FREE ACTIONS ON PRODUCT OF SPHERES AT HIGH DIMENSIONS

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

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July, 2012

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ABSTRACT

FREE ACTIONS ON PRODUCT OF SPHERES AT HIGH DIMENSIONS

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A classical conjecture in the theory of transformation groups states that if $G = (\mathbb{Z}/p)^r$ acts freely on a product of k spheres $S^{n_1} \times \cdots \times S^{n_k}$, then $r \leq k$. We prove a special case of this conjecture. We show that given positive integers k, l and $G = (\mathbb{Z}/p)^r$, there is an integer N such that if G acts freely and cellularly on a CW-complex homotopy equivalent to $S^{n_1} \times \cdots \times S^{n_k}$ where $n_i > N$ for all i and $|n_i - n_j| < l$ for all i, j, then $r \leq k$.

Keywords: Free Actions, Product of Spheres, Rank Conjecture.

ÖZET

YUKSEK BOYUTLU KURELERIN CARPIMI UZERINE SERBEST ETKILER

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July, 2012

$G = (\mathbb{Z}/p)^r$ grubu k tane kürenin çarpımı $S^{n_1} \times \cdots \times S^{n_k}$ üzerine serbest etki ediyorsa, dönüşüm grupları teorisindeki klasik bir sanıya gore $r \leq k$'dır. Bu tezde bu sanının özel bir hali olan şu önermeyi ispatladık: k, l pozitif tamsayilari ve $G = (\mathbb{Z}/p)^r$ verildiğinde, öyle bir N tamsayısı vardır ki, eğer G grubu $S^{n_1} \times \cdots \times S^{n_k}$'ye homotopik olan bir CW-kompleksine serbest etki ediyorsa öyle ki her i için $n_i > N$ ve her i, j için $|n_i - n_j| < l$ ise, $r \leq k$'dır.

Anahtar sözcükler: Serbest Etkiler, Kürelerin Çarpımı, Rank Sanısı.

Acknowledgement

I would like to express my sincere gratitude to my supervisor Prof. Dr. Ergün Yalçın for his excellent guidance, valuable suggestions, encouragement, patience and conversations full of motivation.

I would like to thank Prof. Dr. Turgut Önder and Assist. Prof. Dr. Özgün Ünlü for accepting to read and review my thesis.

I would like to thank my wife Esra, my parents Kezban and Necat as this thesis would never be possible without their encouragement, support, and love.

The work that form the content of the thesis is supported financially by TÜBİTAK through the graduate fellowship program, namely "TÜBİTAK-BİDEB 2228-Yurt İçi Yüksek Lisans Burs Programı". I am grateful to the council for their kind support.

I thank to Nesin Matematik Köyü for changing my understanding of mathematics and making me hopeful about future of mathematics in Turkey.

I thank to my office mates Akif, İpek, and Serdar and all my friends who offered help without any hesitation, cared about my studies and, increased my motivation.

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Chapter 1

Introduction

Let G be a finite group. The rank of G, denoted by $rk(G)$, is defined to be the largest integer r such that $(\mathbb{Z}/p)^r \subseteq G$ for some prime p. Due to results of Smith [12] and Swan [13], we know that G acts freely and cellularly on a finite CW-complex homotopy equivalent to a sphere S^n if and only if $rk(G) = 1$.

Homotopy rank of G, denoted by $hrk(G)$, is defined to be the smallest integer k such that G acts freely and cellularly on a finite complex homotopy equivalent to a product of k spheres $S^{n_1} \times \cdots \times S^{n_k}$ for some $n_1, \dots, n_k \geq 1$. Benson-Carlson [2] conjectured that $hrk(G) = rk(G)$. Note that this implies the result in the previous paragraph. The weaker argument $rk(G) \leq hrk(G)$ is a classical conjecture that can be equivalently written as follows.

Conjecture 1.1. *If $G = (\mathbb{Z}/p)^r$ acts freely and cellularly on a finite CW-complex X homotopy equivalent to a product of spheres $S^{n_1} \times \cdots \times S^{n_k}$, then $r \leq k$.*

The case $n_1 = \cdots = n_k = n$ is proved by G. Carlsson [5] under the assumption that the action of G on homology groups of X is trivial. Later Adem-Browder [1] proved the same case without assuming the action of G on homology groups is trivial except for $p = 2$ and $n = 1, 3, 7$. The $n = 1, p = 2$ case is proven by Yalçın [15]. More recently, B. Hanke [9] proved Conjecture 1.1 when $p \geq 3 \dim X$.

In this paper we prove another special case of this conjecture. Our main result is the following.

Theorem 1.2. *Let $G = (\mathbb{Z}/p)^r$ and k, l are positive integers. Then there exists an integer N such that if G acts freely and cellularly on a finite dimensional CW-complex homotopy equivalent to $S^{n_1} \times \cdots \times S^{n_k}$ with $n_i \geq N$ for all i and $|n_i - n_j| \leq l$ for all i, j, then $r \leq k$.*

Browder [3] gives another proof of Conjecture 1.1 for the case $n_1 = \cdots = n_k$ where the action of G on homology groups are trivial, with a different approach. His proof is as follows: He shows that if a finite group G acts freely and cellularly on a CW-complex X then the order of the group G divides the product

$$\prod_{j=1}^{\dim X} exp\ H^{j+1}(G, H_j(X))$$

Notice that when X is homotopy equivalent to $(S^n)^k$, it has nonzero homology groups only at dimensions $0, n, 2n, \ldots, kn$. If a $\mathbb{Z}G$-module M has a trivial G-action, then the exponent of $H^i(G, M)$ divides p for all $i > 0$. Hence we get p^r divides p^k and so $r \leq k$. In this paper this idea of Browder will be one of the main tools for proving our result.

If the dimensions of the spheres are not equal, then there are nonzero homology groups of X at more than k dimensions. Therefore, if we apply Browder's idea directly, we do not get $p^r \leq p^k$ but instead we get $p^r \leq p^m$ where m is the number of dimensions where X has nonzero homology groups and $m > k$. To handle this problem, we use a method used by Habegger [8] to glue homologies at different dimensions and decrease the number of dimensions where there are nonzero homology groups. However after gluing, the new homology groups may not have trivial G-action, so the exponents in the Browder's theorem may not divide p. To overcome this difficulty, we use a theorem by Pakianathan [11] to show that for any finitely generated $\mathbb{Z}G$-module M, there is an integer N such that if $i > N$ then $exp\ H^i(G, M)$ divides p. We show that there are finitely many possibilities for homology groups as $\mathbb{Z}G$-modules after gluing so that we can take the largest N coming from the Pakianathan's theorem. To show this finiteness we use a version of Jordan-Zassenhaus Theorem [6] and finiteness of the *Ext*-groups under some conditions.

Chapter 2

Preliminaries

2.1 Homology Groups of Products of Spheres

We know that if $n > 0$, then the homology group $H_i(S^n)$ is isomorphic to integers for $i = 0, n$ and is equal to 0 otherwise. Künneth theorem, which we will just state without a proof, says that the homology groups of a product of spaces is determined by homology groups of those spaces in the product. By using this theorem, we can compute the homology groups of products of spheres.

Theorem 2.1 (Künneth theorem)**.** *If X and Y are CW-complexes, then there are split exact sequences*

$$0 \to \bigoplus_{i=0}^{n}(H_i(X) \otimes H_{n-i}(Y)) \to H_n(X \times Y) \to$$
$$\bigoplus_{i=0}^{n-1} Tor_{\mathbb{Z}}(H_i(X), H_{n-i-1}(Y)) \to 0$$

for all $n > 0$.

In the case of product of spheres, the Tor part disappears since all homology groups of a sphere are $\mathbb{Z}$-free.

Corollary 2.2. *The homology groups of a product of spheres is given by the following isomorphism*

$$H_n(S^{n_1} \times \cdots \times S^{n_k}) \cong \bigoplus_{i_1+\ldots+i_k=n} H_{i_1}(S^{n_1}) \otimes \cdots \otimes H_{i_k}(S^{n_k}).$$

As a consequence, nonzero homology groups of $S^{n_1} \times \cdots \times S^{n_k}$ are $\mathbb{Z}$-free and occurs at dimensions of the form $n_{j_1} + \cdots + n_{j_m}$ where $\{j_1, \ldots, j_m\}$ is a nonempty subset of $\{1, \ldots, k\}$.

Proof of Corollary 2.2. We will prove the corollary by induction on k. If $k = 1$, the statement is obvious. Assume $k > 1$ and the statement is true for all $m \leq k-1$. Let $X = S^{n_1} \times \cdots \times S^{n_{k-1}}$ and $Y = S^{n_k}$. Note that in the short exact sequence in Theorem 2.1, the Tor part is equal to 0 since $H_i(Y)$ is $\mathbb{Z}$-free for all i. Hence the first map in Theorem 2.1 becomes an isomorphism. By using the inductive step, we get the desired result. □

Let us apply this theorem to find homology groups of some products of spheres.

Example 2.3. Let us consider the case $n_1 = ... = n_k > 0$, in other words let $X := \underbrace{S^n \times \cdots \times S^n}_{k\ times}$ and $n > 0$.

By Corollary 2.2 we know that nonzero homology groups of X occur only at dimensions $0, n, ..., kn$ and for $j = 0, 1, ..., k$, we have

$$H_{jn}(X) = \bigoplus_{\binom{k}{j}} \mathbb{Z}.$$

Here is another example:

Example 2.4. Let $X := S^n \times S^{n+1}$ and $n > 0$. By Corollary 2.2 we have

$$H_i(X) = \begin{cases} \mathbb{Z} & \text{for } i = 0, n, n+1, 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

2.2 Group Actions and Cellular Chain Complexes

Let X be a CW-complex with cellular chain complex $(C_*(X), \partial)$ and G be a group acting cellularly on X. If e^n_α is an open n-cell in $C_n(X)$, then $e^n_{g\alpha} := g(e^n_\alpha)$ is again an open n-cell in $C_n(X)$ since the action is cellular. This defines a G-action on $C_n(X)$, hence $C_n(X)$ becomes a $\mathbb{Z}G$-module for all n. We will see that the boundary map ∂ respects this $\mathbb{Z}G$-module structure, i.e. $(C_*(X), \partial)$ is a chain complex of $\mathbb{Z}G$-modules. To see this, we should look what ∂ does.

We will denote the indices of open n-cells in X by α and the indices of open $(n-1)$-cells in X by β. Each open n-cell e^n_α is attached to the $(n-1)$ -skeleton X^{n-1} of X by an attaching map $\phi_\alpha : S^{n-1} \to X^{n-1}$. Since the action of G is cellular, we have $\phi_{g\alpha} = g\phi_\alpha$. For each open $(n-1)$-cell e^{n-1}_β, we have the quotient map $\pi_\beta : X^{n-1} \to S^{n-1}$ where π_β is the composition of the maps $X^{n-1} \to X^{n-1}/(X^{n-1} - e^{n-1}_\beta) \cong S^{n-1}$ where the first map is the quotient map and the second map comes from the embedding of e^{n-1}_β in X^{n-1}. Notice that $\pi_{g\beta} = \pi_\beta g^{-1}$ since the second map takes $e^{n-1}_{g\beta}$ to e^{n-1}_β and collapses all other cells to a point, hence in total it just collapses all cells except $e^{n-1}_{g\beta}$ to a point. The boundary map ∂ is defined by $\partial(e^n_\alpha) = \Sigma_\beta d_{\alpha\beta} e^n_\beta$ where $d_{\alpha\beta}$ denotes the degree of the map $\pi_\beta \circ \phi_\alpha : S^{n-1} \to S^{n-1}$ (see [10, p. 140]). We want to show that $\partial(e^n_{g\alpha}) = g\partial(e^n_\alpha)$. We have $g\partial(e^n_\alpha) = \Sigma_\beta d_{\alpha\beta} e^{n-1}_{g\beta} = \Sigma_\beta d_{\alpha(g^{-1}\beta)} e^{n-1}_\beta$. Hence, to show the desired equality, we need to show $d_{(g\alpha)\beta} = d_{\alpha(g^{-1}\beta)}$. This is true since $d_{(g\alpha)\beta} = deg(\pi_\beta \circ \phi_{g\alpha}) = deg(\pi_\beta \circ g \circ \phi_\alpha) = deg(\pi_{g^{-1}\beta} \circ \phi_\alpha) = d_{\alpha(g^{-1}\beta)}$. Therefore, we have shown that $(C_*(X), \partial)$ is a chain complex of $\mathbb{Z}G$-modules. This implies that homology groups are also $\mathbb{Z}G$-modules as quotients of $\mathbb{Z}G$-modules.

