can be formalized as follows:

$$\max_{\{c_t, d_{t+1}, s_t, r_{t+1}, e_{t+1}\}} \ln c_t + \beta \ln d_{t+1}$$

subject to

$$w_t \ = c_t + s_t + P^r_t r_{t+1} + P^e_t e_{t+1}, \qquad (1)$$

$$d_{t+1} = Q_{t+1}s_t \ + P^r_{t+1}(r_{t+1} + g(r_{t+1})) + P^e_{t+1}e_{t+1}, \qquad (2)$$

$$c_t \geq 0, d_{t+1} \geq 0, r_{t+1} \geq 0, e_{t+1} \geq 0.$$

[6]See, among others, King and Rebelo (1993) and Agnani et al. (2005), for the need to assume comsumer's preferences with CIES in order to have the existence of a BGP.

The following first-order conditions for the consumer's optimization problem follows:

$$\frac{d_{t+1}}{c_t} = \beta Q_{t+1}, \tag{3}$$

$$\frac{P^e_{t+1}}{P^e_t} = Q_{t+1}, \tag{4}$$

$$\frac{P^r_{t+1}}{P^r_t} = \frac{Q_{t+1}}{(1+g'(r_{t+1}))}. \tag{5}$$

Equation (3) gives the equalization of discounted marginal utilities where the marginal rate of substitution between the current and the future consumption is equal to their relative prices. (4) and (5) present no-arbitrage conditions among different types of savings implying that the marginal return on investing in the exhaustible resource is equal to the marginal return on investing in the renewable resource taking the regeneration factor into account. In other words, an increase in the price of the exhaustible resources from period t to $t+1$ is higher than that of the renewable resources reflecting the relative scarcity of the non-renewable resources.

Firms are owned by the old households and produce a homogenous consumption/investment good under perfect competition. Production at the final good sector is made through a Cobb-Douglas constant returns to scale technology[7]:

$$Y_t = {\Xi_t}^{\alpha_1} K_t^{\alpha_2} N_t^{\alpha_3}, \quad \alpha_i > 0 \text{ and } \sum_i \alpha_i = 1, \tag{6}$$

where

$$\Xi_t = A_t X_t^{\rho} Z_t^{1-\rho}, \ 0 \le \rho \le 1, \tag{7}$$

$$A_{t+1} = (1+a)A_t, \quad a > 0. \tag{8}$$

The energy input, Ξ_t is produced from non-renewable (X_t) and renewable energy resources (Z_t) by means of a Cobb-Douglas production technology. The intensity

[7] Taking into account the use of energy, Ayres (2008) shows that Cobb-Douglas production function fits to the economic growth for the US and Japan economy in the 20th century. Also, Serrenho et al. (2010) show that the inclusion of energy-related variables, increases the explanatory power of the models for a panel data of EU-15 countries between 1995 to 2007.

of non-renewable resources in producing energy is captured by ρ. As ρ increases the production of energy becomes more intensive in using non-renewable resources than using renewable resources. The productivity of resources in producing energy is represented by A_t. If the productivity of the resources in producing energy increases, less amount of resources will be needed to produce the same amount of energy. Therefore, the technical progress (increase in productivity) which is captured by a is considered to be energy saving. Generally, technical progress is considered as Hicks-neutral under Cobb-Douglas specification. The importance of distinguishing the energy-saving effect of the technical progress from the input neutral technological progress is that the prospects for sustainability depend on the energy-saving effect of technical progress and not on its global effect on the output levels.

Note that the assumption of perfect substitutability between all inputs does not stem from theoretical considerations only. As a matter of fact, the extent to which capital and energy are substitutes or complements in production is highly debated in the literature. Even in the early literature, Hudson and Jorgenson (1973) and Berndt and Wood (1975) found that capital and energy were complements, while Humphrey and Moroney (1975), Griffin and Gregory (1976) and Halvorsen (1977) concluded that they were substitutes. Apostolakis (1990), suggested that the studies based on time-series data reflect short-term relationships and hence these studies concludes capital and energy to be complements. However, he claims that the cross-sectional analysis reflects the long term relationship thus the studies based on cross-sectional data imply the perfect substitutability between energy and capital inputs.

As with constant returns to scale, the number of firms does not matter and the production is independent of the number of firms that use the same technology, a representative firm is taken. Under this perfectly competitive environment, the representative firm producing at period t maximizes its profit by choosing the amount of labor, physical capital and the energy inputs that will be utilized in the production process[8]:

$$\max_{\{K_t,N_t,Z_t,X_t\}} \pi_t = A_t^{\alpha_1} X_t^{\rho\alpha_1} Z_t^{(1-\rho)\alpha_1} K_t^{\alpha_2} N_t^{\alpha_3} - (q_t+\delta)K_t - w_t N_t - P_t^r Z_t - P_t^m X_t, \quad (9)$$

where $0 \leq \delta \leq 1$ denotes the depreciation rate of capital.

At an interior solution of the firm's optimization problem, where all variables are expressed in per capita terms ($k_t = \frac{K_t}{N_t}$, $z_t = \frac{Z_t}{N_t}$ and $e_t = \frac{E_t}{N_t}$), the following first order conditions are satisfied :

$$\alpha_2 A_t^{\alpha_1} {x_t}^{\alpha_1\rho} {z_t}^{\alpha_1(1-\rho)} k_t^{\alpha_2-1} = q_t + \delta, \quad (10)$$

$$\alpha_3 A_t^{\alpha_1} {x_t}^{\alpha_1\rho} {z_t}^{\alpha_1(1-\rho)} k_t^{\alpha_2} = w_t, \quad (11)$$

$$\alpha_1(1-\rho) A_t^{\alpha_1} {x_t}^{\alpha_1\rho} {z_t}^{\alpha_1(1-\rho)-1} k_t^{\alpha_2} = P_t^r, \quad (12)$$

$$\alpha_1\rho A_t^{\alpha_1} {x_t}^{\alpha_1\rho-1} {z_t}^{\alpha_1(1-\rho)} k_t^{\alpha_2} = P_t^e. \quad (13)$$

Re-arranging equations (12) and (13), the optimal mix between the exhaustible and renewable energy resources can be obtained as:

$$\frac{(1-\rho)}{\rho}\frac{x_t}{z_t} = \frac{P_t^r}{P_t^e}. \quad (14)$$

By Equation (14), the optimal mix between the renewable and non-renewable resources depend on their prices and the elasticity of substitution between the two sources of energy resources.

[8] Maximization problem of the firm is

$$\max_{\{K_t,N_t,Z_t,X_t\}} \pi_t = A_t^{\alpha_1} X_t^{\rho\alpha_1} Z_t^{(1-\rho)\alpha_1} K_t^{\alpha_2} N_t^{\alpha_3} - (q_t+\delta)K_t$$
$$- w_t N_t - P_t^r(R_t + g(R_t) - R_{t+1}) - P_t^m(E_t - E_{i+1}),$$

if the cash flow going through the firm is the explicitly written. Taking into account the dynamics of the energy resources, the consumption of the old individual at $t+1$ can be recast as Equation (9).

2.2. The Competitive Equilibrium

A dynamic competitive equilibrium of this overlapping generations economy is determined by the sequence of prices $\{w_t, q_t, P_t^e, P_t^r\}_{t=0}^{\infty}$, and the feasible allocations $\{c_t, d_t, s_t, r_t, e_t, x_t, z_t, y_t, \Xi_t, k_{t+1,}A_{t+1}\}_{t=0}^{\infty}$ given positive initial values for the state variables $\{k_0, E_0, R_0, A_0\}$ and the law of motion of A_t and N_t such that the consumers maximize their life-time utility, firms maximize their profits and all markets clear at every period t:

$$s_t = k_{t+1}(1+n), \tag{15}$$

$$r_t + g(r_t) = (1+n)r_{t+1} + z_t, \tag{16}$$

$$e_t = (1+n)e_{t+1} + x_t, \tag{17}$$

$$y_t = c_t + d_t(1+n)^{-1} + s_t. \tag{18}$$

Accordingly, a dynamic competitive equilibrium is a solution of the equation system, (1)-(18). Equation (15) indicates that the capital stock at $t+1$ is fully determined by saving decisions made at t, since the output is used either for consumption or investment in capital goods. The following two equations, (16) and (17) reveal the resource constraints for the energy resources. Equation (18) is the market clearing condition in the output market which holds by Walras' law.

2.3. The Balanced Growth Path

In order to analyze the feasibility of positive long run growth in the economy and hence the study focuses on the balanced growth path. To guarantee the analytical solution of the balanced growth path it is assumed assume that the renewable resource regenerates linearly, i.e., $g(R_t) = \Pi R_t$ for some constant regeneration factor $0 < \Pi < 1$.[9]

[9] Mourmouras (1991) also utilizes a constant regeneration rate in an overlapping generations framework. Apart from Mourmouras (1991) linear generation of renewables is widely used in the infinitely lived agents framework (see among others Nguyen and Van, 2008; Maltousoglu, 2009).

In order to characterize the balanced growth path of this competitive economy first growth factors of the variables are defined. The growth factor of any variable a_t is denoted by γ_a which is the ratio a_{t+1}/a_t. Along the balanced growth path $\gamma_a - 1$ will represent the growth rate of the corresponding variable. Then, the system will be reduced in terms of the depletion rates of resources. How much of the energy resources is used in the production compared to the total resource stock is represented by these rates. The depletion rate of the non-renewable resources are defined as $\tau_t = \frac{x_t}{e_t}$ and the depletion rate of the renewables can be defined as $\xi_t = \frac{z_t}{r_t(1+\Pi)}$.

Proposition 1 *Along a balanced growth path of this economy all variables grow at a constant rate. The balanced growth path is described by the stationary depletion rates, $\bar{\tau} = \tau_t = \tau_{t+1}$ and $\bar{\xi} = \xi_t = \xi_{t+1}$ which solve the following non-linear equations*

$$\frac{\gamma(1+n)}{1-\bar{\tau}} = \frac{\alpha_2 \gamma}{\left[\frac{\alpha_3 \beta}{(1+\beta)} - \frac{\alpha_1 \rho (1-\bar{\tau})}{(1+n)\bar{\tau}} - \frac{\alpha_1 (1-\rho)(1-\bar{\xi})}{\bar{\xi}}\right]} + (1-\delta),$$

where

$$\gamma = (1+a)^{\frac{\alpha_1}{(1-\alpha_2)}} \left[\left(\frac{1-\bar{\tau}}{1+n}\right)^{\frac{\alpha_1 \rho}{(1-\alpha_2)}} \left(\frac{(1+\Pi)(1-\bar{\xi})}{1+n}\right)^{\frac{\alpha_1 (1-\rho)}{(1-\alpha_2)}} \right],$$

and the following growth rates

$$\gamma_y = \gamma_k = \gamma_c = \gamma_d = \gamma_s = \gamma_w = \gamma,$$
$$\gamma_A = (1+a), \quad \gamma_N = (1+n),$$
$$\gamma_e = \gamma_x = \frac{1-\bar{\tau}}{1+n},$$
$$\gamma_r = \gamma_z = \frac{(1+\Pi)(1-\bar{\xi})}{1+n},$$
$$\gamma_{en} = \frac{(1+a)(1+\Pi)^{\rho}(1-\bar{\xi})^{\rho}(1-\bar{\tau})^{(1-\rho)}}{1+n},$$
$$\gamma_z = \gamma_x(1+\Pi),$$
$$\gamma_{p^e} = \frac{\gamma(1+n)}{1-\bar{\tau}},$$
$$\gamma_{p^r} = \frac{\gamma(1+n)}{(1+\Pi)(1-\bar{\xi})},$$
$$\gamma_Q = 1, \ \textit{and}\ \bar{\xi} = \bar{\tau}.$$

Proof. The equality of $\gamma_A = (1+a), \ \gamma_N = (1+n)$ follows from the definition of the technological progress and the population growth rate equations.

$\gamma_e = \gamma_x$ is obtained by the ratio of Equation (17) in period $t+1$ and t. After evaluating the resulting equation on the balanced growth path it is first observed that $\bar{\tau} = \tau_t = \tau_{t+1}$ and then $\gamma_x = \frac{1-\bar{\tau}}{1+n}$ is obtained.

To find the growth factor of capital per capita first Equation (13) is substituted into Equation (4):

$$Q_{t+1} = \gamma_A^{\alpha_1}\gamma_x^{\alpha_1\rho}\gamma_z^{\alpha_1(1-\rho)} \tag{19}$$

Then, using Equation (10) :

$$\gamma_A^{\alpha_1}\gamma_x^{\alpha_1\rho}\gamma_z^{\alpha_1(1-\rho)} - (1-\delta) = \alpha_2(A_{t+1}x_{t+1})^{\alpha_1\rho}(A_{t+1}z_{t+1})^{\alpha_1(1-\rho)}k_{t+1}^{\alpha_2-1}.$$

By evaluating this expression at $t+1$ and t and taking the ratio one gets

$$1 = \gamma_A^{\alpha_1}\gamma_x^{\alpha_1\rho}\gamma_z^{\alpha_1(1-\rho)}\gamma_k^{\alpha_2-1}, \tag{20}$$

$$\gamma \equiv \gamma_k = \gamma_A^{\frac{\alpha_1}{(1-\alpha_2)}} \gamma_x^{\frac{\alpha_1 \rho}{(1-\alpha_2)}} \gamma_z^{\frac{\alpha_1 (1-\rho)}{(1-\alpha_2)}}.$$

$\gamma_q = \gamma_Q = 1$ or $Q_{t+1} = Q_t = \bar{Q}$ are obtained from taking the ratio of Equation (4) in period $t+1$ and t, evaluating on the balanced growth path and then substituting Equation (19).

Evaluating Equation (5) along the balanced growth path, it is observed that to guarantee a constant growth in renewable prices $g'(r_{t+1})$ as to be constant. That is why the growth of the renewable resource is $g(r_t)$ is assumed to be a linear function of the previous period's stock.

$\gamma_r = \gamma_z$ is obtained by the ratio of Equation (16) in period $t+1$ and t. After evaluating the resulting equation on the balanced growth path first it is observed that $\bar{\xi} = \xi_t = \xi_{t+1}$ and this yields $\gamma_r = \frac{(1-\bar{\xi})(1+\Pi)}{1+n}$.

The growth factor of capital is equal to the output per capita i.e.$\gamma_k = \gamma_y$ from taking the ratio of the production function in period $t+1$ and t and then substituting Equation (20). Similarly, the equality of the growth factor of capital and the wages i.e.$\gamma_k = \gamma_w$ can be shown by taking the ratio of Equation (12) in period $t+1$ and t and then substituting Equation (20). The equality of $\gamma_k = \gamma_s$ is obtained through taking the ratio of Equation (15) in period $t+1$ and t.

The growth factor of the energy resource is obtained through taking the ratio of Equation (7) in period $t+1$:

$$\gamma_\Xi = \gamma_A \gamma_x^\rho \gamma_z^{(1-\rho)} = \frac{(1+a)(1+\Pi)^\rho (1-\bar{\xi})^\rho (1-\bar{\tau})^{(1-\rho)}}{1+n} \tag{21}$$

For the growth factor of price of the exhaustible resources taking the ratio of Equation (14) in period $t+1$ and t, evaluating it on the balanced growth path and then substituting Equation (20). $\gamma_{P_t^e} = \frac{\gamma_k}{\gamma_x} = \frac{\gamma(1+n)}{1-\bar{\tau}}$ follows. In parallel with the growth factor of the price of the exhaustible resources, the ratio of Equation (13) in period $t+1$ and t is taken, evaluating it on the balanced growth path and then substituting Equation (20) yields $\gamma_{p^r} = \frac{\gamma(1+n)}{(1+\Pi)(1-\bar{\xi})}$.

From dividing Equation (12) to Equation (13) and substituting the growth factors of the prices of the energy resources $\gamma_z = \gamma_x(1+\Pi)$ and $\bar{\xi} = \bar{\tau}$.

$\gamma_d = \gamma_c$ from taking the ratio of Equation (3) in period $t+1$ and t and evaluating it on the balanced growth path. Moreover to show$\gamma_d = \gamma_c = \gamma_k$ first substitute Equations (12), (13) and (15) into Equation (2) and obtain

$$d_{t+1} - Q_{t+1}k_{t+1}(1+n) = \alpha_1 A_{t+1}^{\alpha_1\rho} {x_{t+1}}^{\alpha_1\rho} A_{t+1}^{\alpha_1(1-\rho)} {z_{t+1}}^{\alpha_1(1-\rho)} k_{t+1}^{\alpha_2} \left(\frac{1}{\bar{\xi}}(1-\rho) + \rho\frac{1}{\bar{\tau}} \right).$$

Then taking the ratio of Equation (**??**) in period $t+1$ and t and evaluating it on the balanced growth path:

$$\frac{d_{t+1} - Q_{t+1}k_{t+1}(1+n)}{d_t - Q_t k_t (1+n)} = \gamma_A^{\alpha_1}\gamma_x^{\alpha_1\rho}\gamma_z^{\alpha_1(1-\rho)-1}\gamma_k^{\alpha_2}$$

Using Equation (20) and the definition of growth factors yields

$$\frac{d_t\gamma_d}{\gamma_k} - Q_{t+1}k_t(1+n) = d_t - Q_t k_t (1+n).$$

$Q_{t+1} = Q_t = \bar{Q}$ implies $\gamma_d = \gamma$.

As a final step the growth factor of capital is characterized as follows. Substituting Equations (15), (4), (5) and (6) into Equation (2) and dividing both sides by k_t to obtain;

$$\gamma_k(1+n) = A_t^{\alpha_1} {x_t}^{\alpha_1\rho} {z_t}^{\alpha_1(1-\rho)} k_t^{\alpha_2 - 1} [\frac{\alpha_3\beta}{(1+\beta)} - \alpha_1(1-\rho)\frac{\gamma_r}{(1+\Pi) - \gamma_r(1+n)} - \alpha_1\rho\frac{\gamma_e}{(1-\gamma_e(1+n)}]$$

From Equation (19) and Equation (4):

$\gamma_k(1+n) =$

$$\left[\gamma_A^{\alpha_1}\gamma_x^{\alpha_1\rho}\gamma_z^{\alpha_1(1-\rho)}\gamma_k^{\alpha_2}-(1-\delta)\right]\frac{1}{\alpha_2}\left[\frac{\alpha_3\beta}{(1+\beta)}-\alpha_1(1-\rho)\frac{1-\bar{\xi}}{\bar{\xi}(1+n)}-\alpha_1\rho\frac{1-\tau}{\tau(1+n)}\right]$$

$$\gamma_k(1+n) = \left[\frac{\alpha_2\gamma}{\left(\frac{\alpha_3\beta}{(1+\beta)}-\alpha_1(1-\rho)\frac{1-\bar{\xi}}{\bar{\xi}(1+n)}-\alpha_1\rho\frac{1-\tau}{\tau(1+n)}\right)}+(1-\delta)\right]$$

■

From the dynamics of the non-renewable resource stock (17), it can be inferred that e_t is decreasing as long as there is a positive amount of extraction. Along the balanced growth path, a constant decrease in non-renewable resource stock is only possible with a constant depletion rate: $\bar{\tau} = \tau_t = \tau_{t+1}$. Similarly, to guarantee a constant growth rate for the renewable resource stock, the depletion rate of renewables should also be constant $\bar{\xi} = \xi_t = \xi_{t+1}$ along the balanced growth path. Therefore, the energy resources used in production will decline over time indicating an asymptotic depletion. However, the rate of decrease in renewable resource stock used in production will be smaller than that of the non-renewable resources due to the regeneration factor. As the non-renewable stock is declining along the balanced growth path, the price non-renewables are growing at a higher rate than income and that of the renewable resources. However, the comparison between the price of renewables and income depends on the relationship between the growth rate of the population and regeneration factor. As the regeneration rate decreases or the population growth rate increases, the increase in the price of renewables will be higher than that of income. In parallel with Agnani et al. (2005), Proposition 1 shows that income, capital, consumption, savings and wages grow at the same rate and the interest rate is constant along the balanced growth path. It should be noted that, in line with Agnani et al. (2005), although the technological progress is modeled as exogenous, the growth rate of the economy depends on all of the parameters of the model, actually a feature of endogenous growth models. In contrast with the standard ILA economies with non-renewable resources, where the stationary depletion rate depends exclusively on the subjective discount factor, the depletion rate depends on all of the parameters of the model. In addition to this striking result, the setting allows analyzing the effects of the regeneration factor and the intensity of non-renewables in energy production explicitly.

In order to prove the balanced growth path of this model described by the above system has a unique solution with a constant depletion rate, the system of equations will be recast in terms of a single depletion rate.

Corollary 2 *Any balanced growth path of this economy is characterized by a stationary depletion rate, $\bar{\tau}$ which is the solution of the following non-linear equation*

$$\frac{\gamma(1+n)}{1-\bar{\tau}} = \frac{\alpha_2\gamma}{\left[\frac{\alpha_3\beta}{(1+\beta)} - \frac{\alpha_1(1-\bar{\tau})}{(1+n)\bar{\tau}}\right]} + (1-\delta),$$

where

$$\gamma = (1+a)^{\frac{\alpha_1}{(1-\alpha_2)}}\left[\frac{(1-\bar{\tau})}{(1+n)}\right]^{\frac{\alpha_1}{(1-\alpha_2)}}(1+\Pi)^{\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}},$$

and the following growth rates:

$$\begin{aligned}
\gamma_y &= \gamma_k = \gamma_c = \gamma_d = \gamma_s = \gamma_w = \gamma, \\
\gamma_{p^e} &= \frac{\gamma}{\gamma_x} = \frac{\gamma(1+n)}{1-\bar{\tau}}, \\
\gamma_{p^r} &= \frac{\gamma}{\gamma_z} = \frac{\gamma(1+n)}{(1+\Pi)(1-\bar{\tau})}, \\
\gamma_e = \gamma_x &= \frac{\gamma_r}{(1+\Pi)} = \frac{\gamma_z}{(1+\Pi)} = \frac{1-\bar{\tau}}{1+n}, \\
\gamma_{en} &= \gamma_A\gamma_x^{\rho}\gamma_z^{(1-\rho)} = \frac{(1+a)(1+\Pi)^{\rho}(1-\bar{\tau})}{1+n}, \\
\gamma_A &= (1+a) \;\; , \;\gamma_n = (1+n), \;\; \textit{and } \gamma_Q = 1.
\end{aligned}$$

Proposition 3 *A unique stationary equilibrium exists if*

$$\frac{\alpha_1}{\alpha_3(1+n)\frac{\beta}{(1+\beta)} + \alpha_1} < \bar{\tau} < 1.$$

Proof. Substituting γ, the system can be solved from solving the following equation involving only $\bar{\tau}$:

$$\frac{(1+a)^{\frac{\alpha_1}{(1-\alpha_2)}}\left[\frac{(1-\bar{\tau})}{(1+n)}\right]^{\frac{\alpha_1}{(1-\alpha_2)}}(1+\Pi)^{\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}}(1+n)}{1-\bar{\tau}} = \frac{\alpha_2(1+a)^{\frac{\alpha_1}{(1-\alpha_2)}}\left[\frac{(1-\bar{\tau})}{(1+n)}\right]^{\frac{\alpha_1}{(1-\alpha_2)}}(1+\Pi)^{\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}}}{\left[\frac{\alpha_3\beta}{(1+\beta)}-\frac{\alpha_1(1-\bar{\tau})}{(1+n)\bar{\tau}}\right]}+(1-\delta). \quad (22)$$

It could be easily checked that left hand side of the above equation is increasing with respect to $\bar{\tau}$ in [0,1) and right hand side of the equation is decreasing with respect to $\bar{\tau}$ in $[\frac{\alpha_1}{\alpha_3(1+n)\frac{\beta}{(1+\beta)}+\alpha_1}, 1]$. Thus, there exists a unique $\tau^* \epsilon \left(\frac{\alpha_1}{\alpha_3(1+n)\frac{\beta}{(1+\beta)}+\alpha_1}, 1\right)$.
■

Proposition 3 indicates that a unique balanced growth path exists if the depletion rate is higher than some critical level which is positively related with the share of energy input in production and inversely related with the share of the labor in production, the growth rate of population and the discount factor. The positive growth along the balanced growth path is not guaranteed under Proposition 3. In fact, it is shown that even without the incorporation of scarce natural resources the OLG economies may contract. To illustrate, Galor and Ryder (1989) examines an OLG economy without natural resources and show that unless restrictions on the nature of the interaction between technology and preferences are satisfied the economy may contract. There are also studies indicating the possibility of contraction if natural resources are taken into account. For instance, Agnani et al. (2005) show that a high enough labor share is a necessary condition for the economy to exhibit positive growth. In particular, they mention that the young generations need to earn high enough wages to do savings on capital and exhaustible resources whose prices are increasing along the balanced growth path. They conclude a minimum amount of labor share is necessary to guarantee such a wage income.

