FIRST PRICE AUCTIONS UNDER PROSPECT THEORY WITH LINEAR PROBABILITY WEIGHTING

A Master's Thesis

by
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The Graduate School of Economics and Social Sciences
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by

KERİM KESKİN

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

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Overbidding in first-price sealed-bid auctions is a well-known result in the auction theory literature. For the possible reasons behind this phenomenon, economists provided many explanations; such as risk aversion, regret theory, and subjective probability weighting. However, for subjective probability weighting to explain overbidding, the probability weighting function (PWF) is needed to be underweighting all probabilities. Such a function is not in accord with PWFs in the prospect theory literature as it suggests a specific function which satisfies certain properties. In this paper we investigate, to what extent prospect theory is able to explain overbidding by using a linear PWF satisfying all of the axiomatic properties (Currim and Sarin, 1989). Moreover, we introduce a non-zero reference point, fully utilizing prospect theory. Our results show that, subjective probability weighting alone is unable to explain overbidding. However, with the non-zero reference point assumption, we obtain partial overbidding.

Keywords: First-price auctions, Subjective probability weighting, Prospect theory, Reference point, Overbidding

ÖZET

DOĞRUSAL OLASILIK AYARLAMALI BEKLENTİ KURAMI VARSAYIMLARI ALTINDA BİRİNCİL FİYAT İHALELERİ

KESKİN, Kerim

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Yard. Doç Dr. Tarık Kara

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Birincil fiyat kapalı zarf ihalelerinde gözlemlenen aşırı fiyat verme, ihale kuramı literatüründe bilinen bir sonuçtur. Ekonomistlerin, bu sonucun arkasında yatan gerekçeler için getirdikleri açıklamalardan bazıları; riskten kaçınma, pişmanlık kuramı ve subjektif olasılık ayarlamasıdır. Fakat, subjektif olasılık ayarlamasının, bu durumu açıklayabilmesi için olasılık ayarlama fonksiyonu (OAF) her olasılığı küçümsemelidir. Bu tip bir fonksiyon ise, beklenti kuramı literatüründeki öneriler ile uyuşmamaktadır. Çünkü, beklenti kuramında OAFnun belli özellikleri sağlaması önerilmektedir (Currim and Sarin, 1989). Bu çalışmada biz, doğrusal bir OAF kullanarak, beklenti kuramının aşırı fiyat vermeyi ne ölçüde açıklayabildiğini araştırıyoruz. Ayrıca, pozitif referans noktası öneriyor, ve böylece beklenti kuramını tam anlamıyla kullanmış oluyoruz. Sonuçlarımız, sadece subjektif olasılık ayarlaması altında, düşük fiyat vermeyi göstermektedir. Fakat, pozitif referans noktası varsayımı altında bazı oyuncuların aşırı fiyat verdiği görülmüştür.

Anahtar Kelimeler: Birincil fiyat ihalesi, Subjektif olasılık ayarlaması, Beklenti kuramı, Referans noktası, Aşırı fiyat verme.

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CHAPTER 1

INTRODUCTION

As Krishna (2002) mentions there are many strong results in the current auction theory literature: (i) Revenue-equivalence principle states that any standard auction[1] yields the same ex-ante revenue to the seller. (ii) In second-price sealed-bid auctions, telling the truth is a dominant strategy for each bidder. (iii) Dutch auctions and first-price sealed-bid auctions are strategically equivalent. However, some of these theoretical findings are not in accord with the experimental studies. Arguably, the most interesting discordance is overbidding in first-price auctions. To be more clear, bids observed in experimental studies are greater than the risk neutral Nash equilibrium (RNNE) (Cox et al., 1988; Kagel and Levin, 2008).

In the literature, many studies try to explain overbidding in first-price auctions. Although risk aversion seems enough to explain overbidding, it requires bidders to be excessively risk averse. Thus, it is argued that risk aversion cannot be the only factor behind bidding above the RNNE (Kagel and Roth, 1992). To support such an idea; loss aversion (Lange and Ratan, 2010), and regret theory (Filiz-Ozbay and Ozbay, 2007) are provided.[2] Eisen-

[1] An auction is called a standard auction if the bidder who bids the highest amount becomes the winner.

[2] Lange and Ratan (2010) use a multi-dimensional reference dependent model of Kőzsegi and Rabin (2006), and examine both first-price and second-price auctions. Filiz-Ozbay and Ozbay (2007) define two types of regret (winner and loser), and conclude that loser regret dominates winner regret which leads to overbidding.

huth (2010) also utilizes loss aversion and looks for an efficient mechanism.[3] As an alternative explanation to overbidding, there are some studies that use prospect theory (Kahneman and Tversky, 1979). Goeree et al. (2002) examine auctions with subjective probability weighting, and use a probability weighting function (PWF) which is suggested by Prelec (1998). Ratan (2009) also uses the same PWF together with a multi-dimensional reference dependent model. Armantier and Treich (2009a) state that any star-shaped PWF is able to explain overbidding, and they relate this result to the experimental study (Armantier and Treich, 2009b).

In this paper, we introduce prospect theory into the first-price auction framework, and try to answer the question: to what extent, prospect theory is able to explain overbidding in first-price auctions. We use a linear PWF together and assume that gains are evaluated with respect to an exogenous reference point. Our results suggest that, with linear probability weighting there always exists a bidder who bids less than the RNNE. Besides each bidder underbids for some values of the variables, concluding that prospect theory fails to explain overbidding. Thus, we impose restrictions for these variables so that most of the bidders overbid. With this method, we verify overbidding for bidders with high valuation, thus manage to partially explain overbidding.[4]

In addition, we suggest two different probability weighting methods which are motivated from Metzger and Rieger (2009). As bidders calculate their winning probabilities by compounding the probabilities of events, the following two methods emerge: weighting probabilities *after compounding* and *before compounding*. We show that, former method is less desirable as it suggests unreasonable bidding behavior. With both of these methods, with zero reference point, linear probability weighting causes underbidding. Intro-

[3]The findings of Eisenhuth (2010) suggest that bidders who values the object more overbids, whereas the other bidders bid less than RNNE.

[4]Thus, our result is similar to the results of Eisenhuth (2010).

ducing a positive reference point is not sufficient to explain overbidding as well. However, we claim that imposing more assumptions, such as assigning a lower bound for the reference point, will form a proper model. Moreover, such restrictions are not effective under weighting after compounding method.

Our paper is structured as follows: Chapter 2 includes some aspects of prospect theory, and some literature review about subjective probability weighting in auctions. In chapter 3, we examine the first-price sealed-bid auctions under prospect theory; in section 3.1, we have risk neutral bidders who weights after compounding, whereas, in section 3.2, risk neutral bidders who weights before compounding. In chapter 4, we provide a discussion about using a non-linear probability weighting function. Finally, chapter 5 concludes.

CHAPTER 2

PROSPECT THEORY

Prospect theory suggests the use of a pair of functions to represent the preference relation of an agent, a value function and a probability weighting function (PWF).[1] In this chapter, we define the functions we used throughout this paper.

