MEDIAN RULE AND MAJORITARIAN COMPROMISE

A Master's Thesis

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MEDIAN RULE AND MAJORITARIAN COMPROMISE

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by

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

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In this thesis, we analyze the relationship between Majoritarian Compromise (Sertel & Yılmaz, 1984) and the Median Rule (Basset & Persky, 1999). We show that, for the populations with odd size, these two rules are equivalent and we describe the relationship for the case where population size is even. Then, we explore some axiomatic properties of Median Rule. It turns out that Median Rule satisfies all properties that Majoritarian Compromise satisfies in Sertel and Yılmaz (1999) and it fails all properties that Majoritarian Compromise fails in Sertel and Yılmaz (1999). We, then, introduce two axioms which differentiate these rules. We conclude that, the Median Rule can be considered as a viable alternative to Majoritarian Compromise, as it satisfies all axioms that Majoritarian Compromise is known to satisfy except one particular axiom.

Keywords: Social Choice, Majoritarian Compromise, Median Rule, Subgame Perfect Implementability

ÖZET

ORTANCA KURALI VE ÇOĞUNLUKÇU UZLAŞI

POLAT, Ali Oğuz

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Prof. Semih Koray

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Bu tezde, Çoğunlukçu Uzlaşı (Sertel & Yılmaz, 1984) ile Ortanca Kuralı (Basset & Persky, 1999) arasındaki ilişkiyi inceliyoruz. Tek sayıda topluluklar için, bu iki kuralın birbirine denk olduğunu gösteriyor ve çift sayıda topluluklar için aralarındaki ilişkiyi tarif ediyoruz. Devamla, Ortaca Kuralı'nın bazı aksiyomatik özelliklerini inceliyoruz. Ortanca Kuralı'nın Sertel ve Yılmaz (1999)'da Çoğunlukçu Uzlaşı'nın sağladığı gösterilen her aksiyomu sağladığı, Sertel ve Yılmaz(1999)'da Çoğunlukçu Uzlaşı'nın sağlamadığı gösterilen hiç bir aksiyomu sağlamadığını gösteriyoruz. Daha sonra, iki aksiyom öne sürerek, bu kuralları birbirinden ayırt ediyoruz. Sonuç olarak, Ortanca Kuralı, Çoğunlukçu Uzlaşı'nın bir aksiyom hariç bilinen bütün aksiyomlarını sağladığından, Çoğunlukçu Uzlaşı'ya geçerli bir alternatif olarak değerlendirilebilir.

Anahtar Kelimeler: Sosyal Seçim, Çoğunlukçu Uzlaşı, Ortanca Kuralı, Alt Oyun Yetkin Uygulanabilirliği

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CHAPTER 1

INTRODUCTION

In the literature of Social Choice, how to measure the degree of the support that is attained by an alternative has been one of the main questions that researchers have been trying to answer. In the search for an answer, to endorse a social choice rule, one can argue for some axiomatic properties pertaining the definition of degree of support from a particular perspective. For instance, the well-known plurality rule measures the degree of support to an alternative in terms of sole quantity of individuals considering that alternative to be the best one. However, plurality winner might be an undesired outcome from a different perspective for the degree of support. For example for the cases where every other individual ranks the plurality winner as their worst outcome, a measure of degree of support which respects the dislikes of majority coalitions may not agree with plurality rule. By this motivation, Sertel's (1986) Majoritarian Compromise introduced a new way to measure the degree of support which would focus on the support of majority coalitions. In this perspective, the degree of support granted by a majority coalition to a particular alternative is measured by simply taking the ranking of that alternative by the individual who ranked the alternative worst among the members of that majority coalition. Using this definition for the degree of support for alternatives, Majoritarian Compromise chooses the alternative which gains the maximal support from some majority coalition.

Basset and Persky (1999) introduced[1] another social choice rule that is Median Rule, which is argued to measure the degree of support in a more "robust" way compared to Borda rule. According to them, robustness of a social choice rule is the insensitivity of the social choice rule to the changes in the preferences of the "outliers" whose preferences are quite different than those whose preferences are similar to some majority. This idea of "robustness" is not very different than the majority support idea of Sertel's (1986) Majoritarian Compromise, as both suggested, instead of some "outlier" minorities, focusing on relatively homogeneous majorities, when measuring degree of support.

Both of these two social choice rules are explored in many studies. Majoritarian Compromise has been shown to be Subgame Perfect implementable by Sertel and Yılmaz (1999) along with other minor results. Sanver and Sanver (2003) introduced a new axiom that is efficient compromise, in order to compare Majoritarian Compromise to Borda and Condorcet rule. Merlin et al. (2006) analysed similar types of social choice rules including Basset and Persky's (1999) Median Rule and the Majoritarian Compromise.[2] Giritligil and Sertel (2005) carried out an empirical study for Majoritarian Compromise[3]. The line of research for Basset and Persky's Median Rule is also productive. Gehrlein and Lepelley (2003) demonstrated that although Median Rule without a tie is less manipulable than Borda's rule, it is more manipulable than

[1] Median Rule is not a novel idea, yet as far as we aware there is no earlier work which defines a median rule on the space of linear preferences.

[2] To our knowledge, this is the only paper which acknowledged the similarity between Median Rule and the Majoritarian Compromise, yet they define the Median Rule in a way that Median Rule is the Majoritarian Compromise without tie-breaking rule no matter what the size of the population is.

