# ESSAYS ON NON-COOPERATIVE INVENTORY GAMES

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DOCTOR OF PHILOSOPHY

By
Evren Körpeoğlu
January, 2012

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assist. Prof. Dr. Alper Şen(Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. M. Selim Aktürk

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Mustafa Ç. Pınar

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assist. Prof. Dr. Doğan Serel

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assist. Prof. Dr. Seçil Savaşaneril

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

# ABSTRACT

## ESSAYS ON NON-COOPERATIVE INVENTORY GAMES

Evren Körpeoğlu
Ph.D. in Industrial Engineering
Supervisor: Assist. Prof. Dr. Alper Şen
January, 2012

In this thesis we study different non–cooperative inventory games. In particular, we focus on joint replenishment games and newsvendor duopoly under asymmetric information. Chapter 1 contains introduction and motivation behind the research. Chapter 2 is a preliminary chapter which introduce basic concepts used in the thesis such as Nash equilibrium, Bayesian Nash equilibrium and mechanism design.

In Chapter 3, we study a non-cooperative game for joint replenishment of multiple firms that operate under an EOQ–like setting. Each firm decides whether to replenish independently or to participate in joint replenishment, and how much to contribute to joint ordering costs in case of participation. Joint replenishment cycle time is set by an intermediary as the lowest cycle time that can be financed with the private contributions of participating firms. We consider two variants of the participation-contribution game: in the single–stage variant, participation and contribution decisions are made simultaneously, and, in the two-stage variant, participating firms become common knowledge at the contribution stage. We characterize the behavior and outcomes under undominated Nash equilibria for the one-stage game and subgame-perfect equilibrium for the two-stage game.

In Chapter 4, we extend the private contributions game to an asymmetric information counterpart. We assume each firm only knows the probability distribution of the other firms' adjusted demand rates (demand rate multiplied by inventory holding cost rate). We show the existence of a pure strategy Bayesian Nash equilibrium for the asymmetric information game and provide its characterization. Finally, we conduct some numerical study to examine the impact of information asymmetry on expected and interim values of total contributions, cycle times and total costs.

In Chapter 5, we study a three–stage non-cooperative joint replenishment game. In this model, we assume that the intermediary is also a decision maker. In the first stage, each firm announces his contribution for the ordering cost. In the second stage, based on the contributions, the replenishment service provider determines a common cycle time that he can serve the firms. Finally, each firm decides whether to be a part of the coalition and served under this cycle time or act independently with an EOQ cost. We analyze each stage and give the conditions for equilibrium. We show that the subgame-perfect equilibrium cycle time is not unique. Although minimum and maximum cycle times that arise in equilibrium straddle the efficient cycle time, in general, whether efficient cycle time can be reached in equilibrium depends on the parameters of the joint replenishment environment. For symmetric joint replenishment environments, we show that whether efficient cycle time is a subgame-perfect equilibrium outcome depends only on the number of firms and is independent of all other parameters of the environment.

In Chapter 6, we focus on finding a mechanism that would allocate the joint ordering costs to the firms based on their reported adjusted demand rates. We first provide an impossibility result showing that there is no direct mechanism that simultaneously achieves efficiency, incentive compatibility, individual rationality and budget-balance. We then propose a general, two-parameter mechanism in which one parameter is used to determine the joint replenishment frequency; another is used to allocate the order costs based on firms' reports. We show that efficiency cannot be achieved in this two-parameter mechanism unless the parameter governing the cost allocation is zero. When the two parameters are identical (a single parameter mechanism), we find the equilibrium share levels and corresponding total cost. We finally investigate the effect of this parameter on equilibrium behavior.

In Chapter 7, we study the newsboy duopoly problem under asymmetric cost information. We extend the Lippman and McCardle [30] model of competitive newsboys to allow for private cost information. The market demand is initially split between two firms and the excess demand for each firm is reallocated to the rival firm. We show the existence and uniqueness of pure strategy equilibrium and characterize its structure. The equilibrium conditions have an interesting recursive structure that enables an easy computation of the equilibrium order quantities. Presence of strategic interactions creates incentives to increase order

quantities for all firm types except the type that has the highest possible unit cost, who orders the same quantity as he would as a monopolist newsboy. Consequently, competition leads to higher total inventory in the industry. A firm's equilibrium order quantity increases with a stochastic increase in the total industry demand or with an increase in his initial allocation of the total industry demand. Finally, we provide full characterization of the equilibrium, corresponding payoffs and comparative statics for a parametric special case with uniform demand and linear market shares.

*Keywords:* Joint replenishment problem, Newsvendor problem, Game theory, Mechanism design, Asymmetric information.

# ÖZET

# İŞBİRLİKÇİ OLMAYAN ENVANTER OYUNLARI ÜZERİNE MAKALELER

Evren Körpeoğlu
Endüstri Mühendisliği Bölümü, Doktora
Tez Yöneticisi: Yrd. Doç. Dr. Alper Şen
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Bu tezde işbirlikçi olmayan ortak tedarik oyunları ve asimetrik bilgi altında gazete satıcısı duopolisini de içeren değişik rekabetçi envanter oyunları incelenmektedir. Birinci bölüm girişi ve araştırmanın ardındaki motivasyonu içermektedir. İkinci bölüm bu tezde kullanılmış Nash dengesi, Bayes Nash dengesi ve mekanizma tasarımı gibi bir takım ekonomik konuların kısa bir özetini kapsamaktadır.

Üçüncü bölümde ekonomik sipariş miktarına benzer bir modelde birden çok firmanın işbirlikçi olmayan ortak tedarik oyunu incelenmektedir. Her firma siparişini kendi başına mı vereceğine yoksa ortak tedariğe mi katılacağına ve ortak tedariğe katılırsa tedarik için ne kadar katkıda bulunacağına karar vermektedir. Ortak tedarik çevrim süresi bir aracı tarafından verilen katkılarla finanse edilebilecek en düşük çevrim süresi olarak belirlenmektedir. Bu oyun için iki farklı model incelenmektedir: tek aşamalı modelde katılma ve katkı miktarı kararları bir arada verilirken, iki aşamalı modelde katılan firmalar ilk aşamada belirlenip ikinci aşamada ortak tedariğe katılan firmalar katkı miktarlarını açıklamaktadır. Bu iki model için de firmaların Nash dengesi altındaki davranışları ve maliyetleri bulunmuştur.

Dördüncü bölümde bir önceki bölümde bulunan bireysel katkı miktarı oyununun asimetrik bilgi içeren bir modeli incelenmektedir. Her firma sadece diğer firmalara ait uyarlanmış talep hızlarının (Talep hızı ile envanter maliyet hızının çarpımı) olasılıksal bir dağılımını bilmektedir. Asimetrik bilgi altındaki bu modelde yalın stratejili Nash dengesinin varlığı gösterilmiş ve bu denge için gerekli olan şartlar verilmiştir. Ayrıca, bilgi asimetrisinin toplam katkı miktarı, ortak çevrim süresi ve toplam maliyetlerin beklenen ve ara değerleri üzerine etkilerini

incelemek için sayısal bir çalışma yapılmıştır.

Beşinci bölümde üç aşamalı bir işbirlikçi olmayan ortak tedarik oyunu incelenmektedir. Bu modelde aracının da karar verici olduğu varsayılmaktadır. Birinci aşamada her firma sipariş için katkı miktarını açıklamaktadır. İkinci aşamada verilen katkı miktarları doğrultusunda aracı firmalara servis sağlayacağı ortak çevrim süresini belirlemektedir. Üçüncü aşamada ise verilen çevrim süresine bakan firmalar bu çevrim süresiyle ortak tedarikten mi faydalanacaklarına yoksa ekonomik sipariş miktarı altındaki maliyetle bağımsız mı hareket edeceklerine karar vermektedir. Bu oyunda her aşama ayrı ayrı analiz edilip kusursuz altoyun dengesi için gerekli şartlar verilmiştir. Ayrıca dengenin eşsiz olmadığı gösterilmiştir. En verimli çevrim süresi dengede oluşabilecek çevrim sürelerinin en düşüğü ve en yükseğinin arasında kalsa da dengede bu süreye ulaşılıp ulaşılamayacağı oyunun parametrelerine bağlıdır. Bütün firmaların özdeş olduğu durumda en verimli çevrim süresinin dengenin bir sonucu olup olmadığı sadece oyundaki firma sayısına bağlı olup diğer parametrelerden bağımsızdır.

Altıncı bölümde firmaların rapor ettiği uyarlanmış talep hızlarına bağlı olarak ortak tedarik maliyetlerini paylaştıracak bir mekanizma bulmaya yoğunlaşılmıştır. Öncelikle verimlilik, caziplik, bireysel rasyonellik ve denk bütçeyi sağlayan direk bir mekanizmanın mümkün olmadığı gösterilmiş, sonrasında ise birinci parametresi ortak tedarik frekansını ikinci parametresi ise sipariş maliyetlerinin firma raporlarına göre paylaştırılmasını sağlayan iki parametreli genel bir mekanizma önerilmiştir. Bu mekanizmada maliyet paylaşımını kontrol eden parametrenin sıfır olmadığı durumlarda verimliliğin sağlanamayacağı gösterilmiştir. İki parametrenin de eşit olduğu durumda (tek parametreli bir mekanizma) dengedeki paylaşım seviyeleri ve bunlara karşılık gelen toplam maliyet bulunmuş ve ayrıca bu parametrenin dengedeki firma davranışlarına olan etkisi incelenmiştir.

Yedinci bölümde asimetrik bilgi altındaki gazete satıcısı duopolisi problemi incelenmektedir. Lippman ve McCardle'daki [30] rekabetçi gazete satıcıları modeli asimetrik bilgi de içerecek şekilde genişletilmiştir. Öncelikle toplam pazar talebi iki firma arasında paylaştırıldıktan sonra firmaların karşılayamadıkları talepleri rakip firmaya atanmaktadır. Bu model için yalın stratejili Nash dengesinin varlığı ve eşsiz olduğu gösterilmiş ve dengenin yapısı karakterize edilmiştir. Denge koşullarının özyinelemeli yapısı dengedeki sipariş miktarlarının kolayca hesaplanmasını sağlamaktadır. Stratejik etkileşimlerin varlığı tekel bir firma gibi hareket

eden en yüksek birim maliyete sahip firma tipi dışındaki bütün firma tiplerini daha fazla sipariş vermeye teşvik etmektedir. Bunun bir sonucu olarak, rekabet endüstride daha yüksek toplam envanter miktarlarına sebep olmaktadır. Bir firmanın dengedeki sipariş miktarı toplam talepteki olasılıksal artışla ve kendine verilen baslangıç market payındaki artışla yükselmektedir. Son olarak tekbiçimli dağılım ve doğrusal market talebi paylaşımı altında Nash dengesinin tam karakterizasyonu, karşılık gelen maliyetler ve model parametrelerinin maliyetler ve sipariş miktarları üzerine etkileri verilmiştir.

*Anahtar sözcükler*: Ortak tedarik problemi, Gazete satıcısı problemi, Oyun teorisi, Mekanizma dizaynı, Asimetrik bilgi.

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# Chapter 1

# INTRODUCTION

Inventory management is one of the most important functions in a business since inventories usually tie up a significant portion of a company's capital. Inventory is a necessary evil as almost all firms need to position inventory at various stages of their supply chain to satisfy customer demand. Moreover, it can help the organization achieve economies of scale and creates a buffer against demand uncertainty. On the other hand, if not managed properly, it may lead to a huge financial burden for the business due to product handling, warehouse and capital costs, obsolescence, rework and returns. Both Nike and Cisco experienced major decrease in their stock prices due to unsuccessful inventory management. In 2001, Nike could not establish the necessary inventory levels for its footwear line and the result was shortages in some footwear models and surpluses in others. This in turn cost the company over $100 million in a single quarter. Similarly, after not being able to keep up with the demand, inflated sales forecasts and the economic downturn in 2001, Cisco had a $2.2 billion inventory write–down for the components that were ordered but never used. This resulted in a decrease of its stock prices from $82 to $14 in just thirteen months [40]. As these examples clearly demonstrate, inventory management is critical for the success of a firm and this is the motivation for the increasing amount of academic research on inventory management in the past four decades.

Firms usually coexist with many other firms in the same market which in turn requires them to assess their inventory decisions more carefully considering the competition and possible cooperation opportunities. Thus, the success of a firm depends on taking the right decisions in the marketplace and exploiting any potential to reduce costs. One major source of inventory competition is caused by the demand spillovers due to the stockouts. According to a survey provided by Proctor & Gamble [47], in case of a stockout 50% of the customers switch to another retailer. Another study by Gruen et al. [18] combines the studies over different retailers over the world with a total of 71000 customer surveys for certain FMCG products and concludes that when a stockout situation occurs, 32% of the customers substitute brand and 34% buy the same product at another store both of which drives inventory competition between manufacturers and retailers respectively. However, it is also possible that the firms in the same market can benefit from each other. Recently, BMW started an auto-parts purchasing partnership with Daimler to purchase more than 10 parts together and looking for ways to expand this partnership. BMW is hoping to generate cost savings of around 100 million euros per year in 2012 and 2013 through this venture [17].

In this thesis, we consider non–cooperative inventory games. We mainly focus on the extensions of the economic order quantity (EOQ) problem and the newsboy problem both of which are well–studied problems in the literature. The EOQ model is a deterministic demand model where total cost is comprised of two parts. The first part is the setup or ordering cost associated with production, procurement or transportation of the lot for each order. The second part is the holding cost of inventory which includes the cost of capital, handling and warehouse costs. Smaller lot sizes leads to lower average inventory but higher ordering costs. On the contrary, larger lot sizes lead to lower setup costs but increase the average inventory cost. Considering this trade–off, the firms determine the efficient lot size. In real world, any business with a fairly stable and deterministic demand and a well–defined setup cost may use the EOQ model since the optimal lot size is not very sensitive to the minor demand changes.

The newsboy problem is a single period model in which a firm should decide on order or production quantity of a perishable product which has stochastic demand. Each unit of the product has a purchasing or production cost and predetermined revenue. There is only one ordering opportunity so the firm must decide on the inventory level before the season starts. This assumption is usually justified by long lead times, capacity restrictions and relatively short sales seasons. The sales level of the firm is the smaller of the demand and order quantity. At the end of the season, the firm either has excess demand which leads to lost sales and may be penalized by a unit lost sales cost or excess inventory which either perishes or salvaged at a salvage value lower than the purchasing cost. The objective of the firm is to determine the inventory level that will maximize his expected profit. At the optimal inventory level the marginal cost is equal to marginal revenue. There are many examples for newsboy type products in addition to the newspapers and magazines. Fashion goods should be sold in a single season since each season has a different line of clothing. Moreover, they usually have a long lead time since most of the fashion goods are imported from overseas. Similarly, high–tech equipment should be sold in a relatively short amount of time due to the risk of obsolescence. Again, they may have long lead times due to capacity restrictions of major suppliers.

One major strand of the literature on inventory theory is the joint replenishment problem. Joint replenishment is the problem of coordinating or consolidating the replenishment of multiple items or multiple retailers that are ordered from the same supplier to minimize total ordering and inventory costs using the economies of scale. In case of multiple firms or retailers, coordination requires some type of a centralized decision making by independent firms. However, firms that are subjects of joint replenishment may be competitors in the same market or in some cases they may not be in communication so a cooperative solution is not always viable. In such cases, using a non–cooperative mechanism that coordinates the firms with joint replenishment potential could help them to reduce inventory and ordering costs without a centralized decision making process. For example, recently, Istanbul Textile and Apparel Exporters Union founded a joint ordering platform which aims to decrease the purchasing costs of its members by 25% [43]. This portal for joint purchasing is not only limited to textile supplies but also includes provisions related to energy, logistics and communication.

Also, Koç group of companies in Turkey has a subdivision called Zer which aims to coordinate the purchases of different subgroups under Koç conglomerate and any outside member [54]. It provides services such as purchasing raw materials, logistics and services even for the firms that are not part of Koç group. These examples show that there exist many non-cooperative initiatives to benefit from the advantages of joint replenishment. There is very limited research in the literature about the joint replenishment problem that use non-cooperative models. Thus, in this thesis we attempt to fill this gap with different approaches to this problem.

It is fair to assume that in systems where joint decisions have to rely on information reported by the participants, firms may act strategically and misreport their characteristics to improve their payoffs. Non–cooperative game theory approach focuses on how to characterize the equilibrium behavior of self–interested players in games where each player's information and strategic options as well as the outcomes that result from each combination of decisions are explicitly specified. The non–cooperative approach enables analyses of several broad sets of research questions: First set concerns analysis of equilibrium outcomes. How do equilibrium outcomes for a given game relate to players' characteristics and how do they vary across environments with different player characteristics? How do equilibrium outcomes of two games compare for a given environment? How do outcomes induced by equilibrium behavior under various alternative game rules perform with respect to a system–optimal solution? Second set deals with questions such as how can one design rules of the non–cooperative interaction to achieve "better" outcomes where the notion of "better" reflects concerns related to system–optimality? As observed by Cachon and Netessine [8], in decentralized decision making settings obtaining efficiency is regarded as the exception rather than the rule. Following this philosophy, we consider various non–cooperative joint replenishment games that differ based on their cost allocation schemes. A cost allocation scheme distributes the total cost among the firms based on a reported attribute which may be the independent order frequency, cycle time, holding cost rate or demand rate. We use some allocation schemes that determine the joint cycle time only based on monetary contributions of the firms for the major setup cost. We also use direct mechanisms to allocate the total cost based on the reports from the firms. However, these reports may not reflect a firm's true characteristics since misreporting an attribute may be beneficial for the firm. Thus, a direct mechanism should enforce truth–telling among the firms which can be achieved by using incentive compatibility. Another important property of a mechanism is individual rationality which guarantees a non-negative profit for the firms that participate in the mechanism. Incentive compatibility is not necessary if monetary contributions are used to allocate the total cost allocation however individual rationality is always essential.

In Chapter 3, we study a non–cooperative game for joint replenishment of multiple firms that operate under a deterministic demand setting. Each firm decides whether to participate in joint replenishment or to replenish independently, and each participating firm decides how much to contribute to joint ordering costs. Joint replenishment cycle time is set by an intermediary as the lowest cycle time that can be financed with the private contributions of participating firms. We consider two participation-contribution games: in the single–stage variant, participation and contribution decisions are made simultaneously, and, in the two-stage variant, participating firms becomes known at the contribution stage. We characterize the behavior and outcomes under undominated Nash equilibria for the one-stage game and subgame-perfect equilibrium for the two-stage game. Our results show that the joint replenishment is mostly financed by the firm or group of firms with the highest adjusted demand rate which is the multiplication of inventory holding cost rate and demand rate and the other firms just pay the minimum entree fee.

An important factor in non–cooperative games is the information structure. Information asymmetry is an essential assumption since not all of the game parameters are known by all the parties. Firms usually do not have complete information about the demand and cost parameters of the other firms in the same market. There are companies such as Nielsen, Kantar and Ipsos which provide market data up to an extent but even this information is not exact. Similarly, vertical partnerships and manufacturer–supplier relations may involve information asymmetry since suppliers may not be willing to share their cost information in order not to loose their bargaining position and the demand of the manufacturer may not be known by the supplier. A player may know what kind of player he is i.e., his type, but he may have only some idea about his rivals' types where the type of a player may include any parameter such as cost or demand. Thus, in Chapter 4, we extend the private contributions game to an asymmetric information counterpart. We assume each firm only knows the probabilistic distribution of the other firms' adjusted demand rates. We assume a continuous type distribution and all the other parameters are common knowledge. Consequently, each firm decides on his contribution level without knowing the exact type of his rivals. Asymmetric information games are modeled as Bayesian games. We show the existence of a pure strategy Bayesian Nash equilibrium for the asymmetric information game. We provide conditions for a Bayesian Nash equilibrium. Finally, we conduct a numerical study to examine the impact of information asymmetry on expected and interim values of total contributions, cycle times and total costs. Even though Chapters 3 and 4 focus on non–cooperative joint replenishment solutions and the total cost under these models are lower than the decentralized total cost, they are unable to deliver an efficient solution i.e., the centralized solution.

In Chapter 5, we study a three–stage non–cooperative joint replenishment game aiming for a solution with higher efficiency. In this model, we assume that the intermediary is also a decision maker. In the first stage, each firm announces his contribution for the ordering cost. In the second stage, based on the contributions, the intermediary determines a common cycle time that he can serve the firms. Finally, each firm decides whether to be a part of the coalition and served under this cycle time or act independently with an EOQ cost. We analyze each stage and derive the conditions for an equilibrium. We show that the subgame–perfect equilibrium cycle time is not unique. Although the minimum and maximum cycle times that arise in equilibrium straddle the efficient cycle time, in general, whether efficient cycle time can be reached in equilibrium depends on the parameters of the joint replenishment environment. For symmetric joint replenishment environments, we show that whether efficient cycle time is a subgame–perfect equilibrium outcome depends only on the number of firms and is independent of all other parameters of the environment. Furthermore, this dependence on the number of firms exhibits a highly non–monotone pattern.

In Chapter 6, we consider parametric mechanisms to allocate the setup costs associated with the joint replenishment problem and measure their performance for different parameters. First, we first provide an impossibility result showing that there is no direct mechanism that simultaneously achieves efficiency, incentive compatibility, individual rationality and budget-balance. We then consider a two–parameter mechanism where initially the firms decide on their contribution levels. The first parameter determines the corresponding joint replenishment frequency and the second parameter governs the order cost shares. We show that a non-cooperative joint replenishment mechanism leads to lower order frequencies than the efficient frequency unless the second parameter is zero. Following this, we consider a mechanism where the two parameters are equal (a single parameter mechanism). We derive the best response equations and equilibrium conditions for a constructive equilibrium. We characterize the equilibrium contributions and the corresponding comparative statics.

Finally, in Chapter 7 we take a different direction and consider a competitive newsboy problem under stochastic demand with asymmetric cost information. In our model, we assume two firms where the stochastic market demand for the product is initially allocated to the two firms by some split function. The split function may be linear such as firm 1 gets 60% of market share and firm 2 gets 40% or it can take any form depending on the market share structure. In case that a firm cannot satisfy his share of the market, all excess demand is re-allocated to the other firm if the other firm has any available inventory. Thus, while considering the amount to order or produce, a firm should also consider the potential excess demand coming from the rival firm. This implies that the effective demand of a firm depends on the inventory decision of the rival firm. Hence, we have a competition between the two firms over each others unsatisfied excess demands.

Similarly, information asymmetry in this setting is also a fair assumption. A firm knows his exact cost type but only knows the distribution of his rival's cost type since a firm may not know his rival's cost but may have an idea on their cost level depending on his own cost, the technological capacity of his rival or some general market indications. Companies such as ACNielsen tracks the purchasing and sales information of many firms and sells them to their rivals. However these results are not always comprehensive since companies like Walmart no longer shares their purchasing and sales information with any other company leading to an information asymmetry between competing retailers [22]. Thus, we investigate the impact of information asymmetry on the competitive newsboy problem. For this model we show the existence of pure strategy Bayesian Nash equilibrium under fairly general assumptions on demand distribution and split function. This is followed by a characterization of the equilibrium and proof of its uniqueness under a continuous and strictly increasing probability distribution function for the demand and a deterministic, increasing split function. Comparative statics are also derived. Lastly, we provide the full characterization of the equilibrium, corresponding payoffs and comparative statics for the case of uniform demand.

In the following chapter, we summarize some important game theory concepts we use throughout the thesis such as Nash equilibrium, Bayesian games and mechanism design.

# Chapter 2

# GAME THEORY REVIEW

## 2.1 Introduction

Game theory concepts are used in other disciplines for over fifty years but its use in operations management is relatively new. Game theory provides some powerful tools to improve on the classical views of the inventory management area. This chapter reviews some of the concepts we use throughout the thesis. However, we do not attempt to provide a comrehensive review of the game theory concepts here and only review the material relevant to the thesis. This chapter is heavily based on Fudenberg and Tirole [16].

## 2.2 Definition of a game

A game has three important features: the set of rational players $i \in N$ where $N = 1, 2, .., n$, the set of pure strategies for each player $s_i \in S_i$ where $S = S_1 \times \cdots \times S_n$ is the strategy space and a payoff function for each player $u_i(s)$ where $s = (s_1, .., s_n)$.

The players may choose their strategies simultaneously or sequentially depending on the game form. When the players act simultaneously we have a normal form game and when they act sequentially we have an extensive form game. However, each normal form game can be expressed as an extensive form game where decision points are played simultaneously. One of the major assumptions is the rationality of the player. A rational player would try to maximize his payoff regardless of other circumstances. Without the rationality assumption, it is impossible to predict a player's move so game theoretic notions cannot find an answer. Another important assumption is the common knowledge assumption which states that each player knows the set of players, their strategy sets and the corresponding payoffs. In other words, as Fudenberg and Tirole [16] state "Each player knows the structure of the normal form game and know that their opponents know it, and know that their opponents know that they know, and so on ad infinitum."

### 2.2.1 Mixed Strategies

A mixed strategy $\psi_i$ is a probability distribution over strategy set $S_i$ of a player $i$. We denote the mixed strategy space of player $i$ by $\Psi_i$ and $\Psi = \Psi_1 \times \cdots \times \Psi_n$. Player $i$'s payoff for a mixed strategy profile $\psi$ is:

$$\sum_{s \in S} \left( \prod_{j \in N} \psi_j(s_j) \right) u_i(s).$$

Roughly speaking, we can think of a mixed strategy as a randomization of all strategies of a player since being unpredictable may benefit the player. Clearly, mixed strategies also include pure strategies.

### 2.2.2 Dominated Strategies

In order to predict the outcome of a game, one of the useful tools is elimination of dominated strategies. We can define a dominated strategy as follows:

**Definition 2.1.** *A pure strategy $s_i$ is strictly dominated if there exists a mixed strategy $\psi_i \in \Psi_i$ such that*

$$u_i(\psi_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

*where $-i$ denotes the set of players other than player $i$.*

A rational player would never use a dominated strategy since using an undominated strategy would guarantee a higher payoff. Thus, iterated elimination of the dominated strategies is a common tool that is used for dominated strategies for refinement. It proceeds by eliminating dominated strategies and considering the new strategy space. This process continues until none of the strategy points in the current set is dominated.

### 2.2.3 Best Response functions

Another important concept in game theory is the best response functions. Assume that all the players play before player $i$ and player $i$ can observe their strategies. Now, a best response can be thought as the best possible strategy of player $i$ with the knowledge of other player's strategies.

**Definition 2.2.** *Player $i$'s best response (function) to the strategies $s_{-i}$ of the other players is the strategy $s_i^*$ that maximizes player $i$'s payoff $u_i(s_i, s_{-i})$ i.e.,*

$$s_i^* = argmax_{s_i}\ u_i(s_i, s_{-i}).$$

### 2.2.4 Nash Equilibrium

Using the best response functions we obtain our first important equilibrium concept which is the famous Nash Equilibrium.

**Definition 2.3.** *A strategy profile $(s_1^*, s_2^*, ..., s_n^*)$ is a Nash equilibrium of the game if $s_i^*$ is a best response to $s_{-i}^*$ for all $i = 1, 2, ..., n$ i.e.,*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \forall i \in N, \forall s_i \in S_i.$$

A Nash equilibrium is a point in strategy space where none of the players could profit from unilaterally changing his strategy. It is a point where the strategies of each player is a best response to the strategies of the other players. Nash [39] shows that there exists at least one Nash equilibrium in mixed strategies for all games. However, a Nash equilibrium in pure strategies does not always exist. In this thesis, we use pure strategy equilibria and prove the corresponding existence theorem when necessary.

In order to prove the existence of a pure strategy Nash equilibrium, we need some further definitions. Vives [50] states that a binary relation $\geq$ on a nonempty space $S$ is a partial order if it is transitive, reflexive and anti-symmetric. A supremum (infimum) of $S$ is a least upper bound (greatest lower bound). A lattice is a partially ordered set $(S, \geq)$ in which any two elements has a supremum and an infimum and it is complete if every nonempty subset of $S$ has a supremum and an infimum in $S$. Any compact (closed and bounded) interval in real line with the usual order or product of compact intervals with vector order is a complete lattice.

A function $u$ is supermodular if $u(x_1, x_2) + u(y_1, y_2) \geq u(x_1, y_2) + u(y_1, x_2)$ for all $(x_1, x_2) \geq (y_1, y_2)$. A twice continuously differentiable function $u_i(s_1, .., s_n)$ is supermodular iff $\partial^2 u_i / \partial s_i \partial s_j \geq 0$ for all $s_i, s_j$ where $i \neq j$ [8]. The corresponding game is supermodular if the payoffs of all the players are supermodular. In a supermodular game, a player's best response is increasing in the strategies of other players.

Topkis [48] states that a game has a pure strategy Nash equilibrium if the strategy profile $S$ is a complete lattice, the joint payoff function $u$ is upper–semicontinuos and the payoff function of each player is supermodular.

There can be many equilibria in a game. A good refinement for the Nash equilibrium in case of multiple equilibrium is a dominant–strategy equilibrium. A dominant–strategy equilibrium is an equilibrium point that survives the iterated elimination of dominated strategies which was explained previously.

### 2.2.5 Extensive Form Games

A game may contain more than one stage. In this case each stage is played sequentially in an extensive form game. An extensive form can be thought as a decision tree where at each stage or decision node the corresponding players decide on their new strategies and these strategies are observable by all players. The outcome or payoff is determined after the final stage. A strategy in an extensive form consists of the actions at all the decision points.

Assume that we have $T$ stages in a game. At any stage $t$, the players know the history $h^t$ of the actions by all the players. Thus, we can assume from the stage $t$ on there is game on its right which can be denoted by $\Gamma(h^t)$. These games are called the subgames. Thus, the strategy profile in the subgame $(s|h^t)$ is just a restriction of the original profile $s$ using the history of the game until $t$.

A good example of the games in extensive form is the Stackelberg game where there are two stages and players act sequentially. There is a leader which plays in the first stage and there is a follower which plays in the second stage after observing the action of the leader. Thus, the leader chooses the best possible strategy considering the best response of the follower. Most of the vertical supply chain games between suppliers and manufacturer or manufacturers and retailers are formed of Stackelberg games and the leader is usually the party with more competitive power or the party that prepares the purchasing contract.

The equilibrium concept used in extensive form games is the subgame perfect equilibrium. As the name implies, a strategy profile is in subgame perfect equilibrium if at any stage the corresponding subgame played with the same profile is a Nash equilibrium.

**Definition 2.4.** *A strategy profile $s^*$ of a multi-stage game with observed actions is a sub-game perfect equilibrium if at every decision node $t$ the restricted profile $(s^*|h^t)$ is a Nash equilibrium of the restricted game $\Gamma(h^t)$.*

We use the subgame perfect equilibrium concept in Chapter 3.

## 2.3 Bayesian Games

Most of the games studied in the supply chain management literature assume that all the firms involved in the game have common knowledge about the payoff functions of all the firms. This type of games are called the full information games. The games where all the information is not common knowledge are called incomplete (asymmetric) information games. These games are also called Bayesian games.

Usually, in incomplete information games, the players do not know the payoff functions of other players. Nevertheless, each player has some kind of indication for his payoff function which we call the type of the player. Players' types $\theta = (\theta_1, .., \theta_n)$ are drawn from a probability distribution $f(\theta_1, .., \theta_n)$ over the type space $\Theta = \Theta_1 \times \cdots \times \Theta_n$. The major assumption of the Bayesian games is that the type distributions of the players are common knowledge, i.e., each player knows his own type but only knows the distribution of the type of his opponents. Thus, $\theta_i$ is only observed by player $i$ and we denote $f(\theta_{-i}|\theta_i)$ as the conditional type distribution of other players for given $\theta_i$. This assumption is viable since each firm in a market may estimate the parameters of rival firms based on their own cost, the cost of technology required for production and potential market research.

In case of Bayesian games, a pure strategy of player $i$ is a function $s_i : \Theta_i \to S_i$ from the type space to the strategy space of player $i$.

For each realization of types $\theta$ the ex-post payoff function of player $i$ is $u_i((s_i(\theta_i), s_{-i}(\theta_{-i})), \theta)$. Thus, the interim payoff function of player $i$ is:

$$U_i(s_i, \theta_i; s_{-i}) = \int_{\theta_{-i}} u_i((s_i(\theta_i), s_{-i}(\theta_{-i})), \theta) f(\theta_{-i}|\theta_i) d\theta_{-i}.$$

The payoff function of player $i$ can be thought as some kind of expectation over the types of other players given the conditional probability distribution of the rivals' types.

Similarly, we can define the ex–ante payoff of player $i$, $\mathcal{U}_i$, for a given strategy profile $(s_i, s_{-i})$ as the expected payoff of player $i$ over all type realizations including his own:

$$\mathcal{U}_i(s_i, s_{-i}) = \int_\theta u_i((s_i(\theta_i), s_{-i}(\theta_{-i})), \theta) f(\theta) d\theta.$$

### 2.3.1 Bayesian Nash Equilibrium

**Definition 2.5.** *A strategy profile $s^*(\cdot)$ is a Bayesian Nash equilibrium if for all $i \in N$*

$$\mathcal{U}_i(s_i^*, s_{-i}^*) \geq \mathcal{U}_i(s_i, s_{-i}^*) \qquad \forall s_i \in S_i^{\Theta_i},$$

*where $S_i^{\Theta_i}$ is the set of maps from $\Theta_i$ to $S_i$. Since each type has positive probability, this is equivalent to*

$$U_i(s_i^*, \theta_i; s_{-i}^*) \geq U_i(s_i, \theta_i; s_{-i}^*) \qquad \forall s_i \in S_i, \ \forall \theta_i \in \Theta_i$$

The proof for the existence of pure strategy Bayesian Nash Equilibrium is more tedious than its full information counterpart and is given in the Chapters 4 and 7. We insist on pure strategy equilibriums since it is not straightforward to implement a mixed strategy equilibrium in real life situations.

## 2.4 Mechanism Design

The objective of mechanism design is to implement a given allocation of resources or costs when the relevant information is not common knowledge in the economy. A mechanism is basically a specification of how economic decisions are determined as a function of the information that is known by the players.

In a mechanism design problem, we usually have a resource to allocate. As in the Bayesian games, each player has a type which is drawn from a probabilistic distribution. Depending on his type, a player sends a message to the mechanism and based on these messages the mechanism allocates the resource. Thus, the mechanism is a function which maps the messages to an allocation scheme.

Mechanism design problem usually consists of three steps. In step 1, the mechanism is designed. In step 2, the players accept or reject the mechanism. A player who rejects the mechanism gets some exogenously specified reservation utility. In step 3, the players play the game specified by the mechanism.

A *direct mechanism* is a mechanism where each players sends his true type as his message. A truth–telling strategy is to report true information about preferences for all preference possibilities. A direct mechanism should satisfy incentive compatibility and individual rationality.

**Definition 2.6.** *A mechanism is incentive compatible if for any player $i \in N$ truth-telling is the dominant-strategy.*

Thus, incentive compatibility is essential for players to reveal their true types.

**Definition 2.7.** *A mechanism is individually rational if for any player $i \in N$ the mechanism's resource allocation provides a payoff level that is at least as much as his reservation utility.*

Thus, individual rationality is required for a player to participate in the mechanism. Finally, we give an important result about the mechanism design problem which states that any resource allocation is possible using only direct mechanisms.

**Theorem 2.1.** *Revelation Principle (Dasgupta et al. [12]): Any equilibrium outcome of an arbitrary mechanism can be replicated by an incentive-compatible direct mechanism.*

Revelation Principle guarantees that one can only focus on direct mechanism and not be distracted by any other mechanism.

## 2.5 Game Theory Applications

There is a significant amount of existing research using game theory models in inventory and supply chain management. Leng and Parlar [28] provide an excellent review of more than 130 papers that use game theoretic models and summarize them in five categories including inventory games with fixed unit purchase cost, inventory games with quantity discounts, production and pricing competition, games with other attributes and games with joint decisions on inventory, production/pricing and other attributes. Dror and Hartman [13] provide another survey which mainly concentrates on the cooperative inventory games and explain some of the important concepts such as Shapley value and core allocations.

There are many papers that explain how game theory is used to study inventory, supply chain and operations management problems. Both Cachon and Netessine [8] and Chinchulum et al. [10] summarize the tools of game theory that can be used for competitive and cooperative models. These papers mainly focus on the existence and uniqueness of pure strategy Nash equilibrium and cooperative games. In addition to a game theory review, Erhun and Keskinocak [14] explain game theory can be used in traditional supply chain contracting models such as revenue sharing, buyback and quantity discount contracts and two–part tariffs. Li et al. [29] give a more economic perspective and provides extentions of the well-known operations management and information systems problems using game theory.

# Chapter 3

# A PRIVATE CONTRIBUTIONS GAME FOR JOINT REPLENISHMENT

## 3.1 Introduction

One of the most fundamental trade–offs in operations is between inventory holding costs and ordering costs as they both change as a function of lot sizes used in production, transportation or procurement. Larger lot sizes lead to higher inventory costs, while smaller lot sizes result in higher ordering costs. Beginning with Harris's [20] study of classical economic order quantity (EOQ), a vast body of literature examined these trade–offs. A second major strand in this literature focused on the joint replenishment problem – exploring opportunities to exploit the economies of scale by consolidating or coordinating replenishment of different items or locations to minimize total ordering and inventory costs. For recent surveys of these two strands of literature the reader is referred to the reviews by Jans and Degraeve [23] on lot sizing, and by Aksoy and Erenguc [1] and Khouja and Goyal [26] on the joint replenishment problem.

When joint replenishment involves a group of items or locations that are not controlled centrally, issues arise regarding sharing of joint costs among the parties. In a series of recent papers, Meca et al. [35], Hartman and Dror [21], Anily and Haviv [2] and Zhang [56] analyze cooperative game theory formulations to investigate whether a fair allocation of total costs is possible and if so, how. Meca et al. [35] show that it is possible to obtain the minimum total joint cost when the firms share their order frequencies. They propose a cost allocation mechanism which distributes the total replenishment cost in proportion to the square of individual order frequencies and show that this allocation is in the core of the game, i.e., no coalition can decrease its costs by defecting from the grand coalition. Minner [38] studies a similar problem using a bargaining model which has only two firms, excludes inventory holding costs and uses net present value rather than average costs.

In this chapter, we study joint replenishment in the context of non–cooperative games. It is well–known that, in systems where joint decisions have to rely on information reported by the participants, firms may act strategically and misreport their characteristics. In the last two decades, game theory has been applied in the analysis of a variety of supply–chain related problems (see Cachon and Netessine [8]; Leng and Parlar [28]; Chinchulum et al. [10] for recent comprehensive surveys). Central question of non–cooperative game theory approach is characterization of equilibrium behavior of self–interested players in games where each player's information and strategic options as well as the outcomes that result from each combination of decisions are explicitly specified.

Game theoretic formulations of the joint replenishment problem seem to have adopted almost exclusively the paradigm of cooperative games with transferable utility. Fiestras-Janeior et al. [15] and Dror and Hartman [13] provide excellent surveys of cooperative game theory applications in centralized inventory management. Despite dozens of papers reviewed in Fiestras-Janeior et al. [15] and Dror and Hartman [13] using cooperative game formulations, non-cooperative analysis of joint inventory problems is still in its infancy with many interesting problems that remain to be explored using the machinery of non-cooperative game theory. In fact, Bauso et al. [5] and Meca et al. [34] are the only two exceptions that look at the joint replenishment problem from a non–cooperative point of view.

Bauso et al. [5] study a finite horizon, periodic setting in which multiple firms need to determine their order quantities in each period to satisfy their deterministic, time varying customer demands. The fixed order cost is shared among multiple firms that order in the same period. Bauso et al. [5] show that this game admits a set of pure strategy Nash equilibria, one of which is Pareto optimal. The authors present a consensus protocol that leads the firms converge to one of Nash equilibria, but not necessarily a Pareto optimal one.

Meca et al. [34] (MGB in the sequel) is more closely related to our work. MGB studies a non–cooperative reporting game where stand–alone order frequencies of the firms are observable but not verifiable. Each firm reports an order frequency (that may be different from its true order frequency) and the joint order frequency is determined to minimize the total joint costs based on all reports. Each firm incurs holding cost individually and pays a share of the joint replenishment cost in proportion to the squares of reported order frequencies. MGB shows that, while this rule leads to core allocations under cooperative formulations, it entails significant misreporting and inefficient joint decisions in a non–cooperative framework.

In this chapter, we consider $n$ firms with arbitrary inventory holding cost and demand rates. The firms' characteristics are common knowledge, but they are not verifiable. Each firm decides whether to participate in joint replenishment or to replenish independently, and each participating firm reports the level of his private contribution to the joint ordering costs. An intermediary determines the joint cycle time. The intermediary selects the lowest joint cycle time that can be financed with the participating firms' contributions.

We consider two variants of our basic game with respect to the timeline of participation and contribution decisions. In the single–stage game, each firm makes participation and contribution decisions simultaneously. In this game we seek to characterize the Nash equilibria in undominated strategies. In the two-stage game, the set of firms participating firms becomes known before each participating firm decides how much to contribute. The equilibrium notion we use for the two–stage game is subgame–perfect Nash equilibrium (SPE).

The games we study differs from the one in MGB in several important ways with respect to messages the firms can use and with respect to the outcome functions that specify how joint decisions and individual cost shares are determined based on firms' messages. MGB considers a game where firms' messages are their stand–alone order frequencies. We study games where each firm decides whether to replenish independently or to participate in joint replenishment and then, if he participates, reports the level of his private contribution to the joint ordering cost. With respect to the outcomes functions, while the joint frequency decision in MGB is the efficient joint decision assuming truthful reporting by the firms, in our game joint replenishment frequency is determined to cover the replenishment cost based on the private contributions of participating firms. A participating firm's replenishment cost depends on all the reports through a proportional sharing rule in MGB, whereas, in our setting, it is determined by his report directly.

For the one–stage game, we find that equilibrium behavior and outcomes are determined by a simple property of joint replenishment environment: If there is a single firm with the lowest stand–alone cycle time, then there is a unique undominated Nash equilibrium. For the two–stage game with a positive but small minimum required contribution, participation by all firms is a dominant–strategy equilibrium in the participation stage. Subgame–perfect equilibrium path is unique if and only if the lowest stand–alone cycle time among the firms is strictly less than the second–lowest stand–alone cycle time. For both games, if there are multiple firms with the lowest stand–alone cycle time, there are multiple equilibria. However, the only indeterminacy caused by multiple equilibria concerns how a given aggregate cost share (which is unique) is divided among participating firms with the lowest stand–alone cycle time. Aggregate contributions, joint cycle time, aggregate cost rates, as well as cost rates for firms whose stand–alone cycle times are higher than the lowest stand–alone cycle time are all unique. Some of the proofs are given in the chapter as they are necessary to follow the analysis and the rest of the proofs are contained in the Appendix A.

## 3.2 The Model and Preliminaries

We consider a stylized EOQ environment with a set of firms $N = \{1, ..., n\}$. Demand rate for firm $j$ is constant and deterministic at $\beta_j$ per unit of time. Time rate of inventory holding cost for firm $j$ is $\lambda_j$ per unit. Major ordering cost is fixed at $\kappa$ per order regardless of order size. We assume minor ordering costs (ordering costs associated with firms included in an order) are zero. [1] Although each firm is characterized by two parameters $(\lambda_j, \beta_j)$, an alternative representation $(\alpha_j, \beta_j)$, obtained by a re–parametrization where $\alpha_j = \lambda_j\beta_j$, will be convenient in all the settings that we consider below. For lack of a more natural term, we refer to the parameter $\alpha$ as the adjusted demand rate. We assume a strictly positive lower bound, $\underline{\alpha} > 0$,for the adjusted demand rates, so that $\alpha_j \geq \underline{\alpha}$ for all $j \in N$ to rule out trivial replenishment environments where either the demand rate or the holding cost rate is zero.

For $j \in N$, the ratio

$$\theta_j = \alpha_j / \sum_{k \in N} \alpha_k, \tag{3.1}$$

will prove useful to simplify some comparisons in the sequel.

In a stylized replenishment problem the objective is to minimize the total cost rate, denoted $C$, i.e., the sum of replenishment cost rate ($R$) and holding cost rate ($H$): $C = R + H$. The decision variable can be taken as order cycle time, $t$, or order frequency, $f = 1/t$ (number of orders per time unit). We take cycle time as the decision variable in the sequel.

We use upper–case letters, $N, M, L$ etc., to refer to sets of firms, and use the lower–case version of the same letter for the cardinality of a set. The letters $i, j, k$ are used for firm indices. We label the firms so that $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$. This ordering of firm indices is retained for subsets of $N$. For $M \subseteq N$, denote the

[1]Following a stylized EOQ environment, such as one given in Zipkin (2000, §3.2), it is assumed that the outside supplier that replenishes the orders has no capacity restrictions, delivers the complete order at once after a deterministic lead time and has perfect yield. It is also assumed that the outside supplier is not a strategic player. The firms aim to minimize their long–run *average* costs over time and backorders are not allowed.

set of firms in $M$ with the highest values of the parameter $\alpha$ by $L(M) = \{j \in M | \alpha_j \geq \alpha_i \text{ for all } \; i \in M\}$.

We denote vectors by lower–case letters in bold typeface. For a generic $m-$tuple $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $j \in \{1, \ldots, m\}$, the notation $(y, \boldsymbol{x}_{-j})$ stands for the vector $\boldsymbol{x}$ with its $j$th entry $x_j$ replaced by $y$, and the $(m-1)$-tuple $\boldsymbol{x}_{-j}$ stands for the vector $\boldsymbol{x}$ with its $j$th entry $x_j$ removed.

For an endogenous variable $X$, by $X_M^a$ we refer to the value of $X$ when the set of firms is $M$ and replenishment operations are governed by $a \in \{c, d, g\}$, where $c$ stands for centralized, $d$ stands for decentralized (or independent) replenishment, and $g$ stands for joint replenishment under rules of the non–cooperative game $g$. For instance, $T_M^c$ is the joint cycle time of the firms in $M$ when replenishment is centralized. When the set $M$ is a singleton, e.g., $M = \{j\}$, we use $X_j^a$ instead of $X_{\{j\}}^a$. When we need to refer to the value of an endogenous variable $X_M^a$ faced by firm $j \in M$ we use $X_{Mj}^a$. Thus, for instance, $R_{Mj}^c$ is the replenishment cost faced by firm $j \in M$ when the firms in $M$ replenish jointly.

The vector $e = (N, \kappa, \boldsymbol{\alpha}, \boldsymbol{\beta})$ summarizes the essential data of the inventory environment.

### 3.2.1 Independent (decentralized) replenishment

When the replenishment of the items is controlled by firms operating independently, firm $j$'s total cost rate ($C_j$) is the sum of replenishment cost rate ($R_j$) and the holding cost rate ($H_j$):

$$C_j(t) = R_j(t) + H_j(t) = \frac{\kappa}{t} + \frac{t}{2}\,\alpha_j. \tag{3.2}$$

It is well known that firm $j$'s optimal cycle time is $T_j^d = \sqrt{2\kappa/\alpha_j}$. Hence, optimal frequency and optimal order quantity are $F_j^d = \sqrt{\alpha_j/2\kappa}$ and $Q_j^d = \beta_j\sqrt{2\kappa/\alpha_j}$, respectively. This leads to a replenishment cost rate of $R_j^d = \sqrt{\kappa\alpha_j/2}$. Firm $j$'s holding cost rate is also $H_j^d = \sqrt{\kappa\alpha_j/2}$. Thus firm $j$'s total cost per unit of time is $C_j^d = \sqrt{2\kappa\alpha_j}$. The aggregate total cost rates for $n$ firms under independent replenishment are $C_N^d = \sum_{k\in N}\sqrt{2\kappa\alpha_k}$, and $R_N^d = H_N^d = \sum_{k\in N}\sqrt{\kappa\alpha_k/2}$.

### 3.2.2 Centralized joint replenishment

Efficient joint replenishment requires the replenishment decisions to be taken centrally to minimize the aggregate total cost. It is well known that when there are no minor setup costs, all firms will be replenished in each cycle leading to a common cycle time (see, for example, Meca et al. [35]). The aggregate cost for $n$ firms as function of the common cycle time $t$ can be written as

$$C_N(t) = R_N(t) + H_N(t) = \frac{\kappa}{t} + \frac{t}{2}\sum_{k\in N}\alpha_k\,. \tag{3.3}$$

The optimal cycle time and the corresponding optimal frequency are $T_N^c = \sqrt{2\kappa\,/\,\sum_{k\in N}\alpha_k}$ and $F_N^c = \sqrt{\sum_{k\in N}\alpha_k/2\kappa}$, respectively. Then, the optimal cost rates are $C_N^c = \sqrt{2\kappa\sum_{k\in N}\alpha_k}$, and $R_N^c = H_N^c = C_N^c/2$. At each cycle, firm $j$ orders $Q_{Nj}^c = \beta_j\, T_N^c$.