Proposition 4 *The economy will contract unless the labor share is high enough,* *i.e.*,

$$\alpha_3 \geqslant \frac{\alpha_1 (1+a)^{-1} \frac{(1+\beta)}{\beta} (1+\Pi)^{\rho-1}}{\left[1-(1+n)(1+a)^{-1}(1+\Pi)^{\rho-1}\right]}. \tag{23}$$

Proof. Proposition 3 establishes a lower bound for the depletion rate in the economy by $\frac{\alpha_1}{\alpha_3(1+n)\frac{\beta}{(1+\beta)}+\alpha_1} < \bar{\tau}$, necessary for the existence of a balanced growth path. Moreover, the upper bound for the depletion rate is established when $\gamma > 1$. Thus, $\bar{\tau} \in \left[\frac{\alpha_1}{\alpha_3(1+n)\frac{\beta}{(1+\beta)}+\alpha_1}, 1 - \frac{(1+n)}{(1+a)^{\rho}(1+b)^{1-\rho}(1+\Pi)^{1-\rho}}\right]$. Such a lower bound does not appear in characterizing ■

the stationary depletion rate in the ILA economies. The economy will contract if the the depletion rate is higher than its upper bound. In fact, the economy will not exhibit a positive growth if the lower bound for the depletion rate is higher than its upper bound.

It is shown that the share of labor in production has to be sufficiently high in order to yield positive growth along the balanced growth path. This condition highlights that, a minimum amount of labor share is necessary for the young to earn high enough wages to finance their investments. However, the option of saving on renewables other than just capital and non-renewable resources, relaxes the constraint on the labor share. As a result, compared with Agnani et al. (2005), the constraint on labor share is less binding. This result stems from the fact that Agnani et al. (2005) does not take into account the presence of renewables. With Proposition 3, it is demonstrated that the share of renewables in energy production, the technological progress in producing energy and the regeneration factor are among the key variables affecting the required labor share and hence the possibility of long run growth. Comparing this result with ILA economies one observes that the labor share does not appear in the above equation. Therefore, an economy having the same parameters except the share of labor in production may contract in the OLG framework but grow in the ILA setup.

2.4. Sustainability and Optimality

After demonstrating the existence of the unique competitive balanced growth path, the following propositions analyze whether this unique path is sustainable and/or optimal. In line with recent literature, a sustainable path is defined to be a path along which welfare is non-declining over time[10].

Proposition 5 *A necessary and sufficient condition for sustainability in this economy is to yield positive growth along the BGP* $\gamma_y \geqslant 1$, *so that*

$$(1+a)(\frac{1-\bar{\tau}}{1+n})(1+\Pi)^{1-\rho} \geqslant 1.$$

Proof. Using Equations (2), (4) and (5) yields

$$d_{t+1} = Q_{t+1}\left[s_t \ + P_t^r r_{t+1} + P_t^e e_{t+1}\right] \tag{24}$$

through substituting Equations (1) and (4) ,

$$c_t = \frac{\alpha_3 y_t}{(1+\beta)} \tag{25}$$

Plugging Equation (25) and Equation (10) into Equation (24),

$$d_t = \beta\frac{\alpha_2\alpha_3 y_t}{(1+\beta)}\frac{y_{t+1}}{k_{t+1}}$$

Then, it is clear that

$$U_t(c_t, d_{t+1}) = \log(\frac{\alpha_3^2\alpha_{2\beta}}{(1+\beta)^2}) + (1+\beta)\log y_t + \beta\log y_{t+1} - \beta\log k_{t+1},$$

[10]Specifically, if U_t denotes the lifetime utility of an agent born in period t, sustainability requires

$$U_{t+1}(c_{t+1}, d_{t+2}) \geqslant U_t(c_t, d_{t+1})$$

and the sustainability condition $U_{t+1}(c_{t+1}, d_{t+2}) \geqslant U_t(c_t, d_{t+1})$ reduces to

$$(1+\beta)\log\gamma_y > 0$$

along the balanced growth path. A necessary and sufficient condition for sustainability in this economy is to yield positive growth along the BGP so that $\gamma_y \geqslant 1$, *i.e.*,

$$(1+a)(\frac{1-\bar{\tau}}{1+n})(1+\Pi)^{1-\rho} \geqslant 1.$$

. ■

The above condition clearly shows that the sustainability of the economy depends on the energy saving technological progress but not on the total factor productivity. In addition, it depends on the depletion rate which in turn depends on all of the structural parameters of the economy. It can be observed that the higher the patience of the individuals, the higher the share of renewables in production and the regeneration rate, the more sustainable is the economy. However if the population growth rate increases, as there are more future generations it will be more difficult to sustain growth.

To derive the conditions for intergenerational optimality, the social planners problem is studied as in De La Croix and Michel (2002). The existence of a social planner whose maximizes a discounted sum of the life-cycle utility of all current and future generations with respect to the resource constraints of the economy is assumed. The planners objective function is social welfare function whereas the planner's discount factor is the social discount factor. The optimal balanced growth path is characterized by:

(a) Income, capital, consumption growing at the same rate γ so that $\gamma_y = \gamma_k = \gamma_c = \gamma_d = \gamma$.

(b) Energy resources used in production will decline over time indicating an asymptotic depletion:

$$\gamma_e = \gamma_x = \frac{1}{1+R},$$

where R denotes the subjective discount factor of the social planner.

(c)The rate of decrease in renewable resource stock used in production will be smaller than that of the exhaustible resources: $\gamma_r = \gamma_z = (1+\Pi)\gamma_x$. In accordance with these, the below Proposition on the optimality of the competitive equilibrium follows.

Proposition 6 *The competitive balanced growth path is pareto optimal as long as* $\bar{\tau} = \frac{R}{1+R}$.

Proof. A social planner solves the following problem[11]:

$$\max_{\{c_t,d_t,x_t,z_t,k_{t+1}\}_{t=0}^{\infty}} \beta \ln \mathrm{d}_0 + \sum_{t=0}^{\infty} (\frac{1}{(1+R)})^t \left[\ln c_t + (1+R)^{-1}\beta \ln d_t\right]$$

subject to the aggregate resource constraints of the economy

$$y_t = c_t + d_t(1+n)^{-1} + (1+n)k_{t+1} - (1-\delta)k_t \tag{26a}$$

$$y_t = A_t(x_t)^{\rho}(z_t)^{(1-\rho)}k_t^{\alpha_2} \tag{26b}$$

$$r_t + g(r_t) = (1+n)r_{t+1} + z_t \tag{26c}$$

$$e_t = (1+n)e_{t+1} + x_t \tag{26d}$$

$$e_0 \geqslant \sum_{t=0}^{\infty} x_t \tag{26e}$$

$$r_0 \geqslant \sum_{t=0}^{\infty} z_t - \sum_{t=0}^{\infty}(1+\Pi)r_{t-1} \tag{26f}$$

The first order conditions with respect to c_t and d_t yield

$$\frac{d_{t+1}}{c_t} = \beta(1+n)R_{t+1} \tag{27}$$

[11] See De La Croix and Michel (2002, pp.91).

, then using Equation (26a) and (26b)

$$\frac{(1+R)^{-t-1}}{A_t^{\alpha_1}(e_t-e_{t+1})^{\alpha_1\rho}(r_t(1+\Pi)-r_{t+1})^{\alpha_1(1-\rho)}k_t^{\alpha_2}-\frac{d_t}{(1+n)}-k_{t+1}(1+n)-(1-\delta)k_t} = (1+R)^{-t-1-1}\beta d_t^{-1}. \quad \text{(app7)}$$

From the first order conditions with respect to e_{t+1} and r_{t+1}

$$\frac{(1+R)^{-t-1}\left[A_t^{\alpha_1}(e_t-e_{t+1})^{\alpha_1\rho-1}(r_t(1+\Pi)-r_{t+1})^{\alpha_1(1-\rho)}k_t^{\alpha_2}\right]}{A_t^{\alpha_1}k_t^{\alpha_2}en_t^{\alpha_1}-\frac{d_t}{(1+n)}-k_{t+1}(1+n)-(1-\delta)k_t} = \frac{\left[\alpha_1\rho A_{t+1}^{\alpha_1}(e_{t+1}-e_{t+2})^{\alpha_1\rho-1}(r_{t+1}(1+\Pi)-r_{t+2})^{\alpha_1(1-\rho)}k_{t+1}^{\alpha_2}\right](1+R)^{-t-1-1}}{A_{t+1}^{\alpha_1}k_{t+1}^{\alpha_2}en_{t+1}^{\alpha_1}-\frac{d_{t+1}}{(1+n)}-k_{t+2}(1+n)-(1-\delta)k_{t+1}},$$

and

$$\frac{(1+R)^{-t-1}\left[A_t^{\alpha_1}(e_t-e_{t+1})^{\alpha_1\rho}(r_t(1+\Pi)-r_{t+1})^{\alpha_1(1-\rho)-1}k_t^{\alpha_2}\right]}{A_t^{\alpha_1}k_t^{\alpha_2}en_t^{\alpha_1}-\frac{d_t}{(1+n)}-k_{t+1}(1+n)-(1-\delta)k_t} = \frac{\left[\alpha_1\rho A_{t+1}^{\alpha_1}(e_{t+1}-e_{t+2})^{\alpha_1\rho}(r_{t+1}(1+\Pi)-r_{t+2})^{\alpha_1(1-\rho)-1}k_{t+1}^{\alpha_2}\right](1+R)^{-t-1-1}}{A_{t+1}^{\alpha_1}k_{t+1}^{\alpha_2}en_{t+1}^{\alpha_1}-\frac{d_{t+1}}{(1+n)}-k_{t+2}(1+n)-(1-\delta)k_{t+1}}.$$

From the first order conditions with respect to k_{t+1}

$$A_{t+1}^{\alpha_1}(e_{t+1}-e_{t+2})^{\alpha_1\rho}(r_{t+1}(1+\Pi)-r_{t+2})^{\alpha_1(1-\rho)}k_{t+1}^{\alpha_2}- \frac{d_{t+1}}{(1+n)}-k_{t+2}(1+n)-(1-\delta)k_{t+1} = [A_t^{\alpha_1}k_t^{\alpha_2}en_t^{\alpha_1}-\frac{d_t}{(1+n)}-k_{t+1}(1+n)- (1-\delta)k_t]\left[A_{t+1}^{\alpha_1}k_{t+1}^{\alpha_2-1}en_{t+1}^{\alpha_1}+(1-\delta)\right](1+R)^{-1}.$$

After some algebra from first order conditions

$$(1+\delta)+\alpha_2\mathrm{A}_{t+1}^{\alpha_1}\mathrm{k}_{t+1}^{\alpha_2-1}\mathrm{en}_{t+1}^{\alpha_1}=\frac{c_{t+1}}{c_t}(1+\mathrm{R}), \quad (28)$$

$$\frac{c_{t+1}}{c_t}=\frac{1}{(1+R)}\left(\frac{k_{t+1}}{k_t}\right)^{\alpha_2}\left(\frac{x_{t+1}}{x_t}\right)^{\alpha_1\rho-1}\left(\frac{z_{t+1}}{z_t}\right)^{\alpha_1(1-\rho)}\left(\frac{A_{t+1}}{A_t}\right)^{\alpha_1}, \tag{29}$$

$$\frac{c_{t+1}}{c_t}=\frac{(1+\Pi)}{(1+R)}\left(\frac{k_{t+1}}{k_t}\right)^{\alpha_2}\left(\frac{x_{t+1}}{x_t}\right)^{\alpha_1\rho}\left(\frac{z_{t+1}}{z_t}\right)^{\alpha_1(1-\rho)-1}\left(\frac{A_{t+1}}{A_t}\right)^{\alpha_1}. \tag{30}$$

The equality of $\gamma_A = (1+b)$, $\gamma_N = (1+n)$ follows from the definition of the technological progress and the population growth rate equations. In addition along the balanced growth path $\gamma_R = 1$ or $R_{t+1} = R_t = \bar{R}$.

Equality of $\gamma_e = \gamma_x$ is obtained by the ratio of Equation (26c) in period $t+1$ and t. After evaluating the resulting equation on the balanced growth path first observe $\bar{\tau} = \tau_t = \tau_{t+1}$ and then obtain $\gamma_x = \frac{1-\bar{\tau}}{1+n}$. The equality of $\gamma_r = \gamma_z$ is obtained by the ratio of Equation (26d) in period $t+1$ and t. After evaluating the resulting equation on the balanced growth path it first observed that $\bar{\xi} = \xi_t = \xi_{t+1}$ and then $\gamma_r = \frac{(1-\bar{\xi})(1+\Pi)}{1+n}$. By means of Equation (28), the below equation follows

$$\gamma_c(1+R) - (1-\delta) = \alpha_2(A_{t+1}x_{t+1})^{\alpha_1\rho}(A_{t+1}z_{t+1})^{\alpha_1(1-\rho)}k_{t+1}^{\alpha_2-1}. \tag{31}$$

By evaluating this expression at $t+1$ and t and taking the ratio:

$$\begin{aligned} 1 &= \gamma_A^{\alpha_1}\gamma_x^{\alpha_1\rho}\gamma_z^{\alpha_1(1-\rho)}\gamma_k^{\alpha_2-1}, \\ \gamma &\equiv \gamma_k = \gamma_A^{\frac{\alpha_1}{(1-\alpha_2)}}\gamma_x^{\frac{\alpha_1\rho}{(1-\alpha_2)}}\gamma_z^{\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}}. \end{aligned} \tag{32}$$

Observe that the growth factor of capital is equal to the output per capita, i.e., $\gamma_k = \gamma_y$ from taking the ratio of Equation (26b) in period $t+1$ and t and then substituting in Equation (32). The growth factor of the energy resource is obtained by taking the ratio of Equation (7) in period $t+1$:

$$\gamma_\Xi = \gamma_A\gamma_x^\rho\gamma_z^{(1-\rho)} = \frac{(1+a)(1+\Pi)^\rho(1-\bar{\xi})^\rho(1-\bar{\tau})^{(1-\rho)}}{1+n}.$$

By dividing Equation (29) to Equation (30), $\gamma_z = \gamma_x(1+\Pi)$ and $\bar{\xi} = \bar{\tau}$.

Note that $\gamma_d = \gamma_c$ by taking the ratio of Equation (27) in period $t+1$ and t and evaluating it on the balanced growth path. Moreover by Equation (29) it is clear that $\gamma_d = \gamma_c = \gamma_k$.

From Equation (29) and using the equality of γ_c and γ_k :

$$\gamma_x = \frac{1}{(1+R)} \text{ and } \bar{\tau} = \frac{R}{1+R}.$$

■

2.5. Comparative Statics

The effects of the following parameters on the depletion rates of the resources and on the long run growth is analytically proved: (i) a change in the depreciation rate, (ii) a change in the discount rate, (iii) a change in the resource saving technological progress (iv) a change in the intensity of non-renewables in energy production, and (v) a change in the regeneration factor.

Proposition 7 *(i) Higher depreciation of capital brings about lower depletion rates and higher growth along the balanced growth path:*

$$\frac{\partial \bar{\tau}}{\partial \delta} < 0, \quad \frac{\partial \gamma}{\partial \delta} > 0.$$

(ii) More patient generations will deplete their natural resources less and benefit from higher growth along the balanced growth path:

$$\frac{\partial \bar{\tau}}{\partial \beta} < 0, \; \frac{\partial \gamma}{\partial \beta} > 0.$$

(iii) Economies with higher resource saving technologies grow faster along the balanced growth path ($\frac{\partial \gamma}{\partial a} > 0$) *and deplete their resources less* ($\frac{\partial \bar{\tau}}{\partial a} < 0$).

(iv) Economies with higher share of renewables in energy production have lower depletion rates ($\frac{\partial \bar{\tau}}{\partial \rho} < 0$) *and will exhibit higher growth* ($\frac{\partial \bar{\tau}}{\partial \gamma} > 0$).

(v) Economies with higher regeneration rates in the renewable resources grow faster along the balanced growth path ($\frac{\partial\gamma}{\partial\Pi} > 0$) and deplete their resources less ($\frac{\partial\bar{\tau}}{\partial\Pi} < 0$).

Proof. Equation (22) can be reduced into a implicit equation involving only τ as

$$A(\bar{\tau}) = \frac{(1+a)^{\frac{\alpha_1}{(1-\alpha_2)}}\left[\frac{(1-\bar{\tau})}{(1+n)}\right]^{\frac{\alpha_1}{(1-\alpha_2)}}(1+\Pi)^{\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}}(1+n)}{1-\bar{\tau}} - \frac{\alpha_2(1+a)^{\frac{\alpha_1}{(1-\alpha_2)}}\left[\frac{(1-\bar{\tau})}{(1+n)}\right]^{\frac{\alpha_1}{(1-\alpha_2)}}(1+\Pi)^{\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}}}{\left[\frac{\alpha_3\beta}{(1+\beta)} - \frac{\alpha_1(1-\bar{\tau})}{(1+n)\bar{\tau}}\right]} - (1-\delta).$$

Thus, $A[\delta, \beta, a, \alpha_1, \alpha_2, \alpha_{3,}\rho, \Pi, n, \tau] = 0$ and

$$A_{\bar{\tau}} = \frac{(1+a)^{\frac{\alpha_1}{(1-\alpha_2)}}\left[\frac{(1-\bar{\tau})}{(1+n)}\right]^{\frac{\alpha_1}{(1-\alpha_2)}}(1+\Pi)^{\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}}(1+n)}{1-\bar{\tau}} \left(\begin{array}{c} -\frac{\alpha_3}{(1-\bar{\tau})^2(1-\alpha_2)} - \frac{(1+\beta)^2\alpha_1\alpha_2}{((1+\beta)(-1+\bar{\tau})\alpha_1+(1+n)\alpha_3\beta\bar{\tau})^2} \\ -\frac{(1+\beta)\tau\alpha_1\alpha_2}{(-1+\bar{\tau})(-1+\alpha_2)((1+\beta)(-1+\bar{\tau})\alpha_1+(1+n)\alpha_3\beta\bar{\tau})} \end{array}\right)$$

$$\mathrm{A}_\tau < 0 \quad \text{if} \quad \bar{\tau} > \frac{\alpha_1}{\alpha_3(1+n)\frac{\beta}{(1+\beta)} + \alpha_1}.$$

Taking the total derivative and looking for the comparative statistics with respect to any parameter z ,: $\frac{\partial\tau}{\partial z} = -\frac{A_z}{A_\tau}$.Accordingly, one only needs to check the sign of A_z.

(i) Since $A_\delta = -1$, $\frac{\partial\tau}{\partial\delta} < 0$. Moreover,

$$\frac{\partial\gamma}{\partial\delta} = -\gamma\alpha_1\frac{1}{(-1+\bar{\tau})(-1+\alpha_2)}\frac{\partial\tau}{\partial\delta} > 0.$$

(ii) Since

$$A_\beta = -\frac{(1+a)^{\frac{\alpha_1}{(1-\alpha_2)}}\left[\frac{(1-\bar{\tau})}{(1+n)}\right]^{\frac{\alpha_1}{(1-\alpha_2)}}(1+\Pi)^{\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}}(1+n)^2\alpha_3\alpha_2\bar{\tau}^2}{((1+\beta)(-1+\bar{\tau})\alpha_1+(1+n)\alpha_3\beta\bar{\tau})^2},$$

, $A_\beta < 0$ and hence $\frac{\partial \tau}{\partial \beta} < 0$. Moreover,

$$\frac{\partial \gamma}{\partial \beta} = -\gamma \alpha_1 \frac{1}{\left(-1+\bar{\tau}\right)\left(-1+\alpha_2\right)} \frac{\partial \tau}{\partial \beta} > 0.$$

(iii) Since

$$A_a = -\frac{(1+a)^{\frac{\alpha_1 \rho}{(1-\alpha_2)}-1} (1+a)^{\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}} \left[\frac{(1-\bar{\tau})}{(1+n)}\right]^{\frac{\alpha_1}{(1-\alpha_2)}} (1+\Pi)^{\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}} (1+n)\rho\alpha_1}{(1-\alpha_2)} \left(\frac{1}{-1+\bar{\tau}} + \frac{(1+\beta)\bar{\tau}\alpha_2}{(1+\beta)\left(-1+\bar{\tau}\right)\alpha_1 + (1+n)\alpha_3\beta\bar{\tau}}\right)$$

,

$$A_a < 0 \quad if \quad \frac{1}{-1+\bar{\tau}} + \frac{(1+\beta)\bar{\tau}\alpha_2}{(1+\beta)\left(-1+\bar{\tau}\right)\alpha_1 + (1+n)\alpha_3\beta\bar{\tau}} < 0.$$

This condition can be recast as

$$\alpha_3(1+n)\frac{\beta}{(1+\beta)} > \frac{(1-\bar{\tau})}{\bar{\tau}}\alpha_1 + (1-\bar{\tau})\alpha_2.$$

Thus,

$$\frac{\partial \tau}{\partial a} < 0 \quad if \;\; \alpha_3(1+n)\frac{\beta}{(1+\beta)} > \frac{(1-\bar{\tau})}{\bar{\tau}}\alpha_1 + (1-\bar{\tau})\alpha_2.$$

Moreover,

$$\frac{\partial \gamma}{\partial a} = -\gamma \alpha_1 \frac{1}{\left(-1+\bar{\tau}\right)\left(-1+\alpha_2\right)} \frac{\partial \tau}{\partial a} + \frac{\alpha_1 \rho}{(1-\alpha_2)} (1+a)^{-1} \gamma > 0 \text{ if}$$

$$\alpha_3(1+n)\frac{\beta}{(1+\beta)} > \frac{(1-\bar{\tau})}{\bar{\tau}}\alpha_1 + (1-\bar{\tau})\alpha_2.$$

(iv) Since $A_\rho < 0$ if

$$\alpha_3(1+n)\frac{\beta}{(1+\beta)} > \frac{(1-\bar{\tau})}{\bar{\tau}}\alpha_1 + (1-\bar{\tau})\alpha_2, \text{ and } \log\left(\frac{1}{(1+\Pi)}\right) < 0$$

Thus, $\frac{\partial \tau}{\partial \rho} < 0$ if

$$\alpha_3(1+n)\frac{\beta}{(1+\beta)}>\frac{(1-\bar{\tau})}{\bar{\tau}}\alpha_1+(1-\bar{\tau})\alpha_2, \text{ and } \log\left(\frac{1}{(1+\Pi)}\right)<0.$$

Moreover,

$$\gamma_\rho= -\gamma\alpha_1\frac{1}{(-1+\bar{\tau})(-1+\alpha_2)}\frac{\partial\tau}{\partial\rho}+\frac{\alpha_1}{(1-\alpha_2)}\gamma\log\left(\frac{1}{(1+\Pi)}\right).$$

$$\gamma_\rho=\frac{1}{(-1+\alpha_2)^2}\gamma\log\left(\frac{(1+a)}{(1+a)(1+\Pi)}\right)\alpha_1$$
$$\left[1-\alpha_2+\frac{\gamma(1+n)\left((1+\beta)(-1+\bar{\tau})\alpha_1+\bar{\tau}((1+\beta)(-1+\bar{\tau})\alpha_2+(1+n)\alpha_3\beta)\right)}{(-1+\bar{\tau})^2\left((1+\beta)(-1+\bar{\tau})\alpha_1+(1+n)\alpha_3\beta\bar{\tau}\right)}\right]$$

so that $\gamma_\rho>0$ if

$$\alpha_3(1+n)\frac{\beta}{(1+\beta)}>\frac{(1-\bar{\tau})}{\bar{\tau}}\alpha_1+(1-\bar{\tau})\alpha_2 \ \&\log\left(\frac{1}{(1+\Pi)}\right)<0$$

(v) Since

$$A_\Pi=-\frac{(1+a)^{\frac{\alpha_1}{(1-\alpha_2)}}\left[\frac{(1-\bar{\tau})}{(1+n)}\right]^{\frac{\alpha_1}{(1-\alpha_2)}}(1+\Pi)^{\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}-1}(1+n)(1-\rho)\alpha_1}{(1-\alpha_2)}$$
$$\left(\frac{1}{-1+\bar{\tau}}+\frac{(1+\beta)\bar{\tau}\alpha_2}{(1+\beta)(-1+\bar{\tau})\alpha_1+(1+n)\alpha_3\beta\bar{\tau}}\right),$$

, $A_\Pi<0$ if

$$\alpha_3(1+n)\frac{\beta}{(1+\beta)}>\frac{(1-\bar{\tau})}{\bar{\tau}}\alpha_1+(1-\bar{\tau})\alpha_2.$$

Thus, , $\frac{\partial\tau}{\partial\Pi}<0$ if

$$\alpha_3(1+n)\frac{\beta}{(1+\beta)}>\frac{(1-\bar{\tau})}{\bar{\tau}}\alpha_1+(1-\bar{\tau})\alpha_2.$$

Moreover, under the above mentioned assumption,

$$\frac{\partial\gamma}{\partial\Pi}=-\gamma\alpha_1\frac{1}{(-1+\bar{\tau})(-1+\alpha_2)}\frac{\partial\tau}{\partial a}+\frac{\alpha_1(1-\rho)}{(1-\alpha_2)}(1+\Pi)^{-1}\gamma>0.$$

■

As the depreciation rate δ increases, capital becomes scarce compared to the energy resources. This scarcity will result in an increase in the price of capital relative to the prices of energy resources. Thus, the agents will demand more resource assets than capital for their savings. There will be less resource for production which will in turn yield a lower depletion rate. As a result, the economy grows at a higher rate along the balanced growth path.

The higher the discount factor β – i.e., the more patient the generations are– when young households will consume less and save more. Since agents save more, the depletion rate along the balanced growth path will decrease and therefore the economy will grow at a higher rate along the balanced growth path.