[3] There are many other papers, demonstrating various properties of Majoritarian Compromise(such as Altuntas (2011)) or studying it in different setups (such as Laffond and Laine (2011))

Condorcet rule as well as Copeland rule. Balinski and Laraki (2007) introduced a tie-breaking rule to Median rule which is, surprisingly, no different than the tie-breaking rule of Majoritarian Compromise. Using an extended framework, which includes the possibility of ranking cardinally, Felsenthal and Machover (2009) studied some properties of this Median Rule . The most striking observation for these two lines of research is that, with one notable exception of Merlin et al. (2006), the studies for these two social choice rules are dealing with completely different properties of these similar rules.

In this study, we will demonstrate that these two social choice rules are not very different when same tie-breaking rules are applied. Actually, we will show that, for odd-numbered sets of individuals, these two rules are equivalent. To study the equivalence more thoroughly, we will introduce a third rule, that is strict Majoritarian Compromise, which is only different than the Majoritarian Compromise in definition of majority coalition. Then we will show that many properties of Majoritarian Compromise also applies for Median Rule which is hardly a surprise for what our equivalence results suggest. Yet, these results have a valuable contribution to literature, since not only this study demonstrates that these two lines of study can be merged but also it gives alternative proofs for the results of for example subgame Nash implementability of Majoritarian Compromise by getting the same results for Median Rule. As for the outline of the paper, next section will deal with basic notions and the following section will introduce above mentioned social choice rules with two alternatives: one of which will also incorporate a tie-breaking rule. In the fourth section, we will prove that the properties that Sertel and Yılmaz (1999) demonstrated for Majoritarian Compromise also holds for Median Rule. Finally the study will conclude with a summary and suggestions for further work.

CHAPTER 2

PRELIMINARIES

Let $N = \{1, 2, ..., n\}$ be a finite set of individuals, $A = \{a_1, a_2, ..., a_m\}$ be a finite set of alternatives and $\mathcal{L}(A)$ denote the set of all linear orderings on A where $R_i \in \mathcal{L}(A)$ is the linear ordering of individual $i \in N$ over A. $\mathcal{L}(A)^N = \prod_{i \in N} \mathcal{L}(A)$ denotes the set of all linear orderings for the set of individuals N over the alternative set A where $R \in \mathcal{L}(A)^N$ is said to be a preference profile for the individual set N over alternative set A. $L_i(a, R) = \{b \in A | aR_ib\}$ is the lower contour set of the alternative a with respect to R_i and $U_i(a, R) = \{b \in A | bR_ia\}$ is the upper contour set of the alternative a with respect to R_i. We define the degree $\pi_i(a, R)$ of alternative a in the individual profile R_i such that $\pi_i(a, R) = card(L_i(a, R))$. A social choice rule (SCR) is a function $F : \mathcal{L}(A)^N \to 2^A$.

For every $R \in \mathcal{L}(A)^{N_1}$ and for every $R' \in \mathcal{L}(A)^{N_2}$ where $N_1 \cap N_2 = \emptyset$ define $R + R' \in \mathcal{L}(A)^{N_1 \cup N_2}$ such that for all $i \in N_1$ for all $a \in A$ we have $L_i(a, R + R') = L_i(a, R)$ and for all $i \in N_2$ for all $a \in A$ we have $L_i(a, R + R') = L_i(a, R')$. For every $R \in \mathcal{L}(A)^N$ and for every $i \in N$ define $R_{-i} \in \mathcal{L}(A)^{N/\{i\}}$ such that $\forall j \in N/\{i\}$ and $\forall a \in A$, $L_j(a, R_{-i}) = L_j(a, R)$

Let $F : \mathcal{L}(A)^N \to 2^A$ be a social choice rule, F is Maskin monotonic iff

$\forall a \in F(R)$ we have;

$$\forall R' \in \mathcal{L}(A)^N : [\forall i \in N \; L_i(a,R) \subset L_i(a,R') \Rightarrow a \in F(R')]$$

We define degrees of alternatives with respect to coalitions of a given cardinality la Rawlsian that is; firstly we set the degree of an alternative with respect to a coalition to the degree of that alternative in the individual profile of a member of that coalition whose ordering for that alternative is the worst among the members of the coalition, and then we take the maximal degree for the alternative among the coalitions of same cardinality as the degree of that alternative with respect to coalitions of the given cardinality. Formally, let $\mathcal{K}_k = \{K \in 2^N | card(K) = k\}$ denote the collection of coalitions with cardinality k where $k \in N$, we define the degree of an alternative a with respect to the coalitions of cardinality k as the following,

