MACKEY GROUP CATEGORIES AND THEIR SIMPLE FUNCTORS

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By

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September, 2012

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Constructing the Mackey group category $\mathcal{M}$ using axioms which are reminiscent of fusion systems, the simple $R\mathcal{M}$-functors (the simple functors from the R-linear extension of $\mathcal{M}$ to R-modules, where R is a commutative ring) can be classified via pairs consisting of the objects of the Mackey group category (which are finite groups) and simple modules of specific group algebras. The key ingredient to this classification is a bijection between some $R\mathcal{M}$-functors (not necessarily simple) and some morphisms of $\mathrm{End}_{R\mathcal{M}}(G)$. It is also possible to define the Mackey group category by using Brauer pairs, or even pointed groups as objects so that this classification will still be valid.

Keywords: Mackey group category, Puig category, Brauer category.

ÖZET

MACKEY GRUP KATEGORİLERİ VE BASİT İZLEÇLERİ

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Mackey grup kategorisi $\mathcal{M}$, füzyon sistemlerini andıran aksiyomlarla inşa edildiğinde, basit $R\mathcal{M}$-izleçleri (değişmeli bir halka olan R için; $\mathcal{M}$'in R-lineer genişlemesinden R-modüllere basit izleçler), Mackey grup kategorisinin nesneleri (bunlar sonlu gruplardr) ve belirli grup cebirlerinin basit modüllerinden oluşan ikililer tarafından sınıflandırılabilir. Bu sınıflandırmanın anahtar noktası, bazı $R\mathcal{M}$-izleçleri (basit olmak zorunda değiller) ile $\mathrm{End}_{R\mathcal{M}}(G)$'nin bazı morfizmaları arasndaki birebir örten eşleşmedir. Mackey grup kategorisi tanımlanırken, Brauer ikilileri ve hatta noktalı gruplar nesne kabul edilse dahi bu sınıflandırma geçerli olacaktır.

Anahtar sözcükler: Mackey grup kategorisi, Puig kategorisi, Brauer kategorisi.

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Contents

Introduction

It is known that one can define representations of categories, just like groups, so that if the category consists of a single group, then these two notions coincide. (cf. [1]) Indeed, if $\mathcal{C}$ is a category, then one can extend this category linearly over a commutative unital ring R, and get a category $R\mathcal{C}$. Then a representation of the category $\mathcal{C}$ is defined to be a functor $F : R\mathcal{C} \to {}_R\mathtt{Mod}$.

In this thesis, I will define a general category, with several examples of it, so that we can classify its "irreducible" representations, *i.e.* simple functors $R\mathcal{C} \to {}_R\mathtt{Mod}$ which we call $R\mathcal{C}$-functors. Our categories will have objects either

- p-subgroups (for "fusion systems")
- p-subgroups indexed by some blocks (for "Brauer category")
- p-subgroups indexed by some simple modules (for "Puig category")

respectively, and morphisms built using maps between them.

Throughout the thesis, $\mathtt{k}$ will always stand for an algebraically closed field of prime characteristic p, and every group will be finite. We will be mostly dealing with an arbitrary fixed group G and its subgroups. Also $\mathtt{k}G$ will denote the p-modular group algebra as usual. Various categories constructed throughout the thesis will have certain group homomorphisms as their morphisms and in those cases composition rule will always be usual composition of group homomorphisms.

In Chapter 1, our only aim is to build the necessary tools for later chapters, while providing some examples which aim to show the reader how simple is the theory despite its complicated -yet expressive- language. At the end of the chapter, we shall have defined two categories to build Mackey group categories on.

In Chapter 2, our aim will be to classify the simple $R\mathcal{M}$-functors of a rather general Mackey group category. We will first define bisets which will be used to define the morphisms of Mackey group categories, and then define the category. Then, we will follow Bouc's work in order to classify the simples in our slightly different setting, showing his work is still valid out of the categories which he sees "admissable".

Chapter 3 aims to show that how the Brauer and some Puig categories can be used to build Mackey group categories, hence combined with the classification in Chapter 2, it shows how one can classify simple functors for these categories.

Chapter 1

Modular Group Algebras

This chapter serves as an introduction to the subject while establishing the notation which will be used throughout the thesis.

1.1 Introduction

Theorem 1.1. *Let A be a $\mathtt{k}$-algebra. Then there is a bijective correspondence between:*

- *Conjugacy classes α of primitive idempotents of A*
- *Isomorphism classes of simple A-modules satisfying $e.V_\alpha \neq \{0\}$ for some (or equivalently for all) $e \in \alpha$ where V_α is any representative of such a class.*

Definition. A conjugacy class α of primitive idempotents of a $\mathtt{k}$-algebra A is called a **point** of A, and we denote by $\mathcal{P}(A)$ the set of points of A.

Remark. Note that Theorem 1.1 also implies a bijective correspondence between points and irreducible Brauer characters.

Definition. The subalgebra $\mathtt{k}G^H$, where $H \leq G$, is defined as the set

$$\mathtt{k}G^H := \left\{ a \in \mathtt{k}G \mid \forall h \in H \quad {}^h a = a \right\}$$

which is spanned by the H-conjugacy class sums in $\mathtt{k}G$.

Definition. We define the **relative trace map** $\mathtt{tr}_K^H : \mathtt{k}G^K \to \mathtt{k}G^H$ on an element $a \in \mathtt{k}G^K$ as

$$\mathtt{tr}_K^H(a) = \sum_{hK \in H/K} {}^h a$$

where h runs over coset representatives.

When I is an ideal of $\mathtt{k}G^K$, then $\mathtt{tr}_K^H(I)$ is an ideal of $\mathtt{k}G^H$, since for $i \in I \subseteq \mathtt{k}G^K$ and $a \in \mathtt{k}G^H \subseteq \mathtt{k}G^K$ we have

$$\mathtt{tr}_K^H(a.i) = \sum_{hK \in H/K} {}^h(ai) = \sum_{hK \in H/K} ({}^h a)({}^h i) = \sum_{hK \in H/K} a({}^h i) = a.\mathtt{tr}_K^H(i)$$

because $a \in \mathtt{k}G$ and similarly $\mathtt{tr}_K^H(i.a) = \mathtt{tr}_K^H(i).a$. Hence $\mathtt{k}G_K^H = \mathtt{tr}_K^H(\mathtt{k}G^K)$ is an ideal of $\mathtt{k}G^H$, making $\mathtt{k}G_{<H}^H = \sum_{K<H} \mathtt{k}G_K^H$ also an ideal of $\mathtt{k}G^H$.

Lemma 1.2. *Let $L \leq K \leq H \leq G$ be subgroups. Then*

1. *If $a \in \mathtt{k}G^H$, then $\mathtt{tr}_K^H(a) = |H : K|.a$.*
2. $\mathtt{tr}_K^H \circ \mathtt{tr}_L^K = \mathtt{tr}_L^H$

Definition. The algebra homomorphism

$$br_H : \mathtt{k}G^H \to \mathtt{k}G^H / \mathtt{k}G_{<H}^H$$

is called **Brauer morphism**.

Notation. If C is an H-conjugacy class of a group $G \geq H$, then we will denote by C^+ the class sum $\sum_{c \in C} c$ in $\mathtt{k}G$.

Theorem 1.3. *The elements C^+ where C ranges over the set of H-conjugacy classes of G containing an element g such that $p \nmid |C_H(g) : C_K(g)|$ form a basis of A_K^H.*

Proof. Let D be a K-conjugacy class of G (*i.e.* D^+ is a basis element for $\mathtt{k}G^K$). Then for any $g \in D$,

$$D^+ = \sum_{d \in D} d = \sum_{k.C_K(g) \in K/C_K(g)} {}^k g = tr_{C_K(g)}^K(g)$$

and hence

$$\begin{aligned} tr_K^H(D^+) &= tr_K^H \circ tr_{C_K(g)}^K(g) \\ &= tr_{C_K(g)}^H(g) \\ &= tr_{C_H(g)}^H \circ tr_{C_K(g)}^{C_H(g)}(g) \\ &= tr_{C_H(g)}^H\left(\left|C_H(g) : C_K(g)\right| .g\right) \\ &= \left|C_H(g) : C_K(g)\right| tr_{C_H(g)}^H(g) \\ &= \left|C_H(g) : C_K(g)\right| .C^+ \end{aligned}$$

where C^+ denotes the H-conjugacy class of G containing g. In particular, $tr_K^H(D^+) = \{0\}$ whenever p divides the index $|C_H(g) : C_K(g)|$ for every $g \in D$. □

Corollary 1.4. *If P is a p-subgroup of G, then $\Bbbk G^P = \Bbbk C_G(P) \oplus \Bbbk G^P_{<P}$.*

Proof. Continuing from the previous proof, for any $g \in G$, p divides the index $|C_P(g) : C_Q(g)|$ unless $C_P(g) = C_Q(g)$, since they are both p-groups by hypothesis. If $g \in C_G(P)$, then $C_P(g) = P \gneq C_Q(g)$ for every $Q<P$ and so any such Q yields zero trace. □

Consider the pairs (H, α) consisting of subgroups $H \leq G$ and points α of subalgebras $\mathtt{k}G^H$. Since $\mathtt{k}G^1 = \mathtt{k}G$, the pairs $(1, \alpha)$ simply correspond to the points of $\mathtt{k}G$ whereas the equality $\mathtt{k}G^G = Z(\mathtt{k}G)$ provides us with the pairs $(G, \mathfrak{b})$ corresponding to central primitive idempotents (or simply, **blocks**) $\mathfrak{b}$ of $\mathtt{k}G$.

Definition. Let $H \leq G$ be a subgroup, and $\alpha \in \mathcal{P}(\mathtt{k}G^H)$. Then the pair (H, α) is called a **pointed group**. Instead of writing (H, α), a pointed group is usually denoted by H_α.