To our knowledge, Armantier and Treich (2009a) and Ratan (2009) are the only theoretical studies which use subjective probability weighting in auctions. Ratan (2009) uses one of the PWFs suggested by Prelec (1998), which is defined as $w : [0,1] \to [0,1]$ as: for any $p \in [0,1]$,

$$w(p) = \exp(-\beta(-\ln p)^{\alpha}) \ .$$

However, Ratan (2009) assumes $\alpha = 1$ which yields $w(p) = p^{\beta}$, and claims that $\beta > 1$. He is motivated by Goeree et al. (2002) who estimate similar values for α and β. Note that, for these values of α and β, every probability is being underweighted, so PWF does not satisfy overweighting property. This fact is emphasized by Goeree et al. (2002) as well. Moreover, Armantier and Treich (2009b) suggest star-shaped PWFs which also underweight each probability. We, however, define a linear PWF satisfying all of the axiomatic properties of PWFs mentioned in Currim and Sarin (1989). Thus, we define

[1] One can find the axiomatic properties of these functions in Currim and Sarin (1989).

$w : [0,1] \to [0,1]$ as: for any $p \in [0,1]$,

$$w(p) = \begin{cases} 0 & \text{, if} \quad p = 0 \\ \mu p + \eta & \text{, if} \quad p \in (0,1) \\ 1 & \text{, if} \quad p = 1 \end{cases} \tag{2.1}$$

Here, in order to satisfy the aforementioned properties of PWFs, we further assume that η and μ are both positive such that $2\eta + \mu < 1$.

In subjective probability weighting, the type of weighting is also important as PWF itself. We use two different weighting methods: *weighting after compounding* and *weighting before compounding.* The following example demonstrates the motivation behind using two different methods.

Example 1: When we flip a coin, each outcome is likely to happen with probability $\frac{1}{2}$. We suggest a lottery such that; we flip the coin 2 times and the lottery yields 1$ if both outcomes are heads, and nothing otherwise. Now, the probability of winning is $\frac{1}{4}$. However, if we consider subjective probability weighting, there are two ways to weight. Either weighting $\frac{1}{4}$, i.e. $w(\frac{1}{4})$; or weighting $\frac{1}{2}$'s first, i.e. $w(\frac{1}{2})^2$.

In our setting, former method corresponds to weighting after compounding. In the latter method, the probabilities of other agents having lower than a certain valuation are weighted first, and then the weighted probabilities are multiplied to obtain the winning probability.[2] At the end, we suggest weighting before compounding as the desirable method in explaining overbidding in first-price auctions.

We also define the value function $V : \mathbb{R} \to \mathbb{R}$, which is suggested by Tversky and Kahneman (1992). Thus, for any $x \in \mathbb{R}$:

$$V(x) = \begin{cases} x^\theta & , \quad x \geq 0 \\ -\lambda(-x)^\theta & , \quad x < 0 \end{cases} \tag{2.2}$$

[2] I.e. bidders weight before compounding.

where $\lambda > 1$, and $\theta \in (0, 1]$. Finally, we assume that reference point is determined exogenously. We first consider zero reference point. By doing so, we will be able to capture the sole effect of linear probability weighting on the equilibrium of first-price auctions. Then, by introducing a positive reference point we analyse first-price auctions under prospect theory. We motivate using a positive reference point by the following example from Metzger and Rieger (2009).

Example 2: Let $\varepsilon > 0$. For lottery A, define the outcomes as $a_i = 1 - i \cdot \varepsilon$, and the probabilities as $\delta_i = \frac{1}{n}$ for any $i \in \{1, ..., n\}$; and let B be the sure lottery with outcome 1. Then, obviously, lottery B first order stochastically dominates lottery A. However, since small probabilities are assumed to be overweighted, for high enough n and small enough ε, an agent with prospect theoretic preferences may prefer lottery A to lottery B.

Preferring lottery A over lottery B does sound irrational, as rationality would not imply preferring a lottery that obviously yields less than 1 over a lottery that yield 1 for sure. Yet, of course, such an irrationality would vanish if there was a proper reference point.

Many studies support that an agent's reference point should be equal to her expectation. Using the fact that, in a first price auction, one should be expecting a positive gain as they are willingly participating the auction, we suggest a positive reference point, r. Notice that, such an assumption eliminates the irrational behavior caused by zero reference point assumption illustrated in Example 2.

CHAPTER 3

FIRST PRICE AUCTIONS

There is a single object to be sold. There are n bidders in the player set N, and each bidder $i \in N$ assigns a monetary value for the object, which is denoted by v_i. The valuation v_i is only known to bidder i. Also, each bidder knows that the valuation of the other bidders are identically and independently distributed according to a cumulative distribution function F. We assume that F is a uniform distribution over $(0, 1)$.[1] Finally, bidders choose their strategies simultaneously.

In first-price auctions; the bidder with the highest bid wins the auction, gets the object, and pays her bid. Throughout the paper, we assume that tie in bids will be broken randomly. In this setting, any outcome can be represented as a lottery. Each bidder $i \in N$ with a valuation v_i faces the lottery space, L^{v_i}, which is induced by $l^{v_i} : [0, \infty) \times \prod_{j \in N \setminus \{i\}} B_j \to (-\infty, v_i] \times [0, 1]$. Here, B_j is the set of all increasing functions[2], $\beta_j : (0, 1) \to [0, \infty)$. Given any strategy profile of competitors, β, we have $l^{v_i}(b, \beta) = (g, p)$ where $g = v_i - b$ and $p = \prod_{j \in N \setminus \{i\}} F(\beta_j^{-1}(b))$. In this context, g denotes the gain and p denotes the winning probability[3].

[1] In the equilibrium analysis, we consider any distribution over (0,1). Then we use uniform distribution and present our findings.

[2] We eliminate non-increasing functions as it is plausible to assume that one would bid higher as her valuation increases.

[3] With the remaining probability $1 - p$ the outcome of the lottery is 0.

As a prospect theoretic approach, bidders weight the probabilities, and evaluate gains of the lottery relative to their reference point via value function.

Throughout the paper, we search for symmetric equilibria and use an equilibrium analysis similar to the ones in the current literature. In order to do that, one should calculate the expected payoffs of the bidders, so we define their reference points first. For now, we make a simplifying assumption that the reference point of each bidder is zero. Moreover, as we search for an explanation for overbidding, we focus on risk neutral bidders[4]. Thus, we assume that $\theta = 1$ in (2.2).

3.1 Weighting After Compounding

In auctions, each bidder calculates her winning probability by compounding the probabilities that every other bidder bids less than her bid. When subjective probability weighting is introduced, weighting the winning probability is an option as we have discussed above. In this section, we utilize that method. We denote the probability of winning the auction by a function, $G : (0, 1) \to (0, 1)$, defined by $G = F^{n-1}$ since we consider symmetric equilibrium.

For equilibrium analysis, take any bidder $i \in N$ with valuation v_i. Her expected payoff of bidding $b \in [0, v_i]$,[5] while all other bidders $j \in N \backslash \{i\}$ follow a symmetric, differentiable strategy $\beta \in B_j$ is,

$$w[G(\beta^{-1}(b))](v_i - b) \ .$$

Our analysis yields the following result, as it is shown in the Appendix.

[4]Here, by risk neutrality, we refer to a value function which is linear in both gains and losses frames. In other words, we would be assuming risk neutrality if there was not subjective probability weighting. With a linear PWF, agents risk preference may be different than risk neutrality.