$$\pi^k(a,R) = \max_{K \in \mathcal{K}_k} \min_{i \in K} \pi_i(a,R)$$

For any $n \in \mathbb{N}$ we may define $\overline{n}$ such that, $\underline{n} \in \mathbb{N}$ where $(n-1)/2 < \underline{n} \leq (n+1)/2$ and define $\underline{n}$ such that, $\overline{n} \in \mathbb{N}$ where $(n+1)/2 \leq \overline{n} < (n+3)/2$. Observe that $\underline{n}$ is the minimum cardinality for a majority coalition in N and $\overline{n}$ is the minimum cardinality for a strict majority coalition in N. Using these definitions we say $\pi^{\underline{n}}(a,R)$ denotes the majority degree of an alternative a in the preference profile R and similarly $\pi^{\overline{n}}(a,R)$ denotes the strict majority degree of an alternative a in the preference profile R. To proceed with the median degree of an alternative, we firstly, define a bijection $\tau^{(a,R)} : N \to N$ for every $R \in \mathcal{L}(A)^N$ such that[1] $\forall i, i' \in N \; \pi_{\tau(i)}(a,R) < \pi_{\tau(i')}(a,R) \Leftrightarrow i < i'$ and if $\pi_{\tau(i)}(a,R) = \pi_{\tau(i')}(a,R)$ then $\tau(i) < \tau(i') \Leftrightarrow i < i'$. Observe that, for a give preference profile $R \in \mathcal{L}(A)^N$ and for alternative $a \in A$, τ orders N

[1] We will simply write τ instead of $\tau^{(a,R)}$ as it is always clear which $a \in A$ and $R \in \mathcal{L}(A)^N$ that τ governs throughout the paper.

with respect to degree of alternative a in the individual profiles in R so that $card\{i \in N | \pi_i(a,R) \leq \pi_{\tau(i')}(a,R)\} = i'$. Then, we may define the median degree $\widehat{\pi}(a,R)$ of an alternative a in the preference profile R as

$$\widehat{\pi}(a,R)) = \frac{1}{2}[\pi_{\tau(\overline{n})}(a,R) + \pi_{\tau(\underline{n})}(a,R)]$$

By the following proposition we will show that the median degree is actually the mean of the majority degree and the strict majority degree.

Proposition 1. *For every $R \in \mathcal{L}(A)^N$ and for all $a \in A$ we have*

$$\widehat{\pi}(a,R) = \frac{1}{2}[\pi^{\overline{n}}(a,R) + \pi^{\underline{n}}(a,R)]$$

Proof. Since $card\{i \in N | \pi_i(a,R) \leq \pi_{\tau(i')}(a,R)\} = i'$, we immediately have for every $R \in \mathcal{L}(A)^N$ and $\forall a \in A$, $\pi^{\overline{n}}(a,R) = \pi_{\tau(\overline{n})}(a,R)$ and $\pi^{\underline{n}}(a,R) = \pi_{\tau(\underline{n})}(a,R)$. Then obviously, $\pi_{\tau(\overline{n})}(a,R) + \pi_{\tau(\underline{n})}(a,R) = \pi^{\overline{n}}(a,R) + \pi^{\underline{n}}(a,R)$. □

Corrolary 1. *For every $R \in \mathcal{L}(A)^N$ and for all $a \in A$ we have*

$$\pi^{\overline{n}}(a,R) \leq \widehat{\pi}(a,R) \leq \pi^{\underline{n}}(a,R)$$

CHAPTER 3

THREE SOCIAL CHOICE RULES

In this section we will introduce three social choice rules, which are Majoritarian Compromise, Strict Majoriatarian Compromise and Median Rule, in two forms. Firstly we will introduce these social choice rules in coarse form in the sense that they will not include a tie-breaking rule. Then, refinements of these social choice rules which also include a tie breaking rule will be introduced.

We begin with the Majoritarian Compromise without a tie breaking rule, which will be called coarse Majoritarian Compromise, $cMC : \mathcal{L}(A)^N \to 2^A$ and be defined [1] as follows:

$$cMC(R) = \arg\max_{a \in A} \pi^{\underline{n}}(a, R)$$

Notice that, coarse Majoritarian Compromise chooses the alternatives which has the maximal majority degree. Similarly coarse Strict Majoritarian Compromise, $cSMC : \mathcal{L}(A)^N \to 2^A$, can be defined as:

$$cSMC(R) = \arg\max_{a \in A} \pi^{\overline{n}}(a, R)$$

[1] Our definition of Majoritarian Compromise will be different than Sertel's (1984) definition. Reader may check that two definitions are equivalent.

Now we may define Median Rule without a tie-breaking rule which will be called coarse Median Rule, $cMR : \mathcal{L}(A)^N \to 2^A$ defined as follows;

$$cMR(R) = \arg\max_{a \in A} \widehat{\pi}(a, R)$$

Using Proposition 1, we may give an alternative definition to coarse Median Rule such that

$$cMR(R) = \arg\max_{a \in A} \frac{1}{2}[\pi^{\overline{n}}(a, R) + \pi^{\underline{n}}(a, R)]$$

Coarse Median Rule chooses the alternatives which has the maximal median degree or as the alternative definition suggests, it picks the alternatives which maximize the mean of majority degree and strict majority degree. Consequently, coarse Median Rule will agree with coarse Majoritarian Compromise whenever coarse Majoritarian Compromise and coarse Strict Majoritarian Compromise coincides. Next proposition extends this observation;

Proposition 2. *For every $R \in \mathcal{L}(A)^N$ if $a \in cMR(R)$, then $\forall b \in A$ either $\pi^{\overline{n}}(a, R) \geq \pi^{\overline{n}}(b, R)$ or $\pi^{\underline{n}}(a, R) \geq \pi^{\underline{n}}(b, R)$.*