Definition. If H_α and K_β are two pointed groups of a group algebra $\mathtt{k}G$ and $K \leq H$, then we say K_β is a **pointed subgroup** of H_α, and write $K_\beta \leq H_\alpha$ if for some (equivalently, for all) $a \in \alpha$, there exist some $b \in \beta$ satisfying the following equivalent conditions:

- b appears in a primitive decomposition of a in A,

- $ab = ba = b$,
- $aba = b$.

Remark. It should be clear that this relation $\leq$ between pointed groups is transitive.

Definition. A point $\alpha \in \mathcal{P}(\mathtt{k}G^H)$ is said to be **local** if $br_H(\alpha) \neq \{0\}$. In this case, we also say that H_α is **local**.

Theorem 1.5. *If P_δ is a local pointed group, then P is a p-group.*

Proof. If $Q{<}P$ satisfies $p \nmid |P:Q|$, then the equality

$$\mathtt{k}G^P \supseteq \mathtt{k}G^P_Q = tr^P_Q(\mathtt{k}G^Q) \supseteq tr^P_Q(\mathtt{k}G^P) = |P:Q|\,\mathtt{k}G^P = \mathtt{k}G^P$$

gives us $\mathtt{k}G^P = \mathtt{k}G^P_Q$ and hence $br_P(\mathtt{k}G^P) = \{0\}$. This forces P to be a p-group since we are asking for $br_P(\delta) \neq \{0\}$. $\square$

Theorem 1.6. *For a p-subgroup $P \leq G$, the Brauer map*

$$br_P : \mathtt{k}G^P \to \mathtt{k}C_G(P)$$

induces a bijection between the local points of $\mathtt{k}G^P$ and the points of $\mathtt{k}C_G(P)$.

To prove this, we will use the fact that a point $\alpha \in \mathcal{P}(\mathtt{k}G^P)$ is local if and only if $ker(br_P) \subseteq m_\alpha$, where m_α is the unique maximal ideal corresponding to the point α such that $\alpha \cap m_\alpha = \emptyset$. We will abuse this notation writing $m_a = m_\alpha$ for any $a \in \alpha$.

Proof. Let e be a primitive idempotent in $\mathtt{k}G^P$. Then the projection $br_P(m_e)$ is maximal in $\mathtt{k}C_G(P)$. Hence there is a unique maximal ideal $\tilde{m}_e = br_P(m_e)$ and as a result, a unique point.

Conversely, let i be a primitive idempotent in $\mathtt{k}C_G(P)$, and moreover let m_α and m_β be two maximal ideals of $\mathtt{k}G^P$, corresponding to points α and β such that $m_i \subseteq m_\alpha$, $m_i \subseteq m_\beta$, $ker(br_P) \subseteq m_\alpha$ and $ker(br_P) \subseteq m_\beta$. Now by the first part, $m_i \subseteq br_p(m_\alpha)$ and $m_i \subseteq br_P(m_\beta)$ are maximal ideals. So, $br_P(m_\alpha) = m_i = br_P(m_\beta)$. Since $\alpha \subseteq m_\beta$ and $\beta \subseteq m_\alpha$ when $\alpha \cap \beta = \emptyset$, we must either have $\alpha = \beta$ or else we will be forced to have either $br_P(\alpha) = \emptyset$ (w.l.o.g.) or $m_i = \mathtt{k}C_G(P)$. $\square$

1.2 An Example

In this section, we will inspect the pointed subgroups of $\mathtt{k}S_3$ over a field $\mathtt{k}$ of characteristic 3.

Note that a block $\mathfrak{b}$ of a group algebra $\mathtt{k}G$ can be decomposed orthogonally into a sum of primitive idempotents, and any conjugate of such an idempotent also decomposes $\mathfrak{b} \in Z(\mathtt{k}G)$. Moreover, such a point α can not appear in the decomposition of another block $\mathfrak{b}'$ of $\mathtt{k}G$, since each $a \in \alpha$ satisfies $\mathfrak{b}.a = a$. Thus, blocks seperates points into disjoint sets. Knowing the bijective correspondence between the points and irreducible Brauer characters, we will seperate characters into corresponding disjoint sets, and call them **blocks**, too. We will write $\chi \in IBr(\mathfrak{b})$ when the point corresponding to the Brauer character χ decomposes the block $\mathfrak{b}$.

Recall from modular character theory that the character table of S_3 is:

S_3	1	2.1	3
ζ_1	1	1	1
ζ_2	1	-1	1
ζ_3	2	$-$	-1
ϕ_1	1	1	$-$
ϕ_2	1	-1	$-$
ψ_1	1	$-$	1
ψ_2	2	$-$	-1

where ϕ_1 & ϕ_2 are two 3-characters both belonging to the same unique 3-block of S_3, and ψ_1 & ψ_2 are two 2-characters belonging to two 2-blocks of S_3.

S_3 in characteristic 3

Remark. A p-group, having a unique p'-conjugacy class, has only one irreducible character and in turn, only one point.

3-subgroups of S_3 are C_3 and 1, with centralizers C_3, S_3 respectively. The subalgebra $\mathtt{k}C_G(C_3) = \mathtt{k}C_3$; since C_3 is a 3-group, has a unique point which corresponds to the unique local point of $\mathtt{k}S_3^{C_3}$.

Remark. Let G be an abelian group such that $\mathtt{k}G$ has a unique point α. Then, commutativity ensures that $\alpha = \{a\}$ for some primitive idempotent a. Knowing that $1 - a$ is also an idempotent (not necessarily primitive) which is orthogonal to a, let us write $1 - a = a_1 + a_2 + \ldots + a_n$ as a sum of mutually orthogonal primitive idempotents. But, $\alpha = \{a\}$ was assumed to be the unique point, forcing $a_i = a \ \forall i$, which is possible only when $a = 1$, since $1 - a$ and a were orthogonal.

Because of these arguments, the unique point of $\mathtt{k}C_3$ should be $\{1\}$. Thus the corresponding local primitive idempotent of $\mathtt{k}S_3^{C_3}$ should be of the form $1 + a$, where $a \in ker(br_{C_3}) = \mathtt{k}(S_3)^{C_3}_{<C_3}$. Hence a is a trace from the unique subgroup 1 of C_3. So, let $a = k.(1+r+r^2)s$ where $k \in \mathtt{k}$, and $S_3 = \langle r, s \mid s^2 = r^3 = 1,\ srs = r \rangle$. But

$$a^2 = k^2(1 + r + r^2)^2 s^2 = 0$$

in characteristic 3, so $1 + a = (1 + a)^2 = 1 + 2a$ implies $a = 0$.

Remark. If $H_{\{1\}}$ is a pointed group of a p-modular group algebra $\mathtt{k}G$, then any pointed group K_β where $K \leq H$ satisfies $b.1 = 1.b = b$ for all $b \in \beta$, and hence satisfies $K_\beta \leq H_{\{1\}}$.

Hence every (local) pointed group of $\mathtt{k}(S_3)^1 = \mathtt{k}S_3$ is a pointed subgroup of $(C_3)_{\{1\}}$. The character table of S_3 tells us also that $\mathtt{k}C_{S_3}(1) = \mathtt{k}S_3$ has two points, corresponding to two local points of $\mathtt{k}(S_3)^1 = \mathtt{k}S_3$. To sum up, $\mathtt{k}S_3$ has three local pointed groups $1_{\alpha_1}, 1_{\alpha_2} \leq (C_3)_{\{1\}}$, where the local points α_1 and α_2 correspond to two irreducible Brauer characters of $\mathtt{k}C_{S_3}(1) = \mathtt{k}S_3$ and the local point $\{1\}$ corresponds to the trivial Brauer character of $\mathtt{k}C_{S_3}(C_3) = \mathtt{k}C_3$.

Remark. Since $\mathtt{k}C_G(1) = G = \mathtt{k}G^1$, any pointed group of 1 on $\mathtt{k}G$ is local.

Remark. A pointed group need not have a unique pointed subgroup, as in the example.

S_3 in characteristic 2

We will momentarily deviate from the main subject in order to build the necessary tools for our next example. The example we will be able to give after this deviation worths the effort by partially answering an important question. Our aim in this part is to render the Theorem 1.8 accessible. First, we will prove an analogous theorem to [2, Theorem 4.4].

Lemma 1.7. *Let e be a block of $\Bbbk G$. Then there is a maximal local pointed group $P_\gamma \leq G_e$ if and only if $P \in Syl_p(C_G(x))$ is maximal among all Sylow p-subgroups of $C_G(x)$'s where these x appear in e.*

Proof. Suppose $P_\alpha \leq G_e$ be a local pointed subgroup of G_e. Then for some $a \in \alpha$ we have $br_P(a) \neq 0$. Since

$$br_P(e).br_P(i) = br_P(ei) = br_P(i) \neq 0$$

we have $br_P(e) \neq 0$. Writing $br_P(\mathtt{k}G^P) = \mathtt{k}C_G(P)$ for the p-subgroup P, since $\mathtt{k}G^G \subseteq \mathtt{k}G^P$, there must be some $x \in G$ (appearing in e) such that $x \in C_G(P)$, implying $P \leq C_G(x)$. Now consider a Sylow p-subgroup $P \leq D \in Syl_p(C_G(x))$. Then $D \leq C_G(x)$ gives $x \in C_G(D)$ where x was assumed to appear in e, and so $br_D(e) \neq 0$. decomposing e into primitives of $\mathtt{k}G^D$, we can obtain a local point $\gamma \in \mathcal{P}(\mathtt{k}G^D)$ which also satisfy $P_\alpha \leq D_\gamma$.