As Smulders (2005) emphasizes, the effect of the exogenous resource saving technological progress on the rate of depletion is accepted to be ambiguous due to the opposing income and substitution effects. Under the productivity gains, households would attach a greater value to resources in future periods since these resources will be more productive. Thus, households save more on these resources which demonstrated the substitution channel. On the other hand, when the productivity increases, more output would be obtained given a resource stock. As a result, the households would expect to have more income in the future. The income effect works through consumption smoothing and households will consume more. In the proposition stated above, it is shown that as long as the depletion rate along the balanced growth path is slightly higher than the lower bound of the depletion rate, the higher the resource saving technological progress, the economy will deplete its corresponding energy resource less and have higher growth rate along the balanced growth path. Thus, the higher the resource saving technological progress through the income effect, the economy will deplete its energy resources less.

Due to the regeneration property of renewable resources, economies with higher share of renewables in energy production (lower ρ) have lower depletion rates as long as the depletion rate at the balanced growth path is slightly higher than the existence lower bound. Comparing identical economies with one having a higher share of renewables reveals that the economy which is using renewables more intensively in energy production will deplete its resources less and this will induce higher growth.

Economies with higher regeneration rate have more renewable resources and the constraint on growth due to limited energy sources become less binding. Thus, through a lower depletion rate economies grow faster at the balanced growth path. Similar to the cases discussed above, the depletion rate at the balanced growth path must be slightly higher than the lower bound of it for this result to hold.

The effect of an increase in the population growth rate (higher n) on the depletion rate is found to be ambiguous. This unambiquity creates further unambiquity for the effect of an increase in population growth on the long run growth. As the population is higher, there is a need for higher consumption. This will result in higher depletion rates which indicates a positive relationship. At the same time, through an increase in population growth rate, there will be an increase in the amount of labor utilized in production. As labor substitutes the resources utilized in production, less resources will be exhausted. As a result, the depletion rate will decrease which reflects a negative relationship.

Corollary 8 *(i) For the more patient economies (higher β), the increase in resource saving technological progress has a higher growth effect.*

(ii) If increasing the share of renewable resources in the energy production yields a higher growth rate in the economy, that effect will be even higher for more patient economies.

From the comparative statics analysis, it can be observed that if an economy is more patient the effect of increasing the resource saving technological progress is higher on growth. To begin with, it is seen that under technological progress the economies deplete their resources less. If the agents in the economy are saving more when they are young then the economies' resources will be depleted even lesser on the balanced growth path. Thus, the effect of developments in the technology will benefit growth more.

For more patient economies, an increase in the share of renewables in the energy production induces a higher growth rate. This result can be interpreted in the following manner. For economies whose renewable energy resources is highly developed with high regeneration and technological progress rates, increasing the share of renewables will induce higher growth if the economy is more patient. If developed economies can be considered as economies with higher saving rates and with fastly growing renewable energy production technologies, one could conclude that it is optimal for that economies to support policies increasing the share of renewables compared to the less patient developing economies.

2.6. Conclusion

In this chapter, the feasibility and the determinants of the long run growth within a two-period overlapping generations model in which the energy is an essential input and technical change is resource augmenting is studied.

This study can be considered as a contribution to the resource economic models literature as the recent literature has mostly focused on just one type of resources or utilizes infinitely lived agents framework. In parallel with the OLG literature (Galor and Ryder, 1989; Agnani et al., 2005) the model necessitates sufficiently high labor share for the economy to exhibit positive growth. However, compared with previous models the condition required is less binding. In fact, the share of renewables in energy production, the technological progress in producing energy and the regeneration factor are found among the key variables affecting the required labor share and hence the possibility of long run growth.

This study is able to offer a simpler analytical setting (i) to investigate the effects of an increase in productivity of resources in producing energy on the depletion rate and (ii) to observe under which circumstances increasing/decreasing the resource intensity of energy production will bring about growth. Indeed, it is shown that the effects of an increase in the intensity of the renewable resources in producing energy promote long run growth. After analyzing the sustainability of the economy under the presence of both resources and it is observed that the sustainability depends on the energy saving effect of the technological progress and the depletion rate of the resources which in turn depend on the rest of the parameters in the economy. Finally, the balanced growth path is optimal in this setup.

CHAPTER 3

THE USE OF NATURAL RESOURCES IN CAPITAL ACCUMULATION IN AN OVERLAPPING GENERATIONS RESOURCE ECONOMY

How does the scarcity of natural resources limit growth and to what extent capital accumulation offsets this constraint on growth? These research questions have been the subject of several scholarly papers in the literature of resource economics dating as back as to Dasgupta and Heal (1974, 1979), Solow (1974), and Stiglitz (1974). In these studies, resources are assumed to be extracted to acquire energy to be used in the production of the final good. By this way, the limit on economic growth is imposed directly by the use of the scarce resources in production. Greater physical capital accumulation is suggested (unless non-renewable resources are substituted with renewables) to overcome this constraint. However, the overwhelming majority of the literature, assuming the same technology for the consumption and capital accumulation sector, tends to contradict with the evidence on energy intensities of these sectors. The data suggests that physical capital production is relatively more energy-intensive than consumption, so that the non-renewable resources can limit growth through the equipment production sector.

Differing energy intensities has only been considered in Barahona (2011) –in an infinitely lived agent general equilibrium model-, but not yet within the overlapping generations (OLG) framework. Against this background, the aim of this chapter is to analyze the effects of differentiating energy intensities of the physical capital and the final good production in a overlapping generations resource model. OLG framework offers a better explanatory power for the discussion of resource problems due to three main reasons. First of all, besides being an input to energy production, resources are store of values between generations (see Koskela et al., 2002; Valente, 2008; Birgit Bednar–Friedl and Farmer; 2011) and are not held by one representative generation forever as infinitely lived representative agent framework assumes. Secondly, current decisions on resource extraction taken by short-lived and selfish individuals have consequences not only on current but on future generations as well. Thus, both intratemporal and intertemporal effects should be considered. Solow (1974), Padilla (2002), Agnani (2005) note that these intergenerational aspects should be taken into account when analyzing environmental issues and/or natural resource economies. Finally, contrary to what the infinitely-lived representative agent models claim, there exists some empirical evidence that agents are not perfectly altruistically linked (Altonji et al., 1992; Balestra, 2003).

As modeled in Barahona (2011), the physical capital accumulation sector is assumed to be more energy intensive than consumption where energy is obtained from the extraction of resources. Thus, going beyond the standard literature, in this study the accumulation of the capital stock is assumed to be determined not only by the savings but also by the energy that it requires. In addition, instead of taking resources to be the only way of saving (Krautkraemer and Batina, 1999; Koskela et al., 2002), the resource stock and man-made capital are considered to be alternative assets (Mourmouras, 1991; Farmer, 2000; Agnani et al., 2005; Birgit Bednar–Friedl and Farmer, 2011).

The data supports the claim that capital accumulation sector is more energy intensive than consumption sector in the following manner. Azomahou et al. (2004, 2006) build an energy intensity measurement (ratio between energy consumption and value added) of 14 sectors of the economy from the Structural Analysis Database of OECD and the Energy Balances and Energy Prices and Taxes of IEA. The evidence shows that energy intensity is higher for sectors closely related to physical capital accumulation (ex: iron and steel sector (0.809), transport and storage (0.85), non-ferrous metals (0.599), and non-metallic minerals (0.507). While, the energy intensity is lower for the consumption goods related sectors (ex: food and tobacco (0.134), textile and leather (0.082), and construction (0.018)).

This study addresses how do the standard results on stability of the dynamics, growth and optimality are modified under differing energy intensities. With respect to the assumptions on technological progress (exogenous or null), the results are compared with the two strands of literature. Under technological progress, the results can be compared with Agnani et. al. (2005) which examines the possibility of positive growth in an OLG economy that use non-renewable resources in production of the consumption/investment good. The results on balanced growth path are quite similar to Agnani et al. (2005). In line with that study, although technological progress' is taken to be exogenous, the growth rate of the economy depends on all of the parameters of the model, as if the model is an endogenous growth model. In addition, the balanced growth path of the economy can be optimal depending on the choice of the depletion rate. However, contrary to the Agnani et. al (2005)'s findings, multiple balanced growth paths are encountered rather than a unique balanced growth path. Going beyond, the possible growth rates can be explicitly solved. Yet, the effect of the share of energy resources in capital accumulation on the growth rate of the economy is not analytically clear and depends on the characteristics of that economy.

Under no technological progress, the results can be compared with the resource literature that studies the long run dynamics. After examining the steady state and the stability properties of the model, the dynamic behavior is fully characterized with respect to the share of energy resources in the production of capital accumulation. The main contribution of this study comes from this characterization described below. As widely accepted in the literature, the models converge to a single steady state or a single balanced growth path with saddle path dynamics under linear regeneration of resources and with exogenous or no technological progress. For instance, Mourmouras (1991) considers interactions between capital accumulation and natural resource exploitation under linear regeneration of resources. Apart from how capital accumulates regarding model is similar to ours while equilibrium dynamics are saddle-path stable (Farmer & Friedl, 2010). In contrast, in this study, local indeterminacy and hopf bifurcations arise in the model for the non-renewable resources if the share of energy resources is low enough (less than 25 %). Multiple steady states, indeterminacy and bifurcations are obtained, without taking non-linearizing (logistic regeneration) assumptions evident in the literature[1].

The chapter proceeds as follows. The model is presented in Section 1. The competitive equilibrium is defined in Section 2. Section 3 characterizes the balanced growth path and Section 4 discusses the optimality of the competitive equilibrium. Section 5 presents the equilibrium dynamics and Section 6 examines the stability of the long run dynamics. Conclusions and broader theoretical implications are discussed in Section 7.

3.1. The Model

A perfect foresight overlapping generations economy without population growth in discrete time with infinite horizon is taken. Apart from the standard OLG framework[2], the model differentiates the energy intensities of the physical capital and the final good production. There are three sectors in the economy: the final good production, the equipment (investment) good production and the extraction sector. A single final good –which can be either consumed or invested– is produced in the economy using physical capital and labor. The physical capital that is used in the production process is produced by the investment sector. The physical capital is

[1] Even if logistic regeneration is taken into account, the long run dynamics exhibit saddle path stability (see Farmer, 2000; Koskela et. al., 2002). Koskela et al. (2008) examines whether renewable resource based OLG economies may have other types of dynamics than saddles or not. It has numerically shown that flip bifurcations may arise if the intertemporal elasticity of substitution of the utility function is less than one half and the regeneration function is logistic.

[2] For the discussion of standart OLG models see De La Croix & Michel (2004).

obtained by means of the installed capital stock from the previous period and the energy resource extracted from the resource stock. Finally, the extraction sector extracts the energy from the natural resource.

All agents have rational expectations and each generation consists of a single representative agent. Moreover, all agents in this economy are price-takers and all the markets are competitive.

The natural resource in the model has serves two purposes. It is both a store of value as an asset and an input in the production of investment good as energy input. Following a standard assumption, the initially old generation possesses the stock of the natural resource. At the beginning of each period t, the old agents (generation $t-1$) own the resource stock $E_t = e_t.N_t$. They choose how much of their stock to extract as energy and sell to the equipment good sector X_t. The remaining part of the natural resource is sold young agents as resource assets, $A_t(= E_t - X_t)$. From period t to $t+1$, the resource stock regenerates at a linear rate Π (Mourmouras, 1991), where $\Pi \geqslant 1$[3]. The transition dynamics of the energy resources in per capita terms can be formalized as follows:

$$e_{t+1} = \Pi(e_t - x_t) = \Pi a_t, \tag{1}$$

where $x_t = \frac{X_t}{N_t}$ and $a_t = \frac{A_t}{N_t}$.

The representative individual receives an income equal to the real wage w_t from supplying his one unit of labor to the firms when young. He allocates his income between the current consumption c_t, the savings of physical capital s_t, the purchase of the ownership rights for the renewable resources a_t. In his last period of life (when old at period $t+1$), the agent is retired and he consumes d_{t+1} out of his entire income and do not leave bequests. His income is generated from the return on his savings made when young: $R_{t+1}s_t$, from extracting the demanded portion of the energy resources and selling it to the firms $Q_{t+1}x_{t+1}$ and selling the rest to the young $P_{t+1}a_{t+1}$ from the prices Q_t and P_t respectively. Accordingly, the budget

[3]Note that as long as $\Pi = 1$ the resource is non-renewable.

constraints facing generation t is as follows:

$$c_t + s_t + P_t a_t = w_t \ , \qquad (2)$$

$$d_{t+1} = R_{t+1} s_t + Q_{t+1} x_{t+1} + P_{t+1} a_{t+1}. \qquad (3)$$

Generations derive utility over consumption where their two–period intertemporal utility function is dependent on the level of consumption when young c_t and when old d_{t+1}. A logarithmic instantaneous utility function u is taken in order to guarantee the existence of the balanced growth path and its qualitative properties[4].

Taking the prices of the energy resource and wages as given, the representative agent born at time t maximizes his utility by choosing the young and the old periods' consumption and the ownership of the energy resource. The optimization problem of the representative consumer born at time t can be formalized as follows:

$$\max_{\{c_t, d_{t+1}, s_t, e_{t+1}\}} \ln c_t + \beta \ln d_{t+1}$$

where $\beta \in (0,1)$ is the subjective discount factor.

subject to

$$\begin{aligned} c_t + s_t + P_t a_t &= w_t \ , \\ d_{t+1} &= R_{t+1} s_t + Q_{t+1} x_{t+1} + P_{t+1} a_{t+1}, \\ e_{t+1} &= \Pi(e_t - x_t) = a_t \\ c_t &\geq 0, d_{t+1} \geq 0, e_{t+1} \geq 0, E_0 > 0 \quad \text{given.} \end{aligned}$$

The first-order conditions are:

[4]See, among others, King and Rebelo (1993) and Agnani et al. (2005), for the need to assume comsumer's preferences with CIES in order to have the existence of a BGP.

$$\frac{d_{t+1}}{c_t} = \beta R_{t+1}, \quad (4)$$

$$\frac{P_{t+1}}{P_t} = \frac{R_{t+1}}{\Pi}, \quad (5)$$

$$P_{t+1} = Q_{t+1}. \quad (6)$$

Equation (4) gives the equalization of discounted marginal utilities where the marginal rate of substitution between the current and the future consumption is equal to their relative prices. Equation (5) is the no-arbitrage condition among different types of savings. Equation (6) is a no-arbitrage condition that implies that in the equilibrium the asset price and the extracted energy price of the resource will be the same.

Firms operating in the final goods sector are owned by the old households. Firms produce the final good with the Cobb-Douglas constant returns to scale technology. Equation (7) presents the production function in the final good sector at any date t. The exogenous disembodied total factor productivity is represented by A_t (Equation 8). Under this perfectly competitive environment, at each period t, taking the prices of inputs, the initial technology level and the initial level of capital stock as given, the representative firm maximizes its profit by choosing the amount of labor and physical capital inputs.

$$\max_{\{K_t,N_t\}_{t=0}^{\infty}} \pi_t = Y_t - P_t^K K_t - w_t N_t,$$

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \qquad 0 < \alpha < 1, \quad (7)$$

$$A_{t+1} = (1+a)A_t \qquad a \geq 0. \quad (8)$$

At an interior solution of the firm's optimization problem,where all variables are expressed in per capita ($k_t = \frac{K_t}{N_t}$) terms, the following first order conditions are satisfied equating the price of the inputs to their marginal benefits:

$$\alpha A_t k_t^{\alpha-1} = P_t^K, \tag{9}$$

$$(1-\alpha) A_t k_t^{\alpha} = w_t. \tag{10}$$

Equation (11) summarizes the goods market clearing condition of the economy. The final good is either consumed by young agents C_t, or by old agents (generation $t-1$) D_t, or invested for the production of the future capital stock, S_t.

$$Y_t = C_t + D_t + S_t \tag{11}$$

In the standard OLG literature, the new capital stock at time $t+1$ is fully determined by the savings made at time t which are equal to the investments. However, since the physical capital production is relatively more energy-intensive than consumption, following Barahona (2011), we model the accumulation of the capital stock to be determined not only by the savings made at time t, but also by the energy that it requires. So, the resources put a constraint on the growth through the capital accumulation sector. Thus, in Equation (12) the new capital at $t+1$, K_{t+1} is produced from the natural energy resources X_{t+1} and investment made at time t, I_t with the following Cobb-Douglas technology:

$$K_{t+1} = B_{t+1}^{\theta} X_{t+1}^{\theta} I_t^{1-\theta}, \tag{12}$$

$$S_t = I_t, \tag{13}$$

$$B_{i+1} = (1+b) B_t, \; b \geq 0. \tag{14}$$

It is worthwhile to mention that the savings are still equal to the investments (Equation (13)) but only a fraction of the investments can generate the new capital stock. B_t is the technological progress in the equipment good sector. In contrast to A_t, B_t is energy saving and specific to the accumulation of the capital goods. If B_t increases the productivity of the renewable resources in producing new capital increases, less amount of resources will be needed to produce the same amount of new capital. Therefore, the technical progress (increase in productivity) which is captured by b is considered to be energy saving. On the other hand, changes in B_t represent investment-specific technological change, which is assumed to affect equipment sector only.

Under Cobb- Douglas specification substitutability between energy and investment is assumed. Following the argument of Dasgupta (1979), in the model I_t is interpreted as final good service. Thereby, as Barahona (2011) indicates the provision of a flow of final good I_t, implies the provision of a certain energy flow.

In the equipment good production sector at each period t, the representative firm maximizes its profit by choosing the amount of non-renewable resource input that will be utilized in the production process:

$$\begin{aligned} \max_{\{X_t\}_{t=0}^{\infty}} \pi_t &= P_t^K K_t - Q_t X_t, \\ s.t \quad K_t &= B_t^\theta X_t^\theta I_{t-1}^{1-\theta}. \end{aligned}$$

taking the prices of capital and resource input and the initial level of capital stock as given. At an interior solution of the equipment firm's optimization problem the following first order condition is satisfied:

$$Q_t = \theta \alpha Y_t X_t^{-1}. \tag{15}$$

The profit on investing on capital $R_t S_{t-1}$ at time t should be equal to the profit on producing new capital $(1-\theta) P_t^K K_t$[5] to prevent arbitrage opportunities. The below equation are follows from this no arbitrage condition:

[5]By choosing the optimal non-renewable resource stock (Equation (15)) maximum profit that the representative firm can obtain is as follows:

$$\pi_t^* = P_t^K K_t - \theta P_t^K B_t^\theta X_t^{\theta-1} I_{t-1}^{1-\theta} X_t = (1-\theta) P_t^K K_t$$

$$R_t = (1-\theta) P_t^K B_t^{\frac{\theta}{1-\theta}} X_t^{\frac{\theta}{1-\theta}} K_t^{\frac{\theta}{\theta-1}}. \tag{16}$$

3.2. The Competitive Equilibrium

A dynamic competitive equilibrium for this OLG resource economy is determined by the sequence of prices $\{w_t, R_t, P_t^K, P_t, Q_t\}_{t=0}^{\infty}$ and feasible allocations $\{c_t, d_t, s_t, e_t, a_t, x_t, i_t, y_t, k_{t+1}$ given the positive initial values for $S_{-1}, E_0, A_0, B_0, N_0 > 0$ and the law of motion of exogenous technological progresses A_t and B_t such that the consumers maximize their life-time utility, firms maximize their profits and all markets clear at every period t. The competitive equilibrium of this OLG resource economy is a solution of the non-linear system of equations, (1)–(17) with the following market clearing conditions:

$$\begin{aligned} I_t &= S_{t,} \\ N_t &= N_0, \\ k_{t+1} &= B_{t+1}^{\theta} x_{t+1}^{\theta} s_t^{1-\theta}, \\ y_t &= c_t + d_t + s_t, \\ e_{t+1} &= \Pi(e_t - x_t). \end{aligned} \tag{17}$$

The economy at time t can be summarized in the Figure 1.

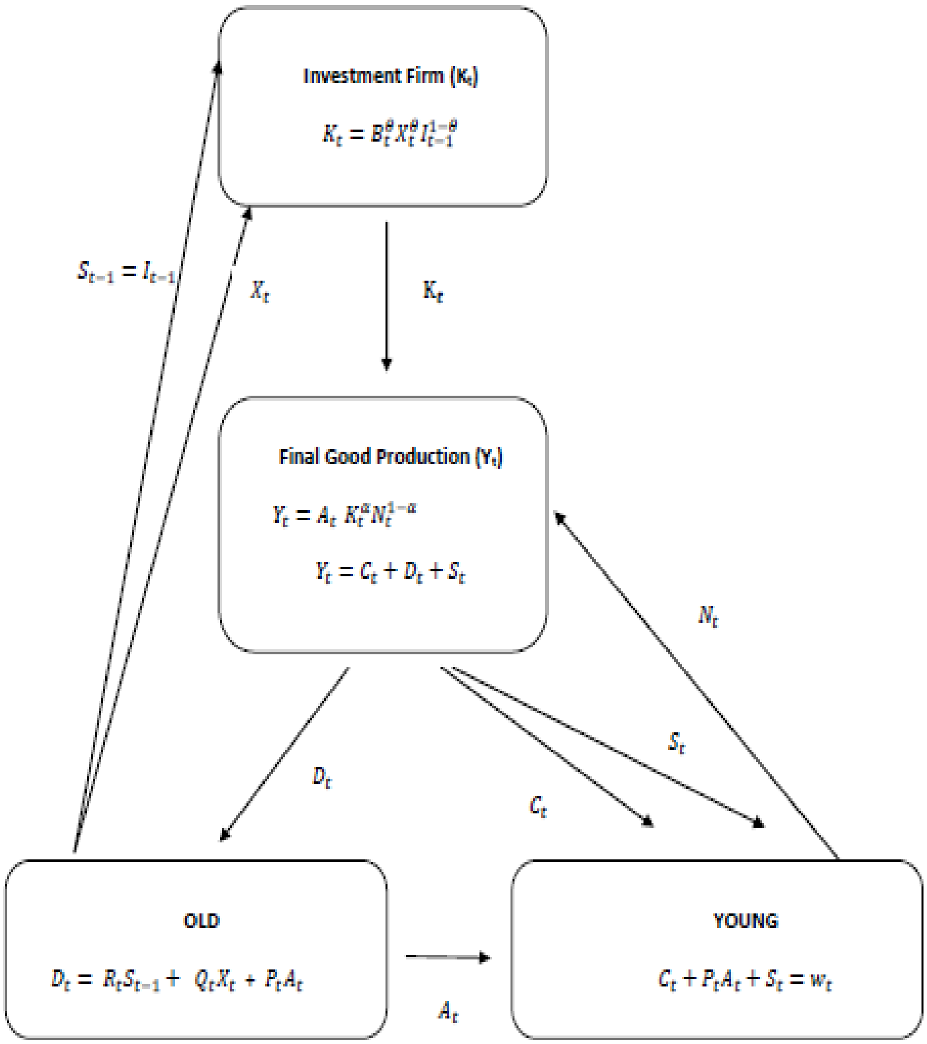


Figure 1: The Economy at Time t.

3.3. The Balanced Growth Path:

Firstly, suppose there is exogenous technological progress in the production and the equipment goods sectors $a, b > 0$. Let us characterize the BGP as the situation where all the endogenous variables grow at a constant rate. The growth factor of any variable g_t will be by γ_g which is the ratio g_{t+1}/g_t. Along the balanced growth path $\gamma_g - 1$ will represent the growth rate of the corresponding variable.

Proposition 1 *Suppose there is exogenous technological progress in the production and the equipment goods sectors $a, b > 0$. Along a balanced growth path of this economy all variables grow at a constant rate. If capital grows at a rate γ,then all the other variables grows at the following rates:*

(i)

$$\begin{aligned}
\gamma_B &= (1+b), \qquad \gamma_A = (1+a) \\
\gamma_k &= \gamma, \\
\gamma_y &= \gamma_c = \gamma_d = \gamma_w = (1+a)\gamma^{\alpha}, \\
\gamma_s &= \gamma_x^{\frac{\theta}{\theta-1}}(1+b)^{\frac{\theta}{\theta-1}}\gamma^{\frac{1}{1-\theta}}, \\
\gamma_A &= (1+a), \quad \gamma_B = (1+b), \ \gamma_N = 1, \\
\gamma_e &= \gamma_x, \\
\gamma_{p^K} &= (1+a)\gamma^{\alpha-1}, \\
\gamma_p &= \frac{\gamma^{\alpha}(1+a)}{\gamma_x}, \\
\gamma_R &= 1 \\
\gamma_x &= \gamma^{\frac{\alpha\theta-\alpha+1}{\theta}}(1+a)^{\frac{\theta-1}{\theta}}(1+b)^{-1} \\
\gamma_x &= \left[\frac{\beta(1-\alpha)}{(1+\beta)} + \theta\alpha - \frac{\Pi\theta\alpha}{(\Pi+\gamma_x)}\right]\frac{\Pi}{\alpha(1-\theta)}.
\end{aligned}$$

(ii) Balanced growth path of this economy exists. There can be multiple growth paths with the following growth rates:

$$\gamma = \gamma_x^{\frac{\theta}{\alpha\theta-\alpha+1}}(1+a)^{\frac{\theta-1}{\alpha\theta-\alpha+1}}(1+b)^{-\frac{\theta}{\alpha\theta-\alpha+1}}$$

$$\gamma_{x_{1,2}} = (2\alpha(1-\theta))^{-1} *$$
$$\left\{ \begin{array}{c} \Pi\left(\alpha(-1+2\theta-\frac{\beta}{(1+\beta)})+\frac{\beta}{(1+\beta)}\right) \pm \\ \sqrt{\Pi^2\left(\alpha(-1+2\theta-\frac{\beta}{(1+\beta)})+\frac{\beta}{(1+\beta)}\right)^2 - 4\Pi^2(1-\alpha)\alpha(1-\theta)\frac{\beta}{(1+\beta)}} \end{array} \right\}$$

Proof. $\gamma_A = (1+a)$ and $\gamma_B = (1+b)$ follows from the definition of the technological progress. $\gamma_n = 1$ since there is no population growth.