Proof. Assume not true then $\exists R \in \mathcal{L}(A)^N$ and $\exists a \in cMR(R)$ such that $\exists b \in A$ where $\pi^{\overline{n}}(a, R) < \pi^{\overline{n}}(b, R)$ and $\pi^{\underline{n}}(a, R) < \pi^{\underline{n}}(b, R)$. But then $\pi^{\overline{n}}(a, R) + \pi^{\underline{n}}(a, R) < \pi^{\overline{n}}(b, R) + \pi^{\underline{n}}(b, R)$, which implies $a \notin cMR(R)$. Contradiction. □

Corrolary 2. *If $cMC(R) = cSMC(R)$, then $cMC(R) = cMR(R)$*

For the next two theorems , we define $i(a) \in N$ for every $a \in A$ such that $\pi_{i(a)}(a, R) = \min_{i \in N} \pi_i(a, R)$. Now we are ready to state equivalence theorems for coarse forms.

Theorem 1. *$\forall R \in \mathcal{L}(A)^N$, we have*

$$cMC(R) = \begin{cases} cMR(R) & \textit{if } n \textit{ is odd} \\ \arg\max_{a\in A} \widehat{\pi}(a, R_{-i(a)}) & \textit{if } n \textit{ is even} \end{cases}$$

Proof. When n is odd, $\overline{n} = \underline{n}$ implying $cMC(R) = cSMC(R)$, then by Corollary 2 we have $cMC(R) = cMR(R)$. If n is even, observe that $\forall a \in A\ \pi^{\underline{n}}(a, R) = \widehat{\pi}(a, R_{-i(a)})$, since $\forall a \in A,\ \pi_{\tau(\underline{n})}(a, R) = \widehat{\pi}(a, R_{-i(a)})$. But then $\forall a \in A, \max_{a\in A} \pi^{\underline{n}}(a, R) = \max_{a\in A} \widehat{\pi}(a, R_{-i(a)})$ which implies $cMC(R) = \arg\max_{a\in A} \widehat{\pi}(a, R_{-i(a)})$. □

Theorem 2. $\forall R \in \mathcal{L}(A)^N$, *we have;*

$$cSMC(R) = \begin{cases} cMR(R) & \textit{if } n \textit{ is odd} \\ \arg\max_{a\in A} \overline{\pi}(a, R + R_{-i(a)}) & \textit{if } n \textit{ is even} \end{cases}$$

Proof. When n is odd, again $\overline{n} = \underline{n}$ implying $cMC(R) = cSMC(R)$, then by Corollary 2 we have $cMC(R) = cMR(R)$. If n is even, observe that $\forall a \in A, \pi^{\underline{n}}(a, R) = \widehat{\pi}(a, R + R_{-i(a)})$, since $\forall a \in A, \pi_{\tau(\overline{n})}(a, R) = \widehat{\pi}(a, R + R_{-i(a)})$. But then $\forall a \in A \max_{a\in A} \pi^{\overline{n}}(a, R) = \max_{a\in A} \widehat{\pi}(a, R_{-i(a)})$ which implies $cMC(R) = \arg\max_{a\in A} \widehat{\pi}(a, R_{-i(a)})$. □

Now we will introduce refinements of these social choice rules which will include a tie-breaking rule for the cases where above defined social choice rules pick more than one alternative. Tie-breaking rule will differentiate alternatives that have same majority (or strict majority or median) degree by picking those, individual degrees of which are greater than the tied majority (or strict majority or median) degree for individuals of a greater coalition in terms of cardinality. To formally state this, firstly define for any given $R \in \mathcal{L}(A)^N$, $a \in A$ and $k \in N$ $\varphi^k(a, R) = card\{i \in N | \pi_i(a, R) \geq \pi^k(a, R)\}$ and $\widehat{\varphi}(a, R) = card\{i \in N | \pi_i(a, R) \geq \widehat{\pi}(a, R)\}$. Note that by definition, $\forall a \in A,\ \varphi^{\overline{n}}(a, R) \geq \overline{n},\ \varphi^{\underline{n}}(a, R) \geq \underline{n}$ and $\widehat{\varphi}(a, R) \geq \underline{n}$. Now we can define

Majoritarian Compromise :

$$MC(R) = \{a \in cMC(R) | \forall b \in cMC(R), \varphi^{\underline{n}}(a,R) \geq \varphi^{\underline{n}}(b,R)\}$$

Similarly Strict Majoritarian Compromise is defined as follows;

$$SMC(R) = \{a \in cSMC(R) | \forall b \in cSMC(R), \varphi^{\overline{n}}(a,R) \geq \varphi^{\overline{n}}(b,R)\}$$

And the Median Rule is defined such that;

$$MR(R) = \{a \in cMR(R) | \forall b \in cMR(R), \widehat{\varphi}(a,R) \geq \widehat{\varphi}(b,R)\}$$

Next, we will provide an important result stating that for any profile where Majoritarian Compromise differs from coarse Majoritarian compromise, Majoritarian Compromise coincides with Median Rule.