Conversely for any $P \in Syl_p(C_G(x))$ where x appears in e, we have $P \leq C_G(x)$, and so $x \in C_G(P)$ implying $br_P(e) \neq 0$. So any primitive idempotent $\hat{a}$ of $\mathtt{k}C_G(P)$ satisfying $br_P(e)\hat{a} = \hat{a}$ corresponds to a primitive idempotent a with its point α in $\mathtt{k}G^P$. Fixing one such point, we get $P_\alpha \leq G_e$. □

Theorem 1.8. *If P_α is a maximal local pointed group of a p-modular algebra $\Bbbk G$, and $P_\alpha \leq G_\mathfrak{b}$ then the order of P is the value d in*

$$p^{a-d} = min\{\chi(1)_p \mid \chi \in IBr(\mathfrak{b})\}$$

where $\mathfrak{b}$ is the corresponding block of $\Bbbk G$, $\chi(1)_p$ stands for the p-part of the value $\chi(1)$ of the modular character χ, and $|G|_p = p^a$.

Proof. In view of the previous Lemma and [2, Theorem 4.4], the proof is given by [2, Corollary 3.17]. □

Using the same notation as the previous example, the two characters ψ_1 and ψ_2 were previously noted to lie in two Brauer 2-blocks, say $\mathfrak{b}_1$ and $\mathfrak{b}_2$ respectively. By Theorem 1.8, we see that the block $\mathfrak{b}_1$ hosts a maximal local pointed group of order 2, and $\mathfrak{b}_2$ hosts one with order 1. Thus all unique local pointed groups $(C_2^n)_{\alpha_n}$ lie in the block $\mathfrak{b}_1$, leaving 1_{ψ_2} alone.

Note. Jacquez Thèvenaz, right after stating the Theorem 1.6 [3, Corollary 37.6, pp. 22-323] comments on the pointed subgroup relations that "... it is not clear whether it is possible to define this partial order relation directly in terms of irreducible representations". We can now see that such a relation can not be given simply by taking restriction and induction in a straightforward fashion. An immediate counterexample is given by S_3 that we just had a brief inspection. We have $C_{S_3}(C_2) = C_2$, $C_{S_3}(1) = S_3$, and the induced character

S_3	1	2.1	3
ζ_1	1	1	1
ζ_2	1	-1	1
ζ_3	2	–	-1
ψ_1	1	–	1
ψ_2	2	–	-1
$ind_{C_2}^{S_3}(1)$	3	–	0

is a sum of other two. In other words, the $\mathtt{k}C_{S_3}(1) = \mathtt{k}S_3$-module induced from the simple (actually, trivial) $\mathtt{k}C_{S_3}(C_2) = \mathtt{k}C_2$-module which corresponds to the local pointed group $(C_2)_\alpha$ affords both simple $\mathtt{k}C_{S_3}(1) = \mathtt{k}S_3$-modules, corresponding to local pointed groups 1_{ψ_1} and 1_{ψ_2}, but the local pointed group 1_{ψ_2} is not a pointed subgroup of $(C_2)_\alpha$. The counterexample for restriction is even simpler, since both $\mathtt{k}C_{S_3}(1) = \mathtt{k}S_3$-module would restrict to (a multiple of) the trivial $\mathtt{k}C_{S_3}(C_2) = \mathtt{k}C_2$-module.

1.3 Brauer Pairs and the Brauer Category

Definition. A **Brauer pair** (P, e) of $\mathtt{k}G$ consists of a p-subgroup $P \leq G$, and a block e of $\mathtt{k}C_G(P)$.

Let e be a block of $\mathtt{k}C_G(P)$, and consider a primitive decomposition

$$e = e_1 + e_2 + \ldots + e_P$$

in $\mathtt{k}C_G(P)$. Then by Theorem 1.6, there are local points δ_i of $\mathtt{k}G^P$ such that $e_i \in br_P(\delta_i)$. In this case, we say that these pointed groups P_{δ_i} are **associated** with the Brauer pair (P, e).

Given two Brauer pairs (P, e) and (Q, f), if $\left\{P_{\delta_j}\right\}_j$ and $\{Q_{\alpha_i}\}_i$ are two sets consisting of all pointed groups (of $\mathtt{k}G$) which are associated with the Brauer pairs (P, e) and (Q, f), respectively, then we assign a partial order relation between (P, e) and (Q, f) via these sets as:

$$(Q, f) \leq (P, e) \quad if \quad \exists i, j \quad Q_{\alpha_i} \leq P_{\delta_j}.$$

Note that the transitivity of this relation follows from the transitivity of pointed groups.

Brauer subpairs are defined uniquely as in the following lemma:

Lemma 1.9. *[3, Corollary 40.9] If $Q \leq P$ and (P, e) is a Brauer pair, then there exists a unique pair (Q, f) such that $(Q, f) \leq (P, e)$.*

Notation. Because of the previous lemma, using its notation, we will simply write $f = e_Q$.

Note that each block b of $\mathtt{k}C_G(1) = \mathtt{k}G$ defines a unique Brauer pair $(1, b)$, and vice versa. By Lemma 1.9, any Brauer pair (P, e) has a unique subpair $(1, e_1) \leq (P, e)$ so that e_1 is the unique block of $\mathtt{k}G$ corresponding to e. We say in this case (P, e) is **associated with** e_1. Also for any subpair $(Q, f) \leq (P, e)$, we have $f_1 = e_1$ by transitivity and uniqueness in the sense of Lemma 1.9.

Example 1.1. Let C_2^i, $i = 1,2,3$ be 2-subgroups of S_3. Each of them, being p-groups, leads to only one block in each $\mathtt{k}C_{S_3}(C_2^i) = \mathtt{k}C_2^i$. Let us write $\mathfrak{b}^{C_2^i}$ for these blocks. As in the previous example, in its notation, $(C_2^i)_{\alpha_i}$, $i = 1,2,3$ are all pointed subgroups of $(S_3)_{\mathfrak{b}_1}$. Thus, $(1, \mathfrak{b}_1) \leq (C_2^i, b^{C_2^i})$ for all $i = 1,2,3$, establishing our example. Moreover, $(1, \mathfrak{b}_2)$ is not a Brauer subpair for any pair, simply reflecting the case for pointed groups.

If (P, e) is a Brauer pair of G, and $g \in G$, then ${}^g e$ is a block of ${}^g C_G(P) = C_G({}^g P)$. So we define the action of G on Brauer pairs via ${}^g(P, e) = ({}^g P, {}^g e)$.

Definition (Brauer Category)**.** The Brauer category $\mathcal{B}_{\mathfrak{b}}$ on a block $\mathfrak{b}$ of $\mathtt{k}G$ is a category consisting of:

- objects: Brauer pairs associated with $\mathfrak{b}$
- morphisms $(Q, f) \to (P, e)$: all group homomorphisms $\phi : Q \to P$ such that
$$\exists g \in G \quad {}^g(Q, f) \leq (P, e) \quad \& \quad \forall u \in Q \quad \phi(u) = {}^g u.$$

Remark. The stabilizer

$$N_G(P, e) = \{g \in G \mid {}^g(P, e) = (P, e)\} = \{g \in N_G(P) \mid {}^g e = e\}$$

of (P, e) satisfies $PC_G(P) \leq N_G(P, e)$.

We previously noted that Brauer pairs are nice in the sense that subpairs are unique (cf. Theorem 1.9), but they still have some peculiarities that which we should note. If $\mathfrak{b}$ is an arbitrary block of $\mathtt{k}G$, then a Brauer pair (Q, g) may satisfy $(Q, g) \leq (P, e)$ and $(Q, g) \leq (P, f)$ simultaneously for some pairs (P, e) and (P, f).

On the other hand, if the block $\mathfrak{b}$ is the principal block (*i.e.* the block containing trivial representation), then this is not a concern anymore:

Theorem 1.10. *[3, Theorem 40.14] Let $\mathfrak{b}$ be the principal block of $\mathtt{k}G$ and let Q be any p-subgroup of G.*

1. *The idempotent* $br_Q(\mathfrak{b})$ *is primitive in* $Z(\boldsymbol{k}C_G(Q))$ *and is equal to the principal block of* $\boldsymbol{k}C_G(Q)$.

2. *If* e *is a block of* $\boldsymbol{k}C_G(Q)$, *then the Brauer pair* (Q, e) *is associated with* $\mathfrak{b}$ *if and only if* e *is the principal block of* $\boldsymbol{k}C_G(Q)$.

3. *The map* $(Q, e) \mapsto Q$ *is an isomorphism between the poset of Brauer pairs associated with* $\mathfrak{b}$ *and the poset of all* p*-subgroups of* G.

Definition (Frobenious category.)**.** We define the **Frobenious category** $\mathcal{F}(G)$ of G to be the category with

- objects; all p-subgroups of G,
- morphisms $Q \to P$; all group monomorphisms induced by conjugation by some element $g \in G$ (which must therefore satisfy ${}^gQ \leq P$).

Corollary 1.11. *Frobenious categories are equivalent to Brauer categories on principal blocks.*

Chapter 2

Mackey Group Categories

In this chapter we will first build a framework and set up our notation regarding bisets without going much into details. Then we will introduce Mackey group categories, and show how the classification of simple functors works on these subcategories of biset categories.

2.1 Bisets

Let us have a quick glance at bisets. A detailed take on the subject can be found in [4].

An (H, K)-**biset** X is defined to be a left H, right K-set such that these actions are compatible in the sense that

$$h(xk) = (hx)k$$

for $h \in H$, $x \in X$, $k \in K$.