$\gamma_e = \gamma_x$ is obtained by the ratio of Equation (1) in period $t+1$and t. After evaluating the resulting equation on the balanced growth the equality of $\bar{\tau} = \tau_t = \tau_{t+1}$ is observed and then $\gamma_x = \Pi(1-\bar{\tau})$ follows.

$\gamma_{P^K} = \gamma_A \gamma_k^{\alpha-1}$ comes from the evaluation of Equation(9) at t and t+1.

Using Equations (**??**), (6), (9) and (12) ,

$$P_t = \theta\alpha A_t k_t^{\alpha} x_t^{-1} \tag{18}$$

To find the growth factor of capital per capita substituting Equation (9) into the Equation (5) and evaluating along the balanced growth path

$$R_{t+1} = \gamma_{P^K} \gamma_k \gamma_x^{-1} \tag{19}$$

Then, using Equation (16) ,

$$\gamma_{P^K} \gamma_k \gamma_x^{-1} = (1-\theta) P_t^K B_t^{\frac{\theta}{\theta-1}} x_t^{\frac{\theta}{\theta-1}} k_t^{\frac{\theta}{\theta-1}}$$

By evaluating this expression at $t+1$ and t and taking the ratio ,

$$1 = \gamma_{P^K} \gamma_B^{\frac{\theta}{1-\theta}} \gamma_x^{\frac{\theta}{1-\theta}} \gamma_k^{\frac{\theta}{\theta-1}}$$

Since $\gamma_{P^K} = \gamma_A \gamma_k^{\alpha-1}$ the growth factor of capital per capita is

$$
\begin{aligned}
1 &= \gamma_A \gamma_B^{\frac{\theta}{1-\theta}} \gamma_x^{\frac{\theta}{1-\theta}} \gamma_k^{\frac{\alpha\theta-\alpha+1}{\theta-1}} \qquad (20)\\
\gamma &\equiv \gamma_k = \gamma_A^{\frac{1-\theta}{\alpha\theta-\alpha+1}} \gamma_B^{\frac{\theta}{\alpha\theta-\alpha+1}} \gamma_x^{\frac{\theta}{\alpha\theta-\alpha+1}} = (1+a)^{\frac{1-\theta}{\alpha\theta-\alpha+1}} (1+b)^{\frac{\theta}{\alpha\theta-\alpha+1}} (1-\bar{\tau})^{\frac{\theta}{\alpha\theta-\alpha+1}}
\end{aligned}
$$

Taking the ratio of Equation(19), evaluating along the balanced growth path and then substituting $\gamma_{P^K} = \gamma_A \gamma_k^{\alpha-1}$ and Equation (20) , $\gamma_R = 1$.

I tis observed that $\gamma_y = \gamma_A \gamma_k^{\alpha}$ from taking the ratio of Equation (**??**) in period $t+1$ and t. Similarly, $\gamma_w = \gamma_A \gamma_k^{\alpha}$ can be shown by taking the ratio of Equation (10) in period $t+1$ and t .

$\gamma_k = \gamma_B^{\theta} \gamma_s^{1-\theta} \gamma_x^{\theta}$ is obtained through taking the ratio of Equation (12) in period $t+1$ and t.

For the growth factor of price of the non-renewable resources taking the ratio of Equation(18) in period $t+1$ and t, evaluating it on the balanced growth path. Thus , $\gamma_P = \frac{\gamma_A \gamma_k \alpha}{\gamma_x} = \frac{(1+a)\gamma^{\alpha}}{1-\bar{\tau}}$.

Observe that $\gamma_d = \gamma_c$ from taking the ratio of Equation(4) in period $t+1$ and t and evaluating it on the balanced growth path. Moreover to show $\gamma_d = \gamma_c = \gamma_w$. first, substitute Equation (2), Equation (4) and Equation (10) into Equation (3) and obtain

$$
c_t = \frac{w_t}{(1+\beta)} \qquad (21)
$$

Evaluating the above equation along the balanced growth path yields the result.

As a final step the growth factor of capital is found as follows. Substituting Equations (**??**), (21), (10) and (15) into the Equation (12);

$$
k_{t+1}^{\frac{1}{1-\theta}} = A_t k_t^{\alpha} x_t^{\frac{\theta}{1-\theta}} B_t^{\frac{\theta}{1-\theta}} \left[\frac{(1-\alpha)\beta}{(1+\beta)} - \alpha\theta \frac{e_{t+1}}{\Pi x_t} \right]
$$

and dividing both sides by k_t:

$$
\gamma_k^{\frac{1}{1-\theta}} \gamma_B^{\frac{\theta}{\theta-1}} \gamma_x^{\frac{\theta}{\theta-1}} = A_t k_t^{\alpha-1+\frac{\theta}{\theta-1}} B_t^{\frac{\theta}{1-\theta}} x_t^{\frac{\theta}{1-\theta}} \left[\frac{(1-\alpha)\beta}{(1+\beta)} - \alpha\theta \frac{e_{t+1}}{\Pi x_t} \right] \qquad (22)
$$

From Equations (15), (9) and (19) ,

$$\frac{[\gamma_A \gamma_x^{-1} \gamma_k^{\alpha} - (1-\delta)]}{\alpha(1-\theta)} = \mathrm{A}_{t+1} \mathrm{k}_{t+1}^{\alpha-1+\frac{\theta}{\theta-1}} \mathrm{B}_{t+1}^{\frac{\theta}{1-\theta}} \mathrm{x}_{t+1}^{\frac{\theta}{1-\theta}}$$

Substituting into Equation (22) ,:

$$\gamma_k^{\frac{1}{1-\theta}} \gamma_x^{\frac{\theta}{\theta-1}} \gamma_B^{\frac{\theta}{\theta-1}} = \left[\gamma_A \gamma_x^{-1} \gamma_k^{\alpha} - (1-\delta)\right] \frac{1}{\alpha(1-\theta)} \left[\frac{(1-\alpha)\beta}{(1+\beta)} - \alpha\theta \frac{e_{t+1}}{x_t}\right]$$

Assuming full depreciation ,:

$$\gamma_k^{\frac{1}{1-\theta}} \gamma_x^{\frac{\theta}{\theta-1}} \gamma_B^{\frac{\theta}{\theta-1}} = \left[\gamma_A \gamma_x^{-1} \gamma_k^{\alpha}\right] \frac{1}{\alpha(1-\theta)} \left[\frac{(1-\alpha)\beta}{(1+\beta)} - \alpha\theta \frac{\Pi e_t}{x_t} + \Pi\alpha\theta\right]$$

After some algebra ,:

$$\gamma_x = \left[\frac{\beta(1-\alpha)}{(1+\beta)} + \theta\alpha - \frac{\Pi\theta\alpha}{(\Pi+\gamma_x)}\right] \frac{\Pi}{\alpha(1-\theta)}$$

Therefore, the BGP of this economy exists. There can be multiple growth paths with the following growth rates:

$$\begin{aligned}
\gamma &= \gamma_x^{\frac{\theta}{\alpha\theta-\alpha+1}} (1+a)^{\frac{\theta-1}{\alpha\theta-\alpha+1}} (1+b)^{-\frac{\theta}{\alpha\theta-\alpha+1}} \\
\gamma_{x_{1,2}} &= \frac{\Pi\Omega \pm \sqrt{\Pi^2\Omega^2 - 4\Pi^2(1-\alpha)\alpha(1-\theta)\frac{\beta}{(1+\beta)}}}{2\alpha(1-\theta)} \\
\Omega &= \alpha(-1+2\theta - \frac{\beta}{(1+\beta)}) + \frac{\beta}{(1+\beta)}
\end{aligned}$$

■

In parallel with the literature, Proposition (1) shows that income, consumption and wages grow at the same rate and the interest rate is constant along the balanced growth path. Indeed, the results on balanced growth path are quite similar to Agnani et al. (2005) where the constraint on growth is evolving due to the presence of exhaustible resources in production. In fact, in line with their study, although technological progress' is taken to be exogenous, the growth rate of the economy depends on all of the parameters of the model, actually a feature of endogenous growth models. However, as the resource stock is used in the production of capital, the growth rate of the economy is not equal to the growth rate the capital stock. Also, in contrast with the previous studies the savings do not grow at the same rate with capital. The growth rate of the savings along the BGP is depending on the relationship between the improvement of the investment-specific technology, the growth rate of capital and the extraction rate of the resources. This result indicates that investment specific technological progress along with the savings offset the limits that the non-renewable resource stock impose on the growth of the capital stock.

Contrary to the Agnani et. al (2005)'s findings instead of a unique balanced growth path, multiple balanced growth paths are encountered. Going beyond, the possible growth rates can explicitly be solved. Yet, the effect of the share of energy resources in capital accumulation on the growth rate of the economy is not analytically on one direction, it depends on the characteristics of that economy.

3.4. Optimality

To derive the conditions for intergenerational optimality, the existence of a social planner whose maximizes a discounted sum of utilities for all current and future generations with respect to the resource constraints of the economy is taken. To be restrictive the case of non-renewable resources is discussed, $\Pi = 1$. The optimal balanced growth path is characterized by:

(a) Capital growing at the rate $\gamma_k = \gamma$,

(b) Income and consumption growing at the same rate so that $\gamma_y = \gamma_c = \gamma_d = (1+a)\gamma_k^{\alpha}$,

(b) Energy resources used in production will decline over time indicating an asymptotic depletion: $\gamma_e = \gamma_x = \frac{1}{1+\rho}$, where ρ denotes the subjective discount factor of the social planner.

(c) In addition,

$$\gamma_k = (1+a)^{\frac{1-\theta}{\alpha\theta-\alpha+1}} (1+b)^{\frac{\theta}{\alpha\theta-\alpha+1}} \gamma_x^{\frac{\theta}{\alpha\theta-\alpha+1}} \ .$$

Proposition below discusses the optimality of the competitive equilibrium.

Proposition 2 *The competitive balanced growth path is pareto optimal as long as* $\bar{\tau} = \frac{\rho}{1+\rho}$.

Proof. A social planner solves the following problem:

$$\max_{\{c_t,d_t,x_t,z_t,k_{t+1}\}_{t=0}^{\infty}} \beta \ln \mathrm{d}_0 + \sum_{t=0}^{\infty} (\frac{1}{(1+\rho)})^{t+1} \left[\ln c_t + (1+\rho)^{-1}\beta \ln d_t\right]$$

subject to the aggregate resource constraints of the economy

$$y_t = c_t + d_t + s_t - (1-\delta)k_t \tag{23a}$$

$$y_t = A_t k_t^{\alpha} \tag{23b}$$

$$e_t = e_{t+1} + x_t \tag{23c}$$

$$e_0 \geqslant \sum_{t=0}^{\infty} x_t \tag{23d}$$

$$k_{t+1} = B_t^{\theta} x_t^{\theta} s_t^{\theta} \tag{23e}$$

$$B_{t+1} = (1+b)B_t \tag{23f}$$

$$c_t, d_t > 0 \tag{23g}$$

$$s_{-1}, B_0, e_0 \text{ given.} \tag{23h}$$

The first order conditions with respect to c_t and d_t yield

$$\frac{d_t}{c_t} = \beta(1+\rho)^{-1} \tag{24}$$

, then using Equation (23a) and (23b) ,

$$\frac{1}{\left[A_t k_t^{\alpha} - d_t - k_{t+1}^{\frac{1}{(1-\theta)}} (e_{t+1} - e_{t+2})^{\frac{-\theta}{1-\theta}} B_{t+1}^{\frac{-\theta}{1-\theta}} - (1-\delta) k_t\right]} = (1+\rho)^{-1} \beta d_t^{-1}. \quad (25)$$

From the first order conditions with respect to k_{t+1} ,

$$\frac{1}{\left[A_t k_t^{\alpha} - d_t - k_{t+1}^{\frac{1}{(1-\theta)}} (e_{t+1} - e_{t+2})^{\frac{-\theta}{1-\theta}} B_{t+1}^{\frac{-\theta}{1-\theta}} - (1-\delta) k_t\right]} = \frac{\left[k_{t+1}^{\frac{-\theta}{(1-\theta)}} (e_{t+1} - e_{t+2})^{\frac{\theta}{1-\theta}} B_{t+1}^{\frac{\theta}{1-\theta}} \left[\alpha A_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)\right] (1+\rho)^{-1}\right]}{\left[A_{t+1} k_{t+1}^{\alpha} - d_{t+1} - k_{t+2}^{\frac{1}{(1-\theta)}} (e_{t+2} - e_{t+3})^{\frac{-\theta}{1-\theta}} B_{t+2}^{\frac{-\theta}{1-\theta}} - (1-\delta) k_{t+1}\right]}.$$

From the first order conditions with respect to e_{t+1} ,

$$\frac{k_{t+1}^{\frac{1}{(1-\theta)}} (e_{t+1} - e_{t+2})^{\frac{-\theta}{1-\theta}-1} B_{t+1}^{\frac{-\theta}{1-\theta}}}{\left[A_t k_t^{\alpha} - d_t - k_{t+1}^{\frac{1}{(1-\theta)}} (e_{t+1} - e_{t+2})^{\frac{-\theta}{1-\theta}} B_{t+1}^{\frac{-\theta}{1-\theta}} - (1-\delta) k_t\right]} = \frac{k_{t+2}^{\frac{1}{(1-\theta)}} (e_{t+2} - e_{t+3})^{\frac{-\theta}{1-\theta}-1} B_{t+2}^{\frac{-\theta}{1-\theta}} (1+\rho)^{-1}}{\left[A_{t+1} k_{t+1}^{\alpha} - d_{t+1} - k_{t+2}^{\frac{1}{(1-\theta)}} (e_{t+2} - e_{t+3})^{\frac{-\theta}{1-\theta}} B_{t+2}^{\frac{-\theta}{1-\theta}} - (1-\delta) k_{t+1}\right]}, \quad (26)$$

After some algebra from first order conditions and taking $\delta = 1$,

$$(1-\theta)\alpha \mathrm{A}_{t+1} \mathrm{k}_{t+1}^{\alpha-1-\frac{\theta}{1-\theta}} \mathrm{x}_{t+1}^{\frac{\theta}{1-\theta}} \mathrm{B}_{t+1}^{\frac{\theta}{1-\theta}} = \frac{c_{t+1}}{c_t}(1+\rho), \quad (27)$$

$$\frac{c_{t+1}}{c_t} = \frac{1}{(1+\rho)} \left(\frac{k_{t+2}}{k_{i+1}}\right)^{\frac{1}{1-\theta}} \left(\frac{x_{t+2}}{x_{t+1}}\right)^{\frac{-\theta}{1-\theta}-1} \left(\frac{B_{t+2}}{B_{t+1}}\right)^{\frac{-\theta}{1-\theta}}, \quad (28)$$

The equality of $\gamma_A = (1+a)$, $\gamma_B = (1+b)$ follows from the definitions of the technological progress. In addition, along the balanced growth path $\gamma_\rho = 1$.

Equality of $\gamma_e = \gamma_x$ is obtained by the ratio of Equation (23c) in period $t+1$

and t. After evaluating the resulting equation on the balanced growth path it is first observed $\bar{\tau} = \tau_t = \tau_{t+1}$ and then $\gamma_x = 1 - \bar{\tau}$ follows. By means of Equation (27),

$$\gamma_c(1+\rho) = (1-\theta)\alpha A_{t+1} k_{t+1}^{\alpha-1-\frac{\theta}{1-\theta}} x_{t+1}^{\frac{\theta}{1-\theta}} B_{t+1}^{\frac{\theta}{1-\theta}}. \quad (29)$$

By evaluating this expression at $t+1$ and t and taking the ratio ,

$$\begin{aligned} 1 &= \gamma_A \gamma_B^{\frac{\theta}{1-\theta}} \gamma_x^{\frac{\theta}{1-\theta}} \gamma_k^{\frac{\alpha\theta-\alpha+1}{\theta-1}} \quad (30) \\ \gamma &\equiv \gamma_k = \gamma_A^{\frac{1-\theta}{\alpha\theta-\alpha+1}} \gamma_B^{\frac{\theta}{\alpha\theta-\alpha+1}} \gamma_x^{\frac{\theta}{\alpha\theta-\alpha+1}} = (1+a)^{\frac{1-\theta}{\alpha\theta-\alpha+1}} (1+b)^{\frac{\theta}{\alpha\theta-\alpha+1}} (1-\bar{\tau})^{\frac{\theta}{\alpha\theta-\alpha+1}} \end{aligned}$$

Observe that the growth factor of capital is equal to the output per capita, i.e., $\gamma_y = \gamma_A \gamma_k^\alpha$ from taking the ratio of Equation (23b) in period $t+1$ and t. Note that $\gamma_d = \gamma_c$ by taking the ratio of Equation (24) in period $t+1$ and t and evaluating it on the balanced growth path.

From Equation (28) ,

$$\gamma_c(1+\rho) = \gamma_B^{\frac{-\theta}{1-\theta}} \gamma_x^{\frac{-\theta}{1-\theta}-1} \gamma_k^{\frac{1}{1-\theta}} \quad (31)$$

Re-arranging and using Equation (30) ,

$$\gamma_c(1+\rho) = \gamma_A \gamma_k^\alpha \gamma_x$$

From Equations (23a), (23b) and (23e) ,

$$k_{t+1}^{\frac{1}{1-\theta}} x_{t+1}^{\frac{-\theta}{1-\theta}} B_{t+1}^{\frac{-\theta}{1-\theta}} = A_t k_t^\alpha - c_t - d_t$$

Dividing by $k_t^{\frac{1}{1-\theta}} x_t^{\frac{-\theta}{1-\theta}} B_t^{\frac{-\theta}{1-\theta}}$ and evaluating along the BGP ,

$$\gamma_c = \gamma_B^{\frac{-\theta}{1-\theta}} \gamma_x^{\frac{-\theta}{1-\theta}} \gamma_k^{\frac{1}{1-\theta}}$$

Substituting γ_c from Equation (31) , $_x = \frac{1}{(1+\rho)}$ and $= \frac{\rho}{1+\rho}$.. ■

The result on optimality is in parallel with the previous studies (see Agnani et al. (2005)).

3.5. Equilibrium Dynamics

Under no technological progress the long run dynamics around the steady state is analyzed. As Mourmouras (1991), Farmer (2000) and Bednar–Friedl and Farmer (2011) suggests, the intertemporal equilibrium dynamics can be reduced into a two–dimensional system which represents the law of motions of E_t and X_t. Using the market clearing conditions (Equations (12) and (13)) and first order conditions oh households and firms maximization problems (Equations (2),(3),(4),(5), and (10)) the below equation for k_{t+1} is derived.

$$\mathrm{k}_{t+1}^{\frac{1}{1-\theta}} = \mathrm{B}_{t+1}^{\frac{\theta}{1-\theta}} \mathrm{x}_{t+1}^{\frac{\theta}{1-\theta}} \left[\sigma(1-\alpha) + \theta\alpha(1 - \frac{e_t}{x_t}) \right] \mathrm{y}_t \tag{32}$$

In addition, plugging from the Equations (5), (6), (9) , (15) and (15) the following difference equation for k_{t+1} is obtained:

$$\frac{\Pi k_{t+1}^{\frac{1}{1-\theta}} x_t}{y_t x_{t+1}} = \alpha(1-\theta) \mathrm{B}_{t+1}^{\frac{\theta}{1-\theta}} \mathrm{x}_{t+1}^{\frac{\theta}{1-\theta}} \tag{33}$$

Substituting for $k_{t.+1}$ in the above equations yields the the equation of motion for resource harvest Equation (34). In order to study the dynamics the law of motion of the natural resource Equation (35) is re-written.

$$x_{t+1} = \Pi \left(\frac{\frac{\beta}{(1+\beta)}(1-\alpha) + \alpha\theta}{\alpha(1-\theta)} \right) x_t - \frac{\Pi\theta}{(1-\theta)} e_t \tag{34}$$

$$e_{t+1} = \Pi(e_t - x_t) \tag{35}$$

The planar system describing the dynamics are Equation (34) and Equation (35).

The Jacobian matrix of the partial derivatives of the system will be

$$\begin{aligned}
\begin{bmatrix} x_{t+1} \\ e_{t+1} \end{bmatrix} &= \begin{bmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{bmatrix} \begin{bmatrix} x_t \\ e_t \end{bmatrix} \\
\text{with } \psi_1 &= \frac{\Pi}{(1-\theta)} \frac{\beta}{(1+\beta)} \frac{(1-\alpha)}{\alpha} + \frac{\Pi\theta}{(1-\theta)}, \\
\psi_2 &= -\frac{\Pi\theta}{(1-\theta)} \\
\psi_3 &= -\Pi \\
\psi_4 &= \Pi
\end{aligned}$$

Lemma 3 *(Steady States) The steady states of equations (34) and (35) are characterized by the following steady state equations:*

$$x^* = \psi_1 x^* + \psi_2 e^*, \tag{36}$$

$$e^* = \psi_3 x^* + \psi_4 e^* \tag{37}$$

These equations have two sets of steady states:

(i) If $\psi_1 = 1$ or $(\psi_1 \neq 1$ and $\psi_4 = 1)$ the steady state is $(x^*, e^*) = (0, 0)$.

(iii) If $\psi_1 \neq 1, \psi_4 \neq 1$ and $(1 - \psi_4) = \frac{\psi_2\psi_3}{(1-\psi_1)}$ any $(x, e) \in R$ will be a steady state. Then, there is a continuum of steady states.

Proof. Obvious. ■

Note that if resources are non-renewable so that $\Pi = 1$, the only possible steady state is $(x^*, e^*) = (0, 0)$.

3.6. Local Dynamics

In this section, the stability of the system and the occurrence of local indeterminacy and bifurcations is analyzed. With the focus on non-renewable resources, the local dynamics in the neighborhood of the steady state (x, e) = (0,0) is studied. The share of energy in capital accumulation is found to play an important role in the occurrence of indeterminacy and bifurcations.

Proposition 4 *(Stability of Dynamics) For non-renewable resources* $\Pi = 1$, *for different parameter combinations with* $\tilde{\beta} = \frac{\beta}{(1+\beta)} \frac{(1-\alpha)}{\alpha}$, *the stability of the zero steady state changes such that*

Parameter	*Description*
α	*share of capital in final good production* $\alpha \in (0,1)$
θ	*share of resources in equipment good production* $\theta \in (0,1)$
β	*discount factor* $\beta \in (0,1)$
Π	*regeneration rate* $\Pi \geq 1$
$\tilde{\beta}$	*increases with patience rate decreases with the share of capital*

1. *If* $\frac{1}{2} < \theta < 1$, *the dynamics are non-complex and eigenvalues (in absolute value) are on the different side of one, so that the steady state is a saddle.*

2. *If* $\theta = \frac{1}{2}$, *the dynamics are non-complex and*

   (a) *if* $\tilde{\beta} < \frac{1}{\sqrt{3}}$, *eigenvalues (in absolute value) are on the different side of one, so that the steady state is a saddle.*

   (b) *if* $\tilde{\beta} \geqslant \frac{1}{\sqrt{3}}$,*both eigenvalues (in absolute value) are greater than one so that the equilibrium dynamics are monotone unstable.*

3. *If* $0 < \theta < \frac{1}{2}$ *and* $2(1-2\theta) < \tilde{\beta}$ *the dynamics are non-complex eigenvalues (in absolute value) and are on the different side of one, so that the steady state is a saddle.*

   (a) *If* $\tilde{\beta} \geqslant (1-2\theta) + \sqrt{2\theta^2 - 3\theta + 1}$,*eigenvalues (in absolute value) are on the different side of one, so that the steady state is a saddle.*

   (b) *If* $2(1-2\theta) < \tilde{\beta} < (1-2\theta) + \sqrt{2\theta^2 - 3\theta + 1}$,*eigenvalues (in absolute value) are on the different side of one, so that the steady state is a saddle.*

4. *If* $\frac{1}{4} \leq \theta < \frac{1}{2}$ *and* $\tilde{\beta} < 2(1-2\theta)$, *the dynamics are non-complex and both eigenvalues (in absolute value) are greater than one, so that the equilibrium dynamics are monotone unstable.*

5. *If* $0 < \theta < \frac{1}{4}$ *and* $\tilde{\beta} + \frac{1}{\tilde{\beta}} < 2(1 - 2\theta)$*, the dynamics are complex such that:*

   (a) *If* $\theta < 1 - \tilde{\beta}$*, both eigenvalues (in absolute value) are smaller than one, so that local indeterminacy occurs where the the steady state is stable.*

   (b) *If* $\theta > 1 - \tilde{\beta}$*, both eigenvalues (in absolute value) are greater than one so that the equilibrium dynamics are monotone unstable.*

Proof. For $\Pi = 1$, the two real eigenvalues of the system are defined by $\lambda_{1,2} = \frac{(1+\psi_1)\pm\sqrt{(1+\psi_1)^2-4(\psi_1+\psi_2)}}{2}$ and the complex eigenvalues are

$$\lambda_{1,2} = \frac{(1 + \psi_1)}{2} \pm \frac{1}{2} \sqrt{(1 + \psi_1)^2 - 4(\psi_1 + \psi_2)} i.$$

The complexity of the dynamics are proved in the Appendix B. The stability analysis is presented first. Note that, the trace T and the determinant D of the associated Jacobian matrix, which respectively represent the sum and the product of the two eigenvalues of the characteristic polynomial are: $T = \psi_1 + 1$ and $D = \Pi(\psi_1 + \psi_2)$ with $T > 1$ and $D > 0$.