If X is a connected CW-complex, then any zero cell generates $H_0(X) \cong \mathbb{Z}$ as a $\mathbb{Z}$-module and they are all in the same homology class, hence the action of G on $H_0(X)$ is trivial. For a nonzero chain complex C_* of $\mathbb{Z}G$-modules, we will call C_* connected if $H_0(C) = \mathbb{Z}$ with trivial G-action.

If the action of G is free and cellular, then $(C_*(X), \partial)$ becomes a chain complex of free $\mathbb{Z}G$-modules as we see in the following argument: Let E denote the set of all n-cells of X. Then E becomes a G-set under the G-action we defined above. Since $C_n(X)$ is free abelian group generated by E, it is enough to show that the action of G on E is free. This is true since by the freeness of the action of G on X, we have $ge^n_\alpha = e^n_\alpha$ implies $g = 1$.

If X is an n-dimensional CW-complex, then the cellular chain complex $C_*(X)$ satisfies $C_n(X) \neq 0$ and $C_i(X) = 0$ for all $i > n$. A nonnegative chain complex satisfying these conditions is called an n-dimensional chain complex.

2.3 Tate Cohomology

The Tate cohomology of a finite group G with coefficients in a $\mathbb{Z}G$-module M is defined by using complete resolutions. A *complete resolution* of a finite group G is an acyclic complex (F_*, ∂_*) of free $\mathbb{Z}G$-modules together with maps $\varepsilon : F_0 \to \mathbb{Z}$, $\delta : \mathbb{Z} \to F_{-1}$ such that ε is a surjection, δ is an injection, and $\partial_0 = \delta \circ \varepsilon$ (see [4, p. 132]). Note that by exactness of F_* we get $\cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is a free resolution and $0 \to \mathbb{Z} \xrightarrow{\delta} F_{-1} \xrightarrow{\partial_{-1}} F_{-2} \to \cdots$ is an inverse free resolution (a free resolution in inverse direction). Conversely if we have a free resolution and an inverse free resolution, we can obtain a complete resolution by taking $\partial_0 = \delta \circ \varepsilon$. We already know that every $\mathbb{Z}G$-module has a free $\mathbb{Z}G$-resolution. Hence the existence of a complete resolution of a finite group G depends on the existence of an inverse free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Such a resolution can be obtained by taking a free $\mathbb{Z}G$-resolution F_* of $\mathbb{Z}$ such that all F_i's are finitely generated $\mathbb{Z}G$-modules (we will see that this is possible when G is finite) and applying $Hom_{\mathbb{Z}}(-, \mathbb{Z})$ to it (see [4, p. 133]). The Tate cohomology group of G is defined by $\hat{H}^*(G, M) = H^*(Hom_{\mathbb{Z}G}(F_*, M))$ where F_* is a complete resolution of G (see [4, p. 134]). Since there is a homotopy between any two complete resolutions of G (see [4, p. 132]), this definition is independent from the complete resolution F_* that we have chosen.

We have

$$\hat{H}^i(G,M) = \begin{cases} H^i(G,M) & \text{for } i \geq 1 \\ H_{-i-1}(G,M) & \text{for } i \leq -2 \end{cases}$$

Multiplying an element in $H^i(G,M)$ by the order of G, we obtain zero for $i \geq 1$, hence the group $H^i(G,M)$ has a finite exponent for $i \geq 1$. This follows from the composition of transfer and restriction maps and proved in [4, p. 84]. If we consider the Tate cohomology groups $\hat{H}^i(G,M)$, then we do not need to make an exception for $i = 0$ since $\hat{H}^i(G,M)$ has a finite exponent for all i. It appears that to obtain some facts about exponents, it is better to use Tate cohomology groups. Another advantage of Tate cohomology that simplifies calculations is that if P is a projective $\mathbb{Z}G$-module, then $\hat{H}^i(G,P) = 0$ (or equivalently we can say that *exp* $\hat{H}^i(G,P) = 1$) for all i. This fact is proved as follows: Let F_* be a complete resolution of G. An exact sequence $K \xrightarrow{i} L \xrightarrow{\pi} M$ of $\mathbb{Z}G$-modules is called an admissible exact sequence if the inclusion map $\mathrm{Im}\pi \hookrightarrow M$ is $\mathbb{Z}$-split (see [4, p. 129]). A $\mathbb{Z}G$-module M is called relatively injective if $Hom_G(-,M)$ takes admissible exact sequences of $\mathbb{Z}G$-modules to exact sequences of abelian groups. Projective $\mathbb{Z}G$ modules are relatively injective (see [4, p. 130]). Since F_* is an exact sequence of free $\mathbb{Z}G$ modules, the exact sequence $F_{i+1} \to F_i \to F_{i-1}$ is admissible exact for all i. Hence, for a projective module P, we have $\hat{H}^i(G,P) = 0$ for all i.

For a given $\mathbb{Z}G$-module M and an integer $m > 0$, we say that a $\mathbb{Z}G$-module N is the m-th syzygy of M if there is an exact sequence of $\mathbb{Z}G$-modules of the form $0 \to N \to P_m \to \cdots \to P_1 \to M \to 0$, where P_i's are projective $\mathbb{Z}G$-modules (see [14, p. 47]). We denote the m-th syzygy by $\Omega^m M$. For $m = 0$ we take $\Omega^0 M = M$. Notice that $\Omega^m M$ depends on projective modules we choose, but we handle this situation as follows. We choose and fix a free resolution for every $\mathbb{Z}G$-module and define $\Omega^m M$ according to that resolution. Let $\cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \to 0$ be a free resolution of M. We let $\Omega^m M = \mathrm{Im}(\partial_m)$. Furthermore, if G is finite and M is finitely generated as a $\mathbb{Z}G$-module (equivalently as a $\mathbb{Z}$-module), then we can choose F_m's finitely generated hence $\Omega^m M$ becomes finitely generated for all $m \geq 0$. We show this as follows: We construct F_m's inductively. Let $m_1, ..., m_k$ be a generating set for M. Let $F_0 = \bigoplus_{i=1}^k \mathbb{Z}G$ and $\partial_0 : F_0 \to M$ be the surjection taking the identity element of i-th summand to m_i for $i = 1, \ldots, k$. Now assume that (F_m, ∂_m) is defined. Since G is finite and F_m is finitely generated as a $\mathbb{Z}G$-module, F_m is finitely generated as a $\mathbb{Z}$-module. Hence if we let the $\mathbb{Z}G$-module K be the kernel of the map ∂_m, it is finitely generated as a $\mathbb{Z}$-module since $\mathbb{Z}$ is Noetherian. Therefore K is finitely generated as a $\mathbb{Z}G$-module. Hence we can find finitely generated free module F_{m+1} surjecting onto K by a map ∂_{m+1} as we found for M. Continuing this process we can obtain (F_*, ∂_*) which is a free $\mathbb{Z}G$-resolution of M with F_m's are finitely generated for all m.

If we fix resolutions as above, then the syzygies $\Omega^m M$ are completely determined by m and M, it is finitely generated if M is. Fixing resolutions in these ways simplifies some results we show later in the thesis. Syzygies satisfy the following nice properties.

Theorem 2.5. *If G is a finite group and M, N are $\mathbb{Z}G$-modules, then*

(i) $\hat{H}^i(G, M) \cong \hat{H}^{i+m}(G, \Omega^m M)$ *for all* $i \in \mathbb{Z}$*,*

(ii) $Ext^i_{\mathbb{Z}G}(\Omega^m M, N) \cong Ext^{i+m}_{\mathbb{Z}G}(M, N)$ *for all* $i \geq 1$*.*

Proof. Let $\cdots \to F_1 \to F_0 \to M \to 0$ be the free resolution of M that we fixed. Notice that there is a short exact sequence of the form $0 \to \Omega^{m+1} M \to F_m \to \Omega^m M \to 0$ for all $m \geq 0$. Corresponding long exact sequences for Tate cohomology and Ext groups are:

$$\cdots \to \hat{H}^i(G, F_m) \to \hat{H}^i(G, \Omega^m M) \to \hat{H}^{i+1}(G, \Omega^{m+1} M) \to \hat{H}^{i+1}(G, F_m) \to \cdots \tag{2.1}$$

$$\cdots \to Ext^i_{\mathbb{Z}G}(F_m, N) \to Ext^i_{\mathbb{Z}G}(\Omega^{m+1} M, N) \to Ext^{i+1}_{\mathbb{Z}G}(\Omega^m M, N) \to Ext^{i+1}_{\mathbb{Z}G}(F_m, N) \to \cdots. \tag{2.2}$$

For a projective $\mathbb{Z}G$-module P, we know that $\hat{H}^i(G, P) = 0$ for all i. Hence by (2.1) we have $\hat{H}^i(G, \Omega^m M) \cong \hat{H}^{i+1}(G, \Omega^{m+1} M)$, so $\hat{H}^i(G, M) \cong \hat{H}^{i+m}(G, \Omega^m M)$ for all i. Also if $i \geq 1$, then $Ext^i_{\mathbb{Z}G}(P, N) = 0$. Similarly, by (2.2) we get $Ext^i_{\mathbb{Z}G}(\Omega^{m+1} M, N) \cong Ext^{i+1}_{\mathbb{Z}G}(\Omega^m M, N)$, so $Ext^i_{\mathbb{Z}G}(\Omega^m M, N) \cong Ext^{i+m}_{\mathbb{Z}G}(M, N)$ for all $i \geq 1$. $\square$

Chapter 3

A Theorem of Browder and Habegger's Method

3.1 A Theorem of Browder

In Chapter 2 we have seen that if a group G acts freely and cellulary on a finite dimensional connected CW-complex X, then the cellular chain complex $C_*(X)$ becomes a nonnegative, connected, finite dimensional chain complex of free $\mathbb{Z}G$-modules. Browder proves the following theorem for such chain complexes.

Theorem 3.1 (Browder [3], p.599)**.** *Let G be a finite group and C_* be a non-negative, connected, n-dimensional chain complex of free $\mathbb{Z}G$-modules. Then the order of G divides $\prod_{j=1}^{n} exp\ H^{j+1}(G, H_j(C_*))$.*

We prove this theorem by using the following lemma.

Lemma 3.2. *If $K \xrightarrow{f} L \xrightarrow{g} M$ is an exact sequence of abelian groups where K, L, M has finite exponents e_K, e_L, e_M respectively, then e_L divides $e_K e_M$.*

Proof. Let $l \in L$. We need to show $(e_K e_M)l = 0$. The element $e_M l$ is in the kernel of the map g since $g(e_M l) = e_M g(l) = 0$. Since the sequence is exact, there exist a $k \in K$ such that $f(k) = e_M l$. Therefore, $(e_K e_M)l = e_K f(k) = f(e_K k) = 0$. $\square$

Now, we can give a proof of Theorem 3.1.