Definition. Given an (H, K)-biset X and a (K, L)-biset Y, their **tensor product** $X \times_K Y$ is defined to be the set of K-orbits of the action given by

$$\forall (h, l) \in H \times L \quad \forall k \in K \quad k.(h, l) = (hk, k^{-1}l).$$

If $L \leq G \times H$ and $M \leq H \times K$ are two subgroups, we define the **star product** $L * M$ as

$$L * M = \{(g,k) \in G \times K \mid \exists h \in H \quad (g,h) \in L \quad \& \quad (h,k) \in M\}$$

Lemma 2.1 (Mackey formula for bisets)**.** *Let $H, K, L \leq G$ be groups. If $Y \leq L \times K$ and if $X \leq K \times H$, then there is an isomorphism of (L,H)-bisets*

$$\left(\frac{L \times K}{Y}\right) \times_K \left(\frac{K \times H}{X}\right) \cong \coprod_{k \in [p_2(Y) \backslash K / p_1(X)]} \left(\frac{L \times H}{Y * {}^{(k,1)}X}\right)$$

where $[p_2(Y) \backslash K / p_1(X)]$ is a set of representatives of double cosets.

Definition. The **biset category** $\mathcal{C}$ associated with a finite set $\mathcal{K}$ of finite groups is defined as follows:

- The objects of $\mathcal{C}$ are the elements of $\mathcal{K}$.
- If H and K are finite groups, then $\mathrm{Hom}_{\mathcal{C}}(H,K)$, is the Grothendieck group of the category of finite (K,H)-bisets.
- If G, H, and K are finite groups, then the composition $v \circ u$ of the morphisms $u \in \mathrm{Hom}_{\mathcal{C}}(G,H)$ and $v \in \mathrm{Hom}_{\mathcal{C}}(H,K)$ is equal to $v \times_H u$. Here, if $v = (G \times H)/L$ and $u = (H \times K)/M$, then we define $[v] \times_H [u] = [v \times_H u]$.
- For any finite group G, the identity morphism of G in $\mathcal{C}$ is equal to $[1_G]$.

Thus, $\mathrm{Hom}_{\mathcal{C}}(H,K)$ is the $\mathbb{Z}$-module generated by the isomorphism classes $[(H \times K)/M]$ of bisets having the form $(H \times K)/M$.

Remark. Two basis elements $[P \times Q/L]$ and $[P \times Q/M]$ are equal if and only if L and M are conjugate under $P \times Q$.

2.2 Mackey Group Category

Definition. A category $\mathcal{D}$ is said to be **preadditive** provided, for all $X, Y, Z \in obj(\mathcal{D})$, each $\mathrm{Hom}_{\mathcal{D}}(X,Y)$ is a $\mathbb{Z}$-module, and the composition

$$\mathrm{Hom}_{\mathcal{D}}(X,Y) \times \mathrm{Hom}_{\mathcal{D}}(Y,Z) \to \mathrm{Hom}_{\mathcal{D}}(X,Z)$$

is bilinear over $\mathbb{Z}$.

Definition. A **group category** $\mathcal{D}$ on a set of finite groups $\mathcal{K}$ which is closed under subgroups is defined to be a preadditive subcategory of a biset category $\mathcal{C}$ on $\mathcal{K}$.

In other words, $\mathrm{Hom}_{\mathcal{D}}(G, H)$ is a $\mathbb{Z}$-submodule of $\mathrm{Hom}_{\mathcal{C}}(G, H)$ and the composition

$$\mathrm{Hom}_{\mathcal{C}}(G, H) \times \mathrm{Hom}_{\mathcal{C}}(H, K) \to \mathrm{Hom}_{\mathcal{C}}(G, K)$$

restricts to

$$\mathrm{Hom}_{\mathcal{D}}(G, H) \times \mathrm{Hom}_{\mathcal{D}}(H, K) \to \mathrm{Hom}_{\mathcal{D}}(G, K),$$

which is bilinear.

Notation. From now on $\mathcal{V}$ will denote a category satisfying the axioms

A1 objects of $\mathcal{V}$ are finite groups, closed under subgroups

A2 All the morphisms in $\mathcal{V}$ are group monomorphisms.

A3 If $h \in H$, and $W \leq H$, then ${}^hW \in obj(\mathcal{M})$ and the conjugation map $con^h : W \to {}^hW$ is in $\mathrm{Hom}_{\mathcal{V}}(W, {}^hW)$.

A4 Given a morphism $\phi \in \mathrm{Hom}_{\mathcal{V}}(V, U)$ and subgroups $U' \leq U$, and $V' \leq V$ then the restriction $\phi\, |_{V' \to U'} \in \mathrm{Hom}_{\mathcal{V}}(V', U')$.

Example 2.1. An example for such a category would be a **fusion system** $\mathcal{F}$ on a set of finite groups $\mathcal{K}$ closed under subgroups and conjugations. This category is defined to be a category with

- objects; all groups in $\mathcal{K}$
- morphisms; group monomorphisms

such that a hom-set $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ is closed under restrictions, *i.e.*

$$\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q) \implies \exists \psi \in \mathrm{Hom}_{\mathcal{F}}(P, \phi(P))\ \forall p \in P\ \phi(p) = \psi(p)$$

and $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ contains all conjugation morphisms ${}_Pcon^u_Q$ where $u \in P$.

Remark. Note that the Frobenious category defined in the last section is a fusion system. Hence a Brauer category on a principal block also satisfies the conditions for $\mathcal{V}$. In fact, as we will see in the next section, a Brauer category on any block satisfies said conditions.

We are interested mainly in bisets which are of the form $(H \times K)/\Delta$ where the subgroup

$$\Delta = \Delta(U, \phi, V) = \{(\phi(v), v) \mid v \in V\} \leq H \times K$$

is determined by a group isomorphism $\phi : V \to U$ from a subgroup $V \leq K$ to a subgroup $U \leq H$.

Note that by the way our bisets are constructed, if

$$\Delta = \Delta(U, \phi, V) \leq H \times K$$

is a subgroup as above, the subgroups

$$p_1(\Delta) := \{h \in H \mid \exists k \in K \ (h, k) \in \Delta\}, \ \textit{and}$$
$$p_2(\Delta) := \{k \in K \mid \exists h \in H \ (h, k) \in \Delta\},$$

satisfy $p_1(\Delta) = U \cong V = p_2(\Delta)$.

Definition. Given such a category $\mathcal{V}$, we define a **Mackey group category** $\mathcal{M}_{\mathcal{V}}$ to be the group category with

- objects; all groups in $\mathcal{V}$
- morphisms; $\mathbb{Z}$-linear combinations of the isomorphism classes

$$\left[\frac{P \times Q}{\Delta(U, \phi, V)}\right] \in \mathrm{Hom}_{\mathcal{M}}(Q, P)$$

  of bisets, where $\Delta(U, \phi, V) = \{(\phi(v), v) \mid v \in V\}$ for some U, V satisfying $P \geq U \cong V \leq Q$, and an isomorphism $\phi \in \mathrm{Hom}_{\mathcal{V}}(V, U)$.
- compositions; tensor products of bisets.

In order to see that $\mathcal{M}_\mathcal{V}$ is really a category, we must make sure that morphisms are closed under composition. We will make use of the Mackey formula for bisets. Let $L = \Delta(U, \phi, V) \leq G \times H$, and $M = \Delta(W, \psi, X) \leq H \times K$, and $h \in H$. Then

$$\begin{aligned} {}^{(h,1)}M &= {}^{(h,1)}\{(\psi(k), k) \mid k \in X\} \\ &= \left\{({}^h\psi(k), k) \mid k \in X\right\} \\ &= \Delta({}^hW, con^h \circ \psi, X) \end{aligned}$$

where the last equality needs the axiom **A3**. As for the star product, we have

$$\begin{aligned} L * M &= \{(g, k) \in G \times K \mid \exists h \in H \ (g, h) \in \Delta(U, \phi, V) \ \& \ (h, k) \in \Delta(W, \psi, X)\} \\ &= \{((\phi \circ \psi)(k), k) \in G \times K \mid k \in X, \quad \psi(k) \in V\} \\ &= \Delta(U', \zeta, X') \end{aligned}$$

where $X' = \psi^{-1}(V \cap W)$, $U' = \phi(V \cap W)$, and $\zeta(x) = (\phi \circ \psi)(x)$ for all $x \in X'$. Note the axiom **A4**, together with the axiom **A3** ensures that these objects and morphisms are in the category $\mathcal{V}$.

2.3 $R\mathcal{M}$-functors

Let $\mathcal{M}$ be any Mackey group category, and R be a ring with identity. We define the category $R\mathcal{M}$ as the category with

- objects: objects of $\mathcal{M}$,
- morphisms: $\mathrm{Hom}_{R\mathcal{M}}(G, H) = R \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{M}}(H, G)$,
- composition: R-linear extension of the composition in $\mathcal{D}$.

An $R\mathcal{M}$**-functor** is a preadditive functor $M : R\mathcal{M} \to {}_R\mathtt{Mod}$. We write $\mathfrak{F}^{R\mathcal{M}}$ for the category of $R\mathcal{M}$-functors, with natural transformations as morphisms.