### 3.2.3 MGB: a direct mechanism for joint replenishment

MGB considers a *a direct mechanism* where the message set of each player coincides with the set of all possible characteristics a player may have and the outcome function assigns the core allocation for the environment reported by the players. Specifically, the firms' stand–alone order frequencies are used as the message space – each firm reports an order frequency that may be different from its true order frequency. Each firm $j$ either reports a positive frequency $f_j$ and joins the coalition for joint replenishment or reports $f_j = 0$ and orders independently. Each firm incurs holding cost individually and the joint replenishment cost is allocated by a proportional sharing rule whereby firms share the joint ordering cost in proportion to the squares of reported order frequencies. For any profile of reported frequencies $(f_1, \ldots, f_n)$, if the number of firms reporting strictly positive frequencies is one or less, all firms replenish independently. With two or more firms reporting positive frequencies, the joint frequency is determined as the efficient frequency for the reported stand–alone frequencies.

However, as MGB find, equilibrium behavior in this game entails significant misreporting. The authors show that the game has multiple equilibria. The strategy profile $(f_1, \ldots, f_n) = (0, \ldots, 0)$ is always an equilibrium resulting in all firms replenishing independently. An equilibrium (dubbed "constructive equilibrium" by the authors) in which all firms participate in joint replenishment exists if, and only if, the firms are sufficiently homogeneous, i.e., if and only if

$$\theta_n < \frac{2}{2n-1}. \tag{3.4}$$

With straightforward translation of MGB's notation to our setting, when a constructive equilibrium exists, it yields the following cycle time and aggregate total cost:

$$T_N^{MGB} = \sqrt{\frac{2\kappa(2n-1)}{\sum_{k\in N}\alpha_k}} = \sqrt{2n-1}\,T_N^c, \tag{3.5}$$

and

$$C_N^{MGB} = \sqrt{\frac{2\kappa n^2 \sum_{k\in N}\alpha_k}{(2n-1)}} = \frac{n}{\sqrt{2n-1}}C_N^c. \tag{3.6}$$

Although the rules of the MGB game would give rise to core allocations with desirable efficiency and fairness properties under truthful reporting, under non–cooperative behavior, we get substantial efficiency loss. In the remainder of this chapter, we investigate the equilibrium outcomes and whether more efficient outcomes can be achieved under an alternative set of rules governing the interaction of the potential participants in joint replenishment.

## 3.3 One–Stage private contributions game for joint replenishment

The participation–contribution game we consider have the following elements: each firm makes two decisions: (1) whether to replenish independently or to participate in joint replenishment, and (2) how much to contribute to joint ordering cost in case of participation. We assume a small but strictly positive lower bound $\delta$ on the contributions for participation in joint replenishment.[2] Specifically, we assume

$$0 < \delta < \bar{\delta} = \sqrt{\kappa \underline{\alpha}/2}/n. \tag{3.7}$$

Formally, the strategy set of players is represented by non–negative real numbers, $\mathcal{M} = \mathbb{R}_+$. A message $r_j$ from player $j$ codes the participation and contribution decisions of firm $j$ as follows: If $r < \delta$, firms $j$ stays out and replenishes independently, if $r_j \geq \delta$, it represents time rate of private contribution to the joint ordering cost.

We denote the vector of messages of the $n$ firms $\boldsymbol{r} = (r_1, \ldots, r_n)$. The set of firms who selected to participate in joint replenishment are denoted by $M(\boldsymbol{r}) = \{i \in N \mid r_i \geq \delta\}$. For $M \subseteq N$, the tuple $\boldsymbol{r}_M$ collects the components of the vector $\boldsymbol{r}$ that correspond to the coordinates in $M$.

Players move simultaneously and each decides his message. For any message profile $\boldsymbol{r}$, the intermediary selects the lowest cycle time that can be financed with the aggregate collection from the participating firms $\sum_{k \in M(\boldsymbol{r})} r_k$, i.e.,

$$\tau(\boldsymbol{r}) = \frac{\kappa}{\sum_{k \in M(\boldsymbol{r})} r_k}. \tag{3.8}$$

[2] The assumed bound on $\delta$ is tighter than needed for the characterization results we present to hold. However, assuming weaker bounds amounts to assuming that the intermediary has more detailed information on the firm-specific details of the replenishment environment, specifically, about the parameter vector $\boldsymbol{\alpha}$. The bound $\bar{\delta}$ involves minimal information about the environment, namely, $n, \kappa$ and $\underline{\alpha}$. Furthermore, under weaker bounds, equilibrium characterization involves complications with many cases and subcases to be considered. If the minimum contribution $\delta$ were to be completely independent of the parameter vector $\boldsymbol{\alpha}$, one could always find replenishment environments where, in the unique equilibrium, no firm participates in joint replenishment.

Implicit in the intermediary's decision rule is an assumption regarding the structure of information held by the firms and the intermediary. The intermediary cannot make use of firm–specific information beyond the contribution decisions reported by individual firms. To be able to decide the joint cycle time, she also needs to know the fixed ordering cost $\kappa$, in addition to the private contributions from the participating firms (and, hence, the set of participating firms).

For given $n-$tuple of messages $\boldsymbol{r}$, the outcome is determined as follows: If $r_j < \delta$, firm $j$ replenishes independently, and his cost is $C_j^d$. All firms in $M(\boldsymbol{r})$ replenish together with joint cycle time $\tau(\boldsymbol{r})$ selected by the intermediary, and firm $j \in M(\boldsymbol{r})$ pays $r_j$ per unit of time as his contribution to joint replenishment cost. [3] A participating firm's replenishment cost rate ($R_j$) is determined directly by his private contribution, $R_j = r_j$, while his holding cost rate ($H_j$) depends on the joint cycle time, $H_j = \alpha_j \tau(\boldsymbol{r})/2$.

The rules of the private contributions mechanism are common knowledge. The parameters of the replenishment environment, i.e., the elements of the list $(\kappa, \boldsymbol{\alpha}, \boldsymbol{\beta})$, are also common knowledge among the firms (but not verifiable).

We can now state the total cost per unit of time for firm $j$, denoted $\phi_j$, as a function of the firms' messages:

$$\phi_j(\boldsymbol{r}) \quad = \quad \begin{cases} \sqrt{2\kappa\alpha_j} & \text{if } r_j < \delta, \\ r_j + \frac{1}{2}\,\alpha_j\,\tau(\boldsymbol{r}) & \text{if } r_j \geq \delta. \end{cases} \tag{3.9}$$

Taking other firms' strategies $\boldsymbol{r}_{-j}$ as given, firm $j$'s decision problem is

$$\min_{r_j} \;\; \phi_j(\boldsymbol{r}),$$

and his best response function, denoted $\rho_j$ is

$$\rho_j(\boldsymbol{r}_{-j}) = \arg\min_{r_j} \phi_j(r_j, \boldsymbol{r}_{-j}).$$

[3] Operationally, the payments for replenishment can be made at the time of the ordering with firm $j \in M(\boldsymbol{z})$ paying $r_j \tau_{M(\boldsymbol{z})}(\boldsymbol{r})$ independent of his order size. Or, firm $j$ can pay a flow of $r_j$ per unit of time without any additional payment at replenishment points.

A Nash equilibrium is a profile $\boldsymbol{r}^* = (r_1^*, \ldots, r_n^*)$ such that $r_j^* = \rho_j(\boldsymbol{r}_{-j}^*)$ for all $j \in N$. A strategy $y$ is said to *strictly dominate* strategy $x$ for player $j$ if $\phi_j(y, \boldsymbol{r}_{-j}) < \phi_j(x, \boldsymbol{r}_{-j})$ for all $(n-1)-$tuple $\boldsymbol{r}_{-j}$ of other players' strategies. A strategy $y$ is said to *weakly dominate* strategy $x$ for player $j$ if $\phi_j(y, \boldsymbol{r}_{-j}) \leq \phi_j(x, \boldsymbol{r}_{-j})$ for all $(n-1)-$tuple $\boldsymbol{r}_{-j}$ of other players' strategies, with strict inequality for at least one $\boldsymbol{r}_{-j}$. A strategy $x$ is said to be an *undominated strategy* for player $j$ if there is no other strategy that weakly dominates it. A profile of strategies $\boldsymbol{r}^* = (r_1^*, \ldots, r_m^*)$ is a *Nash equilibrium in undominated strategies* or *undominated Nash equilibrium* (UNE) if $r_j^*$ is an undominated strategy for player $j$.

Substituting the rule that determines the joint cycle time, firm $j$'s total cost per unit becomes:

$$\phi_j(\boldsymbol{r}) = \phi_j(r_j, \boldsymbol{r}_{-j}) \quad = \quad \begin{cases} \sqrt{2\kappa\alpha_j} & \text{if } r_j < \delta, \\ r_j + \frac{\kappa\alpha_j}{2(r_j + \sum_{k \in M(\boldsymbol{r}) \setminus \{j\}} r_k)} & \text{if } r_j \geq \delta. \end{cases} \tag{3.10}$$

Before we proceed, we collect several observations each with simple proofs.

**Claim 3.1.** *For all replenishment environments, any strategy profile $\boldsymbol{r}$ with $M(\boldsymbol{r}) = \emptyset$, that is, $r_j < \delta$ for all $j \in N$, is a Nash equilibrium.*

**Proof:** Given that other firms are not participating, no strategy $r \geq \delta$ yields a better cost to a player than the cost he gets from independent replenishment.□

**Claim 3.2.** *If $\boldsymbol{r}$ is a Nash equilibrium, then $M(\boldsymbol{r}) \in \{\emptyset, N\}$. That is, unless $\boldsymbol{r}$ yields full participation or no participation, it cannot be a Nash equilibrium.*

**Proof:** Suppose $M(\boldsymbol{r})$ is a non–empty strict subset of $N$, and consider a firm $j \in N \setminus M(\boldsymbol{r})$. Since $j \notin M(\boldsymbol{r})$ player's cost is $C_j^d$. Let $w = \sum_{k \in M(\boldsymbol{r})} r_k$. Since $M(\boldsymbol{r}) \neq \emptyset$, it must be that $w > 0$. If player $j$ deviates from $r_j$ to $R_j^d$ he gets

$$\phi_j(R_j^d, \boldsymbol{r}_{-j}) = R_j^d + \frac{\kappa\alpha_j}{2(R_j^d + w)} < R_j^d + \frac{\kappa\alpha_j}{2(R_j^d)} = 2R_j^d = C_j^d = \phi_j(r_j, \boldsymbol{r}_{-j}). \tag{3.11}$$

where the inequality follows from the fact that $w > 0$, and subsequent equalities follow from the facts $R_j^d = \sqrt{\kappa\alpha_j/2}$ and $C_j^d = 2R_j^d$.□

**Claim 3.3.** *Any strategy $\hat{r}_j < \delta$ is weakly dominated by the strategy $\tilde{r}_j = R_j^d$.*

**Proof:** This follows from observing that the cost strategy $\hat{r}_j$ yields is exactly $C_j^d = \sqrt{2\kappa\alpha_j}$ while the strategy $\tilde{r}_j$ yields a cost that is equal to $C_j^d$ when other players all stay out of joint replenishment, and a cost that is strictly better in all other cases. □

**Claim 3.4.** *Any strategy $\hat{r}_j > R_j^d$ is strictly dominated by the strategy $\tilde{r}_j = R_j^d$.*

**Proof:** Let $w = \sum_{k \in M(\boldsymbol{r})\setminus\{j\}} r_k$. Since $\phi_j(r, \boldsymbol{r}_{-j}) = \Phi_j(r, w) = r + \frac{\kappa\alpha_j}{2(r+w)}$ is strictly convex in $r$, and since the cross–partial $\frac{\partial^2 \Phi_j}{\partial r \partial w} = \frac{\kappa\alpha}{(r+w)^3} > 0$, it follows from the Implicit Function Theorem that $r(w) = \arg\min_r \Phi_j(r, w)$ is unique and strictly decreasing in $w$. Thus, for $w > 0$, we get

$$r(w) < r(0) = R_j^d < \hat{r}_j,$$

which implies, because $\Phi_j(r, w)$ is strictly convex in $r$, that

$$\Phi_j(r(w), w) < \Phi_j(R_j^d, w) < \Phi_j(\hat{r}_j, w).$$

Hence $R_j^d$ strictly dominates $\hat{r}_j$. □

From Claims 3.3 and 3.4 it follows that the set of undominated strategies is the interval $[\delta, R_j^d]$. From Claims 3.1 and 3.3 it follows that if a Nash equilibrium in undominated strategies exists, it involves full participation in joint replenishment. We record these observations in the following proposition.

**Proposition 3.1.** *If $\boldsymbol{r}^*$ is a Nash equilibrium in undominated strategies, then*

1. $M(\boldsymbol{r}^*) = N$ *and*
2. $r_j^* \in [\delta, R_j^d]$.

It remains to characterize the finer details of structure of best response functions and the equilibrium contribution levels. The foregoing observations greatly simplify our task in that they allow us to focus on the second–piece of the cost function and take $M(\boldsymbol{r}) = N$ in the remainder of our investigation. That is,

$$\rho_j(\boldsymbol{r}_{-j}) = \arg\min_{r_j \geq \delta} r_j + \frac{\kappa \alpha_j}{2(r_j + \sum_{k \in N \setminus \{j\}} r_k)}.$$

In order to find the best response of firm $j$, we take the derivative of $\phi_j(r_j, \boldsymbol{r}_{-j})$ with respect to $r_j$ and re–arrange terms:

$$\frac{\partial \phi_j}{\partial r_j} = 1 - \frac{\kappa \alpha_j}{2(r_j + \sum_{k \in N \setminus \{j\}} r_k)^2}. \tag{3.12}$$

Solving $\partial \phi_j / \partial r_j = 0$, and incorporating the minimum contribution requirement, we get:

$$\rho_j(\boldsymbol{r}_{-j}) = \max \left\{ \delta, \sqrt{\frac{\kappa \alpha_j}{2}} - \sum_{k \in N \setminus \{j\}} r_k \right\}. \tag{3.13}$$

Rewriting (3.13), we obtain:

$$\rho_j(\boldsymbol{r}_{-j}) = \begin{cases} R_j^d - \sum_{k \in N \setminus \{j\}} r_k, & \text{if } \sum_{k \in N \setminus \{j\}} r_k \leq R_j^d - \delta, \\ \delta, & \text{if } \sum_{k \in N \setminus \{j\}} r_k > R_j^d - \delta. \end{cases} \tag{3.14}$$

which states that firm $j$'s best response is to contribute such that the aggregate contributions are equal to firm $j$'s stand–alone ordering cost, if the aggregate contributions of other firms are less than firm $j$'s stand–alone ordering cost minus the minimum required amount, and contribute the minimum required amount, otherwise. If firms in $N \setminus \{j\}$ each contributed $\delta$, firm $j$'s best response would be to contribute $R_j^d - (n-1)\delta$ leading to an aggregate contribution of $R_j^d$ from $n$ firms and a cycle time $\tau_N = T_j^d$. Note that $R_j^d - (n-1)\delta = \sqrt{\kappa \alpha_j / 2} - (n-1)\delta$ is strictly larger than $\delta$ since $\delta < \sqrt{\kappa \underline{\alpha}/2}/n \leq \sqrt{\kappa \underline{\alpha}/2}/n$ and $\underline{\alpha} \leq \alpha_j$. For every dollar of contribution from firms in $N \setminus \{j\}$, firm $j$ reduces his contribution dollar for dollar until he reaches the minimum required contribution.

The first pieces of the piecewise–linear best response functions in (3.14) have the same slope (i.e., $-1$) and their intercepts ($R_j^d$ for firm $j$) are ordered. Equilibrium lies in the intersection of best response functions (i.e., solution of $r_j = \rho_j(\sum_{k \in N \setminus \{j\}} r_k)$ for all $j$).

In equilibrium, aggregate contributions must be $R_n^d = \max_{j\in N} R_j^d$. Otherwise, if aggregate contributions were such that $R_n^d - \sum_{j\in N} r_j = R_n^d - \sum_{j\in N\setminus\{m\}} r_j - r_n = \Delta > 0$, firm $n$ would increase his contribution from $r_n$ to $r_n + \Delta$, and using (3.17), this would lead his total cost to decrease from $2R_n^d + \Delta^2/(R_n^d - \Delta) - \sum_{j\in N\setminus\{n\}} r_j$ to $2R_n^d - \sum_{j\in N\setminus\{n\}} r_j$.

In the next proposition we provide a complete characterization of the Nash equilibria in undominated strategies.

**Proposition 3.2.** *In the private contributions joint replenishment game with* $\delta < \sqrt{\kappa\underline{\alpha}/2}/n$*:*

1. *A profile of strategies* $\boldsymbol{r}^* = (r_1^*, \dots, r_{n-\ell}^*, r_{n-\ell+1}^*, \dots r_n^*)$ *is a Nash equilibrium in undominated strategies (UNE) if and only if*
   (a) $r_j^* = \delta$ *for all* $j \in N \setminus L(N)$*, and*
   (b) $(r_{n-\ell+1}^*, \dots r_n^*) \in \left\{ \boldsymbol{x} \in \mathbb{R}^\ell | x_i \geq \delta, \ \text{for } i = 1, \dots, \ell, \ \text{and } \sum_{i\in L(N)} x_i = \sqrt{\kappa\alpha_n/2} - (n-\ell)\delta \right\}$.
2. *The equilibrium is unique if and only if* $L(N)$ *is a singleton, i.e., if and only if* $\alpha_{n-1} < \alpha_n$*. In the unique equilibrium,* $r_j^* = \delta$ *for* $j = 1, \dots, n-1$ *and* $r_n^* = R_n^d - (n-1)\delta$.
3. *In all equilibria, aggregate contributions and the joint cycle time are unique:*
   (a) *Aggregate contributions:* $\sum_{k\in N} r_k^* = \sqrt{\kappa\alpha_n/2} = R_n^d$
   (b) *Cycle time:* $T_N^g = \tau_N(\boldsymbol{r}^*) = \sqrt{2\kappa/\alpha_n} = T_n^d$.
4. *Equilibrium aggregate cost rates are also unique:*
   (a) *Aggregate replenishment cost:* $R_N^g = \sum_{k\in N} r_k^* = \sqrt{\kappa\alpha_n/2} = R_n^d$
   (b) *Aggregate holding cost:* $H_N^g = (\sum_{k\in N} \alpha_k)\sqrt{\kappa/2\alpha_n}$
   (c) *Aggregate total cost:* $C_N^g = \sqrt{\kappa/2\alpha_n}\left(\alpha_n + \sum_{k\in N} \alpha_k\right)$.
5. *In equilibrium firm* $j$ *faces the following cost rates*

(a) *Replenishment cost:* $R^g_{Nj} = \delta$ *if* $j \in N \setminus L(N)$*, and* $R^g_{Nj} \in [\delta, R^d_n - (n-1)\delta]$ *if* $j \in L(N)$*.*

(b) *Holding cost:* $H^g_{Nj} = \alpha_j \sqrt{\kappa/2\alpha_n}$

(c) *Total cost:* $C^g_{Nj} = \delta + \alpha_j \sqrt{\frac{\kappa}{2\alpha_n}}$ *if* $j \in N \setminus L(N)$*, and* $C^g_{Nj} \in [\sqrt{\kappa\alpha_n/2} + \delta, \sqrt{\kappa\alpha_n/2} + R^d_n - (n-1)\delta]$ *if* $j \in L(N)$*.*

Equilibrium cycle time depends on the $2n$-vector $(\alpha_1, ..., \alpha_n, \lambda_1, ..., \lambda_n)$ of the firms' characteristics only through $\alpha_n$ – it is invariant to the number of firms and to the finer details of the firms' characteristics as long as $\alpha_n$ remains fixed. Similarly, equilibrium total cost depends only on two statistics, namely $\alpha_n$ and $\sum_{k \in N} \alpha_k$, of the firms' characteristics.

In the absence of a minimum contribution requirement (i.e., if $\delta = 0$), the order cost is paid by the firms in $L(N)$. If the set $L(N)$ is a singleton, i.e., $L(N) = \{n\}$, in the unique Nash equilibrium, firm $n$ (the firm with the highest stand–alone replenishment rate in $N$) pays $\kappa$ per order and incurs a total cost equal to his stand–alone cost. Other firms ride free and enjoy free deliveries. A free–rider's equilibrium payoff is better than his stand–alone payoff since he does not contribute to the ordering cost and the joint cycle time is strictly better than his stand–alone cycle time. When there are multiple firms with the highest stand–alone replenishment rate, we have multiple equilibria. In some of these equilibria, free–riding can be at its extreme – one of the firms in $L(N)$ finances the entire replenishment cost and others ride free which may also mean that the small firms leave the bigger share of the ordering cost to the larger firms. In any equilibrium that involves more than one contributor, all firms are strictly better off compared to independent replenishment.

## 3.4 Two-stage private contributions game for joint replenishment

Next, we investigate a two–stage model where we separate the participation and contribution decisions to two stages with the following time line. In stage 1, firms move simultaneously and each firm decides whether to replenish jointly through an intermediary or independently. Formally, each firm chooses an action $z \in \{\text{“in”}, \text{“out”}\}$ where “in” stands for participation in joint replenishment through the intermediary and “out” stands for replenishing independently. We denote the vector of first–stage actions of the $n$ firms by $\boldsymbol{z} = (z_1, \ldots, z_n)$, the set of firms who selected to participate in joint replenishment by $M(\boldsymbol{z}) = \{i \in N \mid z_i = \text{“in”}\}$.

If a firm plays “out” in stage 1, he replenishes independently. In stage 2, the set of participating firms, $M = M(\boldsymbol{z})$, becomes common knowledge and firms in $M$ simultaneously submit their time rate of private contributions to the joint ordering cost. Firm $j$'s time rate of private contribution is denoted by $r_j$. Again, we restrict $r_j$ to be at least $\delta$ with the same properties as in the previous section. For any profile $\boldsymbol{r}$ of private contributions submitted by the firms in $M$, all firms in $M$ replenish together with joint cycle time $\tau_M(\boldsymbol{r})$ selected by the intermediary.

Given the contributions $\boldsymbol{r} = (r_1, \ldots, r_m)$ submitted by firms in $M$, the intermediary selects the lowest cycle time that can be financed with the aggregate collection $\sum_{k \in M} r_k$, i.e.,

$$\tau_M(\boldsymbol{r}) = \frac{\kappa}{\sum_{k \in M} r_k}. \tag{3.15}$$

Let $g(\delta)$ denote the two–stage private contributions joint replenishment game with minimum contribution $\delta$ and let $g(\delta, M)$ denote the second stage of the game with participating firms $M$. Let $NE\big(g(\delta, M)\big)$ be the set of Nash equilibria in game $g(\delta, M)$. A subgame perfect equilibrium (SPE) (Selten [45]) for the two–stage game is a profile of strategies $(\boldsymbol{z}^*, \boldsymbol{r}^*(M(\boldsymbol{z}))$ that induces a Nash equilibrium in every subgame – including the subgames that are not reached due to first–stage actions.

### 3.4.1 Stage 2: Equilibrium contributions in subgame $g(\delta, M)$

A participating firm's replenishment cost rate is determined directly by his private contribution, $R_j = r_j$, while his holding cost rate depends on the joint cycle time, $H_j = \alpha_j \tau_M(\boldsymbol{r})/2$. Hence, total cost per unit of time for firm $j$, denoted $\phi_j$, as a function of the private contributions is

$$\phi_j(\boldsymbol{r}) = r_j + \frac{1}{2}\,\alpha_j\,\tau_M(\boldsymbol{r}). \tag{3.16}$$

Let $g(\delta, M)$ represent the private contributions joint replenishment game with minimum contribution $\delta$ among participating firms $M$.

In this section we will use the index set $\{1, \ldots, m\}$ instead of $\{i_1, \ldots, i_m\}$ for the set of participating firms $M \subseteq N$. A participating firm's replenishment cost rate is determined directly by his private contribution, $R_j = r_j$, while his holding cost rate depends on the joint cycle time, $H_j = \alpha_j \tau_M(\boldsymbol{r})/2$. Hence, total cost per unit of time for firm $j$, denoted $\phi_j$, as a function of the private contributions is

$$\phi_j(\boldsymbol{r}) = r_j + \frac{1}{2}\,\alpha_j\,\tau_M(\boldsymbol{r}). \tag{3.17}$$

Taking other firms' contributions $\boldsymbol{r}_{-j}$ as given, firm $j$'s optimization problem is $\min_{r_j \geq \delta}\ \phi_j(\boldsymbol{r})$, and his best response function is $\rho_j(\boldsymbol{r}_{-j}) = \arg\min_{r_j \geq \delta} \phi_j(r_j, \boldsymbol{r}_{-j})$. A Nash equilibrium is a profile $\boldsymbol{r}^* = (r_1^*, \ldots, r_m^*)$ such that $r_j^* = \rho_j(\boldsymbol{r}_{-j}^*)$ for all $j \in M$. Noting that $\tau_M(\boldsymbol{r})$ and firm $j$'s cost depends on $\boldsymbol{r}_{-j}$ only through $\sum_{k \in M \setminus \{j\}} r_k$, aggregate contributions to joint replenishment from the other firms, we re–write firm $j$'s objective function as:

$$\phi_j(r_j, \sum_{k \in M \setminus \{j\}} r_k) = r_j + \frac{\kappa \alpha_j}{2(r_j + \sum_{k \in M \setminus \{j\}} r_k)}. \tag{3.18}$$

The best response function of this stage is the same as the one–stage game for $N = M$. In the next proposition we provide a complete characterization of the Nash equilibria of the game $g(\delta, M)$.

**Proposition 3.3.** *In the private contributions joint replenishment game $g(\delta, M)$ with $M = \{1, \ldots, m\}$ and $\delta < \sqrt{\kappa\underline{\alpha}/2}/n$, the Nash equilibrium $NE\big(g(\delta, M)\big)$ is the Nash equilibrium of the one–stage game with $M = N$.*

PROOF: By definition $M \neq \emptyset$. Thus, for any $M$ we can treat this as a one–stage game where $N = M$. Thus, the results of the one–stage game when all the firms participate in the joint replenishment holds here. $\square$

### 3.4.2 Stage 1: Equilibrium participation

If a firm plays "out" in stage 1, he acts independently and selects his optimal stand–alone cycle time $T_j^d$ and incurs a total cost rate $C_j^d = \sqrt{2\kappa\alpha_j}$. For a firm who selects "in", the payoff depends on the set of other firms who participate in joint replenishment and on the equilibrium bidding strategies in stage 2. If firm $j$ is the only firm who selects "in", in the resulting subgame $g(\delta, \{j\})$ there is a unique bidding equilibrium: firm $j$ submits a contribution equal to his stand–alone replenishment cost $R_j^d = \sqrt{\kappa\alpha_j/2}$, and incurs a total cost rate $C_j^d = \sqrt{2\kappa\alpha_j}$. When there are two or more firms in $M$ but $L(M)$ is a singleton, we have a unique equilibrium in stage 2. In cases where $L(M)$ has multiple firms, we have multiple equilibria in stage 2. Although the stage–2 payoff for a player in $M \setminus L(M)$ is unique (same in all equilibria), for the players in $L(M)$, the payoff to participation depends on which of the stage–2 equilibria is expected to be played. Formally, for $j \in N$, firm $j$'s payoff in the participation stage is:

$$\Phi_j(z_j, \boldsymbol{z}_{-j}) = \begin{cases} C_j^d & \text{if } z_j = \text{“out”} \\ \{\phi_j(\boldsymbol{r}) \,|\, \boldsymbol{r} \in NE\big(g\big(\delta, M(\boldsymbol{z})\big)\big)\} & \text{if } z_j = \text{“in”}. \end{cases} \tag{3.19}$$

In the discussion of first–stage strategies it will be necessary to keep track of the firm indices more carefully in the set $N$ and in the subsets $M(\boldsymbol{z})$ and $L(M(\boldsymbol{z}))$. Thus, we use subscripted indices $M(\boldsymbol{z}) = \{i_1, .., i_{m-\ell}, i_{m-\ell+1}, .., i_m\}$, and $L(M(\boldsymbol{z})) = \{i_{m-\ell+1}, .., i_m\}$.

Using part 5.(c) of Proposition 3.1, we can write the first–stage game payoffs as:

$$\Phi_j(z_j, \boldsymbol{z}_{-j}) \;=\; \begin{cases} \sqrt{2\kappa\alpha_j} & \text{if } z_j = \text{“out” or } M(\boldsymbol{z}) = \{j\} \\ \delta + \alpha_j\sqrt{\kappa/2\alpha_{i_m}} & \text{if } j \in M(\boldsymbol{z}) \setminus L(M(\boldsymbol{z})) \\ \hat{\Phi}, \text{ such that } \hat{\Phi} \in [\underline{\Phi}, \overline{\Phi}] & \text{if } j \in L(M(\boldsymbol{z})) \end{cases} \tag{3.20}$$

where $[\underline{\Phi}, \overline{\Phi}]$ with $\underline{\Phi} = \delta + \sqrt{\kappa\alpha_{i_m}/2}$ and $\overline{\Phi} = \sqrt{2\kappa\alpha_{i_m}} - (m-1)\delta$ denotes the closed interval for the second–stage payoffs for firms in $L(M(\boldsymbol{z}))$ as any value in this interval can arise as an equilibrium outcome in stage 2.

**Proposition 3.4.** *For private contribution games $g(\delta)$ with $0 < \delta < \sqrt{\kappa\underline{\alpha}/2}/n$, the strategy profile $\boldsymbol{z}^* = (z_1^*, \ldots, z_n^*)$, where $z_j^* =$ “in” for all $j \in N$, is a dominant–strategy equilibrium in the participation stage.*

### 3.4.3 Subgame–Perfect Equilibria

The following proposition characterizes the SPE of the two–stage game. We omit the proof as it is straightforward from Propositions 3.3 and 3.4 above.

**Proposition 3.5.** *Let $0 < \delta < \sqrt{\kappa\underline{\alpha}/2}/n$ in the second–stage private contributions game. SPE of the two–stage game have the following properties:*

1. *On the SPE path, all firms participate in stage 1 and play a strategy profile in $NE\big(g(\delta, N)\big)$ in stage 2. SPE path is unique if and only if $\alpha_n > \alpha_{n-1}$.*

2. *SPE outcomes: Cycle time and aggregate cost rates are unique. Equilibrium payoffs of individual firms are unique if and only if $\alpha_n > \alpha_{n-1}$. Otherwise, while the payoffs of firms in $\{j \in N \,|\, \alpha_j < \alpha_n\}$ are unique, equilibrium payoff for a firm in the set $\{j \in N \,|\, \alpha_j = \alpha_n\}$ varies across equilibrium plays.*

    *(a) Cycle time: $T_N^g = \sqrt{2\kappa/\alpha_n}$.*

    *(b) Aggregate total cost: $C_N^g = \sqrt{\kappa/2\alpha_n}\big(\alpha_n + \sum_{k\in N}\alpha_k\big)$.*

3. *In subgames off the SPE path, firms in $M \subsetneq N$ play a strategy profile in $NE\big(g(\delta, M)\big)$.*

Several remarks are in order on the role of minimum contributions, sub–game perfection and the two–stage structure of the game. These two features play complementary roles to reduce the set of outcomes to a unique one with full participation.

Without subgame perfection refinement, Nash equilibrium outcomes of the two–stage game include outcomes that involve participation by a strictly proper subset of the firms. All Nash equilibria that are not subgame–perfect are supported by non–Nash contribution behavior in the second–stage games that are not reached. For example, a strategy profile in which all firms stay out in the participation stage is a Nash equilibrium of the two–stage game. Similarly, one can obtain an arbitrary strict subset $M$ of $N$ as the Nash equilibrium set of participants in the first stage by using second–stage strategies $r_j = \delta$ for all $j \in M'$ and for all $M' \neq M$. With these contribution strategies, the resulting cycle time $\kappa/(m'\delta)$ would be too large since $\delta$ is small and the cost for firm $j$ would be $\delta \frac{1}{2} \frac{\kappa \alpha_j}{m'\delta} > \sqrt{2\kappa\alpha_j}$ i.e., resulting cost would be higher than his stand–alone cost, $C_j^d$. Subgame perfection eliminates such Nash equilibria in the two–stage game by requiring that in every subgame, including the ones not reached, players use Nash equilibrium strategies.

While a minimum contribution requirement in the contribution stage limits free–riders' advantage to some extent, its real significance is due to its role in eliminating a plethora of subgame–perfect equilibria in the two–stage game. Among these equilibria is an equilibrium with no participation in joint replenishment. In absence of a minimum contribution requirement, at least one firm would be indifferent between participation and staying out, and we would lose the dominant strategy property of first stage equilibrium. To take an example, consider environments with strictly ordered $\alpha$s, i.e., $\alpha_1 < \cdots < \alpha_n$. For this case, we have a unique Nash equilibrium in every subgame $M \subseteq N$, and in this equilibrium, the firm with the highest $\alpha$, gets his stand–alone payoff. Firm $j$ is indifferent between the stage 1 strategies "in" and "out" if the firms in $\{j+1, \ldots, n\}$ all choose "out". Thus, the set of first–stage equilibria have the following form: for any $k \in N$, firms $\{1, \ldots, k-1\}$ select "in" and firms $\{k, \ldots, n\}$ select "out". In particular, there exists an sub–game perfect equilibrium in which all firms choose to stay “out”.

## 3.5 Comparison of cycle times and aggregate costs

We can now perform a four–way comparison of cycle times and aggregate total costs under the four modes of joint replenishment: independent, centralized, and non–cooperative joint replenishment under the private contribution game and the direct revelation game studied in MGB. Since both one–stage and two–stage game has the same equilibrium cycle times and agregate costs we do not consider them separately.

As noted above, the equilibrium cycle time depends on the details of the replenishment environment only through $\alpha_n$, the maximum of the $n$ $\alpha$s. Similarly, equilibrium total cost depends only on two statistics, namely $\alpha_n$ and $\sum_{k\in N}\alpha_k$, of the firms' characteristics. For comparisons of cycle times and aggregate costs we obtain a further simplification. Namely, the comparisons depend on the ratios $\theta_j = \alpha_j/\sum_{k\in N}\alpha_k$, rather than the levels of the parameters. Note that the ordering of these $n$ ratios is the same as that of the $\alpha_j s$, that is, $\theta_n = \max\{\theta_j : j \in N\}$. Furthermore, $\theta_n$ takes values in the interval $[1/n, 1]$, and the two limits are obtained for $n$ firms with common $\alpha$s and for $n = 1$, respectively. In particular, $\theta_n < 1$ for $n \geq 2$.

Straightforward algebraic manipulations yield the following ordering of the cycle times under independent, centralized and non–cooperative replenishment:

$$T_1^d \geq T_2^d \geq \cdots \geq T_n^d = T_N^g = T_N^c/\sqrt{\theta_n} > T_N^c. \tag{3.21}$$

For comparison of aggregate costs, after similar algebraic manipulations, we get

$$\begin{aligned} C_N^d > \left(\left(\sqrt{\theta_n} + 1/\sqrt{\theta_n}\right)/2\sum_{k\in N}\sqrt{\theta_k}\right) C_N^d \quad &= \quad C_N^g \qquad (3.22) \\ &= \quad \left(\frac{1}{2}\right)\left(\sqrt{\theta_n} + 1/\sqrt{\theta_n}\right) C_N^c > C_N^c. \end{aligned}$$

To explore how the degree of dispersion in firm characteristics affects the ratio of aggregate cost under cooperative replenishment to that under the participation *ante* contribution game, we observe that the ratio

$$\frac{C_N^g}{C_N^c} = \left(\frac{1}{2}\right)\left(\sqrt{\theta_n} + 1/\sqrt{\theta_n}\right)$$

is strictly decreasing in $\theta_n$. Thus, for fixed $n$, the ratio is largest when the firms have a common $\alpha$. In this case, the ratio becomes

$$\frac{C_N^g}{C_N^c} = \left(\frac{1}{2}\right)\left(\sqrt{n} + 1/\sqrt{n}\right),$$

which increases indefinitely with the number of firms.

Finally we compare the equilibrium cycle times and total cost rates under the private contribution game and the MGB direct revelation game for environments where the MGB game has an equilibrium with full participation. Recall, from (3.4) above, that full participation under the MGB game requires $\theta_n < 2/(2n-1)$. Under this restriction, using (3.4)

$$T_N^{MGB} = \sqrt{2n-1}T_N^c = \sqrt{2n-1}\sqrt{\theta_n}T_n^g > T_n^g$$

since $\theta_n \geq 1/n > 1/(2n-1)$ for $n > 1$. The condition for existence of an equilibrium with full participation under the MGB game yields the following upper bound:

$$\sqrt{2}T_N^g > T_N^{MGB}.$$

To compare the aggregate total cost rates that obtain in the constructive equilibrium of the MGB game and the undominated Nash equilibrium of the private contributions game we use (3.6) and (3.22) to get

$$C_N^{MGB} = \frac{n}{\sqrt{2n-1}}C_N^c = \frac{n}{\sqrt{2n-1}}\frac{2}{\sqrt{\theta_n} + 1/\sqrt{\theta_n}}C_N^g,$$

Hence,

$$\frac{C_N^{MGB}}{C_N^g} = \frac{2n}{\sqrt{2n-1}}\frac{1}{\sqrt{\theta_n} + 1/\sqrt{\theta_n}}. \tag{3.23}$$

For fixed $n$, the right–hand–side of (3.23) is strictly increasing in $\theta_n$, and, it reaches its minimum and maximum when $\theta_n = 1/n$ and $\theta_n = 2/(2n-1)$, respectively. Substituting these values for $\theta_n$ and simplifying we get the following bounds:

$$\frac{2n}{\sqrt{2n-1}}\frac{1}{\sqrt{n}+1/\sqrt{n}} < \frac{C_N^{MGB}}{C_N^g} < \frac{2\sqrt{2}n}{2n+1}. \tag{3.24}$$

To establish that the lower bound is strictly greater than 1, we note the fact that $x(n) = \frac{2n}{\sqrt{2n-1}}\frac{1}{\sqrt{n}+1/\sqrt{n}}$ is strictly increasing in $n$ and $x(2) = 1.0866$. Finally, taking limits of the lower and upper bounds, we find that as $n$ increases indefinitely, the lower and upper bounds both converge to $\sqrt{2}$. That is, for large $n$, total cost under the direct mechanism studied in MGB is more than 40% higher than the total cost under the private contribution mechanism. We conclude by noting that the comparisons would be much more dramatic for situations in which the players' adjusted demand shares are more dispersed than condition (3.4) allows.

## 3.6 Concluding Remarks

In this chapter, we consider a non–cooperative private contributions game for joint replenishment of $n$ firms that operate under an infinite horizon deterministic demand model. Firms may replenish independently or participate in joint replenishment. In case of participation, the firms should decide how much to contribute to the joint ordering cost. The joint cycle time is determined by an intermediary as the lowest cycle time that can be achieved using the collected contributions. We study two variations of this problem: in the single–stage variant, participation and contribution decisions are made simultaneously, and, in the two-stage variant, participating firms becomes known at the contribution stage. We characterize the behavior and outcomes under undominated Nash equilibria for the one-stage game and subgame-perfect equilibrium for the two-stage game. Our results show that the joint replenishment is mostly financed by the firm or group of firms with the highest adjusted demand rate which is the multiplication of inventory holding cost rate and demand rate and the other firms just pay the minimum entree fee. However, even this result is better than the MGB result in most of the cases.

In the following chapter, we explore an extension of the model in this chapter to study situations where the firms are asymmetrically informed about each other's $\alpha$ values and characterize the Bayesian equilibrium, along with a numerical study that investigates the impact of information asymmetry on equilibrium contributions.

# Chapter 4

# PRIVATE CONTRIBUTIONS GAME WITH ASYMMETRIC INFORMATION

## 4.1 Introduction

An important assumption used in the analysis of non-cooperative games is that all information is common knowledge. This assumption is used in many of the articles in supply chain management literature, as well as in Chapter 3. However, information asymmetries exist in many practical settings due to lack of communication or incentives of hiding information especially among competing firms. Neglecting the impact of incomplete information among different parties may misguide the decision makers in supply chain which would affect the overall performance of the business. Moreover, the results of the Chapter 3 indicate that in equilibrium the firm with the highest adjusted demand pays for most of the replenishment cost (all of it if the minimum contribution is 0). Thus, if a firm knows that he does not have the highest adjusted demand rate, he tends to ride free. However, if we have asymmetric information i.e., the firms do not know the ranking of demands, then we may expect positive contributions from the firms with lower adjusted demand rates. Thus, examining the effects of asymmetric information is important.

We extend the one–stage game in Chapter 3 and introduce private information regarding adjusted demand rates. In this case, we assume that the minimum necessary contribution is 0 to focus only on the role of asymmetric information where each firm's adjusted demand can take values from a continuum of types. Each firm learns its type prior to announcing its contribution level, but does not reveal this information to other firms. Our solution concept in this case is Bayesian Nash equilibrium. A Bayesian Nash equilibrium is a Nash equilibrium where each player, given its type, selects a best response against the average best responses of the competing players. We show the existence of a pure–strategy Bayesian Nash equilibrium and derive the necessary equilibrium conditions. In this case, the gain from the contribution game is due to the fact that more information about the demand rates is making its way to the joint replenishment decisions of the intermediary. A numerical study is conducted to show that the performance of the competitive solution behaves similar to the case of full information as $n$ increases, but information asymmetry tends to offer improvements as $n$ and variability in demand rates increase.

The rest of this chapter is organized as follows. In Section 4.2, we simplify the model in Chapter 3 for the case of full information and derive the equilibrium conditions. In Section 4.3, we model the competitive game under asymmetric information, show the equilibrium existence and derive the equilibrium conditions. In Section 4.4, we report the findings of a numerical study that compares the full information and asymmetric information models to the decentralized model. The proofs for the propositions are contained in the Appendix B.

## 4.2 Preliminaries

We consider a stylized EOQ environment with a set of firms $N = \{1, ..., n\}$ ($|N| = n$). Each firm is facing a constant deterministic demand with rate $\beta_j$ per unit of time. Inventory holding cost rate is $\gamma_j$ per unit per unit of time. Major ordering cost is fixed at $\kappa$ per order regardless of order size and we assume minor ordering costs are zero. We define $\alpha_j = \gamma_j \beta_j$, which will be convenient in all the settings that we consider below. We will refer $\alpha_j$ as adjusted demand rate. We assume that $\alpha_j > 0$ to rule out trivial replenishment environments where either the demand rate or the holding cost rate is zero. We label the firms so that $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$. Let $L = \{j \in N \,|\, \alpha_j = \alpha_n\}$ and $\ell = |\, L|$. We follow the same notation as the Chapter 3 where we also show the optimal cycle times and total costs for both independent and centralized models.

We now briefly review the model and results for the competitive game. This is simply the one–stage game described in Chapter 3 with minimum contribution $\delta = 0$. The following mechanism is proposed. There is an intermediary who has a simple role of coordinating the replenishment. Each firm submits a private contribution $r_j$ to the intermediary. This contribution specifies the amount of money the firm will be paying per unit of time for the joint replenishment service. Based on these contributions, the intermediary determines the minimum feasible cycle length. Let $r_1, r_2, ..., r_n$ be the contributions that are submitted by firms. Then the cycle length that is determined by the intermediary will be

$$t = \frac{\kappa}{\sum_{k=1}^{n} r_k}.$$

Proposition 3.2 in Chapter 3 can be used in this game by setting $\delta = 0$.

**Proposition 4.1.** *In the private contributions joint replenishment game,*

1. *A profile of strategies $\boldsymbol{r}^* = (r_1^*, \ldots, r_{n-\ell}^*, r_{n-\ell+1}^*, \ldots r_n^*)$ is a NE if and only if*
   (a) *$r_j^* = 0$ for all $j \in N \setminus L$, and*
   (b) *$(r_{n-\ell+1}^*, \ldots r_n^*) \in \left\{\boldsymbol{x} \in \mathbb{R}^\ell \mid \sum_{i \in L} x_i = \sqrt{\kappa \alpha_n / 2}\right\}$.*
2. *The equilibrium is unique if and only if $L$ is a singleton, i.e., if and only if $\alpha_{n-1} < \alpha_n$. In the unique equilibrium, $r_j^* = 0$ for $j = 1, \ldots, n-1$ and $r_n^* = \sqrt{\kappa \alpha_n / 2}$.*

3. *In all equilibria, aggregate contributions and the joint cycle time are unique:*

   (a) *Aggregate contributions:* $\sum_{j \in N} r_j^* = \sqrt{\kappa \alpha_n / 2} = R_n^d$

   (b) *Cycle time:* $T^f = \sqrt{2\kappa / \alpha_n}$.

4. *Equilibrium aggregate cost rates are also unique:*

   (a) *Aggregate replenishment cost:* $R^f = \sum_{j \in N} r_j^* = \sqrt{\kappa \alpha_n / 2}$

   (b) *Aggregate holding cost:* $H^f = (\sum_{j \in N} \alpha_j) \sqrt{\kappa / 2\alpha_n}$

   (c) *Aggregate total cost:* $C^f = \sqrt{\kappa / 2\alpha_n} \left( \alpha_n + \sum_{j \in N} \alpha_j \right)$.

5. *In equilibrium firm $j$ faces the following cost rates*

   (a) *Replenishment cost:* $R_j^f = 0$ *if* $j \in N \setminus L$, *and* $R_j^f \in [0, \sqrt{\kappa \alpha_n / 2}]$ *if* $j \in L$.

   (b) *Holding cost:* $H_j^f = \alpha_j \sqrt{\kappa / 2\alpha_n}$

   (c) *Total cost:* $C_j^f = \alpha_j \sqrt{\kappa / 2\alpha_n}$ *if* $j \in N \setminus L$, *and* $C_j^f \in [\sqrt{\kappa \alpha_n / 2}, \sqrt{2\kappa \alpha_n}]$ *if* $j \in L$.

The proposition gives the equilibrium cycle times and total costs for the non–cooperative joint replenishment game. Thus, we can move to the non–cooperative joint replenishment game with asymmetric information.

## 4.3 Asymmetric Information

We now turn our attention to the case of private information. We assume that each firm's adjusted demand rate $\alpha_j$ is an independent draw from a common continuous prior distribution function $F$ with support $A = [\underline{\alpha}, \overline{\alpha}]$ with $0 < \underline{\alpha} < \overline{\alpha} < +\infty$ and density function $f$. Note that this captures having uncertainty on demand rate or inventory holding cost rate (given that the other is same across firms) or on both demand rate and inventory holding cost rate.

We first review the impact of uncertainty of adjusted demand rates on independent replenishment, cooperative joint replenishment and non–cooperative joint replenishment under full information. In the case of independent replenishment, each firm learns its adjusted demand rate (type) prior to determining its cycle length. This leads to the following expected cycle length, expected aggregate total cost and expected aggregate replenishment cost expressions:

$$\begin{aligned}
\mathbb{E}[T_j^d] &= \int_A \sqrt{2\kappa/\alpha} f(\alpha) d\alpha, \quad \forall j \in N, \\
\mathbb{E}[C^d] &= n \int_A \sqrt{2\kappa\alpha} f(\alpha) d\alpha, \\
\mathbb{E}[R^d] &= \frac{1}{2}\, \mathbb{E}[C^d].
\end{aligned}$$

In the case of joint ordering with cooperation, we assume that adjusted demand rates of all firms are known prior to establishing the joint replenishment cycle length. Under this assumption, expected joint cycle length, expected aggregate total cost and expected aggregate replenishment cost can be calculated as follows:

$$\begin{aligned}
\mathbb{E}[T^c] &= \int_{A^n} \sqrt{\frac{2\kappa}{\sum_{j\in N} \alpha_j}}\, f^n(\boldsymbol{\alpha}) d\boldsymbol{\alpha}, \\
\mathbb{E}[C^c] &= \int_{A^n} \sqrt{2\kappa \sum_{j\in N} \alpha_j}\, f^n(\boldsymbol{\alpha}) d\boldsymbol{\alpha}, \\
\mathbb{E}[R^c] &= \frac{1}{2}\, \mathbb{E}[C^c]
\end{aligned}$$

where $A^n$ is the $n$th Cartesian power of the interval $A$, $\boldsymbol{\alpha} = (\alpha_1, .., \alpha_n)$ and $f^n(\boldsymbol{\alpha}) = \prod_{j\in N} f(\alpha_j)$.