Proof of Theorem 3.1. For each integer j, there are following short exact sequences of $\mathbb{Z}G$-modules

$$0 \to Z_j \to C_j \to B_{j-1} \to 0$$

$$0 \to B_j \to Z_j \to H_j(C_*) \to 0$$

where Z_j denotes the j-cycles and B_j denotes the j-boundaries of C_*. The long exact sequence of Tate cohomology groups corresponding to the first short exact sequence above is

$$\cdots \to \hat{H}^i(G, C_j) \to \hat{H}^i(G, B_{j-1}) \to \hat{H}^{i+1}(G, Z_j) \to \hat{H}^{i+1}(G, C_j) \to \cdots$$

Since C_j is a free $\mathbb{Z}G$-module, $H^n(G, C_j) = 0$ for all n, so $\hat{H}^i(G, B_{j-1})$ is isomorphic to $\hat{H}^{i+1}(G, Z_j)$ for all i, j.

The long exact sequence of Tate cohomology groups corresponding to the second short exact sequence above is

$$\cdots \to \hat{H}^i(G, B_j) \to \hat{H}^i(G, Z_j) \to \hat{H}^i(G, H_j(C_*)) \to \cdots$$

In this sequence we can replace $\hat{H}^i(G, Z_j)$ with $\hat{H}^{i-1}(G, B_{j-1})$ since they are isomorphic by the above argument. Now, by Lemma 3.2 we have

$$\frac{exp\ \hat{H}^{i-1}(G, B_{j-1})}{exp\ \hat{H}^i(G, B_j)} \text{ divides } exp\ \hat{H}^i(G, H_j(C_*))$$

Notice that the quotient above may not be an integer but what we mean is that the right-hand side is an integer multiple of left-hand side. Letting $i = j + 1$ and multiplying both sides of the expression above through $j = 1, ..., n$, we get

$$\frac{exp\ \hat{H}^1(G, B_0)}{exp\ \hat{H}^{n+1}(G, B_n)} \text{ divides } \prod_{j=1}^{n} exp\ \hat{H}^{j+1}(G, H_j(C_*))$$

Since C_* is n-dimensional, we have $B_n = 0$, so the denominator of the left hand side of the above expression is 1. Also, the Tate cohomology groups on the right hand side of the above expression is the same as the ordinary cohomology groups since $j+1>1$ for $j=1,...,n$. Therefore to prove the theorem, it is enough to show $exp\ \hat{H}^1(G,B_0)=|G|$. We will show that $\hat{H}^1(G,B_0)\cong\mathbb{Z}/|G|$.

Since C_* is a nonnegative chain complex, we have $Z_0=C_0$ and there is a short exact sequence

$$0\to B_0\to C_0\to H_0(C_*)\to 0$$

where $H_0(C_*)\cong\mathbb{Z}$. As above, by considering the long exact Tate cohomology sequence and using the freeness of C_0, we get $\hat{H}^1(G,B_0)\cong\hat{H}^0(G,\mathbb{Z})\cong\mathbb{Z}/|G|$. This completes the proof. □

If we have some upper bounds on the exponents of $H^{j+1}(G,H_j(C_*))$ in Theorem 3.1, we can obtain restrictions on the order of the group G. The following theorem gives us an upper bound for the exponents of Tate cohomology groups in a particular case.

Theorem 3.3. *If $G=(\mathbb{Z}/p)^r$ and M is a $\mathbb{Z}G$-module where G acts trivially on M, then $exp\ H^i(G,M)$ divides p for all $i\geq 1$.*

Proof. We will prove by induction on r. If $r=1$, the statement is true since $|G|=p$ and the exponent of the Tate cohomology groups divides the order of the group.

Assume $r>1$ and the statement is true for rank strictly less than r. We know that $H^i(-,-)$ is a contravariant functor from the category of pairs (K,N) where K is a group and N is a $\mathbb{Z}K$-module (see [4, p. 78]). In this category, a morphism from (K,N) to (K',N') is a pair (α,f) such that $\alpha:K\to K'$ a group homomorphism, $f:N'\to N$ is a $\mathbb{Z}$-module map with $f(\alpha(k)n')=k\alpha(n')$ for all $k\in K,\ n'\in N'$. In other words, f is a $\mathbb{Z}K$-module map if we consider N' as a $\mathbb{Z}K$-module by defining $kn':=\alpha(k)n'$. Now, let $H=(\mathbb{Z}/p)^{r-1}$, $j:H\to G$ be the inclusion map and $\pi:G\to H$ be the projection map such that $\pi\circ j=id_H$. M is also a $\mathbb{Z}H$-module with trivial H action and $\phi:=(j,id_M)$ is a morphism from (H,M) to (G,M). Since the action of G is trivial on M, $\psi:=(\pi,id_M)$ is a morphism from (G,M) to (H,M). Notice that $\psi\circ\phi=id_{(H,M)}$. If we let ϕ^* and ψ^* be the maps between cohomology groups obtained by applying the contravariant functor $H^*(-,-)$ to ϕ and ψ respectively, we get $\phi^* = \mathrm{res}^G_H : H^i(G,M) \to H^i(H,M)$ and $\phi^* \circ \psi^* = (\psi \circ \phi)^* = id_{H^i(H,M)}$. Therefore the restriction map splits and $H^i(G,M) \cong \mathrm{Ker}(\mathrm{res}^G_H) \bigoplus H^i(H,M)$. By induction we know that the exponent of $H^i(H,M)$ divides p, hence it is enough to show that the exponent of $Ker(\mathrm{res}^G_H)$ divides p.

Take any element x in $H^i(G,M)$. We know that $\mathrm{tr}^G_H \mathrm{res}^G_H(x) = [G:H]x = px$ (see [4, p. 82]). Hence if $x \in Ker(\mathrm{res}^G_H)$, then $px = 0$. Therefore, the exponent of $Ker(\mathrm{res}^G_H)$ divides p. $\square$

Corollary 3.4. *Let $G = (\mathbb{Z}/p)^r$ and X be a CW-complex homotopy equivalent to $\underbrace{S^n \times \cdots \times S^n}_{k\ times}$ with $n \geq 1$. If G acts freely and cellularly on X with trivial action on homology groups of X, then $r \leq k$.*

Proof. Let $C_*(X)$ denote the cellular chain complex of X. In Chapter 2 we have seen that $C_*(X)$ is a nonnegative, connected, finite chain complex of free $\mathbb{Z}G$ modules. Homology groups of this chain complex are nonzero at dimensions $0, n, 2n, ..., kn$. Hence by Theorem 3.1, $|G| = p^r$ divides $\prod_{j=1}^k exp\ H^{jn+1}(G, H_{jn}(X))$. By Theorem 3.3, the last expression divides p^k. Therefore, p^r divides p^k and hence $r \leq k$. $\square$

3.2 Habegger's Method

In previous section we have used Theorem 3.1 to show that if $G = (Z/p)^r$ acts freely and cellularly on a CW-complex X homotopy equivalent to $S^{n_1} \times \cdots \times S^{n_k}$ where $n_1 = \cdots = n_k$ and the action of G on homology groups of X is trivial, then $r \leq k$. However, if the dimensions of spheres are not equal, then their product has nonzero homology groups at more than k-many dimensions, hence we can not obtain $r \leq k$ by applying Theorem 3.1. In this section we present a method such that for a given chain complex we can glue homologies at different dimensions and decrease the number of dimensions where the homology groups are nonzero.

We say that a chain complex C_* is freely equivalent to D_* if there is a short exact sequence of chain complexes of the form $0 \to C_* \to D_* \to F_* \to 0$ or $0 \to F_* \to C_* \to D_* \to 0$, where F_* is a finite complex of free $\mathbb{Z}G$-modules. In this case, if C_* is a finite chain complex, then D_* is also finite chain complex and if C_* is a chain complex of free $\mathbb{Z}G$-modules, then also D_* is.

Now we can state the main theorem of this section that gives us a method such that for a given chain complex C_*, we can obtain a new chain complex whose nonzero homologies occurs at fewer dimensions while it is still very similiar to C_*. This method can be found in Habegger's article [8, p. 433-434].

Theorem 3.5. *Let C_* be a chain complex and n, m are integers such that $n < m$. If for all k with $n < k < m$ we have $H_k(C_*) = 0$, then C_* is freely equivalent to a chain complex D_* such that*

(i) $D_i = C_i$ for every $i \leq n$ or $i > m$;

(ii) $H_i(D_*) = H_i(C_*)$ for every $i \neq n, m$;

(iii) $H_n(D_*) = 0$;

(iv) there is an exact sequence of $\mathbb{Z}G$-modules

$$0 \to H_m(C_*) \to H_m(D_*) \to \Omega^{m-n} H_n(C_*) \to 0.$$

Proof. Let $F_{m-1} \to ... \to F_n \to H_n(C_*) \to 0$ be an exact sequence where all F_i's are free $\mathbb{Z}G$-modules. Let Z_n be the set of cycles in C_n, which also a subgroup of C_n. Consider the following diagram:

$$\begin{array}{ccccccccccccccc}
... & \longrightarrow & 0 & \longrightarrow & F_{m-1} & \longrightarrow & ... & \longrightarrow & F_n & \longrightarrow & H_n(C_*) & \longrightarrow & 0 & \longrightarrow & ... \\
 & & & & & & & & & & {\scriptstyle id}\downarrow & & \downarrow & & \\
... & \longrightarrow & C_m & \longrightarrow & C_{m-1} & \longrightarrow & ... & \longrightarrow & Z_n & \longrightarrow & H_n(C_*) & \longrightarrow & 0 & \longrightarrow & ...
\end{array}$$

Since all F_i's are projective and the bottom row has no homology below dimension m, the identity map extends to a chain map between rows.

$$\begin{array}{ccccccccccccccc}
... & \longrightarrow & 0 & \longrightarrow & F_{m-1} & \longrightarrow & ... & \longrightarrow & F_n & \longrightarrow & H_n(C_*) & \longrightarrow & 0 & \longrightarrow & ... \\
 & & & & {\scriptstyle f_{m-1}}\downarrow & & & & {\scriptstyle f_n}\downarrow & & {\scriptstyle id}\downarrow & & \downarrow & & \\
... & \longrightarrow & C_m & \longrightarrow & C_{m-1} & \longrightarrow & ... & \longrightarrow & Z_n & \longrightarrow & H_n(C_*) & \longrightarrow & 0 & \longrightarrow & ...
\end{array}$$

Notice that this chain map is still a chain map if we consider it between $f_* : F_* \to C_*$, as shown in the following diagram.

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & F_{m-1} & \longrightarrow & \dots & \longrightarrow & F_n & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow{\scriptstyle f_{m-1}} & & & & \downarrow{\scriptstyle f_n} & & \downarrow & & \\
\dots & \longrightarrow & C_m & \longrightarrow & C_{m-1} & \longrightarrow & \dots & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \dots
\end{array}$$

Now let D_* be the mapping cone of f_*. We can immediately see that $D_i = C_i$ if $i \leq n$ or $i > m$. We have the following short exact sequence:

$$0 \to C_* \to D_* \to \Sigma F_* \to 0$$

where ΣF_* denotes the chain complex $(\Sigma F_*)_i = F_{i-1}$ and the boundary map is equal to -1 times the boundary of F_*. So C_* is freely equivalent to D_*. Corresponding long exact sequence of homology groups is

$$\dots \longrightarrow H_i(F_*) \xrightarrow{f_*} H_i(C_*) \longrightarrow H_i(D_*) \longrightarrow H_{i-1}(F_*) \longrightarrow \dots$$

Notice that $f_* : H_n(F_*) \to H_n(C_*)$ is a surjection, furthermore it is an isomorphism if $m > n+1$.