For any object $G \in obj(\mathcal{M})$, define the functor

$$\mathrm{Res}_G^{R\mathcal{M}} : \mathfrak{F}^{R\mathcal{M}} \to {}_{\mathrm{End}_{R\mathcal{M}}(G)}\mathtt{Mod}$$

via $\mathrm{Res}_G^{R\mathcal{M}}(M) = M(G)$. The R-module $M(G)$ becomes an $\mathrm{End}_{R\mathcal{M}}(G)$-module via the action of $\phi \in \mathrm{End}_{R\mathcal{M}}(G)$ over $m \in M(G)$ defined as $\phi.m = M(\phi)(m)$. Define another functor

$$\mathrm{Ind}_G^{R\mathcal{M}} : {}_{\mathrm{End}_{R\mathcal{M}}(G)}\mathtt{Mod} \to \mathfrak{F}^{R\mathcal{M}}$$

via $\mathrm{Ind}_G^{R\mathcal{M}}(V) = L_{G,V}$ which is defined for $H \in obj(\mathcal{M})$ and $\phi \in \mathrm{Hom}_{R\mathcal{M}}(H,K)$ as

$$\begin{aligned} L_{G,V}(H) &= \mathrm{Hom}_{R\mathcal{M}}(G,H) \otimes_{\mathrm{End}_{R\mathcal{M}}(G)} V, \ \textit{and} \\ L_{G,V}(\phi)(\alpha \otimes v) &= (\phi\alpha) \otimes v \end{aligned}$$

for any $H \in obj(\mathcal{M})$ and any $\alpha \in \mathrm{Hom}_{R\mathcal{M}}(G,H)$.

Theorem 2.2. *The functors $\mathrm{Res}_G^{R\mathcal{M}}$ and $\mathrm{Ind}_G^{R\mathcal{M}}$ gives rise to a bijection*

$$\mathrm{Hom}_{\mathfrak{F}^{R\mathcal{M}}}(L_{G,V}, M) \cong \mathrm{Hom}_{\mathrm{End}_{R\mathcal{M}}(G)}(V, M(G)) \tag{2.1}$$

for any $G \in obj(\mathcal{M})$, $M \in obj(\mathfrak{F}^{R\mathcal{M}})$, and any simple $\mathrm{End}_{R\mathcal{M}}(G)$-module V.

Proof. If $\tau : L_{G,V} \to M$ is a morphism in $\mathfrak{F}^{R\mathcal{M}}$ (*i.e.* a natural transformation), then it provides us with an R-module homomorphism $\tau_G : L_{G,V}(G) \to M(G)$, which can be made into an $\mathrm{End}_{R\mathcal{M}}(G)$-module homomorphism as explained above. Throughout the proof, we will identify the isomorphic modules $L_{G,V}(G) \cong V$.

Conversely, let $\tau_G : V \to M(G)$ be an $\mathrm{End}_{R\mathcal{M}}(G)$-module homomorphism, which also gives an R-module homomorphism. We will construct τ_H for an arbitrary $H \in obj(\mathcal{M})$ making the following diagram commutative for any $K \in obj(\mathcal{M})$ and any $\alpha \in \mathrm{Hom}_{R\mathcal{M}}(K,H)$, thus giving a natural transformation $\tau \in \mathrm{Hom}_{\mathfrak{F}^{R\mathcal{M}}}(L_{G,V}, M)$:

$$\begin{array}{ccccc} K & & L_{G,V}(K) & \xrightarrow{\tau_K} & M(K) \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle L_{G,V}(\alpha)}\downarrow & & \downarrow{\scriptstyle M(\alpha)} \\ H & & L_{G,V}(H) & \xrightarrow{\tau_H} & M(H) \end{array}$$

First, take $K = G$, and note that for any $H \in obj(\mathcal{M})$ and any $\alpha \in \mathrm{Hom}_{R\mathcal{M}}(G,H)$ we have $L_{G,V}(\alpha)\,(1 \otimes v) = \alpha \otimes v$ where $v \in V$ and $1 = 1_{\mathrm{End}_{R\mathcal{M}}(G)}$.

Define τ_H on an element $\sum_i \phi_i \otimes v_i \in L_{G,V}(H)$ as

$$\tau_H\left(\sum_i \phi_i \otimes v_i\right) = \sum_i M(\phi_i)(\tau_G(1 \otimes v_i)).$$

Let now

$$l^1 = \sum_i \phi_i^1 \otimes v_i^1 \quad \& \quad l^2 = \sum_i \phi_i^2 \otimes v_i^2$$

be two elements of $L_{G,V}(H)$. Since V is simple, we can rewrite a basis element as

$$\phi_i^j \otimes v_i^j = \phi_i^j \rho_i^j \otimes v$$

for some fixed $v \in V$ and appropriate $\rho_i^j \in \mathrm{End}_{R\mathcal{M}}(G)$ satisfying the equation. If $l^1 = l^2$, then we have

$$\left(\sum_i \phi_i^1 \rho_i^1\right) \otimes v = \left(\sum_i \phi_i^2 \rho_i^2\right) \otimes v$$

and hence $\sum_i \phi_i^1 \rho_i^1$ is only a multiple of $\sum_i \phi_i^2 \rho_i^2$ by some $\xi \in \mathrm{End}_{R\mathcal{M}}(G)$ satisfying $\xi v = v$. This gives

$$\begin{aligned}
\tau_H(l^1) &= \sum_i M(\phi_i^1)(\tau_G(1 \otimes v_i^1)) \\
&= \sum_i M(\phi_i^1 \rho_i^1)(\tau_G(1 \otimes v)) && \text{since } M \text{ is a functor} \\
&= M\left(\sum_i \phi_i^1 \rho_i^1\right)(\tau_G(1 \otimes v)) && \text{since } M \text{ is } R\text{-linear} \\
&= M\left(\sum_i \phi_i^2 \rho_i^2 \xi\right)(\tau_G(1 \otimes v)) && \text{since } l^1 = l^2 \\
&= M\left(\sum_i \phi_i^2 \rho_i^2\right) M(\xi)(\tau_G(1 \otimes v)) \\
&= M\left(\sum_i \phi_i^2 \rho_i^2\right) \tau_H(\xi \otimes v)) \\
&= M\left(\sum_i \phi_i^2 \rho_i^2\right) \tau_H(1 \otimes v)) \\
&= \tau_H(l^2)
\end{aligned}$$

making τ_H well-defined. In particular, $\tau_H(l^1) + \tau_H(l^2) = \tau_H(l^1 + l^2)$ since M is preadditive.

It is now left to check τ_H is an R-map. Consider

$$\begin{aligned}
r.\tau_H\left(\sum_i \phi_i \otimes v_i\right) &= r.\sum_i M(\phi_i)(\tau_G(1\otimes v_i)) \\
&= \sum_i r.M(\phi_i)(\tau_G(1\otimes v_i)) \\
&= \sum_i M(\phi_i)(r.\tau_G(1\otimes v_i)) && M(\phi_i) \text{ are } R\text{-morphisms} \\
&= \sum_i M(\phi_i)(\tau_G(r.1\otimes v_i)) && \tau_G \text{ is an } R\text{-morphism} \\
&= \sum_i \tau_H(r.\phi_i \otimes v_i) \\
&= \tau_H\left(r.\sum_i \phi_i \otimes v_i\right).
\end{aligned}$$

Thus τ_H is an R-map.

Now consider this bigger diagram in which the upper and outer squares commute:

$$\begin{array}{ccccc}
G & & L_{G,V}(G) & \xrightarrow{\tau_G} & M(G) \\
\downarrow{\scriptstyle \beta} & & {\scriptstyle L_{G,V}(\beta)}\downarrow & & \downarrow{\scriptstyle M(\beta)} \\
K & & L_{G,V}(K) & \xrightarrow{\tau_K} & M(K) \\
\downarrow{\scriptstyle \alpha} & & {\scriptstyle L_{G,V}(\alpha)}\downarrow & & \downarrow{\scriptstyle M(\alpha)} \\
H & & L_{G,V}(H) & \xrightarrow{\tau_H} & M(H)
\end{array}$$

Seeing that lower digram commutes needs nothing but following the following equalities:

$$\begin{aligned}
M(\alpha)\circ\tau_K(\beta\otimes v) &= M(\alpha)\circ\tau_K\circ L_{G,V}(\beta)(1\otimes v) && \text{by definition} \\
&= M(\alpha)\circ M(\beta)\circ\tau_G(1\otimes v) && \text{upper square is commutative} \\
&= M(\alpha\beta)\circ\tau_G(1\otimes v) && M \text{ is a functor} \\
&= \tau_H\circ L_{G,V}(\alpha\beta)(1\otimes v) && \text{outer square is commutative} \\
&= \tau_H\circ L_{G,V}(\alpha)\circ L_{G,V}(\beta)(1\otimes v) && L_{G,V} \text{ is a functor} \\
&= \tau_H\circ L_{G,V}(\alpha)(\beta\otimes v) && \text{by definition}
\end{aligned}$$

It is obvious that for each natural transformation $\tau : L_{G,V} \to M$, we have exactly one corresponding $\mathrm{End}_{R\mathcal{M}}(G)$-module homomorphism $\tau_G : V \to M(G)$.

To show the converse is true, assume $\tau_H, \tau'_H : L_{G,V}(H) \to M(H)$ be two homomorphisms. For any element, say $l = \sum_i \alpha_i \otimes v_i \in L_{G,V}(H)$, we have

$$\begin{aligned} \tau_H(\alpha_i \otimes v_i) &= \tau_H \circ L_{G,V}(\alpha_i)(1 \otimes v) \\ &= M(\alpha_i) \circ \tau_G(1 \otimes v) \\ &= \tau'_H \circ L_{G,V}(\alpha_i)(1 \otimes v) \\ &= \tau'_H(\alpha_i \otimes v) \end{aligned}$$

which implies $\tau_H(l) = \tau'_H(l)$, completing the proof. □

As a result, whenever we have an object M of $\mathfrak{F}^{R\mathcal{M}}$ so that $V = M(G) = \mathrm{Res}_G^{R\mathcal{M}}(M)$, then the map $\phi : V \to \mathrm{Res}_G^{R\mathcal{M}}(M)$ corresponds to some other map $\bar{\phi} : L_{G,V} \to M$ by Theorem 2.1. Again by Theorem 2.1, since ϕ is non-zero, $\bar{\phi}$ is also not. If moreover M is simple, then $\bar{\phi}$ is surjective.