1. Suppose $\frac{1}{2} \leq \theta < 1$. Then since the dynamics are non-complex as the discriminant $\Delta = (1 + \psi_1)^2 - 4(\psi_1 + \psi_2) > 0$. Furthermore as $D > 0$, $sign(\lambda_1) = sign(\lambda_2)$. Since $T > 1$, one can conclude $\lambda_{1,2} > 0$. Comparing λ_1 and λ_2 with 1, one can observe that the dynamics are stable iff both eigenvalues are smaller than one, iff $\frac{(1+\psi_1)}{2} < 1 - \frac{\sqrt{\Delta}}{2}$. However, as $1 - \frac{\sqrt{\Delta}}{2} < 0$ (see Appendix B for the proof) and $\psi_1 > 0$, stability case can not occur. The steady state is saddle iff eigenvalues are on the different side of one. As $\lambda_{1,2} > 0$, this implies $\lambda_1 > 1$ and $\lambda_2 < 1$. This reduces to $1 - \frac{\sqrt{\Delta}}{2} < \frac{(1+\psi_1)}{2} < 1 + \frac{\sqrt{\Delta}}{2}$. Finally, for $\frac{1}{2} < \theta < 1$,the equilibrium dynamics are monotone unstable iff $\lambda_1 > 1$ and $\lambda_2 \geqslant 1$ iff $\frac{(1+\psi_1)}{2} \geq 1+\frac{\sqrt{\Delta}}{2}$. Yet, unstability is impossible as $1+\frac{\sqrt{\Delta}}{2} > \frac{(1+\psi_1)}{2}$(see the Appendix B for the proof). As $1 - \frac{\sqrt{\Delta}}{2} < \frac{(1+\psi_1)}{2}$ (see the Appendix B for the proof), for $\frac{1}{2} < \theta < 1$ the steady state is a saddle.

2. Suppose $\theta = \frac{1}{2}$. From above it is known that stability case can not occur. Moreover, the steady state is saddle iff $1 - \frac{\sqrt{\Delta}}{2} < \frac{(1+\psi_1)}{2} < 1 + \frac{\sqrt{\Delta}}{2}$ whereas the equilibrium dynamics are monotone unstable iff $\frac{(1+\psi_1)}{2} \geq 1 + \frac{\sqrt{\Delta}}{2}$. For $\theta = \frac{1}{2}$, it is shown in the Appendix B that $1 - \frac{\sqrt{\Delta}}{2} < \frac{(1+\psi_1)}{2}$. Finally, $\frac{(1+\psi_1)}{2} \geq 1 + \frac{\sqrt{\Delta}}{2}$ $\Longleftrightarrow \tilde{\beta} \geqslant \frac{1}{\sqrt{3}}$. which completes the proof of Case 2.

3. Suppose $0 < \theta < \frac{1}{2}$ and $\tilde{\beta} > 2(1 - 2\theta)$. The dynamics are stable iff both eigenvalues are smaller than one, iff $\frac{(1+\psi_1)}{2} < 1 - \frac{\sqrt{\Delta}}{2}$. For the stability case, $1 - \frac{\sqrt{\Delta}}{2} > 0$[6]. As shown in the Appendix B, if $\tilde{\beta} > (1 - 2\theta) - \sqrt{2\theta^2 - 3\theta + 1}$ or $\tilde{\beta} < (1 - 2\theta) + \sqrt{2\theta^2 - 3\theta + 1}$, then $1 - \frac{\sqrt{\Delta}}{2} < 0$, so that the stability is impossible. In addition, if

   $(1 - 2\theta) < \tilde{\beta} < (1 - 2\theta) + \sqrt{2\theta^2 - 3\theta + 1}$, stability is possible. However, one still have to check whether $\frac{(1+\psi_1)}{2} < 1 - \frac{\sqrt{\Delta}}{2}$ or not. It is shown in the Appendix B that $\frac{(1+\psi_1)}{2} > 1 - \frac{\sqrt{\Delta}}{2}$, for all $0 < \theta < \frac{1}{2}$ and $\tilde{\beta} > 2(1 - 2\theta)$ thus stability case can not occur. The steady state is saddle iff eigenvalues are on the different side of one. As $\lambda_{1,2} > 0$, this implies $\lambda_1 > 1$ and $\lambda_2 < 1$. This reduces to $1 - \frac{\sqrt{\Delta}}{2} < \frac{(1+\psi_1)}{2} < 1 + \frac{\sqrt{\Delta}}{2}$. Finally, the equilibrium dynamics are monotone unstable iff $\lambda_1 > 1$ and $\lambda_2 \geqslant 1$ iff $\frac{(1+\psi_1)}{2} \geq 1 + \frac{\sqrt{\Delta}}{2}$. Yet, unstability is impossible as $1 + \frac{\sqrt{\Delta}}{2} > \frac{(1+\psi_1)}{2}$(see the Appendix B for the proof). For $0 < \theta < \frac{1}{2}$ the steady state is a saddle.

4. Suppose$\frac{1}{4} \leq \theta < \frac{1}{2}$ and $\tilde{\beta} < 2(1 - 2\theta)$. The dynamics are stable iff both eigenvalues are smaller than one, iff $\frac{(1+\psi_1)}{2} < 1 - \frac{\sqrt{\Delta}}{2}$. First, as shown in the Appendix B $1 - \frac{\sqrt{\Delta}}{2} > 0$ for all $\frac{1}{4} \leq \theta < \frac{1}{2}$ and $\tilde{\beta} < 2(1 - 2\theta)$. However, as $1 - \frac{\sqrt{\Delta}}{2} < \frac{(1+\psi_1)}{2}$ (see Appendix B for the proof) and $\psi_1 > 0$, stability case can not occur. The steady state is saddle iff eigenvalues are on the different side of one. As $\lambda_{1,2} > 0$, this implies $\lambda_1 > 1$ and $\lambda_2 < 1$. This reduces to $1 - \frac{\sqrt{\Delta}}{2} < \frac{(1+\psi_1)}{2} < 1 + \frac{\sqrt{\Delta}}{2}$. Finally,the equilibrium dynamics are monotone unstable iff $\lambda_1 > 1$ and $\lambda_2 \geqslant 1$ iff $\frac{(1+\psi_1)}{2} \geq 1 + \frac{\sqrt{\Delta}}{2}$. Yet, unstability is impossible as

[6] If this equation holds with equality, it is proved that the system can not be stable.

$1+\frac{\sqrt{\Delta}}{2} > \frac{(1+\psi_1)}{2}$(see the Appendix B for the proof). Then, the steady state is a saddle.

5. $|\lambda_1| = |\lambda_2| = \sqrt{\frac{\tilde{\beta}^2}{(1-\theta)}}$. Comparing $|\lambda_1| = |\lambda_2|$ with 1, one can observe that the dynamics are stable iff both eigenvalues are smaller than one, iff $|\lambda_1| < 1$ iff $\tilde{\beta} < (1-\theta)$. Similarly, one can observe that the dynamics are stable iff both eigenvalues are larger than one, iff $|\lambda_1| > 1$ iff $\tilde{\beta} > (1-\theta)$.

■

The dynamic behavior of the economy is fully characterized with respect to the share of energy resources in the production of capital accumulation. In addition to this parameter, the stability of the system changes with respect to the share of capital in production and the discount rate. Proposition (4), shows that if the share of energy resources in the production of capital accumulation is higher than one half, the steady state is a saddle. However, if regarding share is equal to one half, the stability of the system changes with respect to the $\tilde{\beta}$ parameter. Note that $\tilde{\beta}$, increases with the discount factor β whereas decreases with the share of capital in production. If $\tilde{\beta}$ is low enough (lower than one half), the steady state is a saddle and unstable otherwise. One can interpret this finding as ceteris paribus for capital intensive (or less patient) economies the long run dynamics can turn out to be saddle rather than unstable. On the other hand, if the share of energy resources in the production of capital accumulation is less than one half and $\tilde{\beta}$ is higher than some threshold level $2(1-2\theta)$, the steady state is a saddle. If $\tilde{\beta}$ is lower than this threshold level $2(1-2\theta)$ and the share of energy resources in the production of capital accumulation is higher than one fourth but less than one half the dynamics are unstable. On for small shares of energy resources in the production of capital accumulation (less than 0.25) with small enough $\tilde{\beta}$, the dynamics are complex.

Corollary 5 *(Hopf Bifurcation) Assume that* $0 < \theta < \frac{1}{4}$ *and* $\tilde{\beta} < 2(1-2\theta)$. *The steady state is locally indeterminate for*

$\theta < 1 - \tilde{\beta}$, a Hopf Bifurcation occurs for $\theta = 1 - \tilde{\beta}$, and the steady state is unstable for $\theta > 1 - \tilde{\beta}$.

Parameter	*Description*
θ	*share of resources in equipment good production* $\theta \in (0, 1)$
$\tilde{\beta}$	*increases with patience rate decreases with the share of capital*

As the corollary presents, for small values of the share of energy in capital accumulation, depending on the relationship between this share, capital share and the discount rate, the steady state can be locally indeterminate. Moreover, a Hopf Bifurcation occurs for $\theta = 1 - \tilde{\beta}$. To summarize, in contrast to the Mourmouras (1991), for the non-renewable resources if the share of energy resources is low enough (less than 25 %) local indeterminacy and hopf bifurcations arise in the model.

3.7. Conclusion

Although the literature widely assumes the same technology for the consumption and capital accumulation sector, the data suggests that physical capital production is relatively more energy-intensive than consumption and the non-renewable resources can limit growth through the equipment production sector. Using a overlapping generations resource model, this chapter examines the implications of differentiating energy intensities of the physical capital and the final good production. The model assumes the physical capital accumulation sector to be more energy intensive than consumption where energy is obtained from the extraction of resources. Apart from the standard literature, the accumulation of the capital stock is determined not only by the savings but also by the energy that it requires.

Introduction of such a differentiation has important implications for the standard results in the area. In line with Agnani et al. (2005), although technological progress' is taken to be exogenous, the growth rate of the economy depends on all of the parameters of the model, as if the model is an endogenous growth model. The balanced growth path of the economy can be optimal depending on the choice of the depletion rate. Contrary to the Agnani et. al (2005)'s findings, multiple balanced growth paths are encountered rather than a unique balanced growth path whose growth rate can be explicitly solved. The main finding of the study suggests that when taking the intensities into consideration dynamics other than saddle arise. In fact, for the non-renewable resources local indeterminacy and hopf bifurcations are found if the share of energy resources is low enough (less than 25 %).

For future research several issues can be considered. First, costless harvest assumption could be replaced by introducing harvest costs. In addition, the logistic regeneration could be introduced rather than the linear specification. The dynamics can be even more complex if logistic regeneration is allowed.

CHAPTER 4

EXTRACTION COSTS AND DIFFERENTIATED ENERGY INTENSITIES IN AN OVERLAPPING GENERATIONS RESOURCE ECONOMY

This paper stems from two stylized facts regarding resource economic modeling:

(i) Regardless of the type of the resource (windmills, fisheries, coal, petroleum, hydrocarbon, etc.), natural resource extraction incur costs. These costs depend on the size of the available resource stock and the amount extracted.

(ii) Equipment sector and the final good sector do not have the same resource intensities. There is evidence that capital accumulation sector is more energy intensive than consumption sector. In Azomahou et al. (2004, 2006), an energy intensity measurement (ratio between energy consumption and value added) of 14 sectors of the economy is constructed from the Structural Analysis Database of OECD and the Energy Balances and Energy Prices and Taxes of IEA. Their findings indicate that energy intensity is higher for sectors closely related to physical capital accumulation (eg: iron and steel sector (0.809), transport and storage (0.85), non-ferrous metals (0.599), and non-metallic minerals (0.507). On the other hand, the energy intensity for the consumption goods related sectors (eg: food and tobacco (0.134), textile and leather (0.082), and construction (0.018)) are found to be lower compared to the capital intensive ones.

Acknowledging these fundamental observations, one can question whether costly resource extraction and differentiating energy intensities induce non-linear dynamics is a worthwhile research question to analyze. Costly resource extraction has been analyzed in partial equilibrium framework frequently (e.g., see Clark, 1990; Hartwick and Olewiler, 1986; Munro and Scott, 1985; Hanley et al., 1997; for a survey see Brown, 2000), there are still limited number of studies conducted in infinitely lived agents framework (Krutilla and Reuveny, 2004; Eliasson and Turnovsky, 2004; Valente, 2005). Regarding the overlapping generation economy, Bednar–Friedl and Farmer (2010, 2011) are the only studies focusing on this question due to several limitations. As Farmer (2011) and Krutilla and Reuveny (2004) also notes the analytical complexity driven by harvest costs limits the proper representation of this natural fact in dynamic general equilibrium economic growth models (Berck, 1981). On the other hand, although there is a vast literature analyzing the offsetting effect of capital accumulation on the constraint that limited natural resources put on growth, there is only one paper (Barahona,2011) considering differing energy intensities of consumption good and capital accumulation sector. Motivated by these stylized facts, this paper is an attempt to explain non-linear dynamics (if any) that could arise by integrating costly extraction, different technologies for equipment good and final good sector in an overlapping generations resource economy.

As mentioned in Chapter 3, considering the use of resources in production or more generally the constraint that resources put on growth, the OLG framework appears to have an explanatory power. Within the OLG framework, resources can be considered as store of values between generations (see Koskela et al., 2002; Valente, 2008; Bednar–Friedl and Farmer; 2011) in addition to their role in the production process. On the other hand, the ILA models cannot fully represent the selfishness of short-lived individuals and the effects of their extraction decisions on the future generations. As highlighted by Solow (1974), Padilla (2002), Agnani (2005), both intratemporal and intertemporal effects should be considered to have better understanding of environmental problems and/or natural resource economies. Last but not least, in contrast to what the infinitely-lived representative agent models assert, there exists some empirical evidence that agents are not perfectly altruistically linked (Altonji et al., 1992; Balestra, 2003).

In this study, in a similar vein with the (2011), the technologies of consumption good sector and the final good sector is differentiated in an overlapping generations resource economy with three sectors. The final good sector produces the consumption with physical capital and labor. Physical capital is accumulated by means of the investments (or pre-installed capital stock / savings) and the extracted energy resources. By this way, the capital accumulation sector is assumed to be more energy intensive. The energy input is extracted from the natural resources with some extraction costs.

There are two lines of literature modeling extraction costs. On the one hand, as some claim (such as Krutilla and Reuveny, 2004; Eliasson and Turnovsky, 2004; Bednar–Friedl and Farmer; 2011), harvesting may incur labor costs. In these studies, labor input is used to represent the effort so that the young decides how to allocate the labor across the resource sector and final good sector. In Krutilla and Reuveny (2004) and Eliasson and Turnovsky (2004), the resource extraction costs affect resource dynamics, whereas in Bednar–Friedl and Farmer (2011) the harvest costs are modeled as time costs affecting the budget constraint of the young individuals through the reducing the time spent in production of the final good and hence their earnings from production. On the other hand, the other line of literature suggests that the extraction could incur costs in terms of the resource stock. This way of modeling is evident in as Bednar–Friedl and Farmer (2010)'s study where the resource extraction is integrated into the resource dynamics. Armstrong and Sumaila (2001) and Escapa and Prellezo (2003) both assume that the technology of extraction negatively affects the natural growth rate of the resource. In this study, Bednar–Friedl and Farmer (2010)'s assumption, which gross extraction turn only partly into resource stock during the extraction process, is adopted. . This assumption leads the model to resemble Hayashi (1982) where gross investment turns only partly into capital. In addition, it is assumed that the extraction costs depend on the availability (amount) of the resource. Specifically, it is assumed that the scarcer the resource is, the harder it is to extract and more harvest costs it incurs. Moreover, in the numerical simulations a specific functional form evident in Bednar–Friedl and Farmer (2010) is taken to compare the resulting dynamics.

Following Mournmouras (1991), the natural resource is assumed to regenerate linearly. To visualize the net effect of the differing intensities as well as the harvest costs, the logistic regeneration function are avoided which has the complex dynamics generating property. Koskela et al. (2008) looks for other types of dynamics than saddles in renewable resource based OLG economies. It has numerically shown that flip bifurcations may arise only if the intertemporal elasticity of substitution of the utility function is less than one half and the regeneration function is logistic. Besides, the findings can be compared with the linear specification that is discussed in Chapter 2 where only differing intensities are considered. Such a comparison can reveal the net effect of harvest costs.

The uniqueness of the steady state as well as the dynamics around the steady state is analyzed. The importance of analyzing this dynamics lies behind the fact that if private savings may not be sufficient to sustain capital and resource accumulation in the long run so that a non-trivial steady state might not exist (Bednar–Friedl and Farmer, 2010). Regarding the uniqueness, multiple equilibria can be found after the introduction of harvest costs (see Krutilla and Reuveny).

To isolate the effect of harvest cost the findings are compared to the findings of the model in Chapter 2. In Chapter 2, it is shown that multiple equilibria exist and local indeterminacy and hopf bifurcations arise for the non-renewable resources depending on the share of energy resources if one considers differing technologies. However, as there are infinitely many non-zero steady states the stability analysis could not be performed for that model. Now, with the inclusion of harvest costs multiple equilibria exist and through the numerical simulations, it is shown that indeterminacy, transcritical and hopf bifurcations can arise in the model for the non-zero steady state.

To observe the net effect of differing energy intensities the model is compared with Bednar–Friedl and Farmer (2010). In the model, it is shown that multiple equilibria exist and dynamics are saddle. While, taking the identical cost function with the differentiation of technologies through the numerical simulations it is shown that dynamics other than saddle –indeterminacy, transcritical and hopf bifurcations– can arise in the model for the non-zero steady state. Besides, although Bednar–Friedl and Farmer (2011) model harvest costs in a more complicated fashion, they find a unique steady state. More interestingly, they show that if regeneration is assumed to be linear, then the dynamics are saddle.

The paper proceeds as follows. The model is presented in Section 2. The competitive equilibrium is defined in Section 3. Section 4 examines the stability of the long run dynamics. Numerical simulations are carried on in Section 5. Conclusions and further research opportunities are discussed in Section 6.

4.1. The Model

The model is an extension of Chapter 3, studying an overlapping generations economy with manmade capital and a consumption good have a different technology but extraction is assumed to be costless. In this chapter, it is assumed that resource extraction causes cost to the resource stock as extraction/harvest costs. As in Chapter 3, as opposed to the standard OLG framework[1], the model differentiates the energy intensities of the physical capital and the final good production. There are three sectors in the economy: the final good production, the equipment (investment) good production and the extraction sector. The extraction sector harvests the energy input from the natural resource. During the extraction process gross extraction turn only partly into resource stock. The obtained energy resource and the installed capital stock from the previous period from the physical good in the investment sector. The final good –which can be either consumed or invested– is

[1]For the discussion of standart OLG models see De La Croix & Michel (2004).

produced the using physical capital and labor.

All agents have rational expectations and each generation consists of a single representative agent. Moreover, all agents in this economy are price-takers and all the markets are competitive.

The natural resource can be either saved as an alternative asset to capital or extracted to be used as an input to form energy. The initially old generation is assumed to own the initial stock of the natural resource. At the beginning of each period t, the old agents (generation $t-1$) own the resource stock e_t[2]. They choose how much of their stock to extract as energy and sell to the equipment good sector x_t. The remaining part of the natural resource is sold young agents as resource assets. From period t to $t+1$, the resource stock regenerates at a linear rate Π (Mourmouras, 1991), where $\Pi \geqslant 1$[3].Cost of the extracting x_t units of resource is represented by $h(e_t)x_t$.This function h satisfies the following properties: $h' < 0$, $h'' > 0$ and $h(0) > 0$ (as in Farmer & Friedl, 2006). These hypotheses imply that the scarcer the resource is, the harder it is to extract and the more the harvest costs it incurs. Also, note that if $h(e_t) = 1$, the extraction is costless. Then, the transition dynamics of the energy resource can be summarized as below:

$$e_{t+1} = \Pi(e_t - h(e_t)x_t).$$

Under a perfect foresight overlapping generations economy[4] in discrete time with infinite horizon the representative household's two-period intertemporal utility is defined over the level of consumption when young c_t and when old $d_{t+1}$with the additively separable life-cycle utility function

$$U(c_t, d_{t+1}) = u(c_t) + \beta u(d_{t+1})$$

, where $\beta \in (0, 1)$ is the subjective discount factor.

[2]As population size is taken to be constant all variables are in per capita terms.
[3]Note that as long as $\Pi = 1$ the resource is non-renewable.
[4]without population growth

When young, the resource stock e_t, is bought at the beginning of period t, from the price P_t. The total income of the representative individual is generated from supplying his one unit of labor to the firms when young receiving the real wage w_t and selling the extracted resource stock x_t to the equipment firms from the price Q_t. The representative individual spends his income on the current consumption c_t, the savings of physical capital s_t, the savings on resource stock $P_t e_t$. When the individual gets old (at period $t+1$), the agent is retired and he consumes d_{t+1} out of his entire income and do not leave bequests. His income is generated from the return on his savings made when young: $R_{t+1}s_t$, selling the remaining resource to the young $P_{t+1}e_{t+1}$. Accordingly, the budget constraints facing generation t is as follows:

$$c_t + s_t + P_t e_t \;=\; w_t \; + Q_t x_t, \tag{1}$$

$$d_{t+1} \;=\; R_{t+1}s_t + P_{t+1}e_{t+1}. \tag{2}$$

Taking the prices of the energy resource and wages as given, the representative agent born at time t maximizes his utility by choosing the young and the old periods' consumption and the ownership of the energy resource subject to Equations (1,2) and the appropriate nonnegativity constraints. The first-order conditions follows:

$$\frac{d_{t+1}}{c_t} \;=\; \beta R_{t+1}, \tag{3}$$

$$R_{t+1} \;=\; \frac{P_{t+1}}{P_t}(1 - h'(e_t)x_t), \tag{4}$$

$$\frac{Q_t}{\Pi h(e_t)} \;=\; \frac{P_{t+1}}{R_{t+1}}. \tag{5}$$

Equation (3) gives the equalization of discounted marginal utilities where the marginal rate of substitution between the current and the future consumption is equal to their relative prices. Equation (4) is the no-arbitrage condition between return on capital and the adjusted return on resource assets. Equation (5) can be interpreted as the return on net harvest is the discounted return on the resource stock.

The representative firm in the final goods sector is owned by the old households. The firms in final goods sector produces the consumption good with capital K_t and labor N_t under the Cobb-Douglas constant returns to scale technology. Equation (6) presents the production function in the final good sector at any date t.