Under non–cooperative joint replenishment, we adopt the game in Chapter 3 which is briefly reviewed in Section 4.2. First, each firm learns its adjusted demand rate (type). Then the firms submit their private contributions that specify their payment rate for the replenishment service. Based on the contributions, the intermediary determines the minimum cycle length of the joint replenishment such that would finance the fixed cost $\kappa$. Finally, the firms incur their costs according to this cycle length. If the firms reveal their types to other firms before they disclose their contributions, then we have a full information game. Note that the equilibrium described in Proposition 4.1 is determined by the largest adjusted demand rate. Since adjusted demand rates are independent and identically distributed random variables, this correspond to $\alpha_{(n)} = \max_{j \in N} \alpha_j$ , the largest order statistic. Thus we have the following expressions for the expected joint cycle length, expected aggregate replenishment cost, and expected aggregate total cost:

$$\mathbb{E}[T^f] = n \int_A \sqrt{\frac{2\kappa}{\alpha}} f(\alpha)[F(\alpha)]^{n-1} d\alpha, \tag{4.1}$$

$$\mathbb{E}[R^f] = n \int_A \sqrt{\frac{\kappa\alpha}{2}} f(\alpha)[F(\alpha)]^{n-1} d\alpha, \tag{4.2}$$

$$\mathbb{E}[C^f] = n! \int_{\underline{\alpha}}^{\overline{\alpha}} \int_{\underline{\alpha}}^{\alpha_n} \dots \int_{\underline{\alpha}}^{\alpha_2} \sum_{j \in N} \alpha_j \sqrt{\kappa/2\alpha_n} f^n(\boldsymbol{\alpha}) d\boldsymbol{\alpha} + \mathbb{E}[R^f] \tag{4.3}$$

The expressions in (4.1) and (4.2) are due to the fact that largest order statistic $\alpha_{(n)}$ has a probability density function equal to $nf(\alpha)[F(\alpha)]^{n-1}$. The expression in (4.3) is due to the fact that $\sum_{j \in N} \alpha_j \sqrt{\kappa/2 \max_{j \in N} \alpha_j} = \sum_{j \in N} \alpha_{(j)} \sqrt{\kappa/2\alpha_{(n)}}$ and $\alpha_{(1)}, \alpha_{(2)}, \dots, \alpha_{(n)}$ have a joint density $n!f^n(\boldsymbol{\alpha})$.

If the firms do not reveal their type, then we have an asymmetric information game which is the main topic of this chapter. Let $r_j : A \to \Theta$ be the contribution function where $\Theta = [0, \overline{r}]$ and $r_j(\alpha_j)$ is the contribution that firm $j$ makes if its type is $\alpha_j$. We assume an upper bound $\overline{r} = \sqrt{2\kappa\overline{\alpha}}$ on the action space since a contribution level higher than this value gives a payoff worse than the stand–alone payoff regardless of the replenishment rate realizations. Moreover, we exclude negative contributions. Then, for a given $\boldsymbol{\alpha}$ the intermediary will set the cycle length

$$t(\boldsymbol{\alpha}) = \frac{\kappa}{\sum_{k=1}^{n} r_j(\alpha_j)}.$$

Consider a firm $j$ with type $\alpha_j$. Denote $\mathbf{r}_{-\mathbf{j}}(\boldsymbol{\alpha}_{-\mathbf{j}})$ as the vector of contributions of the firms except that of firm $j$ under realization $\boldsymbol{\alpha}_{-\mathbf{j}}$. The payoff for firm $j$ under this realization can be written as

$$\phi_j(r_j, \boldsymbol{r}_{-\mathbf{j}}, \alpha_j, \boldsymbol{\alpha}_{-\mathbf{j}}) = \frac{1}{2} \alpha_j t(\alpha_j, \boldsymbol{\alpha}_{-\mathbf{j}}) + r_j(\alpha_j), \tag{4.4}$$

and the expected payoff for this firm is

$$\begin{aligned}\Phi_j(r_j(\alpha_j), \boldsymbol{r}_{-\mathbf{j}}) &= \int_{A^{n-1}} \phi(r_j, \boldsymbol{r}_{-\mathbf{j}}, \alpha_j, \boldsymbol{\alpha}_{-\mathbf{j}}) f^{n-1}(\boldsymbol{\alpha}_{-\mathbf{j}}) d\boldsymbol{\alpha}_{-\mathbf{j}} \\ &= \frac{1}{2}\kappa\,\alpha_j \int_{A^{n-1}} \frac{1}{\sum_{k=1}^{n} r_j(\alpha_j)} f^{n-1}(\boldsymbol{\alpha}_{-\mathbf{j}}) d\boldsymbol{\alpha}_{-\mathbf{j}} + r_j(\alpha_j). \quad (4.5)\end{aligned}$$

We establish the existence of a pure–strategy Nash equilibrium with the next proposition (All proofs are provided in Appendix).

**Proposition 4.2.** *A pure–strategy Bayesian Nash equilibrium exists for the joint replenishment game under asymmetric information.*

The next proposition characterizes the Bayesian Nash equilibrium for the asymmetric information game.

**Proposition 4.3.** *Any collection of functions* $(r_1^*(\alpha_1), r_2^*(\alpha_2), ..., r_n^*(\alpha_n))$ *that satisfy (4.6) is a Bayesian Nash equilibrium.*

$$\int_{A^{n-1}} \frac{1}{(r_1^*(\alpha_1) + r_2^*(\alpha_2) + ... + r_n^*(\alpha_n))^2} f^{n-1}(\boldsymbol{\alpha}_{-\mathbf{j}}) d\boldsymbol{\alpha}_{-\mathbf{j}} = \frac{2}{\kappa\alpha_j}, \quad \textit{for all } j \in N. \quad (4.6)$$

As stated in Proposition 4.3, finding an equilibrium requires solving $n$ functional equations simultaneously.

The characterization in (4.6) of Proposition 4.3 allows multiple equilibria with different contribution functions for each player. However, if we restrict ourselves to symmetric equilibrium, we have the following lemma.

**Lemma 4.1.** *The symmetric Bayesian Nash equilibrium* $r$ *satisfies the following*

$$\int_{A^{n-1}} \frac{1}{(r^*(\alpha_j) + \sum_{i \neq j} r^*(\alpha_i))^2} f^{n-1}(\boldsymbol{\alpha}_{-\mathbf{j}}) d\boldsymbol{\alpha}_{-\mathbf{j}} = \frac{2}{\kappa\alpha_j} \quad \textit{for all } \alpha_j. \qquad (4.7)$$

Now consider the symmetric equilibrium $r^*$. For a given realization $\boldsymbol{\alpha} = (\alpha_1, .., \alpha_n)$, the cycle length that is set by the intermediary is given as

$$T^a(\boldsymbol{\alpha}) = \frac{\kappa}{\sum_{j \in N} r^*(\alpha_j)}.$$

This leads to an aggregate total cost expression as follows

$$C^a(\boldsymbol{\alpha}) = \frac{1}{2}\frac{\kappa \sum_{j\in N}\alpha_j}{\sum_{j\in N} r^*(\alpha_j)} + \sum_{j\in N} r^*(\alpha_j).$$

Therefore expected cycle length, expected replenishment cost, and expected aggregate total cost rate can be written as:

$$\begin{aligned}
\mathbb{E}[T^a] &= \int_{A^n} \frac{\kappa}{\sum_{j\in N} r^*(\alpha_j)} f^n(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}, \\
\mathbb{E}[R^a] &= n\int_{\underline{\alpha}}^{\overline{\alpha}} r^*(\alpha) f(\alpha) d\alpha, \text{ and} \\
\mathbb{E}[C^a] &= \frac{1}{2}\kappa \int_{A^n} \frac{\sum_{j\in N}\alpha_j}{\sum_{j\in N} r^*(\alpha_j)} f^n(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} + \mathbb{E}[R^a].
\end{aligned}$$

## 4.4 Numerical Study

We conduct a computational study to understand the impact of competition and information asymmetry on firm behavior and total costs. We first start with understanding equilibrium contributions under non–cooperative asymmetric information game. In Figure 4.1, we assume that the adjusted demand rate $\alpha$ has a discrete uniform distribution between $\underline{\alpha} = 1$ and $\overline{\alpha} = 5$. The fixed cost $\kappa = 10$. We consider only the symmetric equilibrium. The figure shows the contribution of a single firm as a function of its adjusted demand rate when there are 1, 2, 3, or 4 firms with 1–firm case corresponding to independent ordering.

Clearly, a firm's contribution increases in equilibrium as its adjusted demand rate increase regardless of the number of firms participating in joint replenishment. Also, as expected, the firms reduce their contributions as there are more firms in the joint replenishment. The marginal reductions, as also expected, are diminishing in the number of firms.

Figure 4.2 shows the impact of asymmetric information on equilibrium under the same settings when there are 2 firms. The solid line in Figure 4.2 represents the expected contribution by a firm as a function of its own adjusted demand

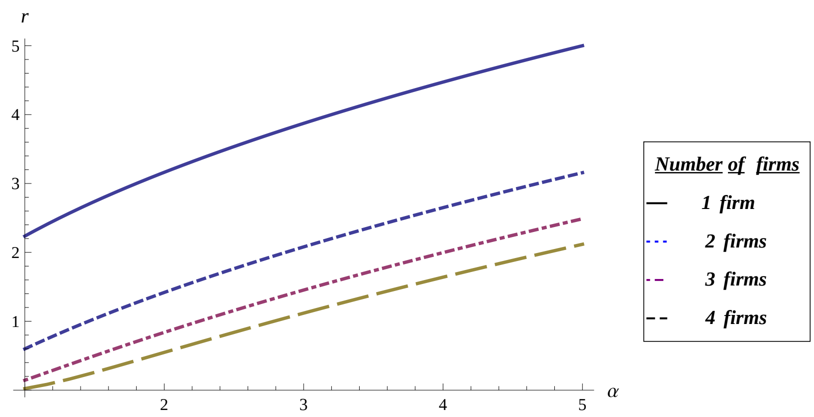


Figure 4.1: Equilibrium contribution vs. demand rate with 1, 2, 3 and 4 firms under asymmetric information

rate, given that it knows the adjusted demand rates of other firms in the joint replenishment program (full information). The dotted line shows the equilibrium contributions under asymmetric information.

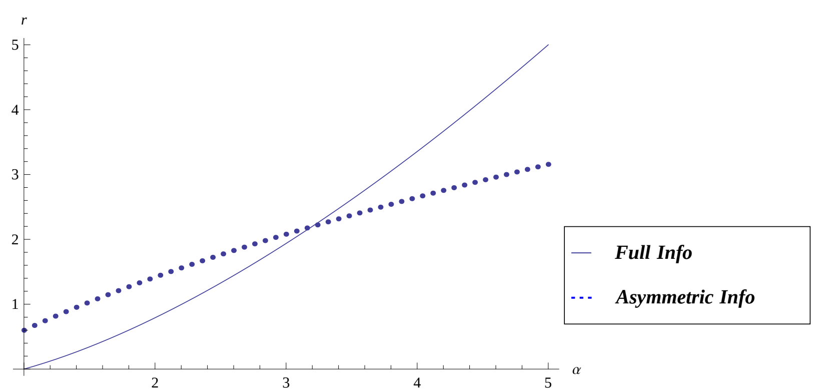


Figure 4.2: The graph of contribution vs. demand rate for two firms under full and asymmetric information

For lower values of adjusted demand rate, a firm that is not informed about its rivals' adjusted demand rates would contribute more than what it would contribute on the average under full information. However, the full information contribution surpass asymmetric information for higher levels of adjusted demand rate.

Figure 4.3 shows the impact of asymmetric information on equilibrium under the same settings for 3 firms. We observe that the rate of increase of contributions is even higher for 3 firms and for lower adjusted demand values asymmetric information contributions are closer to full information contributions.

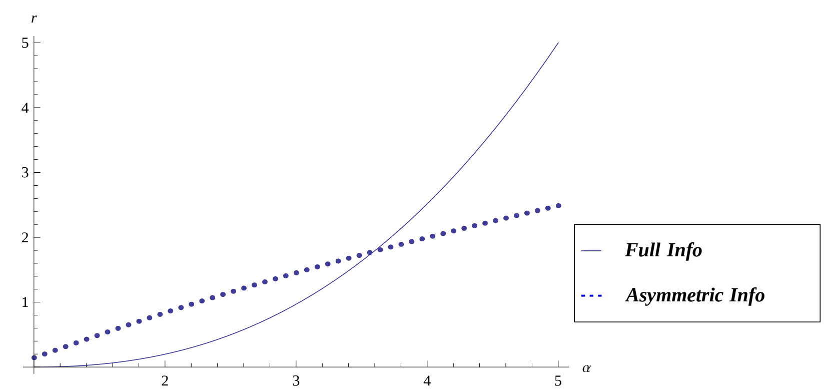


Figure 4.3: The graph of contribution vs. demand rate for three firms under full and asymmetric information

In order to understand the impact of competition and information asymmetry on cycle times, aggregate contributions to replenishment service and aggregate total costs, we carried out a more detailed study in Table 4.1. We assume that the adjusted demand rate of each firm is independently and identically distributed with a discrete uniform distribution in the interval $[\mu - \Delta, \mu + \Delta]$ with 51 points. The mean $\mu$ takes on 3 values, 3, 6 and 9. $\Delta$ takes on various values up to 2/3's of the mean. We consider cases with 2, 3, and 4 players. In order to provide a benchmark, we also show the results for cooperative joint replenishment and independent ordering. Since cooperative joint replenishment leads to lowest aggregate total costs, we use its expected aggregate replenishment cost, expected cycle length and expected aggregate total costs in Columns 4-6. of Table 4.1. Columns 7-9, 10-12, 13-15 show the percentage deviation from the base case, of independent ordering, non–cooperative joint replenishment under asymmetric information and non–cooperative joint replenishment under full information, respectively. In Table 4.1 we provide ex–ante performance comparisons, i.e., if $X^y$ is the performance variable $X$'s performance under policy $y$, we report

$$100 \times \frac{\mathbb{E}[X^y] - \mathbb{E}[X^c]}{\mathbb{E}[X^c]}.$$

As also demonstrated in Section 3.5, independent ordering leads to higher cycle times, and higher aggregate costs than cooperative joint ordering. The gaps increase as the number of firms $n$ increases. Using the results in Section 3.5, when $\Delta = 0$, the gap can be represented as $\sqrt{n}-1$ where $n$ denotes the number of firms. While the mean demand has no effect, increasing the uncertainty (captured by $\Delta$) reduces the gaps in costs and increases the gap in cycle times.

As expected, under asymmetric information, firms contribute less than what they would in a cooperative setting. This leads to a cycle length larger than the cooperative (and optimal) case. As a result, aggregate total costs are also higher. The gap increases as the number of firms increase. Increasing uncertainty leads to expected aggregate contributions that are closer to the cooperative case. The expected aggregate total costs also decline as uncertainty increases. The impact of (scaled) uncertainty is more pronounced, when the mean demands are larger.

Expected aggregate contributions under full information are larger than those under the asymmetric information case. This leads to cycle times closer the cooperative case and a better expected aggregate total cost performance. The performance of non–cooperative joint ordering under full information compared to asymmetric information (and compared to cooperative joint ordering) improves as uncertainty increases.

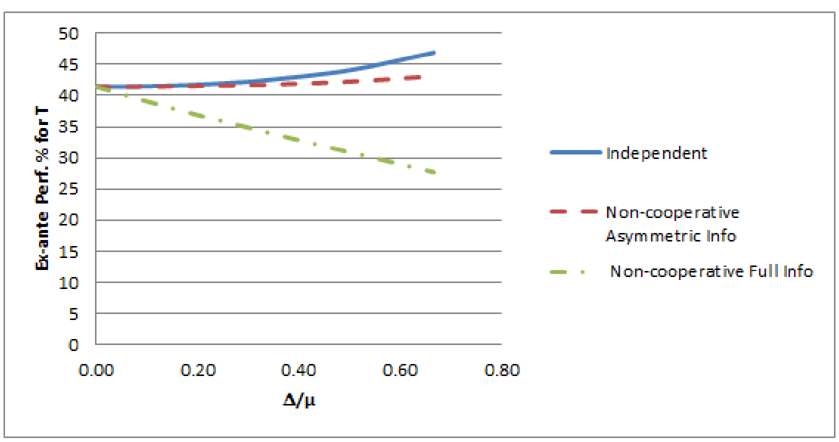


Figure 4.4: Ex-ante performance of cycle times vs. $\Delta/\mu$ for $n = 2$

We can also see the comparison of independent, asymmetric information and full information ex-ante cycle times with respect to the $\Delta/\mu$ ratio for 2 and 3

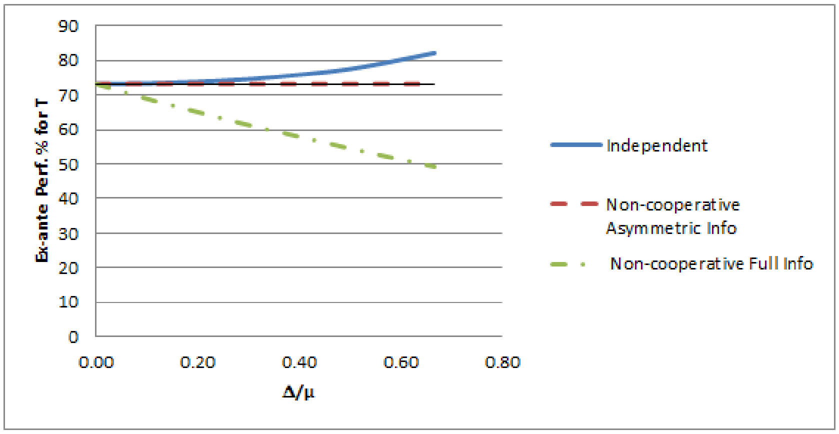


Figure 4.5: Ex-ante performance of cycle times vs. $\Delta/\mu$ for $n = 3$

firm cases in Figures 4.4 and 4.5. These comparisons are given as a percentage of the efficient cycle times. $\Delta/\mu$ ratio gives an idea about the variance of the type distributions. We see that for both figures as $\Delta/\mu$ increases, the full information cycle time decreases but asymmetric information cycle time increases.

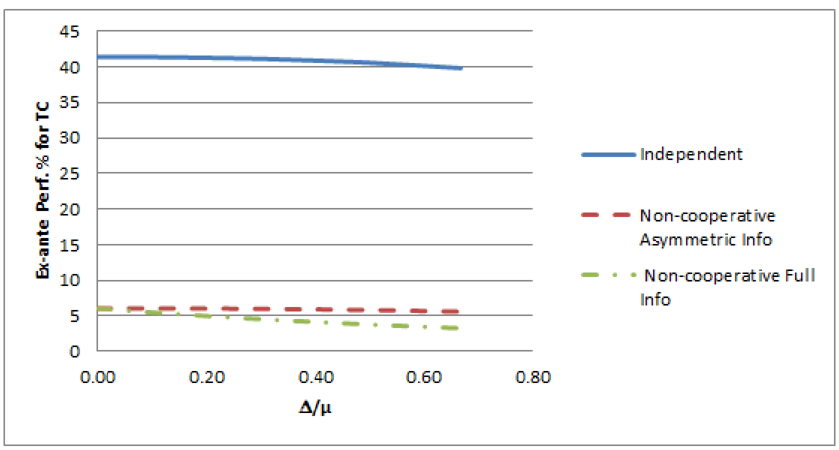


Figure 4.6: Ex-ante performance of total costs vs. $\Delta/\mu$ for $n = 2$

We also compare the expected total cost under independent, full information and asymmetric information cases in Figures 4.6 and 4.7. The performance of asymmetric information case is slightly worse than that of the full information case but both outperforms the independent ordering case. For asymmetric information, even though as $\Delta/\mu$ ratio increases the cycle time increases, we see that the total cost decreases.

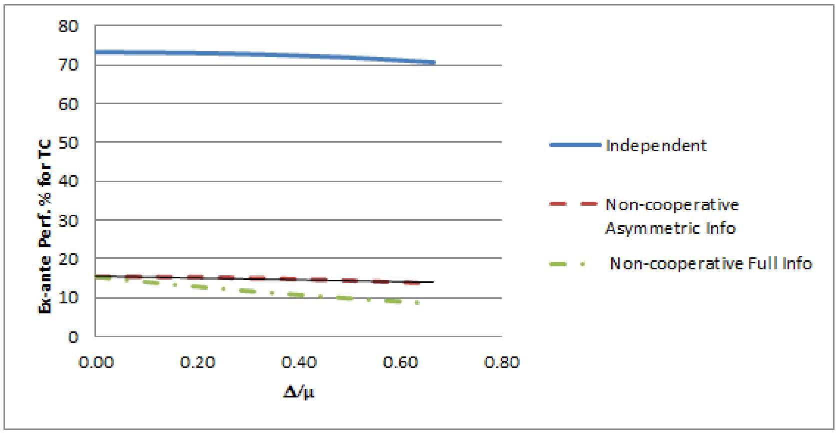


Figure 4.7: Ex-ante performance of total costs vs. $\Delta/\mu$ for $n = 3$

It can be observed both from the table and the figures that the costs for asymmetric information model are getting closer to optimal as $\Delta$ increases while in fact cycle time is getting further away from the optimal.

In Table 4.2, we provide interim performance comparisons, i.e., if $X^y$ is the performance variable $X$'s performance under policy $y$, we report

$$100 \times \mathbb{E}\left[\frac{X^y - X^c}{X^c}\right].$$

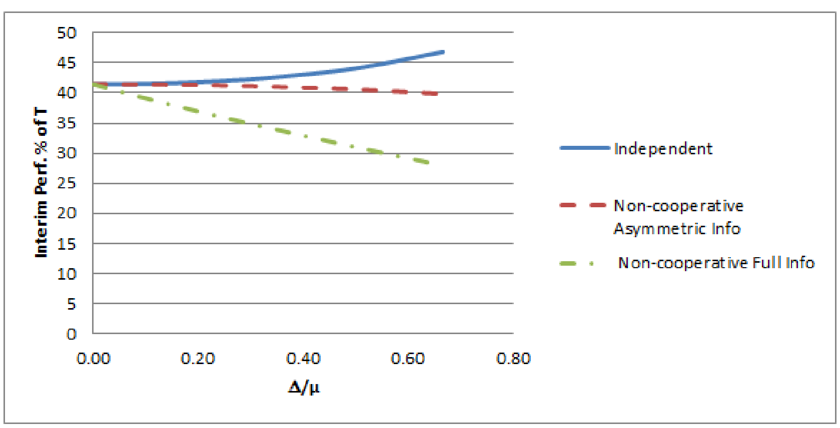


Figure 4.8: Interim performance of cycle times vs. $\Delta/\mu$ for $n = 2$

The results are mostly similar to those obtained in Table 4.1. However, the uncertainty now has a less pronounced impact on performance gaps. In addition, while increasing uncertainty consistently leads to better expected aggregate total cost performance in Table 4.1, this is not the case in Table 4.2. For $n = 2$,

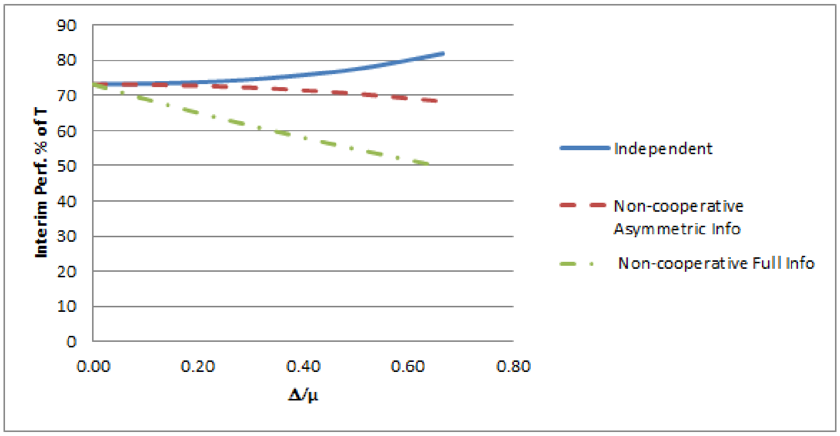


Figure 4.9: Interim performance of cycle times vs. $\Delta/\mu$ for $n = 3$

increasing uncertainty leads to worse performance for non-cooperative joint replenishment under asymmetric information.

Figures 4.8 and 4.9 show that for asymmetric information contrary to the ex-ante case, the cycle time decreases as the $\Delta/\mu$ ratio increases. Moreover, the cycle time for full information case also decreases and the cycle time for the independent case increases.

For the total cost comparison shown in Figures 4.10 for the asymmetric information case we observe that the total cost slowly increases as the ratio increases. However, in Figure 4.11 we see that it has a similar structure to ex-ante case but the decrease in total cost with respect to $\Delta/\mu$ is slower.

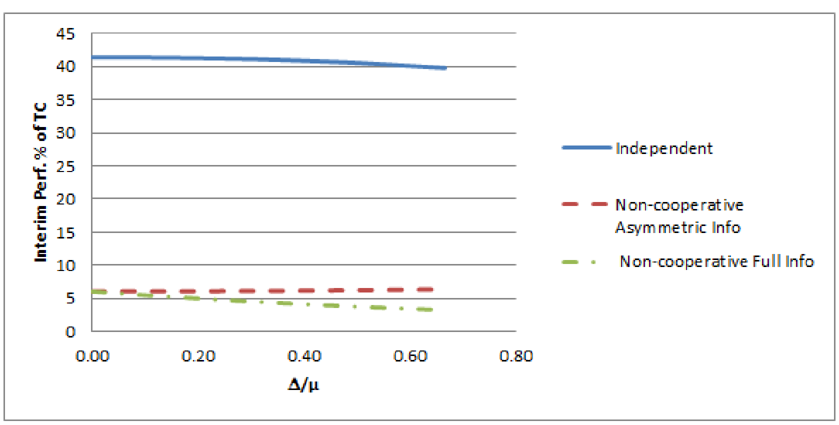


Figure 4.10: Interim performance of total costs vs. $\Delta/\mu$ for $n = 2$

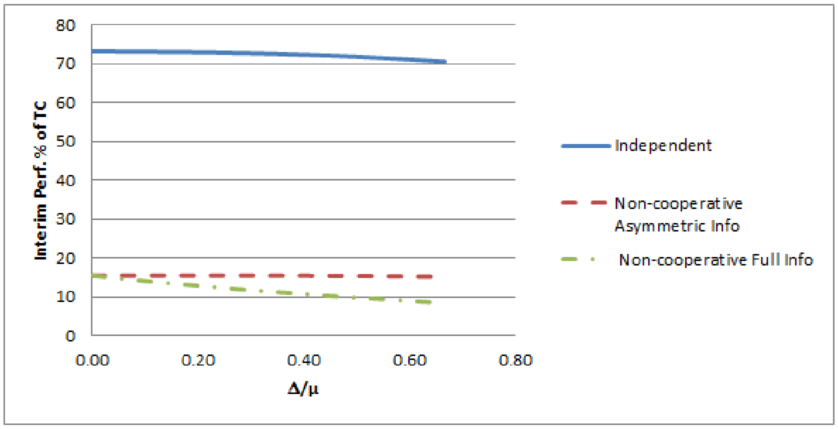


Figure 4.11: Interim performance of total costs vs. $\Delta/\mu$ for $n = 3$

## 4.5 Concluding Remarks

In this chapter, we extend the game in Chapter 3 to incorporate the asymmetric information on adjusted demands of the firms. We do not assume any minimum contribution level and consider a one–stage game. Even though low type firms tend to contribute more, we see that the on average full information costs are lower than the asymmetric information costs. Moreover, we do not observe significant improvements in the total contribution levels due to information asymmetry. Finally, when we increase the variance of the type distribution, we see that the ex-ante cycle time for asymmetric information increases for the case with two firms. In both full information and asymmetric information cases, the private contribution game performs better as the variance increases. However, there is still a gap between the efficient total costs and the equilibrium total costs.

In the next chapter, the investigate a three–stage game for joint replenishment where intermediary is also a decision maker and analyze the implications of this assumption and observe whether the efficient cycle time is attainable in this setting.

## Table 4.1: Ex-ante Performance Comparisons

| | | | Cooperative | | | Independent | | | Non-cooperative AI | | | Non-cooperative FI | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $\mu$ | $\Delta$ | $\mathbb{E}[R^c]$ | $\mathbb{E}[T^c]$ | $\mathbb{E}[C^c]$ | $\mathbb{E}[R^d]$ | $\mathbb{E}[T_j^d]$ | $\mathbb{E}[C^d]$ | $\mathbb{E}[R^a]$ | $\mathbb{E}[T^a]$ | $\mathbb{E}[C^a]$ | $\mathbb{E}[R^f]$ | $\mathbb{E}[T^f]$ | $\mathbb{E}[C^f]$ |
| 2 | 3 | 0 | 5.477 | 1.826 | 10.955 | 41.42 | 41.42 | 41.42 | -29.29 | 41.42 | 6.07 | -29.29 | 41.42 | 6.07 |
| | | 0.5 | 5.474 | 1.829 | 10.948 | 41.34 | 41.68 | 41.34 | -29.08 | 41.51 | 6.05 | -27.32 | 37.64 | 5.16 |
| | | 1 | 5.464 | 1.839 | 10.928 | 41.07 | 42.51 | 41.07 | -28.43 | 41.78 | 5.97 | -25.41 | 34.19 | 4.42 |
| | | 1.5 | 5.447 | 1.858 | 10.893 | 40.59 | 44.09 | 40.59 | -27.32 | 42.29 | 5.84 | -23.53 | 30.93 | 3.80 |
| | | 2 | 5.421 | 1.888 | 10.842 | 39.85 | 46.93 | 39.84 | -25.74 | 43.16 | 5.60 | -21.65 | 27.73 | 3.28 |
| | 6 | 0 | 7.746 | 1.291 | 15.492 | 41.42 | 41.42 | 41.42 | -29.29 | 41.42 | 6.07 | -29.29 | 41.42 | 6.07 |
| | | 0.5 | 7.745 | 1.292 | 15.490 | 41.40 | 41.49 | 41.40 | -29.24 | 41.44 | 6.06 | -28.30 | 39.48 | 5.59 |
| | | 1 | 7.741 | 1.293 | 15.483 | 41.34 | 41.68 | 41.34 | -29.08 | 41.51 | 6.04 | -27.32 | 37.64 | 5.16 |
| | | 1.5 | 7.735 | 1.296 | 15.471 | 41.23 | 42.02 | 41.23 | -28.81 | 41.62 | 6.02 | -26.36 | 35.88 | 4.77 |
| | | 2 | 7.727 | 1.301 | 15.454 | 41.07 | 42.51 | 41.07 | -28.43 | 41.78 | 5.97 | -25.41 | 34.19 | 4.42 |
| | | 3 | 7.703 | 1.314 | 15.405 | 40.59 | 44.09 | 40.59 | -27.32 | 42.29 | 5.84 | -23.53 | 30.93 | 3.80 |
| | | 4 | 7.666 | 1.335 | 15.333 | 39.84 | 46.93 | 39.85 | -25.74 | 43.16 | 5.60 | -21.65 | 27.73 | 3.28 |
| | 9 | 0 | 9.487 | 1.054 | 18.974 | 41.42 | 41.42 | 41.42 | -29.29 | 41.42 | 6.07 | -29.29 | 41.42 | 6.07 |
| | | 0.5 | 9.486 | 1.054 | 18.972 | 41.41 | 41.45 | 41.41 | -29.27 | 41.43 | 6.06 | -28.63 | 40.11 | 5.74 |
| | | 1 | 9.484 | 1.055 | 18.969 | 41.38 | 41.54 | 41.38 | -29.19 | 41.46 | 6.06 | -27.97 | 38.85 | 5.44 |
| | | 1.5 | 9.481 | 1.056 | 18.962 | 41.34 | 41.68 | 41.34 | -29.08 | 41.51 | 6.04 | -27.32 | 37.64 | 5.16 |
| | | 2 | 9.477 | 1.058 | 18.953 | 41.27 | 41.89 | 41.27 | -28.91 | 41.58 | 6.03 | -26.68 | 36.46 | 4.90 |
| | | 3 | 9.464 | 1.062 | 18.927 | 41.07 | 42.51 | 41.07 | -28.43 | 41.78 | 5.97 | -25.41 | 34.19 | 4.42 |
| | | 4.5 | 9.434 | 1.073 | 18.867 | 40.59 | 44.09 | 40.59 | -27.32 | 42.29 | 5.84 | -23.53 | 30.93 | 3.80 |
| | | 6 | 9.389 | 1.090 | 18.779 | 39.84 | 46.93 | 39.85 | -25.74 | 43.16 | 5.60 | -21.65 | 27.73 | 3.28 |
| 3 | 3 | 0 | 6.708 | 1.491 | 13.416 | 73.20 | 73.21 | 73.21 | -42.27 | 73.21 | 15.47 | -42.27 | 73.21 | 15.47 |
| | | 0.5 | 6.706 | 1.493 | 13.411 | 73.07 | 73.63 | 73.07 | -41.89 | 73.20 | 15.38 | -39.87 | 66.32 | 13.25 |
| | | 1 | 6.697 | 1.498 | 13.395 | 72.63 | 74.98 | 72.63 | -40.80 | 73.20 | 15.08 | -37.56 | 60.19 | 11.41 |
| | | 1.5 | 6.683 | 1.508 | 13.367 | 71.86 | 77.55 | 71.86 | -39.02 | 73.20 | 14.53 | -35.30 | 54.56 | 9.86 |
| | | 2 | 6.663 | 1.523 | 13.326 | 70.66 | 82.13 | 70.66 | -36.77 | 73.21 | 13.64 | -33.09 | 49.22 | 8.56 |
| | 6 | 0 | 9.487 | 1.054 | 18.974 | 73.21 | 73.21 | 73.21 | -42.27 | 73.21 | 15.47 | -42.26 | 73.21 | 15.47 |
| | | 0.5 | 9.486 | 1.054 | 18.972 | 73.17 | 73.31 | 73.17 | -42.17 | 73.20 | 15.45 | -41.05 | 69.65 | 14.30 |
| | | 1 | 9.483 | 1.055 | 18.966 | 73.06 | 73.63 | 73.07 | -41.90 | 73.20 | 15.38 | -39.87 | 66.32 | 13.25 |
| | | 1.5 | 9.478 | 1.057 | 18.956 | 72.89 | 74.17 | 72.89 | -41.43 | 73.20 | 15.26 | -38.70 | 63.18 | 12.29 |
| | | 2 | 9.471 | 1.059 | 18.943 | 72.63 | 74.98 | 72.63 | -40.80 | 73.20 | 15.08 | -37.56 | 60.19 | 11.41 |
| | | 3 | 9.452 | 1.066 | 18.903 | 71.86 | 77.55 | 71.86 | -39.02 | 73.20 | 14.53 | -35.30 | 54.56 | 9.86 |
| | | 4 | 9.423 | 1.077 | 18.846 | 70.66 | 82.13 | 70.66 | -36.77 | 73.21 | 13.64 | -33.09 | 49.22 | 8.56 |
| | 9 | 0 | 11.619 | 0.861 | 23.238 | 73.21 | 73.20 | 73.20 | -42.26 | 73.20 | 15.47 | -42.26 | 73.20 | 15.47 |
| | | 0.5 | 11.618 | 0.861 | 23.237 | 73.19 | 73.25 | 73.19 | -42.22 | 73.20 | 15.46 | -41.45 | 70.81 | 14.68 |
| | | 1 | 11.617 | 0.861 | 23.234 | 73.14 | 73.39 | 73.14 | -42.10 | 73.20 | 15.43 | -40.65 | 68.52 | 13.94 |
| | | 1.5 | 11.614 | 0.862 | 23.229 | 73.06 | 73.63 | 73.06 | -41.90 | 73.21 | 15.38 | -39.87 | 66.32 | 13.25 |
| | | 2 | 11.611 | 0.863 | 23.221 | 72.95 | 73.97 | 72.95 | -41.61 | 73.20 | 15.30 | -39.09 | 64.21 | 12.60 |
| | | 3 | 11.600 | 0.865 | 23.200 | 72.63 | 74.98 | 72.63 | -40.80 | 73.20 | 15.08 | -37.56 | 60.19 | 11.41 |
| | | 4.5 | 11.576 | 0.871 | 23.152 | 71.86 | 77.55 | 71.86 | -39.02 | 73.20 | 14.53 | -35.30 | 54.56 | 9.86 |
| | | 6 | 11.541 | 0.879 | 23.082 | 70.66 | 82.13 | 70.66 | -36.77 | 73.21 | 13.64 | -33.09 | 49.22 | 8.56 |
| 4 | 3 | 0 | 7.746 | 1.291 | 15.492 | 100.00 | 100.00 | 100.00 | -50.00 | 100.00 | 25.00 | -50.00 | 100.00 | 25.00 |
| | | 0.5 | 7.744 | 1.292 | 15.487 | 99.81 | 100.57 | 99.81 | -49.49 | 99.81 | 24.79 | -47.47 | 90.38 | 21.48 |
| | | 1 | 7.736 | 1.296 | 15.472 | 99.23 | 102.40 | 99.23 | -48.00 | 99.22 | 24.12 | -45.05 | 81.96 | 18.59 |
| | | 1.5 | 7.724 | 1.302 | 15.448 | 98.19 | 105.88 | 98.19 | -45.77 | 98.15 | 22.92 | -42.71 | 74.39 | 16.17 |
| | | 2 | 7.706 | 1.312 | 15.412 | 96.57 | 112.11 | 96.57 | -43.36 | 96.68 | 21.20 | -40.42 | 67.39 | 14.13 |
| | 6 | 0 | 10.954 | 0.913 | 21.909 | 100.00 | 100.00 | 100.00 | -50.00 | 100.00 | 25.00 | -50.00 | 100.00 | 25.00 |
| | | 0.5 | 10.954 | 0.913 | 21.907 | 99.95 | 100.14 | 99.95 | -49.87 | 99.95 | 24.95 | -48.72 | 95.01 | 23.15 |
| | | 1 | 10.951 | 0.914 | 21.902 | 99.81 | 100.57 | 99.81 | -49.49 | 99.81 | 24.79 | -47.47 | 90.38 | 21.48 |
| | | 1.5 | 10.947 | 0.915 | 21.893 | 99.57 | 101.31 | 99.57 | -48.86 | 99.57 | 24.51 | -46.25 | 86.05 | 19.97 |
| | | 2 | 10.941 | 0.916 | 21.881 | 99.23 | 102.39 | 99.23 | -48.00 | 99.22 | 24.12 | -45.05 | 81.96 | 18.59 |
| | | 3 | 10.923 | 0.921 | 21.846 | 98.19 | 105.88 | 98.19 | -45.77 | 98.15 | 22.92 | -42.71 | 74.39 | 16.17 |
| | | 4 | 10.898 | 0.928 | 21.795 | 96.57 | 112.11 | 96.57 | -43.36 | 96.68 | 21.20 | -40.42 | 67.39 | 14.13 |
| | 9 | 0 | 13.416 | 0.745 | 26.833 | 100.00 | 100.00 | 100.00 | -50.00 | 100.00 | 25.00 | -50.00 | 100.00 | 25.00 |
| | | 0.5 | 13.416 | 0.745 | 26.832 | 99.98 | 100.06 | 99.98 | -49.94 | 99.98 | 24.98 | -49.14 | 96.63 | 23.75 |
| | | 1 | 13.415 | 0.746 | 26.829 | 99.92 | 100.25 | 99.92 | -49.77 | 99.92 | 24.91 | -48.30 | 93.43 | 22.58 |
| | | 1.5 | 13.412 | 0.746 | 26.824 | 99.81 | 100.57 | 99.81 | -49.49 | 99.81 | 24.79 | -47.47 | 90.38 | 21.48 |
| | | 2 | 13.409 | 0.747 | 26.818 | 99.66 | 101.03 | 99.66 | -49.09 | 99.66 | 24.62 | -46.65 | 87.46 | 20.46 |
| | | 3 | 13.399 | 0.748 | 26.799 | 99.23 | 102.39 | 99.23 | -48.00 | 99.22 | 24.12 | -45.05 | 81.96 | 18.59 |
| | | 4.5 | 13.378 | 0.752 | 26.756 | 98.19 | 105.88 | 98.19 | -45.77 | 98.15 | 22.92 | -42.71 | 74.39 | 16.17 |
| | | 6 | 13.347 | 0.758 | 26.694 | 96.57 | 112.11 | 96.57 | -43.36 | 96.67 | 21.20 | -40.42 | 67.39 | 14.13 |

## Table 4.2: Interim Performance Comparisons

| | | | Cooperative | | | Independent | | | Non-cooperative AI | | | Non-cooperative FI | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $\mu$ | $\Delta$ | $\mathbb{E}[R^c]$ | $\mathbb{E}[T^c]$ | $\mathbb{E}[C^c]$ | $\mathbb{E}[R^d]$ | $\mathbb{E}[T_j^d]$ | $\mathbb{E}[C^d]$ | $\mathbb{E}[R^a]$ | $\mathbb{E}[T^a]$ | $\mathbb{E}[C^a]$ | $\mathbb{E}[R^f]$ | $\mathbb{E}[T^f]$ | $\mathbb{E}[C^f]$ |
| 2 | 3 | 0 | 5.477 | 1.826 | 10.955 | 41.42 | 41.42 | 41.42 | -29.29 | 41.42 | 6.07 | -29.29 | 41.42 | 6.07 |
| | | 0.5 | 5.474 | 1.829 | 10.948 | 41.34 | 41.68 | 41.34 | -29.16 | 41.34 | 6.09 | -27.32 | 37.64 | 5.16 |
| | | 1 | 5.464 | 1.839 | 10.928 | 41.07 | 42.51 | 41.07 | -28.77 | 41.07 | 6.15 | -25.39 | 34.21 | 4.41 |
| | | 1.5 | 5.447 | 1.858 | 10.893 | 40.58 | 44.07 | 40.58 | -28.09 | 40.59 | 6.25 | -23.46 | 31.01 | 3.77 |
| | | 2 | 5.421 | 1.888 | 10.842 | 39.80 | 46.83 | 39.80 | -27.08 | 39.83 | 6.37 | -21.48 | 27.93 | 3.22 |
| | 6 | 0 | 7.746 | 1.291 | 15.492 | 41.42 | 41.42 | 41.42 | -29.29 | 41.42 | 6.07 | -29.29 | 41.42 | 6.07 |
| | | 0.5 | 7.745 | 1.292 | 15.490 | 41.40 | 41.49 | 41.40 | -29.26 | 41.40 | 6.07 | -28.30 | 39.48 | 5.59 |
| | | 1 | 7.741 | 1.293 | 15.483 | 41.34 | 41.68 | 41.34 | -29.16 | 41.34 | 6.09 | -27.32 | 37.64 | 5.16 |
| | | 1.5 | 7.735 | 1.296 | 15.471 | 41.23 | 42.02 | 41.23 | -29.00 | 41.23 | 6.11 | -26.36 | 35.89 | 4.77 |
| | | 2 | 7.727 | 1.301 | 15.454 | 41.07 | 42.51 | 41.07 | -28.77 | 41.07 | 6.15 | -25.39 | 34.21 | 4.41 |
| | | 3 | 7.703 | 1.314 | 15.405 | 40.58 | 44.07 | 40.58 | -28.09 | 40.59 | 6.25 | -23.46 | 31.01 | 3.77 |
| | | 4 | 7.666 | 1.335 | 15.333 | 39.80 | 46.83 | 39.80 | -27.08 | 39.83 | 6.37 | -21.48 | 27.93 | 3.22 |
| | 9 | 0 | 9.487 | 1.054 | 18.974 | 41.42 | 41.42 | 41.42 | -29.29 | 41.42 | 6.07 | -29.29 | 41.42 | 6.07 |
| | | 0.5 | 9.486 | 1.054 | 18.972 | 41.41 | 41.45 | 41.41 | -29.28 | 41.41 | 6.07 | -28.63 | 40.11 | 5.74 |
| | | 1 | 9.484 | 1.055 | 18.969 | 41.38 | 41.54 | 41.38 | -29.23 | 41.38 | 6.08 | -27.97 | 38.86 | 5.44 |
| | | 1.5 | 9.481 | 1.056 | 18.962 | 41.34 | 41.68 | 41.34 | -29.16 | 41.34 | 6.09 | -27.32 | 37.64 | 5.16 |
| | | 2 | 9.477 | 1.058 | 18.953 | 41.27 | 41.89 | 41.27 | -29.06 | 41.27 | 6.10 | -26.68 | 36.47 | 4.89 |
| | | 3 | 9.464 | 1.062 | 18.927 | 41.07 | 42.51 | 41.07 | -28.77 | 41.07 | 6.15 | -25.39 | 34.21 | 4.41 |
| | | 4.5 | 9.434 | 1.073 | 18.867 | 40.58 | 44.07 | 40.58 | -28.09 | 40.59 | 6.25 | -23.46 | 31.01 | 3.77 |
| | | 6 | 9.389 | 1.090 | 18.779 | 39.80 | 46.83 | 39.80 | -27.08 | 39.83 | 6.37 | -21.48 | 27.93 | 3.22 |
| 3 | 3 | 0 | 6.708 | 1.491 | 13.416 | 73.21 | 73.21 | 73.21 | -42.27 | 73.21 | 15.47 | -42.27 | 73.21 | 15.47 |
| | | 0.5 | 6.706 | 1.493 | 13.411 | 73.06 | 73.63 | 73.06 | -41.99 | 72.93 | 15.47 | -39.86 | 66.34 | 13.24 |
| | | 1 | 6.697 | 1.498 | 13.395 | 72.63 | 74.97 | 72.63 | -41.16 | 72.06 | 15.45 | -37.52 | 60.28 | 11.38 |
| | | 1.5 | 6.683 | 1.508 | 13.367 | 71.84 | 77.51 | 71.84 | -39.80 | 70.55 | 15.37 | -35.20 | 54.78 | 9.79 |
| | | 2 | 6.663 | 1.523 | 13.326 | 70.59 | 81.97 | 70.59 | -38.07 | 68.33 | 15.13 | -32.85 | 49.69 | 8.42 |
| | 6 | 0 | 9.487 | 1.054 | 18.974 | 73.21 | 73.21 | 73.21 | -42.27 | 73.21 | 15.47 | -42.27 | 73.21 | 15.47 |
| | | 0.5 | 9.486 | 1.054 | 18.972 | 73.17 | 73.31 | 73.17 | -42.20 | 73.14 | 15.47 | -41.05 | 69.66 | 14.30 |
| | | 1 | 9.483 | 1.055 | 18.966 | 73.06 | 73.63 | 73.06 | -41.99 | 72.93 | 15.47 | -39.86 | 66.34 | 13.24 |
| | | 1.5 | 9.478 | 1.057 | 18.956 | 72.89 | 74.17 | 72.89 | -41.64 | 72.57 | 15.46 | -38.68 | 63.23 | 12.27 |
| | | 2 | 9.471 | 1.059 | 18.943 | 72.63 | 74.97 | 72.63 | -41.16 | 72.06 | 15.45 | -37.52 | 60.28 | 11.38 |
| | | 3 | 9.452 | 1.066 | 18.903 | 71.84 | 77.51 | 71.84 | -39.80 | 70.55 | 15.37 | -35.20 | 54.78 | 9.79 |
| | | 4 | 9.423 | 1.077 | 18.846 | 70.59 | 81.97 | 70.59 | -38.07 | 68.33 | 15.13 | -32.85 | 49.69 | 8.42 |
| | 9 | 0 | 11.619 | 0.861 | 23.238 | 73.21 | 73.21 | 73.21 | -42.27 | 73.21 | 15.47 | -42.27 | 73.21 | 15.47 |
| | | 0.5 | 11.618 | 0.861 | 23.237 | 73.19 | 73.25 | 73.19 | -42.23 | 73.17 | 15.47 | -41.45 | 70.81 | 14.68 |
| | | 1 | 11.617 | 0.861 | 23.234 | 73.14 | 73.39 | 73.14 | -42.14 | 73.08 | 15.47 | -40.65 | 68.53 | 13.94 |
| | | 1.5 | 11.614 | 0.862 | 23.229 | 73.06 | 73.63 | 73.06 | -41.99 | 72.93 | 15.47 | -39.86 | 66.34 | 13.24 |
| | | 2 | 11.611 | 0.863 | 23.221 | 72.95 | 73.97 | 72.95 | -41.77 | 72.70 | 15.47 | -39.07 | 64.24 | 12.59 |
| | | 3 | 11.600 | 0.865 | 23.200 | 72.63 | 74.97 | 72.63 | -41.16 | 72.06 | 15.45 | -37.52 | 60.28 | 11.38 |
| | | 4.5 | 11.576 | 0.871 | 23.152 | 71.84 | 77.51 | 71.84 | -39.80 | 70.55 | 15.37 | -35.20 | 54.78 | 9.79 |
| | | 6 | 11.541 | 0.879 | 23.082 | 70.59 | 81.97 | 70.59 | -38.07 | 68.33 | 15.13 | -32.85 | 49.69 | 8.42 |
| 4 | 3 | 0 | 7.746 | 1.291 | 15.492 | 100.00 | 100.00 | 100.00 | -50.00 | 100.00 | 25.00 | -50.00 | 100.00 | 25.00 |
| | | 0.5 | 7.744 | 1.292 | 15.487 | 99.81 | 100.57 | 99.81 | -49.58 | 99.43 | 24.93 | -47.46 | 90.41 | 21.47 |
| | | 1 | 7.736 | 1.296 | 15.472 | 99.22 | 102.39 | 99.22 | -48.36 | 97.71 | 24.67 | -45.01 | 82.09 | 18.54 |
| | | 1.5 | 7.724 | 1.302 | 15.448 | 98.17 | 105.83 | 98.17 | -46.51 | 94.77 | 24.13 | -42.59 | 74.71 | 16.06 |
| | | 2 | 7.706 | 1.312 | 15.412 | 96.49 | 111.90 | 96.49 | -44.53 | 90.87 | 23.17 | -40.18 | 68.03 | 13.92 |
| | 6 | 0 | 10.954 | 0.913 | 21.909 | 100.00 | 100.00 | 100.00 | -50.00 | 100.00 | 25.00 | -50.00 | 100.00 | 25.00 |
| | | 0.5 | 10.954 | 0.913 | 21.907 | 99.95 | 100.14 | 99.95 | -49.89 | 99.86 | 24.98 | -48.72 | 95.02 | 23.15 |
| | | 1 | 10.951 | 0.914 | 21.902 | 99.81 | 100.57 | 99.81 | -49.58 | 99.43 | 24.93 | -47.46 | 90.41 | 21.47 |
| | | 1.5 | 10.947 | 0.915 | 21.893 | 99.57 | 101.31 | 99.57 | -49.06 | 98.72 | 24.83 | -46.23 | 86.12 | 19.94 |
| | | 2 | 10.941 | 0.916 | 21.881 | 99.22 | 102.39 | 99.22 | -48.36 | 97.71 | 24.67 | -45.01 | 82.09 | 18.54 |
| | | 3 | 10.923 | 0.921 | 21.846 | 98.17 | 105.83 | 98.17 | -46.51 | 94.77 | 24.13 | -42.59 | 74.71 | 16.06 |
| | | 4 | 10.898 | 0.928 | 21.795 | 96.49 | 111.90 | 96.49 | -44.53 | 90.87 | 23.17 | -40.18 | 68.03 | 13.92 |
| | 9 | 0 | 13.416 | 0.745 | 26.833 | 100.00 | 100.00 | 100.00 | -50.00 | 100.00 | 25.00 | -50.00 | 100.00 | 25.00 |
| | | 0.5 | 13.416 | 0.745 | 26.832 | 99.98 | 100.06 | 99.98 | -49.95 | 99.94 | 24.99 | -49.14 | 96.63 | 23.75 |
| | | 1 | 13.415 | 0.746 | 26.829 | 99.92 | 100.25 | 99.92 | -49.81 | 99.75 | 24.97 | -48.30 | 93.44 | 22.57 |
| | | 1.5 | 13.412 | 0.746 | 26.824 | 99.81 | 100.57 | 99.81 | -49.58 | 99.43 | 24.93 | -47.46 | 90.41 | 21.47 |
| | | 2 | 13.409 | 0.747 | 26.818 | 99.66 | 101.03 | 99.66 | -49.26 | 98.99 | 24.87 | -46.64 | 87.51 | 20.44 |
| | | 3 | 13.399 | 0.748 | 26.799 | 99.22 | 102.39 | 99.22 | -48.36 | 97.71 | 24.67 | -45.01 | 82.09 | 18.54 |
| | | 4.5 | 13.378 | 0.752 | 26.756 | 98.17 | 105.83 | 98.17 | -46.51 | 94.77 | 24.13 | -42.59 | 74.71 | 16.06 |
| | | 6 | 13.347 | 0.758 | 26.694 | 96.49 | 111.90 | 96.49 | -44.53 | 90.87 | 23.17 | -40.18 | 68.03 | 13.92 |

# Chapter 5

# A THREE-STAGE GAME FOR JOINT REPLENISHMENT WITH PRIVATE CONTRIBUTIONS

## 5.1 Introduction

In this chapter we study a three–stage non–cooperative game of joint replenishment where the intermediary is also a decision maker. We follow the same direction as the other chapters and consider non–cooperative behavior. Our goal is to understand what the impact of a profit-maximizer intermediary on equilibrium behavior is and whether this new approach would lead to equilibrium total cost levels closer to efficient total cost.