[5]It is straightforward that, bidding any amount higher than own valuation is a dominated strategy as bidding 0 dominates such a strategy. Thus, we do not consider those values of b for the expected payoff, although they are in the strategy set of bidder i.

Proposition 1. *The unique risk neutral symmetric equilibrium in first-price auctions is characterized by,*

$$\beta_A(v_i) = \frac{1}{w(v_i^{n-1})} \int_0^{v_i} y \frac{\partial w(y^{n-1})}{\partial y} dy = v_i - \frac{\int_0^{v_i} w(y^{n-1}) dy}{w(v_i^{n-1})} \tag{3.1}$$

when bidders are assumed to weight probabilities after compounding, and evaluate payoffs relative to zero reference point.

Proof. See Appendix. □

Notice that, assuming $w(p) = p$ would yield the equilibrium function for the expected utility case. In that case, the bidding function becomes $\beta^{RN}(v_i) = \frac{n-1}{n} v_i$.

Now, using the linear PWF, (2.1), yields the unique symmetric equilibrium of our model. Corollary 1, below, states that our setting predicts underbidding at the equilibrium.

Corrolary 1. *In first-price auctions, risk neutral bidders bid,*

$$\beta_A^*(v_i) = \frac{\mu(n-1)v_i^n}{\eta n + \mu n v_i^{n-1}}$$

if they weight probabilities after compounding according to the linear PWF, (2.1), and evaluate payoffs relative to zero reference point. Thus, agents bid less aggressively than RNNE.

Proof. It directly follows that $\beta_A^*(v_i) = \frac{\mu(n-1)v_i^n}{\eta n + \mu n v_i^{n-1}}$. Since $\mu > 0$, for any $v_i \in (0,1)$, $\beta_A^*(v_i)$ decreases in η. If $\eta = 0$, then $\beta_A^*(v_i) = \frac{n-1}{n} v_i = \beta^{RN}(v_i)$. As we assume $\eta > 0$, we have $\beta_A^*(v_i) < \beta^{RN}(v_i)$ for any $v_i \in (0,1)$. Thus we conclude that any bidder bids less aggressively. □

With this result, we deduce that weighting after compounding method predicts a bidding strategy less than the RNNE. Thus, overbidding in first-price auctions cannot be explained by this model.

Then, we introduce the other notion, namely reference point, of prospect theory in our setting, and assume that $r > 0$. Hence, we have solved the problem presented in Example 2, however, there may emerge a new problem: Consider a bidder i with valuation $v_i < r$. No matter how small we take the reference point, such a bidder may exist. But now, how can someone expect to win more than her valuation in a first-price auction? This argument implies that reference point cannot be constant in valuations. Thus we suggest a function $r : (0, 1) \to (0, 1)$ such that $r(v)$ gives the reference point of a bidder with valuation v. We define r such that for any $v \in (0, 1)$:

$$r(v) = \varphi \frac{1}{n} v \ . \tag{3.2}$$

where $\varphi \in (0, 1)$. Notice that, the reference point is assumed to be increasing in valuation of the bidder and decreasing in the number of competitors.

For equilibrium analysis, take any bidder $i \in N$ with valuation v_i, again. Assuming that any bidder $j \in N \backslash \{i\}$ bids according to a symmetric, differentiable strategy $\beta \in B_j$, expected payoff of bidder i from bidding $b \in [0, v_i]$ is,[6]

$$w[G(\beta^{-1}(b))]\lambda_1(v_i - b - r(v_i)) - w[1 - G(\beta^{-1}(b)]\lambda r(v_i) \ ,$$

where λ_1 equals to 1 if $b \leq v_i - r(v_i)$, and λ if otherwise. Under these assumptions, we have the following proposition.

Proposition 2. *The unique risk neutral symmetric equilibrium in first-price auctions is characterized by,*

$$\beta_A^+(v_i) = \begin{cases} \left(1 + \frac{\varphi(\lambda-1)}{n}\right)\beta_A(v_i) & , \ if \quad \frac{v_i}{\beta_A(v_i)} > 1 + \frac{\varphi\lambda}{n-\varphi} \\ v_i - \frac{\varphi}{n} v_i & , \ if \quad o/w \end{cases} \tag{3.3}$$

when bidders are assumed to weight probabilities after compounding, and eval-

[6]It is still a dominated strategy to bid higher than own valuation. Under these assumptions, a bidder gains a negative amount from winning if she bids higher than $v_i - r(v_i)$.

uate payoffs relative to positive reference point in (3.2).

Proof. See Appendix. □

Corollary 2 uses the linear PWF, (2.1), and states the equilibrium bidding function under aforementioned assumptions. Moreover, the failure in explaining overbidding is presented.

Corrolary 2. *In first-price auctions, risk neutral bidders bid,*

$$\beta_A^{+,*}(v_i) = \begin{cases} \left(1 + \frac{\varphi(\lambda-1)}{n}\right)\beta_A^*(v_i) & , \textit{ if } \ \frac{v_i}{\beta_A^*(v_i)} > 1 + \frac{\varphi\lambda}{n-\varphi} \\ v_i - \frac{\varphi}{n}v_i & , \textit{ if } \ o/w \end{cases}$$

if they weight probabilities after compounding according to the linear PWF, (2.1), and evaluate payoffs relative to positive reference point in (3.2). Thus, agents bid more aggressively than those with zero reference point. Besides, for any λ, η and μ; there is $k^* \in \mathbb{R}$ such that when $\frac{\varphi}{n} \leq k^*$, any bidder bids less aggressively than the RNNE.

Proof. It is trivial that, linear PWF, (2.1), returns the suggested $\beta_A^{+,*}(v_i)$. As $(1 - \frac{\varphi}{n})v_i > \beta^{RN}(v_i)$, $\varphi > 0$, $n > 0$, and $\lambda > 1$; $\beta_A^{+,*}(v_i) > \beta_A^*(v_i)$.

For the existence of $k^* \in \mathbb{R}$, we consider the case where $\frac{v_i}{\beta_A^*(v_i)} > 1 + \frac{\varphi\lambda}{n-\varphi}$, and simply solve $\beta_A^{+,*}(v_i) < \beta^{RN}(v_i)$ for v_i. This inequality holds if $\frac{\eta}{\mu v_i^{n-1}} > \frac{\varphi(\lambda-1)}{n}$. Since $v_i < 1$, we also have $\frac{\eta}{\mu v_i^{n-1}} > \frac{\eta}{\mu}$. Now, fix λ. If $\frac{\varphi(\lambda-1)}{n} \leq \frac{\eta}{\mu}$, that is to say, if $\frac{\varphi}{n} \leq \frac{\eta}{\mu(\lambda-1)} = k^*$; then for any $v_i \in (0,1)$, $\beta_A^{+,*}(v_i) < \beta^{RN}(v_i)$ holds. Thus, under such values of $\frac{\varphi}{n}$, any bidder bids less aggressively than the RNNE.