If $i > m$ or $i < n$, then $H_i(F_*) = H_{i-1}(F_*) = 0$, hence $H_i(C_*) = H_i(D_*)$.

If $n < i < m$, then we have $0 \to H_i(D_*) \to H_{i-1}(F_*) \to H_{i-1}(C_*)$ exact. If $n+1 < i < m$, then $H_{i-1}(F_*) = 0$, so $H_i(D_*) = 0$. If $i = n+1$, then $m > n+1$, hence $f_* : H_n(F_*) \to H_n(C_*)$ is an isomorphism. This implies that $H_i(D_*) = 0$. Therefore, if $n < i < m$, then $H_i(D_*) = H_i(C_*) = 0$. By combining with the above paragraph, we conclude that $H_i(D_*) = H_i(C_*)$ for all $i \neq m, n$.

If $i = n$, then we have the exact sequence $H_n(F_*) \to H_n(C_*) \to H_n(D_*) \to 0$. Since the first map is a surjection, $H_n(D_*) = 0$. It remains to show that we have an exact sequence $0 \to H_m(C_*) \to H_m(D_*) \to \Omega^{m-n} H_n(C_*) \to 0$. If $m = n+1$, we have $0 \to H_m(C_*) \to H_m(D_*) \to F_n \to H_n(C) \to 0$. Hence the result follows. If $m > n+1$, then the sequence $0 \to H_m(C_*) \to H_m(D_*) \to H_{m-1}(F_*) \to 0$ is exact, and this proves the result since $H_{m-1}(F_*) = \Omega^{m-n}(H_n(C_*))$. $\square$

Chapter 4

Tate Hypercohomology

In this chapter we give another proof of Theorem 3.1 by using Habegger's method. To do this, we generalize the concept of Tate cohomology and obtain Tate hypercohomology where coefficients of the cohomology groups comes from a chain complex. One can skip this chapter and read the last chapter to see the proof the main theorem since material of this chapter will not be used in the last chapter.

Many definitions and theorems that we will prove for chain complexes of $\mathbb{Z}G$-modules in this chapter are valid for arbitrary chain complexes, but for our purposes we will restrict our attention to chain complexes of $\mathbb{Z}G$-modules. Throughout this section, every chain complex will be a chain complex of $\mathbb{Z}G$-modules.

4.1 Extended Hom Functor

Recall that for a finite group G and a $\mathbb{Z}G$-module M, the i-th Tate cohomology group is defined by $\hat{H}^i(G, M) = H^i(Hom_{\mathbb{Z}G}(F_*, M))$ where F_* is a complete resolution of G (see [4, p. 134]). Notice that $Hom_{\mathbb{Z}G}(-, M)$ is a functor from the category of chain complexes of $\mathbb{Z}G$-modules to the category of cochain complexes of abelian groups. If we can generalize this functor to the functor $\mathcal{H}om_{\mathbb{Z}G}(-, C_*)$ from the category of chain complexes of $\mathbb{Z}G$-modules to the category of cochain complexes of abelian groups where C_* is a chain complex, then we obtain Tate cohomology groups with coefficients in a chain complex.

A graded module homomorphism f_* of degree n from a chain complex C_* to a chain complex D_* is a family of module homomorphisms $(f_k)_{k=-\infty}^{\infty}$ such that $f_k : C_k \to D_{k+n}$ for all k. The group $\mathcal{H}om^n_{\mathbb{Z}G}(C_*, D_*)$ is defined to be the set of all graded module homomorphisms of degree $-n$ from C_* to D_*. This set has an abelian group structure under addition of graded module homomorphisms. Define the boundary map $\delta^n : \mathcal{H}om^n_{\mathbb{Z}G}(C_*, D_*) \to \mathcal{H}om^{n+1}_{\mathbb{Z}G}(C_*, D_*)$ by $\delta^n(f) = f\partial - (-1)^n \partial f$ (see [4, p. 5]). By these definitions, $(\mathcal{H}om_{\mathbb{Z}G}(C_*, D_*), \delta)$ becomes a cochain complex of abelian groups.

Let us show that $\mathcal{H}om_{\mathbb{Z}G}(C_*, -)$ is a covariant functor from the category of chain complexes of $\mathbb{Z}G$-modules to the category of cochain complexes of abelian groups. Let E_*, E'_* be two chain complexes of $\mathbb{Z}G$-modules and f_* be a chain map from E_* to E'_*. Let g_* be a graded module homomorphism of degree n from C_* to E_*. Define the graded module homomorphism $(fg)_* : C_* \to E'_*$ such that $(fg)_k = f_{k+n} \circ g_k$. If we define $\mathcal{H}om_{\mathbb{Z}G}(C_*, f_*)$ in this way, then $\mathcal{H}om_{\mathbb{Z}G}(C_*, -)$ becomes a covariant functor from the category of chain complexes of $\mathbb{Z}G$-modules to the category of cochain complexes of abelian groups. Similarly, $\mathcal{H}om_{\mathbb{Z}G}(-, D_*)$ is a contravariant functor from the category of chain complexes of $\mathbb{Z}G$-modules to the category of cochain complexes of abelian groups.

If D_* is a chain complex concentrated at 0, then $\mathcal{H}om_{\mathbb{Z}G}(C_*, D_*) \cong Hom_{\mathbb{Z}G}(C_*, D_0)$. Hence the contravariant functor $\mathcal{H}om_{\mathbb{Z}G}(-, D_*)$ extends the functor $Hom_{\mathbb{Z}G}(-, M)$ if we consider a module as a chain complex concentrated at 0. Now let us define Tate hypercohomology of a finite group G with coefficients in a $\mathbb{Z}G$-module C_* as $\hat{H}^*(G, C_*) := H^*(\mathcal{H}om_{\mathbb{Z}G}(F_*, C_*))$ where F_* is a complete resolution of G. This is well defined since if F'_* is another complete resolution of G then it is homotopic to F_* and by functoriality of $\mathcal{H}om_{\mathbb{Z}G}(-, C_*)$ the cochain complex $\mathcal{H}om_{\mathbb{Z}G}(F_*, C_*)$ is homotopic to the cochain complex $\mathcal{H}om_{\mathbb{Z}G}(F'_*, C_*)$. Similarly, Tate hypercohomology extends Tate cohomology if we consider a module as a chain complex concentrated at 0. Now let us obtain some properties of $\mathcal{H}om$ and Tate hypercohomology.

For a chain complex (C_*, ∂_*), the *n-fold suspension* of C_* is the chain complex denoted by $(\Sigma^n C_*, \Sigma^n \partial)$ such that $(\Sigma^n C)_k := C_{k-n}$ and $(\Sigma^n \partial)_k := (-1)^n \partial_{k-n}$. We write ΣC_* instead of $\Sigma^1 C_*$. With this notation we have the equality $\Sigma^n C_* = \Sigma(\Sigma^{n-1} C_*)$ (see [4, p. 5]). The n-fold suspension of a cochain complex is defined similarly.

Proposition 4.1. *Let G be a group and C_*, D_* be chain complexes of $\mathbb{Z}G$-modules.*

(i) $\mathcal{H}om_{\mathbb{Z}G}(\Sigma^n C_*, D_*) = \Sigma^n \mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)$,

(ii) $\mathcal{H}om_{\mathbb{Z}G}(C_*, \Sigma^n D_*) \cong \Sigma^{-n} \mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)$.

Proof. (i) Let $f_* : \Sigma^n C_* \to D_*$ be a graded module homomorphism of degree $-i$. The $\mathbb{Z}G$-module homomorphism $f_k : \Sigma^n C_p \to D_{p-i}$ can be considered as $f_k : C_{p-n} \to D_{p-i}$. Hence f_* is a graded module homomorphism of degree $-(i-n)$ from C_* to D_*, implying $\mathcal{H}om^i_{\mathbb{Z}G}(\Sigma^n C_*, D_*) = \mathcal{H}om^{i-n}_{\mathbb{Z}G}(C_*, D_*) = (\Sigma^n \mathcal{H}om_{\mathbb{Z}G}(C_*, D_*))^i$. If we denote the boundary map of $\mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)$ by δ, then $\Sigma^n \delta^i(f) = (-1)^n \delta^{i-n}(f) = (-1)^n [f\partial - (-1)^{i-n} \partial f] = f \Sigma^n \partial - (-1)^i \partial f$, which is equal to the boundary map of $\mathcal{H}om_{\mathbb{Z}G}(\Sigma^n C_*, D_*)$. This proves (i).

(ii) Let $f_* : C_* \to \Sigma^n D_*$ be a graded module homomorphism of degree $-i$. The $\mathbb{Z}G$-module homomorphism $f_k : C_p \to (\Sigma^n D_*)_{p-i}$ can be considered as $f_k : C_p \to D_{p-i-n}$. Hence f_* is a graded module homomorphism of degree $-(i+n)$ from C_* to D_*, implying f_* is an element of $\mathcal{H}om^{i+n}_{\mathbb{Z}G}(C_*, D_*) = (\Sigma^{-n} \mathcal{H}om_{\mathbb{Z}G}(C_*, D_*))^i$. Define $\Phi^* : \mathcal{H}om_{\mathbb{Z}G}(C_*, \Sigma^n D_*) \to \Sigma^{-n} \mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)$ such that $\Phi^i : \mathcal{H}om^i_{\mathbb{Z}G}(C_*, \Sigma^n D_*) \to (\Sigma^{-n} \mathcal{H}om_{\mathbb{Z}G}(C_*, D_*))^i$ is the isomorphism sending f to $(-1)^{in} f$. It is enough to show that Φ^* is a chain map. Let α^*, β^* denote the boundary maps of $\mathcal{H}om_{\mathbb{Z}G}(C_*, \Sigma^n D_*)$ and $\Sigma^{-n} \mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)$ respectively and let δ^* denote the boundary map of $\mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)$. We need to show that $\Phi^{i+1} \circ \alpha^i = \beta^i \circ \Phi^i$, in other words, $(-1)^{in+n} \alpha^i = (-1)^{in} \beta^i$ that means

$\alpha^i = (-1)^n \beta^i$. But this is true since

$$\begin{aligned}
\beta^i(f) &= (-1)^n \delta^{i+n}(f) \\
&= (-1)^n [f\partial + (-1)^{i+n} \partial f] \\
&= (-1)^n [f\partial + (-1)^i (\Sigma^n \partial) f] \\
&= (-1)^n \alpha^i(f).
\end{aligned}$$

□

Let us consider the cycles, boundaries, and the cohomology groups of the cochain complex $\mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)$. We shall start with cycles and boundaries at dimension zero. Let $f_* : C_* \to D_*$ be a graded module homomorphism of degree 0. It is a 0-cycle if $\delta^0(f) = f\partial - \partial f = 0$, in other words if it is a chain map. A 0-cycle is a boundary if it is equal to $\delta^1(h) = h\partial + \partial h$ for some $h_* : C_* \to D_*$ a graded module homomorphism of degree -1. Since two 0-cycles (or equivalently chain maps) f and g belongs to the same homology class if $f - g = \delta h = h\partial + \partial h$ for some $h : C_* \to D_*$ a graded module homomorphism of degree 1, they have the same homology class if they are homotopic. Hence there is a bijection between $H^0(\mathcal{H}om_{\mathbb{Z}G}(C_*, D_*))$ and the homotopy classes of chain maps from C_* to D_*. Homotopy classes of chain maps from C_* to D_* is denoted by $[C_*, D_*]$ (see [4, p. 5]). There is a natural way to give an abelian group structure to this set since if a chain map f is homotopic to f' and a chain map g is homotopic to g' then $f + g$ is homotopic to $f' + g'$. With this abelian group structure we have $H^0(\mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)) \cong [C_*, D_*]$. By using this result and Proposition 4.1, we have the following corollary.