Theorem 2.3. *Let R be a commutative ring with identity element, and $\mathcal{M}$ be a Mackey group category. If F is a simple object of $\mathfrak{F}^{R\mathcal{M}}$, and G is an object of $\mathcal{M}$ such that $F(G) \neq \{0\}$, then $F(G)$ is a simple* $\mathrm{End}_{R\mathcal{M}}(G)$*-module.*

Proof. If S is a simple $\mathrm{End}_{R\mathcal{M}}(G)$-submodule of $F(G)$, then the inclusion morphism $S \hookrightarrow F(G)$ yields a non-zero morphism $\tau : L_{G,S} \to F$ under the bijection proven above. The image of τ is a non-zero subfunctor of F, which is simple as our hypothesis, and hence it is equal to F. This makes $\tau_G : L_{G,S}(G) \to F(G)$ surjective. But $L_{G,S} \cong S$, so τ_G is isomorphic to the inclusion map, which is now forced to be surjective, providing $S = F(G)$. □

Definition. Let F be an $R\mathcal{M}$-functor. Then we say S is a **subfunctor** of F if $S(H) \leq F(H)$ for all $H \in obj(\mathcal{M})$, and $S(\phi)$ is the restriction of $F(\phi)$ to $S(H)$ for all $\phi \in \mathrm{Hom}_{R\mathcal{M}}(H, K)$ and for all $K \in obj(\mathcal{M})$.

Theorem 2.4. *Let R be a commutative ring with identity element, and $\mathcal{M}$ be a Mackey group category. If G is an object of $\mathcal{M}$, and V is a simple* $\mathrm{End}_{R\mathcal{M}}(G)$*-module, then the functor $L_{G,V}$ has a unique proper maximal subfunctor $J_{G,V}$ and the quotient $S_{G,V} = L_{G,V}/J_{G,V}$ is a simple object of $\mathfrak{F}^{R\mathcal{M}}$, such that $S_{G,V}(G) \cong V$.*

Proof. Let M be a subfunctor of $L_{G,V}$. That is, for all $H \in obj(\mathcal{M})$, $M(H)$ is an $\mathrm{End}_{R\mathcal{M}}(H)$-submodule of

$$L_{G,V}(H) = \mathrm{Hom}_{R\mathcal{M}}(G,H) \otimes_{\mathrm{End}_{R\mathcal{M}}(G)} V,$$

and $M(\phi)$ is the restriction of $L_{G,V}(\phi)$ to $M(H)$ for all $\phi \in \mathrm{Hom}_{R\mathcal{M}}(H,K)$ for any $K \in obj(\mathcal{M})$.

Then $M(G)$ is an $\mathrm{End}_{R\mathcal{M}}(G)$-submodule of $L_{G,V}(G) \cong V$. Thus by simplicity of V, either $M(G) \cong V$ or $M(G) = \{0\}$.

In the former case, if $H \in obj(\mathcal{M})$, $\phi \in \mathrm{Hom}_{R\mathcal{M}}(G,H)$, and $v \in V$, then

$$L_{G,V}(\phi)(id \otimes v) = \phi \otimes v \in L_{G,V}(H).$$

So since $id \otimes v \in M(G)$, and since $M(\phi)$ is the restriction of $L_{G,V}(\phi)$ to $M(G)$, we have

$$M(H) \ni M(\phi)(id \otimes v) = \phi \otimes v$$

for all $\phi \in \mathrm{Hom}_{R\mathcal{D}}(G,H)$. Hence $L_{G,V}(H) = M(H)$ for any object $H \in obj(\mathcal{M})$, which implies $M = L_{G,V}$ by very definition of a subfunctor.

Thus if M is a proper subfunctor of $L_{G,V}$ then $M(G) = \{0\}$. Then if $\sum_i \phi_i \otimes v_i \in M(H)$, and $\psi \in \mathrm{Hom}_{R\mathcal{M}}(H,G)$, then

$$\begin{aligned} M(\psi)\left(\sum_i \phi_i \otimes v_i\right) &= \psi\left(\sum_i \phi_i \otimes v_i\right) && \text{by definition, since } M \leq L_{G,V} \\ &= \sum_i \psi\phi_i \otimes v_i \in M(G) = \{0\} && \text{since } \psi\phi \in \mathrm{End}_{R\mathcal{M}}(G) \end{aligned}$$

that is, $M(H) \subseteq ker(M(\psi)) \subseteq ker(L_{G,V}(\psi))$ for all $\psi \in \mathrm{Hom}_{R\mathcal{M}}(H,G)$ where the latter inclusion comes from the hypothesis $M \leq L_{G,V}$. Hence if we define

$$J(H) = \bigcap_{\psi \in \mathrm{Hom}_{R\mathcal{M}}(H,G)} ker(L_{G,V}(\psi))$$

for all $H \in obj(\mathcal{M})$, then $M(H) \subseteq J(H) \subseteq L_{G,V}(H)$.

In order to construct J as a subfunctor $J \leq L_{G,V}$, it is enough to show that $L_{G,V}(\phi)(J(H)) \subseteq J(K)$ for any $\phi \in \mathrm{Hom}_{R\mathcal{M}}(H,K)$, and $K \in obj(\mathcal{M})$. Indeed, if $K \in obj(\mathcal{M})$, $\phi \in \mathrm{Hom}_{R\mathcal{M}}(H,K)$, and $\sum_i \phi_i \otimes v_i \in J(H)$, then

$$L_{G,V}(\phi)\left(\sum_i \phi_i \otimes v_i\right) = \sum_i \phi\phi_i \otimes v_i,$$

so for all $\psi \in \mathrm{Hom}_{R\mathcal{M}}(K,G)$,

$$L_{G,V}(\psi)\left(\sum_i \phi\phi_i \otimes v\right) = \sum_i \psi\phi\phi_i \otimes v_i = \sum_i (\psi\phi)\phi_i \otimes v_i = 0$$

since $\sum_i \phi_i \otimes v_i \in J(H)$, and $\psi\phi \in \mathrm{Hom}_{R\mathcal{M}}(H,G)$. Thus $\sum_i \phi\phi_i \otimes v_i \in J(K)$.

Since $M \leq L_{G,V}$, $J \leq L_{G,V}$ and $M(H) \leq J(H)$ for all $H \in obj(\mathcal{M})$, we have $M \leq J$ for any proper subfunctor M of $L_{G,V}$. J itself is also proper since $J(G) = \{0\}$, so J is the unique proper maximal subfunctor of $L_{G,V}$. In particular, the quotient functor $S_{G,V} := L_{G,V}/J$ is a simple object of $\mathfrak{F}^{R\mathcal{M}}$, and $S_{G,V}(G) \cong V$, since $J(G) = \{0\}$.

□

Notation. We denote by I_G the free R-submodule of $\mathrm{End}_{R\mathcal{M}}(G)$ generated by all endomorphisms of G which can be factored through some object H of $\mathcal{M}$ with $|H| < |G|$. Note that I_G is a two-sided ideal of $\mathrm{End}_{R\mathcal{M}}(G)$.

If S is a non-zero object of $\mathfrak{F}^{R\mathcal{M}}$, then there must be some $G \in obj(\mathcal{M})$ satisfying $S(G) \neq \{0\}$. Hence a **minimal group** G for S is defined to be a group $G \in obj(\mathcal{M})$ satisfying $S(G) \neq \{0\}$, but $S(H) = \{0\}$ for all $H \in obj(\mathcal{M})$ where $|H| < |G|$.

Let S be a simple $R\mathcal{M}$-functor. Thus S is non-zero, and there exists a minimal group G for S, and $S(G)$ is a simple $\mathrm{End}_{R\mathcal{M}}(G)$-module by Theorem 2.3. If $f \in \mathrm{End}_{R\mathcal{M}}(G)$ factors through an object $H \in obj(\mathcal{M})$ with $|H| < |G|$, then $S(f) = 0$, since then $S(f)$ factors through $S(H) = \{0\}$. It follows that I_G acts as 0 on $S(G)$, *i.e.* that $S(G)$ is a simple module for the quotient algebra $Q_G = \mathrm{End}_{R\mathcal{M}}(G)/I_G$.

Conversely, suppose that G is an object of $\mathcal{M}$ and V is a simple RQ_G-module. Then V becomes a simple $\mathrm{End}_{R\mathcal{M}}(G)$-module via the algebra homomorphism

$\mathrm{End}_{R\mathcal{M}}(G) \to Q_G$. Then by Theorem 2.4, $L_{G,V}$ has a unique simple quotient $S_{G,V}$ satisfying $S_{G,V}(G) \cong V$.

Definition. Let R be a commutative ring with unity. The pairs (G, V) of groups $G \in obj(\mathcal{M})$ and simple RQ_G-modules V, are called **seeds** of $R\mathcal{M}$.

Note that the argument above points out a way of associating seeds of $R\mathcal{M}$ with simple $R\mathcal{M}$-functors.

Definition. If (G, V) is a seed of $R\mathcal{M}$, the associated simple functor is the unique simple quotient $S_{G,V}$ described in Theorem 2.4.