Note that, k^* is positive by our assumptions on η, μ and λ. Also, notice that, under such φ values, $\beta_B^{+,*}(v_i)$ is less than the RNNE, thus the other case of the equilibrium function is never realized. □

To see the sole effect of positive reference point assumption, consider the case where $\frac{v_i}{\beta_A^*(v_i)} > 1 + \frac{\varphi\lambda}{n-\varphi}$. Notice that, taking the identity function as

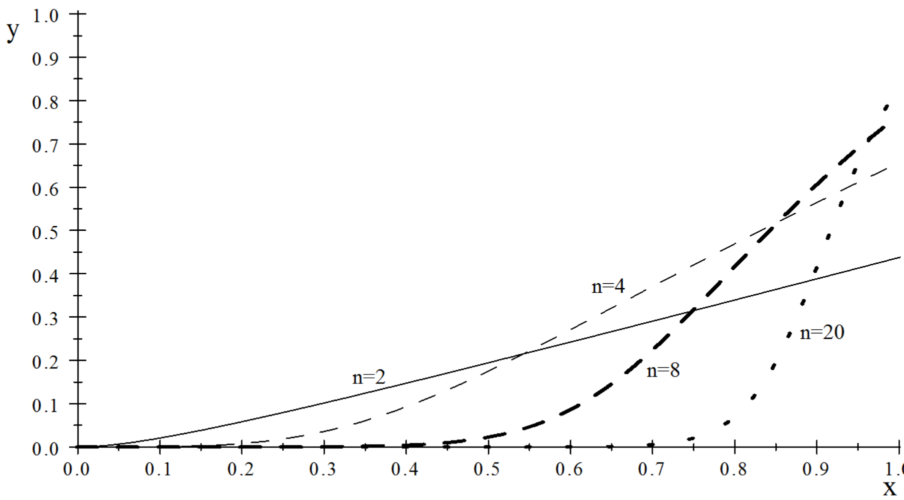

Figure 3.1: Equilibrium strategies, $\beta^*_A(v_i)$, for different values of n

PWF, i.e. taking $w(p) = p$, would make the equilibrium $\left(1 + \frac{\varphi(\lambda-1)}{n}\right) \frac{n-1}{n} v_i$ which is greater than the RNNE.[7] That means, the value function in (2.2) with respect to the reference point in (3.2) is enough to explain overbidding in first-price auctions. Thus, we provide a simple and strong explanation for overbidding. However, as it is shown in Corollary 2, introducing subjective probability weighting causes our model to lose this explanatory power.

Beside the failure in providing the explanation, our model has another problematic result: bids are decreasing in the number of bidders, i.e. more competition leads to less aggressive bidding behavior (See Figure 1). In addition; as the number of bidders increase, the bidders bid closer to 0. This result is shown in Corollary 3 below.

Corrolary 3. *Under the assumption that bidders weight probabilities after compounding, the equilibrium bidding function converges to* 0 *as* n *goes to infinity, for both zero reference point and positive reference point assumptions.*

Proof. Considering $\beta^{+,*}_A(v_i)$ is enough since it equals to $\beta^*_A(v_i)$ when $\varphi = 0$. Since $v_i \in (0, 1)$, it is straightforward that,

$$\lim_{n\to\infty} w(p) = \eta \ .$$

[7] In the other case, $\beta^{+,*}_B(v_i) = v_i - r(v_i)$ which is trivially greater than the RNNE.

Thus,

$$\lim_{n\to\infty}\int_0^{v_i} w(y)dy = \eta v_i \ .$$

Hence, $\lim_{n\to\infty}\beta_A^{+,*}(v_i) = \left(1+\frac{\varphi(\lambda-1)}{n}\right)\left(v_i - \frac{\eta v_i}{\eta}\right) = 0$ for any $v_i \in (0,1)$. $\square$

This result shows that weighting after compounding causes unreasonable bidding behavior, which is also the reason for this model to be unsuccessful in explaining overbidding in first-price auctions. In the following section, we utilize the weighting before compounding method.

3.2 Weighting Before Compounding

In this section, we assume that the bidders weight probabilities before compounding. The idea of weighting is based on agents' realization of the winning probability. It may not be that obvious in an auction setting, but the calculation of the winning probabilities in auctions is similar to the case in Example 1. In our model, latter method corresponds to weighting before compounding which returns $w(F(p))^{n-1}$ as the weighted winning probability, whereas the former method corresponds to weighting after compounding, $w(F(p)^{n-1})$.[8]

As Proposition 3 and its corollary suggest, the bids are still less aggressive than the RNNE. However, more competition leads to higher bid amounts now. To obtain equilibrium, take any bidder $i \in N$, with valuation v_i. The expected payoff of her from bidding $b \in [0, v_i]$ while other bidders, $j \in N\backslash\{i\}$, follow a symmetric, differentiable strategy $\beta \in B_j$ is,

$$w[F(\beta^{-1}(b)]^{n-1}(v_i - b) \ .$$

Proposition 3. *The unique risk neutral symmetric equilibrium in first-price*

[8]Notice that, these two methods coincide when $n = 2$.

auctions is characterized by,

$$\beta_B(v_i) = \frac{1}{w(v_i)^{n-1}} \int_0^{v_i} y \frac{\partial w(y)^{n-1}}{\partial y} dy = v_i - \frac{\int_0^{v_i} w(y)^{n-1} dy}{w(v_i)^{n-1}} \tag{3.4}$$

when bidders are assumed to weight probabilities before compounding, and evaluate payoffs relative to zero reference point.

Proof. See Appendix. □

Corollary 4, below, states that bids are less aggressive than the RNNE if the PWF is linear as defined in (2.1).

Corrolary 4. *In first-price auctions, risk neutral bidders bid,*

$$\beta_B^*(v_i) = v_i - \frac{\eta + \mu v_i}{\mu n} + \frac{\eta^n}{\mu n(\eta + \mu v_i)^{n-1}}$$

if they weight probabilities before compounding according to the linear PWF, (2.1), and evaluate payoffs relative to zero reference point. Thus, agents bid less aggressively than the RNNE. Besides, more competition leads to more aggressive bidding.

Proof. If we put (2.1) in the equation, we get $\beta_B^*(v_i) = v_i - \frac{\eta+\mu v_i}{\mu n} + \frac{\eta^n}{\mu n(\eta+\mu v_i)^{n-1}} = \frac{n-1}{n} v_i - \frac{\eta}{\mu n} + \frac{\eta^n}{\mu n(\eta+\mu v_i)^{n-1}} = \frac{n-1}{n} v_i - \frac{\eta}{\mu n} + \frac{\eta}{\mu n}\left(\frac{\eta}{\eta+\mu v_i}\right)^{n-1}$. Since η and μ are both positive, $\beta_B^*(v_i) < \frac{n-1}{n} v_i = \beta^{RN}(v_i)$ follows.