We consider $n$ firms with arbitrary inventory holding cost and demand rates, which are publicly known by all parties in the game. Each firm bids how much he is willing to contribute for the replenishment to an intermediary, henceforth referred to as the "replenishment service provider" (RSP) to prevent confusion with the intermediary in the previous chapters which is not a decision maker. The RSP may be a transportation service provider if the setup costs are due to transportation, or a manufacturing company if the setup costs are due to switchovers in manufacturing. The RSP sets the order frequency to maximize her profits and the firms are allowed to opt out consequently. Since this is a multi-stage game, we analyze the characteristics of the subgame–perfect equilibrium outcomes.

In this chapter, we show that the subgame–perfect equilibrium cycle time is not unique. Additional cycle times – including inefficiently low and inefficiently high cycle times – can arise as subgame–perfect equilibrium outcomes. Although the minimum and maximum cycle times that arise in equilibrium straddle the efficient cycle time, in general, whether efficient cycle time can be reached in equilibrium depends on the parameters of the joint replenishment environment. For symmetric joint replenishment environments, whether efficient cycle time is a subgame–perfect equilibrium outcome depends only on the number of firms – it is independent of all other parameters of the environment. Furthermore, this dependence on the number of firms exhibits a highly non–monotone pattern – e.g. efficient joint replenishment is possible with three firms but not with four firms; eleven firms cannot cooperate efficiently but twelve firms can, etc. All the proofs are contained in the Appendix C.

## 5.2 The Model and Preliminaries

We consider a stylized EOQ environment with a set of firms $N = \{1, ..., n\}$. Demand rate for firm $j$ is constant and deterministic at $\beta_j$ per unit of time. Time rate of inventory holding cost for firm $j$ is $\gamma_j$ per unit. Major ordering cost is fixed at $\kappa$ per order regardless of order size. We assume minor ordering costs are zero. Although each firm is characterized by two parameters $(\gamma_j, \beta_j)$, an alternative representation $(\alpha_j, \beta_j)$, obtained by a re–parametrization where $\alpha_j = \gamma_j \beta_j$, will be convenient in all the settings that we consider below. We assume a strictly positive lower bound $\underline{\alpha} > 0$ such that $\alpha_j \geq \underline{\alpha}$ for all $j \in N$ to rule out trivial replenishment environments where either the demand rate or the holding cost rate is zero.

We investigate a three–stage model where first, the firms simultaneously declare their private contributions for each replenishment cycle. Then, the RSP selects a cycle time to maximize her profit, and each firm is allowed to opt out if they are not satisfied with the RSP's cycle time offer. The time line of the game is as follows. In stage 1, firms move simultaneously and each firm $j \in N$ announces his private contribution $r_j$. In stage 2, the RSP decides on the cycle time $T$ that will maximize her profit for given contributions. In stage 3, firms again move simultaneously and each firm chooses an action $\omega \in \{0, 1\}$ where 0 denotes "Out" and 1 denotes "In". We denote the vector of third–stage actions of the $n$ firms by $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_n)$. If firm $j$ plays 1 in the third stage, he accepts the cycle time $T$ and is served by the RSP with cost $r_j + \frac{1}{2}\alpha_j T$ which the sum of his contribution and the corresponding inventory holding cost. If he plays 0, he replenishes independently with cost $C_j^d = \sqrt{2\kappa\alpha_j}$ which is the EOQ or stand–alone cost. We denote the set of firms that choose to be served by the RSP in stage 3 as $\mathcal{M}$.

Second–stage subgames are parameterized by the vector of contributions $\mathbf{r} = (r_1, .., r_n)$ that may be selected by the firms in the first stage. Similarly, the third–stage subgames are parameterized by the actions of the players in the preceding stages, that is, by $(\mathbf{r}, T)$. We denote firm $j$'s third–stage strategy as a function that assigns an action in $\{0, 1\}$ to every third–stage subgame $(\mathbf{r}, T)$: $\omega_j(\mathbf{r}, T) \in \{0, 1\}$. Similarly, a strategy for the RSP specifies a cycle time $T(\mathbf{r})$ for each second–stage subgame $\mathbf{r}$. A subgame–perfect equilibrium for the three–stage game is a profile of strategies that induces a Nash equilibrium in every subgame –including the subgames not reached due to actions taken in previous stages. We start with some observations on the equilibrium strategies in the third– and second–stage subgames.

### 5.2.1 Stage 3: Participation

In a generic third–stage subgame $(\mathbf{r}, T)$, firm $j$'s total cost rate, denoted $\psi_j(\mathbf{r}, T, \boldsymbol{\omega})$, is:

$$\psi_j(\mathbf{r}, T, \boldsymbol{\omega}) = \begin{cases} \sqrt{2\kappa\alpha_j} & \text{if} \quad \omega_j = 0, \\ r_j + \frac{1}{2}\alpha_j T & \text{if} \quad \omega_j = 1. \end{cases} \tag{5.1}$$

Once the RSP's cycle time $T$ is fixed, firm $j$'s cost rate depends only on his first–stage contribution $r_j$ and his third–stage action $\omega_j$. Hence his optimal third–stage action depends only on $r_j$ and the RSP's cycle time choice $T$. Suppressing the obvious dependence on exogenous model variables $\kappa$ and $\alpha$, we define

$$\omega_j^*(r_j, T) = 1 \qquad \Leftrightarrow \qquad r_j + \frac{1}{2}\alpha_j T \leq \sqrt{2\kappa\alpha_j}\ , \tag{5.2}$$

and

$$\tau_j^*(r_j) = \max\{T \mid \omega_j^*(r_j, T) = 1\} \ = 2\sqrt{\frac{2\kappa}{\alpha_j}} - 2\,\frac{r_j}{\alpha_j}. \tag{5.3}$$

By definition, $\tau_j^*(r_j)$ is the threshold cycle time below which the firm $j$ plays 1 given his price, i.e., $\omega_j^*(r_j, T) = 1 \Leftrightarrow T \leq \tau_j^*(r_j))$. $\tau_j^*(r_j)$ is the highest acceptable cycle time offer for firm $j$. Since $\psi_j$ is non–decreasing with $T$, any cycle time offer above $\tau_j$ will be rejected.

Two straightforward properties of firm $j$'s threshold cycle time, $\tau_j^*(r_j)$, are worth noting. First, $\tau_j^*(0) = 2\sqrt{\frac{2\kappa}{\alpha_j}} = 2T_j^d$, i.e., twice his stand–alone cycle time. Thus, firm $j$'s third–stage response will be "out" if $T$ exceeds $2T_j^d$. Second, $\tau_j^*(r_j) < 0$ if $r_j$ exceeds $\sqrt{2\kappa\alpha_j} = 2r_j^d$, i.e., his stand–alone per–unit replenishment price. Hence, firm $j$'s optimal third–stage response will also be "out" if his first–stage contribution $r_j$ exceeds $2r_j^d$.

To summarize, for all third–stage subgames $(\mathbf{r}, T)$, the equilibrium strategies are given by the vector $\boldsymbol{\omega}^*(\mathbf{r}, T) = (\omega_1^*(r_1, T), \omega_2^*(r_2, T), ..., \omega_n^*(r_n, T))$.

### 5.2.2 Stage 2: RSP's cycle time decision

In a second–stage subgame $\mathbf{r}$, the RSP's profit, denoted by $\pi_{RSP}(\mathbf{r}, T, \boldsymbol{\omega})$, anticipating the firms' optimal behavior in stage 3, is

$$\pi_{RSP}(\mathbf{r}, T, \boldsymbol{\omega}^*(\mathbf{r}, T)) = \sum_{k=1}^{n} r_k \omega_k^*(r_k, T) - \frac{\kappa}{T}. \tag{5.4}$$

which is the sum of all the contributions from the firms accepting the cycle time offer minus the average serving cost. Thus, the RSP's optimization problem in the second–stage subgame $\mathbf{r}$ is

$$\max_T \sum_{k=1}^{n} r_k\, \omega_k^*(r_k, T) - \frac{\kappa}{T}. \tag{5.5}$$

In equilibrium, the RSP's optimal cycle time decision in subgame $\mathbf{r}$ is

$$T^*(\mathbf{r}) = \arg\max_T \sum_{k=1}^{n} r_k\, \omega_k^*(r_k, T) - \frac{\kappa}{T}. \tag{5.6}$$

To simplify the explicit characterization of $T^*(\mathbf{r})$ we introduce a fictitious player $n+1$ with $r_{n+1} = 0$, $\alpha_{n+1} = 0$, $\omega_{n+1}^*(r_{n+1}, T) = 0$, and $\tau_{n+1}^*(r_{n+1}) = \infty$. We first note that any finite $T$ that exceeds $\max_{j \in N} \tau_j^*(r_j)$ yields zero revenue for the RSP since all firms stay out in stage 3. Thus such $T$ yields negative profit for the RSP and it is dominated by $\tau_{n+1}^*$ which guarantees zero profit. Second, any $T$ that falls strictly between two consecutive thresholds, say $\tau_i^*(r_i)$ and $\tau_j^*(r_j) > \tau_i^*(r_i)$, is strictly dominated by $\tau_j^*(r_j)$ since $\tau_j^*(r_j)$ yields the same revenue as $T$ but costs less than $T$. Therefore, the RSP selects either one of the firms' threshold cycle times as her cycle time and serves all firms with higher threshold cycle times or selects $\tau_{n+1}^* = \infty$ and does not serve any firm. Thus, the optimal cycle time offer of the RSP in stage 2 given the bids $\mathbf{r}$ (the maximizer of (5.5)) can be written formally as

$$T^*(\mathbf{r}) = \{\tau_\ell^*(r_\ell) |\, \ell = \arg \max_{j \in \{1,\ldots,n+1\}} \sum_{k | \tau_k^*(r_k) \geq \tau_j^*(r_j)} r_k - \frac{\kappa}{\tau_j^*(r_j)}\}. \tag{5.7}$$

If the RSP is indifferent between several threshold cycle times we assume that she selects the lowest among these cycle times.

### 5.2.3 Stage 1: Private Contribution

In stage 1, each firm $j$ decides on the replenishment bid $r_j$ that will minimize his cost. Stage 1 payoff of firm $j$ taking the equilibrium behavior in later stages into account becomes:

$$\psi_j(r_j, \mathbf{r}_{-j}) = \frac{1}{2}\,\alpha_j\,T(r_j, \mathbf{r}_{-j}) + r_j = \begin{cases} \sqrt{2\kappa\alpha_j} & if \quad \tau_j(r_j) \leq T(r_j, \mathbf{r}_{-j}), \\ \frac{1}{2}\,\alpha_j\,T(r_j, \mathbf{r}_{-j}) + r_j & if \quad \tau_j(r_j) > T(r_j, \mathbf{r}_{-j}). \end{cases} \tag{5.8}$$

By taking the second and third stage responses into account, any first–stage contribution $r_j$ that exceeds $2r_j^d$ is dominated by $r_j = 0$ for firm $j$.

Each firm $j$'s payoff depends on other firms' bids $\mathbf{r}_{-j}$ only through $\boldsymbol{\tau}_{-j} = (\tau_1(r_1), .., \tau_{j-1}(r_{j-1}), \tau_{j+1}(r_{j+1}), .., \tau_n(r_n), \tau_{n+1}(r_{n+1}))$. Given that the RSP's optimal behavior is to select one of $\tau_i(r_i)$, $i \neq j$ or $\tau_j(r_j)$, firm $j$'s best response problem reduces to selecting a price to induce the RSP to choose a cycle time in $(\tau_j(r_j), \boldsymbol{\tau}_{-j})$ that is best from firm $j$'s point of view. If $r_j$ induces the RSP to select $\tau_j(r_j)$, by definition, firm $j$'s payoff is his stand–alone payoff.

### 5.2.4 Subgame–Perfect Equilibrium

We collect the observations above in the following proposition that characterizes the subgame–perfect equilibria of the three–stage game.

**Proposition 5.1.** *A strategy profile $(\mathbf{r}^*, T^*(\mathbf{r}), \boldsymbol{\omega}^*(\mathbf{r}, T))$ is a subgame–perfect equilibrium if and only if the following conditions are satisfied*

*i.* $\omega_j^*(\mathbf{r}, T) = 1 \;\Leftrightarrow r_j + \frac{1}{2}\alpha_j T \leq \sqrt{2\kappa\alpha_j}\,, \qquad \forall j \in N,$

*ii.* $T^*(\mathbf{r}) = \{\tau_\ell^*(r_\ell) |\, \ell = \arg\max_j \sum_{k|\tau_k^*(r_k) \geq \tau_j^*(r_j)} r_k - \kappa/\tau_j^*(r_j)\},$

*iii. (a)* $\forall\, i, j \in N$ *such that* $r_i^* > 0$ *and* $\tau_i^*(r_i^*) \leq 2T_j^d$

$r_j^* + \sum_{k\neq j, \tau_k^*(r_k^*)\geq \tau_i^*(r_i^*)} r_k^*$
$$\leq \tfrac{1}{2}\alpha_j(\tau_i^*(r_i^*) - T^*(\mathbf{r}^*)) + \kappa/\tau_i^*(r_i^*) \quad \textit{if } \tau_i^*(r_i^*) \geq \tau_j^*(r_j^*) \geq T^*(\mathbf{r}^*), \quad \textit{or } \tau_j^*(r_j^*) \geq T^*(\mathbf{r}^*) \geq \tau_i^*(r_i^*),$$

$\sum_{k\neq j, \tau_k^*(r_k^*)\geq \tau_i^*(r_i^*)} r_k^*$
$$\leq \tfrac{1}{2}\alpha_j\tau_i^*(r_i^*) - \sqrt{2\kappa\alpha_j} + \kappa/\tau_i^*(r_i^*) \quad \textit{if } \tau_j^*(r_j^*) \leq T^*(\mathbf{r}^*),$$

*(b)* $\sum_{j\in N} r_j^*\, \omega_j^*(\mathbf{r}^*, T(\mathbf{r}^*)) = \kappa/T^*(\mathbf{r}^*)$.

Condition iii(b), shows that in equilibrium the RSP makes zero profit (The RSP serves for a fixed fee). This is straightforward since for any $\boldsymbol{r}$ vector if the RSP makes a positive profit, then at least one of the firms may reduce his contribution and still get the same cycle time.

For any firm, inducing his own $\tau$ results in the same payoff with the stand–alone payoff thus he will be indifferent between choosing "in" or "out" in the third stage. However, a firm is forced to induce his own $\tau$ if there is no better alternative i.e., all the other $\tau$ levels result in worse payoffs. A firm may induce the $\tau$ of another firm by adjusting his contribution level $r$ however this depends on the system parameters and the actions of the other players. Thus, in equilibrium none of the firms should want to change his current $\tau$ level and induce a $\tau$ other than the equilibrium. Condition iii(a) guarantees that none of the firms has any incentive to do so.

A wide range of equilibria is possible under Proposition 5.1. Each equilibrium involves a "coalition" of firms that accept to be served by the RSP by playing "In" in stage 3 of the game. Next, we characterize the minimum and maximum cycle times that can be obtained for a given coalition.

**Proposition 5.2.** *The minimum and maximum SPE cycle times for a given coalition $S$ are given by*

$$T_S^{\min} = \sqrt{2\kappa}\frac{\sum_{j\in S}\sqrt{\alpha_j} - \sqrt{(\sum_{j\in S}\sqrt{\alpha_j})^2 - \sum_{j\in S}\alpha_j}}{\sum_{j\in S}\alpha_j}, \tag{5.9}$$

$$T_S^{\max} = \min\left\{\min_{j\in S}\left\{2\sqrt{\frac{2\kappa}{\alpha_j}}\right\}, \sqrt{2K}\frac{\sum_{j\in S}\sqrt{\alpha_j} + \sqrt{(\sum_{j\in S}\sqrt{\alpha_j})^2 - \sum_{j\in S}\alpha_j}}{\sum_{j\in S}\alpha_j}\right\}. \tag{5.10}$$

The minimum equilibrium cycle time in Proposition 5.2 is supported by first stage contributions $r_j^* = 0$ for $j \notin S$ and $r_j^* = \sqrt{2\kappa\alpha_j} - \frac{1}{2}\alpha_j T_S^{\min}$ for $j \in S$. In particular, for $S = \{i\}$ we see that $T_{\{i\}}^{\min} = T_i^d$ is an SPE outcome. In this equilibrium, firm $i$ finances the order cost and other firms either ride free or replenish independently. Similarly, the maximum cycle time is supported as an SPE outcome by the first stage bids $r_j^* = \sqrt{2K\alpha_j} - \frac{1}{2}\alpha_j T_S^{\max}$ for $j \in S$ and $r_j^* = 0$ for $j \notin S$.

We can now characterize the minimum and maximum SPE cycle times that can be obtained in Game 2. We define $\mathcal{P}_k$ as the set of firms in $N$ with the $k$ smallest $\alpha_j$ where $1 \leq k \leq n = |N|$ (i.e., $\mathcal{P}_n = N$).

**Proposition 5.3.** *The minimum and maximum SPE cycle times that can be obtained in Game 2 are given by*

$$T^{\min} = \sqrt{2\kappa}\frac{\sum_{j\in N}\sqrt{\alpha_j} - \sqrt{(\sum_{j\in N}\sqrt{\alpha_j})^2 - \sum_{j\in N}\alpha_j}}{\sum_{j\in N}\alpha_j}, \tag{5.11}$$

$$T^{\max} = \max_{1\leq k\leq n}\left\{\sqrt{2\kappa}\frac{\sum_{j\in\mathcal{P}_k}\sqrt{\alpha_j} + \sqrt{(\sum_{j\in\mathcal{P}_k}\sqrt{\alpha_j})^2 - \sum_{j\in\mathcal{P}_k}\alpha_j}}{\sum_{j\in\mathcal{P}_k}\alpha_j}\right\}. \tag{5.12}$$

Using $\Gamma(N) = \sum_{i,j\in N,\, i\neq j}\alpha_i\alpha_j$, $T^{\min}, T_N^{\max}$ and $T_N^c$ can be written as

$$T^{\min} = \frac{\sqrt{2\kappa}}{\sum_{j\in N}\alpha_j + \sqrt{2\Gamma(N)}},$$

$$T_N^{\max} = \min\left\{2\sqrt{\frac{2\kappa}{\alpha_n}}, \frac{\sqrt{2K}}{\sum_{j\in N}\alpha_j - \sqrt{2\Gamma(N)}}\right\},$$

$$T_N^c = \frac{\sqrt{2\kappa}}{\sqrt{\sum_{j\in N}\alpha_j^2}}.$$

Since $\alpha_j > 0$ for all $j \in N$, we have $(\sum_{j\in N}\alpha_j)^2 > \sum_{j\in N}\alpha_j^2$. Thus, $\sum_{j\in N}\alpha_j - \sqrt{2\Gamma(N)} < \sqrt{\sum_{j\in N}\alpha_j^2} < \sum_{j\in N}\alpha_j + \sqrt{2\Gamma(N)}$. We also have $\alpha_n/2 < \sqrt{\sum_{j\in N}\alpha_j^2}$. Therefore, $T_N^C$ is in the interval $[T^{\min}, T_N^{\max}]$. Since $T^{\max} \geq T_N^{\max}$, we establish that $T_N^C$ is in the interval $[T^{\min}, T^{\max}]$.

The observation above may suggest a conjecture that any cycle time in the interval $[T^{\min}, T^{\max}]$, in particular the efficient cycle time $T_N^c$, can arise as an SPE outcome. However, in general, this is not the case; the SPE cycle times do not form a connected interval and the efficient cycle time may or may not be an SPE outcome.

We demonstrate this using symmetric joint replenishment environments where $\alpha_i = \alpha$ and $\beta_i = \beta$ for all $i \in N$. For this setting the efficient, minimum and maximum SPE cycle times are $\frac{1}{\sqrt{n}}\sqrt{\frac{2\kappa}{\alpha}}$, $\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}}\sqrt{\frac{2\kappa}{\alpha}}$, and $\frac{\sqrt{n}+\sqrt{n-1}}{\sqrt{n}}\sqrt{\frac{2\kappa}{\alpha}}$, respectively. Asymptotically, both efficient and minimum SPE cycle times go to zero and the maximum SPE cycle time approaches twice the stand–alone cycle time.

We seek necessary and sufficient conditions for the efficient cycle time $T_N^c$ to arise as an SPE outcome. In Proposition 5.4, we show that, for symmetric joint replenishment environments, whether efficient cycle time is a subgame–perfect equilibrium outcome depends only on the number of firms – it is independent of all other parameters of the environment.

**Proposition 5.4.** *For all symmetric joint replenishment environments with $n$ firms, efficient cycle time $T_N^c$ is an SPE outcome if and only if*

$$(b(n)-\lfloor b(n)\rfloor)(1-\frac{1}{2\sqrt{n}})(1+\frac{1}{n-\lfloor b(n)\rfloor})-\frac{1}{4(1-\frac{b(n)-\lfloor b(n)\rfloor}{n-\lfloor b(n)\rfloor}(1-\frac{1}{2\sqrt{n}}))} \leq 0, \quad (5.13)$$

*where*

$$b(n) = \frac{n}{2\sqrt{n}-1}. \quad (5.14)$$

Interestingly, this dependence on the number of firms exhibits a highly non–monotone pattern – e.g. efficient joint replenishment is possible with three firms but not with four firms; eleven firms cannot cooperate efficiently but twelve firms can, etc. The set of industry sizes for which efficient joint replenishment arises as an SPE outcome for $n$ less than 100 is as follows: {2, 3, 12, 13, 14, 15, 30, 31, 32, 33, 34, 35, 56, 57, 58, 59, 60, 61, 62, 63, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99}. Although the efficient cycle time and minimum SPE cycle time asymptotically converge, the efficient cycle time is not guaranteed to be an SPE. For any $N$ such that the $T_N^c$ is (respectively, is not) an SPE outcome, there exists $\widehat{N}$ with $|\widehat{N}| > |N|$ such that $T_{\widehat{N}}^c$ is not (respectively, is) an SPE outcome, i.e., SPE property of the efficient cycle time oscillates indefinitely.

## 5.3 Concluding Remarks

In this chapter, we considered a three–stage non–cooperative joint replenishment game where the intermediary is also a decision maker. In the first stage the firms announce their contribution levels. In the second stage, the intermediary announces the cycle time he is willing to provide given the contributions and in the final stage the firms announce whether they will joint the coalition or act independently. We see that this game leads to many equilibrium cycle times and a list of conditions that the contributions should satisfy for an equilibrium. The minimum cycle time that as the result of the equilibrium is smaller than the efficient cycle time and the maximum cycle time larger than any stand–alone cycle time of the firms. At the minimum and maximum, all the firms served by the intermediary have cost levels equal to their stand–alone costs. Moreover, we show that for the identical firms case, whether the efficient cycle time is an outcome of the game depends only on the number of firms.

In the next chapter, we consider direct and parametric mechanisms for non–cooperative joint replenishment.

# Chapter 6

# DESIGN AND ANALYSIS OF MECHANISMS FOR DECENTRALIZED JOINT REPLENISHMENT

## 6.1 Introduction

In the previous chapters we have considered direct contribution schemes for financing the setup or transportation costs. The firms announce only a per order monetary contribution to an intermediary and intermediary decides on the cycle time. In Chapters 3 and 4 intermediary is not a profit maximizer. In Chapter 5 intermediary is also a player in the game and tries to maximize his profit. We observed that the first approach never leads to an efficient joint cycle time and the second approach may lead to an efficient cycle time depending on the number of firms.

In this chapter, instead of relying on direct contribution methods, we consider direct and parametric mechanisms that will allocate the setup costs associated with the joint replenishment problem and investigate their performance for different parameters.

We generalize the non–cooperative reporting game studied by Meca et al. [34] (MGB in the sequel) which is embedded in the cooperative joint replenishment game where stand-alone order frequencies of the firms are observable but not verifiable. Each firm reports an order frequency (that may be different from its true order frequency) and the joint order frequency is determined to minimize the total joint costs based on all the reports. Each firm incurs holding cost individually and pays a share of the joint replenishment cost in proportion to the squares of reported order frequencies as in Meca et al. [35]. MGB show that, while this rule leads to core allocations under cooperative formulations, it entails significant misreporting and inefficient joint decisions in a non-cooperative framework. The authors show that the game has multiple equilibria. In one equilibrium none of the firms participate in joint replenishment. If the firms are sufficiently homogeneous, there also exists a (unique) "constructive equilibrium" (an equilibrium in which all firms participate in joint replenishment).

In this chapter, we study the mechanism design problem for the joint replenishment of decentralized firms which have private information about their adjusted demand rates. We first use a direct mechanism where each firm reports an adjusted demand rate and joint replenishment cycle time and allocation of the joint order costs between the firms are decided based on these reports. We show that a direct mechanism which satisfies the efficiency, incentive compatibility and individual rationality constraints cannot satisfy the budget–balance constraint, i.e., a truth telling direct mechanism cannot finance the joint replenishment for efficient cycle times. Next, we study other mechanisms and generalize the non–cooperative reporting game studied by MGB where stand-alone order frequencies of the firms are observable but not verifiable. While the mechanism in MGB determines the joint order frequency and the order cost allocation both based on the *squares* of the reported stand–alone order frequencies, we use a general formulation in which two separate parameters govern these decisions. For this two–parameter sharing mechanism, we show that the joint frequency is always lower than the efficient frequency unless the order cost is allocated uniformly. We then study the one-parameter mechanism, where the parameters are same. This is a generalization of the game considered in MGB which uses a parameter value 2. We find the conditions necessary and for a constructive equilibrium and characterize this equilibrium. We also provide necessary conditions for convexity at the equilibrium point. We analyze the comparative statics of the one–parameter model and show that using smaller values of this single parameter leads to better mechanisms in terms of fairness and efficiency. All proofs as well as detailed derivations are contained in the Appendix D.

## 6.2 The Model and Preliminaries

We consider a stylized EOQ environment with a set of firms $N = \{1, ..., n\}$. Demand rate for firm $i$ is constant and deterministic at $\beta_i$ per unit of time. Inventory holding cost per unit time for firm $i$ is $\gamma_i$ per unit. We denote the adjusted demand rate of firm $i$ as $\alpha_i = \gamma_i \beta_i$. We assume that adjusted demand rates are strictly positive, $\alpha_i > 0$ for all $i \in N$ to rule out trivial replenishment environments where either the demand rate or the holding cost rate is zero. Major ordering cost is fixed at $\kappa$ per order regardless of order size. Minor ordering costs (ordering costs associated with firms included in an order) are assumed to be zero. We assume that the outside supplier that replenishes the orders has infinite capacity. The firms aim to minimize their long–run *average* costs over time and backorders are not allowed.

In any setting, the objective is to minimize the total cost rate, denoted by $C$, i.e., the sum of replenishment cost rate ($R$) and holding cost rate ($H$): $C = R + H$. The decision variable can be taken as order cycle time, $t$, or order frequency, $f = 1/t$ (number of orders per time unit). We take frequency as the decision variable in the sequel.

Vectors are denoted by lower–case letters in bold typeface. For a generic $m-$tuple vector $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $i \in \{1, \ldots, m\}$, the notation $(y, \boldsymbol{x}_{-i})$ stands for the vector $\boldsymbol{x}$ with its $i$th entry $x_i$ replaced by $y$, and the $(m-1)$-tuple $\boldsymbol{x}_{-i}$

stands for the vector $\boldsymbol{x}$ with its $i$th entry $x_i$ removed which in our case includes all the firms but firm $i$.

For an endogenous variable $X$, by $X^a_M$ we refer to the value of $X$ when the set of firms is $M$ and replenishment operations are governed by $a \in \{c, d, dm, 2p, 1p\}$, where $c$ stands for centralized, $d$ stands for decentralized (or independent) replenishment, $dm$ stands direct mechanism for joint replenishment, $2p$ stands for two–parameter mechanism and $1p$ stands for the single–parameter mechanism. For instance, $T^c_M$ is the joint cycle time of the firms in $M$ when replenishment is centralized. When the set $M$ is a singleton, e.g., $M = \{i\}$, we use $X^a_i$ instead of $X^a_{\{j\}}$. Exceptions to this notation are used for $f_i$, the optimal frequency of the decentralized replenishment for firm $i$ and for $f_*$, the optimal frequency of centralized replenishment.

### 6.2.1 Independent (decentralized) replenishment

When the replenishment of the items is controlled by firms operating independently, firm $i$'s total cost rate ($C_i$) is the sum of replenishment cost rate ($R_i$) and the holding cost rate ($H_i$):

$$C_i(f) = R_i(f) + H_i(f) = \kappa f + \frac{\alpha_i}{2f}. \tag{6.1}$$

Using the first order condition and convexity, it can be found that firm $i$'s optimal frequency is $f_i = \sqrt{\alpha_i/2\kappa}$. With this frequency, optimal replenishment cost rate and optimal inventory holding cost rate are equal at $R^d_i = H^d_i = \kappa f_i$. The aggregate total cost rate for all firms under independent replenishment is therefore $C^d_N = \sum_{i\in N} 2\kappa f_i$.

### 6.2.2 Centralized joint replenishment

When all firms cooperate, they order with a joint order frequency to achieve the efficiency. [35] show that when there are no minor setup costs, it is optimal for all firms to be replenished in each cycle and this leads to a common order frequency. Denoting the joint order frequency by $f$, the total cost under cooperation is given by

$$C_N(f) = R_N(f) + H_N(f) = \kappa f + \frac{\sum_{i \in N} \alpha_i}{2f} = \kappa f + \kappa \frac{\sum_{i \in N} f_i^2}{f}.$$

Using the first order condition, we obtain the efficient frequency as $f_* = (f_1^2 + ... + f_n^2)^{1/2}$. The efficient total cost is then $C_N^c = 2\kappa f_*$.

We use the proportional rule of [35] which simply allocates the order costs based on the proportion of adjusted demand rate of firm $i$ to the sum of adjusted demand rates. This rule is in the core of the cooperative game. With this proportional rule, the cost share of firm $i$ is $\alpha_i / (\alpha_1 + ... + \alpha_n)$. Since, $f_i^2 = \alpha_i/(2\kappa)$, we can rewrite the cost share as $f_i^2 / (f_1^2 + ... + f_n^2)$. Thus the cost of firm $i$ under cooperation is given by

$$C_i^c = 2\kappa \frac{f_i^2}{\sqrt{f_1^2 + ... + f_n^2}}.$$

## 6.3 Direct Mechanisms

We consider the design of a mechanism for the joint replenishment problem. A mechanism is a specification of how economic decisions should be taken for a set of players who are privately informed about their preferences based on the messages they provide to an intermediary. Mechanism design problem usually consists of three steps. In step 1, the mechanism is designed. In step 2, the players accept or reject the mechanism. If a firm rejects the mechanism, it gets an exogenously specified reservation utility. In step 3, the players play the game specified by the mechanism and economic outcomes and payoffs for each player are determined. A mechanism is *efficient* if it maximizes the sum of player's payoffs. A truth–telling strategy is to report true information about preferences, for all possible preferences. A mechanism is *incentive compatible* if for any player, truth–telling is a dominant–strategy. A mechanism is *individually rational* if for any player the mechanism leads to a payoff that is at least as much as his reservation utility. A *direct mechanism* is a mechanism where each player sends a message regarding his preference.

We consider designing a mechanism to allocate the jointly incurred setup costs. We assume that each firm's adjusted demand rate, $\alpha_i$ for firm $i$, is observable, but not verifiable. Each firm's reservation utility is equal to its independent optimal ordering cost, $C_i^d = 2\kappa f_i$ for firm $i$. We consider a direct mechanism, therefore firms report their adjusted demand rates simultaneously and the joint cycle time and the allocation of joint setup costs is accomplished using these reports. An efficient mechanism for this problem should generate total costs to be equal to the total costs for the centralized problem, i.e., $2\kappa f_*$ where $f_* = (f_1^2 + ... + f_n^2)^{1/2}$ is the optimal frequency for the centralized problem. A necessary condition for a mechanism in this setting is *budget–balance*. This condition requires that the sum of allocations through the mechanism should finance the joint setup or ordering cost. The main question that we investigate in this section is whether there is a direct mechanism for the joint replenishment problem that is efficient, incentive compatible, individually rational and budget–balanced.

Let $\hat{\alpha}_i$ be firm $i$'s report of its adjusted demand rate (which can be different from the true adjusted demand rate $\alpha_i$) and let $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n)$ be the vector of reported adjusted demand rates. We denote $\sigma_i(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i})$ to be the setup cost allocated to firm $i$ if its own reported adjusted demand rate is $\hat{\alpha}_i$ and its competitors' reported adjusted demand rates are given by the vector $\hat{\boldsymbol{\alpha}}_{-i}$. Since firms' adjusted demand rates are not verifiable, the allocation function should be identical for all firms, i.e., $\sigma_i = \sigma$ for all $i = 1, 2, \ldots, n$. In this setting, the allocation function $\sigma$ alone defines the direct mechanism that we use for the joint replenishment problem. Since we are pursuing a mechanism to achieve efficiency, we use $\sqrt{\frac{2\kappa}{\sum_{i \in N} \hat{\alpha}_i}}$ for the cycle time. Note that if all the firms report their adjusted demand rates truthfully i.e. $\hat{\alpha}_i = \alpha_i$ for all $i \in N$ this formulation would give the efficient cycle time.

The cost of firm $i$ as a function of its own report $\hat{\alpha}_i$ and competing firms' reports $\hat{\boldsymbol{\alpha}}_{-i}$ can be written as

$$
\begin{aligned}
C_i^{dm}(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i}) &= \frac{1}{2}\alpha_i\sqrt{\frac{2\kappa}{\hat{\alpha}_i + \sum_{j\neq i}\hat{\alpha}_j}} + \sigma(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i})\sqrt{\frac{(\hat{\alpha}_i + \sum_{j\neq i}\hat{\alpha}_j)}{2\kappa}} \\
&= \sqrt{\frac{\kappa}{2}}\alpha_i(\hat{\alpha}_i + \sum_{j\neq i}\hat{\alpha}_j)^{-1/2} + \sqrt{\frac{1}{2\kappa}}\sigma(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i})(\hat{\alpha}_i + \sum_{j\neq i}\hat{\alpha}_j)^{1/2}.
\end{aligned}
$$

The first equation on the right hand side is the average inventory holding cost of firm $i$ which is found by multiplying $\alpha_i/2$ by the joint cycle time calculated using the reported adjusted demand rates. The second equation is the average replenishment cost share of firm $i$ which is determined by multiplying the share function $\sigma_i(\hat{\alpha}_i, \hat{\boldsymbol{\alpha}}_{-i})$ by the average order cost.

The next proposition states that does not exist a function $\sigma$, thus no direct mechanism, that simultaneously satisfies the efficiency, incentive compatibility, individual rationality and budget balance constraints (All proofs are provided in Appendix).

**Proposition 6.1.** *There is no direct mechanism for the joint replenishment problem that simultaneously satisfies efficiency, incentive compatibility, individual rationality and budget balance constraints.*

Given this impossibility result for direct mechanisms, we explore alternative mechanisms and investigate their efficiency in the next two sections.

## 6.4 Two–Parameter Mechanisms

In the previous section we showed that there is no truth–telling direct mechanism that can achieve efficiency, individual rationality and budget–balance simultaneously. In this section we consider a class of indirect mechanisms and investigate their ability to reach an efficient outcome. We again assume that adjusted demand rates, thus independent frequencies are observable by all firms, but not verifiable. We assume that each firm reports a frequency denoted by $\hat{s}_i$ for firm

$i$ and a mechanism determines the joint order frequency and the allocation of the setup cost based on these reports. We consider a two–parameter mechanism where one parameter ($\xi \geq 0$) governs the joint order frequency decision and another parameters ($\theta \geq 0$) governs the allocation decision. In particular, the joint frequency under the two parameter mechanism is $\left(\hat{s}_1^{\xi} + ... + \hat{s}_n^{\xi}\right)^{1/\xi}$, and replenishment setup cost share of firm $i$ is $\hat{s}_i^{\theta} / \left(\hat{s}_1^{\theta} + ... + \hat{s}_n^{\theta}\right)$. Since we allocate all of the setup cost using the parameter $\xi$, the budget–balance condition is trivially satisfied for this mechanism.

Using these values we can easily find the total cost rate $C_i^{2p}$ for firm $i$ as

$$C_i^{2p}(\hat{\mathbf{s}}) = \frac{1}{2}\alpha_i \left(\sum_{j\in N} \hat{s}_j^{\xi}\right)^{-\frac{1}{\xi}} + \frac{\kappa \hat{s}_i^{\theta} \left(\sum_{j\in N} \hat{s}_j^{\xi}\right)^{\frac{1}{\xi}}}{\sum_{j\in N} \hat{s}_j^{\theta}}. \tag{6.2}$$

The first term on the right hand side of (6.2) is the average inventory holding cost and is found by multiplying adjusted demand rate $\alpha_i$ (the demand rate multiplied by the holding cost rate) by the joint order frequency. The second term is the time averaged order cost that is allocated to firm $i$. Note that the cost of firm $i$ depends on its reported frequency as well as its rivals'. Therefore, we have a non–cooperative game where each firm's strategy is its reported frequency and we can use Nash equilibrium as a solution concept.

In order to find the best response function of firm $i$ to the strategies of other firms, we obtain the first order condition. Denoting the equilibrium strategy vector as $\boldsymbol{s} = \{s_1, .., s_n\}$, the first order condition at the equilibrium is given by:

$$\begin{aligned}\left.\frac{\partial C_i^{2p}(\hat{\mathbf{s}})}{\partial \hat{s}_i}\right|_{\hat{\mathbf{s}}=\mathbf{s}} &= -\frac{1}{2}\alpha_i s_i^{\xi-1}\left(\sum_{j\in N} s_j^{\xi}\right)^{-\frac{1}{\xi}-1} - \kappa\theta s_i^{2\theta-1}\left(\sum_{j\in N} s_j^{\xi}\right)^{\frac{1}{\xi}}\left(\sum_{j\in N} s_j^{\theta}\right)^{-2} \\ &+ \kappa\theta s_i^{\theta-1}\left(\sum_{j\in N} s_j^{\xi}\right)^{\frac{1}{\xi}}\left(\sum_{j\in N} s_j^{\theta}\right)^{-1} + \kappa s_i^{\theta+\xi-1}\left(\sum_{j\in N} s_j^{\xi}\right)^{\frac{1}{\xi}-1}\left(\sum_{j\in N} s_j^{\theta}\right)^{-1} = 0.\end{aligned}$$

We can simplify this equation by multiplying by $\kappa^{-1} s_i^{1-\theta}\left(\sum_{j\in N} s_j^{\theta}\right)^2 \left(\sum_{j\in N} s_j^{\xi}\right)^{1-\frac{1}{\xi}}$ and substituting for $f_i^2 = \alpha_i / 2\kappa$ which yields

$$f_i^2 s_i^{\xi-\theta} \left(\sum_{j\in N} s_j^{\theta}\right)^2 \left(\sum_{j\in N} s_j^{\xi}\right)^{-\frac{2}{\xi}} = \theta\left(\sum_{j\in N} s_j^{\theta}\right)\left(\sum_{j\in N} s_j^{\xi}\right) + s_i^{\xi}\left(\sum_{j\in N} s_j^{\theta}\right) - \theta s_i^{\theta}\left(\sum_{j\in N} s_j^{\xi}\right).$$

By rearranging the terms, we obtain

$$f_i^2 = \theta s_i^{\theta-\xi}\left(\sum_{j\in N} s_j^\theta\right)^{-1}\left(\sum_{j\in N} s_j^\xi\right)^{\frac{2+\xi}{\xi}} + s_i^\theta\left(\sum_{j\in N} s_j^\theta\right)^{-1}\left(\sum_{j\in N} s_j^\xi\right)^{\frac{2}{\xi}} - \theta s_i^{2\theta-\xi}\left(\sum_{j\in N} s_j^\xi\right)^{\frac{2+\xi}{\xi}}\left(\sum_{j\in N} s_j^\theta\right)^{-2} \tag{6.3}$$

This implicit function gives the equilibrium reported frequencies $s_i$, but no further simplification is possible and a closed form solution for the equilibrium is not available. However, we can determine the performance (with respect to its ability to reach the efficient solution) of the two–parameter mechanism by the following proposition.

**Proposition 6.2.** *The ratio of the efficient frequency and the equilibrium frequency under the two–parameter mechanism is given by:*

$$\left(\frac{(\sum_{i\in N} f_i^2)^{1/2}}{\left(\sum_{i\in N} s_i^\xi\right)^{1/\xi}}\right)^2 = 1 + \left(\sum_{i\in N} s_i^\theta\right)^{-2}\theta\left(2\sum_{i\neq j} s_i^\theta s_j^\theta + \sum_{i\neq j} s_i^{\theta+\xi} s_j^{\theta-\xi} + \sum_{i\neq j, j\neq k} s_i^\theta s_j^\xi s_k^{\theta-\xi}\right). \tag{6.4}$$

Proposition 6.2 shows that unless $\theta = 0$, the efficient joint frequency is always larger than the joint frequency in the constructive equilibrium (if it exists) which in turn implies that cooperative solutions would give smaller costs for all firms. This is formally given in the following corollary.

**Corollary 6.1.** *For the two–parameter allocation mechanism, the joint frequency is always less than the efficient frequency unless the order cost allocation parameter $\theta = 0$, i.e., the order cost is allocated uniformly.*

However, an equilibrium under a uniform cost allocation is not guaranteed as we will show next.

## A special case: $(\xi, \theta) = (2, 0)$

We consider a two–parameter mechanism with joint frequency parameter as ($\xi = 2$) and sharing parameter as ($\theta = 0$) which corresponds to a uniform sharing (replenishment cost share of firm $i = 1/n$).

In this case, the payoff for firm $i$ is:

$$C_i^{2p}(\hat{\mathbf{s}}) = \frac{1}{n}\kappa\left(\sum_{j\in N}\hat{s}_j^2\right)^{\frac{1}{2}} + \frac{\alpha_i}{2}\left(\sum_{j\in N}\hat{s}_j^2\right)^{-\frac{1}{2}} = \frac{\kappa}{n}\left(\left(\sum_{j\in N}\hat{s}_j^2\right)^{\frac{1}{2}} + nf_i^2\left(\sum_{j\in N}\hat{s}_j^2\right)^{-\frac{1}{2}}\right).$$

First order condition for optimal response is:

$$\begin{aligned}
\left.\frac{\partial C_i^{1p}(\hat{\mathbf{s}})}{\partial \hat{s}_i}\right|_{\hat{\mathbf{s}}=\mathbf{s}} &= \frac{\kappa s_i}{n}\left(\left(\sum_{j\in N}s_j^2\right)^{-\frac{1}{2}} - nf_i^2\left(\sum_{j\in N}s_j^2\right)^{-\frac{3}{2}}\right) \\
&= \frac{\kappa s_i}{n}\left(\sum_{j\in N}s_j^2\right)^{-\frac{3}{2}}\left(s_i^2 + \sum_{j\neq i}s_j^2 - nf_i^2\right) = 0.
\end{aligned}$$

We obtain the best responses as $s_i^2 = nf_i^2 - \sum_{j\neq i}s_j^2$ and derive the equilibrium frequency as

$$\begin{aligned}
& \sum_{j\in N}s_j^2 = n\sum_{j\in N}f_j^2 - (n-1)\sum_{j\in N}s_j^2 \\
\Rightarrow\quad & \sum_{j\in N}s_j^2 = \sum_{j\in N}f_j^2 \\
\Rightarrow\quad & \sqrt{\sum_{j\in N}s_j^2} = \sqrt{\sum_{j\in N}f_j^2} \\
\Rightarrow\quad & f_\xi = f_*,
\end{aligned}$$

which is equal to the cooperative joint frequency. However, the major drawback here is, in order to have an equilibrium, all firms should have the same stand–alone frequency $f$ since using the best response function of firm $i$ we should have $-\sum_{j\in N}s_j^2 = nf_i^2$ which is true for all $i \in N$. Otherwise, there is no constructive equilibrium and each firm replenishes independently.