At each period t, taking the prices of inputs, the initial technology level and the initial level of capital stock as given, the representative firm maximizes its profit by choosing the amount of labor and physical capital inputs.

$$\max_{\{K_t,N_t\}_{t=0}^{\infty}} \pi_t = Y_t - P_t^K K_t - w_t N_t,$$

$$Y_t = K_t^{\alpha} N_t^{1-\alpha} \qquad 0 < \alpha < 1 \tag{6}$$

At an interior solution of the firm's optimization problem,where all variables are expressed in per capita ($k_t = \frac{K_t}{N_t}$) terms, the following first order conditions are satisfied equating the price of the inputs to their marginal benefits.

$$\alpha y_t = P_t^K k_t, \tag{7}$$

$$(1-\alpha) y_t = w_t. \tag{8}$$

Apart from the standard literature where the consumption good and capital accumulation have the same technology, it is assumed that the physical capital production is relatively more energy-intensive than consumption. As in (2011), the capital stock is accumulating by means of the investments –installed capital stock from time t- and by the energy extracted. Thus, in Equation (9) the new capital at $t+1$, K_{t+1} is produced from the natural energy resources X_{t+1} and investment made at time t, I_t with the following Cobb-Douglas technology.

$$K_{t+1} = X_{t+1}^{\theta} I_{t.}^{1-\theta} \tag{9}$$

In the equipment good production sector at each period t, the representative firm maximizes its profit by choosing the amount of non-renewable resource input that will be utilized in the production process:

$$\begin{aligned} \max_{\{X_t\}_{t=0}^{\infty}} \pi_t &= P_t^K K_t - Q_t X_t, \\ s.t \quad K_t &= X_t^{\theta} I_{t-1}^{1-\theta}. \end{aligned}$$

At an interior solution of the equipment firm's optimization problem the following first order condition is satisfied:

$$Q_t = \theta \alpha Y_t X_t^{-1}. \tag{10}$$

The profit on investing on capital $R_t S_{t-1}$ at time t should be equal to the profit on producing new capital $(1-\theta) P_t^K K_t$[5] to prevent arbitrage opportunities.The below equation are follows from this no arbitrage condition:

$$\mathrm{R}_t = (1-\theta) \mathrm{P}_t^K \mathrm{X}_t^{\frac{\theta}{1-\theta}} \mathrm{K}_t^{\frac{\theta}{\theta-1}}. \tag{11}$$

4.2. The Competitive Equilibrium

Equation (15) summarizes the goods market clearing condition of the economy. The final good is either consumed by young agents C_t, or by old agents (generation $t-1$)

[5] By choosing the optimal non-renewable resource stock maximum profit that the representative firm can obtain is as follows:

$$\pi_t^* = P_t^K K_t - \theta P_t^K B_t^{\theta} X_t^{\theta-1} I_{t-1}^{1-\theta} X_t = (1-\theta) P_t^K K_t$$

D_t, or invested for the production of the future capital stock, S_t.

$$\mathrm{y}_t = \mathrm{c}_t + \mathrm{d}_t + \mathrm{s}_t \tag{12}$$

As in Chapter 2 savings are still equal to the investments (Equation (13)) but only a fraction of the investments can generate the new capital stock. Labor market clears as labor supply is given by N_0. Finally, resource market clears.

$$
\begin{aligned}
I_t &= S_{t,} && (13)\\
N_t &= N_0, && (14)\\
e_{t+1} &= \Pi(e_t - h(e_t)x_t). && (15)
\end{aligned}
$$

Definition 1 *A dynamic competitive equilibrium for this OLG resource economy is determined by the sequence of prices* $\left\{w_t, R_t, P_t^K, P_t, Q_t\right\}_{t=0}^{\infty}$ *and feasible allocations* $\{c_t, d_t, s_t, e_t, x_t, i_t, y_t, k_{t+1}\}$ *given the positive initial values for* $S_{-1}, E_0, N_0, h(0) > 0$ *such that the consumers maximize their life-time utility, firms maximize their profits. The endogenous prices are obtained from clearing of the markets at every period* t. *The competitive equilibrium of this OLG resource economy is a solution of the non-linear system of equations, (1)–(15).*

4.3. Equilibrium Dynamics

The long run dynamics around the steady state is analyzed. As Mourmouras (1991), Farmer (2000) and Bednar–Friedl and Farmer (2011) suggests, the intertemporal equilibrium dynamics can be reduced into a two–dimensional system which represents the law of motions of E_t and X_t.

Substituting Equations((15),(3),(4),(5),(8)) into the Equation (2), the consumption when young can be re-written as:

$$c_t = \frac{1}{(1+\beta)}\left((1-\alpha)y_t + \frac{q_t h'(e_t)x_t e_t}{h(e_t)}\right) \tag{16}$$

Plugging Equation (16) into the goods market clearing and using Equations ((11),(7),(10)) yields:

$$s_t = \frac{y_t}{x_t h(e_t)}\left[\left((\theta\alpha + \frac{\beta}{(1+\beta)})h(e_t) + \theta\alpha\frac{\beta}{(1+\beta)}h'(e_t)e_t\right)x_t - \theta\alpha e_t\right] \tag{17}$$

On the other hand, plugging Equations ((5),(10),(11),(7)) into the Equation (4) the following equation is derived for s_t :

$$s_t = \frac{(1-\theta)\alpha}{\Pi}\frac{x_{t+1}}{x_t}\frac{h(e_{t+1})}{h(e_t)}\frac{y_t}{(1-h'(e_{t+1})x_{t+1})} \tag{18}$$

Equating Equation (17) to Equation (18) the first law of motion equation is obtained:

$$x_{t+1} = \Pi\left[\left((\theta\alpha + \frac{\beta}{(1+\beta)})h(e_t) + \theta\alpha\frac{\beta}{(1+\beta)}h'(e_t)e_t\right)x_t - \theta\alpha e_t\right] * \frac{1}{\left\{(1-\theta)\alpha h(e_{t+1}) + \Pi h'(e_{t+1})\left[\begin{array}{c}\left((\theta\alpha + \frac{\beta}{(1+\beta)})h(e_t) + \theta\alpha\frac{\beta}{(1+\beta)}h'(e_t)e_t\right)x_t \\ -\theta\alpha e_t\end{array}\right]\right\}}$$

The second law of motion equation comes from the dynamics of the resource stock. In order to study the dynamics we rewrite the law of motions as follows.

$$x_{t+1} = \Pi\left[\left((\theta\alpha + \frac{\beta}{(1+\beta)})h(e_t) + \theta\alpha\frac{\beta}{(1+\beta)}h'(e_t)e_t\right)x_t - \theta\alpha e_t\right] * \frac{1}{\left\{(1-\theta)\alpha h(e_{t+1}) + \Pi h'(e_{t+1}) * \left[\left(\begin{array}{c}(\theta\alpha + \frac{\beta}{(1+\beta)})h(e_t) + \\ \theta\alpha\frac{\beta}{(1+\beta)}h'(e_t)e_t\end{array}\right)x_t - \theta\alpha e_t\right]\right\}} \tag{19}$$

$$e_{t+1} = \Pi(e_t - h(e_t)x_t) \tag{20}$$

The planar system describing the dynamics are Equation (19) and Equation (20).

Lemma 2 *(Steady States) The steady states of equations (19) and (20) are the solutions of the following equations:*

$$\text{x} = \frac{(\Pi - 1)}{\Pi}\frac{e}{h(e)} \tag{21}$$

$$x^2\left[\Pi(\theta\alpha + \frac{\beta(1-\alpha)}{(1+\beta)})h(e)h'(e) + \theta\alpha\frac{\beta\Pi}{(1+\beta)}h'(e)^2 e\right] - \left[\Pi\theta\alpha e h'(e) + \Pi(\theta\alpha + \frac{\beta(1-\alpha)}{(1+\beta)})h(e) + \theta\alpha\frac{\beta\Pi}{(1+\beta)}h'(e)e\right] + \theta\alpha\Pi e = 0$$

Proof. Obvious. ■

Note that if resources are non-renewable so that $\Pi = 1$, the only possible steady state is $(x^*, e^*) = (0, 0)$.

From now on, the harvest function will be specified. To control how the results on dynamics change with respect to the specification of diffrerent intensities, the harvest cost function in Farmer-Friedl (2010) is adopted where:

$$\text{h}(\text{e}_t) = 1 + \frac{\lambda}{e_t}, \qquad \lambda > 0.$$

In above specified function λ represents the difficulty of extraction. Indeed, with $\lambda = 0$, the exrtaction is costless. Now, the steady states are found in Lemma (3):

Lemma 3 *(Steady States) Assume that $h(e_t) = 1 + \frac{\lambda}{e_t}$ with $\lambda > 0$. The steady states of equations in Lemma (2) are the solutions of the following equations:*

$$e^3\left[e^2+\left(\frac{\lambda^2\Xi\theta\alpha\frac{\beta}{(1+\beta)}+((\theta\alpha+\frac{\beta(1-\alpha)}{(1+\beta)})-\Pi\frac{1}{(1+\beta)})}{\theta\alpha\frac{\beta}{(1+\beta)}\Xi}\right)e-\frac{1}{\lambda^2\Xi}\right]=0,$$

$$with \quad \Xi=\frac{(\theta\alpha\Pi-(\Pi-1)(\theta\alpha+\frac{\beta(1-\alpha)}{(1+\beta)}))\Pi}{\theta\alpha\frac{\beta}{(1+\beta)}(\Pi-1)^2}$$

These equations have three steady states:

(i) One of the steady states is "Zero Resource Steady State"

$$\begin{aligned} x^* &= 0, \\ e^* &= 0. \end{aligned}$$

(ii) The second steady state is "Positive Resource Low Steady State":

$$e^*=-\left(\frac{\lambda^2\Xi\theta\alpha\frac{\beta}{(1+\beta)}+((\theta\alpha+\frac{\beta(1-\alpha)}{(1+\beta)})-\Pi\frac{1}{(1+\beta)})}{2\theta\alpha\frac{\beta}{(1+\beta)}\Xi}\right)-\sqrt{\left(\frac{\lambda^2\Xi\theta\alpha\frac{\beta}{(1+\beta)}+((\theta\alpha+\frac{\beta(1-\alpha)}{(1+\beta)})-\Pi\frac{1}{(1+\beta)})}{2\theta\alpha\frac{\beta}{(1+\beta)}\Xi}\right)^2+\frac{1}{\lambda^2\Xi}}$$

$$x^*=\frac{(\Pi-1)}{\Pi}\frac{e^{*2}}{(e^*+\lambda)}$$

Lemma 4 *(iii) The third steady state is "Positive Resource High Steady State":*

$$e^*=\sqrt{\left(\frac{\lambda^2\Xi\theta\alpha\frac{\beta}{(1+\beta)}+((\theta\alpha+\frac{\beta(1-\alpha)}{(1+\beta)})-\Pi\frac{1}{(1+\beta)})}{2\theta\alpha\frac{\beta}{(1+\beta)}\Xi}\right)^2+\frac{1}{\lambda^2\Xi}}-\left(\frac{\lambda^2\Xi\theta\alpha\frac{\beta}{(1+\beta)}+((\theta\alpha+\frac{\beta(1-\alpha)}{(1+\beta)})-\Pi\frac{1}{(1+\beta)})}{2\theta\alpha\frac{\beta}{(1+\beta)}\Xi}\right),$$

$$x^*=\frac{(\Pi-1)}{\Pi}\frac{e^{*2}}{(e^*+\lambda)}.$$

Proof. Solving Equation in Lemma (3): yields

$$\left[e^2 + \frac{(\theta\alpha + \frac{\beta(1-\alpha)}{(1+\beta)}) - \Pi\frac{1}{(1+\beta)})}{\theta\alpha\frac{\beta}{(1+\beta)}h'(e)}e + \frac{(\theta\alpha\Pi - (\Pi-1)(\theta\alpha + \frac{\beta(1-\alpha)}{(1+\beta)}))}{\theta\alpha\frac{\beta}{(1+\beta)}(\Pi-1)^2}\frac{h(e)}{h'(e)^2}\right]e = 0$$

The proof follows from plugging $h(e_t) = 1 + \frac{\lambda}{e_t}$. ■

Lemma 5 *The Jacobian matrix of the partial derivatives of the system will be*

$$\begin{aligned}
J &= \begin{bmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{bmatrix} \\
\textit{with } \psi_1 &= \frac{\Pi(1-h'(e)x)}{(1-\theta)\alpha h(e)} * \\
&\left[(1-\theta)\alpha h'(e)h(e)x + \Pi\nabla(x_t,e_t)h'(e)h(e)x + \frac{\partial\nabla(x_t,e_t)}{\partial x}(1-h'(e)x)\right], \\
\psi_2 &= \frac{\Pi(1-h'(e)x)^2}{(1-\theta)\alpha h(e)}\left[\frac{\partial\nabla(x_t,e_t)}{\partial e} - (1-\theta)\alpha h'(e)x - \Pi\nabla(x_t,e_t)h''(e)x\right] \\
\psi_3 &= -h(e)\Pi \\
\psi_4 &= \Pi(1-h'(e)x) \\
\nabla(x_t,e_t) &= \left[\left((\theta\alpha + \frac{\beta}{(1+\beta)})h(e_t) + \theta\alpha\frac{\beta}{(1+\beta)}h'(e_t)e_t\right)x_t - \theta\alpha e_t\right]
\end{aligned}$$

Proof. From Equation (19) and defining

$$\nabla(x_t,e_t) = \left[\left((\theta\alpha + \frac{\beta}{(1+\beta)})h(e_t) + \theta\alpha\frac{\beta}{(1+\beta)}h'(e_t)e_t\right)x_t - \theta\alpha e_t\right]$$

, along the steady state $\nabla(x_t,e_t) = \frac{(1-\theta)\alpha h(e)x}{\Pi(1-h'(e))}$ re-writing

$$(1-\theta)\alpha h(e) + \nabla(x,e)h'(e)\Pi = \frac{(1-\theta)\alpha h(e)}{(1-h'(e))} \tag{22}$$

.

$$\psi_1 = \frac{\partial x_{t+1}}{\partial x}$$

$$= \frac{\Pi}{((1-\theta)\alpha h(e) + \nabla(x,e)h'(e)\Pi)^2} *$$

$$\left[\begin{array}{c} \frac{\partial \nabla(x,e)}{\partial x}((1-\theta)\alpha h(e) + \nabla(x,e)h'(e)\Pi) - \\ \nabla(x,e)\left((1-\theta)\alpha \frac{\partial h(e_{t+1})}{\partial x} + \frac{\partial \nabla(x,e)}{\partial x}h'(e)\Pi + \nabla(x,e)\frac{\partial h'(e_{t+1})}{\partial x}\Pi\right) \end{array} \right]$$

$$= \frac{\Pi}{((1-\theta)\alpha h(e) + \nabla(x,e)h'(e)\Pi)^2} *$$

$$\left[\frac{\partial \nabla(x,e)}{\partial x}(1-\theta)\alpha h(e) + \nabla(x,e)(1-\theta)\alpha h'(e)h(e)\Pi + \nabla(x,e)^2 h''(e)h(e)\Pi^2\right]$$

$$= \frac{\Pi(1-h'(e)x)^2}{((1-\theta)\alpha h(e))^2} *$$

$$\left[\frac{\partial \nabla(x,e)}{\partial x}(1-\theta)\alpha h(e) + \nabla(x,e)(1-\theta)\alpha h'(e)h(e)\Pi + \nabla(x,e)^2 h''(e)h(e)\Pi^2\right],$$

from Equation(22)

$$= \frac{\Pi(1-h'(e)x)}{(1-\theta)\alpha h(e)} *$$

$$\left[\frac{\partial \nabla(x,e)}{\partial x}(1-h'(e)x) + \nabla(x,e)h'(e)\Pi(1-h'(e)x) + \frac{\nabla(x,e)^2 h''(e)\Pi^2}{(1-\theta)\alpha}\right]$$

$$= \frac{\Pi(1-h'(e)x)}{(1-\theta)\alpha h(e)} *$$

$$\left[\frac{\partial \nabla(x,e)}{\partial x}(1-h'(e)x) + (1-\theta)\alpha h(e)xh'(e) + \nabla(x,e)h''(e)\Pi h(e)x\right]$$

from Equation(22).

$$
\begin{aligned}
\psi_2 &= \frac{\partial x_{t+1}}{\partial e} \\
&= \frac{\Pi}{((1-\theta)\alpha h(e) + \nabla(x,e)h'(e)\Pi)^2} * \\
&\left[\begin{array}{c} \frac{\partial \nabla(x,e)}{\partial e}((1-\theta)\alpha h(e) + \nabla(x,e)h'(e)\Pi) - \\ \nabla(x,e)\left((1-\theta)\alpha \frac{\partial h(e_{t+1})}{\partial e} + \frac{\partial \nabla(x,e)}{\partial e}\frac{\partial h(e_{t+1})}{\partial e}\Pi + \nabla(x,e)\Pi\frac{\partial h'(e_{t+1})}{\partial e}\right) \end{array} \right] \\
&= \frac{\Pi}{((1-\theta)\alpha h(e) + \nabla(x,e)h'(e)\Pi)^2} * \\
&\left[\frac{\partial \nabla(x,e)}{\partial e}(1-\theta)\alpha h(e) - \nabla(x,e)(1-\theta)\alpha\frac{\partial h(e_{t+1})}{\partial e} - \nabla(x,e)^2\frac{\partial h'(e_{t+1})}{\partial e}\Pi\right] \\
\text{Since } \frac{\partial h(e_{t+1})}{\partial e_t} &= \frac{\partial h(e_{t+1})}{\partial e_{t+1}}\frac{\partial e_{t+1}}{\partial e_t} = h'(e)\frac{\partial e_{t+1}}{\partial e_t} = h'(e)\Pi(1-h'(e)x) \\
&\text{and from Equation(22);} \\
\psi_2 &= \frac{\Pi(1-h'(e)x)^2}{((1-\theta)\alpha h(e))^2} * \\
&\left[\begin{array}{c} \frac{\partial \nabla(x,e)}{\partial x}(1-\theta)\alpha h(e) - (1-\theta)\alpha\nabla(x,e)h'(e)\Pi(1-h'(e)x) \\ -\nabla(x,e)^2\Pi^2 h''(e)(1-h'(e)x) \end{array} \right] \\
&= \frac{\Pi(1-h'(e)x)^2}{(1-\theta)\alpha h(e)} * \\
&\left[\begin{array}{c} \frac{\partial \nabla(x,e)}{\partial x} - \frac{\nabla(x,e)(1-\theta)\alpha h'(e)\Pi(1-h'(e)x)}{(1-\theta)\alpha h(e)} \\ -\frac{\nabla(x,e)^2\Pi^2 h''(e)(1-h'(e)x)}{(1-\theta)\alpha h(e)} \end{array} \right] \\
&= \frac{\Pi(1-h'(e)x)^2}{(1-\theta)\alpha h(e)} * \\
&\left[\frac{\partial \nabla(x,e)}{\partial x} - (1-\theta)\alpha h'(e)x - \nabla(x,e)\Pi h''(e)x\right] \\
&\text{from Equation(22).}
\end{aligned}
$$

$$
\begin{aligned}
\psi_3 &= \frac{\partial e_{t+1}}{\partial x} = -h(e)\Pi \\
\psi_4 &= \frac{\partial e_{t+1}}{\partial e} = (1-h'(e)x)\Pi
\end{aligned}
$$

■

4.4. Local Dynamics

This subsection examines the stability of the system and the occurrence of local indeterminacy and bifurcations. Firstly, the local dynamics in the neighborhood of the steady state (x, e) = (0,0) is presented in the following proposition.

Proposition 6 *Suppose* $\frac{\Pi\beta}{(1+\Pi)(1+\beta)\alpha} < 1$[6]*.For different parameter combinations, the stability of the zero steady state changes such that,*

Parameter	*Description*
α	*share of capital in final good production*
θ	*share of resources in equipment good production*
β	*discount factor*
Π	*regeneration rate*

i. *If* $\theta < 1 - \frac{\Pi\beta}{(1+\Pi)(1+\beta)\alpha}$ *eigenvalues (in absolute value) are on the different side of one, then the steady state is a saddle point stable;*

iii. *If* $\theta > 1 - \frac{\Pi\beta}{(1+\Pi)(1+\beta)\alpha}$ *, eigenvalues (in absolute value) are on the same side of one, both eigenvalues are greater than one so that the equilibrium dynamics are monotone unstable.*

Proof. The Jacobian around (0,0) steady state is as follows:

$J = \begin{bmatrix} \frac{\Pi(\theta\alpha+\frac{\beta(1-\alpha)}{(1+\beta)})}{(1-\theta)\alpha} & 0 \\ \psi_3 & \Pi \end{bmatrix}$ so that the corresponding eigenvalues are

$$\Phi_1 = \frac{\Pi(\theta\alpha + \frac{\beta(1-\alpha)}{(1+\beta)})}{(1-\theta)\alpha}$$

and $\Phi_2 = \Pi$. Clearly, the second eigenvalue is gerater than one. So, the dynamics are saddle for $\Phi_1 < 1$ and unstable otherwise. Re-writing $\Phi_1 = \frac{\Pi(\theta\alpha+\frac{\beta(1-\alpha)}{(1+\beta)})}{(1-\theta)\alpha} < 1$, yields $\theta < 1-\frac{\Pi\beta}{(1+\Pi)(1+\beta)\alpha}$. Similarly, $\Phi_1 = \frac{\Pi(\theta\alpha+\frac{\beta(1-\alpha)}{(1+\beta)})}{(1-\theta)\alpha} > 1$, yields $\theta > 1-\frac{\Pi\beta}{(1+\Pi)(1+\beta)\alpha}$.

[6] If this assumption on parameters is not made, then the primary non-negativity assumptions on parameters ($\theta > 0$, $\alpha > 0, \Pi > 0$ or $\beta > 0$) will not hold.

In any case, to have a positive share of resources in capital accumulation sector (for $\theta > 0$), $\frac{\Pi\beta}{(1+\Pi)(1+\beta)\alpha} < 1$ must be satisfied. ■

For the zero steady state the stability of the system can be fully characterized in terms of the share of resources in capital accumulation. As Proposition (6) states if the regarding share is below some critical level the system follows saddle path dynamics and unstability otherwise.

Proposition 7 *The Jacobian matrix of the partial derivatives of the system around the positive steady states is*

$$
\begin{aligned}
J &= \begin{bmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{bmatrix} \\
\textit{with } \psi_1 &= \frac{-\lambda[\Pi(e^*+\lambda)+\lambda(\Pi-1)]}{(e^*+\lambda)^2}\frac{(\Pi-1)}{\Pi} + \\
&\left[\begin{array}{c} \frac{2\lambda[\Pi(e^*+\lambda)+(\lambda-1)]}{(e^*+\lambda)^2(1-\theta)\alpha}\frac{(\Pi-1)}{\Pi} * \\ \left\{ \begin{array}{c} \left((\Pi-1)(\theta\alpha+\frac{\beta(1-\alpha)}{(1+\beta)}\right) - \\ \left(\frac{\lambda(\Pi-1)\theta\alpha\frac{\beta}{(1+\beta)}}{(e^*+\lambda)}\right) - \theta\alpha\Pi \end{array} \right\} \end{array} \right] \\
&+ \frac{[\Pi(e^*+\lambda)+\lambda(\Pi-1)]^2\left[\left(\theta\alpha+\frac{\beta(1-\alpha)}{(1+\beta)}\right)(e^*+\lambda)-(1-\theta)\alpha\frac{\beta}{(1+\beta)}\lambda\right]}{(e^*+\lambda)^3(1-\theta)\alpha\Pi}, \\
\psi_2 &= \frac{(\Pi(e^*+\lambda)+\lambda(\Pi-1))^2}{\Pi e^*(1-\theta)\alpha\,((e^*+\lambda))^3} * \\
&\left[\left\{\left(\frac{-\theta\alpha}{(1+\beta)}-\theta\alpha+(1-\theta)\alpha\right)\lambda-\theta\alpha {e^*}^2+\frac{(\Pi-1)}{\Pi}\frac{{e^*}^2\lambda}{(e^*+\lambda)}\right\}\right] + \\
&\frac{(\Pi(e^*+\lambda)+\lambda(\Pi-1))^2}{\Pi(e^*+\lambda)^4} 2\lambda e^*(\Pi-1)^2 * \\
&\left[-\theta\alpha-\frac{\beta(1-\alpha)}{(1+\beta)}+\frac{\lambda\theta\alpha\frac{\beta}{(1+\beta)}}{(e^*+\lambda)}+\frac{(\Pi-1)}{\Pi}\frac{e^*2\lambda\theta\alpha}{(e^*+\lambda)}\right] \\
\psi_3 &= -\frac{(e^*+\lambda)\Pi}{e^*} \\
\psi_4 &= \frac{\Pi e^*+2\lambda\Pi-\lambda}{(e^*+\lambda)}
\end{aligned}
$$

Proof. The proof involves long algebraic operations but it is basically derived by substituting

$$x^* = \frac{(\Pi - 1)}{\Pi} \frac{e^{*2}}{(e^* + \lambda)}$$

into the (5). ■

Since this Jacobian too complex to analyze, the stability for the positive steady states will be carried on through numerical simulations presented in the following section.

4.5. Numerical Simulations

To pursue the dynamics, values for the parameters are specified. In parametrization, two criterias are taken into considertion: i) to be consistent with empirical facts and ii) to be standard in the literature. The parametrization is as follows:

α	β	Π	λ
0.3	0.99	1.007	0.08

Parameter	Description
α	share of capital in final good production $\alpha \in (0, 1)$
θ	share of resources in equipment good production $\theta \in (0, 1)$
β	discount factor $\beta \in (0, 1)$
Π	regeneration rate $\Pi \geq 1$
λ	parameter representing the accessibility of the resource $\lambda \in (0, 1)$

In the OLG economy, each period is taken to be 30 years long. The subjective discount rate (β) was chosen to meet the annual discount factor 0.99, which is pretty standard in calibration exercises. Under Cobb-Douglas technology with only capital and labor as inputs, the values for capital and labor shares are taken to be standard in the literature. Specifically, the share of capital (α) is 0.30, while the share of labor is 0.70. The assumption on the regeneration rate is taken as 1.007. The parameter lamda reflects the accessibility of the resource and it is a scale factor. As there is no emprical evidence on lamda it is taken to be arbitrary $\lambda = 0.08$ to quarantee positive and real resource stock and extraction.

As one of the purpose of this study is to analyze the effect differing technologies, the parameter θ representing the share of resources in capital accumulation is of critical importance. The dynamics of the system is characterized as θ starts from low intensity (0.05) to higher intensities (0.9) with 0.01 incrementals.

The below figure (2) shows how the magnitude of the eigenvalues change as θ starts from low intensity (0.05) to higher intensities (0.9). The x-axis represents the value of the real part and y-axis shows the value of the complex part of the eigenvalues of the corresponding caharcterictic polynomial of the Jacobian introduced in the previous section. In Figure 2, the stability diagram is presented where regions 1 and 3 represent saddle, regions 5 and 6 represent complex eigenvalues, regions 6 and 7 represent stable solutions, regions 2, 4, 5 and 8 represent unstable solutions.