Now, we prove the claim that more competition leads to more aggressive bidding. Take any $n \in \mathbb{N}$. We will show that,

$$v_i - \frac{\eta + \mu v_i}{\mu n} + \frac{\eta^n}{\mu n(\eta + \mu v_i)^{n-1}} < v_i - \frac{\eta + \mu v_i}{\mu(n+1)} + \frac{\eta^{n+1}}{\mu(n+1)(\eta + \mu v_i)^n}$$

so that, for any $v_i \in (0, 1)$, as the number of bidders increases, so does $\beta_B^*(v_i)$. Above inequality simplifies into $\frac{\eta^n - (\eta+\mu v_i)^n}{n} < \frac{\eta^{n+1} - (\eta+\mu v_i)^{n+1}}{(n+1)(\eta+\mu v_i)}$, which is $(\eta + \mu v_i)^{n+1} > \eta^{n+1} + \eta^n(n+1)\mu v_i$. Noting that η and μ are both positive, and n is a natural number; this inequality always holds since the RHS is the first two terms of the binomial expansion of the LHS. Thus, more competition leads to more aggressive bidding. □

For our next result, our assumptions is similar to those we have used in the previous section: bidders weight probabilities according to the linear PWF, (2.1), and they have positive reference points, (3.2). Then, take any bidder $i \in N$ with v_i, again. Now, her 'weighted' probability of losing from bidding $b \in [0, v_i]$ is,

$$\sum_{k=0}^{n-2} \left(\frac{(n-1)!}{k!(n-k-1)!} w[1 - F(\beta^{-1}(b))]^{n-k-1} w[F(\beta^{-1}(b))]^k \right) .$$

where any bidder $j \in N \backslash \{i\}$ follows a symmetric, differentiable strategy $\beta \in B_j$. Since it is a part of a binomial expansion, we can write the expected payoff of bidder i as,

$$\begin{aligned} & w[F(\beta^{-1}(b)]^{n-1} \lambda_1 (v_i - b - r(v_i)) \\ & \qquad -\lambda r(v_i) \left(\left(w[1 - F(\beta^{-1}(b))] + w[F(\beta^{-1}(b))] \right)^{n-1} - w[F(\beta^{-1}(b)]^{n-1} \right) . \end{aligned}$$

where λ_1 equals to 1 if $b \leq v_i - r(v_i)$, and λ if otherwise.

Proposition 4. *The unique risk neutral symmetric equilibrium in first-price auctions is characterized by,*

$$\beta_B^+(v_i) = \begin{cases} \left(1 + \frac{\varphi(\lambda-1)}{n}\right) \beta_B(v_i) & , \ \frac{v_i}{\beta_B(v_i)} > 1 + \frac{\varphi\lambda}{n-\varphi} \\ v_i - \frac{\varphi}{n} v_i & , \ o/w \end{cases} \tag{3.5}$$

when bidders are assumed to weight probabilities before compounding, and evaluate payoffs relative to positive reference point in (3.2).

Proof. See Appendix. □

According to the next corollary, there is a real number k^* such that any bidder bids less aggressively if $\varphi \leq k^*$.

Corrolary 5. *In first-price auctions, risk neutral bidders bid,*

$$\beta_B^{+,*}(v_i) = \begin{cases} \left(1+\frac{\varphi(\lambda-1)}{n}\right)\beta_B^*(v_i) & , \quad \frac{v_i}{\beta_B^*(v_i)} > 1+\frac{\varphi\lambda}{n-\varphi} \\ v_i - \frac{\varphi}{n}v_i & , \quad o/w \end{cases}$$

if they weight probabilities before compounding according to the linear PWF, (2.1), and evaluate payoffs relative to positive reference point in (3.2). Thus, agents bid more aggressively than those with zero reference point. Besides, for any values of λ, η, μ and n; there is $k^* \in \mathbb{R}$ such that when $\varphi \leq k^*$, any bidder bids less aggressively than the RNNE.

Proof. As it is also utilized in the proof of Corollary 2, $\frac{\varphi(\lambda-1)}{n} > 0$, so that $\beta_B^{+,*}(v_i) > \beta_B^*(v_i)$.

For the existence of $k^* \in \mathbb{R}$, we consider the case where $\frac{v_i}{\beta_A^*(v_i)} > 1+\frac{\varphi\lambda}{n-\varphi}$, and simply solve $\beta_B^{+,*}(v_i) < \beta^{RN}(v_i)$ for v_i. This inequality holds if

$$\varphi(\lambda-1) < \frac{v_i}{\beta_B^*(v_i)}\left(\frac{\eta}{\mu v_i} - \frac{\eta^n}{\mu v_i(\eta+\mu vi)^{n-1}}\right) \quad .$$

We have $\beta_B^*(v_i) < v_i$, so the RHS is always greater than $\left(\frac{\eta}{\mu v_i} - \frac{\eta^n}{\mu v_i(\eta+\mu vi)^{n-1}}\right)$, which is decreasing in v_i. Thus, we also have

$$\frac{\eta}{\mu v_i} - \frac{\eta^n}{\mu v_i(\eta+\mu vi)^{n-1}} > \frac{\eta}{\mu} - \frac{\eta^n}{\mu(\eta+\mu)^{n-1}} \quad .$$

Now, we conclude that if

$$\varphi \leq \frac{\eta}{\mu(\lambda-1)}\left(1-\left(\frac{\eta}{\eta+\mu}\right)^{n-1}\right) = k^* \quad ,$$

then bidders bid less than the RNNE. Note that, k^* is positive by our assumptions on η, μ and λ. Also, notice that, under such φ values, $\beta_B^{+,*}(v_i)$ is less than RNNE, thus the other case of the equilibrium function is never realized. □

Although this result implies that overbidding in first-price auctions cannot be explained by our prospect theoretic model, it shows a way to build a proper model.[9] Trivially, if φ is too small, then β_B^+ is too close to β_B. We have showed in Corollary 4 that $\beta_B^*(v_i) < \beta^{RN}(v_i)$ for any valuation. However, imposing further restrictions on φ under which the inequality in the proof of Corollary 5 is never satisfied, completes a proper model that provides a partial explanation for overbidding. In order to do that, we impose a lower bound for φ.

Corrolary 6. *When bidders weight probabilities before compounding according to the linear PWF, (2.1), and evaluate gains with relative to positive reference point in (3.2), any agent with a sufficiently low valuation underbids. Moreover, if $\mu > 6\eta$, $\lambda \geq 2$, and $\varphi \in \left(\frac{8}{9}, 1\right)$; any bidder $i \in N$ with a valuation $v_i \in [\frac{1}{2}, 1)$ bids more aggressively than the RNNE.*

Proof. For the existence of an underbidder, we simply take the derivative of $\beta_B^{+,*}(v_i)$ with respect to v_i and evaluate at $v_i = 0$. The result is 0, which is less than $\frac{n-1}{n}$ (the derivative of $\beta^{RN}(v_i)$ with respect to v_i). Since $\beta_B^+(0) = 0 = \beta^{RN}(0)$, we conclude that, $\beta_B^{+,*}(v_i) < \beta^{RN}(v_i)$ for small enough v_i.