Since further analysis of the two–parameter mechanisms is not tractable, in the next section, we explore one parameter mechanisms in detail.

## 6.5 One–Parameter Mechanisms

In this section, we consider a single parameter mechanism where we set the value of the parameters for determining the joint order frequency and allocating the ordering costs equal to each other. When we assume that $\theta = \xi$, the resulting cost function for a given vector of reports $\hat{\mathbf{s}}$ is

$$C_i^{1p}(\hat{\mathbf{s}}) = \kappa f_i \left( \sum_{j \in N} \hat{s}_j^{\xi} \right)^{-\frac{1}{\xi}} + \kappa \hat{s}_i^{\xi} \left( \sum_{j \in N} \hat{s}_j^{\xi} \right)^{\frac{1}{\xi}-1}.$$

In this case, equation (6.3) simplifies to

$$f_i^2 = \xi \left( \sum_{j \in N} s_j^{\xi} \right)^{\frac{2}{\xi}} + s_i^{\xi}(1-\xi) \left( \sum_{j \in N} s_j^{\xi} \right)^{\frac{2}{\xi}-1}, \tag{6.5}$$

and (6.4) can be written as

$$\begin{aligned} \left( \frac{(\sum_{i \in N} f_i^2)^{1/2}}{\left(\sum_{i \in N} s_i^{\xi}\right)^{1/\xi}} \right)^2 &= 1 + \left( \sum_{i \in N} s_i^{\xi} \right)^{-2} \xi \left( 2 \sum_{i \neq j} s_i^{\xi} s_j^{\xi} + (n-1) \sum_{i \in N} s_i^{2\xi} + 2(n-2) \sum_{i \neq j} s_i^{\xi} s_j^{\xi} \right) \\ &= 1 + \left( \sum_{i \in N} s_i^{\xi} \right)^{-2} \xi (n-1) \left( \sum_{i \in N} s_i^{2\xi} + 2 \sum_{i,j \in N, i \neq j} s_i^{\xi} s_j^{\xi} \right) \\ &= 1 + \xi(n-1). \end{aligned} \tag{6.6}$$

Denoting the joint frequency in equilibrium $f_{\xi} = \left(\sum_{i \in N} s_i^{\xi}\right)^{1/\xi}$, we obtain

$$\frac{f_*}{f_{\xi}} = \sqrt{\xi(n-1)+1}, \tag{6.7}$$

which shows that the deviation of the equilibrium joint frequency from the efficient joint frequency depends only on the parameter $\xi$ and $n$. In particular, $f_* > f_{\xi}$ for all $\xi > 0$ and $f_*/f_{\xi}$ is an increasing function of $\xi$. This means that the one parameter mechanisms are never perfectly efficient in general, but their efficiency improves as $\xi$ gets smaller.

### Best Response Functions

In order to find the equilibrium allocation in the model, we first obtain the best response function for firm $i$. The expression in (6.6) can be written as:

$$\left(\frac{\sum_{j\in N} f_j^2}{\xi(n-1)+1}\right)^{\frac{\xi}{2}} = \sum_{j\in N} s_j^{\xi}. \tag{6.8}$$

Therefore, the best response of firm $i$ is given by

$$s_i^{\xi} = \left(\frac{\sum_{j\in N} f_j^2}{\xi(n-1)+1}\right)^{\frac{\xi}{2}} - \sum_{j\in N\setminus\{i\}} s_j^{\xi}, \text{ for } i = 1,\dots,n. \tag{6.9}$$

### Constructive Equilibrium

Clearly, there can be equilibria in which a firm reports 0 and stays out of the joint replenishment. However, since our focus is efficiency, we are mainly interested in constructive equilibria where each firm reports a positive frequency.

We can use the best response functions (6.8) in (6.5) and re–arrange the terms to get the following equality for the equilibrium reports:

$$s_i^{\xi} = \frac{\xi\sum_{j\in N} f_j^2 - ((n-1)\xi+1)\, f_i^2}{((n-1)\xi+1)\,(\xi-1)}\left(\frac{\sum_{j\in N} f_j^2}{(n-1)\xi+1}\right)^{\xi/2-1}. \tag{6.10}$$

If $\xi \geq 1$, the argument in (6.10) is positive if and only if $\xi\sum_{j\in N} f_j^2 - ((n-1)\xi+1)f_i^2 \geq 0$. On the other hand, if $\xi < 1$, the argument in (6.10) is positive if and only if $\xi\sum_{j\in N} f_j^2 - ((n-1)\xi+1)f_i^2 < 0$. We formalize these conditions in the following proposition without proof.

**Proposition 6.3.** *The necessary and sufficient condition for a constructive equilibrium for the one–parameter mechanism is given by*

$$\frac{f_i^2}{\sum_{j\in N} f_j^2} \leq \frac{\xi}{(n-1)\xi+1} \quad \textit{for all } i = 1,\dots,n,$$

*if* $\xi \geq 1$, *and*

$$\frac{f_i^2}{\sum_{j\in N} f_j^2} > \frac{\xi}{(n-1)\xi+1} \quad \textit{for all } i = 1,\ldots,n,$$

*if* $\xi < 1$.

Proposition 6.3 shows that the constructive equilibrium exists if firms' stand–alone optimal frequencies are close to each other. In fact, one can simplify the conditions in Proposition 6.3 such that the maximum (minimum) frequency among $n$ frequencies should have a bounded from above (below) for $\xi > 1$ ($\xi < 1$). Thus, instead of $n$ conditions for each case, we can guarantee constructive equilibrium with only one condition using the following corollary.

**Corollary 6.2.** *The necessary and sufficient condition for a constructive equilibrium for the one–parameter mechanism is given by*

$$\frac{\max_{j\in N} f_j^2}{\sum_{j\in N} f_j^2} \leq \frac{\xi}{(n-1)\xi+1},$$

*if* $\xi \geq 1$, *and*

$$\frac{\min_{j\in N} f_j^2}{\sum_{j\in N} f_j^2} > \frac{\xi}{(n-1)\xi+1},$$

*if* $\xi < 1$.

## Convexity of Payoff Function

In order to show that the the solution in (6.10) is in fact the equilibrium, we need to show that the payoff function is convex at this point. We provide the conditions for this in the following proposition.

**Proposition 6.4.** *The cost function is convex at (6.10) and the solution in (6.10) is a Nash equilibrium if and only if*

$$\xi \sum_{j\in N} f_j^2 - (\xi-2)\left((n-1)\xi+1\right) f_i^2 \geq 0, \quad \textit{for all } i = 1,\ldots,n. \tag{6.11}$$

An consequence of this result is that for $\xi > 3$, we do not have convexity at the equilibrium point regardless of the frequency distribution and for $\xi \leq 2$ we always have convexity.

## Equilibrium Payoffs

For the single parameter joint replenishment mechanism, the cost of firm $i$ in equilibrium can be found by using the equilibrium reports $\mathbf{s} = \{s_1, .., s_n\}$.

$$C_i^{1p}(\mathbf{s}) = \kappa f_i^2 \left(\sum_{j\in N} s_j^\xi\right)^{-\frac{1}{\xi}} + \kappa s_i^\xi \left(\sum_{j\in N} s_j^\xi\right)^{\frac{1}{\xi}-1}.$$

In equilibrium, using (6.8) and (6.10):

$$C_i^{1p}(\mathbf{s}) = \kappa f_i^2 \left(\frac{\sum_{j\in N} f_j^2}{(n-1)\xi+1}\right)^{-\frac{1}{2}} + \kappa \frac{\xi \sum_{j\in N} f_j^2 - ((n-1)\xi+1)\, f_i^2}{((n-1)\xi+1)\,(\xi-1)} \left(\frac{\sum_{j\in N} f_j^2}{(n-1)\xi+1}\right)^{-\frac{1}{2}}.$$

Taking the terms to $\left(\frac{\sum_{j\in N} f_j^2}{(n-1)\xi+1}\right)^{-\frac{1}{2}}$ parenthesis and rearranging the terms gives the equilibrium cost of firm $i$ as:

$$C_i^{1p}(\mathbf{s}) = \kappa \left(\frac{\xi \sum_{j\in N} f_j^2 + (\xi-2)\,((n-1)\xi+1)\, f_i^2}{((n-1)\xi+1)\,(\xi-1)}\right) \left(\frac{\sum_{j\in N} f_j^2}{(n-1)\xi+1}\right)^{-\frac{1}{2}}.$$

Summing over all the firms, we obtain the total cost as

$$C_N^{1p}(\mathbf{s}) = \sum_{j\in N} C_j^{1p}(\mathbf{s}) = \kappa \left(\frac{\xi n + (\xi-2)\,((n-1)\xi+1)}{\xi-1}\right) \left(\frac{\sum_{j\in N} f_j^2}{\xi(n-1)+1}\right)^{\frac{1}{2}},$$

and the cost ratio of firm $i$ is given by

$$\frac{C_i^{1p}(\mathbf{s})}{C_N^{1p}(\mathbf{s})} = \left(\frac{\xi \sum_{j\in N} f_j^2 + (\xi-2)\,((n-1)\xi+1)\, f_i^2}{\xi n + (\xi-2)\,((n-1)\xi+1)}\right) \left(\sum_{j\in N} f_j^2\right)^{-1}.$$

## A special case: $\boldsymbol{\xi = 2}$

A special case of our one–parameter mechanisms is the mechanism used in [34] where the parameter is $\xi = 2$. In this case, the necessary and sufficient condition for a constructive equilibrium given in Proposition 6.3 simplifies to

$$f_i^2 \leq \frac{2}{2n-3} \sum_{j\neq i} f_j^2, \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

as is also shown in Theorem 2 of [34]. The equilibrium joint frequency simplifies to:

$$f_\xi = \frac{1}{\sqrt{2n-1}} f_* < f_*.$$

The cost of firm $i$ in this case is:

$$C_i^{1p} = \kappa \left( \frac{2\sum_{j\in N} f_j^2}{(2(n-1)+1)} \right) \left( \frac{\sum_{j\in N} f_j^2}{2(n-1)+1} \right)^{-\frac{1}{2}} = 2\kappa \left( \frac{\sum_{j\in N} f_j^2}{2(n-1)+1} \right)^{\frac{1}{2}},$$

which shows that each firm has the same cost under joint replenishment regardless of their stand–alone frequencies or adjusted demand rates.

## Impact of $\xi$ and Comparative Statics

We now investigate how the equilibrium behavior and efficiency change as a function of $\xi$ and stand–alone frequencies. For this purpose we obtain the comparative statics for the game.

First remember that Equation (6.7) states $\frac{f_*}{f_\xi} = \sqrt{\xi(n-1)+1}$, and therefore we know that the efficiency of the one parameter mechanism improves as $\xi$ gets smaller. One can also derive an expression for the difference between reported frequencies of two firms $i$, $k$ with $f_i > f_k$ as follows:

$$s_i^\xi - s_k^\xi = \frac{f_k^2 - f_i^2}{(\xi-1)} \left( \frac{\sum_{j\in N} f_j^2}{(n-1)\xi+1} \right)^{\xi/2-1}, \tag{6.12}$$

which shows that for $\xi > 1$, we have $s_i < s_j$. Therefore, the firm with higher stand–alone frequency reports a lower frequency than a firm with lower stand–alone frequency. For $\xi < 1$, the firm with higher stand–alone frequency reports a higher frequency. A similar expression can be derived for equilibrium cost of two firms as follows:

$$C_i^{1p} - C_k^{1p} = \kappa \left( \frac{(\xi-2)(f_i^2 - f_k^2)}{(\xi-1)} \right) \left( \frac{\sum_{j\in N} f_j^2}{(n-1)\xi+1} \right)^{-\frac{1}{2}}. \tag{6.13}$$

Equation (6.13) can be used to show that for $1 < \xi < 2$, $C_i^{1p} < C_k^{1p}$, i.e., the firm with higher stand–alone frequency has a lower equilibrium cost. For $1 < \xi$ or $\xi > 2$, the reverse is true and we have $C_i^{1p} > C_k^{1p}$.

We demonstrate these results in a test problem with three firms with $(f_1, f_2, f_3)$=$(0.95, 1, 1.05)$ in Figures 6.1 and 6.2 as $\xi$ varies between 0 and 3. Note that the efficient joint frequency for this problem is $f^* = 1.733$. Figure 6.1 shows the equilibrium frequency reports and resulting joint frequency as a function of $\xi$. Notice that we have a region of $\xi$ for which there is no constructive equilibrium.

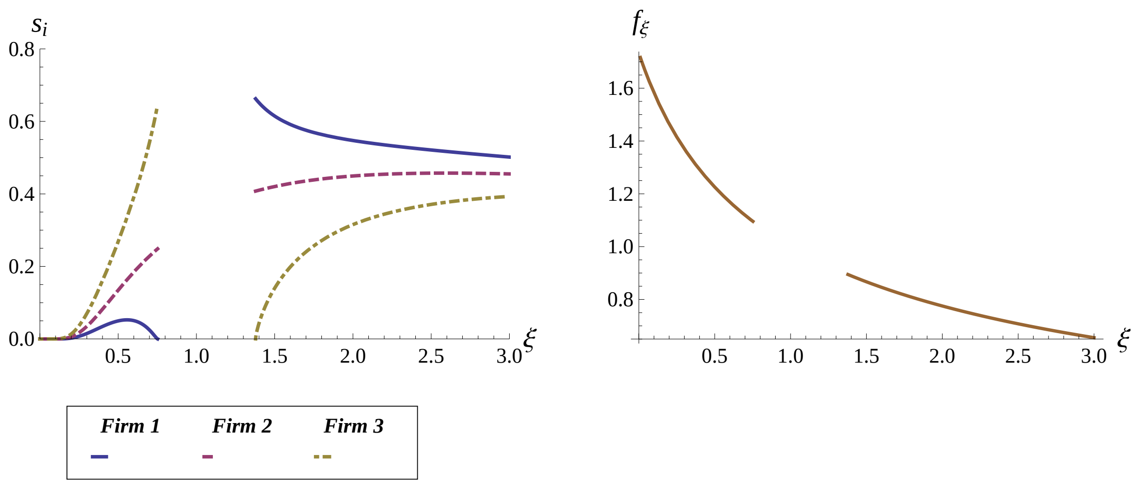


Figure 6.1: Reported Frequencies and Equilibrium Joint frequency as a function of $\xi$ for $(f_1, f_2, f_3)$=$(0.95, 1, 1.05)$

Corresponding costs (as a percentage of total efficient costs) for each firm and total costs are shown in Figure 6.2. Since the equilibrium joint frequency approaches the efficient joint frequency as $\xi$ gets smaller, total costs also approaches to the efficient total costs in this direction. Also notice that in the first region of $\xi$ which contains constructive equilibrium ($\xi < 1$), the equilibrium cost of a higher stand–alone frequency (or higher adjusted demand rate) firm is always larger than the equilibrium cost of a firm with a lower stand–alone frequency. This simple sense of "fairness" is not guaranteed in the second region ($\xi > 1$).

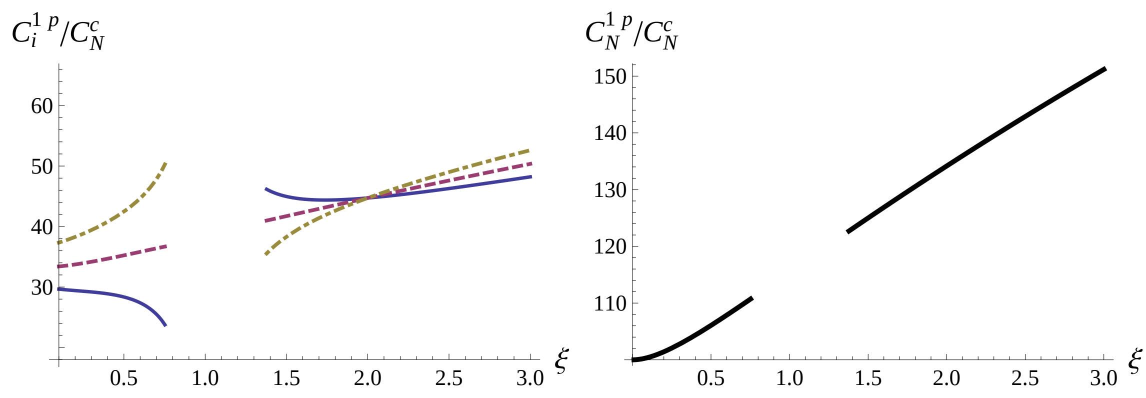


Figure 6.2: Equilibrium individual costs and total cost as a percentage of efficient cost as a function of $\xi$ for $(f_1, f_2, f_3)$=$(0.95, 1, 1.05)$

Based on the equations (6.7), (6.12), and (6.13), and Figures 6.1 and 6.2, we can conclude that, if one can guarantee a constructive equilibrium, using smaller values of $\xi$ than 2 (as used in [34]) is more desirable from an efficiency and fairness perspective.

Figures 6.3, 6.4 and 6.5 show equilibrium reported frequencies, individual firm costs and total costs, respectively, for two other test problems: $(f_1, f_2, f_3) = (0.9, 1, 1.1)$ and $(f_1, f_2, f_3) = (1, 1.05, 1.1)$. The results are similar to the results for the first problem, except that the region for which no constructive equilibrium can be obtained expands (shrinks) as stand–alone frequencies get closer to (further away from) each other.

It is also important to understand how a firm's equilibrium frequency report changes as its own true stand–alone frequency or its competitor's stand–alone frequency changes. We can derive the partial derivative of the equilibrium reported frequency of firm $i$, $s_i$ with respect to its own stand–alone frequency $f_i$ as follows:

$$\frac{\partial s_i}{\partial f_i} = \frac{f_i s_i}{\xi}\left(\frac{\left(\xi^2 - 2\left((n-1)\xi+1\right)\right)\sum_{j\in N} f_j^2 - (\xi-2)\left((n-1)\xi+1\right) f_i^2}{\xi\sum_{j\in N} f_j^2 - \left((n-1)\xi+1\right) f_i^2}\right)\left(\sum_{j\in N} f_j^2\right)^{-1} \tag{6.14}$$

Similarly, the partial derivative with respect to a rival firm $j$'s true frequency is

$$\frac{\partial s_i}{\partial f_k} = \frac{f_k s_i}{\xi}\left(\frac{\xi^2\sum_{j\in N} f_j^2 - (\xi-2)\left((n-1)\xi+1\right) f_i^2}{\xi\sum_{j\in N} f_j^2 - \left((n-1)\xi+1\right) f_i^2}\right)\left(\sum_{j\in N} f_j^2\right)^{-1}. \tag{6.15}$$

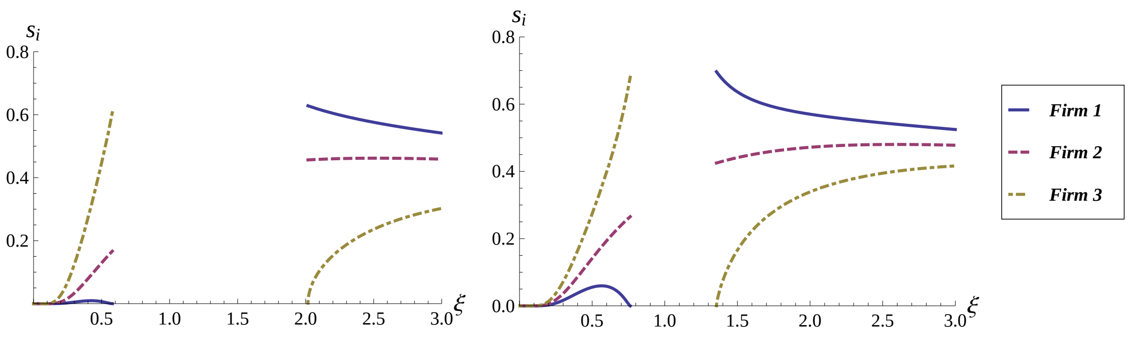


Figure 6.3: Reported frequencies as a function of $\xi$ for $(f_1, f_2, f_3) = (0.9, 1, 1.1)$ and $(f_1, f_2, f_3) = (1, 1.05, 1.1)$

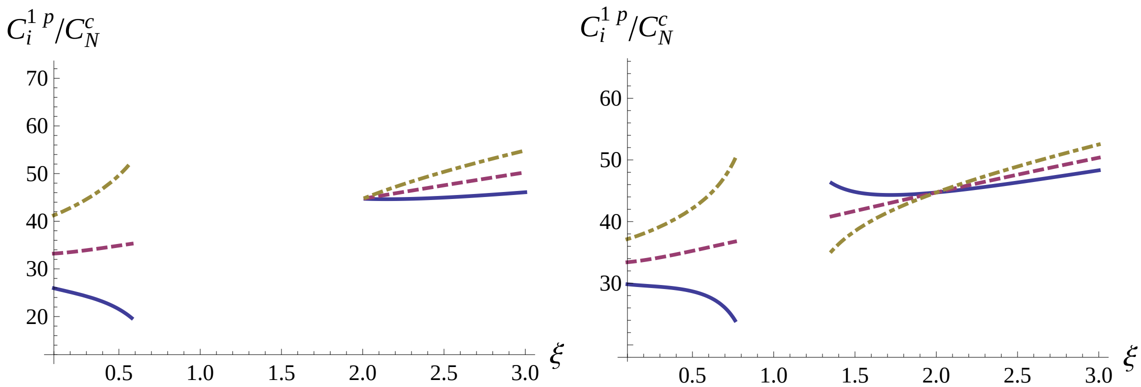


Figure 6.4: Equilibrium firms costs as a percentage of efficient cost as a function of $\xi$ for $(f_1, f_2, f_3) = (0.9, 1, 1.1)$ and $(f_1, f_2, f_3) = (1, 1.05, 1.1)$

Corresponding changes in equilibrium costs are given by the following

$$\frac{\partial C_i^{1p}}{\partial f_i} = \kappa f_i \left( \frac{(\xi + 2(\xi - 2)\left((n-1)\xi + 1\right)) \sum_{j \in N} f_j^2 - (\xi - 2)\left((n-1)\xi + 1\right) f_i^2}{\left((n-1)\xi + 1\right)^{1/2} (\xi - 1)} \right) \left( \sum_{j \in N} f_j^2 \right)^{-\frac{3}{2}} \tag{6.16}$$

$$\frac{\partial C_i^{1p}}{\partial f_j} = \kappa f_j \left( \frac{\xi \sum_{j \in N} f_j^2 - (\xi - 2)\left((n-1)\xi + 1\right) f_i^2}{\left((n-1)\xi + 1\right)^{1/2} (\xi - 1)} \right) \left( \sum_{j \in N} f_j^2 \right)^{-\frac{3}{2}}. \tag{6.17}$$

In Figure 6.6, we compute the comparative statics given in (6.14) and (6.15) for the test problem with $(f_1, f_2, f_3)$=$(0.95, 1, 1.05)$. Figure 6.6 shows that when $\xi < 1$, the firm should report higher frequencies as its true frequency increases. This is in contrast to the second region of constructive equilibrium, where the firm report lower frequency as its true frequency increases. For the same problem,

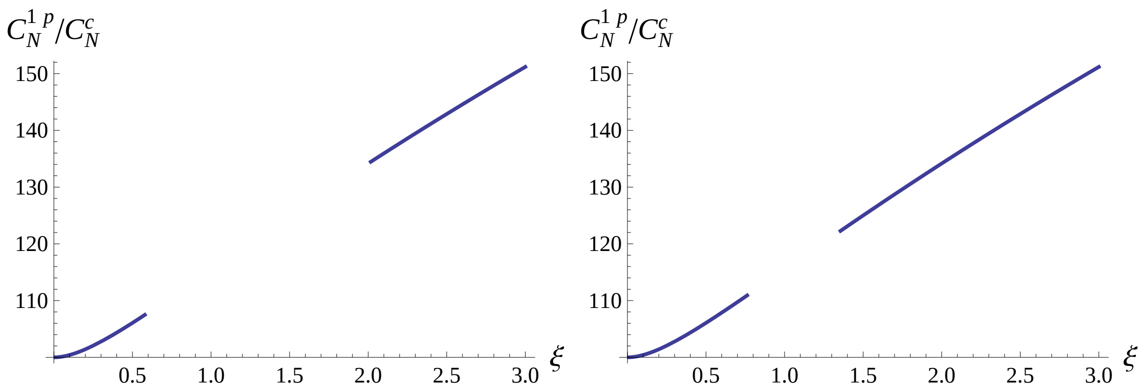


Figure 6.5: Equilibrium total cost as a percentage of efficient cost as a function of $\xi$ for for $(f_1, f_2, f_3) = (0.9, 1, 1.1)$ and $(f_1, f_2, f_3) = (1, 1.05, 1.1)$

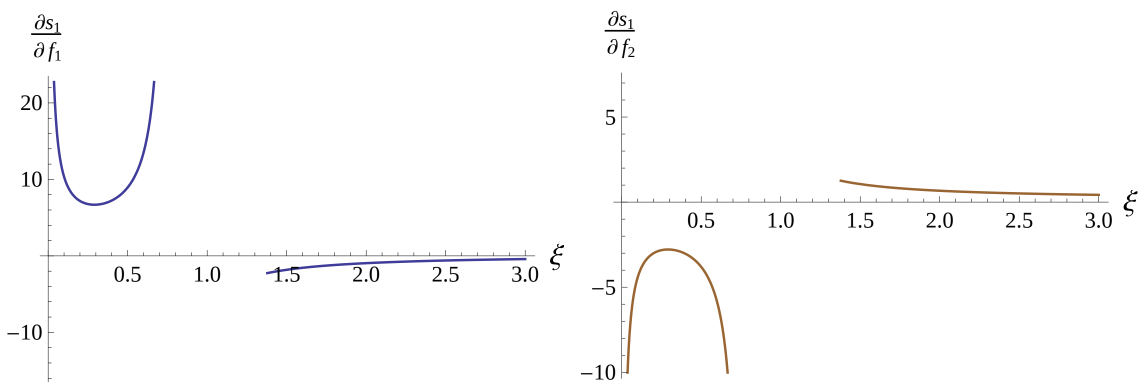


Figure 6.6: Rate of change of firm 1's equilibrium reports with $f_1$ and $f_2$ as a function of $\xi$ for $(f_1, f_2, f_3)$=$(0.95, 1, 1.05)$

the comparative statics given in (6.16) and (6.17) are shown in Figure 6.7. Figure 6.7 shows that equilibrium cost for a firm is increasing in its own frequency and decreasing in its rival's frequency when $\xi < 1$ and and the signs are reversed when $\xi > 1$. The results in Figures 6.6 and 6.7 confirm that using $\xi < 1$ leads to a more desirable mechanism in terms of fairness.

One can also consider the effect of an additional firm, firm $n+1$, entering the joint replenishment, to the reported frequency of firm $i$. For brevity, we only consider the difference of the $\xi^{th}$ power of the reported frequencies.

$$
\begin{aligned}
s_i^{\xi}(N \cup \{n+1\}) - s_i^{\xi}(N) \quad = \quad & \frac{\xi(\sum_{j\in N} f_j^2 + f_{n+1}^2) - (n\xi+1)\, f_i^2}{(n\xi+1)\,(\xi-1)} \left( \frac{\sum_{j\in N} f_j^2 + f_{n+1}^2}{n\xi+1} \right)^{\xi/2-1} \\
& - \frac{\xi \sum_{j\in N} f_j^2 - ((n-1)\xi+1)\, f_i^2}{((n-1)\xi+1)\,(\xi-1)} \left( \frac{\sum_{j\in N} f_j^2}{(n-1)\xi+1} \right)^{\xi/2-1} . \quad (6.18)
\end{aligned}
$$

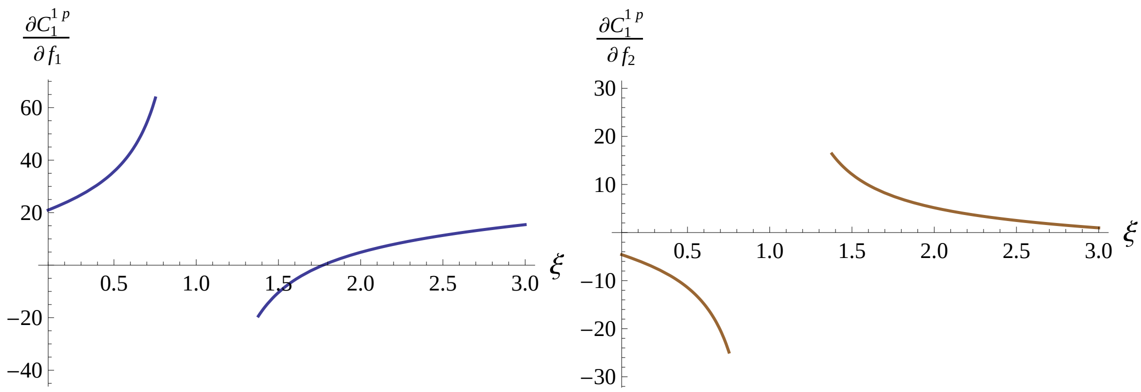


Figure 6.7: Rate of change of firm 1's cost with $f_1$ and $f_2$ as a function of $\xi$ for $(f_1, f_2, f_3)$=$(0.95, 1, 1.05)$

Correspondingly, the change in equilibrium costs can be shown as follows

$$\begin{aligned} C_i^{1p}(N \cup \{n+1\}) - C_i^{1p}(N) \quad &= \quad \kappa \left( \frac{\xi(\sum_{j\in N} f_j^2 + f_{n+1}^2) + (\xi-2)\,(n\xi+1)\, f_i^2}{(n\xi+1)^{1/2}\,(\xi-1)} \right) \left( \sum_{j\in N} f_j^2 + f_{n+1}^2 \right)^{-\frac{1}{2}} \\ &\quad -\kappa \left( \frac{\xi \sum_{j\in N} f_j^2 + (\xi-2)\,((n-1)\xi+1)\, f_i^2}{((n-1)\xi+1)^{1/2}\,(\xi-1)} \right) \left( \sum_{j\in N} f_j^2 \right)^{-\frac{1}{2}}. \qquad (6.19) \end{aligned}$$

## 6.6 Concluding Remarks

In this chapter, we consider jointly replenishing multiple, decentralized firms under an EOQ like environment. We assume that the adjusted demand rates are observable, but not verifiable and therefore investigate the use of direct and indirect mechanisms to determine a joint replenishment frequency and allocate setup costs. First, we show that there is no direct mechanism that is efficient, incentive compatible, individually rational, and budget–balanced. Hence, we explore indirect mechanisms where each firm reports its stand–alone replenishment frequency and propose general, two–parameter mechanisms in which one parameter governs the joint frequency decision and the other governs the setup cost allocation. We show that it is not possible to achieve efficiency unless the setup costs are allocated uniformly. When these two parameters are equal, we derive conditions for the constructive equilibrium and characterize the equilibrium and comparative statics. We show that mechanisms with smaller values of this single parameter leads to more efficient outcomes and are more defendable in terms of fairness.

# Chapter 7

# NEWSBOY DUOPOLY WITH ASYMMETRIC INFORMATION

## 7.1 Introduction

The newsboy problem has played a central role at the conceptual foundations of stochastic inventory theory, and variants of it have been used in analysis of decision problems – such as capacity, allocation and overbooking – under demand uncertainty. In the classical newsboy problem, a firm facing uncertain demand orders a quantity of a perishable item prior to observing demand. If the demand realization is less than the ordered quantity, then the firm will have excess inventory in hand that will perish. If demand turns out to be more than the ordered quantity, then the firm will miss the opportunity of additional profit. In the well–known characterization, the optimal order quantity, which balances the marginal expected cost of ordering one more unit against the marginal expected revenue from satisfying an additional demand, is a critical quantile of the demand distribution.

In the standard newsboy model, strategic interactions are assumed away by taking the demand faced by a firm as a model primitive. In many practical situations, however, the details of the market interaction does matter for the order quantity decisions. Some or all of a firm's unsatisfied demand can be served by other firms offering substitutes; and, vice versa, a firm may be able to sell more than its initial market share in case the rival firm is understocked. Under such conditions, a firm's payoff depends on rival firms', as well as its own, order quantities and appropriate analysis of optimal inventory decisions requires a game theoretic approach. The resulting model, dubbed the competitive newsboy model, has been studied in the literature starting with the seminal works of Parlar [44], who study the case where the firms' initial demands are statistically independent, and Lippman and McCradle [30], who study the cases where the demands faced by competing firms are derived from a general class of rationing rules applied to the total industry demand.

A natural extension of the competitive newsboy analysis involves incorporating information asymmetry. Asymmetric information adds a new dimension to the competitive newsboy problem. Firms may be asymmetrically informed in a competitive newsboy setting due to two broad reasons. The firms may be privately informed about their cost and/or revenue structures. Alternatively, there may be asymmetric information regarding the market demand. Alternative specifications for the key structural elements – e.g., the nature of information asymmetry, the structure of the market and firm demands – span a number of interesting classes of models. Among these are models of newsboy oligopoly, and models that allow arbitrary statistical dependence in firm demands, and in cost structures.

In this chapter, we study the competitive newsboy problem with asymmetric cost information. The competitive newsboy model we study is built on Parlar [44] and Lippman and McCardle [30]. The industry demand is random. There are two firms among whom the industry demand is split. Each firm has private information about their costs. If the demand that is allocated to one firm exceeds the order quantity of that firm, a portion of the excess demand spills over to the rival firm. As standard in analysis of games of incomplete information, we use the Bayesian–Nash equilibrium as the solution concept. In a Bayesian–Nash equilibrium each player's strategy is a best response against the strategies of the competing players.

The rest of this chapter is organized as follows. In Section 7.2, we review the related literature. In Section 7.3, we introduce a model of inventory competition under asymmetric information. Section 7.4 presents our main results on the characterization of equilibrium and comparative statics analysis. We present the full characterization of equilibrium in a parametric version of the model under uniform demand distribution and a linear split rule in Section 7.5. All proofs as well as detailed derivations are contained in the Appendix E.

## 7.2 Literature Review

The literature on multiple item inventory problem with substitution dates back to the paper by Mcgillivray and Silver [33]. However, the role of competition has not been studied until the pioneering work of Parlar [44]. Parlar studies a competitive newsboy problem with two firms managing two substitutable items facing independent demands. A deterministic fraction of unsatisfied demand for each item can be substituted to the other item, if that item has excess stock. It is shown that a unique Nash equilibrium exists. It is also shown that total profits of two competing firms are less than that would have been obtained if they were to cooperate. [51] and Karjalainen [25] generalize the results of Parlar for the 3 and $n$ firms cases, respectively.

Lippman and McCardle [30] consider the competitive newsboy problem under a general setting with respect to how initial demands are generated and how excess demand is reallocated. It is assumed that each firm's initial demand is a result of an allocation of the industry demand which is a random variable. In deterministic rules, a specific deterministic function of the industry demand is allocated to each firm in competition. In stochastic rules, a firm's initial allocation depends on the outcome of a random variable (independent demands as in [44] can be shown to be a special case of stochastic splitting). If a firm's initial demand exceeds its order quantity, a non–decreasing function of the excess demand is reallocated to each other firm. Lippman and McCardle [30] show the existence of an equilibrium in the general setting. For the case of symmetric firms and continuous distributions of effective demand for each firm, they also show the uniqueness of the equilibrium. For the case of two firms, they show that competition leads to higher inventory in the system.

Netessine and Rudi [41] characterize the equilibrium for the case of $n$ firms when the initial demands follow a multi–variate continuous distribution and excess demands spill over fractionally to other firms. The uniqueness of the equilibrium is shown with further conditions and a comparison of centralized and competitive order quantities is provided.

Mahajan and Van Ryzin [32] study a model where the firms' demands are generated by a dynamic process – heterogeneous consumers arrive sequentially and choose a vendor based on a utility maximization criterion and availability at the time of their arrival. They characterize the equilibrium and show its uniqueness for the case of symmetric firms. They also show that competition leads to overstocking.

Serin [46] considers the possibility of a Stackelberg game in the competitive newsboy problem. She considers both Nash equilibrium solutions and Stackelberg equilibrium solution and gives conditions under which these two lead to the same inventory levels.

Anupindi and Bassok [3] study the impact of competition and centralization among two retailers on the performance of a supplier in the upper echelon. Under the optimal wholesale pricing mechanism, they show that there is a threshold for the level of substitution, above which the supplier may prefer a decentralized system.

There are other papers in operations literature where competition carries on for multiple periods and backordering is possible. In Hall and Porteus [19] and Liu et al. [31], two firms compete on product availability which impacts the market share in future periods. However, within each period that is modeled as a newsboy problem, no substitution occurs. Netessine et al. [42] model substitution to a competing firm in the current period as well as backordering in future periods.

We restricted our literature review on the horizontal inventory competition where the competition is between the parties in the same echelon. There is a growing body of operations literature where inventory competition takes place between different echelons in the supply chain (vertical inventory competition). These models are usually solved using a principal-agent model and menu of contracts that the leader party offers to the follower. Examples include Cachon [7], Cachon and Zipkin [9], Corbett [11], Zhang et al. [55] and Kostamis and Duenyas [27].

Jiang et al. [24] consider a horizontal inventory competition setting under asymmetric demand information. They use an absolute regret minimization objective from the robust optimization literature. They show the existence of the equilibrium and give a close form solution. Yan and Zhao [53] also consider the asymmetric demand information in a decentralized inventory-sharing system consisting of a manufacturer and two independent retailers.

Our model focuses on horizontal inventory competition model under asymmetric cost information. In a recent paper, Wu and Parlar [52] study the games of asymmetric information with inventory management applications. They review static and dynamic games under asymmetric information. They extend the Parlar [44] model for each different setting they use. They only give the equilibrium conditions however do not focus on the existence and uniqueness of the equilibrium. They also do not pursue a detailed investigation of the equilibrium.

Our model in spirit is similar to Parlar [44] and Lippman and McCardle [30]. We extend the model in Lippman and McCardle [30] for the case of non–identical firms and asymmetric cost information. We show the existence of an equilibrium and show its uniqueness under fairly general assumptions.

The asymmetric information newsboy duopoly game we study can be transformed to a supermodular game. Supermodular games were first introduced by

Topkis [48] who show that there exists at least one pure strategy Nash equilibrium in a full information supermodular game. Milgrom and Roberts [37] show that a large class of games in economics literature are supermodular and thus have equilibrium. Supermodularity is also used recently to study games in operations literature. Examples include [30], [6] and [7]. Vives [50] uses supermodularity to show the existence of pure strategy Nash equilibrium for compact action spaces and complete separable metric type spaces. This work is recently extended by Athey [4] to include a larger class of type and strategy spaces which satisfy the single crossing condition. Van Zandt and Vives [49] shows the existence of Bayesian–Nash equilibrium for supermodular asymmetric information games when type sets are discrete and action sets are continuous. Our model of asymmetric information newsboy duopoly is an instance of the general class of incomplete information games studied in [49].

## 7.3 A Model of Newsboy Duopoly

We consider an industry served by two firms $i = 1, 2$ that offer two substitutable items. Throughout, we assume that the two firms are risk–neutral.

### 7.3.1 Industry and Firm Demands

The total industry demand $D$ is a continuous positive random variable with an everywhere positive density function $g()$. Thus, the distribution function $G()$, and the survival function $\overline{G}()$, where $\overline{G}(x) = 1 - G(x) = Pr(D \geq x)$, are strictly monotonic.

As in Lippman and McCardle [30], demand faced by each firm is determined in a two-step rationing process. First, for any realization, $d$, of random market demand, initial market shares of the two firms are determined by a deterministic function $s$ such that firm 1's initial market share is $s(d)$ and that of firm 2 is

$\hat{s}(d) = d - s(d)$. The share function $s$ satisfies $0 \leq s(d) \leq d$ for all $d$. To guarantee that both market shares are non–decreasing in market demand realization, we assume $0 \leq s'(d) \leq 1$.

A given initial market share function $s$ induces random demands faced by firm 1, $D_1 = s(D)$, and firm 2, $D_2 = \hat{s}(D) = D - s(D)$. By construction, the initial demands faced by the two firms, $(D_1, D_2)$, are comonotonic since both are deterministic monotone functions of the industry demand.

In the second step, given realized market demand and the order quantities of the two firms, if firm $j$ is stocked out, then some portion, $a_i$, of firm $j$'s underage goes to firm $i$. Thus, the effective demand $R_i$ for firm $i$ is the sum of initial allocation and the reallocation:

$$R_i(Q_j) = D_i + a_i(D_j - Q_j)^+.$$

where $(x)^+$ denotes $\max\{x, 0\}$ and $a_i \in [0, 1]$ for $i = 1, 2$ is the demand substitution rate from firm $j$ to $i$ and is assumed to be deterministic. For notational simplicity, we suppress the dependence of the effective demand on other arguments. The effective demand of firm $i$, $R_i$, is a continuous random variable and its distribution is induced by the distributions of initial demands.

As a first attempt to incorporate private information into the competitive newsboy problem, we take the two items produced by the two firms as perfect substitutes: $a_1 = a_2 = 1$. Despite obvious reduction in model dimensions and notational economy that come with this assumption, this is not without loss of generality. We leave many interesting and important issues related to finer details of the substitution possibilities to future work. However, our main findings (equilibrium existence and qualitative features of the equilibrium) are not affected by this assumption[1].

[1]For example, by taking share functions parameterized by the substitution parameters, $z_1(D, a_1) = s(D) + a_1\hat{s}(D)$ and $z_2(D, a_2) = \hat{s}(D) + a_2 s(D)$, the analysis below can be extended to the more general case.

### 7.3.2 Cost and Information Structures

Firm $i$ pays a unit cost for the items that he purchases. We take the type set of firm $i$, denoted $\mathcal{C}_i$, as the set of values his unit cost can take. Firm $i$'s type is governed by a probability measure $p_i$ over $\mathcal{C}_i$. Type distributions of the two firms are independent. Each firm observes his own cost prior to deciding his order quantity, but he does not observe the other firm's cost. From firm $j$'s perspective, firm $i$'s unit cost is a random variable $C_i$ with support $\mathcal{C}_i$ and distribution $p_i$.

In this chapter, we focus on the case with discrete type sets. Specifically, the unit cost of each firm can take one of two values, i.e., $\mathcal{C}_i = \{c_{iL}, c_{iH}\}$ with $c_{iL} < c_{iH}$. We assume that firm 1's unit cost is $c_{1H}$ with probability $p_1(c_{1H}) = p$ and $c_{1L}$ with probability $p_1(c_{1L}) = 1 - p_1(c_{1H}) = (1-p)$ and firm 2's unit cost is $c_{2H}$ with probability $p_2(c_{2H}) = q$ and $c_{2L}$ with probability $p_2(c_{2L}) = 1 - p_2(c_{2H}) = (1-q)$. With appropriate relabeling of the players, we take $c_{1H} \leq c_{2H}$.

We assume that salvage prices and back–order costs are 0. (The analysis can easily be extended to relax this assumption.) We also assume, without loss of generality, that each firm earns a normalized revenue of 1 per unit of good he sells. This normalization can be achieved by changing the unit of measurement for costs. Under this normalization, we have $c_{2H} \leq 1$. In fact, all our results remain unchanged if one were to take per unit revenues, instead of unit costs, as the source of private information.

Finally, all elements of the model except the cost realizations such as split function, unit revenues and total market demand distributions are common knowledge at the time the order quantity decisions are made.

### 7.3.3 Actions, Strategies and Payoffs

For each player $i$ the order quantities are the action sets, $\mathcal{Q}_i = [0, \overline{Q}_i]$, where $\overline{Q}_i$ is the optimal order quantity of firm $i$ assuming that he gets all of the industry demand $D$ with the smallest possible value of $c_i$. Finally, firm $i$'s expected payoff is $\Pi_i : \mathcal{Q} \times \mathcal{C} \to \Re$ where $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ and $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2$.

A pure strategy for player $i$ is a function which maps his type into his action set, $Q_i : \mathcal{C}_i \to \mathcal{Q}_i$ where $Q_i(c_i)$ is the strategy choice for type $c_i$ of player $i$. Player $i$'s *interim*[2] expected payoff $\Pi_i$ is his expected profit conditional on his realized type $c_i$ and order quantity $Q$, when his rival follows the strategy $Q_j()$:

$$\Pi_i(c_i, Q) = E_{C_j}[\pi_i(Q, Q_j(C_j), c_i)] = \sum_{c_j \in \mathcal{C}_j} p_j(c_j)\pi_i(Q, Q_j(c_j), c_i),$$

where, conditional on $C_j = c_j$,

$$\pi_i(Q, Q_j(c_j), c_i) = E_{R_i(Q_j(c_j))}\big[\min\{R_i(Q_j(c_j)), Q\}\big] - c_i Q$$

is the player's *ex post* profit when his unit cost is $c_i$ and his order quantity $Q$.

## 7.4 Equilibrium Order Quantities

A strategy profile $Q^* = (Q_1^*(), Q_2^*())$ is a Bayesian–Nash equilibrium if, for each player $i$, and each type $c_i \in \mathcal{C}_i$ of player $i$,

$$Q_i^*(c_i) \in \arg\max_{Q \in \mathcal{Q}_i} \sum_{c_j \in \mathcal{C}_j} p_j(c_j)\pi_i(Q, Q_j(c_j), c_i).$$

Let $Q_{iL} = Q_i(c_{iL})$ be the order quantity of player $i$ if his cost is $c_{iL}$ and let $Q_{iH} = Q_i(c_{iH})$ be the order quantity of player $i$ if his cost is $c_{iH}$. Let $(Q_{1L}^*, Q_{1H}^*, Q_{2L}^*, Q_{2H}^*)$ denote a Bayesian–Nash equilibrium. *Interim* expected

[2]The terms *ex ante, interim* and *ex post* refer to conditioning with respect to the realizations of firm types. Throughout, demand remains uncertain. That is, no new information becomes available about market demand, and, thus, all expressions are *ex ante* with respect to demand.

payoffs conditional on own cost realizations are:

$$\begin{aligned}
\Pi_1(c_{1L}, Q_{1L}) &= q\, E[\min\{R_1(Q_{2H}), Q_{1L}\}] + (1-q)\, E[\min\{R_1(Q_{2L}), Q_{1L}\}] - c_{1L}Q_{1L}, \\
\Pi_1(c_{1H}, Q_{1H}) &= q\, E[\min\{R_1(Q_{2H}), Q_{1H}\}] + (1-q)\, E[\min\{R_1(Q_{2L}), Q_{1H}\}] - c_{1H}Q_{1H}, \\
\Pi_2(c_{2L}, Q_{2L}) &= p\, E[\min\{R_2(Q_{1H}), Q_{2L}\}] + (1-p)\, E[\min\{R_2(Q_{1L}), Q_{2L}\}] - c_{2L}Q_{2L}, \\
\Pi_2(c_{2H}, Q_{2H}) &= p\, E[\min\{R_2(Q_{1H}), Q_{2H}\}] + (1-q)\, E[\min\{R_2(Q_{1L}), Q_{2H}\}] - c_{2H}Q_{2H}.
\end{aligned}$$

A standard property used in newsboy models is that $\partial E_R[\min\{R, Q\}]/\partial Q = Pr(R \geq Q)$. Thus, taking the derivative of each type's payoff with respect to his action, the Bayesian–Nash equilibrium order quantities $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ satisfy the following conditions:

$$q\, Pr(R_1(Q_{2H}) \geq Q_{1L}) + (1-q)Pr(R_1(Q_{2L}) \geq Q_{1L}) - c_{1L} = 0, \quad (7.1)$$

$$q\, Pr(R_1(Q_{2H}) \geq Q_{1H}) + (1-q)Pr(R_1(Q_{2L}) \geq Q_{1H}) - c_{1H} = 0, \quad (7.2)$$

$$p\, Pr(R_2(Q_{1H}) \geq Q_{2L}) + (1-p)Pr(R_2(Q_{1L}) \geq Q_{2L}) - c_{2L} = 0, \quad (7.3)$$

$$p\, Pr(R_2(Q_{1H}) \geq Q_{2H}) + (1-p)Pr(R_2(Q_{1L}) \geq Q_{2H}) - c_{2H} = 0. \quad (7.4)$$

### 7.4.1 Equilibrium Existence

Van Zandt and Vives [49] show the existence of Bayesian–Nash equilibrium for supermodular asymmetric information games when type sets are discrete and action sets are continua. Our model of asymmetric information newsboy duopoly is an instance of the general class of incomplete information games studied in Van Zandt and Vives [49]. To establish the existence of pure strategy equilibrium we verify that the equilibrium existence conditions in Van Zandt and Vives [49] are satisfied in our setting. These conditions are: (i) the payoff function $\pi_i$ is supermodular in $Q_i$, (ii) it has increasing differences in $(Q_i, Q_j)$, and (iii) it has increasing differences in $(Q_i, t_i)$, where $t_i = -c_i$.