2.4 Structure of $\mathrm{End}_{R\mathcal{M}}(G)$

Lemma 2.5. *Let f, g be automorphisms of $H \in obj(\mathcal{V})$. Then $f^{-1} \circ g$ is an inner automorphism of H if and only if the subgroups $\Delta = \Delta(H, f, H)$ and $\Delta' = \Delta(H, g, H)$ of $H \times H$ are conjugate in $H \times H$.*

Proof. Assume ${}^{(q,r)}\Delta = \Delta'$ for some $(q, r) \in H \times H$. Then

$$\begin{aligned}
\Delta' &= {}^{(q,r)}\Delta \\
&= {}^{(q,r)}\{(f(p), p) \mid p \in H\} \\
&= \left\{(qf(p)q^{-1}, rpr^{-1}) \mid p \in H\right\} \\
&= \left\{(qf(r^{-1}p'r)q^{-1}, p') \mid p' \in H\right\} && \text{writing } rpr^{-1} = p' \in H \\
&= \left\{(qf(r^{-1})f(p')f(r)q^{-1}, p') \mid p' \in H\right\}
\end{aligned}$$

implies $g(p) = {}^{qf(r^{-1})}f(p)$ for all $p \in H$. Since $q, r \in H$, they are conjugate in H. Conversely if $g \circ con^q = f$ for some $q \in H$, then

$$\begin{aligned}
\Delta &= \{(f(p), p) \mid p \in H\} \\
&= \left\{g(qpq^{-1}), p) \mid p \in H\right\} \\
&= \left\{(g(q)g(p)g(q^{-1}), p) \mid p \in H\right\} \\
&= {}^{(g(q),1)}\{(g(p), p) \mid p \in H\} \\
&= {}^{(g(q),1)}\Delta'
\end{aligned}$$

showing the conjugacy by $(g(q), 1) \in H \times H$. $\square$

Now denote by A_G the free R-submodule of $\mathrm{End}_{R\mathcal{M}}(G)$ generated by all endomorphisms of the form $[G \times G/\Delta]$ for $\Delta \in \Sigma(G)$ where

$$\Sigma(G) = \{\Delta(G, \phi, G) \mid \phi \in \mathrm{Aut}_{\mathcal{V}}(G)\}.$$

Since

$$\left[\frac{G \times G}{\Delta(G, \phi, G)}\right] \times_G \left[\frac{G \times G}{\Delta(G, \phi', G)}\right] = \left[\frac{G \times G}{\Delta(G, \phi\phi', G)}\right]$$

for all automorphisms ϕ, ϕ' of G, it follows that A_G is an R-subalgebra of $\mathrm{End}_{R\mathcal{M}}(G)$. Moreover, there is an R-algebra isomorphism $\rho : A_G \to R\,\mathrm{Out}_{\mathcal{V}}(G)$ given by

$$\rho\left(\frac{G \times G}{\Delta(G, \phi, G)}\right) = \pi_G(\phi),$$

where $\pi_G : \mathrm{Aut}_{\mathcal{V}}(G) \to \mathrm{Out}_{\mathcal{V}}(G)$ is the projection map. Indeed by Lemma 2.5,

$$\left[\frac{G \times G}{\Delta(G, \phi, G)}\right] = \left[\frac{G \times G}{\Delta(G, \phi', G)}\right] \quad \Longleftrightarrow \quad \pi_G(\phi) = \pi_G(\phi').$$

Write J_G for the R-submodule of $\mathrm{End}_{R\mathcal{M}}(G)$ generated by all endomorphisms $[G \times G/\Delta]$ of G with $|q(\Delta)| < |G|$. Then we should also note the decomposition $\mathrm{End}_{R\mathcal{M}}(G) = A_G \oplus J_G$. Indeed, any representative $G \times G/\Delta$ in A_G must have $q(\Delta) \cong G$, and conversely $q(\Delta) \cong G$ implies $\Delta = \Delta(G, \phi, G)$.

Lemma 2.6. *Let R be a commutative ring with identity, and let $\mathcal{M}$ be a Mackey group category. If $G \in obj(\mathcal{M})$, then the following two free R-submodules of* $\mathrm{End}_{R\mathcal{M}}(G)$ *are equal:*

- *The R-module I_G generated by all endomorphisms of G which can be factored through some object H of $\mathcal{M}$ with $|H| < |G|$*
- *The R-module J_G generated by all endomorphisms $[(G \times G)/L]$ of G with $|p_1(L)| = |p_2(L)| < |G|$.*

Proof. Let $B = (G \times G)/\Delta(U, \phi, V)$ so that $[B] \in \mathrm{End}_{R\mathcal{M}}(G)$, and write $\Delta(U, \phi, V) = L$. Then since $obj(\mathcal{M})$ is closed under subgroups, $p_1(L) \leq G$ must be in $obj(\mathcal{M})$, and so B factors through $p_1(L)$. So if $B \in J_G$, *i.e.* $|p_1(L)| < |G|$, then $B \in I_G$. Thus, $J_G \subseteq I_G$.

Conversely any element α of I_G, is generated by morphisms of the form $\psi\phi$ where

$$\psi = \left[\frac{G \times H}{M}\right] \quad \text{and} \quad \phi = \left[\frac{H \times G}{L}\right]$$

where $H \in obj(\mathcal{M})$ satisfy $|H| < |G|$. And so by Mackey formula, α is a linear combination of morphisms of the form $\left[\frac{G \times G}{M * L'}\right]$ where L' is some conjugate of L in $H \times G$. Now the group $p_1(M * L')$ is isomorphic in $\mathcal{V}$ to a subgroup of H, and hence $|p_1(M * L')| < |G|$, thus $I_G \subseteq J_G$. □

Let us summarize what we have shown up to this point:

Corollary 2.7. *Let R be a commutative ring with identity, and let $\mathcal{M}$ be a Mackey group category. For $G \in obj(\mathcal{M})$, I_G is a two-sided ideal of* $\mathrm{End}_{R\mathcal{M}}(G)$*, and there is a decomposition*

$$\mathrm{End}_{R\mathcal{M}}(G) = A_G \oplus I_G$$

where A_G is an R-subalgebra which is isomorphic to the group algebra $R\,\mathrm{Out}_{\mathcal{V}}(G)$.

2.5 Classification of Simple $R\mathcal{M}$-functors

Definition. Two seeds (G, V) and (G', V') are said to be equivalent if there is a group isomorphism $\phi \in \mathrm{Hom}_{R\mathcal{M}}(G, G')$ and an R-module isomorphism $\psi : V \to V'$ such that

$$\forall v \in V, \quad \forall a \in Q_G, \quad \psi(a.v) = (\phi a \phi^{-1}).\psi(v).$$

In this case, we write $(G, V) \sim (G', V')$.

Lemma 2.8. *Let R be a commutative ring with identity element, and let $\mathcal{M}$ be a Mackey group category. Let $S_{G,V}$ denote the simple functor associated to the seed (G, V) of $R\mathcal{M}$. If $H \in obj(\mathcal{M})$ such that $S_{G,V}(H) \neq \{0\}$, then G is isomorphic to a subgroup of H.*

Proof. Let $H \in obj(\mathcal{M})$ such that $S_{G,V}(H) \neq \{0\}$. Note $S_{G,V}(H) \neq \{0\}$ implies $L_{G,V} \neq J_{G,V}$. Then by definition of $J_{G,V}$ there must be some $\sum_i \phi_i \otimes v_i \in L_{G,V}$ and some $\psi \in \mathrm{Hom}_{R\mathcal{M}}(H,G)$ such that $\sum_i \psi\phi_i \otimes v_i \neq 0$. So we can pick a $\phi \in \mathrm{Hom}_{R\mathcal{M}}(G,H)$ satisfying $(\psi\phi).v_i \neq 0$, and so $(\psi\phi)V \neq \{0\}$. In particular, $\psi\phi \notin I_G$ since otherwise non-zero $\psi\phi$ would factor through a group strictly smaller than G, which would contradict with G being minimal for $S_{G,V}$.

It follows that there exists groups

$$\Delta := \Delta(U,\rho,V) \leq H \times G \quad \& \quad \Delta' := \Delta(U',\rho',V') \leq G \times H$$

appearing in some summands of ϕ and ψ respectively, such that the product

$$\frac{G \times H}{\Delta'} \times_H \frac{H \times G}{\Delta} = \sum_h \frac{G \times G}{\Delta' * ^{(h,1)}\Delta}$$

is not in I_G. This implies, choosing without loss of generality $h = 1$ that $p_1(\Delta' * \Delta) = G = p_2(\Delta' * \Delta)$. So $p_1(\Delta' * \Delta) \leq p_1(\Delta') \leq G$ implies $p_1(\Delta') = G$, and since $G = p_1(\Delta') \cong p_2(\Delta') \leq H$, then $\rho' : p_2(\Delta') \to G$ is an isomorphism between G and the subgroup $p_2(\Delta') = V'$ of H. $\square$

Theorem 2.9. *Let R be a commutative ring with unity, and let $\mathcal{M}$ be a Mackey group category. Then there is a one-to-one correspondence between*

- *the set $\mathfrak{f}$ of simple objects of $\mathfrak{F}^{R\mathcal{M}}$*
- *the set $\mathfrak{s}$ of equivalence classes of seeds of $R\mathcal{M}$*

sending the isomorphism class of a simple functor $S \in \mathfrak{f}$ to the equivalence class of a seed $(G, S(G)) \in \mathfrak{s}$, where G is any minimal group for S. The inverse correspondence maps the class of the seed (G,V) to the class of the functor $S_{G,V}$.

Proof. Let S be a simple $R\mathcal{M}$-functor. Since S is non-zero, it has a minimal group $G \in obj(\mathcal{M})$ with respect to the property $S(G) \neq \{0\}$. Now set $S(G) := V$. Then (G,V) is a seed of S. Since $S(G) = V$, we can form a non-zero morphism $L_{G,V} \to S$ and this morphism is surjective since S is simple. But, $L_{G,V}$ has a unique simple quotient $S_{G,V}$ and thus $S \cong S_{G,V}$. If G' is another such minimal group, and we write $S(G') = V'$, then again $S \cong S_{G',V'}$. Since $S_{G,V}(G') \neq \{0\}$, it follows from the previous lemma that G is isomorphic to a subgroup of G'. Similarly G' is isomorphic to a subgroup of G and hence there exists a group isomorphism $\phi \in \mathrm{Hom}_{\mathcal{V}}(G', G)$.