For the latter part, we again consider the case, $\frac{v_i}{\beta_A^*(v_i)} > 1 + \frac{\varphi\lambda}{n-\varphi}$, of the bidding function. If the claim holds for this case, it would hold for the other case as well, since $v_i - \frac{\varphi}{n} v_i > \beta^{RN}(v_i)$. Now, we suppose there is some bidder $i \in N$ with valuation $v_i \in [\frac{1}{2}, 1)$ who underbids. Then, $\beta_B^*(v_i) \geq \left(\frac{1}{2} - \frac{\eta}{2\eta+\mu}\right) v_i$ for any values of n. As the maximum value of $\left(1 - \left(\frac{\eta}{\eta+\mu v_i}\right)^{n-1}\right)$ is 1, the inequality in (8) becomes $\varphi(\lambda - 1) \leq \frac{4\eta+2\mu}{\mu} \frac{\eta}{\mu v_i}$. Using the lowest possible values for λ, v_i and μ leads us to our conclusion: $\varphi \leq \frac{8}{9}$, a contradiction to $\varphi \in \left(\frac{8}{9}, 1\right]$. Thus, there is no such bidder. Hence, any bidder $i \in N$ with a valuation $v_i \in [\frac{1}{2}, 1)$ bids more aggressively than the RNNE. $\square$

Now, we focus on the reason for weighting before compounding to be more

[9] We are unable to do so with weighting after compounding method because of the result in Corollary 3.

successful than weighting after compounding. Notice that, under weighting before compounding method, the number of overweighters are not affected by the number of competitors: For any n, the bidders with valuation in $(0, \frac{a}{1-b})$ are the overweighters. However, when we consider weighting after compounding, for $n = 2$, the bidders with valuation in $(0, \frac{a}{1-b})$ are the overweighters whereas for $n = 3$, the bidders with valuation in $(0, \sqrt{\frac{a}{1-b}})$ are the overweighters. More generally, the number of overweighters are increasing in n. In other words, when n is too high, almost every bidder overweights her winning probability. As the shape[10] of the PWF changes with n, so does the behavior of equilibrium bidding function which is remarked by Corollary 3.

Our results show that prospect theoretic approach with a linear PWF fails to explain overbidding in first-price auctions. However, under suggested restrictions, our model is successful in explaining overbidding in first-price auctions, for bidders with high valuations. We also show that, subjective probability weighting with zero reference point suggests that any bidder bids less aggressively than the RNNE, whereas positive reference point without subjective probability weighting leads to overbidding for any bidder. In addition, we suggest that weighting before compounding is more successful in explaining overbidding in first-price auctions.

[10]By shape, we mean the amount of overweighting. Recall that, prospect theory suggests overweighting for small probabilities only. Thus, under weighting after compounding method, we move away from the axiomatic properties of prospect theory as the number of bidders increases.

CHAPTER 4

NON-LINEAR PROBABILITY WEIGHTING

To our knowledge; Goeree et al. (2002), Armantier and Treich (2009a), Armantier and Treich (2009b), and Ratan (2009) are the studies that adapt subjective probability weighting into an auction framework. The estimation of Goeree et al. (2002) suggests a PWF function which underweights any probability as the most successful PWF to explain overbidding. However, as Goeree et al. (2002) argue, their findings are not in accord with the suggested PWF of prospect theory literature. Although, there is such discordance, their suggested PWF is verified by a later experimental study. In the experiment conducted by Armantier and Treich (2009b), subjects are asked to tell their winning probabilities given their valuations, the number of bidders, and the distribution. Armantier and Treich (2009b) observe that subjects' winning probabilities are greater than their answers, and claim that bidders underweight probabilities in a first-price auction setting.

The findings of Goeree et al. (2002) and Armantier and Treich (2009b) also mean that, using a non-linear PWF that satisfies the suggested properties of PWFs[1] (Currim and Sarin, 1989) would be less successful. In fact, our result about the existence of an agent who underbids is also valid under reverse-s

[1] In the current literature, PWFs are suggested to be reverse-s shaped.

shaped PWFs.

From the equilibrium results of this paper, one can deduce that the effect of η is significant. For example, if we violate the overbidding property by choosing a negative η we would obtain overbidding for each bidder and any values of μ. Under the assumption of a reverse-s shaped PWF, right-handed limit of the PWF at 0 equals to 0, thus such an effect would not be observed. On the other hand, the unreasonable bidding behavior under weighting after compounding vanishes as Corollary 3 would not be holding any more. Notice that, this result is also related with above η argument.

With these observations and the results of Goeree et al. (2002), one can claim that overweighting leads to underbidding. Besides, underweighting leads to overbidding unless the underweighters are not affected by the decrease in the bids of overweighters. Such an effect can be too significant, so that each bidder bids less than the RNNE although most of them are underweighters, as we can see in our results.

To sum up, with a reverse-s shaped PWF, one can provide a better explanation for overbidding. However, it would still be a partial explanation since an underbidder always exists.

CHAPTER 5

CONCLUSION

Our paper is motivated by the experimental studies in which overbidding is observed in first-price auctions. As it is mentioned earlier, there are several auction models trying to explain overbidding. In this paper, we use prospect theory with linear probability weighting. For subjective probability weighting, we suggest two different methods. We deduce that weighting before compounding is more successful in providing an explanation, since weighting after compounding leads to unreasonable bidding behavior. To be more clear, bidding function converges to 0 as the number of bidders increases, if weighting after compounding method is used.

We show that, if bidders weight before compounding and consider outcomes with respect to a zero reference point, they bid less aggressively. Introducing a positive reference point also does not work, as too small reference points are not strong enough to increase the bids sufficiently. According to our results, for both reference point assumptions, bidders who weight after compounding bid even less aggressively.

To obtain overbidding, we suggest imposing further assumptions on the reference point. To be more specific, if the reference points of the bidders are sufficiently high, then most of the bidders bid more aggressively than the RNNE. Thus, prospect theory with linear probability weighting provides a partial explanation for overbidding. Moreover, because of the result that there always exists a bidder who underbids, linear probability weighting is unable to provide a stronger explanation.

In addition, we show that value function with a positive reference point is enough to explain overbidding in first-price auctions, if bidders do not weight probabilities subjectively. Thus, although the models with subjective probability weighting mostly fail, we provide a strong and simple explanation for overbidding in first-price auctions.

As a final remark, the literature on prospect theory suggests a non-linear PWF, thus assuming a non-linear PWF would provide a better answer to our question. The weighting methods and reference point assumptions would still be valid under the assumption of a non-linear PWF. Moreover, one can combine the assumptions of this model with other assumptions, such as risk aversion or regret theory, and obtain the collective effect of these assumptions on the equilibrium strategies. After all, Kagel and Roth (1992) may be right: "... risk aversion cannot be the only factor and may not well be the most important factor behind bidding above the RNNE".

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APPENDIX

Proof of Proposition 1. First order condition with respect to b is,

$$\frac{\partial w[G(\beta^{-1}(b))]}{\partial \beta^{-1}(b)} \frac{\partial \beta^{-1}(b)}{\partial b}(v_i - b) - w[G(\beta^{-1}(b))] = 0 \ .$$

Arranging terms yield,

$$\frac{\partial w[G(\beta^{-1}(b))]}{\partial \beta^{-1}(b)} \frac{1}{\beta'(\beta^{-1}(b))}(v_i - b) = w[G(\beta^{-1}(b))] \ .$$

As we assume symmetric equilibrium, $b = \beta(v_i)$ should be the maximizer of the objective function, i.e. would solve the above equality. Thus,

$$\frac{\partial w(G(v_i))}{\partial v_i} \frac{1}{\beta'(v_i)}(v_i - \beta(v_i)) = w(G(v_i)) \ .$$

Arranging terms yield,

$$w(G(v_i))\beta'(v_i) + \frac{\partial w(G(v_i))}{\partial v_i}\beta(v_i) = \frac{\partial w(G(v_i))}{\partial v_i} v_i \ .$$