**Theorem 7.1.** *A pure strategy Nash equilibrium exists for the newsboy duopoly game with asymmetric information.*

Equilibrium exists under more general assumptions than we make. For instance, the theorem above is valid for arbitrary type sets, not only discrete types since the existence theorem in Van Zandt and Vives [49] can be generalized for any type set. Furthermore, as noted by Lippman and McCardle [30] in their model of complete information, the existence of equilibrium does not require any assumption on the split functions, or on the joint distribution of the initial demands.

### 7.4.2 Preliminary Observations on the Equilibrium

In characterizing the structure of equilibrium, some preliminary remarks will be useful. We start with some observations on the best response functions. We then examine optimal order quantities in the absence of strategic interactions to establish a baseline.

Our first claim exploits the assumption that the split functions $s(\cdot)$ and $\hat{s}(\cdot)$ are deterministic and increasing, thus invertible.

**Claim 7.1.** $\min\{s^{-1}(x), \hat{s}^{-1}(y)\} \leq x + y \leq \max\{s^{-1}(x), \hat{s}^{-1}(y)\}$.

The best response functions of the two types of firm 1, $(Q^*_{1L}(Q_{2L}, Q_{2H}), Q^*_{1H}(Q_{2L}, Q_{2H}))$, and those of firm 2, $(Q^*_{2L}(Q_{1L}, Q_{1H}), Q^*_{2H}(Q_{1L}, Q_{1H}))$, solve:

$$
\begin{aligned}
q\, Pr(R_1(Q_{2H}) \geq Q^*_{1L}) + (1-q)\, Pr(R_1(Q_{2L}) \geq Q^*_{1L}) - c_{1L} = 0, \\
q\, Pr(R_1(Q_{2H}) \geq Q^*_{1H}) + (1-q)\, Pr(R_1(Q_{2L}) \geq Q^*_{1H}) - c_{1H} = 0, \\
p\, Pr(R_2(Q_{1H}) \geq Q^*_{2L}) + (1-p)\, Pr(R_2(Q_{1L}) \geq Q^*_{2L}) - c_{2L} = 0, \\
p\, Pr(R_2(Q_{1H}) \geq Q^*_{2H}) + (1-p)\, Pr(R_2(Q_{1L}) \geq Q^*_{2H}) - c_{2H} = 0.
\end{aligned}
$$

Since $R_i(Q)$ and, hence, $Pr(R_i(Q) \geq Q_i)$ are non–increasing in $Q$, best response functions for both types of both players are non–increasing in both arguments.

Stand–alone order quantities in the absence of competitive interactions will play a useful role as a baseline. We denote by $(Q^o_{1L}, Q^o_{1H}, Q^o_{2L}, Q^o_{2H})$ the vector of optimal order quantities for the case with no spillovers (i.e., no competitive interaction).

**Lemma 7.1.** *The vector of stand–alone order quantities* $(Q^o_{1L}, Q^o_{1H}, Q^o_{2L}, Q^o_{2H})$ *is the unique solution to the system of equations:*

$$\begin{aligned} Pr(D_1 \geq Q_{1L}) = c_{1L}, &\qquad Pr(D_1 \geq Q_{1H}) = c_{1H}, \\ Pr(D_2 \geq Q_{2L}) = c_{2L}, &\qquad Pr(D_2 \geq Q_{2H}) = c_{2H}. \end{aligned}$$

The ranking of optimal order quantities of the two types of a player is straightforward – the higher a firms' unit cost the lower his stand–alone order quantity: $Q^o_{1L} \geq Q^o_{1H}$ and $Q^o_{2L} \geq Q^o_{2H}$.

In contrast, comparison of the order quantities across firms is complicated by the fact that relative rankings of the firms' market shares and unit costs are not a priori restricted. In general, depending on the relative orderings of market shares and unit costs, all rankings of the four order quantities $(Q^o_{1L}, Q^o_{1H}, Q^o_{2L}, Q^o_{2H})$ that are compatible with the orderings $Q^o_{1L} \geq Q^o_{1H}$ and $Q^o_{2L} \geq Q^o_{2H}$ are possible.

One needs further assumptions on market shares and unit costs to be able to rank the stand–alone order quantities of the two firms. For example, if unit costs and initial market shares are perfectly negatively correlated (so that the initial market share of the firm with the lower unit cost exceeds that of the firm with higher unit cost for all demand realizations) then stand–alone order quantities are ordered in the same way as initial market shares.

Note, on the other hand, that stock–out levels, $(Pr(D_i \geq Q^o_{ix}) : i \in \{1, 2\}, x \in \{L, H\})$, are ordered the same way as the unit costs. This simple observation, combined with our assumption that initial demands of the two firms are monotone functions of a common market demand, allows a complete ordering of the transformed order quantities:

**Claim 7.2.** *For* $x, y \in \{L, H\}$, $c_{1x} \leq c_{2y}$ *if and only if* $s^{-1}(Q^o_{1x}) \geq \hat{s}^{-1}(Q^o_{2y})$.

Returning to the analysis of the equilibrium conditions, we first note an observation on the stock–out probability of firm $i$ with order level $Q_i$. For firm 1:

$$\begin{aligned}
Pr(R_1(Q_2) \geq Q_1) &= Pr(D_1 + (D_2 - Q_2)^+ \geq Q_1) \\
&= Pr(s(D) + (\hat{s}(D) - Q_2)^+ \geq Q_1) \\
&= Pr(D \geq \hat{s}^{-1}(Q_2), D \geq Q_1 + Q_2) + Pr(D \leq \hat{s}^{-1}(Q_2), D \geq s^{-1}(Q_1)).
\end{aligned}$$

Similarly, for firm 2:

$$\begin{aligned}
Pr(R_2(Q_1) \geq Q_2) &= Pr(D_2 + (D_1 - Q_1)^+ \geq Q_2) \\
&= Pr(\hat{s}(D) + (s(D) - Q_1)^+ \geq Q_2) \\
&= Pr(D \geq s^{-1}(Q_1), D \geq Q_2 + Q_1) + Pr(D \leq s^{-1}(Q_1), D \geq \hat{s}^{-1}(Q_2)).
\end{aligned}$$

Second, we observe that low–cost type of each player orders a larger quantity than his high–cost type in equilibrium.

**Claim 7.3.** *(i)* $Q^*_{1L} > Q^*_{1H}$, *(ii)* $Q^*_{2L} > Q^*_{2H}$.

Using stand–alone order quantities as a baseline, the next claim shows that order quantities strictly less than the stand–alone order quantities are dominated. Thus, presence of spillovers leads to order quantities that are no less than the order quantities without spillovers. This means that competition does not lead to a decrease in total industry inventory.

**Claim 7.4.** *(i)* $Q^*_{1L} \geq Q^o_{1L}$, *(ii)* $Q^*_{1H} \geq Q^o_{1H}$, *(iii)* $Q^*_{2L} \geq Q^o_{2L}$, *(iv)* $Q^*_{2H} \geq Q^o_{2H}$.

The following lemma identifies a useful boundary condition that ties the equilibrium order quantity of one of the players to the stand–alone order quantity for the high–cost type of that player.

**Lemma 7.2.** *In a Bayesian–Nash equilibrium either (i)* $Q^*_{2H} = Q^o_{2H}$ *or (ii)* $Q^*_{1H} = Q^o_{1H}$.

Next, equilibrium order quantities of high–cost types of the two firms are ordered up to transformation by initial market shares:

**Lemma 7.3.** *If* $c_{1H} \leq c_{2H}$*, then* $s^{-1}(Q^*_{1H}) \geq \hat{s}^{-1}(Q^*_{2H})$*.*

Finally, in equilibrium, the firm with highest possible unit cost orders his optimal quantity under no competition.

**Lemma 7.4.** *If* $c_{1H} \leq c_{2H}$*, then* $Q^*_{2H} = Q^o_{2H} = \hat{s}(\overline{G}^{-1}(c_{2H}))$*.*

When $c_{1H} = c_{2H}$, high–cost types of both firms order their optimal quantities under no competition, i.e., $Q^*_{2H} = Q^o_{2H}$ and $Q^*_{1H} = Q^o_{1H}$.

As a final observation, we note that the best response function of the second firm's high–cost type is flat at its stand–alone level when the order quantities of the first firm's two types exceed their respective stand–alone levels:

**Lemma 7.5.** *For* $c_{1H} \leq c_{2H}$*,* $Q^*_{2H}(x, y) = Q^o_{2H}$ *for all* $(x, y) \geq (Q^o_{1L}, Q^o_{1H})$*.*

### 7.4.3 Structure of the Equilibrium

Summarizing the observations in the previous sub–section, under the player labeling with $c_{1H} \leq c_{2H}$, the conditions for equilibrium can be stated as follows:

$$q\,Pr(R_1(\hat{s}(\overline{G}^{-1}(c_{2H}))) \geq Q^*_{1L}) + (1-q)Pr(R_1(Q^*_{2L}) \geq Q^*_{1L}) = c_{1L},$$

$$q\,Pr(R_1(\hat{s}(\overline{G}^{-1}(c_{2H}))) \geq Q^*_{1H}) + (1-q)Pr(R_1(Q^*_{2L}) \geq Q^*_{1H}) = c_{1H},$$

$$p\,Pr(R_2(Q^*_{1H}) \geq Q^*_{2L}) + (1-p)Pr(R_2(Q^*_{1L}) \geq Q^*_{2L}) = c_{2L},$$

$$Q^*_{2H} = \hat{s}(\overline{G}^{-1}(c_{2H})).$$

We can now state the main theorem of this chapter that characterizes the structure of equilibrium order quantities.

**Theorem 7.2.** *Assume, without loss of generality, that* $c_{1H} \leq c_{2H}$. $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ *is a Bayesian–Nash equilibrium if and only if*

**1)**
$$Q^*_{2H} = \hat{s}(\overline{G}^{\,-1}(c_{2H}))$$

**2)** $Q^*_{1L}$,$Q^*_{1H}$ *and* $Q^*_{2L}$ *satisfy one of the following sets of conditions:*

**(i)**
$$q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{\,-1}(c_{2H}))) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1L})) = c_{1L} \tag{$i_1$}$$

$$q\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{\,-1}(c_{2H}))) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1H})) = c_{1H} \tag{$i_2$}$$

$$p\ \overline{G}(Q^*_{2L} + Q^*_{1H}) + (1-p)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_{2L} \tag{$i_3$}$$

$$\hat{s}^{-1}(Q^*_{2L}) \ \geq \ s^{-1}(Q^*_{1L}) \tag{$i_4$}$$

**(ii)**
$$q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{\,-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_{1L} \tag{$ii_1$}$$

$$q\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{\,-1}(c_{2H}))) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1H})) = c_{1H} \tag{$ii_2$}$$

$$p\ \overline{G}(Q^*_{2L} + Q^*_{1H}) + (1-p)\,\overline{G}(\hat{s}^{-1}(Q^*_{2L})) = c_{2L} \tag{$ii_3$}$$

$$s^{-1}(Q^*_{1L}) \ > \ \hat{s}^{-1}(Q^*_{2L}) \ \geq \ (s^{-1}(Q^*_{1H}) \tag{$ii_4$}$$

**(iii)**
$$q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{\,-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{\,-1}(c_{2L}))) = c_{1L} \tag{$iii_1$}$$

$$q\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{\,-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{\,-1}(c_{2L}))) = c_{1H} \tag{$iii_2$}$$

$$Q^*_{2L} = \hat{s}(\overline{G}^{\,-1}(c_{2L})) \tag{$iii_3$}$$

$$s^{-1}(Q^*_{1H}) \ > \ \hat{s}^{-1}(Q^*_{2L}) \tag{$iii_4$}$$

Before we proceed with discussion of properties of the equilibrium, we first show that it is unique.

**Theorem 7.3.** *The vector of order quantities $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ in Theorem 7.2 is unique.*

Uniqueness of solutions for each block of equations is a straightforward consequence of the continuity of the demand distribution. To establish uniqueness of the equilibrium, we rule out the possibility that the two or more blocks of equations may have solutions that also satisfy the corresponding inequality. This is done in the Appendix E.4.

A notable pattern in the equilibria across the model space is the recursive structure of the order quantities. This pattern greatly simplifies the computation of equilibrium order quantities. The order quantity of the player type with highest unit cost is determined based on the demand distribution, the split function and his unit cost, independently of other parameters of the game. The remaining equilibrium quantities are obtained recursively. At each step, substituting for the previously computed equilibrium values, a single equation is solved for a single unknown equilibrium quantity. For example if an equilibrium satisfying the first block can be solved recursively by solving first $Q^*_{1L}$ from $(i_1)$ and $Q^*_{1H}$ from $(i_2)$, since they are the only variables in those equations, and then solving $Q^*_{2L}$ from $(i_3)$ using the values of $Q^*_{1L}$ and $Q^*_{1H}$.

The recursive pattern of the equilibrium quantities reflect the fact that the equilibrium is partially dominance–solvable, which in turn is a consequence of the supermodular structure of the game. By Claim 4 above, any quantity strictly less than the stand–alone order quantity is strictly dominated by the stand–alone order quantity for every type. Given this fact and Lemma 5, order quantities strictly greater than the stand–alone order quantity are also dominated by the stand–alone order quantity for the highest cost type ($c_{2H}$). Thus, a two–step reasoning pins the equilibrium behavior of the highest cost type.

### 7.4.4 Special Cases

In this sub–section we consider several corollaries of Theorem 7.2 for special cases of the general model. Corollary 7.1 considers a model with *ex ante* symmetric cost structures without restricting the initial market shares. Corollary 7.2, we impose a restriction on the initial market share function so that one of the firms has larger initial market share for all demand realizations. Corollary 7.3 presents the equilibrium for the case with fully symmetric firms where both initial market shares and *ex ante* cost structures are identical. In Corollary 7.4, we remove the restrictions on the initial market shares and consider an extreme form of *ex ante* cost asymmetry: one firm's unit costs are uniformly higher than the other firm's unit costs for all type realizations. Finally, in Corollary 7.5, we consider a model with symmetric initial market shares and unrestricted *ex ante* asymmetries in the cost structures. As these corollaries are obtained through straightforward substitutions, we omit the proofs.

**Corollary 7.1.** *Assume that the two firms are* ex ante *symmetric with respect to costs. That is,* $c_{1H} = c_{2H} = c_H$, $c_{1L} = c_{2L} = c_L$, *and* $p = q$. *Then* $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ *is a Bayesian–Nash equilibrium if and only if*

**1)** $Q^*_{2H} = \hat{s}(\overline{G}^{\,-1}(c_H))$ *and* $Q^*_{1H} = s(\overline{G}^{\,-1}(c_H))$

**2)** $Q^*_{1L}$ *and* $Q^*_{2L}$ *satisfy one of the following sets of conditions:*

**(i)**
$$q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{\,-1}(c_H))) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1L})) = c_L \qquad (i_1)$$

$$p\,\overline{G}(Q^*_{2L} + s(\overline{G}^{\,-1}(c_H))) + (1-p)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_L \qquad (i_2)$$

$$\hat{s}(Q^*_{2L}) \;\geq\; s^{-1}(Q^*_{1L}) \qquad (i_3)$$

**(ii)**
$$q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{\,-1}(c_H))) + (1-q)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_L \qquad (ii_1)$$

$$p\,\overline{G}(Q^*_{2L} + s(\overline{G}^{\,-1}(c_H))) + (1-p)\,\overline{G}(\hat{s}^{-1}(Q^*_{2L})) = c_L \qquad (ii_2)$$

$$s^{-1}(Q^*_{1L}) \;>\; \hat{s}(Q^*_{2L}) \qquad (ii_3)$$

Further simplification is possible under the assumption that initial market shares of the two firms are uniformly ranked, i.e., one firm's initial market share is higher than the other's for all demand realizations. By relabeling firms if necessary, we can take initial market shares to favor firm 1: $s(d) \geq d/2$.

**Corollary 7.2.** *Assume that the two firms are* ex ante *symmetric with respect to costs. That is, $c_{1H} = c_{2H} = c_H$, $c_{1L} = c_{2L} = c_L$, and $p = q$. Furthermore, assume $s(d) \geq d/2$ for all demand levels $d$. Then $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ is a Bayesian–Nash equilibrium if and only if*

$$Q^*_{2H} = \hat{s}(\overline{G}^{-1}(c_H)), Q^*_{1H} = s(\overline{G}^{-1}(c_H)) \textit{ and } (Q^*_{1L}, Q^*_{2L}) \textit{ solves:}$$

$$\begin{aligned} q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_H))) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1L})) &= c_L, \\ p\;\overline{G}(Q^*_{2L} + s(\overline{G}^{-1}(c_H))) + (1-p)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) &= c_L. \end{aligned}$$

When the two firms are fully symmetric in terms of cost structures and initial market shares, we get a fully symmetric equilibrium.

**Corollary 7.3.** *Assume that the two firms are* ex ante *symmetric with respect to costs. That is, $c_{1H} = c_{2H} = c_H$, $c_{1L} = c_{2L} = c_L$, and $p = q$. Furthermore, let $s(d) = \hat{s}(d) = d/2$ for all demand levels $d$. Then $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ is a Bayesian–Nash equilibrium if and only if*

$$Q^*_{1H} = Q^*_{2H} = Q^*_H = (1/2)(\overline{G}^{-1}(c_H)) \textit{ and } Q^*_{1L} = Q^*_{2L} = Q^*_L \textit{ where } Q^*_L \textit{ solves}$$

$$q\,\overline{G}(Q^*_L + (1/2)\overline{G}^{-1}(c_H)) + (1-q)\,\overline{G}(2Q^*_L) = c_L.$$

The next corollary looks at the case where one firm has a cost disadvantage for all cost realizations.

**Corollary 7.4.** *Assume that $c_{1H} \leq c_{2L}$. Then $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ is a Bayesian–Nash equilibrium if and only if*

$$\begin{gathered} Q^*_{2H} = \hat{s}(\overline{G}^{-1}(c_{2H})) \\ Q^*_{2L} = \hat{s}(\overline{G}^{-1}(c_{2L})) \\ q\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2L}))) = c_{1L} \\ q\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2L}))) = c_{1H}. \end{gathered}$$

As a final corollary, we present the equilibrium order quantities for symmetric initial market shares. In this special case, the equilibrium conditions can be stated explicitly in terms of the exogenous cost parameters, in contrast to the implicit characterization in Theorem 7.2. For each of the three possible orderings of the unit cost parameters, we have a different set of equilibrium conditions.

**Corollary 7.5.** *Assume that $s(d) = \hat{s}(d) = d/2$ and, without loss of generality, that $c_{1H} \leq c_{2H}$. Then $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ is a Bayesian–Nash equilibrium if and only if*

**1)** $Q^*_{2H} = (1/2)\overline{G}^{-1}(c_{2H})$

**2)** *$Q^*_{1L}$, $Q^*_{1H}$ and $Q^*_{2L}$ satisfy one of the following sets of conditions:*

**(i)** *If* $c_{2L} \leq c_{1L} \leq c_{1H} \leq c_{2H}$

$$q\,\overline{G}(Q^*_{1L} + (1/2)\overline{G}^{-1}(c_{2H})) + (1-q)\,\overline{G}(2Q^*_{1L}) = c_{1L} \qquad (i_1)$$

$$q\,\overline{G}(Q^*_{1H} + (1/2)\overline{G}^{-1}(c_{2H})) + (1-q)\,\overline{G}(2Q^*_{1H}) = c_{1H} \qquad (i_2)$$

$$p\,\overline{G}(Q^*_{2L} + Q^*_{1H}) + (1-p)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_{2L} \qquad (i_3)$$

**(ii)** *If* $c_{1L} \leq c_{2L} \leq c_{1H} \leq c_{2H}$

$$q\,\overline{G}(Q^*_{1L} + (1/2)\overline{G}^{-1}(c_{2H}))) + (1-q)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_{1L} \qquad (ii_1)$$

$$q\,\overline{G}(Q^*_{1H} + (1/2)\overline{G}^{-1}(c_{2H})) + (1-q)\,\overline{G}(2Q^*_{1H}) = c_{1H} \qquad (ii_2)$$

$$p\,\overline{G}(Q^*_{2L} + Q^*_{1H}) + (1-p)\,\overline{G}(2Q^*_{2L}) = c_{2L} \qquad (ii_3)$$

**(iii)** *If* $c_{1L} \leq c_{1H} \leq c_{2L} \leq c_{2H}$

$$q\,\overline{G}(Q^*_{1L} + (1/2)\overline{G}^{-1}(c_{2H})) + (1-q)\,\overline{G}(Q^*_{1L} + (1/2)\overline{G}^{-1}(c_{2L})) = c_{1L} \qquad (iii_1)$$

$$q\,\overline{G}(Q^*_{1H} + (1/2)\overline{G}^{-1}(c_{2H})) + (1-q)\,\overline{G}(Q^*_{1H} + (1/2)\overline{G}^{-1}(c_{2L})) = c_{1H} \qquad (iii_2)$$

$$Q^*_{2L} = (1/2)\overline{G}^{-1}(c_{2L}) \qquad (iii_3)$$

### 7.4.5 Intra–equilibrium Comparisons

As noted in Claim 3 above, equilibrium is monotone: low–cost type of a firm orders a larger quantity than his high–cost type. Without further restrictions on the initial market shares and the level of unit costs, this is about the extent of what can be said regarding intra–equilibrium comparisons. That is, no general ranking of order quantities across firms is possible without imposing further structure on the model. Furthermore, even under normalization an analog of Claim 2 does not hold for equilibrium order quantities. The only possible ranking is the one provided in Lemma 3 that ranks the normalized equilibrium order quantities of the high–cost types of the two firms.

An interesting observation can be made using the characterization in Corollary 4 in the previous section to illustrate a general phenomenon of inter–type externality. The equilibrium characterization there remains valid for a range of unit costs with $c_{2L} < c_{1H} < c_{2H}$. In this equilibrium, both types of firm 2 choose an order quantity equal to his stand–alone quantity while it is common knowledge that firm 1 may have larger unit cost. That is, low–cost type firm 2 ignores spillover from the less efficient type of the rival firm. This is due to the fact that high–cost type of firm 1, while less efficient than the low–cost type firm 2, selects a large order quantity expecting spillover demand from the high cost type of firm 2. The increased order quantity of the firm $1H$ forces firm $2L$ to stick to $Q^o_{2L}$.

### 7.4.6 Comparative Statics

Comparative static analysis of the equilibrium and payoffs with respect to the exogenous parameters of the model is done in two parts. We first establish general comparative statics results with respect to two exogenous functions in the model, namely, the demand and the market share function. Then we derive explicit comparative static expressions for the scalar parameters.

**Theorem 7.4.** *Let $D_A$ and $D_B$ be two positive random variables such that $D_A$ dominates $D_B$ under first order stochastic dominance. Then, the equilibrium* *order quantities with industry demand $D_A$ are larger than the equilibrium order quantities with industry demand $D_B$.*

**Theorem 7.5.** *If $s_A(d) > s_B(d)$ for all positive real numbers d, then the equilibrium order quantities of both types of firm 1 (firm 2) are larger (respectively, smaller) under the split function $s_A$ than the order quantities under $s_B$.*

In Table 7.1, we provide the signs of all first order derivatives of equilibrium order quantities with respect to the exogenous scalar parameters, $c_{1L}$, $c_{1H}$, $p$, $c_{2L}$, $c_{2H}$ and $q$. The explicit expressions for the comparative statics derivatives themselves are provided in Appendix E.7. Cases (*i*), (*ii*) and (*iii*) correspond to the cases in Theorem 7.2.

Table 7.1: Comparative Statics

| Cases | Quantities | Conditions | $\mathbf{c_{1L}}$ | $\mathbf{c_{1H}}$ | $\mathbf{p}$ | $\mathbf{c_{2L}}$ | $\mathbf{c_{2H}}$ | $\mathbf{q}$ |
|---|---|---|---|---|---|---|---|---|
| | $Q^*_{2H}$ | | 0 | 0 | 0 | 0 | $-$ | 0 |
| **(i)** | $Q^*_{1L}$ | | $-$ | 0 | 0 | 0 | $+$ | $+$ |
| | $Q^*_{1H}$ | | 0 | $-$ | 0 | 0 | $+$ | $+$ |
| | $Q^*_{2L}$ | $\overline{G}(Q^*_{1L}+Q^*_{2L}) > 0$ | $+$ | $+$ | $+$ | $-$ | $-$ | $-$ |
| | $Q^*_{2L}$ | $\overline{G}(Q^*_{1L}+Q^*_{2L}) = 0$ | 0 | $+$ | $+$ | $-$ | $-$ | $-$ |
| **(ii)** | $Q^*_{1L}$ | $\overline{G}(Q^*_{1L}+Q^*_{2L}) > 0$ | $-$ | $-$ | $-$ | $+$ | $+$ | $+$ |
| | $Q^*_{1L}$ | $\overline{G}(Q^*_{1L}+Q^*_{2L}) = 0$ | $-$ | 0 | 0 | 0 | $+$ | $+$ |
| | $Q^*_{1H}$ | | 0 | $-$ | 0 | 0 | $+$ | $+$ |
| | $Q^*_{2L}$ | | 0 | $+$ | $+$ | $-$ | $-$ | $-$ |
| **(iii)** | $Q^*_{1L}$ | $\overline{G}(Q^*_{1L}+Q^*_{2L}) > 0$ | $-$ | 0 | 0 | $+$ | $+$ | $+$ |
| | $Q^*_{1L}$ | $\overline{G}(Q^*_{1L}+Q^*_{2L}) = 0$ | $-$ | 0 | 0 | 0 | $+$ | $+$ |
| | $Q^*_{1H}$ | $\overline{G}(Q^*_{1H}+Q^*_{2L}) > 0$ | 0 | $-$ | 0 | $+$ | $+$ | $+$ |
| | $Q^*_{1H}$ | $\overline{G}(Q^*_{1H}+Q^*_{2L}) = 0$ | 0 | $-$ | 0 | 0 | $+$ | $+$ |
| | $Q^*_{2L}$ | | 0 | 0 | 0 | $-$ | 0 | 0 |

As expected, the equilibrium order quantities for both players are non–increasing with respect to their own costs and non–decreasing with respect to their rival's costs. In equilibrium, each player orders more as his rival's probability of being high type increases. Conversely, each player orders less as his own probability of being high type increases. This is due to information asymmetry between players and can be explained as follows. Suppose the probability of being high type for firm 1 is increasing. In this case, firm 2 will be ordering more since he will anticipate a higher chance of low order quantity from firm 1. This will lead firm 1 to expect less spillover from firm 2 and hence order less himself. Whether these monotonicities are strict or not depend on specific cases and conditions as given in Table 7.1. The only exception to these results is that firm 2's (the firm with larger high cost) equilibrium order quantity when his type is high only depends on its own cost as shown in Theorem 7.2.

## 7.5 A Special Case: Uniform Demand and Linear Market Shares

In this section, we present the full explicit characterization of the equilibrium and the corresponding payoff functions for uniformly distributed demand and linear market share functions: $D \sim \text{Uniform}(0, 1)$, and $s(D) = sD$ and $\hat{s}(D) = (1-s)D$. Under uniform demand and linear market shares, an instance of the model is represented by 7 parameters: $(c_{1L}, c_{1H}, c_{2L}, c_{2H}, p, q, s)$.

As shown in Section 7.4, while $Q^*_{2H} = (1-s)(1-c_{2H})$, solution to $Q^*_{1L}$, $Q^*_{1H}$ and $Q^*_{2L}$ (and the corresponding payoffs) requires a detailed analysis.

### 7.5.1 A Partition of the Parameter Space

Detailed analysis, provided in Appendix E.8, lead to 8 regions in the parameter space. In each of the 8 regions, different equilibrium quantities and payoff functions are valid. In other words, in each of these regions the equilibrium structure (functional form) of at least one of endogenous variable is different from its from in other regions. The conditions that determine the partition of the parameter space are as follows: Denoting $\hat{p} = sp$ and $\hat{q} = (1-s)q$,

$$
\begin{array}{rrclr}
 & (1-\hat{q})\,c_{2L} & < & c_{1H} \quad -\hat{q}\,c_{2H} & (C_A) \\
c_{1L} & & < & \hat{q}\,c_{2H} & (C_B) \\
 & (1-\hat{q})\,c_{2L} & < & \hat{p}\,c_{1H} \quad -\hat{p}\,\hat{q}\,c_{2H} & (C_C) \\
 & \hat{q}\,c_{2L} & < & -c_{1H} \quad +\hat{q}\,c_{2H} & (C_D) \\
-(1-\hat{p})\,c_{1L} & +(1-\hat{q})\,c_{2L} & < & \hat{p}\,c_{1H} \quad -\hat{q}\,c_{2H} & (C_E) \\
\hat{p}\,c_{1L} & +(1-\hat{q})\,c_{2L} & < & \hat{p}\,c_{1H} & (C_F) \\
(1-\hat{p})\,(1-\hat{q})\,c_{1L} & +\hat{q}\,(1-\hat{q})\,c_{2L} & < & \hat{p}\,\hat{q}\,c_{1H} \quad +\hat{q}\,(1-\hat{p}-\hat{q})\,c_{2H} & (C_G) \\
c_{1L} & +\hat{q}\,c_{2L} & < & \hat{q}\,c_{2H} & (C_H)
\end{array}
$$

The 8 different regions that these equilibrium conditions lead to are given in Figure 1.

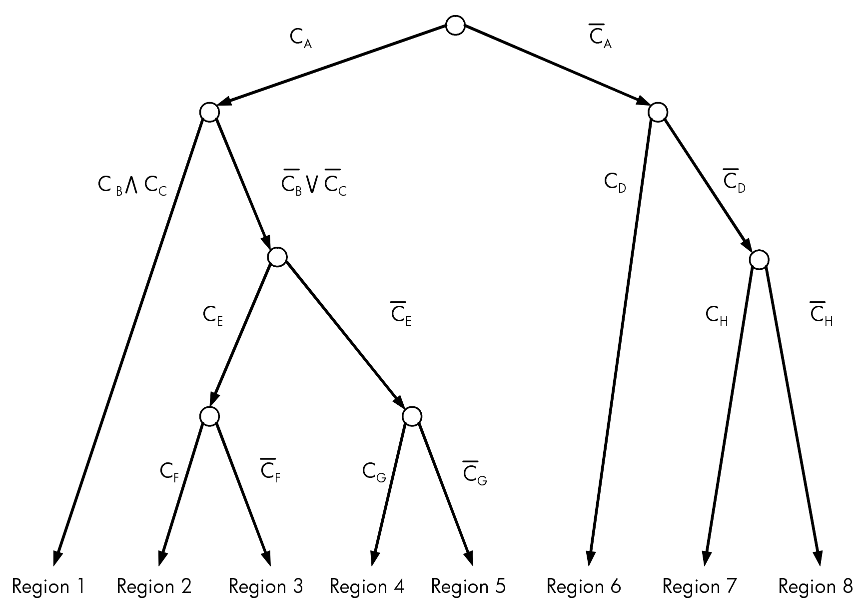


Figure 7.1: Conditions characterizing the partition of the parameter space

### 7.5.2 Equilibrium Order Quantities

$Q^*_{1L}$,$Q^*_{1H}$ and $Q^*_{2L}$ and payoffs $\pi_1(c_{1L}, c_{2L})$, $\pi_1(c_{1H}, c_{2L})$, $\pi_2(c_{1L}, c_{2L})$ and $\pi_2(c_{1H}, c_{2L})$ in these regions can be found using the following table:

Table 7.2: Functional forms of endogenous variables by parameter region

| Region | $\mathbf{Q_{1L}}$ | $\mathbf{Q_{1H}}$ | $\mathbf{Q_{2L}}$ | $\pi_1(\mathbf{c_{1L}, c_{2L}})$ | $\pi_1(\mathbf{c_{1H}, c_{2L}})$ | $\pi_2(\mathbf{c_{1L}, c_{2L}})$ | $\pi_2(\mathbf{c_{1H}, c_{2L}})$ |
|---:|---|---|---|---|---|---|---|
| **1** | $Q^\alpha_{1L}$ | $Q^\alpha_{1H}$ | $Q^\alpha_{2L}$ | $\pi^\alpha_1(c_{1L}, c_{2L})$ | $\pi^\alpha_1(c_{1H}, c_{2L})$ | $\pi^\alpha_2(c_{1L}, c_{2L})$ | $\pi^\alpha_2(c_{1H}, c_{2L})$ |
| **2** | $Q^\beta_{1L}$ | $Q^\alpha_{1H}$ | $Q^\alpha_{2L}$ | $\pi^\beta_1(c_{1L}, c_{2L})$ | $\pi^\alpha_1(c_{1H}, c_{2L})$ | $\pi^\beta_2(c_{1L}, c_{2L})$ | $\pi^\alpha_2(c_{1H}, c_{2L})$ |
| **3** | $Q^\beta_{1L}$ | $Q^\alpha_{1H}$ | $Q^\beta_{2L}$ | $\pi^\beta_1(c_{1L}, c_{2L})$ | $\pi^\alpha_1(c_{1H}, c_{2L})$ | $\pi^\gamma_2(c_{1L}, c_{2L})$ | $\pi^\alpha_2(c_{1H}, c_{2L})$ |
| **4** | $Q^\alpha_{1L}$ | $Q^\alpha_{1H}$ | $Q^\gamma_{2L}$ | $\pi^\gamma_1(c_{1L}, c_{2L})$ | $\pi^\alpha_1(c_{1H}, c_{2L})$ | $\pi^\delta_2(c_{1L}, c_{2L})$ | $\pi^\alpha_2(c_{1H}, c_{2L})$ |
| **5** | $Q^\gamma_{1L}$ | $Q^\alpha_{1H}$ | $Q^\gamma_{2L}$ | $\pi^\delta_1(c_{1L}, c_{2L})$ | $\pi^\alpha_1(c_{1H}, c_{2L})$ | $\pi^\delta_2(c_{1L}, c_{2L})$ | $\pi^\alpha_2(c_{1H}, c_{2L})$ |
| **6** | $Q^\alpha_{1L}$ | $Q^\beta_{1H}$ | $Q^\delta_{2L}$ | $\pi^\gamma_1(c_{1L}, c_{2L})$ | $\pi^\beta_1(c_{1H}, c_{2L})$ | $\pi^\delta_2(c_{1L}, c_{2L})$ | $\pi^\beta_2(c_{1H}, c_{2L})$ |
| **7** | $Q^\alpha_{1L}$ | $Q^\gamma_{1H}$ | $Q^\delta_{2L}$ | $\pi^\gamma_1(c_{1L}, c_{2L})$ | $\pi^\gamma_1(c_{1H}, c_{2L})$ | $\pi^\delta_2(c_{1L}, c_{2L})$ | $\pi^\beta_2(c_{1H}, c_{2L})$ |
| **8** | $Q^\delta_{1L}$ | $Q^\gamma_{1H}$ | $Q^\delta_{2L}$ | $\pi^\delta_1(c_{1L}, c_{2L})$ | $\pi^\gamma_1(c_{1H}, c_{2L})$ | $\pi^\delta_2(c_{1L}, c_{2L})$ | $\pi^\beta_2(c_{1H}, c_{2L})$ |

The equilibrium order quantity for firm 1 when his type is low takes four different functional forms:

$$
\begin{aligned}
Q^\alpha_{1L} &= 1 - \frac{c_{1L}}{q} - (1-s)(1-c_{2H}), \\
Q^\beta_{1L} &= \frac{(1 - c_{1L} - q(1-s)(1-c_{2H}))}{(q + (1-q)/s)}, \\
Q^\gamma_{1L} &= 1 - c_{1L} - q(1-s)(1-c_{2H}) - \frac{(1-q)(1-c_{2L})}{(p + (1-p)/(1-s))} \\
& \quad + \frac{(1-q)p(1-c_{1H} - q(1-s)(1-c_{2H}))}{(q+(1-q)/s)(p+(1-p)/(1-s))}, \\
Q^\delta_{1L} &= 1 - c_{1L} - q(1-s)(1-c_{2H}) - (1-q)(1-s)(1-c_{2L}).
\end{aligned}
$$

When firms 1's type is high, his equilibrium order quantity takes three possible forms:

$$
\begin{aligned}
Q^\alpha_{1H} &= \frac{(1 - c_{1H} - q(1-s)(1-c_{2H}))}{(q + (1-q)/s)}, \\
Q^\beta_{1H} &= 1 - \frac{c_{1H}}{q} - (1-s)(1-c_{2H}), \\
Q^\gamma_{1H} &= 1 - c_{1H} - q(1-s)(1-c_{2H}) - (1-q)(1-s)(1-c_{2L}).
\end{aligned}
$$

Finally, the low type of firm 2 has four different functional forms for his equilibrium order quantity:

$$
\begin{aligned}
Q_{2L}^{\alpha} &= 1-\frac{c_{2L}}{p}-\frac{(1-c_{1H}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)}, \\
Q_{2L}^{\beta} &= 1-c_{2L}-\frac{p\,(1-c_{1H}-q\,(1-s)(1-c_{2H}))}{(q+(1-q)/s)}-\frac{(1-p)(1-c_{1L}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)}, \\
Q_{2L}^{\gamma} &= \frac{(1-c_{2L})}{(p+(1-p)/(1-s))}-\frac{p\,(1-c_{1H}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)(p+(1-p)/(1-s))}, \\
Q_{2L}^{\delta} &= (1-s)(1-c_{2L}).
\end{aligned}
$$

### 7.5.3 Equilibrium Payoffs

When both firms have low costs, Firm 1's *ex post* payoff can take four different functional forms:

$$
\begin{aligned}
\pi_1^{\alpha}(c_{1L},c_{2L}) &= \frac{1}{2}\,s-c_{1L}Q_{1L}, \\
\pi_1^{\beta}(c_{1L},c_{2L}) &= Q_{1L}(1-c_{1L})-\frac{(Q_{1L})^2}{2s}, \\
\pi_1^{\gamma}(c_{1L},c_{2L}) &= \frac{1}{2}+\frac{(Q_{2L})^2}{2(1-s)}-Q_{2L}-c_{1L}Q_{1L}, \\
\pi_1^{\delta}(c_{1L},c_{2L}) &= Q_{1L}(1-c_{1L})+\frac{(Q_{2L})^2}{2(1-s)}-\frac{(Q_{1L}+Q_{2L})^2}{2}.
\end{aligned}
$$

Firm 2's payoff, similarly, has four possible functional forms when both firms have low cost:

$$
\begin{aligned}
\pi_2^{\alpha}(c_{1L},c_{2L}) &= \frac{1}{2}\,(1-s)-c_{2L}Q_{2L}, \\
\pi_2^{\beta}(c_{1L},c_{2L}) &= \frac{1}{2}+\frac{(Q_{1L})^2}{2s}-Q_{1L}-c_{2L}Q_{2L}, \\
\pi_2^{\gamma}(c_{1L},c_{2L}) &= Q_{2L}(1-c_{2L})+\frac{(Q_{1L})^2}{2s}-\frac{(Q_{1L}+Q_{2L})^2}{2}, \\
\pi_2^{\delta}(c_{1L},c_{2L}) &= Q_{2L}(1-c_{2L})-\frac{(Q_{2L})^2}{2(1-s)}.
\end{aligned}
$$

When firms 1 and 2 have low and high costs, respectively, we have three possibilities for the payoff for firm 1's payoff:

$$\begin{aligned}
\pi_1^\alpha(c_{1H}, c_{2L}) &= Q_{1H}(1-c_{1H}) - \frac{(Q_{1H})^2}{2s}, \\
\pi_1^\beta(c_{1H}, c_{2L}) &= \frac{1}{2} + \frac{(Q_{2L})^2}{2(1-s)} - Q_{2L} - c_{1H}Q_{1H}, \\
\pi_1^\gamma(c_{1H}, c_{2L}) &= Q_{1H}(1-c_{1H}) + \frac{(Q_{2L})^2}{2(1-s)} - \frac{(Q_{1H}+Q_{2L})^2}{2};
\end{aligned}$$

and two possible forms for the payoff for firm 2:

$$\begin{aligned}
\pi_2^\alpha(c_{1H}, c_{2L}) &= Q_{2L}(1-c_{2L}) + \frac{(Q_{1H})^2}{2s} - \frac{(Q_{1H}+Q_{2L})^2}{2}, \\
\pi_2^\beta(c_{1H}, c_{2L}) &= Q_{2L}(1-c_{2L}) - \frac{(Q_{2L})^2}{2(1-s)}.
\end{aligned}$$

When firm 2 has a high cost, the payoffs of the two players are same in all regions:

$$\begin{aligned}
\pi_1(c_{1L}, c_{2H}) &= Q_{1L}(1-c_{1L}) + \frac{Q_{2H}^2}{2(1-s)} - \frac{(Q_{1L}+Q_{2H})^2}{2}, \\
\pi_1(c_{1H}, c_{2H}) &= Q_{1H}(1-c_{1H}) + \frac{Q_{2H}^2}{2(1-s)} - \frac{(Q_{1H}+Q_{2H})^2}{2}, \\
\pi_2(c_{1L}, c_{2H}) &= \pi_2(c_{1H}, c_{2H}) = (1-s)(1-c_{2H})^2/2.
\end{aligned}$$

### 7.5.4 Comparative Statics

We present the explicit expressions for comparative static derivatives for the equilibrium order quantities for the uniform demand and linear split case in Appendix E.9. Comparative static sign patterns are summarized in Table 7.3. This is a specific version of Table 7.1 for the uniform demand and linear split function. Since $s$ characterize the whole split function in this case, we also provide the comparative statics with respect to $s$ in this table.

Table 7.3: Comparative Statics for Uniform Demand Case

| | $\mathbf{Q}^{\alpha}_{\mathbf{1L}}$ | $\mathbf{Q}^{\beta}_{\mathbf{1L}}$ | $\mathbf{Q}^{\gamma}_{\mathbf{1L}}$ | $\mathbf{Q}^{\delta}_{\mathbf{1L}}$ | $\mathbf{Q}^{\alpha}_{\mathbf{2L}}$ | $\mathbf{Q}^{\beta}_{\mathbf{2L}}$ | $\mathbf{Q}^{\gamma}_{\mathbf{2L}}$ | $\mathbf{Q}^{\delta}_{\mathbf{2L}}$ | $\mathbf{Q}^{\alpha}_{\mathbf{1H}}$ | $\mathbf{Q}^{\beta}_{\mathbf{1H}}$ | $\mathbf{Q}^{\gamma}_{\mathbf{1H}}$ | $\mathbf{Q}_{\mathbf{2H}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{c}_{\mathbf{1L}}$ | $-$ | $-$ | $-$ | $-$ | 0 | $+$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{c}_{\mathbf{1H}}$ | 0 | 0 | $-$ | 0 | $+$ | $+$ | $+$ | 0 | $-$ | $-$ | $-$ | 0 |
| $\mathbf{p}$ | 0 | 0 | $-$ | 0 | $+$ | $+$ | $+$ | 0 | 0 | $-$ | $-$ | 0 |
| $\mathbf{c}_{\mathbf{2L}}$ | 0 | 0 | $+$ | $+$ | $-$ | $-$ | $-$ | $-$ | 0 | 0 | $+$ | 0 |
| $\mathbf{c}_{\mathbf{2H}}$ | $+$ | $+$ | $+$ | $+$ | $-$ | $-$ | $-$ | 0 | $+$ | $+$ | $+$ | $-$ |
| $\mathbf{q}$ | $+$ | $+$ | $+$ | $+$ | $-$ | $-$ | $-$ | 0 | $+$ | $+$ | $+$ | 0 |
| $\mathbf{s}$ | $+$ | $+$ | $+$ | $+$ | $-$ | $-$ | $-$ | $-$ | $+$ | $+$ | $+$ | $-$ |

## 7.6 Concluding Remarks

We studied a model of inventory competition in a newsboy duopoly under asymmetric cost information. We showed that a pure strategy Bayesian–Nash equilibrium exists under fairly general assumptions. We characterized the equilibrium for the case where the industry demand is allocated between two firms using a deterministic split function and show its uniqueness. We showed that presence of strategic interactions creates incentives to increase order quantities for all firm types except the type that has the highest possible unit cost, who orders the same quantity as he would as a monopolist newsboy facing scaled version of the market demand. Therefore, competition leads to higher total inventory in the industry. The equilibrium conditions have an interesting recursive structure that enables an easy computation of the equilibrium order quantities. Comparative statics analysis shows that a stochastic increase in market demand or an increase in one firm's initial allocation of the total industry demand lead to higher inventory for that firm. We finally derived a complete characterization of the equilibrium and its comparative statics for the case of uniform demand and linear split rule.

# Chapter 8

# CONCLUSIONS

It is imperative to consider the effect of non–cooperative behavior on inventory games since it may lead to results not foreseen by the classical inventory models. In this thesis we investigate the impact of non–cooperative behavior in joint replenishment games and the effect of asymmetric information in newsvendor duopolies. We consider various models through the thesis.

In Chapter 3, we study a non–cooperative private contributions game for multiple firm. The firms contribute to the ordering cost and an intermediary determines the order cycle time as the minimum cycle time that can be financed with these contributions. Our results show that for both the single–stage and two–stage variant of this game the firm or group of firms with the highest adjusted demand rate finance most of the joint replenishment and the other firms just pay the minimum entree fee.

In Chapter 4, we extend the private contributions game to an asymmetric information counterpart to investigate whether asymmetric information would lead the firms with low adjusted demand rates to contribute more. We show the existence of a pure strategy Bayesian Nash equilibrium for the asymmetric information game and provide the equilibrium conditions. Finally, we conduct some numerical study to examine the impact of information asymmetry on expected and interim values of total contributions, cycle times and total costs. The results show that firms with low adjusted demand rates contribute more under asymmetric information. However full information case performs better.

In Chapter 5, we study a three-stage joint replenishment game. In this model, we assume that the intermediary is also a decision maker. We analyze each stage and give the conditions for equilibrium. We show that the subgame-perfect equilibrium cycle time is not unique. We find the minimum and maximum cycle times attainable under equilibrium. Even though efficient cycle time is in between minimum and maximum equilibrium cycle time, it is not always an equilibrium outcome. For symmetric joint replenishment environments, we show that whether efficient cycle time is a subgame-perfect equilibrium outcome depends only on the number of firms and is independent of all other parameters of the environment.

In Chapter 6, we consider finding a mechanism that would allocate the joint ordering costs to multiple firms based on their reported independent order frequencies. We first show that there is no direct mechanism that simultaneously achieves efficiency, incentive compatibility, individual rationality and budget-balance. We then propose a two–parameter mechanism that would take the reported frequencies of the firms and determine the joint replenishment frequency using the first parameter and allocate the order cost using the second parameter. We show that unless the parameter governing the cost allocation is zero efficiency cannot be achieved. For the single parameter mechanism, we find the equilibrium share levels and corresponding total cost. We finally investigate the effect of this parameter on equilibrium behavior. We show that properly adjusting this parameter leads to mechanisms that are better than suggested earlier in the literature in terms of fairness and efficiency.