Theorem 3. *$\forall R \in \mathcal{L}(A)^N$, if $MC(R) \neq cMC(R)$, then $MC(R) = MR(R)$.*

Proof. For the fist part assume that $MC(R) \neq cMC(R)$ so $cMC(R)/MC(R) \neq \emptyset$ as $\forall R \in \mathcal{L}(A)^N, MC(R) \subset cMC(R)$.Then take any $a \in MC(R)$ which implies $\forall b \in cMC(R)$, $\pi^{\underline{n}}(a,R) = \pi^{\underline{n}}(b,R)$ but $\varphi^{\underline{n}}(a,R) > \varphi^{\underline{n}}(b,R) \geq \underline{n}$. So then $\varphi^{\underline{n}}(a,R) \geq \overline{n}$ where $\overline{n} = \underline{n}$ or $\overline{n} = 1 + \underline{n}$ and $\widehat{\pi}\underline{n} \in \mathbb{N}$. That means $\pi^{\overline{n}}(a,R) = \pi^{\underline{n}}(a,R)$. But since $\pi^{\overline{n}}(a,R) \geq \widehat{\pi}(a,R) \geq \pi^{\underline{n}}(a,R)$, we have $\pi^{\overline{n}}(a,R) = \widehat{\pi}(a,R) = \pi^{\underline{n}}(a,R)$. Then we have, $\varphi^{\underline{n}}(a,R) = \varphi^{m}(a,R)$. Now since $\forall b \in A\pi^{\underline{n}}(a,R) \geq \widehat{\pi}(a,R)$, $\forall b \in \pi^{\underline{n}}(a,R) \geq \pi^{\underline{n}}(b,R)$ implies $\forall b \in \widehat{\pi}(a,R) \geq \widehat{\pi}(a,R)$. Hence $a \in MR(R) \cap cMR(R)$ implying that $MC(R) \subset MR(R)$. Conversely assume $\exists b \in MR(R)/MC(R)$, but then since $\exists a \in MR(R) \cap MC(R)$ we have $\widehat{\pi}(b,R) = \widehat{\pi}(a,R)$, but since $\widehat{\pi}(a,R) = \pi^{\underline{n}}(a,R)$ and $\forall c \in A\pi^{\underline{n}}(c,R) \geq \widehat{\pi}(c,R)$ we have $\pi^{\underline{n}}(b,R) = \pi^{\underline{n}}(a,R)$. But then $\varphi^{\underline{n}}(b,R) = \varphi^{m}(b,R)$ implies $b \in MC(R)$ which is a contradiction. Hence we have $MR(R) \subset MC(R)$ implying $MC(R) = MR(R)$.

$\square$

Using above theorem, we can write an equivalence theorem for Majoritarian Compromise and Median Rule as corollary to Theorem 3.

Corrolary 3. *$\forall R \in \mathcal{L}(A)^N$, we have;*

$$MC(R) = \begin{cases} MR(R) & \textit{if } n \textit{ is odd} \\ \arg\max_{a \in A} \operatorname{median}_i \pi(a, R_{-i(a)}) & \textit{if } n \textit{ is even} \end{cases}$$

where $\forall a \in A$ define $i(a)$ such that $\pi_{i(a)}(a, R) = \min_{i \in N} \pi_i(a, R)$.

Finally, as $\pi^{\overline{n}}(a, R) = \widehat{\pi}(a, R) = \pi^{\underline{n}}(a, R)$ implies $\varphi^{\underline{n}}(a, R) = \varphi^m(a, R)$, we can conclude this section by stating Corollary 2 in terms of Majoritarian Compromise and Median Rule instead of coarse forms;

Corrolary 4. *If $MC(R) = SMC(R)$ then $MC(R) = MR(R)$*

CHAPTER 4

AXIOMATIC PROPERTIES of MEDIAN RULE

In the previous section, we have established an analytical relationship between Majoritarian Compromise and Median Rule. In this section we will explore the properties of Majoritarian Compromise and verify that Median Rule satisfies all properties of Majoritarian Compromise except that being a majoritarian rule. So firstly we will start with a property that Majoritarian Compromise satisfies while many other well known social rules fail to satisfy.

We say a social choice rule $F : \mathcal{L}(A)^N \to 2^A$ satisfies Majoritarian Approval if any alternative that is picked by F is ranked in the better half of alternatives by a majority. Formally we say F satisfies Majoritarian Approval if $\forall a \in F(R) \exists K \in \mathcal{K}_{\underline{n}}$ such that $\forall i \in K, \pi_i(a, R) > m/2$. The following property will imply that Median Rule satisfies Majoritarian Approval :

Proposition 3. $\forall R \in \mathcal{L}(A)^N, \forall a \in MR(R),\ \widehat{\pi}(a, R) \geq \frac{m+1}{2}$.

Proof. Assume not true. Then given, $R \in \mathcal{L}(A)^N, \forall a \in A, \widehat{\pi}(a, R) < \frac{m+1}{2}$. Since $\forall a \in A, card\{i \in N | \pi_i(a, R) \leq \widehat{\pi}(a, R)\} \geq (n + 1)/2$ we have $\forall a \in A, card\{i \in N | \pi_i(a, R) < \frac{m+1}{2}\} \geq (n + 1)/2$ which is a contradiction. $\square$

We may read this proposition as the following: Median degree of the winner alternative according to Median rule cannot be less than $\frac{m+1}{2}$ which is the median ranking. This proposition directly gives us the following corollary:

Corrolary 5. *Median Rule satisfies Majoritarian Approval.*

Majoritarian Compromise satisfies Weak No Veto Power which is a necessary condition for Subgame Perfect Nash Implementability. We say a social choice rule $F : \mathcal{L}(A)^N \to 2^A$ satisfies Weak No Veto Power if all but one of the individuals agree on their best options, than that option is surely picked. Formally we say that F satisfies Weak No Veto Power if $\exists K \in \mathcal{K}_{n-1}$ and $a \in A$ such that $\forall i \in K$ $L_i(a, R) = A$ then $a \in F(R)$.