Then, we obtain,

$$\frac{\partial}{\partial v_i}[w(G(v_i)\beta(v_i)] = v_i \frac{\partial w(G(v_i))}{\partial v_i}$$

which implies,

$$\beta_A(v_i) = \frac{1}{w(G(v_i))} \int_0^{v_i} y \frac{\partial w(G(y))}{\partial y} dy = v_i - \frac{\int_0^{v_i} w(G(y))dy}{w(G(v_i))} \ .$$

Now, β_A is the only candidate for the symmetric equilibrium. It is straightforward that it is less than v_i for any $v_i \in (0,1)$. By differentiating β_A with respect to v_i we conclude that it is increasing in v_i, since

$$1 - \frac{w^2(G(v_i)) - \frac{\partial w(G(v_i))}{\partial v_i} \int_0^{v_i} w(G(y))dy}{w^2(G(v_i))} > 0 \ ,$$

by noting that both the derivative and the integral in the numerator are positive terms. The following argument states that bidding $\beta_A(v_i)$ is a best response for bidder i with valuation v_i given that the others follow β_A as well. Now, suppose that bidder i acts as if her valuation is z. Note that, bidding higher than $\beta_A(1)$ is not a best response, since bidding slightly less yields more. Then her expected payoff from bidding $\beta_A(z) < v_i$ is,

$$EP_A(z) = w(G(z))(v_i - \beta_A(z)) = w(G(z))(v_i - z) + \int_0^z w(G(y))dy \ .$$

Then, $EP_A(v_i) - EP_A(z)$ is,

$$w(G(z))(z - v_i) - \int_{v_i}^z w(G(y))dy \ ,$$

which is,

$$\int_{v_i}^z w(G(z))dy - \int_{v_i}^z w(G(y))dy \ .$$

Since, G is an increasing function, this difference is non-negative for any values of z, concluding that bidding $\beta_A(v_i)$ is a best response. Thus, β_A is the unique symmetric equilibrium. Since we assume that F is a uniform distribution,

$$\beta_A^*(v_i) = v_i - \frac{\int_0^{v_i} w(y^{n-1})dy}{w(v_i^{n-1})} \ .$$

□

Proof of Proposition 2. First order condition with respect to b is,

$$\frac{\partial w[G(\beta^{-1}(b))]}{\partial \beta^{-1}(b)} \frac{\partial \beta^{-1}(b)}{\partial b} \lambda_1 (v_i - b - r(v_i)) \\ - \frac{\partial w[1 - G(\beta^{-1}(b))]}{\partial \beta^{-1}(b)} \frac{\partial \beta^{-1}(b)}{\partial b} \lambda r(v_i)) = \lambda_1 w[G(\beta^{-1}(b))] \ .$$

As we assume symmetric equilibrium, $b = \beta(v_i)$ would solve the above equality. Thus,

$$\frac{\partial w(G(v_i))}{\partial v_i} \frac{1}{\beta'(v_i)} \lambda_1 (v_i - \beta(v_i) - r(v_i)) \\ - \frac{\partial w(1 - G(v_i))}{\partial v_i} \frac{1}{\beta'(v_i)} \lambda r(v_i)) = \lambda_1 w(G(v_i)) \ .$$

Arranging terms yield,

$$\begin{aligned} \frac{\partial w(G(v_i))}{\partial v_i} \lambda_1 (v_i - r(v_i)) - \frac{\partial w(1 - G(v_i))}{\partial v_i} \lambda r(v_i) & \\ = \lambda_1 \left(w(G(v_i))\beta'(v_i) + \frac{\partial w(G(v_i))}{\partial v_i} \beta(v_i) \right) & \\ = \lambda_1 \frac{\partial}{\partial v_i} [w(G(v_i))\beta(v_i)] & \ . \end{aligned}$$

Then, we have,

$$\beta(v_i) = \frac{1}{w(G(v_i))} \left[\int_0^{v_i} y \frac{\partial w(G(y))}{\partial y} dy - \int_0^{v_i} r(y) \frac{\partial w(G(y))}{\partial y} dy \right] \\ - \frac{\lambda}{\lambda_1} \frac{1}{w(G(v_i))} \int_0^{v_i} r(y) \frac{\partial w(1 - G(y))}{\partial y} dy \ ,$$

and this yields below result when $r(y)$ is replaced with $\frac{\varphi}{n} y$ as defined above,

$$\beta(v_i) = \left(1 - \frac{\varphi}{n}\right) \frac{1}{w(G(v_i))} \int_0^{v_i} y \frac{\partial w(G(y))}{\partial y} dy \\ - \frac{\lambda}{\lambda_1} \frac{\varphi}{n} \frac{1}{w(G(v_i))} \int_0^{v_i} y \frac{\partial w(1 - G(y))}{\partial y} dy \ .$$

We, now, check whether the equilibrium bidding function can be greater than or equal to $v_i - r(v_i)$. Suppose that $b \geq v_i - r(v_i)$ implying that $\lambda_1 = \lambda$.[1]

[1] We abuse notation here, to be able to study with a compact set. Although we have defined λ_1 as being equal to 1 when $b = v_i - r(v_i)$, we use $\lambda_1 = \lambda$. Such an action will have no consequence on the result, because when $b = v_i - r(v_i)$, λ_1 will be multiplied by 0 in the expected payoff that we are maximizing.

Then, above equality becomes,

$$\beta(v_i) = \beta_A(v_i) < v_i - \frac{v_i}{n} < v_i - r(v_i) \ ,$$

which concludes that there is no interior solution. Thus, $\beta(v_i) = v_i - r(v_i)$, if we assume that $b \geq v_i - r(v_i)$. Now, suppose that $b \leq v_i - r(v_i)$, aiming to check whether b can be less than or equal to $v_i - r(v_i)$. Then, $\lambda_1 = 1$. Thus,

$$\beta(v_i) = \left(1 + \frac{\varphi(\lambda - 1)}{n}\right) \frac{1}{w(G(v_i))} \int_0^{v_i} y \frac{\partial w(G(y))}{\partial y} dy \ ,$$

which may be greater than $v_i - r(v_i)$, for high values of v_i and λ. For those values, we conclude that there is a corner solution which is $\beta(v_i) = v_i - r(v_i)$. To sum up,

$$\beta_A^+(v_i) = \begin{cases} \left(1 + \frac{\varphi(\lambda-1)}{n}\right) \beta_A(v_i) & , \quad \frac{v_i}{\beta_A(v_i)} > 1 + \frac{\varphi\lambda}{n-\varphi} \\ v_i - \frac{\varphi}{n} v_i & , \quad \text{o/w} \end{cases}$$