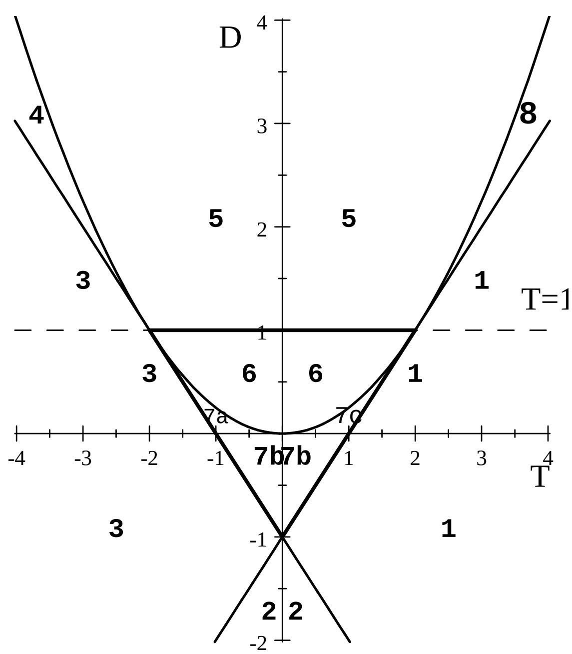


Figure 2: Asymptotic Stability on the Plane.

Proposition 8 *Given these parameter combinations, the stability of the non-zero steady state changes such that,*

Parameter	*Description*
θ	*share of resources in equipment good production,* $\theta \in (0,1)$

i) *If* $\theta < 0.09$, *eigenvalues (in absolute value) are on the same side of one, both eigenvalues are greater than one so that the equilibrium dynamics are unstable. Moreover, eigenvalues are complex and oscillatory behavior is observed such that unstable spirals arise. The economy falls into Region 5 in Figure 1.*

ii) *If* $0.09 \leq \theta < 0.1$, *both eigenvalues (in absolute value) are smaller than one, so that local indeterminacy occurs. Moreover, eigenvalues are complex and oscillatory behavior is observed such that stable spirals arise. The economy falls into Region 6 in Figure 1.*

iii) *If* $0.1 \leq \theta < 0.12$, *both eigenvalues (in absolute value) are smaller than one, so that local indeterminacy occurs. Moreover, eigenvalues are real and stable nodes is observed such that stable nodes arise. The economy moves falls into Region 7a in Figure 1.*

iv) $0.12 \leq \theta < 0.2$ *eigenvalues (in absolute value) are on the different side of one, then the steady state is a saddle point stable. Moreover, eigenvalues are real and the economy falls into Region 1 in Figure 1.*

v) *If* $\theta \geq 0.20$,*eigenvalues (in absolute value) are on the same side of one, both eigenvalues are greater than one so that the equilibrium dynamics are unstable. Moreover, eigenvalues are real and steady state is a source. The economy falls into Region 8 in Figure 1.*

Proposition 9 *Given these parameter combinations,*

i) a Hopf Bifurcation occurs for $\theta = 0.09$.

ii) a Transcritical Bifurcation occurs for $\theta = 0.12$.

To isolate the effect of harvest cost the findings are compared to the findings of the model in Chapter 2. In Chapter 2, it is shown that multiple equilibria exist and local indeterminacy and hopf bifurcations arise for the non-renewable resources depending on the share of energy resources if one considers differing technologies. However, as there are infinitely many non-zero steady states the stability analysis could not be performed for that model. Now, with the inclusion of harvest costs multiple equilibria exist and through the numerical simulations it is shown that indeterminacy, transcritical and hopf bifurcations can arise in the model for the non-zero steady state. To observe the net effect of differing energy intensities the model is compared with Bednar–Friedl and Farmer (2010). In the regarding model, it is shown that multiple equilibria exist and dynamics are saddle. While, taking the identical cost function with the differentiation of technologies through the numerical simulations it is shown that dynamics other than saddle –indeterminacy, flip and hopf bifurcations– can arise in the model for the non-zero steady state. Besides, although Bednar–Friedl and Farmer (2011) model harvest costs in a more complicated fashion, they find a unique steady state. More interestingly, they show that if regeneration is assumed to be linear, the dynamics are saddle.

4.6. Conclusion

In this study, whether costly resource extraction and diffrentiating energy intensities induce dynamics other than saddles or not is examined in an overlapping generations resource economy. Following Barahona (2011) and Chapter 2, the technologies of consumption good sector and the final good sector is differentiated where the capital accumulation sector is asssumed to be more energy intensive. The extraction costs are inversely related with the amount of the resource. Specifically, it is assumed that the scarcer the resource is, the harder it is to extract and the more the harvest costs it incurs.

The net effect of modelling harvest costs and as well as differing technologies on the standard dynamics are revealed independently. The main finding of the paper is that both naturally evident assumptions contribute to the richness of the dynamics. Multiple equilibria exists in the model. In addition, depending on the share of resurces in capital accumulation dynamics other than saddle –indeterminacy, transcritical and hopf bifurcations– can arise in the model for the non-zero steady state.

For future research several issues can be considered. First, the functional form of the harvest cost may be diversified. It will be interesting to analyze the effects of harvest costs increasing (or hump-shaped) with the resource stock. In a similar fashion, harvest costs could ve modeled in terms of labor effort as in previous studies. Second, the effects of shocks to the cost parameters can be explored. Finally, the logistic regeneration could be introduced rather than the linear specification.

CHAPTER 5

INDETERMINACY AND BIFURCATIONS IN AN OVERLAPPING GENERATIONS RESOURCE ECONOMY WITH ENDOGENOUS POPULATION GROWTH RATE

Multiplicity of the steady states, indeterminacy and bifurcations have been obtained in overlapping generations models (OLG) with natural resources. Yet, dynamics in these papers rest on the logistic function regeneration rate and some assumptions on the intertemporal elasticity of substitution in consumption. The aim of the paper is to show that multiple steady states, indeterminacy and bifurcations may arise by endogenizing population growth even in the absence of logistic regeneration or shocks and independent of intertemporal elasticity of substitution.

Vast of the standard economic growth literature assumes labor force grows at a constant rate, following exponential growth. Allowing population to grow in an exponential manner is not realistic, as scarce environmental resources will put a constraint on growth. Smith (1974), describes such a constraint on population growth by defining a feedback mechanism between population growth and carrying capacity of the environment. As indicated by Michetti et.al. (2007, pp.2), Smith (1974) claims that population growth should posses the following properties in a more realistic growth model: "1) when population is small in proportion to environmental carrying capacity, then it grows at a positive constant rate, 2) when population is larger in proportion to environmental carrying capacity, the resources become relatively more scarce and as result this must affect the population growth rate negatively". As the carrying capacity of the environment is directly linked with the avaliability of natural resources, this paper is inspired by Smith's (1974) idea that a feedback mechanism between population and natural resources is essential where population dynamics should be analyzed along with resource dynamics. Motivated by this idea, this study evaluates whether a feedback mechanism between the population growth rate and per capita resource extraction and resource availability modifies the standard results in the area. Specifically, the possibilities of non-linearities in an OLG growth model where the natural resource is essential in production is investigated.

Under infinitely lived agents framework several authors described the population growth using the logistic growth function (Verhulst, 1938; Schtickzelle and Verhulst, 1981; Faria, 2004; Accinelli and Brida, 2005) instead of assuming constant growth rates. While some papers (Verhulst, 1938; Schtickzelle and Verhulst, 1981; Faria, 2004) models population growth as a logistic function, other papers (see Accinelli and Brida, 2005) models growth of population a generalized logistic equation (Richard's law). In this paper, the population growth rate is assumed to be a function of per capita resource extraction. A specific functional form or relationship between population growth rate and the extraction of the renewable natural resources is not assumed in order to allow for different relationships evident in the empirical literature. However, the final good is assumed to be produced from the renewable resources and the labor so that the effect of resource extraction rate per capita on population growth is, in fact, reflecting the effect of output per capita.

In contrast to the model in this paper, the overwhelming majority of natural resource-growth models under OLG framework assume that population size is constant (see among others, Mourmouras, 1991; Farmer, 2000; Krautkramer and Raymond, 1999) or grows at a given rate (see among others, Mourmouras, 1993; Valente, 2008; Kemp and Van Long, 1979). Under linear regeneration of renewable resources and with exogenous or no technological progress, the models converge to a single steady state or a single balanced growth path with the saddle path dynamics (see Mourmouras, 1991). Allowing for logistic regeneration, Farmer (2000) is the first to show the existence of the steady states and the saddle path dynamics depending on the assumptions with respect to the parameters of the utility, production and regeneration functions. Koskela et al. (2002) employing quasilinear utility and logistic regeneration functions shows that the long run dynamics exhibit saddle path stability. Koskela et al. (2008) examines, whether renewable resource based OLG economies may have other types of dynamics than saddles. They have numerically shown that flip bifurcations may arise if the intertemporal elasticity of substitution of the utility function is less than one half and the regeneration function is logistic.

Compared to these integrating the endogenized population growth rate function into the a standard model of renewable resource based OLG economy with logarithmic preferences, multiple steady states, indeterminacy and bifurcations are obtained, without taking logistic regeneration or assuming that intertemporal elasticity of substitution is less than one half, even in the absence of shocks. Transcritical bifurcations, as well as local indeterminacy may arise in the model varying the rate of constant regeneration with respect to population growth rate. In addition, for the nonzero steady state of the economy, it is shown that if the elasticity of the growth rate of the population with respect to the extraction rate is positive and lower than some critical value –in other words inelastic enough– the long run dynamics are saddle path stable and unstable otherwise. Thus, it is shown that richer dynamics can occur under weaker conditions than previous studies if the feedback mechanism between population growth rate and natural resource extraction is taken into account.

The paper is structured as follows. The model is introduced in the following section. The equilibrium dynamics and the local stability of the system is analyzed in Section . Section concludes.

5.1 The Model

A two-period OLG model in discrete time with an infinite horizon is taken. The setting differs from the standard framework in two respects. First, the renewable resources are considered to be essential to production and they can act as stores of values. Second, under the presence of limited natural resources, the growth rate of the population is allowed to change over time. In fact, the effects of endogenizing population growth by setting population growth rate as a function of per capita extraction rate of the natural resources are analyzed.

At each period t, a generation of agents appears and lives for two periods, young and old. The population in period t, consists of N_t young and N_{t-1} old individuals. The rate of population growth $(1 + n(x_t))$ is assumed to be related with the natural resource x_t, used in production. However, the functional form of $n(x_t)$ is left unspecified in order not to limit the model with specific forms.

$$\mathrm{N}_{t+1} = (1 + \mathrm{n}(\mathrm{x}_t))\mathrm{N}_t. \tag{1}$$

The economy is initially endowed with a positive amount of the natural resource E_0 which belongs to the first generation of old agents. It is assumed that at the beginning of each period t, the old agents (generation $t - 1$) own the stock of the natural resource, E_t. Incurring no extraction costs (see Dasgupta and Heal (1979)), old agents decide on how much of this resource will be extracted for production X_t and how much would be sold to the young (generation t) as assets $A_t(= E_t - X_t)$, in line with Valente (2008). From period t to $t + 1$, the assets bought by the young regenerates at a rate $\Pi \geqslant 1$. Therefore, the natural resource accumulates as a result of both the natural regeneration[1] and the depletion for production which can be formalized as follows:

[1]Note that if $\Pi = 1$, the resource turns out to be renewable.

$$E_{t+1} = \Pi A_t, \quad (2)$$

$$e_t = a_t + x_t, \quad (3)$$

$$(1 + n(x_t))e_{t+1} = \Pi (e_t - x_t), \quad (4)$$

where quantities of resource assets and extracted resources per young individual are denoted by, $e_t = \frac{E_t}{N_t}$, $a_t = \frac{A_t}{N_t}$ and $x_t = \frac{X_t}{N_t}$, respectively.

At a given date, each agent is endowed with one unit of labor when she is young and supplies it to firms inelastically. Young households receives a wage w_t, which is allocated between consumption of the good produced by the representative firm and the purchase of the ownership rights for the natural resource. When old, they consume their entire income generated by selling their stock of natural resources to the firms from the price P_t and to the young from the price Q_t. It is assumed that the life-time well-being of the representative individual is measured by the logarithmic function over young and old periods consumption, i.e., $U(c,d) = u(c) + \beta u(d)$, where $\beta \in (0,1)$ is the subjective discount factor. Accordingly, the representative agent born in period t, maximizes his utility with respect to the young and old periods' consumption, taking wages and the price of the natural resource as given.

$$\max_{\{c_t, d_{t+1}, a_t\}} \ln c_t + \beta \ln d_{t+1}$$

subject to

$$c_t + Q_t a_t = w_t, \tag{5}$$

$$d_{t+1} = P_{t+1}(1 + n(x_t))x_{t+1} + Q_{t+1}(1 + n(x_t))a_{t+1}, \tag{6}$$

$$(1 + n(x_t))e_{t+1} = \Pi a_t, \tag{7}$$

$$a_t = e_t - x_t, \tag{8}$$

$$c_t \geq 0, d_{t+1} \geq 0,$$

$$e_{t+1} \geq 0, E_0 \ > 0, \text{ given.} \tag{9}$$

The first order conditions for an interior solution of the maximization problem of the representative household is as follows:

$$\frac{d_{t+1}}{\beta c_t} = \Pi \frac{Q_{t+1}}{Q_t}, \tag{10}$$

$$P_{t+1} = Q_{t+1}. \tag{11}$$

Equation (10) guarantees the equalization of the intertemporal marginal rate of substitution and the change in prices taking the regeneration factor into account, whereas the latter condition is the no-arbitrage condition.

Firms are owned by the old households and produce a homogenous consumption good under perfect competition. At each period, a single final good Y_t is produced in the economy by means of labor N_t and the natural resource X_t according to the following technology:

$$Y_t = {X_t}^{\alpha} N_t^{1-\alpha}, 0 < \alpha < 1. \tag{12}$$

Under the perfectly competitive environment, the representative firm producing at period t maximizes its profit by choosing the amount of labor and the resource input that will be utilized in the production process:

$$\max_{\{N_t, X_t\}} \pi_t = \mathrm{X}_t^{\alpha} \mathrm{N}_t^{1-\alpha} - \mathrm{w}_t \mathrm{N}_t - \mathrm{P}_t \mathrm{X}_t \tag{13}$$

At an interior solution of the firm's optimization problem, where all variables are expressed in per capita terms ($y_t = \frac{Y_t}{N_t}$), profit maximization implies :

$$(1-\alpha)y_t = w_t, \tag{14}$$

$$\alpha y_t = P_t x_t. \tag{15}$$

Intertemporal equilibrium requires the clearing of the resource market, the clearing of the labor market and the clearing of the goods market for all t:

$$(1+n(x_t))e_{t+1} = \Pi\left(e_t - x_t\right), \tag{16}$$

$$L_t = N_t, \tag{17}$$

$$y_t = c_t + d_t(1+n(x_{t-1}))^{-1}. \tag{18}$$

5.2. Equilibrium Dynamics

As Mourmouras (1991), Farmer(2000) ans Bednar–Friedl and Farmer (2011) suggest, the intertemporal equilibrium dynamics can be reduced into a two–dimensional system which represents the law of motions of e_t and x_t.

From equations (5), (6), (10), (11), (14) and (18), :

$$c_t = \frac{w_t}{(1+\beta)} = \frac{(1-\alpha)y_t}{(1+\beta)}, \tag{19}$$

$$d_{t+1} = \frac{(1+n(x_t))(\alpha+\beta)}{(1+\beta)}y_{t+1}. \tag{20}$$

Plugging equations (19) and (20) into (10), the law of motion for the resource stock follows:

$\mathrm{x}_{t+1} = \Pi\frac{\beta(1-\alpha)}{(1+n(x_t))(\alpha+\beta)}\mathrm{x}_t.$

The second equation of motion, is the dynamics of the natural resource stock:

$(1+\mathrm{n}(\mathrm{x}_t))\mathrm{e}_{t+1} = \Pi\left(e_t - x_t\right).$

The economy is governed by the following dynamics:

$$x_{t+1} = \frac{\Pi\beta(1-\alpha)}{(1+n(x_t))(\alpha+\beta)}x_t, \tag{21}$$

$$e_{t+1} = -\frac{\Pi x_t}{(1+n(x_t))} + \frac{\Pi e_t}{(1+n(x_t))}. \tag{22}$$

Lemma 1 *(Steady States) The steady states of equations (21) and (22) are characterized by the following steady state equations:*

$$x\left(\frac{\Pi\beta(1-\alpha)}{(1+n(x))(\alpha+\beta)} - 1\right) = 0, \tag{23}$$

$$\left(\frac{\Pi}{(1+n(x))} - 1\right)e = \frac{\Pi}{(1+n(x))}x. \tag{24}$$

These equations have two sets of steady states: One of the steady states is

$$x = 0,$$
$$e = 0.$$

The other steady state is the solution of the following equations:

$$1+n(x) = \frac{\Pi\beta(1-\alpha)}{\alpha+\beta},$$
$$\frac{\Pi x}{1+n(x)} = \left(\frac{\Pi}{1+n(x)} - 1\right)e,$$

or,equivalently, for the second line,

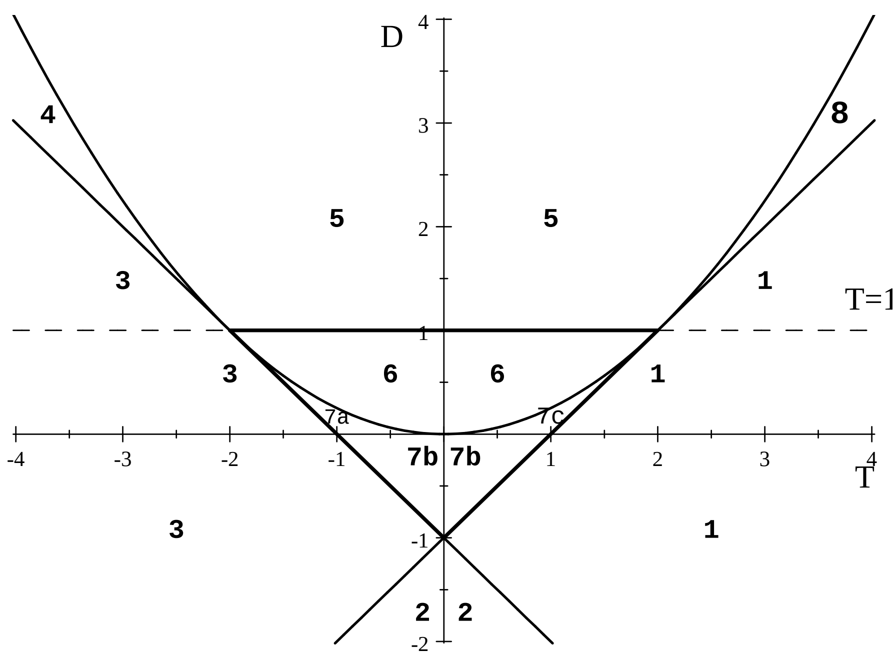
$$e = \frac{(\alpha+\beta)}{\alpha(1-\beta)}x.$$

Proof. Obvious. ■

There may be more than one solution depending on the structure of $n(x)$. The only assumption is that there are finitely many such solutions. The first steady state is named as the zero steady state and the other as the nonzero steady state.

5.3. Stability

To analyze the stability properties of the steady states, the discrete dynamical system of equations (21) and (22) is linearized around these states. The dynamics can be characterized by the elements of the Jacobian matrix, as well as its determinant and trace. Such a characterization can be found in below figure (Figure 2, Chapter 4).



In Figure 2, regions 1 and 3 represent saddle, regions 5 and 6 represent eigenvalues, regions 6 and 7 represent stable solutions, regions 2, 4, 5 and 8 represent unstable solutions.

The linearization around the zero steady state can be represented by the following Jacobian matrix[2]:

$$J \equiv \begin{bmatrix} \left(\frac{\Pi\beta(1-\alpha)}{(1+n(0))(\alpha+\beta)}\right) & 0 \\ \cdot & \frac{\Pi}{1+n(0)} \end{bmatrix}. \tag{25}$$

[2]Since $\left.\frac{\partial x_t}{\partial e_t}\right|_{(x,e)} = 0$, $\frac{\partial e_t}{\partial x_t}$ does not have any effect on eigenvalues. That is why it is left uncomputed.

The two real eigenvalues are

$$\lambda_1 = \frac{\beta(1-\alpha)}{(\alpha+\beta)} \frac{\Pi}{(1+n(0))}, \ \lambda_2 = \frac{\Pi}{1+n(0)}.$$

Proposition 2 *For different parameter combinations, the stability of the zero steady state changes such that,*

Parameter	*Description*
α	*share of resources in final good production* $\alpha \in (0,1)$
β	*discount factor* $\beta \in (0,1)$
Π	*regeneration rate* $\Pi \geq 1$
n	*population growth rate,* $n \in (-1,1)$

i. *If* $\frac{\Pi}{(1+n(0))} < 1$, *both eigenvalues (in absolute value) are smaller than one, so that local indeterminacy occurs;*

ii. *If* $1 < \frac{\Pi}{(1+n(0))} < \frac{\alpha+\beta}{\beta(1-\alpha)}$, *eigenvalues (in absolute value) are on the different side of one, then the steady state is a saddle point stable;*

iii. *If* $1 < \frac{\Pi}{(1+n(0))}$ *and* $\frac{\Pi}{(1+n(0))} > \frac{\alpha+\beta}{\beta(1-\alpha)}$, *eigenvalues (in absolute value) are on the same side of one, both eigenvalues are greater than one so that the equilibrium dynamics are monotone unstable.*

Proposition 3 *Suppose* $n(0)$ *is given. A transcritical bifurcation occurs, for* $\Pi = 1 + n(0))$.

Proof. (of Propositions 2 and 3)The determinant, trace, $p(1)$ and $p(-1)$ are as follows (Denote $\psi = \frac{\Pi}{(1+n(0))}$ and $\theta = \frac{\alpha+\beta}{\beta(1-\alpha)} > 1$):

$$\begin{aligned}
D &= \det J = \left(\frac{\Pi}{1+n(0)}\right)^2 \frac{\beta(1-\alpha)}{(\alpha+\beta)} = \frac{1}{\theta}\psi^2 \in [0,1), \\
T &= trJ = \left(\frac{\Pi}{1+n(0)}\right)\left(\frac{\beta(1-\alpha)}{(\alpha+\beta)}+1\right) = \psi\left(\frac{1}{\theta}+1\right), \\
p(1) &= \left(\frac{\Pi}{1+n(0)}\right)^2 \frac{\beta(1-\alpha)}{(\alpha+\beta)} - \left(\frac{\Pi}{1+n(0)}\right)\left(\frac{\beta(1-\alpha)}{(\alpha+\beta)}+1\right)+1 \\
&= \frac{1}{\theta}\psi^2 - \left(\frac{1}{\theta}+1\right)\psi + 1, \\
p(-1) &= \left(\frac{\Pi}{1+n(0)}\right)^2 \frac{\beta(1-\alpha)}{(\alpha+\beta)} + \left(\frac{\Pi}{1+n(0)}\right)\left(\frac{\beta(1-\alpha)}{(\alpha+\beta)}+1\right)+1 \\
& \frac{1}{\theta}\psi^2 + \left(\frac{1}{\theta}+1\right)\psi + 1, \\
\triangle &= (trJ)^2 - 4\det J \\
&= \left(\frac{\Pi}{1+n(0)}\right)^2 \left(\frac{\beta(1-\alpha)}{(\alpha+\beta)}-1\right)^2 > 0.
\end{aligned}$$

Note that $p(1)$ can be considered as a polynomial with respect to ψ, in which the coefficient of ψ^2 is $\frac{1}{\theta} > 0$ and the roots are $\{1, \theta\}$. Similarly, $p(-1)$ can also be considered as a polynomial with respect to ψ, in which the coefficient of ψ^2 is $\frac{1}{\theta} > 0$ and the roots are $\{-\theta, -1\}$. Table 1 gives the sign of $p(1)$ and $p(-1)$ for different $\psi = \frac{\Pi}{(1+n(0))}$.

[Insert Table 1 in Appendix B here].

Table 2 summarizes the stability regions[3] for the zero steady state:

[Insert Table 2 in Appendix B here].

■

Proposition 2 states that local indeterminacy can occur if the constant regeneration rate is smaller than the population growth rate at the steady state in absolute value. In the proof above, it is shown that if $(1 + n(0)) < \Pi$, then the steady state

[3] For the abbr., SpU: Spiral Unstable; Sa: Saddle; MS: Monotone Stable; FB: Flip Bifurcation; TB: Transcritical Bifurcation.

is monotone stable. This equilibria is represented by Region 7c in Figure 2, where the graphical representation of dynamic equilibria in a planar system is replicated (see, Azeriadis (1993)). The axes are the trace and the determinant of the Jacobian. In addition, it is proved that if $1 < \frac{\Pi}{(1+n(0))} < \frac{\alpha+\beta}{\beta(1-\alpha)}$, then the steady state is saddle which is represented by Region 1. Therefore, a small change in the magnitude of the regeneration factor can cause a stable economy to become saddle defining a transcritical bifurcation. In a similar fashion, it is easy to check that the transcritical bifurcation value of Π is $(1 + n(0))$, where a small change in the magnitude of the regeneration factor can move the economy from Region 7c to Region 1. The results are summarized in Figure 3.