Corollary 4.2. *Let C_*, D_* be a chain complexes of $\mathbb{Z}G$-modules. We have isomorphisms $H^n(\mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)) \cong [\Sigma^{-n} C_*, D_*] \cong [C_*, \Sigma^n D_*]$.*

Proof. By the definition of suspension, we have an isomorphism

$$H^n(\mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)) \cong H^0(\Sigma^{-n} \mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)).$$

Theorem 4.1 implies that

$$\begin{aligned}
H^0(\Sigma^{-n} \mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)) &\cong H^0(\mathcal{H}om_{\mathbb{Z}G}(\Sigma^{-n} C_*, D_*)) \\
&\cong [\Sigma^{-n} C_*, D_*]
\end{aligned}$$

and

$$\begin{aligned} H^0(\Sigma^{-n}\mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)) &\cong H^0(\mathcal{H}om_{\mathbb{Z}G}(C_*, \Sigma^n D_*)) \\ &\cong [C_*, \Sigma^n D_*]. \end{aligned}$$

These prove the statement. □

Corollary 4.3. *Let P_* be a chain complex of projective $\mathbb{Z}G$-modules. Then*

(i) If C_* is an acyclic nonnegative chain complex of $\mathbb{Z}G$-modules, then the cochain complex $\mathcal{H}om_{\mathbb{Z}G}(P_*, C_*)$ is acyclic.

(ii) If P_* is nonnegative and C_* is an acyclic chain complex of $\mathbb{Z}G$-modules, then the cochain complex $\mathcal{H}om_{\mathbb{Z}G}(P_*, C_*)$ is acyclic.

Proof. By Corollary 4.2, it is enough to show that $[P_*, \Sigma^n C_*] = 0$ for all n. This is true in both of the cases (i),(ii) by the *fundamental lemma of homological algebra* (see [4, p. 22]). □

Let $f_* : D_* \to D'_*$ be a chain map. We know that the mapping cone of f_* gives important informations about f_*. The following theorem says that the mapping cone of the $\mathcal{H}om_{\mathbb{Z}G}(C_*, f)$ is isomorphic to the $\mathcal{H}om_{\mathbb{Z}G}(C_*, E_*)$ where E_* is the mapping cone of f_*. In other words it says that it is same if you first take mapping cone and then apply $\mathcal{H}om$ or if you first apply $\mathcal{H}om$ and then take mapping cone.

Theorem 4.4. *Let C_*, D_*, D'_* be chain complexes of $\mathbb{Z}G$-modules and $f_* : D_* \to D'_*$ be a chain map. If we denote the mapping cone of f_* by E_*, then the mapping cone of $\mathcal{H}om_{\mathbb{Z}G}(C_*, f_*)$ is isomorphic to $\mathcal{H}om_{\mathbb{Z}G}(C_*, E_*)$.*

Before proving this theorem let us recall the definition of the mapping cone for chain complexes and cochain complexes. Let $f : D_* \to D'_*$ be a chain map and ∂, ∂' be the boundary maps of D_*, D'_* respectively. The mapping cone of f is a chain complex (E_*, ∂'') such that $E_i = D'_i \bigoplus D_{i-1}$ and $\partial''(d', d) = (\partial' d' + f(d), -\partial d)$. We can write ∂'' in matrix notation as follows (see [4, p. 6])

$$\partial'' = \begin{pmatrix} \partial' & f \\ 0 & -\partial \end{pmatrix}$$

Mapping cones of chain maps between cochain complexes defined similarly. Let $g^* : D^* \to D'^*$ be a chain map and δ, δ' be the boundary maps of D^*, D'^* respectively. The mapping cone of g is a cochain complex E^*, δ'' such that $E^i = D'^i \bigoplus D^{i+1}$ and $\delta(d', d) = (\delta' d' + g(d), -\delta d)$. We can write δ'' in matrix notation as follows

$$\delta'' = \begin{pmatrix} \delta' & g \\ 0 & -\delta \end{pmatrix}$$

Now let us prove Theorem 4.4.

Proof of Theorem 4.4. Let δ, δ' denote the boundary maps of cochain complexes $\mathcal{H}om_{\mathbb{Z}G}(C_*, D_*)$ and $\mathcal{H}om_{\mathbb{Z}G}(C_*, D'_*)$ respectively. We have the chain map

$$\mathcal{H}om_{\mathbb{Z}G}(C_*, f_*) : \mathcal{H}om_{\mathbb{Z}G}(C_*, D_*) \to \mathcal{H}om_{\mathbb{Z}G}(C_*, D'_*)$$

If we denote the mapping cone of this map by (A^*, δ''), then

$$A^i = \mathcal{H}om^i_{\mathbb{Z}G}(C_*, D'_*) \bigoplus \mathcal{H}om^{i+1}_{\mathbb{Z}G}(C_*, D_*)$$

and we can write δ'' in matrix form as follows

$$(\delta'')^i = \begin{pmatrix} (\delta')^i & f \\ 0 & -\delta^{i+1} \end{pmatrix}$$

Let ∂^E denote the boundary map of E_* and γ denote the boundary map of $\mathcal{H}om_{\mathbb{Z}G}(C_*, E_*)$. If $h : C_* \to E_*$ is a graded module homomorphism of degree $-i$, then since $h_p : C_p \to D'_{p-i} \bigoplus D_{p-i-1}$, we can consider h as a pair of graded module homomorphisms (g', g) where $g' : C_* \to D'_*$ a graded module homomorphism of degree $-i$ and $g : C_* \to D_*$ is a graded module homomorphism of degre $-(i+1)$. Under these identifications, we have

$$\mathcal{H}om^i_{\mathbb{Z}G}(C_*, E_*) = \mathcal{H}om^i_{\mathbb{Z}G}(C_*, D'_*) \bigoplus \mathcal{H}om^{i+1}_{\mathbb{Z}G}(C_*, D_*) = A^i$$

and

$$\begin{aligned}\gamma^i(g',g) &= (g',g)\partial - (-1)^i\partial^E(g',g)\\ &= (g'\partial, g\partial) - (-1)^i(\partial g' + fg, -\partial g)\\ &= (g'\partial - (-1)^i\partial g' + (-1)^{i+1}fg, g\partial - (-1)^{i+1}\partial g)\\ &= ((\delta')^i g' + (-1)^{i+1}fg, \delta^{i+1}g)\end{aligned}$$

Therefore, we can write γ in matrix notation as follows

$$\gamma^i = \begin{pmatrix} (\delta')^i & (-1)^{i+1}f \\ 0 & \delta^{i+1}\end{pmatrix}$$

Now define $\Phi^* : A^* \to \mathcal{H}om_{\mathbb{Z}G}(C_*, E_*)$ such that Φ_i is the isomorphism sending (g',g) to $(g', (-1)^{i+1}g)$. It is enough to show this is a chain map, i.e. $\gamma^i \circ \phi^i = \phi^{i+1} \circ (\delta'')^i$. Let us see that this is true by calculating both of them.

$$\begin{aligned}\gamma^i \circ \phi^i(g',g) &= \gamma^i(g', (-1)^{i+1}g)\\ &= ((\delta')^i g' + fg, (-1)^{i+1}\delta^{i+1}g)\end{aligned}$$

and

$$\begin{aligned}\phi^{i+1} \circ (\delta'')^i(g',g) &= \phi^{i+1}((\delta')^i g' + fg, -\delta^{i+1}g)\\ &= ((\delta')^i g' + fg, -(-1)^{i+2}\delta^{i+1}g)\\ &= ((\delta')^i g' + fg, (-1)^{i+1}\delta^{i+1}g)\end{aligned}$$

Therefore $\gamma^i \circ \phi^i = \phi^{i+1} \circ (\delta'')^i$, implying that $A^* \cong \mathcal{H}om_{\mathbb{Z}G}(C_*, E_*)$. □

We have the following corollary (see [4, p. 29]).

Corollary 4.5. *Let D_*, D'_* be nonnegative chain complexes $\mathbb{Z}G$-modules and $f : D_* \to D'_*$ be a weak equivalence. If P_* is a chain complex of projective $\mathbb{Z}G$-modules, then $\mathcal{H}om_{\mathbb{Z}G}(P_*, f_*) : \mathcal{H}om_{\mathbb{Z}G}(P_*, D_*) \to \mathcal{H}om_{\mathbb{Z}G}(P_*, D'_*)$ is a weak equivalence.*

Proof. A chain map is a weak equivalence if and only if its mapping cone is acyclic. Hence by Theorem 4.4 it is enough to show that $\mathcal{H}om_{\mathbb{Z}G}(P_*, E_*)$ is acyclic where E_* is the mapping cone of f. Since f is a weak equivalence, E_* is acyclic. Therefore, by Corollory 4.3, $\mathcal{H}om_{\mathbb{Z}G}(P_*, E_*)$ is acyclic. □

Let us return back to Tate hypercohomology. This corollary implies that if G is a finite group and C_*, D_* are nonnegative chain complexes of $\mathbb{Z}G$-modules such that C_* is weakly equivalent to D_*, then $\hat{H}^i(G, C_*) \cong \hat{H}^i(G, D_*)$.

Proposition 4.6. *If C_* is a nonnegative chain complex of $\mathbb{Z}G$-modules whose homology concentrated at dimension n, then $H^i(G, C_*) = H^{i+n}(G, H_n(C_*))$ for all i.*

Proof. Let Z_n denote the n-cycles of C_*. Define the chain complex D_*, E_* as follows:

$$D_i = \begin{cases} C_i & \text{if } i > n \\ Z_n & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$$

where D_* has same boundary map with C_*, and let E_* be the chain complex concentrated at dimension n with $E_n = H_n(C_*)$. If we consider $H_n(C_*)$ as a chain complex concentrated at 0, then $E_* = \Sigma^n(H_n(C_*))$. Hence by Proposition 4.1, we have $H^i(G, E_*) = H^{i+n}(G, H_n(C_*))$ for all i.

Define a chain map from D_* to C_* as follows

$$\begin{tikzcd}
D_* :\ldots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "id"'] & Z_n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
C_* :\ldots \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

where the map $Z_n \to C_n$ is the inclusion map. This is a weak equivalence, hence $H^i(G, D_*) \cong H^i(G, C_*)$ for all i. Now define a chain map from D_* to E_* as follows

$$\begin{tikzcd}
D_* :\ldots \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & Z_n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
E_* :\ldots \arrow[r] & 0 \arrow[r] & H_n(C_*) \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

where the map $Z_n \to H_n(C_*)$ is the quotient map. This is also a weak equivalence, hence $H^i(G, D_*) \cong H^i(G, E_*)$, implying $H^i(G, C_*) \cong H^i(G, E_*) \cong H^{i+n}(G, H_n(C_*))$ for all i. □

We know that if P is a projective $\mathbb{Z}G$-module, then $Hom_{\mathbb{Z}G}(P, -)$ is an exact functor, i.e., it takes exact sequences to exact sequences. We have a generalization of this fact for $\mathcal{H}om$.