In Chapter 7, we examine the Lippman and McCardle (1997) model of competitive newsboys under private cost information. The stochastic market demand is initially allocated between two firms and any unsatisfied demand is reallocated to the rival firm. We show the existence and uniqueness of pure strategy Bayesian Nash equilibrium and characterize its structure. The equilibrium conditions have an interesting recursive structure that enables an easy computation of the equilibrium order quantities. A firm's equilibrium order quantity increases with a stochastic increase in the total industry demand or with an increase in his initial allocation of the total industry demand. Finally, for a special case with uniform demand and linear market shares, we provide full characterization of the equilibrium, corresponding payoffs and comparative statics.

For Chapters 3 to 6, a number of important research directions remain to be explored to build an analytical foundation that captures the details of realistic operational management settings. First group of research directions include explorations of alternative mechanisms such as sequential contributions, alternative message spaces (e.g. contribution schedules $r(T)$ stating a firm's contribution as a function of joint cycle time), and alternative outcome functions mapping the firms' messages to the joint cycle time and cost allocation decisions. A second group includes extensions along the environment dimension include models that allow minor setup costs, and models that incorporate uncertainty. Also, considering different coalition structures for the firms is also an interesting dimension since we can have models where firms with similar attributes can form coalitions for better performance.

For Chapter 7, the newsvendor duopoly with asymmetric information, certain extensions of the current model are relatively straightforward and not likely to change the structure of the equilibrium qualitatively. For instance, allowing more than two levels for the unit costs, will lead to more complicated but qualitatively similar equilibrium characterization in that many of the claims, the recursive structure of the equilibrium order quantities, and, particularly, the behavior of the highest–cost type will remain valid with this extension. However, continuous type distributions may also be considered. Alternative specifications for the key structural elements of the current model – e.g., the the nature of information asymmetry, and the structure of the market and firm demands – span a number of interesting classes of models we intend to explore in the future. Among these are models of newsboy oligopoly, and models that allow arbitrary statistical dependence in firm demands, and in cost structures.

# APPENDIX A

# A Private Contributions Game For Joint Replenishment

## A.1 Proof of Proposition 3.2:

For part 1 we provide detailed arguments. Parts 2-5 of the proposition are obtained by straightforward algebraic manipulations. For part 1 we provide detailed arguments. Parts 2-5 of the proposition are obtained by straightforward algebraic manipulations.

1. Given other firms' contributions, each firm $j$'s optimization problem is

$$\min_{r_j} \; r_j + \frac{\kappa \alpha_j}{2 \sum_{k \in N} r_k} \quad \text{subject to} \quad r_j \geq \delta. \tag{A.1}$$

Karush–Kuhn–Tucker conditions for optimality are given by

$$1 - \frac{\kappa \alpha_j}{2(\sum_{k \in N} r_k)^2} - \mu_j = 0, \tag{A.2}$$

$$\mu_j (r_j - \delta) = 0, \tag{A.3}$$

$$\mu_j \geq 0, \tag{A.4}$$

$$r_j \geq \delta. \tag{A.5}$$

By definition, any strategy profile $\boldsymbol{r}^* = (r_1^*, \ldots, r_n^*)$ is a Nash equilibrium if and only if it is a solution to (A.2)-(A.5) for $j = 1, \ldots, n$. Conditions (A.2)-(A.5) ensure that there is at least one firm $i$ such that $r_i^* > \delta$ and $\mu_i = 0$. Because, if $r_j^* = \delta$ for all $j$, we would have $\mu_j = 1 - \frac{\kappa\alpha_j}{2n^2\delta^2}$ for all $j$. Since $\mu_j \geq 0$ for all $j$, this requires that $\delta \geq \sqrt{\kappa\alpha_j/2}/n$ for all $j$, which contradicts with the fact that $\delta < \sqrt{\kappa\underline{\alpha}/2}/n$, as $\underline{\alpha} \leq \alpha_j$ for all $j$. Using (A.2),

$$\mu_i = 1 - \frac{\kappa\alpha_i}{2(\sum_{k\in N} r_k^*)^2} = 0. \tag{A.6}$$

Now firm $i$ that satisfies (A.6) has to belong to the set $L(N)$. Otherwise, for any $k$ with $\alpha_k > \alpha_i$, we have $\mu_k < 0$ violating condition (A.4). Conditions (A.6) and (A.2) also show that $\mu_j > 0$ for all $j \in N \setminus L(N)$. Therefore, using (A.3), we have, for $j \in N \setminus L(N)$,

$$r_j^* = \delta,$$

and, for $j \in L(N)$,

$$r_j^* \geq \delta \text{ and } \sum_{i\in L(N)} r_j^* = \sqrt{\frac{\kappa\alpha_n}{2}} - (n-\ell)\delta.$$

The following chain of inequalities show that the conditions on the vector $(r_{n-\ell+1}^*, \ldots r_n^*)$ are consistent:

$$\delta < \frac{\sqrt{\kappa\underline{\alpha}/2}}{n} \leq \frac{\sqrt{\kappa\alpha_1/2}}{n} \leq \frac{\sqrt{\kappa\alpha_n/2}}{n} < \frac{\sqrt{\kappa\alpha_n/2}}{n-1} \leq \frac{\sqrt{\kappa\alpha_n/2}}{n-\ell}. \tag{A.7}$$

2. Straightforward from 1.(b).

3. In equilibrium, aggregate contributions from the $n$ firms is $\sum_{i\in N} r_i^* = \sum_{i\in N\setminus L(N)} r_i^* + \sum_{i\in L(N)} r_i^* = (n-\ell)\delta + \sqrt{\kappa\alpha_n/2} - (n-\ell)\delta = \sqrt{\kappa\alpha_n/2} = R_n^d$. The resulting cycle time is $T_N^g = \tau_N(\boldsymbol{r}^*) = \kappa/\sum_{i\in N} r_i^* = \kappa\sqrt{\kappa\alpha_n/2} = \sqrt{2\kappa/\alpha_n} = T_n^d$.

4. Since equilibrium total replenishment cost for the $n$ firms is equal to the aggregate contributions, the claim in 4.(a) follows from 3.(a) above. The claim in 4.(b) results from straightforward substitution and summing over $n$ firms. Part 4.(c) is obtained by summing the results in parts (a) and (b) and combining terms.

5. Part 5.(a) follows from 1.(a) directly for $j \in N \setminus L(N)$. For a firm $j \in L(N)$, we note that his maximum equilibrium contribution is obtained when other firms in $N$ each contribute $\delta$. Part 5.(b) follows from substituting the equilibrium cycle time in the expression for $j$'s holding cost rate. Part 5.(c) follows from adding the replenishment and holding costs in parts 5.(a) and 5.(b). $\square$

## A.2 Proof of Proposition 3.4:

In order to show that the strategy "out" is weakly dominated by the strategy "in" for all firms, we re–write (3.20) separating the replenishment and holding cost components in $\Phi_j(\boldsymbol{z})$

$$\Phi_j(z_j, \boldsymbol{z}_{-j}) = \begin{cases} \sqrt{\kappa\alpha_j/2} \ + \ \sqrt{\kappa\alpha_j/2} & \text{if } z_j = \text{“out” or } M(\boldsymbol{z}) = \{j\} \\ \delta \ + \ (\sqrt{\alpha_j/\alpha_{i_m}})\sqrt{\kappa\alpha_j/2} & \text{if } j \in M(\boldsymbol{z}) \setminus L(M(\boldsymbol{z})) \\ \hat{R} + \ \sqrt{\kappa\alpha_{i_m}/2}, \text{ such that } \hat{R} \in [\underline{R}, \overline{R}] & \text{if } j \in L(M(\boldsymbol{z})). \end{cases} \tag{A.8}$$

where $[\underline{R}, \overline{R}]$ with $\underline{R} = \delta$ and $\overline{R} = \sqrt{\kappa\alpha_{i_m}/2} - (m-1)\delta$ denotes the closed interval for the replenishment cost of the players in $L(M(\boldsymbol{z}))$.

First, from the first line of (A.8), the strategy "out" yields a payoff independent of other firms' participation strategies. Also from the first line, the two strategies give the same payoff when $z_k =$ "out" for all $k \in N \setminus \{j\}$. To see that participation yields a strictly better total cost for player $j$ in all other cases, we compare lines 2 and 3 to line 1. Take any $\boldsymbol{z}$ with $z_j =$"in". If $\boldsymbol{z}_j$ is such that $j \in M(\boldsymbol{z}) \setminus L(M(\boldsymbol{z}))$, then $\alpha_j < \alpha_{i_m}$. In this case, both components of firm $j$'s total cost in line 2 are strictly less than their counterparts in line 1, because $\delta < \sqrt{\kappa\alpha_j/2}$ by (A.7), and $\sqrt{\alpha_j/\alpha_{i_m}} < 1$. If $\boldsymbol{z}_j$ is such that $j \in L(M(\boldsymbol{z}))$, then $\alpha_j = \alpha_{i_m}$. In this case, holding cost component of firm $j$'s total cost is $\sqrt{\kappa\alpha_{i_m}/2}$ for both strategies. However, the worst possible replenishment cost from participating, $\sqrt{\kappa\alpha_{i_m}/2} - (m-1)\delta$, is strictly lower than the stand–alone replenishment cost, $\sqrt{\kappa\alpha_{i_m}/2}$. Thus, $z_j =$"in" weakly dominates strategy $z_j =$"out". Since this is true for all firms, the claim follows. $\square$

# APPENDIX B

# Private Contributions Game For Joint Replenishment with Asymmetric Information

## B.1 Proof of Proposition 4.2

In order to prove the existence we invoke the following proposition by Meirowitz [36]:

**Proposition B.1.** *A Bayesian game has a pure strategy BNE if for each $j \in N$*

1. *$A$ and $\Theta$ are nonempty, convex and compact subsets of Euclidean space.*
2. *$u_j(\boldsymbol{r}, \boldsymbol{\alpha}) = -\phi_j(\boldsymbol{r}, \boldsymbol{\alpha})$ is continuous.*
3. *For every $\boldsymbol{\alpha}$ and measurable function $f^{n-1}(\boldsymbol{\alpha_{-j}})$*
   $$U_j(r_j(\alpha_j), \boldsymbol{r^*_{-j}}) = -\Phi_j(r_j(\alpha_j), \boldsymbol{r^*_{-j}}) = \int_{A^{n-1}} u_j(r_j, \boldsymbol{r^*_{-j}}, \alpha_j, \boldsymbol{\alpha_{-j}}) f^{n-1}(\boldsymbol{\alpha_{-j}}) d\boldsymbol{\alpha_{-j}}$$
   *is strictly quasi–concave in $r_j$.*
4. *For every $\varepsilon_j > 0$ there exists some constant $\delta_j$ s.t. if*
   $$r^*_j(\alpha_j) \in arg \max_{r_j \in R_j} \left\{ \int_{A^{n-1}} u_j(r_j, \boldsymbol{r^*_{-j}}, \alpha_j, \boldsymbol{\alpha_{-j}}) f^{n-1}(\boldsymbol{\alpha_{-j}}) d\boldsymbol{\alpha_{-j}} \right\}$$

*for some* , $\boldsymbol{r^*_{-j}}(\boldsymbol{\alpha_{-j}})$ *then* $sup_{\{(\alpha_j,\alpha'_j)\in A:|\alpha_j-\alpha'_j|<\delta_j\}}|r^*_j(\alpha_j)-r^*_j(\alpha'_j)|<\varepsilon_j$.

5. $f^{\,n-1}(\boldsymbol{\alpha_{-j}})$ *is continuous.*

Now, $A$ and $\Theta$ are both closed, bounded and consists of a single interval by assumption. Thus, they are nonempty, convex and compact. Thus, condition (1) is satisfied. Similarly, the belief function $f^{\,n-1}(\boldsymbol{\alpha_{-j}})$ is continuous since it is the multiplication of continuous probability density functions $f(\alpha)$ by assumption so condition (5) is satisfied.

Assume that

$$u_j(\boldsymbol{r},\boldsymbol{\alpha}) = -\phi_j(\boldsymbol{r},\boldsymbol{\alpha}) = -\frac{1}{2}\kappa\alpha_j\frac{1}{r_j+\sum_{i\neq j} r_i} - r_j$$

which is the negative of our cost function. This assumption is necessary for a utility maximization model.

$u_j(\boldsymbol{r},\boldsymbol{\alpha})$ is obviously continuous.

The first order condition for $U_j(r_j(\alpha_j),\boldsymbol{r^*_{-j}}) = -\Phi_j(r_j(\alpha_j),\boldsymbol{r^*_{-j}})$ is:

$$\frac{\partial U_j}{\partial r_j} = -\frac{\partial \Phi_j}{\partial r_j} = \frac{1}{2}\,\kappa\,\alpha_j\int_{A^{n-1}}\frac{1}{(r_j+\sum_{i\neq j} r^*_i(\alpha_i))^2}f^{\,n-1}(\boldsymbol{\alpha_{-j}})d\boldsymbol{\alpha_{-j}}\;-\;1\quad\text{for all } j.$$

and the second order condition is:

$$\frac{\partial^2 U_j}{\partial r_j^2} = -\frac{\partial^2 \Phi_j}{\partial r_j^2} = -\,\kappa\,\alpha_j\int_{A^{n-1}}\frac{1}{(r_j+\sum_{i\neq j} r^*_i(\alpha_i))^3}f^{\,n-1}(\boldsymbol{\alpha_{-j}})d\boldsymbol{\alpha_{-j}} < 0\quad\text{for all } j.$$

Thus, $U_j$ is strictly concave in $r_j$ which implies $U_j$ is strictly quasi–concave in $r_j$ which shows that condition (3) is satisfied.

The only remaining condition is (4). In order to prove it, we take two different types $\alpha_j$ and $\alpha'_j$ for firm $j$ and use the difference between their respective first order conditions:

$$\begin{aligned}
\frac{2}{\kappa\alpha_j} - \frac{2}{\kappa\alpha'_j} &= \int_{A^{n-1}} \frac{1}{(r^*_j(\alpha_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2} f^{\,n-1}(\boldsymbol{\alpha}_{-\mathbf{j}})d\boldsymbol{\alpha}_{-\mathbf{j}} \\
&\qquad - \int_{A^{n-1}} \frac{1}{(r^*_j(\alpha'_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2} f^{\,n-1}(\boldsymbol{\alpha}_{-\mathbf{j}})d\boldsymbol{\alpha}_{-\mathbf{j}} \\
&= \int_{A^{n-1}} \frac{1}{(r^*_j(\alpha_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2} - \frac{1}{(r^*_j(\alpha'_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2} f^{\,n-1}(\boldsymbol{\alpha}_{-\mathbf{j}})d\boldsymbol{\alpha}_{-\mathbf{j}} \\
&= \int_{A^{n-1}} \frac{(r^*_j(\alpha'_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2 - (r^*_j(\alpha_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2}{(r^*_j(\alpha_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2 (r^*_j(\alpha'_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2} f^{\,n-1}(\boldsymbol{\alpha}_{-\mathbf{j}})d\boldsymbol{\alpha}_{-\mathbf{j}} \\
&= [r^*_j(\alpha'_j) - r^*_j(\alpha_j)] \int_{A^{n-1}} \frac{r^*_j(\alpha_j) + r^*_j(\alpha'_j) + 2\sum_{i\neq j} r^*_i(\alpha_i)}{(r^*_j(\alpha_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2 (r^*_j(\alpha'_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2} f^{\,n-1}(\boldsymbol{\alpha}_{-\mathbf{j}})d\boldsymbol{\alpha}_{-\mathbf{j}} \\
&= [r^*_j(\alpha'_j) - r^*_j(\alpha_j)] \int_{A^{n-1}} \frac{1}{(r^*_j(\alpha_j) + \sum_{i\neq j} r^*_i(\alpha_i))(r^*_j(\alpha'_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2} \\
&\qquad + \frac{1}{(r^*_j(\alpha_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2 (r^*_j(\alpha'_j) + \sum_{i\neq j} r^*_i(\alpha_i))} f^{\,n-1}(\boldsymbol{\alpha}_{-\mathbf{j}})d\boldsymbol{\alpha}_{-\mathbf{j}}
\end{aligned}$$

Using the first order conditions and $r_j(\alpha_j) \leq \overline{r}$ for all $j \in N$, we can write

$$\begin{aligned}
&\int_{A^{n-1}} \frac{1}{(r^*_j(\alpha'_j) + \sum_{i\neq j} r^*_i(\alpha_i))(r^*_j(\alpha_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2} f^{\,n-1}(\boldsymbol{\alpha}_{-\mathbf{j}})d\boldsymbol{\alpha}_{-\mathbf{j}} \geq \\
&\qquad \int_{A^{n-1}} \frac{1}{2\overline{r}(r^*_j(\alpha_j) + \sum_{i\neq j} r^*_i(\alpha_i))^2} f^{\,n-1}(\boldsymbol{\alpha}_{-\mathbf{j}})d\boldsymbol{\alpha}_{-\mathbf{j}} = \frac{1}{\overline{r}\kappa\alpha_j}
\end{aligned}$$

A similar result can be obtained for $\alpha'_j$. Taking the absolute values on both sides, we have

$$|\frac{2}{\kappa\alpha_j} - \frac{2}{\kappa\alpha'_j}| \geq |r^*_j(\alpha'_j) - r^*_j(\alpha_j)| \left( \frac{1}{\overline{r}\kappa\alpha_j} + \frac{1}{\overline{r}\kappa\alpha'_j} \right)$$

Rearranging the terms and using $\alpha_j \geq \underline{\alpha} > 0$ and $\alpha'_j \geq \underline{\alpha} > 0$, we obtain:

$$|\alpha'_j - \alpha_j| \geq |r^*_j(\alpha'_j) - r^*_j(\alpha_j)| \left( \frac{1}{2\overline{r}}(\alpha'_j + \alpha_j) \right) \geq |r^*_j(\alpha'_j) - r^*_j(\alpha_j)| \left( \frac{\underline{\alpha}}{\overline{r}} \right)$$

Now, assume that $\varepsilon_j > 0$ and let $\delta = \varepsilon(\underline{\alpha}/\overline{r})$. Then $|\alpha'_j - \alpha_j| < \delta_j$ implies $|r^*_j(\alpha'_j) - r^*_j(\alpha_j)|(\underline{\alpha}/\overline{r}) < \delta_j$ or $|r^*_j(\alpha'_j) - r^*_j(\alpha_j)| < \varepsilon_j$. This is true for all $\alpha_j$ and $\alpha'_j$ so condition (4) is satisfied which shows that all the conditions of the proposition are met. $\square$

## B.2 Proof of Proposition 4.3

A given firm $j$'s optimization problem can be formulated as follows, if the firm is type $\alpha_j$

$$\min_{r_j} \qquad \phi_j(r_j(\alpha_j), \mathbf{r_{-j}}) = \int_{A^{n-1}} \frac{\kappa}{\sum_{j\in N} r_j(\alpha_j)} f^{\,n-1}(\boldsymbol{\alpha}_{-\mathbf{j}}) d\boldsymbol{\alpha}_{-\mathbf{j}} + r_j(\alpha_j) \tag{B.1}$$

$$\text{subject to} \qquad r_j(\alpha_j) \geq 0. \tag{B.2}$$

Following the same arguments in the proof of Proposition 3.2, one gets (4.6) as a necessary condition. It can easily be verified that the function $\phi_j$ is convex in $r_j$ which shows that these conditions are also sufficient. $\square$

# APPENDIX C

# A Three-Stage Game for Joint Replenishment With Private Contributions

## C.1 Proof of Proposition 5.1:

The first and second conditions are necessary and sufficient for equilibrium in stages 3 and 2, respectively, as shown in (5.2) and (5.7). The third condition concerns the equilibrium in stage 1. First consider condition *iii (a).* In equilibrium, each firm's price bid should be his best response to others' price bids, i.e., a firm cannot decrease his costs by unilaterally changing his price. First, note that no firm will change his bid price to induce the RSP to select his threshold cycle time as the new replenishment cycle time because inducing his threshold cycle time would lead to a cost equal to his stand–alone cost and he may achieve a lower cost level if the threshold cycle time of another firm is chosen. Moreover, for any $i$ with $r_i^* = 0$, firm $j$ cannot force the RSP to select $T^* = \tau_i^*(0)$ since the RSP cannot improve his payoff by doing so. Similarly, if $\tau_i^*(r_i^*) > 2T_j^d$ then firm $j$ has no incentive to induce $\tau_i^*(r_i^*)$, as this will lead to higher costs than his stand–alone cost. Now consider a firm $j$ and the case in which he is served by the RSP ($\tau_j^*(r_j^*) \geq T^*(\mathbf{r}^*)$). His cost is therefore $r_j^* + \frac{1}{2}\alpha_j T^*(\mathbf{r}^*)$. First, since $T^*(\mathbf{r}^*)$ is optimal for RSP we have

$$\sum_{k|\tau_k^*(r_k^*)\geq\tau_i^*(r_i^*)} r_k^*\lambda_k - \kappa/\tau_i^*(r_i^*) \leq \sum_{k|\tau_k^*(r_k^*)\geq T^*(\mathbf{r}^*)} r_k^* - \kappa/T^*(\mathbf{r}^*) \qquad \forall i \in N. \quad \text{(C.1)}$$

Thus there does not exist a $r_j'$ that will lead to $T^*(r_j', \mathbf{r}^*_{-\mathbf{j}}) = \tau_i^*(r_i^*)$ for which $T^*(\mathbf{r}^*) \leq \tau_i^*(r_i^*) < \tau_j^*(r_j^*)$.

In order for firm $j$ not to deviate from $r_j^*$, each $r_j'$ should lead to a higher cost, so we should have

$$r_j' + \frac{1}{2}\alpha_j T^*(r_j', \mathbf{r}^*_{-\mathbf{j}}) - r_j^* - \frac{1}{2}\alpha_j T^*(\mathbf{r}^*) \geq 0, \quad \text{for all } r_j' \neq r_j^*. \qquad \text{(C.2)}$$

However, $T^*(r_j', \mathbf{r}^*_{-\mathbf{j}})$ is equal to a threshold cycle time i.e., $\tau_i^*(r_i^*)$ for some $i \in N \setminus \{j\}$ as stated in condition *ii*. This implies that,

$$T^*(r_j', \mathbf{r}^*_{-\mathbf{j}}) = \tau_i^*(r_i^*) \Rightarrow r_j' + \sum_{k\neq j, \tau_k^*(r_k^*)\geq\tau_i^*(r_i^*)} r_k^* \geq \kappa/\tau_i^*(r_i^*). \qquad \text{(C.3)}$$

Combining (C.2) and (C.3), we have

$$r_j^* + \sum_{k\neq j, \tau_k^*(r_k^*)\geq\tau_j^*(r_i^*} r_k^* \leq \frac{1}{2}\alpha_j(\tau_i^*(r_i^*) - T^*(\mathbf{r}^*)) + \kappa/\tau_i^*(r_i^*), \qquad \text{(C.4)}$$

for all $i, j$ for which $\tau_i^*(r_i^*) \geq \tau_j^*(r_j^*) \geq T^*(\mathbf{r}^*)$ or $\tau_j^*(r_j^*) \geq T^*(\mathbf{r}^*) \geq \tau_i^*(r_i^*)$.

Now consider a firm which is not served by the RSP ($\tau_j^*(r_j^*) < T^*(\mathbf{r}^*)$). His cost is $\sqrt{2\kappa\alpha_j}$. Since $r_j^*$ is a best response to $\mathbf{r}^*_{-\mathbf{j}}$, we have,

$$r_j' + \frac{1}{2}\alpha_j T^*(r_j', \mathbf{r}^*_{-\mathbf{j}}) - \sqrt{2\kappa\alpha_j} \geq 0, \quad \text{for all } r_j' \neq r_j^*. \qquad \text{(C.5)}$$

Combining (C.5) and (C.3), we have,

$$\sum_{k\neq j, \tau_k^*(r_k^*)\geq\tau_i^*(r_i^*)} r_k^* \leq \frac{1}{2}\alpha_j\tau_i^*(r_i^*) - \sqrt{2\kappa\alpha_j} + \kappa/\tau_i^*(r_i^*) \quad \forall i, j : \tau_j^*(r_j^*) < T^*(\mathbf{r}^*). \qquad \text{(C.6)}$$

Combining (C.4) and (C.6) yields condition *iii (a)*. Now consider condition *iii (b)*. If $T^*(\mathbf{r}^*) = \tau_{n+1}^*(r_{n+1}^*) = \infty$ then condition *iii.(b)* is trivially satisfied.

Now, let $T^*(\mathbf{r}^*) = \tau_\ell^*(r_\ell^*)$ for some $\ell \in N$. Then, $\omega_k^* = 1$ for all $k \in N$ with $\tau_k^*(r_k^*) \geq \tau_\ell^*(r_\ell^*)$. In this case, $\sum_{j \in N} r_j^* \, \omega_j^*(\mathbf{r}^*, T(\mathbf{r}^*)) \geq \kappa / T^*(\mathbf{r}^*)$ should be satisfied since otherwise, the RSP would not select $\tau_\ell^*(r_\ell^*)$ as $T^*(\mathbf{r}^*)$. However, $\sum_{j \in N} r_j^* \, \omega_j^*(\mathbf{r}^*, T(\mathbf{r}^*)) > \kappa / T^*(\mathbf{r}^*)$ cannot be true, since any firm $k$ with $\tau_k^*(r_k^*) \geq \tau_\ell^*(r_\ell^*)$ and $r_k^* > 0$ can decrease his price without changing the cycle time and incur a smaller cost. $\square$

## C.2 Proof of Proposition 5.2:

We first show (5.9). The minimum cycle time for coalition $S$ can be found by solving the following optimization problem:

$$\min \quad T \tag{C.7}$$

$$\text{subject to} \quad T - \frac{\kappa}{\sum_{j \in S} r_j} = 0, \tag{C.8}$$

$$T - 2\sqrt{\frac{2\kappa}{\alpha_j}} + 2\frac{r_j}{\alpha_j} \leq 0, \quad \forall j \in S \tag{C.9}$$

$$-r_j \leq 0, \quad \forall j \in S \tag{C.10}$$

If we relax the constraint (C.10), the Karush–Kuhn–Tucker conditions for optimality are given as follows:

$$1 + \nu + \sum_{j \in S} \mu_j = 0, \tag{C.11}$$

$$\frac{\nu\kappa}{(\sum_{j \in S} r_j)^2} + 2\frac{\mu_j}{\alpha_j} = 0, \ \forall j \in S \tag{C.12}$$

$$T - \frac{\kappa}{\sum_{j \in S} r_j} = 0, \tag{C.13}$$

$$T - 2\sqrt{\frac{2\kappa}{\alpha_j}} + 2\frac{r_j}{\alpha_j} \leq 0, \quad \forall j \in S \tag{C.14}$$

$$\mu_j \left( T - 2\sqrt{\frac{2\kappa}{\alpha_j}} + 2\frac{r_j}{\alpha_j} \right) = 0, \quad \forall j \in S \tag{C.15}$$

$$\mu_j \geq 0, \quad \forall j \in S \tag{C.16}$$

Due to (C.11) and since $\mu_j \geq 0$, we have $\nu \leq -1$. Since $\nu < 0$, $\kappa > 0$ and $\alpha_j > 0$ we have $\mu_j > 0$ due to (C.12). This shows that the constraint (C.9) in the relaxed problem is always binding in an optimal solution. Therefore it suffices to solve

$$\begin{aligned} T - \frac{\kappa}{\sum_{j\in S} r_j} &= 0, && \text{(C.17)} \\ T - 2\sqrt{\frac{2\kappa}{\alpha_j}} + 2\frac{r_j}{\alpha_j} &= 0, \quad \forall j \in S && \text{(C.18)} \end{aligned}$$

to find the minimum equilibrium cycle time for the relaxed problem. The system (C.17-C.18) leads to a quadratic equation for $T$. The smaller root of this equation is

$$\tilde{T}_S^{\min} = \sqrt{2\kappa}\frac{\sum_{j\in S}\sqrt{\alpha_j} - \sqrt{(\sum_{j\in S}\sqrt{\alpha_j})^2 - \sum_{j\in S}\alpha_j}}{\sum_{j\in S}\alpha_j}. \tag{C.19}$$

Now, denote $a_j = \sqrt{\alpha_j}$. We can rewrite $\tilde{T}_S^{\min}$ as

$$\begin{aligned} \tilde{T}_S^{\min} &= \sqrt{2\kappa}\frac{\sum_{j\in S} a_j - \sqrt{2\sum_{i,j\in S,\, i\neq j} a_i a_j}}{(\sum_{j\in S} a_j)^2 - 2\sum_{i,j\in S,\, i\neq j} a_i a_j} \\ &= \sqrt{2\kappa}\frac{1}{\sum_{j\in S} a_j + \sqrt{2\sum_{i,j\in S,\, i\neq j} a_i a_j}}, && \text{(C.20)} \end{aligned}$$

which is clearly smaller than $2\sqrt{\frac{2\kappa}{\alpha_j}} = 2\frac{\sqrt{2\kappa}}{a_j}$ for all $j \in S$ and thus, $\tilde{T}_S^{\min}$ also satisfies the constraint (C.10). This shows that $\tilde{T}_S^{\min}$ is also the optimal solution of the original problem as given in (5.9).

We now turn to proving (5.10). The maximum cycle time for coalition $S$ can be found by solving the same optimization problem in (C.7-C.10), but this time using a maximization objective. Again, if we relax the constraint (C.10), we now have the same Karush–Kuhn–Tucker conditions (C.12–C.16), but now (C.11) is replaced with

$$-1 + \nu + \sum_{j\in S}\mu_j = 0. \tag{C.21}$$

Using (C.21) and (C.12) and denoting $c_j = \frac{\kappa}{(\sum_{j\in S} r_j)^2}$, we have

$$c_j(1 - \sum_{i\in S}\mu_i) + 2\frac{\mu_j}{\alpha_j} = 0, \quad \forall j \in S. \tag{C.22}$$

Summing (C.22) over the set $S$, we get

$$\sum_{j\in S} c_j \left(1 - \sum_{j\in S} \mu_j\right) + 2\sum_{j\in S} \frac{\mu_j}{\alpha_j} = 0, \tag{C.23}$$

which shows that $\sum_{j\in S} \mu_j > 1$ since $h_j > 0$ and $\mu_j \geq 0,\ \ \forall j \in S$. Using this and (C.22), we have $\mu_j > 0,\ \ \forall j \in S$ which shows that the constraint (C.8) in the relaxed maximization problem is also binding for all $j \in S$. Thus, the maximum cycle time for the relaxed problem is the larger root of the system (C.17–C.18) which is given by

$$\tilde{T}_S^{\max} = \sqrt{2\kappa}\frac{\sum_{j\in S}\sqrt{\alpha_j} + \sqrt{(\sum_{j\in S}\sqrt{\alpha_j})^2 - \sum_{j\in S}\alpha_j}}{\sum_{j\in S}\alpha_j}. \tag{C.24}$$

The solution in (C.24) satisfies the constraint (C.10) if $\tilde{T}_S^{\max} \leq 2\sqrt{\frac{2\kappa}{\alpha_j}}$ for all $j \in S$. Otherwise, $\min_{j\in S} 2\sqrt{\frac{2\kappa}{\alpha_j}}$ is the maximum cycle time for coalition $S$ as (C.9) with $j = arg\min_{j\in S}\sqrt{\frac{2\kappa}{\alpha_j}}$ is the tightest constraint for the maximization problem. Then the solution in (5.10) follows. $\square$

## C.3 Proof of Proposition 5.3:

For (5.11), we need to show that $T^{\min} = \min_{S\subset N} T_S^{\min} = T_N^{\min}$. Denoting $a_j = \sqrt{\alpha_j}$, we have the expression (C.20) which is clearly minimized when $S = N$, which results in (5.11).

In order to find the maximum cycle time, we need to solve $T^{\max} = \max_{S\subset N} T_S^{\max}$ which potentially requires to search over $2^n - 1$ subsets of $N$. However, we next show that it suffices to search over the subsets $\mathcal{P}_k$, $k = 1, 2, \ldots, n$. This is equivalent to showing

$$T_{\mathcal{P}_k}^{\max} = \max_{S\subset N, |S|=k} T_S^{\max}. \tag{C.25}$$

We will show (C.25) by induction. First $T_S^{\max}$ can be shown to be equal to

$$T_S^{\max} = \min\left\{\min_{j\in S}\left\{2\frac{\sqrt{2\kappa}}{a_j}\right\}, \sqrt{2\kappa}\frac{1}{\sum_{j\in S} a_j - \sqrt{2\sum_{i,j\in S,\, i\neq j} a_i a_j}}\right\}. \tag{C.26}$$

The statement (C.25) is certainly true for $k=1$. Assume (C.25) is true for $k$. Denoting $\Gamma(\mathcal{P}_k) = \sum_{i,j\in\mathcal{P}_k,\, i\neq j} a_i a_j$, we can write the maximum cycle time for the set $\mathcal{P}_k \cup \{\ell\}$ as

$$T_{\mathcal{P}_k\cup\{\ell\}}^{\max} = \min\left\{2\frac{\sqrt{2\kappa}}{a_\ell}, \sqrt{2\kappa}\frac{1}{\sum_{j\in\mathcal{P}_k} a_j + a_\ell - \sqrt{2\,\Gamma(\mathcal{P}_k) + 2a_\ell\sum_{j\in\mathcal{P}_k} a_j}}\right\}. \tag{C.27}$$

Now let us consider the two cases: $\ell = k+1$ and $\ell = h$ for some $h > k+1$. If $T_{\mathcal{P}_k\cup\{k+1\}}^{\max} = T_{\mathcal{P}_{k+1}}^{\max} = 2\frac{\sqrt{2\kappa}}{a_{k+1}}$. Then, $T_{\mathcal{P}_{k+1}}^{\max} \geq T_{\mathcal{P}_k\cup\{h\}}^{\max}$ since $2\frac{\sqrt{2\kappa}}{a_{k+1}} \geq 2\frac{\sqrt{2\kappa}}{a_h} \geq T_{\mathcal{P}_k\cup\{h\}}^{\max}$. If $T_{\mathcal{P}_{k+1}}^{\max}$ is realized at the second part of the minimum expression in (C.26), then we can write the difference of denominators of the second parts of the minimum expression in (C.26) for $\ell = k+1$ and $\ell = h > k+1$ as

$$a_{k+1} - a_h + \sqrt{2\,\Gamma(\mathcal{P}_k) + 2a_h\sum_{j\in\mathcal{P}_k} a_j} - \sqrt{2\,\Gamma(\mathcal{P}_k) + 2a_{k+1}\sum_{j\in\mathcal{P}_k} a_j}. \tag{C.28}$$

Multiplying with $\sqrt{2\,\Gamma(\mathcal{P}_k) + 2a_h\sum_{j\in\mathcal{P}_k} a_j} + \sqrt{2\,\Gamma(\mathcal{P}_k) + a_{k+1}\sum_{j\in\mathcal{P}_k} a_j})$, we get

$$(a_{k+1}-a_h)\left(\sqrt{2\,\Gamma(\mathcal{P}_k) + 2a_h\sum_{j\in\mathcal{P}_k} a_j}) + \sqrt{2\,\Gamma(\mathcal{P}_k) + 2a_{k+1}\sum_{j\in\mathcal{P}_k} a_j}\right) + 2(a_h - a_{k+1})\sum_{j\in\mathcal{P}_k} a_j, \tag{C.29}$$

which is non–positive since $a_h \geq a_{k+1} \geq \max_{j\in\mathcal{P}_k} a_j$ and $2\,\Gamma(\mathcal{P}_k) + 2a_h\sum_{j\in\mathcal{P}_k} a_j \geq 2\,\Gamma(\mathcal{P}_k) + 2a_{k+1}\sum_{j\in\mathcal{P}_k} a_j \geq (\sum_{j\in\mathcal{P}_k} a_j)^2$. This also leads to $T_{\mathcal{P}_k\cup\{k+1\}}^{\max} \geq T_{\mathcal{P}_k\cup\{\ell\}}^{\max}$, for all $\ell > k+1$ which completes the induction.

Now consider the case where $T_{\mathcal{P}_k}^{\max} = 2\frac{\sqrt{2\kappa}}{a_k}$. This means that $p_k = 0$, and the constraint (C.9) for firm $k$ sets an upper bound for $T_{\mathcal{P}_k}^{\max}$. Since $p_k = 0$, removing firm $k$ will only result in a larger maximum cycle time as this would remove the upper bound. Thus $T_{\mathcal{P}_k}^{\max} \leq T_{\mathcal{P}_{k-1}}^{\max}$. Thus it is sufficient to search over $\tilde{T}_{\mathcal{P}_k}^{\max}$ for $T^{\max}$. However, since $\tilde{T}_{\mathcal{P}_k}^{\max}$ is not monotone in $k$, one needs to find the $k$ value that maximizes $\tilde{T}_{\mathcal{P}_k}^{\max}$ which leads to (5.12). $\square$

## C.4 Proof of Proposition 5.4:

Let $r^* = (r_1^*, ..., r_n^*)$ be the first–stage contributions that yield the efficient cycle time $T_N^c = \sqrt{\frac{2\kappa}{n\alpha}}$ as a SPE outcome. By part (iii.b) of Proposition 5.1, $\sum_{j \in N} r_j^* = \frac{\kappa}{T_N^c}$. By part (ii) of Proposition 5.1, there should be at least one firm $j$ with $\tau_j^*(r_j^*) = T_N^c$. Let $M$ be the set of firms $j$ such that $\tau_j^*(r_j^*) = T_N^c$. For each $j \in M$, firm $j$'s first–stage bid $r_j^*$ must be such that his contribution to the order cost is $r_j^* = \sqrt{2\kappa\alpha}(1 - 1/2\sqrt{n})$.

For a firm in $N \setminus M$, say firm $i$, his first stage contribution $r_i^*$ must be such that $\tau_i^*(r_i^*) > T_N^c$, i.e., firm $i$ selects "In" in stage 3. If this is not the case, that is, if $\tau_i^*(r_i^*) < T_N^c$, firm $i$'s optimal action in stage 3 is "Out" and, thus, his total cost is his stand–alone cost. But this cannot be part of a SPE since firm $i$ can improve his payoff by bidding 0 in stage 1 and selecting "In" in stage 3, getting a better cycle time $T_N^c < T_i^d$ than his stand–alone cycle time at a lower replenishment cost. Therefore, in a SPE outcome that yields $T_N^c$, all firms must be served.

In summary, by relabeling firm indices, we have

$$T^*(\mathbf{r}^*) = T_N^c = \tau_1^*(r_1^*) = ... = \tau_m^*(r_m^*) < \tau_{m+1}^*(r_{m+1}^*) \leq ... \leq \tau_n^*(r_n^*), \quad \text{(C.30)}$$

and

$$\pi_{RSP}(\mathbf{r}^*, T_N^c, \boldsymbol{\omega}^*(\mathbf{r}^*, T_N^c)) = 0 \geq \quad \text{(C.31)}$$

$$\pi_{RSP}(\mathbf{r}^*, \tau_{m+1}^*(r_{m+1}^*), \boldsymbol{\omega}^*(\mathbf{r}^*, \tau_{m+1}^*(r_{m+1}^*))) \geq \quad \text{(C.32)}$$

$$... \geq \pi_{RSP}(\mathbf{p}^*, \tau_n^*(r_n^*), \boldsymbol{\omega}^*(\mathbf{r}^*, \tau_n^*(r_n^*))). \quad \text{(C.33)}$$

Second, no firm $i \in N$ has any incentive to deviate from $r_i^*$ to a higher bid. This is because, for a firm $i \in M$, a bid $\hat{r}_i > r_i^*$ (hence a lower $\tau_i^*(\hat{r}_i)$ leads the RSP to select $\tau_i^*(\hat{r}_i)$ or a strictly higher cycle time than $T_N^c$ since the contributions from other firms are no longer sufficient to cover the cost of $T_N^c$. In either case, firm $i$'s total cost is unaffected. Thus, no firm $i \in M$ can improve his total cost by increasing his bid above $r_i^*$. For a firm $i \in N \setminus M$, deviating to higher bid $\hat{r}_i > r_i^*$, can lead to one of two possible cases depending on $\hat{r}_i$: the RSP's stage–two cycle time response may be $\tau_i^*(\hat{r}_i)$ or $T_N^c$. In either case, firm $i$'s total cost becomes worse. Therefore, no firm in $N \setminus M$ has an incentive to deviate to a higher bid either.

As the RSP's revenue just covers the order cost in a SPE outcome, we can also rule out profitable deviations to a lower price for firm $n$ since the RSP's response to a lower bid $\hat{r}_n < r_n^*$ by firm $n$ would be to select $\tau_{n+1} = \infty$.

We start with identifying the conditions for ruling out possible deviations to lower bids for firms in $M$. Define $\tau_{N\setminus M} = \min_{j \in N\setminus M} \tau_j^*(r_j^*)$. For a firm $j \in M$ inducing $\tau_{N\setminus M}$ is better than inducing any $\tau_i^*(r_i)$ for $i \in N \setminus M$ by lemma C.1. Thus, maximizing $\tau_{N\setminus M}$ minimizes the deviation possibility of firms in $M$. To obtain the maximum possible $\tau_{N\setminus M}$ value, we must assign as many firms $j \in N$ as possible to set $M$ and distribute the remaining payment equally to the firms in $N \setminus M$. The former follows since increasing the number of firms in $M$ decreases the remaining payment to firms in $N \setminus M$ thus increases $\tau_{N\setminus M}$ and latter follows since dividing the remaining payment equally leads to equal $\tau$ values for all firms in $N \setminus M$ thus maximizes $\tau_{N\setminus M}$.

Define $b(n)$ as the solution to

$$\frac{\kappa}{T_N^c} - b(n)\big(\sqrt{2\kappa\alpha}(1 - \frac{1}{2\sqrt{n}})\big) = 0. \tag{C.34}$$

Straightforward substitutions yield $b(n) = n/(2\sqrt{n} - 1)$. Since $b(n)$ is not necessarily an integer, the maximum possible number of firms in set $M$ is $\lfloor b(n) \rfloor$, that is, $M = \{1, .., \lfloor b(n) \rfloor\}$. Since $\lfloor b(n) \rfloor$ firms in $M$ each contribute $\sqrt{2\kappa\alpha}(1 - 1/2\sqrt{n})$, the firms in $N \setminus M$ need to contribute a total payment of $(\sqrt{\frac{n\kappa\alpha}{2}} - \lfloor b(n) \rfloor(\sqrt{2\kappa\alpha}(1 - 1/2\sqrt{n})))\rfloor)$ to satisfy condition (iii)-b in Proposition 5.1. Dividing this total payment equally, each of the $n - \lfloor b(n) \rfloor$ firms in $N \setminus M$, contributes $r_j^* = (\sqrt{\frac{n\kappa\alpha}{2}} - \lfloor b(n) \rfloor(\sqrt{2\kappa\alpha}(1 - 1/2\sqrt{n})))/(n - \lfloor b(n) \rfloor)$ and the corresponding threshold cycle time is

$$\tau_j^*(r_j^*) = \tau_{N\setminus M} = 2\sqrt{\frac{2\kappa}{\alpha}} - 2\frac{(\sqrt{\frac{n\kappa\alpha}{2}} - \lfloor b(n) \rfloor(\sqrt{2\kappa\alpha}(1 - \frac{1}{2\sqrt{n}}))}{(n - \lfloor b(n) \rfloor)\alpha}. \tag{C.35}$$

For any firm $j \in M$ a deviation to $\tau_{N\setminus M}$ is not profitable if and only if condition (iii)-a of Proposition 5.1 is satisfied, i.e.,

$$r_j^* + \sum_{k\in N\setminus M} r_k^* \leq \frac{1}{2}\alpha(\tau_{N\setminus M} - T_N^c) + \frac{\kappa}{\tau_{N\setminus M}}. \tag{C.36}$$

Substituting for $r_j^*$ and $r_k^*$ on the left–hand side of (C.36) and rearranging, we get

$$\sqrt{2\kappa\alpha} + \sqrt{\frac{n\kappa\alpha}{2}} - \lfloor b(n)\rfloor(\sqrt{2\kappa\alpha}(1-\frac{1}{2\sqrt{n}})) - \frac{1}{2}\alpha\tau_{N\setminus M} - \frac{\kappa}{\tau_{N\setminus M}} \leq 0. \tag{C.37}$$

Plugging the value of $\tau_{N\setminus M}$ from (C.35) in (C.37) and rearranging terms, we obtain

$$\begin{aligned}(\sqrt{\frac{n\kappa\alpha}{2}} - \lfloor b(n)\rfloor\sqrt{2\kappa\alpha}(1-\frac{1}{2\sqrt{n}}))(1+\frac{1}{n-\lfloor b(n)\rfloor})& \\ -\frac{\kappa}{2\sqrt{\frac{2\kappa}{\alpha}-2\frac{(\sqrt{\frac{n\kappa\alpha}{2}}-\lfloor b(n)\rfloor(\sqrt{2\kappa\alpha}(1-\frac{1}{2\sqrt{n}}))}{(n-\lfloor b(n)\rfloor)\alpha}}} &\leq 0.\end{aligned}$$

Finally, using $\sqrt{\frac{n\kappa\alpha}{2}} = b(n)(\sqrt{2\kappa\alpha}(1-1/2\sqrt{n}))$ and dividing both sides by $\sqrt{2\kappa\alpha}$ we obtain

$$(b(n)-\lfloor b(n)\rfloor)(1-\frac{1}{2\sqrt{n}}))(1+\frac{1}{n-\lfloor b(n)\rfloor}) - \frac{1}{4(1-\frac{b(n)-\lfloor b(n)\rfloor}{n-\lfloor b(n)\rfloor}(1-\frac{1}{2\sqrt{n}}))} \leq 0. \tag{C.38}$$

□

**Lemma C.1.** *For a firm $j \in M$ inducing $\tau_{N\setminus M}$ is better than inducing any $\tau_i^*(r_i)$ for $i \in N\setminus M$.*

Proof: In order to prove the lemma we only need to show that deviating to the firm with the smallest cycle time is always better than deviating to the others. Assume that firms $k,\ell \in N\setminus M$ satisfy $\tau_k(r_k) < \tau_\ell(r_\ell)$ so $r_k > r_\ell$ and further assume that $\tau_k = \tau_{N\setminus M}$ and $\tau_\ell < \tau_i$ for all $i \in N\setminus(M\bigcup k)$. Thus,

$$\sum_{i\in N\setminus M|\tau_i^*(r_i)\geq\tau_k^*} r_i = (\sqrt{\frac{n\kappa\alpha}{2}} - \lfloor b(n)\rfloor(\sqrt{2\kappa\alpha}(1-1/2\sqrt{n})))\rfloor) \tag{C.39}$$

and

$$\sum_{i\in N\setminus M|\tau_i^*(r_i)\geq\tau_\ell^*} r_i = (\sqrt{\frac{n\kappa\alpha}{2}} - \lfloor b(n)\rfloor(\sqrt{2\kappa\alpha}(1-1/2\sqrt{n})))\rfloor) - r_k \tag{C.40}$$

If we can show that deviating to $k$ is always better than deviating to $\ell$ we are done. For $\tau_k > T_k^d$ this is definitely true since the cost function is increasing in cycle time.