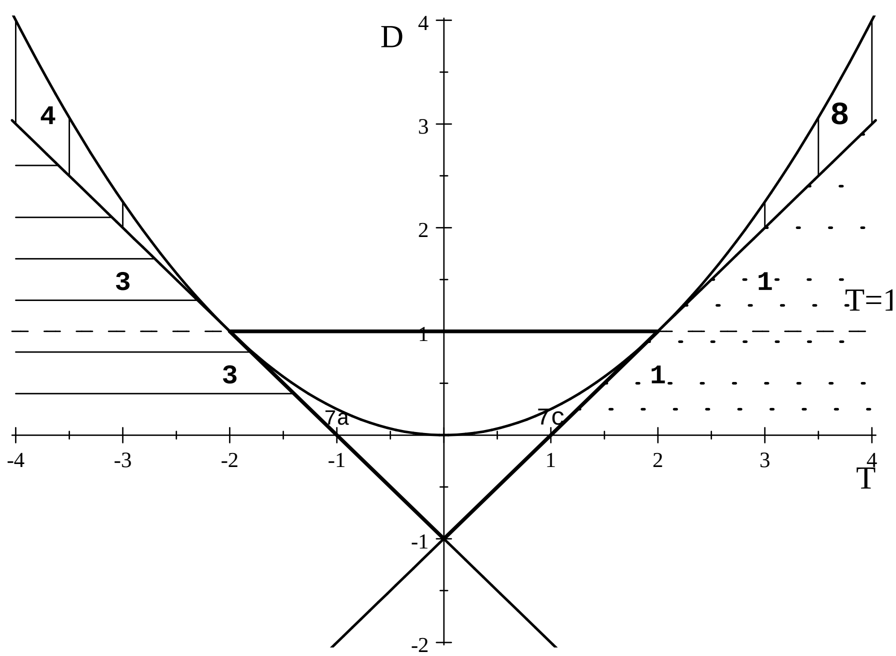


Figure 3: Asymptotic Stability (For the zero-steady state)

Propositions 2 and 3 establishes that transictional bifurcation or local indeterminacy can occur. In contrast to previous studies, theresults are independent from the intertemporal elasticity of substitution of the utility function and without assuming the regeneration function to be logistic (for a comparison, see (**?**)).

The linearization around the nonzero steady state can be represented by the following Jacobian matrix:

$$
\begin{aligned}
J &\equiv \begin{bmatrix} \frac{\Pi\beta(1-\alpha)}{\alpha+\beta}\left(\frac{1+n(x)-xn'(x)}{(1+n(x))^2}\right) & 0 \\ \cdot & \frac{\Pi}{1+n(x)} \end{bmatrix}, \\
&= \begin{bmatrix} \left(1-\frac{xn'(x)}{(1+n(x))}\right) & 0 \\ \cdot & \frac{\Pi}{1+n(x)} \end{bmatrix}.
\end{aligned}
$$

The two real eigenvalues are

$$
\begin{aligned}
\lambda_1 &= \left(1-\frac{xn'(x)}{(1+n(x))}\right), \\
\lambda_2 &= \frac{\Pi}{1+n(x)} = \frac{\alpha+\beta}{\beta(1-\alpha)} > 1.
\end{aligned}
$$

Define the elasticity of the growth rate of the population $(1+n(x))$ with respect to the extraction rate of the resource x at the steady state as

$$
\varepsilon := \varepsilon_{(n(x)+1),x}\Big|_x = \frac{xn'(x)}{(1+n(x))}.
$$

The eigenvalues can be recast as

$$
\begin{aligned}
\lambda_1 &= (1-\varepsilon), \text{ and} \\
\lambda_2 &= \frac{\Pi}{1+n(x)} = \frac{\alpha+\beta}{\beta(1-\alpha)} > 1.
\end{aligned}
$$

The next proposition proves that if the elasticity of the growth rate of the population with respect to the extraction rate is positive but lower than some critical value, then the long run dynamics are saddle path stable and unstable otherwise.

Proposition 4 *For the nonzero steady state, $\lambda_2 > 1$ is already known. Then,*

Parameter	*Description*
ε	*weigheted elasticity of the growth rate of the population with respect to the extr*

i. *If $0 < \varepsilon < 2$, then $\lambda_1 \in (-1, 1)$, saddle path stability occurs,*

ii. *If $\varepsilon > 2$ or $\varepsilon < 0$, then $\lambda_1 < -1$, unstable equilibrium occurs.*

Proof. The determinant, trace, $p(1)$ and $p(-1)$ are as follows

$$\begin{aligned}
\det J &= (1-\varepsilon)\frac{\Pi}{1+n(x)}, \\
trJ &= 1-\varepsilon+\frac{\Pi}{1+n(x)}, \\
p(1) &= \left(1-\frac{\Pi}{1+n(x)}\right)\varepsilon, \\
p(-1) &= (2-\varepsilon)\left(\frac{\Pi}{1+n(x)}+1\right), \\
\triangle &= (trJ)^2-4\det J \\
&= \left(1-\varepsilon-\frac{\Pi}{1+n(x)}\right)^2.
\end{aligned}$$

The stability possibilities[4] with respect to λ_1 are summarized in Table 3:

[Insert Table 3 here].

■

The stability chart with respect to different parameter combinations is shown in

[4]The non-existence of bifurcations is expected, since $\lambda_2 > 1$ for any parameter combination.

Figure 4.

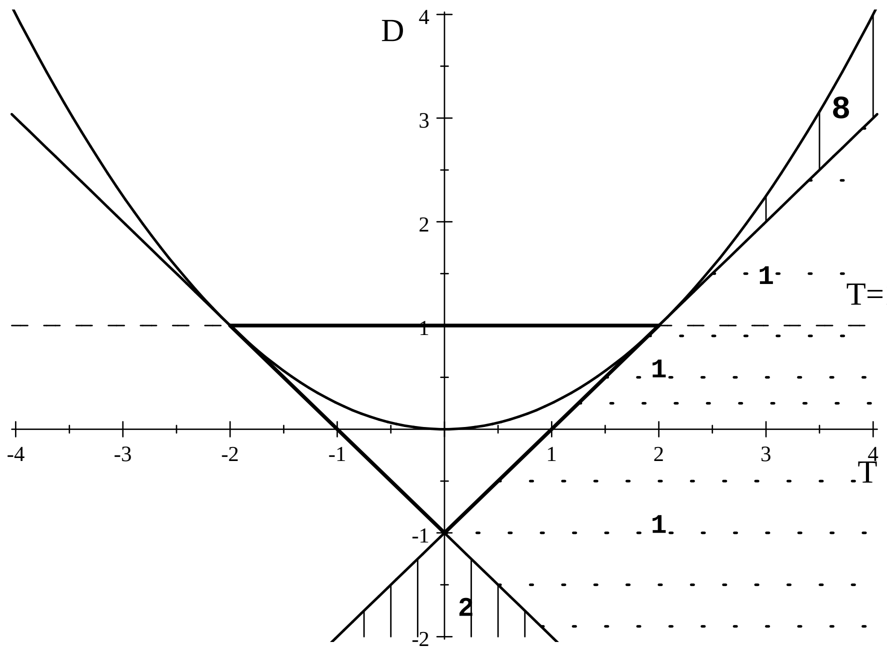


Figure 4: Asymptotic Stability for the Nonzero-Steady State.

5.3. Conclusion

The standard model of renewable resource based OLG economy through a feedback mechanism between population and natural resource is extended to check the stability of the standard results in the area. The population growth rate is assumed to be function of per capita resource extraction, yet a specific functional form or relationship is not taken in order to be as general as possible.

Integrating the endogenized population growth rate function into a standard model of renewable resource based OLG economy with logarithmic preferences, multiple steady states and indeterminacy are obtained, without taking nonlinearizing assumptions common in the literature. In particular, flip and transcritical bifurcations, as well as local indeterminacy may arise in themodel depending on the constant regeneration rate with respect to population growth rate. The elasticity of the growth rate of population with respect to the extraction rate gains special importance for the stability of the the long run dynamics. Thus, it is shown that richer dynamics can occur under weaker conditions than previous studies if the feedback mechanism between population growth rate and natural resource extraction is taken into account.

Last but not the least, it is worthwhile to point out here that the linear regeneration specification in themodel provokes the question of how the stability of the system changes under the non-linear regeneration case. Allowing the renewable resource to regenerate non-linearly (for instance logistically) could bring about even more dynamics. This is on the research agenda.

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APPENDICES

A. PROOF OF PROPOSITION 4 IN CHAPTER 4

To prove this proposition first, some claims will be proved:

Claim 1 *For the non-renewable resources, the discriminant* $\Delta = (1+\psi_1)^2 - 4(\psi_1 + \psi_2)$ *can not be zero. Moreover,*

$$\Delta > 0 \Longleftrightarrow 2\left(1-2\theta\right) - \tilde{\beta} < \frac{1}{\tilde{\beta}} \quad \textit{and} \quad \Delta < 0 \Longleftrightarrow 2\left(1-2\theta\right) - \tilde{\beta} > \frac{1}{\tilde{\beta}}. \qquad (1)$$

Proof. Suppose not, let $\Delta = 0$. Then, $(1+\psi_1)^2 = 4(\psi_1 + \psi_2)$. Since $\psi_1 = \frac{\tilde{\beta}+\theta}{1-\theta}$ and $\psi_2 = -\frac{\theta}{(1-\theta)}$, this condition implies $\left(\frac{1+\tilde{\beta}}{1-\theta}\right)^2 = \frac{4\tilde{\beta}}{1-\theta}$. This holds iff $\tilde{\beta} = (1-2\theta) \pm \sqrt{(1-2\theta)^2 - 1}$. As $(1-2\theta)^2 < 1$; $\tilde{\beta}$ will be complex. This can not be true as $\tilde{\beta}$ is real. If $\Delta > 0$, then, $\left(\frac{1+\tilde{\beta}}{1-\theta}\right)^2 > \frac{4\tilde{\beta}}{1-\theta}$. This equation reduces to $2\left(1-2\theta\right) - \tilde{\beta} < \frac{1}{\tilde{\beta}}$. ■

Claim 2 *Whether the dynamics are complex or not depends on the parameter combinations such that:*

1. *If* $\frac{1}{2} \leq \theta < 1$ *or* $\left[0 < \theta < \frac{1}{2} \text{ and } \tilde{\beta} > 2(1-2\theta)\right]$ *then* $\Delta > 0$, *the dynamics are non-complex.*

2. *If* $\frac{1}{4} \leq \theta < \frac{1}{2}$ *and* $\tilde{\beta} < 2(1-2\theta)$, *then* $\Delta > 0$, *the dynamics are non-complex.*

3. *If* $0 < \theta < \frac{1}{4}$ *and* $\tilde{\beta} + \frac{1}{\tilde{\beta}} < 2(1-2\theta)$, *the dynamics are complex.*

Proof of Claim 6.1:

It is clear that if $\frac{1}{2} \leq \theta < 1$ or $\left[0 < \theta < \frac{1}{2} \text{ and } \tilde{\beta} > 2(1-2\theta)\right]$,the left hand side of Equation(1) is negative. As $\tilde{\beta} > 0$, the right hand side is always positive, so that $\Delta > 0$.

Proof of Claim 6.2:Suppose not. Then, for $\frac{1}{4} \leq \theta < \frac{1}{2}$ and $\tilde{\beta} < 2(1-2\theta)$, $\Delta < 0$ so that $2\left(1-2\theta\right) > \frac{1}{\tilde{\beta}} + \tilde{\beta}$. Then since all of the terms are non-negative this implies $2\left(1-2\theta\right) > \frac{1}{\tilde{\beta}}$ and $2\left(1-2\theta\right) > \tilde{\beta}$. This can be rewritten as $2\left(1-2\theta\right) > \tilde{\beta} > \frac{1}{2(1-2\theta)}$. this holds iff $4\left(1-2\theta\right)^2 > 1$. This reduces to $2\left(1-2\theta\right) > 1$ as $(1-2\theta) > 0$. From that , $\frac{1}{4} > \theta$, contradicting with the initial assumption on θ.

Proof of Claim 6.3: Follows from Equation(1).

Claim 3 *The following hold:*

1. If $\frac{1}{2} \leq \theta < 1$, then $1 - \frac{\sqrt{\Delta}}{2} < 0$.

2. If $\frac{1}{2} < \theta < 1$, then $1 + \frac{\sqrt{\Delta}}{2} > \frac{(1+\psi_1)}{2}$.

3. If $\frac{1}{2} \leq \theta < 1$, then $\frac{(1+\psi_1)}{2} > 1 - \frac{\sqrt{\Delta}}{2}$.

4. Suppose $0 < \theta < \frac{1}{2}$ and $\tilde{\beta} > 2(1-2\theta)$. If $\tilde{\beta} > (1-2\theta) - \sqrt{2\theta^2 - 3\theta + 1}$ or $\tilde{\beta} < (1-2\theta) + \sqrt{2\theta^2 - 3\theta + 1}$, then $1 - \frac{\sqrt{\Delta}}{2} < 0$.

   In addition, if $2(1-2\theta) < \tilde{\beta} < (1-2\theta) + \sqrt{2\theta^2 - 3\theta + 1}$then $1 - \frac{\sqrt{\Delta}}{2} > 0$.

5. If $0 < \theta < \frac{1}{2}$ and $\tilde{\beta} > 2(1-2\theta)$ then $\frac{(1+\psi_1)}{2} > 1 - \frac{\sqrt{\Delta}}{2}$.

6. If $0 < \theta < \frac{1}{2}$ and $\tilde{\beta} > 2(1-2\theta)$ then $\frac{(1+\psi_1)}{2} < 1 + \frac{\sqrt{\Delta}}{2}$.

7. If $\frac{1}{4} \leq \theta < \frac{1}{2}$ and $\tilde{\beta} < 2(1-2\theta)$, then $1 - \frac{\sqrt{\Delta}}{2} > 0$.

8. If $\frac{1}{4} \leq \theta < \frac{1}{2}$ and $\tilde{\beta} < 2(1-2\theta)$, then $\frac{(1+\psi_1)}{2} < 1 + \frac{\sqrt{\Delta}}{2}$.

9. If $\frac{1}{4} \leq \theta < \frac{1}{2}$ and $\tilde{\beta} < 2(1-2\theta)$, then $\frac{(1+\psi_1)}{2} > 1 - \frac{\sqrt{\Delta}}{2}$.

Proof. Proof of Claim 7.1: Let us first show that $1 - \frac{\sqrt{\Delta}}{2} \neq 0$. Suppose the contrary that $1 = \frac{\sqrt{\Delta}}{2}$.Then, $\tilde{\beta} = (1-2\theta) \pm 2\sqrt{2\theta^2 - 3\theta + 1}$. For $\tilde{\beta}$ to be real, $2\theta^2 - 3\theta + 1 \geq 0$. This inequality is binding if and only if $\theta = 0$ or $\theta = 1$. For $\theta = \frac{1}{2}$, , $\tilde{\beta} = 0$. Thus, since $\frac{1}{2} < \theta < 1$, $2\theta^2 - 3\theta + 1 \neq 0$. If $2\theta^2 - 3\theta + 1 > 0$, then rewriting $3 > \frac{1}{\theta} + 2\theta$. As $\frac{1}{2} \leq \theta < 1 \Rightarrow 3 < \frac{1}{\theta} + 2\theta$, contradicting with $\tilde{\beta}$ to be real. In order to show $1 - \frac{\sqrt{\Delta}}{2} < 0$, assume not. Then, for this equality to hold, $4(1-\theta)^2 - 1 > \tilde{\beta}\left[\tilde{\beta} - 2(1-2\theta)\right]$. However, for $\frac{1}{2} \leq \theta < 1$, while RHS of the equality is negative LHS is positive. So, there is a contradiction.

Proof of Claim 7.2: $1 + \frac{\sqrt{\Delta}}{2} > \frac{(1+\psi_1)}{2}$ is equivalent to $(\psi_1 - 1)^2 < \Delta$. Suppose that $\frac{1}{2} < \theta < 1$. If $\frac{1}{2} < \theta < 1$, then $-1 < (1-2\theta) < 0$. This implies $(1-2\theta)^2 < 1$ so that $\tilde{\beta}^2 - 2\tilde{\beta}(1-2\theta) + (1-2\theta)^2 < \tilde{\beta}^2 - 2\tilde{\beta}(1-2\theta) + 1$ so that $(\psi_1 - 1)^2 < \Delta$.

Proof of Claim 7.3: Suppose not. Then $\frac{(1+\psi_1)}{2} < 1 - \frac{\sqrt{\Delta}}{2}$ which is equivalent to $\psi_1 - 1 < -\sqrt{\Delta}$. LHS of this equality is positive with $\left[\frac{\tilde{\beta}-(1-2\theta)}{1-\theta}\right] > 0$ as $(1-2\theta) \leq 0$. Yet, the RHS is negative, yielding a contradiction.

Proof of Claim 7.4: If $1 - \frac{\sqrt{\Delta}}{2} > 0$ then $4(1-\theta)^2 - 1 > \tilde{\beta}\left[\tilde{\beta} - 2(1-2\theta)\right]$. This reduces an equation of a parabola $f(\tilde{\beta}) := \tilde{\beta}^2 - 2\tilde{\beta}(1-2\theta) + 1 - 4(1-\theta)^2$. Thus, $1 - \frac{\sqrt{\Delta}}{2} > 0$ iff $f(\tilde{\beta}) < 0$.This completes the proof.

Proof of Claim 7.5 **:**Suppose not. Then $\frac{(1+\psi_1)}{2} < 1 - \frac{\sqrt{\Delta}}{2}$ which is equivalent to $\psi_1 - 1 < -\sqrt{\Delta}$. LHS of this equality is positive with $\left[\frac{\tilde{\beta}-(1-2\theta)}{1-\theta}\right] > 0$ as $2(1-2\theta) < \tilde{\beta}$. Yet, the RHS is negative, yielding a contradiction.

Proof of Claim 7.6 **:**$1 + \frac{\sqrt{\Delta}}{2} > \frac{(1+\psi_1)}{2}$ is equivalent to $(\psi_1 - 1)^2 < \Delta$. Suppose that $0 < \theta < \frac{1}{2}$. If $0 < \theta < \frac{1}{2}$, then $0 < (1-2\theta) < 1$. This implies $(1-2\theta)^2 < 1$ so that $\tilde{\beta}^2 - 2\tilde{\beta}(1-2\theta) + (1-2\theta)^2 < \tilde{\beta}^2 - 2\tilde{\beta}(1-2\theta) + 1$ so that $(\psi_1 - 1)^2 < \Delta$.

Proof of Claim 7.7 **:**Let us first show that $1 - \frac{\sqrt{\Delta}}{2} \neq 0$. Suppose the contrary that $1 = \frac{\sqrt{\Delta}}{2}$.Then, $\tilde{\beta} = (1-2\theta) \pm 2\sqrt{2\theta^2 - 3\theta + 1}$. For $\tilde{\beta}$ to be real, $2\theta^2 - 3\theta + 1 \geq 0$.Since $\frac{1}{4} \leq \theta < \frac{1}{2}$, $2\theta^2 - 3\theta + 1 \neq 0$. If $2\theta^2 - 3\theta + 1 > 0$, then rewriting $3 > \frac{1}{\theta} + 2\theta$. As $\frac{1}{4} \leq \theta < \frac{1}{2}, \Rightarrow 3 > \frac{1}{\theta} + 2\theta$, thus $\tilde{\beta} = (1-2\theta) \pm 2\sqrt{2\theta^2 - 3\theta + 1}$ can hold. As $0 < \tilde{\beta}$, $0 < (1-2\theta) - 2\sqrt{2\theta^2 - 3\theta + 1} \Leftrightarrow \theta > \frac{1}{2}$. Therefore, $\tilde{\beta} \neq (1-2\theta) - 2\sqrt{2\theta^2 - 3\theta + 1}$.In addition,as $\tilde{\beta} < 2(1-2\theta)$, $(1-2\theta) - 2\sqrt{2\theta^2 - 3\theta + 1} < 2(1-2\theta) \Leftrightarrow \theta > \frac{1}{2}$. So, $1 - \frac{\sqrt{\Delta}}{2} \neq 0$.In order to show $1 - \frac{\sqrt{\Delta}}{2} > 0$, assume not. Then, for this equality to hold, $4(1-\theta)^2 - 1 < \tilde{\beta}\left[\tilde{\beta} - 2(1-2\theta)\right]$. However, for $\frac{1}{4} \leq \theta < \frac{1}{2}$,and $\tilde{\beta} < 2(1-2\theta)$ while RHS of the equality is negative LHS is positive. So there is a a contradiction.

Proof of Claim 7.8: Same proof with Case (6) above.

Proof of Claim 7.9: Suppose not. Then $\frac{(1+\psi_1)}{2} < 1 - \frac{\sqrt{\Delta}}{2}$ which is equivalent to

$\psi_1 - 1 < -\sqrt{\Delta}$. If $\tilde{\beta} \geq 1 - 2\theta$ i.e. $\psi_1 \geq 1$, LHS of this equality is positive with $\left[\frac{\tilde{\beta}-(1-2\theta)}{1-\theta}\right] > 0$. Yet, the RHS is negative, yielding a contradiction. If $\tilde{\beta} < 1 - 2\theta$, $\psi_1 - 1 < 0$. Then, $(\psi_1 - 1)^2 > \Delta$ yielding a contradiction with $\frac{1}{4} \leq \theta < \frac{1}{2}$. ■

B. TABLES IN CHAPTER 5

In this Appendix, the tables in Chapter 5 are presented.

Table 1: The sign of $p(1)$ and $p(-1)$ for varying $\psi = \frac{\Pi}{(1+n(0))}$.

$\psi:$	$<-\theta$	$=-\theta$	$\in(-\theta,-1)$	$=-1$	$\in(-1,0)$	$=0$	$\in(0,1)$	$=1$	$\in(1,\theta)$	$=\theta$	$>\theta$
$p(1):$	>0	>0	>0	>0	>0	>0	>0	$=0$	<0	$=0$	>0
$p(-1):$	>0	$=0$	<0	$=0$	>0	>0	>0	>0	>0	>0	>0

Table 2: Determination of regions for different $\psi = \frac{\Pi}{(1+n(0))}$.

ψ :	$< -\theta$	$= -\theta$	$\in (-\theta, -1)$	$= -1$	$\in (-1, 0)$	$= 0$	$\in (0, 1)$	$= 1$	$\in (1, \theta)$	$= \theta$	$> \theta$
D :	$> \theta$	$= \theta$	$\in (\frac{1}{\theta}, \theta)$	$= \frac{1}{\theta}$	$\in (0, \frac{1}{\theta})$	$= 0$	$\in (0, \frac{1}{\theta})$	$= \frac{1}{\theta}$	$\in (\frac{1}{\theta}, \theta)$	$= \theta$	$> \theta$
T :	< -2	< -2	$-$	$\in (-1, -2)$	$-$	$= 0$	$-$	$\in (1, 2)$	$-$	> 2	> 2
$p(1)$:	> 0	> 0	> 0	> 0	> 0	> 0	> 0	$= 0$	< 0	$= 0$	> 0
$p(-1)$:	> 0	$= 0$	< 0	$= 0$	> 0	> 0	> 0	> 0	> 0	> 0	> 0
$\triangle$:	> 0	> 0	> 0	> 0	> 0	> 0	> 0	> 0	> 0	> 0	> 0
region:	4	$\bar{4} \cap \bar{3}$	3	$\bar{3} \cap \overline{7a}$	$7a$	$\overline{7a} \cap \overline{7c}$	$7c$	$\bar{1} \cap \overline{7c}$	1	$\bar{1} \cap \bar{8}$	8
	SpU		Sa	FB	MS		MS	TB	Sa		MU

Table 3: Determination of regions for different ε.

λ_1 :	$\lambda_1 < -1$	$\lambda_1 = -1$	$-1 < \lambda_1 < 0$	$\lambda_1 = 0$	$0 < \lambda_1 < 1$	$\lambda_1 = 1$	$1 < \lambda_1$
ε :	$2 < \varepsilon$	$\varepsilon = 2$	$1 < \varepsilon < 2$	$\varepsilon = 1$	$0 < \varepsilon < 1$	$\varepsilon = 0$	$\varepsilon < 0$
$\det J$:	< -1	$-\frac{\Pi}{1+n(x)} < -1$	< 0	$= 0$	$(1-\varepsilon)\frac{\Pi}{1+n(x)} > 0$	> 1	> 1
trJ :	$1-\varepsilon+\frac{\Pi}{1+n(x)}$	$\frac{\Pi}{1+n(x)} - 1 > 0$	> 0	> 1	$1-\varepsilon+\frac{\Pi}{1+n(x)} > 1$	> 2	> 2
$p(1)$:	< 0	< 0	< 0	< 0	< 0	$= 0$	> 0
$p(-1)$:	< 0	$= 0$	> 0	> 0	> 0	> 0	> 0
$\triangle$:	> 0	> 0	> 0	> 0	> 0	> 0	≥ 0
region :	2	$\bar{1} \cap \bar{2}$	1	1	1	$\bar{1} \cap \bar{8}$	8
	MU		Sa	Sa	Sa		MU