Proposition 4.7. *Let $C_* \xrightarrow{\alpha} D_* \xrightarrow{\beta} E_*$ be a short exact sequence of chain complexes of $\mathbb{Z}G$-modules. If P_* is a chain complex of projective $\mathbb{Z}G$-modules, then the following sequence of cochain complexes is exact*

$$\mathcal{H}om_{\mathbb{Z}G}(P_*, C_*) \xrightarrow{\mathcal{H}om_{\mathbb{Z}G}(P_*,\alpha)} \mathcal{H}om_{\mathbb{Z}G}(P_*, D_*) \xrightarrow{\mathcal{H}om_{\mathbb{Z}G}(P_*,\beta)} \mathcal{H}om_{\mathbb{Z}G}(P_*, E_*).$$

Proof. Let $f : P_* \to D_*$ be a graded module homomorphism of degree n. We need to show that if $\beta \circ f = 0$, then there is a graded module homomorphism $g : P_* \to C_*$ of degree n such that $\alpha \circ g = f$. For all i, we have the following diagram:

$$\begin{tikzcd} & P_i \arrow[d, "f_i"] \arrow[dr, "0"] & \\ C_{i+n} \arrow[r, "\alpha"] & D_{i+n} \arrow[r, "\beta"] & E_{i+n} \end{tikzcd}$$

By the projectivity of P_i, there is a module homomorphism $g_i : P_i \to C_{i+n}$ such that $\alpha \circ g_i = f_i$. Therefore, there is a graded module homomorphism $g : P_* \to C_*$ of degree n such that $\alpha \circ g = f$. □

By using this proposition, we can obtain the long exact sequence for Tate hypercohomology.

Proposition 4.8. *Let G be a finite group and $0 \to C_* \to D_* \to E_* \to 0$ be a short exact sequence of chain complexes of $\mathbb{Z}G$-modules. Then, there is a long exact sequence of the form*

$$\cdots \to \hat{H}^i(G, C_*) \to \hat{H}^i(G, D_*) \to \hat{H}^i(G, E_*) \to \hat{H}^{i+1}(G, C_*) \to \cdots .$$

Proof. Let F_* be a complete resolution of group G. By Proposition 4.7 we have the following short exact sequence of cochain complexes

$$0 \to \mathcal{H}om_{\mathbb{Z}G}(F_*, C_*) \to \mathcal{H}om_{\mathbb{Z}G}(F_*, D_*) \to \mathcal{H}om_{\mathbb{Z}G}(F_*, E_*) \to 0.$$

Corresponding long exact sequence for cohomology groups is

$$\cdots \to \hat{H}^i(G, C_*) \to \hat{H}^i(G, D_*) \to \hat{H}^i(G, E_*) \to \hat{H}^{i+1}(G, C_*) \to \cdots .$$

□

In Chapter 2, we have mentioned that for a finite group G and a projective $\mathbb{Z}G$-module P, $\hat{H}^i(G, P) = 0$ for all i. We will generalize this result to Tate hypercohomology, not for arbitrary but finite chain complexes of projective $\mathbb{Z}G$-modules.

Proposition 4.9. *Let G be a finite group. If P_* is a finite chain complex of projective modules, then $\hat{H}^i(G, P_*) = 0$ for all i.*

Proof. Without loss of generality we can assume that P_* is nonnegative. Let

$$P_* = \cdots 0 \to P_n \to \cdots \to P_0 \to 0 \to \cdots .$$

We will prove the proposition by induction on n.

If $n = 0$, then we have $\hat{H}^i(G, P_*) = \hat{H}^i(G, P_0) = 0$ for all i.

Assume $n > 0$ and the statement is true for all k with $0 \le k < n$. Let

$$Q_* := \cdots \to 0 \to P_{n-1} \to \cdots \to P_0 \to 0 \to \cdots$$

and $Q'_* = \Sigma^{n-1} P_n$ where we consider the module P_n as a chain complex concentrated at 0. By inductive step $\hat{H}^i(G, Q_*) = \hat{H}^i(G, Q'_*) = 0$ for all i. If ∂_* denote the boundary map of P_*, then we have the following chain map from Q'_* to $Q*$

$$\begin{tikzcd}
Q'_* : & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d, "\partial_n"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
Q_* : & \cdots \arrow[r] & 0 \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

P_* is the mapping cone of this chain map. Hence there is a short exact sequence

$$0 \to Q_* \to P_* \to \Sigma Q'_* \to 0$$

By Proposition 4.8, we have the following long exact sequence

$$\cdots \to \hat{H}^i(G, Q_*) \to \hat{H}^i(G, P_*) \to \hat{H}^{i+1}(G, Q'_*) \to \cdots$$

which gives that $\hat{H}^i(G, P_*) = 0$ for all i. □

This proposition gives us the following corollary.

Corollary 4.10. *Let G be a finite group and C_*, D_* be chain complexes of $\mathbb{Z}G$-modules. If C_* is freely equivalent to D_*, then $\hat{H}^i(G, C_*) \cong \hat{H}^i(G, D_*)$ for all i.*

Proof. Since C_* is freely equivalent to D_*, there is a short exact sequence

$$0 \to C_* \to D_* \to F_* \to 0$$

where F_* is a finite chain complex of free $\mathbb{Z}G$-modules. Corresponding long exact sequence of Tate hypercohomology groups is

$$\cdots \to \hat{H}^{i-1}(G, F_*) \to \hat{H}^i(G, C_*) \to \hat{H}^i(G, D_*) \to \hat{H}^i(G, F_*) \to \cdots$$

which implies $\hat{H}^i(G, C_*) \cong \hat{H}^i(G, D_*)$ since by Proposition 4.9 $\hat{H}^i(G, F_*) = 0$ for all i. $\square$

4.2 Another Proof of Browder's Theorem

A nonnegative chain complex C_* is said to have *homological dimension* n, if $H_i(C_*) = 0$ for $i > n$ and $H_n(C_*) \neq 0$. The following theorem says that for such a chain complex of $\mathbb{Z}G$-modules where G is a finite group, there is a $\mathbb{Z}G$-module M such that the Tate hypercohomology of C_* can be understood in terms of Tate cohomology of M. By using this theorem, we will be able to give a new proof of Browder's Theorem.

Theorem 4.11. *(Habegger [8], p. 433) Let G be a finite group and C_* be a nonnegative chain complex of $\mathbb{Z}G$-modules. If C_* has homological dimension at most n, then there is a $\mathbb{Z}G$-module M such that*

(i) $\hat{H}^i(G, C_*) \cong \hat{H}^{i+n}(G, M)$,

(ii) M has a filtration $0 \subseteq M_0 \subseteq \cdots \subseteq M_n = M$ such that

$$M_i/M_{i-1} \cong \Omega^i H_{n-1}(C_*).$$

Proof. We can apply Theorem 3.5 to C_* for the pair of integers $(n-1, n)$, and obtain the chain complex $C_*^{(1)}$ freely equivalent to C_* with the properties mentioned in Theorem 3.5. Notice that now we can apply Theorem 3.5 to $C_*^{(1)}$ for the pair of integers $(n-2, n)$ and obtain the chain complex $C_*^{(2)}$ again. Continuing this way, we obtain a sequence of chain complexes $C_*^{(1)}, \ldots, C_*^{(n)}$, where $C_*^{(i)}$ is obtained from $C_*^{(i-1)}$ by applying Theorem 3.5 for pair of integeres $(n-i, n)$. Let us denote $C_*^{(0)} := C_*$. By Corollary 4.10 $\hat{H}^k(G, C_*^{(i)}) = \hat{H}^k(G, C_*)$ for all i, k since $C_*^{(i-1)}$ is freely equivalent to $C_*^{(i)}$ by Theorem 3.5. For all i, we have $C_k^{(i)} = C_k$ and $H_k(C_*^{(i)}) = H_k(C_*) = 0$ if k is not in the set $\{0, 1, \ldots, n\}$ by Theorem 3.5.

By the construction above, $C_*^{(n)}$ becomes a chain complex whose homology is concentrated at n. If we let $M := H_n(C_*^{(n)})$, then

$$\begin{aligned} \hat{H}^i(G, C_*) &\cong \hat{H}^i(G, C_*^{(n)}) \\ &\cong \hat{H}^{i+n}(G, M) \end{aligned}$$

by Proposition 4.6, which proves (i).

Let M_i denote the homology group $H_n(C_*^{(i)})$. By Theorem 3.5, there is a short exact sequence

$$0 \to M_{i-1} \to M_i \to \Omega^i H_{n-i}(C_*^{(i-1)}) \to 0$$

We can show $H_k(C_*^{(i)}) = H_k(C_*)$ if $k < n-i$ by induction on i. If $i = 0$, it is obvious. Now assume $i > 0$ and the statement is true up to i. We know that $H_k(C_*^{(i)}) = H_k(C_*^{(i-1)})$ if $k < n-i < n-(i-1)$, hence by inductive step $H_k(C_*^{(i)}) = H_k(C_*)$ if $k < n-i$. This completes the induction. Therefore, we can rewrite the short exact sequence above as follows

$$0 \to M_{i-1} \to M_i \to \Omega^i H_{n-i}(C_*) \to 0$$

If we consider $M_{i-1} \subseteq M_i$ with the injection above, then we have the filtration

$$0 \subseteq M_0 \subseteq \cdots \subseteq M_n = M$$

with sections

$$\Omega^0 H_n(C_*) - \Omega^1 H_{n-1}(C_*) - \cdots - \Omega^n H_0(C_*)$$

which proves (ii) and completes the proof. $\square$

We will give another proof of Theorem 3.1 after proving the following lemma.

Lemma 4.12. *Let G be a finite group and M be a $\mathbb{Z}G$-module. If M has a filtration $0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ with sections $A_0 - A_1 - \cdots - A_n$, then*

$$exp\ \hat{H}^i(G, M) \text{ divides } \prod_{j=0}^{n} exp\ \hat{H}^i(G, A_j)$$

for all i.

Proof. For each $j \geq 0$, we have the following short exact sequence

$$0 \to M_{j-1} \to M_j \to A_j \to 0$$

Corresponding long exact Tate cohomology sequence is

$$\cdots \to \hat{H}^i(G, M_{j-1}) \to \hat{H}^i(G, M_j) \to \hat{H}^i(G, A_j) \to \cdots$$

By Lemma 3.2, we have

$$\frac{exp\ \hat{H}^i(G, M_j)}{exp\ \hat{H}^i(G, M_{j-1})} \text{ divides } exp\ \hat{H}^i(G, A_j)$$

Multiplying both sides through $j = 0$ to n, we get

$$exp\ \hat{H}^i(G, M) \text{ divides } \prod_{j=0}^{n} exp\ \hat{H}^i(G, A_j)$$

□

Theorem 3.1 says that if G is a finite group and C_* is a nonnegative, connected, n-dimensional chain complex of free $\mathbb{Z}G$-modules, then the order of G divides $\prod_{j=1}^{n} exp\ H^{j+1}(G, H_j(C_*))$.