Proposition 4. *Median Rule satisfies Weak No Veto Power when* $n \geq 3$.

Proof. Take any $R \in \mathcal{L}(A)^N$ such that $\exists K \in \mathcal{K}_{n-1}$ and $a \in A$ where $\forall i \in K$ $L_i(a, R) = A$. Now obviously, $\widehat{\pi}(a, R) = m$. Take any $b \in A$, we have $\forall i \in K$ $\pi_i(b, R) < m$ so $\widehat{\pi}(b, R) < m$ where $n \geq 3$ which implies $b \notin MR(R)$. □

Not surprisingly Median Rule does not satisfy some properties that Majoritarian Compromise does not. Both of these rules are not Maskin monotonic, but both of them satisfy ceteris-paribus monotonicity. For an example showing that Median Rule is not Maskin monotonic;

R				R'			
i	ii	iii	iv	i	ii	iii	iv
a	c	d	b	a	b	d	b
b	b	a	a	b	c	a	a
c	a	c	d	c	a	c	d
d	d	b	c	d	d	b	c

Now, $\forall j \in \{i, ii, iii, iv\}$ $L_j(a, R) \subset L_j(a, R')$ where $a \in MR(R)$ but $a \notin MR(R')$. Median Rule also carries Majoritarian Compromise's other undesirable results. Firstly it admits no show paradox which means $\exists R \in$

$\mathcal{L}(A)^N$ such that $\exists a \notin F(R)$ and $R' \in \mathcal{L}(A)^{\{j\}}$ where $L_J(a, R') = \{a\}$ and $a \in F(R + R')$. Now consider following example;

R			R'
i	ii	iii	j
a	a	b	b
x	x	x	d
c	c	c	c
d	d	d	a
b	b	a	x

We have $x \notin MR(R)$ but $x \in MR(R + R')$ where $L_j(x, R') = \{x\}$.

Secondly Median Rule violates consistency. We say $F : \mathcal{L}(A)^N \to 2^A$ satisfies consistency if $\forall R \in \mathcal{L}(A)^{N_1}$ and $\forall R \in \mathcal{L}(A)^{N_2}$ where $N_1 \cap N_2 = \emptyset$ whenever $F(R) \cap F(R') \neq \emptyset$ we have $F(R+R') = F(R) \cap F(R')$. Now consider this example;

R				R'
i	ii	iii	iv	j
a	a	b	b	x
c	d	e	f	a
x	x	x	x	d
d	c	d	d	e
e	e	c	e	c
f	f	f	c	f
b	b	a	a	b

We have $MR(R) = \{x\}$ and $MR(R') = \{x\}$ but $x \notin MR(R + R') = \{a\}$. So Median Rule violates consistency.

Lastly, Median Rule verifies Condorcet Paradox which means that Median Rule may pick a Condorcet loser even in the presence of a Condorcet winner. Consider the following example;

R

i	ii	iii	iv	v	vi	vii
a	b	c	d	a	a	b
x	x	x	x	c	d	c
b	a	a	a	b	c	a
c	c	b	b	d	b	d
d	d	d	c	x	x	x

Observe that for each $y \in \{a, b, c, d\}$ $\exists K \in \mathcal{K}_4$ such that $\forall i \in K, \pi_i(y, R) > \pi_i(x, R)$ meaning that x loses every pairwise voting against any other alternative and for each $y \in \{x, b, c, d\}$ $\exists K \in \mathcal{K}_4$ such that $\forall i \in K, \pi_i(a, R) > \pi_i(y, R)$ meaning that a beats every pairwise voting against any other alternative. However, in this particular example, $a \notin MR(R)$ although a is the Condorcet winner and $x \in MR(R)$ although x is the Condorcet loser.

On the bright side, Median Rule is Subgame Perfect Nash Implementable like Majoritarian Compromise. But, before giving the relevant proof for this property which will conclude this section, we will mention some properties that differentiate these two rules. Firstly, for any $R \in \mathcal{L}(A)^N$, define the inverse preference profile $R^{-1} \in \mathcal{L}(A)^N$ of preference profile R such that $\forall a \in A, i \in N$ we have $L_i(a, R) = U_i(a, R^{-1})$. We argue that, given that for any R, $F(R)$ is the set of alternatives that is considered best in R with respect to F, $F(R^{-1})$ is the set of alternatives that is considered worst in R with respect to F. Now consider $R \in \mathcal{L}(A)^N$ for some social choice rule $F : \mathcal{L}(A)^N \to 2^A$ where $F(R) = F(R^{-1})$. Then, for this one particular preference profile, social choice rule F picks same set of alternatives for the given preference profile and inverse of it. Now, say $\exists b \notin F(R) = F(R^{-1})$. But this means, that particular alternative is considered neither best nor worst even though same set of alternatives are considered both as best and as worst. Then one may require a social choice rule to mark the same set of alternatives as both best and worst alternatives only when this set is equal to set of all alternatives. Formally we say that $F : \mathcal{L}(A)^N \to 2^A$ satisfies Indifference for Polarized Preferences if $\forall R \in \mathcal{L}(A)^N, F(R) = F(R^{-1})$ implies $F(R) = A$. Firstly, let's see that Median Rule satisfies Indifference for Polarized Preferences;