Now let ${}^{\phi}V'$ be the $R\,\mathrm{Out}_{\mathcal{V}}(G)$-module equal to V' as an R-module, with $\mathrm{Out}_{\mathcal{V}}(G)$ action defined for any $\rho \in \mathrm{Out}_{\mathcal{V}}(G)$ by $\rho.v := (\phi^{-1}\rho\phi).v$ for all $v \in V'$. Also since the functors $S_{G,V}$ and $S_{G',V'}$ are isomorphic, there exists an isomorphism $\psi : {}^{\phi}V' \to V$ of R-modules. Then the pair (ϕ, ψ) is an isomorphism from the seed (G', V') to the seed (G, V). Indeed, $\phi : G' \to G$ is a group isomorphism and $\psi : {}^{\phi}V' \to V$ play also the role of an R-module isomorphism $\psi : V' \to V$ such that

$$\forall v \in V',\ \forall \rho \in \mathrm{Out}_{\mathcal{V}}(G),\ \psi(\rho.v) = (\phi\rho\phi^{-1}).\psi(v).$$

Thus we have a well-defined map $\nu : \mathfrak{f} \to \mathfrak{s}$.

Also to each seed (G, V) of $R\mathcal{M}$ we can associate a simple $R\mathcal{M}$-functor $S_{G,V} = L_{G,V}/J_{G,V}$. Noting that an isomorphism $(\phi, \psi) : (G, V) \to (G', V')$ for another seed $(G', V') \in \mathfrak{s}$ provides us with an isomorphism of $R\mathcal{M}$-functors, it becomes clear that we also have an inverse map $\mu : \mathfrak{s} \to \mathfrak{f}$. Indeed, $G' \cong G$ are minimal groups for $S_{G',V'}$ and $S_{G,V}$ with

$$S_{G',V'}(G') = V' \cong V = S_{G,V}(G)$$

making $S_{G,V} \cong S_{G',V'}$.

Now it should be clear by construction that $\mu \circ \nu = id_{\mathfrak{f}}$ and that $\nu \circ \mu = id_{\mathfrak{s}}$. Thus, μ and ν are two mutual inverse bijections between $\mathfrak{f}$ and $\mathfrak{s}$. $\square$

Chapter 3

Mackey Group Categories for Brauer and Puig Categories

In this chapter we will first apply our treatise on Mackey group categories to the Brauer category, building a category which we will call Mackey-Brauer category. Then we will introduce the Puig category, which has pointed groups of a modular group algebra as its objects, and speculate on a Mackey-Puig category.

3.1 Mackey-Brauer Category

Let $\mathcal{B}_{\mathfrak{b}}$ be a Brauer category on a block $\mathfrak{b}$ of a modular p-algebra $\mathtt{k}G$. As we have noted before, although a Brauer pair (P, e) has a unique Brauer subpair (Q, e_Q) for any subgroup $Q \leq P$, it need not have a unique Brauer superpair. That is, there may be two pairs (R, f) and (R, g) satisfying both $(P, e) \leq (R, f)$ and $(P, e) \leq (R, g)$ although $f \neq g$. We can circumvent this problem by simply taking the maximal Brauer pairs in a block, and exploit the uniqueness of subpairs.

Notation. Let $\mathtt{k}G$ be a p-modular group algebra, and $\mathfrak{b}$ be a block of $\mathtt{k}G$. If (D, e) is a maximal Brauer pair in $\mathcal{B}_{\mathfrak{b}}$, then we write $\mathcal{B}_{(D,e)}$ for the full subcategory of $\mathcal{B}_{\mathfrak{b}}$ where objects are subpairs of (and including) (D, e).

Hence we can define the **Mackey-Brauer category** $\mathcal{M}_{\mathcal{B}_{(D,e)}}$, since $\mathcal{B}_{(D,e)}$ satisfies the axioms **A1-A4**. Although the axiom **A1** is not satisfied directly, it is enough to note that the category $\mathcal{B}_{(D,e)}$ is equivalent to a category with subgroups of D as its objects.

Notation. Throughout this chapter, $\mathcal{M}$ will denote a Mackey-Brauer category.

All of the previous results work for $\mathcal{M}$, since it is a Mackey group category, but in this case we have more to say on the structure of $Q_G = \mathrm{End}_{R\mathcal{M}}(G)/I_G$, hence the modules in seeds.

Lemma 3.1. *Conjugacy classes of the subgroups $\Delta_g = \Delta(P, con^g, P) \leq P \times P$ are in one-to-one correspondence with $N_G(P,e)/PC_G(P)$.*

Proof. Let $g = hk$ for some $k \in PC_G(P)$. Then we have

$$\begin{aligned}
\Delta_g &= \{({}^g p, p) \mid p \in P\} = \left\{\left({}^{hk}p, p\right) \mid p \in P\right\} \\
&= \left\{\left({}^h\left(kpk^{-1}\right), p\right) \mid p \in P\right\} \\
&= \left\{\left({}^h\left(k_1k_2pk_2^{-1}k_1^{-1}\right), p\right) \mid p \in P\right\} && k = k_1.k_2 \text{ for } k_1 \in P \& k_2 \in C_G(P) \\
&= \left\{\left({}^h\left(k_1pk_1^{-1}\right), p\right) \mid p \in P\right\} \\
&= {}^{({}^hk_1,1)}\left\{({}^h p, p) \mid p \in P\right\} = {}^{({}^hk_1,1)}\Delta_h
\end{aligned}$$

Since $h \in N_G(P,e) \leq N_G(P)$ and $k_1 \in P$, we have ${}^hk_1 \in P$, and so Δ_g and Δ_h are conjugate in $P \times P$.

This time assume $\Delta_g = {}^{(q,r)}\Delta_h$ for some $(q,r) \in P \times P$. That is to say,

$$\begin{aligned}
\{({}^g p, p) \mid p \in P\} = \Delta_g &= {}^{(q,r)}\Delta_h \\
&= \left\{\left({}^{qh}p, {}^r p\right) \mid p \in P\right\} \\
&= \left\{\left({}^{qhr^{-1}}p, p\right) \mid p \in P\right\} && \text{since } r \in P
\end{aligned}$$

or in other words, ${}^g p = {}^{qhr^{-1}}p \ \forall p \in P$. So $p = {}^{qhr^{-1}g^{-1}}p \ \forall p \in P$. If we write $qhr^{-1}g^{-1} = qr'hg^{-1}$ where $hr^{-1}h^{-1} = r' \in P$, then we get $qr'hg^{-1} \in C_G(P)$ and $qr' \in P$. Thus $hg^{-1} \in PC_G(P)$. □

Thus by Theorem 2.9, we can parametrize the simple objects of $\mathfrak{F}^{R\mathcal{M}}$ via pairs $((P,e),V)$ where $(P,e) \in obj(\mathcal{M})$ is any Brauer pair on b and V is a simple $RN_G(P,e)/PC_G(P)$-module.

Example 3.1. Let us consider the group A_4 in characteristic 2, and the principal block b. So, Brauer pairs are in one-to-one correspondence with p-subgroups of A_4 as we have noted in the first chapter. These subgroups are 1, C_2's and V_4 with centralizers A_4, V_4, V_4 and normalizers A_4, V_4, A_4, respectively. So, the simple $\mathtt{k}\mathcal{M}_{\mathcal{B}_{\mathfrak{b}}}$-functors are characterized by the pairs having one $\mathtt{k}A_4/(1.A_4) = \mathtt{k}1$-module, one $\mathtt{k}V_4/(C_2.V_4) = \mathtt{k}1$-module, and three $\mathtt{k}A_4/(V_4.V_4) = \mathtt{k}C_3$-modules. Since all three seeds corresponding to C_2's are equivalent, we deduce that A_4 has five $\mathtt{k}\mathcal{M}_{\mathcal{B}_{\mathfrak{b}}}$-functors.

3.2 Mackey-Puig Category

Definition. The **Puig category** $\mathcal{L}_p(G)$ of a p-modular group algebra $\mathtt{k}G$ is defined to be the category with

- objects; local pointed groups on $\mathtt{k}G$,
- morphisms $Q_\beta \to P_\alpha$; group homomorphisms $\phi_g : Q \to P$ such that $\phi_g(q) = {}^g q$ for all $q \in Q$, where g satisfies ${}^g(Q_\beta) \leq P_\alpha$.

Also, if we restrict the objects to local pointed groups in a block $\mathfrak{b}$ of $\mathtt{k}G$, then we will write $\mathcal{L}_{\mathfrak{b}}(G)$ for the resulting category.

When we are working with the Puig category the trick we used in the previous section is no longer valid, since a local pointed subgroup need not be unique as we have seen while observing the local pointed subgroup relations for $\mathtt{k}S_3$ when $char(\mathtt{k}) = 3$. But we can define a category based on the Puig category, which satisfies our axioms.

Definition. Given a modular group algebra $\mathtt{k}G$ and a block $\mathfrak{b}$ of $\mathtt{k}G$, we define a category $\mathcal{V}$ as follows:

- objects are pairs (Q, P_α) where P_α is a local pointed group in $\mathfrak{b}$, and $Q \leq P$ is a subgroup,

- morphisms in $\mathrm{Hom}_{\mathcal{V}}((Q, P_\alpha), (Q', P'_{\alpha'}))$ are group monomorphisms $\phi \in \mathrm{Hom}_{\mathcal{L}_{\mathfrak{b}}(G)}(P_\alpha, P'_{\alpha'})$ such that $\phi(Q) \leq Q'$.

A Mackey group category $\mathcal{M}$ constructed using such a category $\mathcal{V}$ would definitely satisfy the axioms, and hence we can classify the simple $R\mathcal{M}$-functors as we have shown.

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