Another proof of Theorem 3.1. Let M be the module obtained from C_* by applying Theorem 4.11. By Theorem 4.11 and Proposition 4.9, we have

$$\hat{H}^i(G, M) \cong \hat{H}^{i-n}(G, C_*) = 0$$

for all i. Furthermore, M has a filtration $0 \subseteq M_0 \subseteq \cdots \subseteq M_n = M$ with sections

$$\Omega^0 H_n(C_*) - \Omega^1 H_{n-1}(C_*) - \cdots - \Omega^n H_0(C_*).$$

There is a short exact sequence

$$0 \to M_{n-1} \to M_n \to \Omega^n H_0(C_*) \to 0$$

and $\Omega^n H_0(C_*) = \Omega^n \mathbb{Z}$ since C_* is connected. Corresponding long exact sequence for Tate cohomology groups is

$$\cdots \to \hat{H}^i(G, M) \to \hat{H}^i(G, \Omega^n \mathbb{Z}) \to \hat{H}^{i+1}(G, M_{n-1}) \to \hat{H}^{i+1}(G, M) \cdots$$

Hence, $\hat{H}^{i+1}(G, M_{n-1}) \cong \hat{H}^{i-n}(G, \mathbb{Z})$ for all i by Theorem 2.5. Letting $i = n$, we get

$$\hat{H}^{n+1}(G, M_{n-1}) \cong \hat{H}^0(G, \mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}.$$

M_{n-1} has a filtration $0 \subseteq M_0 \subseteq \cdots \subseteq M_{n-1}$ with sections

$$\Omega^0 H_n(C_*) - \Omega^1 H_{n-1}(C_*) - \cdots - \Omega^{n-1} H_1(C_*)$$

By Lemma 4.12 we have

$$exp\ \hat{H}^{n+1}(G, M_{n-1}) = |G| \text{ divides } \prod_{j=1}^{n} exp\ \hat{H}^{n+1}(G, \Omega^{n-j} H_j(C_*))$$

and by Theorem 2.5 we have

$$\begin{aligned}\prod_{j=1}^{n} exp\ \hat{H}^{n+1}(G, \Omega^{n-j} H_j(C_*)) &= \prod_{j=1}^{n} exp\ \hat{H}^{j+1}(G, H_j(C_*)) \\ &= \prod_{j=1}^{n} exp\ H^{j+1}(G, H_j(C_*))\end{aligned}$$

This completes the proof. $\square$

Chapter 5

Main Result

5.1 Exponents of the Tate Cohomology Groups

In Theorem 3.3 we have seen that for a $\mathbb{Z}G$-module M with a trivial G action, $exp\ H^i(G, M)$ divides p for all $i \geq 1$. In the previous chapter, we have obtained a method to glue homologies of a chain complex at different dimensions. Even if the original homology groups have trivial G-action, the new homology group at the glued dimension may not be a trivial $\mathbb{Z}G$-module, hence it may not have exponent dividing p. The following is an example of a $\mathbb{Z}G$-module such that $exp\ H^i(G, M)$ does not divide p for some $i \geq 1$.

Example 5.1. Let $G = (\mathbb{Z}/p)^r$ for some $r > 1$ and $M := \Omega\mathbb{Z}$ where $\mathbb{Z}$ is a $\mathbb{Z}G$-module under the trivial action of G. Then we have $H^1(G, M) = \hat{H}^1(G, \Omega\mathbb{Z}) = \hat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/|G|$. Therefore, $exp\ H^1(G, M) = p^r$ does not divide p. Notice that $H^i(G, M) = H^{i-1}(G, \mathbb{Z})$ for $i \geq 2$, hence $H^i(G, M)$ has exponent dividing p for $i \geq 2$.

Although $exp\ H^i(G, M)$ does not divide p for all $i \geq 1$, it divides p for $i \geq 2$ in the example above. We will prove that for a finitely generated $\mathbb{Z}G$-module M, $exp\ H^i(G, M)$ divides p for i large enough. To prove this result, we will use the graded ring structure of $H^*(G, \mathbb{Z})$ and the graded module structure of $H^*(G, M)$.

Let us first review these structures.

A ring R is called a graded ring if there are abelian subgroups $(A_0, A_1, ...)$ of R such that R is isomorphic to $\bigoplus_{i=0}^{\infty} A_i$ as an abelian group and $a_i a_j \in A_{i+j}$ for all $a_i \in A_i, a_j \in A_j$. A nonzero element of a graded ring is called homogeneuous with degree i if it is an element of A_i. An R-module M over a graded ring R is called a graded module if there are abelian subgroups $(M_0, M_1, ...)$ of M such that M is equal to $\bigoplus_{i=0}^{\infty} M_i$ as an abelian group and $r_i m_j \in M_{i+j}$ for $r_i \in A_i, m_j \in M_j$. A nonzero element of a graded module is called homogeneous with degree i if it is an element of M_i.

A graded ring structure on $H^*(G, \mathbb{Z})$ and a graded module structure on $H^*(G, M)$ over $H^*(G, \mathbb{Z})$ are given by cup product (see [4, p. 109]). Cup product is a bilinear map $H^i(G, M) \otimes_{\mathbb{Z}} H^j(G, N) \to H^{i+j}(G, M \otimes_{\mathbb{Z}} N)$. Notice that when we take $M = N = \mathbb{Z}$, then the cup product takes the form $H^i(G, \mathbb{Z}) \otimes_{\mathbb{Z}} H^j(G, \mathbb{Z}) \to H^{i+j}(G, \mathbb{Z})$. If we let $H^*(G, \mathbb{Z}) = \bigoplus_{i=0}^{\infty} H^i(G, \mathbb{Z})$, then it becomes a graded ring. Take $N = \mathbb{Z}$, then the cup product takes the form $H^i(G, M) \otimes_{\mathbb{Z}} H^j(G, \mathbb{Z}) \to H^{i+j}(G, M)$. Similarly if we let $H^*(G, M) = \bigoplus_{i=0}^{\infty} H^i(G, M)$, then it becomes a graded module over $H^*(G, \mathbb{Z})$. The following theorem implies that if G is a finite group and M is finitely generated $\mathbb{Z}G$-module, then $H^*(G, M)$ is a finitely generated as an $H^*(G, \mathbb{Z})$ module.

Theorem 5.2 (Evens [7], p.87)**.** *Let G be a finite group and k a commutative ring on which G acts trivially, and M a kG-module. If M is Noetherian as a k-module, then $H^*(G, M)$ is noetherian over $H^*(G, k)$.*

We will not prove this theorem but use it to prove the following theorem.

Theorem 5.3 (Pakianathan [11])**.** *Let $G = (\mathbb{Z}/p)^r$ and M is a finitely generated $\mathbb{Z}G$ module. There is an integer N such that if $i > N$, then the exponent of $H^i(G, M)$ divides p.*

Proof. M is finitely generated as a $\mathbb{Z}$-module since it is finitely generated as a $\mathbb{Z}G$-module and G is finite. Since all finitely generated $\mathbb{Z}$-modules are Noetherian, M is Noetherian as a $\mathbb{Z}$-module. By Theorem 5.2 the module $H^*(G, M)$ is Noetherian, hence finitely generated over the ring $H^*(G, \mathbb{Z})$.

Let $m_1, ..., m_k$ be elements generating $H^*(G, M)$ over $H^*(G, \mathbb{Z})$. Without loss of generality we can assume that all of them are homogeneous. Let N be the maximum of the degrees of m_i's. Assume $i > N$ and $x \in H^i(G, M)$ is a nonzero element. We want to show $px = 0$. We know that $x = \Sigma_{j=1}^k \alpha_j m_j$ for some α_j's in $H^*(G, \mathbb{Z})$. Since x is homogeneous, we can assume α_j's are homogeneous too and $\alpha_j m_j \in H^i(G, M)$ for all j. The degree of m_j is strictly less than i for all j, so the degree of α_j is greater than or equal to 1. Since $\mathbb{Z}$ is a $\mathbb{Z}G$-module with trivial G action, $p\alpha_j = 0$ for all j by Theorem 3.3. Hence $px = \Sigma_{j=1}^k p\alpha_j m_j = 0$. □

Notice that if we have a finite collection of finitely generated $\mathbb{Z}G$-modules, then we can obtain an integer for each module in that collection by Theorem 5.3. Since there are finitely many, we can take the maximum of these integers and call this maximum N. If M is a $\mathbb{Z}G$-module which is isomorphic to one of the modules in the finite collection and if $i > N$, then *exp* $H^i(G, M)$ divides p. The last two theorems of this section are finiteness theorems that enables us to say that up to isomorphism there are finitely many modules satisfying some certain conditions.

Theorem 5.4 (Curtis and Reiner [6] p.563)**.** *If G is a finite group, then for each $n \geq 1$, there are finitely many $\mathbb{Z}$-free $\mathbb{Z}G$-modules of $\mathbb{Z}$-rank n up to isomorphism.*

We do not prove Theorem 5.4 but use it in the proof of the main theorem. Now, we prove another useful result.

Theorem 5.5. *Let G be a finite group, and M, N are finitely generated $\mathbb{Z}G$ modules. If M is $\mathbb{Z}$-free, then $Ext^i_{\mathbb{Z}G}(M, N)$ is finite for $i > 0$.*

To prove Theorem 5.5, let us review some properties of Ext. Let F_* be a free $\mathbb{Z}G$ resolution of M. The group $Ext^i_{\mathbb{Z}G}(M, N)$ is defined as the i-th cohomology group of the chain complex $Hom_{\mathbb{Z}G}(F_*, N)$. Notice that if F_* is a free $\mathbb{Z}G$ resolution of M, then it is also a free $\mathbb{Z}$ resolution of M. Also if $f : F_i \to N$ is a $\mathbb{Z}G$-module homomorphism, then it is also a $\mathbb{Z}$-module homomorphism. There is a homomorphism $res : Ext^i_{\mathbb{Z}G}(M, N) \to Ext^i_{\mathbb{Z}}(M, N)$, called the restriction map, induced from the inclusion $Hom_{\mathbb{Z}G}(F_*, M) \hookrightarrow Hom_{\mathbb{Z}}(F_*, M)$. If G is a finite group, then we have a map in the reverse direction $tr : Ext^i_{\mathbb{Z}}(M,N) \to Ext^i_{\mathbb{Z}G}(M,N)$, called the transfer map, induced from the homomorphism $Hom_{\mathbb{Z}}(F_*, M) \to Hom_{\mathbb{Z}G}(F_*, M)$ taking f to $\sum_{g\in G} gfg^{-1}$. One can easily see that for all $i \geq 0$ and for all x in $Ext^i_{\mathbb{Z}G}(M,N)$, we have $tr \circ res(x) = |G|x$. Now we can prove Theorem 5.5.

Proof of Theorem 5.5. We will show that if $i \geq 1$, then $Ext^i_{\mathbb{Z}G}(M,N)$ is a finitely generated $\mathbb{Z}$-module and has finite exponent. Notice that by the classification of finitely generated $\mathbb{Z}$-modules such a module has finite order. Actually we can see this without classification. Let $x_1, ..., x_k$ be a generating set and m be the exponent. Then every element can be written in the form $n_1x_1 + \cdots + n_kx_k$, where $0 \leq n_i \leq m$ for all i, and there are finitely many elements in this form. Hence to prove the theorem it is enough to show that $Ext^i_{\mathbb{Z}G}(M,N)$ is finitely generated and has finite exponent for all $i \geq 1$.

Let $i \geq 1$. Since M is $\mathbb{Z}$-free, $Ext^i_{\mathbb{Z}}(M,N) = 0$. Hence for an element x in the $Ext^i_{\mathbb{Z}G}(M,N)$ we have $|G|x = tr \circ res(x) = 0$. Therefore $Ext^i_{\mathbb{Z}G}(M,N)$ has finite exponent. Since M is finitely generated, we can take a free $\mathbb{Z}G$-resolution F_* of M such that all F_i's are finitely generated. Since G is finite, F_i's are finitely generated as a $\mathbb{Z}$-module. Let $F_i \cong \bigoplus \mathbb{Z}G$ be a finite direct sum of $\mathbb{Z}G$'s. Then $Hom_{\mathbb{Z}G}(F_i, N) \cong \bigoplus N$, which is also finitely generated as a $\mathbb{Z}$-module. Therefore, as a quotient module of a finitely generated module, $Ext^i_{\mathbb{Z}G}(M,N)$ is also finitely generated. $\square$

5.2 Explanation of the Main Ideas of the Proof on Small Cases

The aim of this section is to show how the main ideas in the proof of the main theorem evolve from the simple cases. One can skip this section and directly read the proof of the main theorem since the proof does not refer to any material in this section.