Proposition 5. *Median Rule satisfies Indifference for Polarized Preferences.*

Proof. First we will prove following claim;

Claim. $\forall R \in \mathcal{L}(A)^N \ \forall a \in A \ \operatorname{median}_{i\in N} \pi_i(a, R) = m+1-\operatorname{median}_{i\in N} \pi_i(a, R^{-1})$

Proof. Take any $R \in \mathcal{L}(A)^N$ and $a \in A$ we have that $\pi_i(a, R) = m + 1 - \pi_i(a, R^{-1})$. Then $\operatorname{median}_{i\in N} \pi_i(a, R) = \operatorname{median}_{i\in N} m + 1 - \pi_i(a, R^{-1})$, so $\operatorname{median}_{i\in N} \pi_i(a, R) = m + 1 - \operatorname{median}_{i\in N} \pi_i(a, R^{-1})$. □

Now take any $R \in \mathcal{L}(A)^N$ such that $M(R) = M(R^{-1})$. Then $\forall a \in M(R)$ and so $\forall a \in M(R^{-1})$, by proposition 3 we have $\operatorname{median}_{i\in N} \pi_i(a, R) \geq \frac{m+1}{2}$ and $\operatorname{median}_{i\in N} \pi_i(a, R^{-1}) \geq \frac{m+1}{2}$. So by claim we have, $\operatorname{median}_{i\in N} \pi_i(a, R) \geq \frac{m+1}{2}$ and $m+1-\operatorname{median}_{i\in N} \pi_i(a, R) \geq \frac{m+1}{2}$, which implies $\operatorname{median}_{i\in N} \pi_i(a, R) = \frac{m+1}{2}$ and $\operatorname{median}_{i\in N} \pi_i(a, R^{-1}) = \frac{m+1}{2}$. Now assume $\exists b \in A$ such that $b \notin M(R)$. The $\operatorname{median}_{i\in N} \pi_i(b, R) < \operatorname{median}_{i\in N} \pi_i(a, R) = \frac{m+1}{2}$. But then by the first claim we have; $\operatorname{median}_{i\in N} \pi_i(a, R^{-1}) > \frac{m+1}{2}$ which implies $b \in M(R^{-1})$ and $a \notin M(R^{-1})$ which is a contradiction. □

Now we may give an example where Majoritarian Compromise fails Indifference for Polarized Preferences;

R

i	ii
a	b
c	c
b	a

For above example we have $MC(R) = \{a,b\} = MC(R^{-1})$ which clearly violates Indifference for Polarized Preferences.

However, one should observe that not all properties of Majoritarian Compromise is satisfied by Median Rule. For example, Majoritarian Compromise is a majoritarian rule that is if one alternative is considered to be best by some majority coalition then that alternative is surely selected. Following example shows why Median Rule is not a majoritarian rule;

R

i	ii
a	b
c	c
d	d
b	a

For above example we have $MR(R) = \{c\}$ although both a and b are considered to be best for some majority coalition. On the other hand, Median Rule is a strict majoritarian rule that is if an alternative is considered to be best by some strict majority coalition (a coalition with cardinality $\overline{n}$) then that alternative is surely selected.

Finally we may present the last result of this paper, that is the Median Rule is Subgame Perfect Implementable. We will do this firstly by it is true for the coarse form of Median Rule and then extend it for the Median Rule. But before that, let us remind the characterization for Subgame Perfect

Implementability which is due to Abrue and Sen (1990):

Theorem 4. *A social choice rule $F : \mathcal{L}(A)^N \to 2^A$ is Subgame Perfect Implementable if following holds;*

- *F satisfies Weak No Veto Power.*
- *F satisfies condition α that is $\forall R, R' \in \mathcal{L}(A)^N$ and $a_0 \in F(R)/F(R')$ then $\exists (a_0, a_1, .., a_{h+1}) \subset A$ and $\exists (i(0), i(1), ..., i(h)) \subset N$ satisfying the following;*

  1. *$a_{j+1} \in L_{i(j)}(a_j, R) \forall j \in \{0, ..., h\}$*
  2. *$a_h \in L_{i(h)}(a_{h+1}, R')$*
  3. *$L_i(j)(a_j, R') \neq A, \forall j \in \{0, ..., h\}$*
  4. *If $L_{i(j)}(a_{h+1}, R) = A$ $\forall j \in \{0, ..., h-1\}$ then either $h = 0$ or $i(h-1) \neq i(h)$.*

So we are ready to state the last result;

Theorem 5. *Median Rule is Subgame Perfect Implementable when $n \leq 3$.*

Proof. We already have shown that Median Rule satisfies Weak No Veto Power. So it suffices to show that Median Rule satisfies condition α. Now take any $R, R' \in \mathcal{L}(A)^N$ such that $x \in MR(R)/MR(R')$.

Case 1 *$\exists i \in N$ and $y \in A$ such that xR_iy but yR'_ix*

Letting $(a_0, a_1) = (x, y)$ and $(i(0)) = (i)$, condition α is trivially satisfied.