First of all, such a function is not differentiable at a breaking point. However, the value of the function at that point is known by continuity so that the function is well-defined. It is straightforward that it is increasing in v_i, and less than v_i for any $v_i \in (0, 1)$ since β_A is. Now, β_A^+ is a candidate for the symmetric equilibrium. Although we do not prove sufficiency, we refer to a theorem proved by Reny (2011): Under certain conditions (First of all, each player's strategy set of monotone pure strategies should be non-empty and join-closed. Then; the type set is required to be partially ordered, the density function should be continuous, the strategy set is needed to be a compact metric space and a semi-lattice with a closed partial order, and the utility function should be bounded and measurable), a symmetric game has a symmetric monotone pure strategy equilibrium. And, it is easy to show that these conditions hold for our model except compact strategy sets and the bounded utility function. Notice that, the strategy sets we have defined are not compact as bidders are allowed to bid as high as they wish. Because of this, the utility function is not bounded as well. However, with a minor modification, these two properties can be satisfied. For instance, we can narrow the strategy set to $[0, 1]$ and nothing would change in the equilibrium results[2]. Since β_A^+ is the unique candidate, it follows that it is the unique symmetric equilibrium. As we assume that F is a uniform distribution,

$$\beta_A^{+,*}(v_i) = \begin{cases} \left(1 + \frac{\varphi(\lambda-1)}{n}\right) \beta_A^*(v_i) & , \quad \frac{v_i}{\beta_A^*(v_i)} > 1 + \frac{\varphi\lambda}{n-\varphi} \\ v_i - \frac{\varphi}{n} v_i & , \quad \text{o/w} \end{cases}$$

$\square$

Proof of Proposition 3. First order condition with respect to b is,

$$(n-1)w(F(\beta^{-1}(b)))^{n-2} \frac{\partial w(F(\beta^{-1}(b)))}{\partial \beta^{-1}(b)} \frac{\partial \beta^{-1}(b)}{\partial b}(v_i - b) = w(F(\beta^{-1}(b)))^{n-1} \ .$$

Arranging terms yield,

$$(n-1)\frac{\partial w(F(\beta^{-1}(b)))}{\partial \beta^{-1}(b)} \frac{1}{\beta'(\beta^{-1}(b))}(v_i - b) = w(F(\beta^{-1}(b))) \ .$$

As we assume symmetric equilibrium, $b = \beta(v_i)$ would solve the above equality. Thus,

$$(n-1)\frac{\partial w(F(v_i))}{\partial v_i} \frac{1}{\beta'(v_i)}(v_i - \beta(v_i)) = w(F(v_i)) \ .$$

Arranging terms yield,

$$\frac{\partial w(F(v_i))}{\partial v_i} v_i = \frac{1}{n-1} w(F(v_i))\beta'(v_i) + \frac{\partial w(F(v_i))}{\partial v_i} \beta(v_i) \ .$$

[2] Recall that, bidding above the valuation is a dominated strategy.

Now, multiply each component of the equality with $(n-1)w(F(v_i))^{n-2}$,

$$w(F(v_i))^{n-1}\beta'(v_i) + (n-1)\frac{\partial w(F(v_i))}{\partial v_i} w(F(v_i))^{n-2}\beta(v_i)$$
$$= (n-1)\frac{\partial w(F(v_i))}{\partial v_i} w(F(v_i))^{n-2} v_i \ .$$

Then, we obtain,

$$\frac{\partial}{\partial v_i}\left(w(F(v_i))^{n-1}\beta(v_i)\right) = v_i \frac{\partial w(F(v_i))^{n-1}}{\partial v_i} \ .$$

The result follows,

$$\beta_B(v_i) = \frac{1}{w(F(v_i))^{n-1}} \int_0^{v_i} y \frac{\partial w(F(y))^{n-1}}{\partial y} dy = v_i - \frac{\int_0^{v_i} w(F(y))^{n-1} dy}{w(F(v_i))^{n-1}} \ .$$

Now, it is again straightforward that β_B is less than v_i for any values of $v_i \in (0,1)$. By differentiating β_B with respect to v_i we conclude that it is increasing in v_i, since

$$1 - \frac{w(F(v_i))^{2(n-1)} - \frac{\partial w(F(v_i))^{n-1}}{\partial v_i} \int_0^{v_i} w(F(y))^{n-1} dy}{w(F(v_i))^{2(n-1)}} > 0 \ ,$$

by noting that both the derivative and the integral in the numerator are positive terms. For the sufficiency part, we use a similar method as in the proof of Proposition 1. Consider any bidder i with valuation v_i who acts as if her valuation is z. Her expected payoff from bidding $\beta_B(z) < v_i$ while her competitors follow β is,

$$EP_B(z) = w(F(z))^{n-1}(v_i - \beta_B(z)) = w(F(z))^{n-1}(v_i - z) + \int_0^z w(F(y))^{n-1} dy \ .$$

Now, $EP_B(v_i) - EP_B(z)$ is,

$$w(F(z))^{n-1}(z - v_i) - \int_z^{v_i} w(F(y))^{n-1} dy \ ,$$

which is not less than 0 by a similar argument. Thus, β_B is the unique symmetric equilibrium. From the assumption that F is a uniform distribution,

$$\beta_B^*(v_i) = v_i - \frac{\int_0^{v_i} w(y)^{n-1} dy}{w(v_i)^{n-1}} \quad .$$

$\square$

Proof of Proposition 4. First of all, the expected payoff simplifies into

$$w(F(\beta^{-1}(b)))^{n-1}\lambda_1(v_i - b - r(v_i)) - \lambda r(v_i)\left(\kappa - w(F(\beta^{-1}(b)))^{n-1}\right) \quad ,$$

where κ is constant as we consider linear PWFs. For example, with the PWF we suggest, $\kappa = 2\eta + \mu$. First order condition with respect to b is,

$$(n-1)w[F(\beta^{-1}(b))]^{n-2}\frac{\partial w[F(\beta^{-1}(b))]}{\partial \beta^{-1}(b)}\frac{\partial \beta^{-1}(b)}{\partial b}\left(\lambda_1(v_i - b) + (\lambda - \lambda_1)r(v_i)\right)$$
$$= \lambda_1 w[F(\beta^{-1}(b))]^{n-1} \quad .$$

The rest will follow similarly, and thus omitted. The equilibrium bidding function is,

$$\beta_B^+(v_i) = \begin{cases} \left(1 + \frac{\varphi(\lambda-1)}{n}\right)\beta_B(v_i) & , \quad \frac{v_i}{\beta_B(v_i)} > 1 + \frac{\varphi\lambda}{n-\varphi} \\ v_i - \frac{\varphi}{n}v_i & , \quad \text{o/w} \end{cases}$$

First of all, such a function is not differentiable at the breaking point. However, the value of the function at that point is known by continuity so that the function is well-defined. It is straightforward that it is increasing in v_i, and less than v_i for any $v_i \in (0,1)$ since β_B is. Now, β_B^+ is a candidate for the symmetric equilibrium. Although we do not prove sufficiency, we refer to a theorem proved by Reny (2011): Under certain conditions, a symmetric game has a symmetric monotone pure strategy equilibrium.[3] And, it is easy to show that these conditions hold for our model. Since β_B^+ is the unique can-

[3]This is the same theorem mentioned in the proof of Proposition 2. Note also that, the modification for the strategy sets is also valid here.

didate, it follows that it is the unique symmetric equilibrium. As we assume that F is a uniform distribution,

$$\beta_B^{+,*}(v_i) = \begin{cases} \left(1 + \frac{\varphi(\lambda-1)}{n}\right) \beta_B^*(v_i) & , \quad \frac{v_i}{\beta_B^*(v_i)} > 1 + \frac{\varphi\lambda}{n-\varphi} \\ v_i - \frac{\varphi}{n} v_i & , \quad \text{o/w} \end{cases}$$

$\square$