Assume that two positive integers r, k are given. Let us show that there is an integer N such that if $n > N$ and $G = (\mathbb{Z}/p)^r$ act freely and cellularly on a CW-complex X homotopy equivalent to $S^{n_1} \times \cdots \times S^{n_k}$ where $n_1 = \ldots n_k = n$, then $r \leq k$. We know that X has nonzero homologies at dimensions $n, 2n, \ldots, kn$ where $H_{jn}(X)$ is a $\mathbb{Z}$-free $\mathbb{Z}G$-module with $\mathbb{Z}$-rank $\binom{k}{j}$ for $j = 1, \ldots, k$. By Theorem 5.4 there are finitely many $\mathbb{Z}G$-modules of $\mathbb{Z}$-rank $\binom{k}{j}$ up to isomorphism. By Theorem 5.3, there is an integer N_j such that if $i > N_j$ and M is a $\mathbb{Z}G$-module of $\mathbb{Z}$-rank $\binom{k}{j}$, then $exp\ H^i(G, M)$ divides p. Let $N := max\{N_j :\ j = 1, ..., k\}$. If $n > N$ then $jn + 1 > N \geq N_j$, so $exp\ H^{jn+1}(G, H_{jn}(X))$ divides p. Therefore, if $n > N$, then by Theorem 3.1 $|G| = p^r$ divides $\prod_{j=1}^{k} exp\ H^{jn+1}(G, H_{jn}(X))$ which divides p^k. This implies $r \leq k$.

Now let us consider a case where the dimensions of spheres are not equal. Assume that positive integers r, l are given. Let us show that there is an integer N such that if $n > N$ and $G = (\mathbb{Z}/p)^r$ act freely and cellularly on a CW-complex X homotopy equivalent to $S^n \times S^{n+l}$, then $r \leq 2$. The space X has nonzero homologies at dimensions $n, n+l, 2n+l$ and all of the homologies are $\mathbb{Z}$-free and have $\mathbb{Z}$-rank 1. By Theorem 5.4 there are finitely many $\mathbb{Z}G$-modules of $\mathbb{Z}$-rank 1 up to isomorphism. By Theorem 5.3, there is an integer N_1 such that if $i > N_1$ and M is a $\mathbb{Z}G$-module of $\mathbb{Z}$-rank 1, then $exp\ H^i(G, M)$ divides p. Let $C_*(X)$ be the cellular chain complex of X. We can obtain another chain complex $D_*(X)$ by applying Theorem 3.5 to chain complex $C_*(X)$ for tuple of integer $n, n+l$. Hence $D_*(X)$ is a nonnegative, finite dimensional, connected chain complex of free $\mathbb{Z}G$-modules. Furthermore, $D_*(X)$ has nonzero homologies at dimensions $n+l, 2n+l$ where $H_{2n+l}(D_*(X)) = H_{2n+l}(X)$ and there is a short exact sequence of the form

$$0 \to H_{n+l}(X) \to H_{n+l}(D_*(X)) \to \Omega^l H_n(X) \to 0$$

By Theorem 5.4 both $H_{n+l}(X)$ and $\Omega^l H_n(X)$ have finitely many possibilities up to isomorphism. Therefore, to show that there are finitely many possibilities for $H_{n+l}(D_*(X))$ up to isomorphism, it is enough to show that $Ext^1_{\mathbb{Z}G}(\Omega^l H_n(X), H_{n+l}(X))$ is finite. This is true since

$$Ext^1_{\mathbb{Z}G}(\Omega^l H_n(X), H_{n+l}(X)) \cong Ext^{l+1}_{\mathbb{Z}G}(H_n(X), H_{n+l}(X))$$

which is finite by Theorem 5.5. By Theorem 5.3 there is an integer N_2 such that if $i > N_2$, then $exp\ H^i(G, H_{n+l}(D_*(X)))$ divides p for all n (notice that the space X depends on n). Let $N = max\{N_1, N_2\}$. By Theorem 3.1 we have

$$|G| = p^r \text{ divides } exp\ H^{n+l+1}(G, H_{n+l}(D_*(X))).exp\ H^{2n+l+1}(G, H_{2n+l}(D_*(X)))$$

which divides p^2. This implies $r \leq 2$.

By using the result in the previous paragraph, we can prove a generalization of it. Assume that positive integers r, l is given. Let us show that there is an integer N such that if $n > N$ and $G = (\mathbb{Z}/p)^r$ act freely and cellularly on a CW-complex X homotopy equivalent to $S^m \times S^n$ where $|n - m| < l$, then $r \leq 2$. This is true since we can find an integer for all of the cases $S^{n-l} \times S^n$, $S^{n-l+1} \times S^n$, $\ldots$, $S^{n+l} \times S^n$ and then we can take N as the maximum of these integers.

The following case shows us why our methods do not apply for arbitrary $S^n \times S^m$ without an upper bound to the difference $|n - m|$. Consider the case $S^n \times S^{2n}$. Let us further assume that the action of $G = (\mathbb{Z}/p)^r$ on homology groups is trivial, which simplifies our calculations. Similarly, we have $D_*(X)$ but we should change l with n. Hence we have a short exact sequence of the form

$$0 \to H_{2n}(X) \to H_{2n}(D_*(X)) \to \Omega^n H_n(X) \to 0$$

and we want to show that there are finitely many possibilities for $H_{2n}(D_*(X))$ although n may take infinitely many different values. Therefore, it is not enough to show that $Ext^1_{\mathbb{Z}G}(\Omega^n \mathbb{Z}, \mathbb{Z})$, which is isomorphic to $Ext^{n+1}_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$, is finite for all n; but we need to find an integer N_0 such that $|Ext^{n+1}_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})| \leq N_0$ for n is large enough. Let us show this is not the case for $G = (\mathbb{Z}/p)^2$.

Notice that $Ext^n_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \cong H^n(G, \mathbb{Z})$. By Kunneth formula for cohomology groups (see [14, p. 166]) there is a split exact sequence

$$\begin{aligned} 0 \to \bigoplus_{p+q=n} H^p(\mathbb{Z}/p, \mathbb{Z}) \otimes H^q(\mathbb{Z}/p, \mathbb{Z}) \to H^n(\mathbb{Z}/p \times \mathbb{Z}/p, \mathbb{Z}) \to \\ \bigoplus_{p+q=n+1} Tor_1^{\mathbb{Z}}(H^p(\mathbb{Z}/p, \mathbb{Z}), H^q(\mathbb{Z}/p, \mathbb{Z})) \to 0 \end{aligned}$$

This gives us $H^{2k+1}(\mathbb{Z}/p \times \mathbb{Z}/p, \mathbb{Z}) \cong H^{2k}(\mathbb{Z}/p \times \mathbb{Z}/p, \mathbb{Z}) \cong (\mathbb{Z}/p)^k$. Hence there is no upper bound for $|H^n(\mathbb{Z}/p \times \mathbb{Z}/p, \mathbb{Z})|$ as $n \to \infty$. We can easily generalize this result to $(\mathbb{Z}/p)^r$ for $r \geq 2$, since in this case if we apply Kunneth formula by considering $(\mathbb{Z}/p)^r = (\mathbb{Z}/p)^2 \times (\mathbb{Z}/p)^{r-2}$, we can see that there is an injection from $H^n((\mathbb{Z}/p)^2, \mathbb{Z})$ to $H^n((\mathbb{Z}/p)^r, \mathbb{Z})$.

5.3 Proof of the Main Theorem

Let $G = (\mathbb{Z}/p)^r$ and k, l are positive integers. We want to show that there is an integer N such that if G acts freely and cellularly on a CW-complex X homotopy equivalent to $S^{n_1} \times \cdots \times S^{n_k}$ where $n_i > N$ for all i and $|n_i - n_j| < l$ for all i, j, then $r \leq k$.

Let $n := max\{n_1, ..., n_k\}$ and $a_i := n - n_i$. If we let $C_*(X)$ denote the cellular chain complex of X, then it has nonzero homology groups at the following dimensions

$$\begin{array}{cr} kn - (a_1 + \cdots + a_k) & (k) \\ \vdots & \\ jn - (a_1 \cdots + a_j), ..., jn - (a_{k-j+1} + \cdots + a_k) & (j) \\ \vdots & \\ 2n - (a_1 + a_2), 2n - (a_1 + a_3), ..., 2n - (a_{k-1} + a_k) & (2) \\ n - a_1, n - a_2, ..., n - a_k & (1) \end{array}$$

If $n > lk$, then every dimension d on the (j)-th row satisfies $(j-1)n < d \leq jn$. Hence every dimension on the (j')-th row is strictly greater then every dimension on the (j)-th row whenever $j' > j$. By taking $N > lk$, we can guarantee that $n > lk$. In the remaining part of the proof, we will assume that $n > lk$.

By applying Theorem 3.5 to $C_*(X)$, we can glue all the homologies at dimensions on the (j)-th row to the dimension jn. Let $D_*(X)$ denote this new chain complex. Hence $D_*(X)$ is a nonnegative, finite dimensional, connected chain complex of free $\mathbb{Z}G$-modules and it has nonzero homologies at dimensions $0, n, 2n, \ldots, kn$. Let $M_j := H_{jn}(D_*(X))$. We will show that there are finitely many possibilities for M_j up to isomorphism. We know that $|n_i - n_j| < l$ for all i, j, so without loss of generality we can fix a k-tuple $a_1, \ldots, a_k$ since there are finitely many k-tuples of nonnegative integers where each coordinate is less than l.

Let D be the set of dimensions on the (j)-th row and $m = |D|$. The integer m depends only on $a_1, \dots, a_k$ and if we let $D = \{jn - s_1, ..., jn - s_m\}$, then s_i's depends only on $a_1, ..., a_k$. Assume $s_1 < \cdots < s_k$ and let $A_i := H_{jn-s_i}(X)$ for $i = 1, \dots, m$. The $\mathbb{Z}G$-module A_i is finitely generated, $\mathbb{Z}$-free and its $\mathbb{Z}$-rank is less than $\binom{k}{j}$ for all $i = 1, \dots, m$. By Theorem 3.5 M_j has a filtration $0 \subseteq N_1 \subseteq \cdots \subseteq N_m = M_j$ with sections $\Omega^{s_1} A_1 - \cdots - \Omega^{s_k} A_k$. Let us inductively show that N_i is finitely generated and there are finitely many possibilities for N_i up to isomorphism. For $i = 1$, we have $N_1 = \Omega^{s_1} A_1$ which is finitely generated, so it is enough to show that A_1 has finitely many possibilities up to isomorphism. This is true by Theorem 5.4 since A_1 is finitely generated, $\mathbb{Z}$-free and it is $\mathbb{Z}$-rank is independent of n. For $i > 1$, we have a short exact sequence of the form

$$0 \to N_{i-1} \to N_i \to \Omega^{s_i} A_i \to 0$$

By inductive step we know that N_{i-1} is finitely generated and there are finitely many possibilities for N_{i-1} up to isomorphism. Similarly, $\Omega^{s_i} A_i$ is finitely generated and there are finitely many possibilities for it up to isomorphism by rank arguments. Therefore, N_i is finitely generated and to show that there are finitely many possibilites for N_i up to isomorphism, it is enough to show that $Ext^1_{\mathbb{Z}G}(\Omega^{s_i} A_i, N_{i-1})$ is finite. This is true since

$$Ext^1_{\mathbb{Z}G}(\Omega^{s_i} A_i, N_{i-1}) \cong Ext^{1+s_i}_{\mathbb{Z}G}(A_i, N_{i-1})$$

which is finite by Theorem 5.5. This completes the induction.

We have seen that M_j is finitely generated and there are finitely many possibilities for it up to isomorphism. Therefore, by Theorem 5.3 there is an integer N_j such that if $i > N_j$, then $exp\ H^i(G, M_j)$ divides p. Now let $N = max\{N_1, \dots, N_k\}$. By Theorem 3.1 we have

$$|G| = p^r \text{ divides } \prod_{j=1}^{k} exp\ H^{jn+1}(G, H_{jn}(D_*(X))) = \prod_{j=1}^{k} exp\ H^{jn+1}(G, M_j)$$

Hence if $n > N$, then p^r divides p^k, implying $r \leq k$. This completes the proof.

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