Case 2 *$\forall i \in N$ we have $L_i(x, R) \subset L_i(x, R')$*

So $\forall i \in N$ $\forall b \in A$ $\pi_i(x, R) \geq \pi_i(b, R)$ whenever $\pi_i(x, R') \geq \pi_i(b, R')$. Now take any $z \in MR(R')$. We have either $\widehat{\pi}(z, R') > \widehat{\pi}(x, R')$ or $\widehat{\pi}(z, R') = \widehat{\pi}(x, R')$. If the former is true, then $\widehat{\pi}(z, R') > \widehat{\pi}(x, R') \geq$

$\widehat{\pi}(x,R) \geq \widehat{\pi}(z,R)$. But $\widehat{\pi}(z,R') > \widehat{\pi}(z,R)$ implies $\pi_j(z,R') > \pi_j(z,R)$ for some $j \in N$. Then $\exists t \in A$ such that tR_jz but zR'_jt. If the latter is true, since $x \notin MR(R')$, we have $\widehat{\pi}(x,R') \leq \widehat{\pi}(z,R')$ and $\widehat{\pi}(x,R) \geq \widehat{\pi}(z,R)$. Then $\exists t \in A$ such that $\exists j \in N$ where tR_jz, zR_jx and zR'_jt, tR'_jx since $\forall i \in N$ we have $L_i(x,R) \subset L_i(x,R')$.

Case 2.1 $\exists k \in N$ *such that* xR_kt *with* $\pi_k(x,R') \neq 1$

Then we have xR_kt , tR_jz where $\pi_k(x,R') \neq 1$ and $\pi_j(t,R') \neq 1$ so (x,t,z) with (k,j) satisfies α.

Case 2.2 $\nexists k \in N$ *such that* xR_kt *with* $\pi_k(x,R') \neq 1$

Then we have tR_kx whenever $\pi_k(x,R') \neq 1$. But then let $K = \{k \in N | tR_kx\}$ we have $cardK > \frac{n}{2}$, since otherwise we have $card\{k \in N | \pi_k(x,R') = 1\} > \frac{n}{2}$ implying $x \in MR(R')$. Now if $\exists r \in A$ with $l \in K$ where xR_lr and $\exists m \notin K$ where rR_mt then xR_lr, rR_mt and tR_jz where $\pi_l(x,R') \neq 1, \pi_m(r,R') \neq 1$ and $\pi_j(t,R') \neq 1$ implying (x,r,t,z) with (l,m,j) satisfies α. If $\nexists r \in A$ with $l \in K$ where xR_lr and $\nexists m \notin K$ where rR_mt then, $\forall l \in K, \forall r \in L_l(x,R)), \nexists m \in N/K$ where rR_mt. So $\forall l \in K, \forall r \in L_l(x,R)$, we have $\forall m \in N/K$ $r \in L_m(t,R)$. But then $\min_{l \in K} \pi_l(x,R) = \pi_m(t,R)$ $\forall m \in N/K$ and $\pi_l(x,R) < \pi_l(t,R)$ $\forall l \in K$, so then $\mathrm{median}\,(t,R) \geq \mathrm{median}\,(x,R)$ which implies $\mathrm{median}\,(t,R) = \mathrm{median}\,(x,R)$ where $x \in MR(R)$. So then $\widehat{\pi}(t,R) \leq \widehat{\pi}(x,R)$, but given $\min_{l \in K} \pi_l(x,R) = \pi_m(t,R)$ $\forall m \in N/K$ and $\pi_l(x,R) < \pi_l(t,R)$ $\forall l \in K$, this is a contradiction so $\exists r \in A$ with $l \in K$ where xR_lr and $\exists m \notin K$ where rR_mt.

$\square$

CHAPTER 5

CONCLUSION

In this paper we have analysed the relationship between Median Rule and Majoritarian Compromise both in coarse and refined form, i.e. both with tie breaking rule and without tie breaking rule. After stating equivalence theorems, we have proceeded with exploring which properties of Majoritarian Compromise are satisfied by Median Rule. We have shown that, just like Majoritarian Compromise, Median Rule satisfies Majoritarian Approval, Weak No Veto Power and Subgame Perfect Implementability, while it fails consistency and admits Condorcet and no show paradox. Then we have drawn a line between these two rules by introducing two axioms, each of which is satisfied only by one of the rules. While Median Rule satisfies Indifference for Polarized Preferences and fails to be a majoritarian rule, reverse of this statement holds for Majoritarian Compromise. Yet it should be noted that, Median Rule is a a strict majoritarian rule where being strict Majoritarian rule is a slightly weaker condition than being a Majoritarian rule.

This paper have shown that, Median Rule has an appeal for those who favours Majoritarian Compromise for its axiomatic properties the Median Rule satisfies above stated properties. Moreover, this paper provided alternative and shorter proofs for properties of Majoritarian compromise if we restrict ourselves to the set of individuals with odd size.

However, this paper only followed the line of research for Majoritarian Compromise and showed that Median Rule is as good as Majoritarian Compromise in nearly every perspective that Majoritarian Compromise is studied. Reverse of this exercise might be fruitful and for example the question of; is Majoritarian Compromise also less manipulable than Borda rule as Basset and Persky's study suggested for Median Rule, might be studied.

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