Consider the deviation of firm $j \in M$ and let $r_j'$ and $r_j''$ be the necessary contributions of firm $j$ for the RSP to select $\tau_k$ and $\tau_\ell$ respectively. For the firm $j$ to prefer $\tau_\ell$ we must have:

$$r_j'' + \frac{1}{2}\alpha\tau_\ell < r_j' + \frac{1}{2}\alpha\tau_k$$

using $r_k = \sqrt{2\kappa\alpha} - \frac{1}{2}\alpha\tau_k$ and $r_\ell = \sqrt{2\kappa\alpha} - \frac{1}{2}\alpha\tau_\ell$ we obtain

$$0 < r_k - r_\ell < r_j' - r_j'' \tag{C.41}$$

Now, for the RSP to select $\tau_\ell$ we should have

$$r_j'' + \sum_{i \neq j | \tau_i^*(r_i) \geq \tau_\ell^*} r_i - \frac{\kappa}{\tau_\ell^*(r_\ell)} \geq 0$$

and

$$r_j' + \sum_{i \neq j | \tau_i^*(r_i) \geq \tau_k^*} r_i - \frac{\kappa}{\tau_k^*(r_k)} < r_j'' + \sum_{i \neq j | \tau_i^*(r_i) \geq \tau_\ell^*} r_i - \frac{\kappa}{\tau_\ell^*(r_\ell)}$$

By plugging (C.39) and (C.40) and making the obvious simplifications we obtain:

$$r_j' - \frac{\kappa}{\tau_k^*(r_k)} < r_j'' - r_k - \frac{\kappa}{\tau_\ell^*(r_\ell)}$$

Rearranging the terms and adding (C.41) yield:

$$r_k - r_\ell < r_j' - r_j'' < \frac{\kappa}{\tau_k^*(r_k)} - r_k - \frac{\kappa}{\tau_\ell^*(r_\ell)}$$

Further rearranging yields:

$$2r_k - \frac{\kappa}{\tau_k^*(r_k)} < r_\ell - \frac{\kappa}{\tau_\ell^*(r_\ell)}$$

which forms a contradiction. $\square$

# APPENDIX D

# Design and Analysis of Mechanisms for Decentralized Joint Replenishment

## D.1 Proof of Proposition 6.1

In order for firm $i$ to be truth telling its cost should be minimized at $\hat{\alpha}_i = \alpha_i$ when all other firms are truth telling, i.e, $\hat{\alpha}_j = \alpha_j$ for $j \in N \setminus \{i\}$. The first order condition in this case is:

$$\left.\frac{dC_i^{dm}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}{d\hat{\alpha}_i}\right|_{\hat{\alpha}_i=\alpha_i} = -\frac{1}{2}\sqrt{\frac{\kappa}{2}}\alpha_i(\sum_{j\in N}\alpha_j)^{-3/2} + \sqrt{\frac{1}{2\kappa}}\sigma'(\alpha_i, \boldsymbol{\alpha}_{-i})(\sum_{j\in N}\alpha_j)^{1/2}$$
$$+\frac{1}{2}\sqrt{\frac{1}{2\kappa}}\sigma(\alpha_i, \boldsymbol{\alpha}_{-i})(\sum_{j\in N}\alpha_j)^{-1/2} = 0$$

where $\sigma'(\alpha_i, \boldsymbol{\alpha}_{-i}) = \frac{d\sigma(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}{d\hat{\alpha}_i}|_{\hat{\alpha}_i=\alpha_i}$.

Rearranging the terms yield

$$\sqrt{\frac{1}{2\kappa}}\sigma'(\alpha_i, \boldsymbol{\alpha}_{-i})(\sum_{j\in N}\alpha_j)^{1/2} + \frac{1}{2}\sqrt{\frac{1}{2\kappa}}\sigma(\alpha_i, \boldsymbol{\alpha}_{-i})(\sum_{j\in N}\alpha_j)^{-1/2} = \frac{1}{2}\sqrt{\frac{\kappa}{2}}\alpha_i(\sum_{j\in N}\alpha_j)^{-3/2}.$$

Multiplying both sides by $\sqrt{2\kappa}$, we get

$$\sigma'(\alpha_i, \boldsymbol{\alpha}_{-i})(\sum_{j\in N} \alpha_j)^{1/2} + \frac{1}{2}\sigma(\alpha_i, \boldsymbol{\alpha}_{-i})(\sum_{j\in N} \alpha_j)^{-1/2} = \frac{\kappa}{2}\alpha_i(\sum_{j\in N} \alpha_j)^{-3/2}. \tag{D.1}$$

The left hand side of the equation (D.1) is the derivative of $\sigma(\alpha_i, \boldsymbol{\alpha}_{-i})(\sum_{j\in N} \alpha_j)^{1/2}$ with respect to $\alpha_i$. Thus the first order condition takes the form

$$\frac{d[\sigma(\alpha_i, \boldsymbol{\alpha}_{-i})(\sum_{j\in N} \alpha_j)^{1/2}]}{d\alpha_i} = \frac{\kappa}{2}\alpha_i(\sum_{j\in N} \alpha_j)^{-3/2}.$$

Solving the differential equation gives

$$\sigma(\alpha_i, \boldsymbol{\alpha}_{-i})(\alpha_i + \sum_{j\neq i} \alpha_j)^{1/2} = \frac{\kappa}{2}\int_{t=0}^{\alpha_i} t(t + \sum_{j\neq i} \alpha_j)^{-3/2} dt + c = \left.\frac{\kappa(t + 2\sum_{j\neq i} \alpha_j)}{(t + \sum_{j\neq i} \alpha_j)^{1/2}}\right|_{t=0}^{t=\alpha_i} + c.$$

The cost share function $a$ is then given by

$$\sigma(\alpha_i, \boldsymbol{\alpha}_{-i}) \quad = \quad \frac{\kappa(\alpha_i + 2\sum_{j\neq i} \alpha_j)}{(\alpha_i + \sum_{j\neq i} \alpha_j)} - \frac{2\kappa(\sum_{j\neq i} \alpha_j)^{1/2}}{(\alpha_i + \sum_{j\neq i} \alpha_j)^{1/2}} + c.$$

The integration constant $c$ can be found by using the fact $\sigma(0, \boldsymbol{\alpha}_{-i}) = 0$.

$$\sigma(0, \boldsymbol{\alpha}_{-i}) \quad = \quad \frac{\kappa(0 + 2\sum_{j\neq i} \alpha_j)}{(0 + \sum_{j\neq i} \alpha_j)} - \frac{2\kappa(\sum_{j\neq i} \alpha_j)^{1/2}}{(0 + \sum_{j\neq i} \alpha_j)^{1/2}} + c = c,$$

which shows that the constant $c = 0$. Therefore, the cost share function is given by

$$\sigma(\alpha_i, \boldsymbol{\alpha}_{-i}) \quad = \quad \frac{\kappa(\alpha_i + 2\sum_{j\neq i} \alpha_j)}{(\alpha_i + \sum_{j\neq i} \alpha_j)} - \frac{2\kappa(\sum_{j\neq i} \alpha_j)^{1/2}}{(\alpha_i + \sum_{j\neq i} \alpha_j)^{1/2}}.$$

Using the share function $\sigma(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$ we can find the cost of firm $i$ as

$$C_i^{dm}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) = \sqrt{\frac{\kappa}{2}}\frac{(\alpha_i + \hat{\alpha}_i + 2\sum_{j\neq i} \alpha_j)}{(\hat{\alpha}_i + \sum_{j\neq i} \alpha_j)^{1/2}} - \sqrt{2\kappa}(\sum_{j\neq i} \alpha_j)^{1/2}.$$

In order to show that incentive compatibility constraint is globally satisfied, we first need to prove that the cost function is strictly quasi–convex. By definition, $C_i^{dm}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$ is quasi–convex in $\hat{\alpha}_i$ if the set $Q(b) = \{\hat{\alpha}_i : C_i^{dm}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) \leq b\}$ is a convex set for any $b \in \mathbb{R}$. Now, take any $\{\alpha_i', \alpha_i''\} \in Q(b)$. First, since $\alpha_i' \in Q(b)$, we have $C_i^{dm}(\alpha_i', \boldsymbol{\alpha}_{-i}) \leq b$, or

$$\begin{aligned}
& \sqrt{\frac{\kappa}{2}} \frac{(\alpha_i + \alpha_i' + 2\sum_{j\neq i}\alpha_j)}{(\alpha_i' + \sum_{j\neq i}\alpha_j)^{1/2}} - \sqrt{2\kappa}(\sum_{j\neq i}\alpha_j)^{1/2} \leq b \\
\Rightarrow \quad & \sqrt{\frac{\kappa}{2}} \frac{(\alpha_i + \alpha_i' + 2\sum_{j\neq i}\alpha_j)}{(\alpha_i' + \sum_{j\neq i}\alpha_j)^{1/2}} \leq b + \sqrt{2\kappa}(\sum_{j\neq i}\alpha_j)^{1/2} \\
\Rightarrow \quad & \frac{(\alpha_i + \alpha_i' + 2\sum_{j\neq i}\alpha_j)}{(\alpha_i' + \sum_{j\neq i}\alpha_j)^{1/2}} \leq \sqrt{\frac{2}{\kappa}} \left( b + \sqrt{2\kappa}(\sum_{j\neq i}\alpha_j)^{1/2} \right).
\end{aligned}$$

Denoting $\sqrt{\frac{2}{\kappa}} \left( b + \sqrt{2\kappa}(\sum_{j\neq i}\alpha_j)^{1/2} \right) = \bar{b}$ we have

$$\frac{(\alpha_i + \alpha_i' + 2\sum_{j\neq i}\alpha_j)}{(\alpha_i' + \sum_{j\neq i}\alpha_j)^{1/2}} \leq \bar{b}. \tag{D.2}$$

Similarly, $C_i^{dm}(\alpha_i'', \boldsymbol{\alpha}_{-i}) < b$ and using the same steps we obtain:

$$\frac{(\alpha_i + \alpha_i'' + 2\sum_{j\neq i}\alpha_j)}{(\alpha_i'' + \sum_{j\neq i}\alpha_j)^{1/2}} \leq \bar{b}. \tag{D.3}$$

For $Q(b)$ to be a convex set, for any $\lambda \in [0, 1]$, $\lambda\, \alpha_i' + (1-\lambda)\, \alpha_i'' \in Q(b)$ should be satisfied, i.e., we must have

$$\frac{(\alpha_i + \lambda\alpha_i' + (1-\lambda)\alpha_i'' + 2\sum_{j\neq i}\alpha_j)}{(\lambda\alpha_i' + (1-\lambda)\alpha_i'' + \sum_{j\neq i}\alpha_j)^{1/2}} \leq \bar{b}. \tag{D.4}$$

Now, we first take the square of (D.2) and (D.3) and then multiply the first one with $\lambda$ and second one with $1-\lambda$. Finally we sum them up to get:

$$\lambda(\alpha_i + \alpha_i' + 2\sum_{j\neq i}\alpha_j)^2 + (1-\lambda)(\alpha_i + \alpha_i'' + 2\sum_{j\neq i}\alpha_j)^2 \leq (\bar{b})^2(\lambda\alpha_i' + (1-\lambda)\alpha_i'' + \sum_{j\neq i}\alpha_j).$$

Simplifying the left hand side yields

$$(\alpha + \lambda\alpha_i' + (1-\lambda)\alpha_i'' + 2\sum_{j\neq i}\alpha_j)^2 + (\alpha_i' - \alpha_i'')^2\lambda(1-\lambda) \leq (\bar{b})^2(\lambda\alpha_i' + (1-\lambda)\alpha_i'' + \sum_{j\neq i}\alpha_j),$$

which can be written as:

$$\frac{(\alpha + \lambda\alpha_i' + (1-\lambda)\alpha_i'' + 2\sum_{j\neq i}\alpha_j)^2}{(\lambda\alpha_i' + (1-\lambda)\alpha_i'' + \sum_{j\neq i}\alpha_j)} + \frac{(\alpha_i' - \alpha_i'')^2\lambda(1-\lambda)}{(\lambda\alpha_i' + (1-\lambda)\alpha_i'' + \sum_{j\neq i}\alpha_j)} \leq (\bar{b})^2. \tag{D.5}$$

Since the second term on the left hand side of equation (D.5) is non–negative we have

$$\frac{(\alpha + \lambda\alpha_i' + (1-\lambda)\alpha_i'' + 2\sum_{j\neq i}\alpha_j)^2}{(\lambda\alpha_i' + (1-\lambda)\alpha_i'' + \sum_{j\neq i}\alpha_j)} \leq (\bar{b})^2.$$

This shows that (D.4) is satisfied and $\lambda\,\alpha_i' + (1-\lambda)\,\alpha_i'' \in Q(b)$. Thus, $Q(b)$ is a convex set and $C_i^{dm}(\hat{\alpha}_i, \alpha_{-i})$ is a quasi–convex function.

The second order derivative of $C_i^{dm}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$ at the point $\hat{\alpha}_i = \alpha_i$ is

$$\left.\frac{d^2 C_i^{dm}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}{d\hat{\alpha}_i^2}\right|_{\hat{\alpha}_i=\alpha_i} = \frac{1}{2}\sqrt{\frac{\kappa}{2}}(\alpha_i + \sum_{j\neq i}\alpha_j)^{-3/2}$$

which is always positive. Therefore $C_i^{dm}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$ is convex at $\alpha_i$. Since $C_i^{dm}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$ is also strictly quasi–convex, it has a global minimum at $\hat{\alpha}_i = \alpha_i$. Thus, global incentive compatibility is satisfied.

Next, we show that the individual rationality constraint is satisfied. For this purpose, we need to show that a firm should not have a better payoff if it rejects the mechanism. In other words, its decentralized cost should be higher that the cost it would obtain through the mechanism. At the point $\hat{\alpha}_i = \alpha_i$ the cost function takes the form

$$C_i^{dm}(\alpha_i, \boldsymbol{\alpha}_{-i}) = \sqrt{\frac{\kappa}{2}}\alpha_i(\alpha_i + \sum_{j\neq i}\alpha_j)^{-1/2} + \sqrt{\frac{\kappa}{2}}\frac{(\alpha_i + 2\sum_{j\neq i}\alpha_j)}{(\alpha_i + \sum_{j\neq i}\alpha_j)^{1/2}} - \sqrt{2\kappa}(\sum_{j\neq i}\alpha_j)^{1/2}.$$

Further manipulation yields:

$$C_i^{dm}(\alpha_i, \boldsymbol{\alpha}_{-i}) = \sqrt{2\kappa}(\alpha_i + \sum_{j\neq i}\alpha_j)^{1/2} - \sqrt{2\kappa}(\sum_{j\neq i}\alpha_j)^{1/2},$$

which is definitely less than $C_i^d = \sqrt{2\kappa\alpha_i}$ by the concavity of the square root function. Thus, individual rationality is satisfied.

Our final step is to determine whether the budget–balance condition is satisfied, i.e., whether the sum of the contributions determined through the direct mechanism is enough to finance the fixed order cost $\kappa$. Total contributions collected from the firms is

$$\begin{aligned}
\sum_{i\in N}\sigma(\alpha_i,\boldsymbol{\alpha}_{-i}) &= \sum_{i\in N}\frac{\kappa(\alpha_i+2\sum_{j\neq i}\alpha_j)}{(\sum_{j\in N}\alpha_j)}-\sum_{i\in N}\frac{2\kappa(\sum_{j\neq i}\alpha_j)^{1/2}}{(\sum_{j\in N}\alpha_j)^{1/2}}\\
&= (2n-1)\kappa-\sum_{i\in N}\frac{2\kappa(\sum_{j\neq i}\alpha_j)^{1/2}}{(\sum_{j\in N}\alpha_j)^{1/2}}\\
&= (2n-1)\kappa-\sum_{i\in N}\frac{2\kappa(\sum_{j\neq i}\alpha_j)^{1/2}(\sum_{j\in N}\alpha_j)^{1/2}}{(\sum_{j\in N}\alpha_j)}\\
&< (2n-1)\kappa-\sum_{i\in N}\frac{2\kappa(\sum_{j\neq i}\alpha_j)}{(\sum_{j\in N}\alpha_j)}=(2n-1)\kappa-2(n-1)\kappa=\kappa,
\end{aligned}$$

which shows that sum of the allocations is smaller than $\kappa$ so budget–balance constraint is not satisfied. □

## D.2 Proof of Proposition 6.2

Summing (6.3) over all $i\in N$ yields:

$$\begin{aligned}
\sum_{i\in N}f_i^2 &= \left(\sum_{i\in N}s_i^\theta\right)^{-1}\left(\sum_{i\in N}s_i^\xi\right)^{2/\xi}\left(\theta\sum_{i\in N}s_i^\xi\sum_{i\in N}s_i^{\theta-\xi}+\sum_{i\in N}s_i^\theta-\theta\sum_{i\in N}s_i^{2\theta-\xi}\sum_{i\in N}s_i^\xi\left(\sum_{i\in N}s_i^\theta\right)^{-1}\right)\\
&= \left(\sum_{i\in N}s_i^\theta\right)^{-2}\left(\sum_{i\in N}s_i^\xi\right)^{2/\xi}\left(\theta\sum_{i\in N}s_i^\xi\sum_{i\in N}s_i^{\theta-\xi}\sum_{i\in N}s_i^\theta+\left(\sum_{i\in N}s_i^\theta\right)^2-\theta\sum_{i\in N}s_i^{2\theta-\xi}\sum_{i\in N}s_i^\xi\right)\\
&= \left(\sum_{i\in N}s_i^\theta\right)^{-2}\left(\sum_{i\in N}s_i^\xi\right)^{2/\xi}\left(\theta\left(2\sum_{i\neq j}s_i^\theta s_j^\theta+\sum_{i\neq j}s_i^{\theta+\xi}s_j^{\theta-\xi}+\sum_{i\neq j,j\neq k}s_i^\theta s_j^\xi s_k^{\theta-\xi}\right)+\left(\sum_{i\in N}s_i^\theta\right)^2\right)\\
&= \left(\sum_{i\in N}s_i^\xi\right)^{2/\xi}\left(\left(\sum_{i\in N}s_i^\theta\right)^{-2}\theta\left(2\sum_{i\neq j}s_i^\theta s_j^\theta+\sum_{i\neq j}s_i^{\theta+\xi}s_j^{\theta-\xi}+\sum_{i\neq j,j\neq k}s_i^\theta s_j^\xi s_k^{\theta-\xi}\right)+1\right).
\end{aligned}$$

Dividing both sides by $\left(\sum_{i\in N}s_i^\xi\right)^{2/\xi}$ leads to the desired result. □

## D.3 Proof of Proposition 6.4

For the single parameter case the second derivative of the payoff function is as follows:

$$\left.\frac{\partial C_i^{1p}(\hat{\mathbf{s}})}{\partial \hat{s}_i}\right|_{\hat{\mathbf{s}}=\mathbf{s}} = \begin{aligned}[t] & -\kappa f_i^2(\xi-1)s_i^{\xi-2}\left(\sum_{j\in N} s_j^\xi\right)^{-\frac{1}{\xi}-1} - \kappa f_i^2(-1-\xi)s_i^{2\xi-2}\left(\sum_{j\in N} s_j^\xi\right)^{-\frac{1}{\xi}-2} \\ & +\kappa\xi(\xi-1)s_i^{\xi-2}\left(\sum_{j\in N} s_j^\xi\right)^{\frac{1}{\xi}-1} + \kappa\xi(1-\xi)s_i^{2\xi-2}\left(\sum_{j\in N} s_j^\xi\right)^{\frac{1}{\xi}-2} \\ & +\kappa(1-\xi)(2\xi-1)s_i^{2\xi-2}\left(\sum_{j\in N} s_j^\xi\right)^{\frac{1}{\xi}-2} + \kappa(1-\xi)(1-2\xi)s_i^{3\xi-2}\left(\sum_{j\in N} s_j^\xi\right)^{\frac{1}{\xi}-3}. \end{aligned}$$

Factoring the expression, we obtain

$$\left.\frac{\partial C_i^{1p}(\hat{\mathbf{s}})}{\partial \hat{s}_i}\right|_{\hat{\mathbf{s}}=\mathbf{s}} = \kappa s_i^{\xi-2}\left(\sum_{j\in N} s_j^\xi\right)^{-\frac{1}{\xi}-3}$$

$$\left(f_i^2\left(\sum_{j\in N} s_j^\xi\right)\left((1-\xi)\left(\sum_{j\in N} s_j^\xi\right)+(1+\xi)s_i^\xi\right)+(\xi-1)\left(\sum_{j\in N} s_j^\xi\right)^{\frac{2}{\xi}}\left(\sum_{j\neq i} s_j^\xi\right)\left(\xi\left(\sum_{j\in N} s_j^\xi\right)-(2\xi-1)s_i^\xi\right)\right).$$

For convexity, the argument above should be non–negative. Using this, we get the following condition:

$$(\xi-1)\left(\sum_{j\in N} s_j^\xi\right)^{\frac{2}{\xi}-1}\left(\sum_{j\neq i} s_j^\xi\right)\left(\xi\left(\sum_{j\in N} s_j^\xi\right)-(2\xi-1)s_i^\xi\right) \geq f_i^2\left((\xi-1)\left(\sum_{j\in N} s_j^\xi\right)-(1+\xi)s_i^\xi\right).$$

Using (6.8) and (6.10) in the inequality, we get

$$\begin{aligned} &(\xi-1)\left(\left(\frac{\sum_{j\in N} f_j^2}{\xi(n-1)+1}\right)^{\frac{\xi}{2}}\right)^{\frac{2}{\xi}-1}\left(\left(\frac{\sum_{j\in N} f_j^2}{\xi(n-1)+1}\right)^{\frac{\xi}{2}}-\frac{\xi\sum_{j\in N} f_j^2-((n-1)\xi+1)f_i^2}{((n-1)\xi+1)(\xi-1)}\left(\frac{\sum_{j\in N} f_j^2}{(n-1)\xi+1}\right)^{\xi/2-1}\right) \\ &\qquad\left(\xi\left(\frac{\sum_{j\in N} f_j^2}{\xi(n-1)+1}\right)^{\frac{\xi}{2}}-(2\xi-1)\frac{\xi\sum_{j\in N} f_j^2-((n-1)\xi+1)f_i^2}{((n-1)\xi+1)(\xi-1)}\left(\frac{\sum_{j\in N} f_j^2}{(n-1)\xi+1}\right)^{\xi/2-1}\right) \\ &\qquad\geq f_i^2\left((\xi-1)\left(\frac{\sum_{j\in N} f_j^2}{\xi(n-1)+1}\right)^{\frac{\xi}{2}}-(1+\xi)\frac{\xi\sum_{j\in N} f_j^2-((n-1)\xi+1)f_i^2}{((n-1)\xi+1)(\xi-1)}\left(\frac{\sum_{j\in N} f_j^2}{(n-1)\xi+1}\right)^{\xi/2-1}\right). \end{aligned}$$

Simplifying the terms yields

$$\begin{aligned} &(\xi-1)\left(\frac{((n-1)\xi+1)f_i^2-\sum_{j\in N} f_j^2}{((n-1)\xi+1)(\xi-1)}\right)\left(\frac{(2\xi-1)((n-1)\xi+1)f_i^2-\xi^2\sum_{j\in N} f_j^2}{((n-1)\xi+1)(\xi-1)}\right) \\ &\qquad\geq f_i^2\left(\frac{(\xi+1)((n-1)\xi+1)f_i^2-(3\xi-1)\sum_{j\in N} f_j^2}{((n-1)\xi+1)(\xi-1)}\right). \end{aligned}$$

Next, we consider the cases for $\xi > 1$ and $\xi < 1$ separately since the equilibrium conditions for both cases are different. For $\xi > 1$ the condition is:

$$\left(((n-1)\xi+1)\, f_i^2 - \sum_{j\in N} f_j^2\right)\left((2\xi-1)\,((n-1)\xi+1)\, f_i^2 - \xi^2 \sum_{j\in N} f_j^2\right)$$
$$\geq f_i^2\,((n-1)\xi+1)\left((\xi+1)\,((n-1)\xi+1)\, f_i^2 - (3\xi-1)\sum_{j\in N} f_j^2\right).$$

Denote $E = ((n-1)\xi+1)\, f_i^2$ and $F = \sum_{j\in N} f_j^2$ and the condition simplifies to:

$$\begin{aligned}
& (E-F)\left((2\xi-1)E - \xi^2 F\right) \geq E\,((\xi+1)E - (3\xi-1)F) \\
\Rightarrow\quad & (2\xi-1)E^2 - \xi^2 EF - (2\xi-1)EF + \xi^2F^2 \geq (\xi+1)E^2 - (3\xi-1)EF \\
\Rightarrow\quad & (\xi-2)E^2 - (\xi^2-\xi)EF + \xi^2F^2 \geq 0 \\
\Rightarrow\quad & (\xi F - (\xi-2)E)(\xi F - E) \geq 0.
\end{aligned}$$

By Proposition 6.3, $\xi F - E > 0$. Thus we must have:

$$\xi \sum_{j\in N} f_j^2 - (\xi-2)\,((n-1)\xi+1)\, f_i^2 \geq 0.$$

For $\xi < 1$,

$$\begin{aligned}
& (E-F)\left((2\xi-1)E - \xi^2 F\right) \leq E\,((\xi+1)E - (3\xi-1)F) \\
\Rightarrow\quad & (2\xi-1)E^2 - \xi^2 EF - (2\xi-1)EF + \xi^2F^2 \leq (\xi+1)E^2 - (3\xi-1)EF \\
\Rightarrow\quad & (\xi-2)E^2 - (\xi^2-\xi)EF + \xi^2F^2 \leq 0 \\
\Rightarrow\quad & (\xi F - (\xi-2)E)(\xi F - E) \leq 0.
\end{aligned}$$

Again by Proposition 6.3, $\xi F - E < 0$. Thus we must have:

$$\xi \sum_{j\in N} f_j^2 - (\xi-2)\,((n-1)\xi+1)\, f_i^2 \geq 0,$$

which is same as what get for $\xi > 1$. $\square$

# APPENDIX E

# Newsvendor Duopoly With Asymmetric Information

## E.1 Proof of Theorem 7.1

First, define $\mathcal{Y}_2 = -\mathcal{Q}_2$ so that $\mathcal{Q}_1 \times \mathcal{Y}_2$ is a lattice (This order change is necessary to form a supermodular game). Moreover, let $t_1 = -c_1$, $t_2 = c_2$ and define effective demand functions as $R_i : t_j \to \Re$. Then for

$$\pi_1(Q_1, y_2, t_1, t_2) = E[\min\{R_1(t_2), Q_1\}] + t_1 Q_1,$$
$$\pi_2(Q_1, y_2, t_1, t_2) = E[\min\{R_2(t_1), -y_2\}] + t_2 y_2.$$

The supermodularity and continuity of these functions and the increasing differences in $(Q_1, y_2)$ are proved in [30]. The only thing remains is to show that $\pi_1$ has increasing differences in $(Q_1, t_1)$ and $\pi_2$ has increasing differences in $(y_2, t_2)$ (Again, $\pi_i$ is not directly dependent on the type of firm $j$. Hence, increasing differences for $(Q_1, t_2)$ and $(y_2, t_1)$ are trivially satisfied.). Let $\varsigma_1(t_1) = \pi_1(Q_1', y_2, t_1, t_2) - \pi_1(Q_1, y_2, t_1, t_2)$ where $Q_1' \geq Q_1$ for given $y_2, t_2$. Then

$$\varsigma_1(t_1) = E[\min\{R_1(t_2), Q_1'\}] - E[\min\{R_1(t_2), Q_1\}] + t_1[Q_1' - Q_1].$$

Define $t_1'$ such that $t_1' \geq t_1$. It follows that $\varsigma(t_1') - \varsigma(t_1) = [t_1' - t_1][Q_1' - Q_1] \geq 0$. Thus $\pi_1$ has increasing differences in $(Q_1, t_1)$. Similarly, $\varsigma_2(t_2) = \pi_2(Q_1, y_2', t_1, t_2) -$

$\pi_2(Q_1, y_2, t_1, t_2)$ where $y_2' \geq y_2$ for given $Q_1, t_1$. Then

$$\varsigma_2(t_2) = E[\min\{R_2(t_1), -y_2'\}] - E[\min\{R_2(t_1), -y_2\}] + t_2[y_2' - y_2].$$

Define $t_2'$ such that $t_2' \geq t_2$. It follows that $\varsigma(t_2') - \varsigma(t_2) = [t_2' - t_2][y_2' - y_2] \geq 0$. Thus $\pi_2$ has increasing differences in $(y_2, t_2)$. Since our priors over the types are independent, the condition for priors to be increasing with respect to types is trivially satisfied. The existence of pure strategy Nash equilibrium follows. □

## E.2 Proof of Claims 7.1–7.4 and Lemmas 7.2–7.5

**Proof of Claim 7.1:** Let $\min\{s^{-1}(x), \hat{s}^{-1}(y)\} = \hat{s}^{-1}(y)$, i.e., $s^{-1}(x) \geq \hat{s}^{-1}(y)$. Suppose, to get a contradiction, that $s^{-1}(x) < x + y$. Then $x < s(x+y) = x + y - \hat{s}(x+y)$, since $\hat{s}(x) = x - s(x)$. Thus, $\hat{s}(x+y) < y$, and $x + y < \hat{s}^{-1}(y)$. Therefore, $s^{-1}(x) < \hat{s}^{-1}(y)$, yielding a contradiction. The second inequality is established similarly. □

**Proof of Claim 7.2:** $Pr(D_1 \geq Q_{1H}^o) = Pr(D \geq s^{-1}(Q_{1H}^o)) = c_{1H} \leq c_{2H} = Pr(D \geq \hat{s}^{-1}(Q_{2H}^o))$.
Hence, $s^{-1}(Q_{1H}^o) \geq \hat{s}^{-1}(Q_{2H}^o)$. □

**Proof of Claim 7.3:** (i) (1) evaluated at $Q_{1L} = Q_{1H}^*$ is positive.

(ii) Similar argument with (i). □

**Proof of Claim 7.4:** We will only show (i). Other cases are established similarly. Evaluating the left hand side of (1) at $Q_{1L} = Q_{1L}^o$ gives:

$$q\,Pr(D_1 + (D_2 - Q_{2H})^+ \geq Q_{1L}^o) + (1-q)Pr(D_1 + (D_2 - Q_{2L})^+ \geq Q_{1L}^o) - c_{1L}$$
$$\geq q\,Pr(D_1 \geq Q_{1L}^o) + (1-q)Pr(D_1 \geq Q_{1L}^o) - c_{1L} = Pr(D_1 \geq Q_{1L}^o) - c_{1L} = 0$$

Thus, $Q^*_{1L} \geq Q^o_{1L}$. $\square$

**Proof of Lemma 7.2:** Assume that $s^{-1}(Q^*_{1H}) > \hat{s}^{-1}(Q^*_{2H})$. First note that,

$$\begin{aligned}
&Pr(D_2 + (D_1 - Q_1)^+ \geq Q_2) \\
&= Pr(D \geq s^{-1}(Q_1), D \geq Q_1 + Q_2) + Pr(D \leq s^{-1}(Q_1), D \geq \hat{s}^{-1}(Q_2)).
\end{aligned}$$

By substituting this in (4) we obtain:

$$\begin{aligned}
p\,Pr(D \geq s^{-1}(Q^*_{1H}), D \geq Q^*_{2H} + Q^*_{1H}) + p\,Pr(D \leq s^{-1}(Q^*_{1H}), D \geq \hat{s}^{-1}(Q^*_{2H}))& \\
+(1-p)Pr(D \geq s^{-1}(Q^*_{1L}), D \geq Q^*_{2H} + Q^*_{1L})& \\
+(1-p)Pr(D \leq s^{-1}(Q^*_{1L}), D \geq \hat{s}^{-1}(Q^*_{2H})) - c_{2H} =& \quad 0
\end{aligned}$$

Since $s^{-1}(Q^*_{1H}) > \hat{s}^{-1}(Q^*_{2H})$, $s^{-1}(Q^*_{1L}) > \hat{s}^{-1}(Q^*_{2H})$ by Claim 3. By Claim 1, $Pr(D \geq s^{-1}(Q^*_{1H}), D \geq Q^*_{2H} + Q^*_{1H}) = Pr(D \geq s^{-1}(Q^*_{1H})$. In addition, $Pr(D \geq s^{-1}(Q^*_{1H})) + Pr(\hat{s}^{-1}(Q^*_{2H}) \leq D \leq s^{-1}(Q^*_{1H})) = Pr(D \geq \hat{s}^{-1}(Q^*_{2H}))$. Therefore,

$$p\,Pr(D \geq \hat{s}^{-1}(Q^*_{2H})) + (1-p)Pr(D \geq \hat{s}^{-1}(Q^*_{2H})) - c_{2H} = Pr(D \geq \hat{s}^{-1}(Q^*_{2H})) - c_{2H} = 0.$$

Thus, $Q^*_{2H} = Q^o_{2H}$. Using $s^{-1}(Q^*_{1H}) \leq \hat{s}^{-1}(Q^*_{2H}) < \hat{s}^{-1}(Q^*_{2L})$ in (2) in a similar fashion gives the result $Q^*_{1H} = Q^o_{1H}$. $\square$

**Proof of Lemma 7.3:** Assume to the contrary that for $c_{1H} \leq c_{2H}$, $s^{-1}(Q^*_{1H}) < \hat{s}^{-1}(Q^*_{2H})$. Then, by Lemma 1, $Q^*_{1H} = Q^o_{1H}$. By Claims 2 and 3, we get $s^{-1}(Q^*_{1H}) \geq s^{-1}(Q^o_{1H}) \geq \hat{s}^{-1}(Q^o_{2H})$ and

$$\begin{aligned}
&Pr(D \geq Q^*_{2H} + Q^*_{1H}) \leq Pr(D \geq Q^o_{2H} + Q^o_{1H}) \\
&\quad < Pr(D \geq Q^o_{2H} + \hat{s}(s^{-1}(Q^o_{2H}))) = Pr(\hat{s}(D) \geq Q^o_{2H}) = c_{2H}. \qquad (*)
\end{aligned}$$

Now, we have either $s^{-1}(Q^*_{1L}) > \hat{s}^{-1}(Q^*_{2H})$ or $s^{-1}(Q^*_{1L}) \leq \hat{s}^{-1}(Q^*_{2H})$. In the first case equilibrium condition (4) simplifies to:

$$\begin{aligned}
c_{2H} &= p\,Pr(D \geq Q^*_{2H} + Q^*_{1H}) + (1-p)Pr(\hat{s}(D) \geq Q^*_{2H}) \\
&\leq p\,Pr(D \geq Q^*_{2H} + Q^*_{1H}) + (1-p)c_{2H},
\end{aligned}$$

since $Pr(\hat{s}(D) \geq Q^*_{2H}) \leq Pr(\hat{s}(D) \geq Q^o_{2H})$ by Claim 4 and $Pr(\hat{s}(D) \geq Q^o_{2H}) = c_{2H}$ by definition. This leads to

$$\begin{aligned} c_{2H} &\leq p\, Pr(D \geq Q^*_{2H} + Q^*_{1H}) + (1-p)c_{2H} \\ &\leq Pr(D \geq Q^*_{2H} + Q^*_{1H}), \end{aligned}$$

which is a contradiction to $(*)$.

For the second case, the equilibrium condition (4) simplifies to:

$$\begin{aligned} c_{2H} &= p\, Pr(D \geq Q^*_{2H} + Q^*_{1H}) + (1-p)Pr(D \geq Q^*_{2H} + Q^*_{1L}) \\ &< Pr(D \geq Q^*_{2H} + Q^*_{1H}), \end{aligned}$$

since $Q^*_{1L} > Q^*_{1H}$ by Claim 3. Again this contradicts $(*)$. $\square$

**Proof of Lemma 7.4:** By Lemma 2, $c_{1H} \leq c_{2H}$ implies $s^{-1}(Q^*_{1H}) \geq \hat{s}^{-1}(Q^*_{2H})$. Using this condition in Lemma 1 yields the desired result. $\square$

**Proof of Lemma 7.5:** First note that $Q^o_{iL}$ and $Q^o_{iH}$ are stand-alone order levels for firms $i = 1, 2$. It is important to notice that each firm will at least play his stand-alone order quantity in the equilibrium. Now, define $Q^1_{2H}$ as the order level of high type of firm 2 when firm 1 plays his stand-alone quantities for both his types in the equilibrium i.e.,

$$\begin{aligned} &p\, Pr(\hat{s}(D) + (s(D) - Q^o_{1H})^+ \\ &\quad \geq Q^1_{2H}) + (1-p)Pr(\hat{s}(D) + (s(D) - Q^o_{1L})^+ \geq Q^1_{2H}) - c_{2H} = 0. \end{aligned}$$

and $Q^1_{2H} \geq Q^o_{2H}$ since firm 2 will play at least his stand-alone order level. Rewriting the equilibrium condition gives,

$$\begin{aligned} &p\, Pr(D \geq s^{-1}(Q^o_1), D \geq Q^o_{1H} + Q^1_{2H}) + p\, Pr(D \leq s^{-1}(Q^o_{1H}), D \geq \hat{s}^{-1}(Q^1_{2H})) \\ &+(1-p)Pr(D \geq s^{-1}(Q^o_{1L}), D \geq Q^o_{1L} + Q^1_{2H}) \\ &+(1-p)Pr(D \leq s^{-1}(Q^o_{1L}), D \geq \hat{s}^{-1}(Q^1_{2H})) - c_{2H} = 0. \end{aligned}$$

For this equilibrium condition, we have three possibilities: $\hat{s}^{-1}(Q^1_{2H}) \leq s^{-1}(Q^o_{1H})$, $s^{-1}(Q^o_{1H}) < \hat{s}^{-1}(Q^1_{2H}) \leq s^{-1}(Q^o_{1L})$ and $s^{-1}(Q^o_{1L}) < \hat{s}^{-1}(Q^1_{2H})$. First assume

$\hat{s}^{-1}(Q^1_{2H}) \le s^{-1}(Q^o_{1H})$, then the equilibrium condition becomes:

$$p\,Pr(D \ge \hat{s}^{-1}(Q^1_{2H})) + (1-p)Pr(D \ge \hat{s}^{-1}(Q^1_{2H})) - c_{2H} = Pr(D \ge \hat{s}^{-1}(Q^1_{2H})) - c_{2H} = 0.$$

Thus, $Q^1_{2H} = Q^o_{2H}$. Now, we assume that $s^{-1}(Q^o_{1H}) < \hat{s}^{-1}(Q^1_{2H}) < s^{-1}(Q^o_{1L})$. Moreover, if we use the fact that $s^{-1}(Q^o_{1H}) < Q^o_{1H} + Q^1_{2H}$ (by Claim 1), the condition becomes

$$\begin{aligned}
0 &= p\,Pr(D \ge Q^o_{1H} + Q^1_{2H}) + (1-p)Pr(D \ge \hat{s}^{-1}(Q^1_{2H})) - c_{2H} \\
&< p\,Pr(D \ge s^{-1}(Q^o_{1H})) + (1-p)Pr(D \ge s^{-1}(Q^o_{1H})) - c_{2H} \\
&= Pr(D \ge s^{-1}(Q^o_{1H})) - c_{2H} = c_{1H} - c_{2H}
\end{aligned}$$

Thus, $c_{1H} > c_{2H}$ which is a contradiction to our assumption that $c_{1H} \le c_{2H}$. A similar proof can be obtained for $s^{-1}(Q^o_{1L}) \le \hat{s}^{-1}(Q^1_{2H})$. Hence, $Q^1_{2H} = Q^o_{2H}$ which implies that any order quantity of high type of firm 2 satisfies $Q_{2H} \le Q^o_{2H}$. Combining this with the fact that $Q_{2H} \ge Q^o_{2H}$, we obtain $Q_{2H} = Q^o_{2H}$. $\square$

## E.3 Proof of Theorem 7.2

Under an increasing and deterministic split function, we know that there is a unique Bayesian–Nash equilibrium and using Lemma 3, our unique equilibrium conditions take the form:

$$\begin{aligned}
q\,Pr(D \ge Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D_1 + (D_2 - Q^*_{2L})^+ \ge Q^*_{1L}) &= c_{1L}, \\
q\,Pr(D \ge Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D_1 + (D_2 - Q^*_{2L})^+ \ge Q^*_{1H}) &= c_{1H}, \\
p\,Pr(D_2 + (D_1 - Q^*_{1H})^+ \ge Q^*_{2L}) + (1-p)Pr(D_2 + (D_1 - Q^*_{1L})^+ \ge Q^*_{2L}) &= c_{2L}, \\
Q^*_{2H} &= \hat{s}(\overline{G}^{-1}(c_{2H})).
\end{aligned}$$

Now, if we use $D_1 = s(D)$ and $D_2 = \hat{s}(D)$ and use the fact that,

$$\begin{aligned}
&Pr(D_1 + (D_2 - Q_2)^+ \ge Q_1) \\
&\qquad = Pr(D \ge \hat{s}^{-1}(Q_2), D \ge Q_2 + Q_1) + Pr(D \le \hat{s}^{-1}(Q_2), D \ge s^{-1}(Q_1)), \\
&Pr(D_2 + (D_1 - Q_1)^+ \ge Q_2) \\
&\qquad = Pr(D \ge s^{-1}(Q_1), D \ge Q_1 + Q_2) + Pr(D \le s^{-1}(Q_1), D \ge \hat{s}^{-1}(Q_2)),
\end{aligned}$$

which can be obtained using a simple conditional probability argument, equilibrium conditions will become:

$$q\,Pr(D \geq Q_{1L}^* + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq \hat{s}^{-1}(Q_{2L}^*), D \geq Q_{2L}^* + Q_{1L}^*)$$
$$+(1-q)Pr(D \leq \hat{s}^{-1}(Q_{2L}^*), D \geq s^{-1}(Q_{1L}^*)) = c_{1L}, \qquad (A_1)$$

$$q\,Pr(D \geq Q_{1H}^* + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq \hat{s}^{-1}(Q_{2L}^*), D \geq Q_{2L}^* + Q_{1H}^*)$$
$$+(1-q)Pr(D \leq \hat{s}^{-1}(Q_{2L}^*), D \geq s^{-1}(Q_{1H}^*)) = c_{1H}, \qquad (A_2)$$

$$pPr(D \geq s^{-1}(Q_{1H}^*), D \geq Q_{2L}^* + Q_{1H}^*) + pPr(D \leq s^{-1}(Q_{1H}^*), D \geq \hat{s}^{-1}(Q_{2L}^*))$$
$$+(1-p)Pr(D \geq s^{-1}(Q_{1L}^*), D \geq Q_{2L}^* + Q_{1L}^*)$$
$$+(1-p)Pr(D \leq s^{-1}(Q_{1L}^*), D \geq \hat{s}^{-1}(Q_{2L}^*)) = c_{2L}, \qquad (A_3)$$

$$Q_{2H}^* = \hat{s}(\overline{G}^{-1}(c_{2H})). \qquad (A_4)$$

The proof of part 1 follows since $Q_{2H}^* = \hat{s}(\overline{G}^{-1}(c_{2H}))$ is obviously an equilibrium condition.

Part 2 has three separate subsets. To prove $(i)$, let $\hat{s}^{-1}(Q_{2L}^*) \geq s^{-1}(Q_{1L}^*)$. $(A_1)$ becomes $(i_1)$:

$$q\overline{G}(Q_{1L}^* + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq \hat{s}^{-1}(Q_{2L}^*), D \geq Q_{2L}^* + Q_{1L}^*)$$
$$+(1-q)Pr(D \leq \hat{s}^{-1}(Q_{2L}^*), D \geq s^{-1}(Q_{1L}^*))$$
$$= q\overline{G}(Q_{1L}^* + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq s^{-1}(Q_{1L}^*))$$
$$= q\overline{G}(Q_{1L}^* + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\overline{G}^{-1}(s^{-1}(Q_{1L}^*)) = c_{1L}.$$

Similarly, using the fact that $\hat{s}^{-1}(Q_{2L}^*) \geq s^{-1}(Q_{1L}^*)$ implies $\hat{s}^{-1}(Q_{2L}^*) \geq s^{-1}(Q_{1H}^*)$, $(A_2)$ becomes $(i_2)$:

$$q\overline{G}(Q_{1H}^* + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq \hat{s}^{-1}(Q_{2L}^*), D \geq Q_{2L}^* + Q_{1H}^*)$$
$$+(1-q)Pr(D \leq \hat{s}^{-1}(Q_{2L}^*), D \geq s^{-1}(Q_{1H}^*))$$
$$= q\overline{G}(Q_{1L}^* + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)Pr(D \geq s^{-1}(Q_{1H}^*))$$
$$= q\overline{G}(Q_{1H}^* + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\overline{G}^{-1}(s^{-1}(Q_{1H}^*)) = c_{1H}.$$

And combining two inequalities, $(A_3)$ becomes $(i_3)$:

$$
\begin{aligned}
& pPr(D \geq s^{-1}(Q^*_{1H}), D \geq Q^*_{2L} + Q^*_{1H}) + pPr(D \leq s^{-1}(Q^*_{1H}), D \geq \hat{s}^{-1}(Q^*_{2L})) \\
& +(1-p)Pr(D \geq s^{-1}(Q^*_{1L}), D \geq Q^*_{2L} + Q^*_{1L}) + (1-p)Pr(D \leq s^{-1}(Q^*_{1L}), D \geq \hat{s}^{-1}(Q^*_{2L})) \\
& = pPr(D \geq Q^*_{2L} + Q^*_{1H}) + (1-p)Pr(D \geq Q^*_{2L} + Q^*_{1L}) \\
& = p\overline{G}^{-1}(D \geq Q^*_{2L} + Q^*_{1H}) + (1-p)\overline{G}^{-1}(D \geq Q^*_{2L} + Q^*_{1L}) = c_{2L}.
\end{aligned}
$$

The proof for (ii) and (iii) follows similarly under $s^{-1}(Q^*_{1L}) > \hat{s}^{-1}(Q^*_{2L}) \geq s^{-1}(Q^*_{1H})$ and $s^{-1}(Q^*_{1H}) > \hat{s}^{-1}(Q^*_{2L})$. $\square$

## E.4 Proof of Theorem 7.3

First, since the demand has a continuous distribution, the inverse of distribution function $G$ and $\overline{G}$ are well-defined. Only one of the $(i)$,$(ii)$ or $(iii)$ given in Theorem 7.2 can be satisfied since a vector of order quantities satisfying one of the inequality conditions $(i_4)$,$(ii_4)$ or $(iii_4)$ cannot satisfy others.

Take the region $(i)$. There can be only one $Q^*_{1L}$ satisfying condition $(i_1)$ which is:

$$q\overline{G}(Q^*_{1L} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\overline{G}(s^{-1}(Q^*_{1L})) = c_{1L},$$

since $s^{-1}$, $\hat{s}^{-1}$ and $\overline{G}^{-1}$ gives unique results and it does not depend on any other variables. Similarly, only one $Q^*_{1H}$ satisfies $(i_2)$:

$$q\overline{G}(Q^*_{1H} + \hat{s}(\overline{G}^{-1}(c_{2H}))) + (1-q)\overline{G}(s^{-1}(Q^*_{1H})) = c_{1H}.$$

Since both $Q^*_{1L}$ and $Q^*_{1H}$ are unique, $(i_3)$ i.e.,

$$p\overline{G}(Q^*_{2L} + Q^*_{1H}) + (1-p)\overline{G}(Q^*_{2L} + Q^*_{1L}) = c_{2L},$$

also gives a unique $Q^*_{2L}$. Thus, the set of order quantities satisfying region $(i)$ is unique.

Similar arguments are valid for regions $(ii)$ and $(iii)$. The argument so far does not rule out multiple equilibria each of which is the unique solution of one of three blocks of equalities. Finally, we need to show that only one of that three cases can arise.

Assume to the contrary that case $(i)$ and $(ii)$ gives different solutions. Now, let $(Q^*_{1L}, Q^*_{1H}, Q^*_{2L}, Q^*_{2H})$ and $(\hat{Q}_{1L}, \hat{Q}_{1H}, \hat{Q}_{2L}, \hat{Q}_{2H})$ be the solutions of cases $(i)$ and $(ii)$ respectively. First notice that $Q^*_{1H} = \hat{Q}_{1H} = Q_{1H}$ and $Q^*_{2H} = \hat{Q}_{2H} = Q_{2H}$ since they require the same conditions. However, low type quantities should satisfy:

$$\begin{aligned}
q\,\overline{G}(Q^*_{1L} + Q_{2H}) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1L})) &= q\,\overline{G}(\hat{Q}_{1L} + Q_{2H}) + (1-q)\,\overline{G}(\hat{Q}_{2L} + \hat{Q}_{1L}) \\
p\ \overline{G}(Q^*_{2L} + Q_{1H}) + (1-p)\,\overline{G}(Q^*_{2L} + Q^*_{1L}) &= p\ \overline{G}(\hat{Q}_{2L} + Q_{1H}) + (1-p)\,\overline{G}(\hat{s}^{-1}(\hat{Q}_{2L})) \\
\hat{s}^{-1}(Q^*_{2L}) \geq Q^*_{1L} &+ Q^*_{2L} \geq s^{-1}(Q^*_{1L}) \\
\hat{s}^{-1}(\hat{Q}_{2L}) < \hat{Q}_{1L} &+ \hat{Q}_{2L} < s^{-1}(\hat{Q}_{1L})
\end{aligned}$$

where inequalities come from Claim 4. Thus, we have

$$\begin{aligned}
q\,\overline{G}(Q^*_{1L} + Q_{2H}) + (1-q)\,\overline{G}(s^{-1}(Q^*_{1L})) &> q\,\overline{G}(\hat{Q}_{1L} + Q_{2H}) + (1-q)\,\overline{G}(s^{-1}(\hat{Q}_{1L})) \\
p\ \overline{G}(Q^*_{2L} + Q_{1H}) + (1-p)\,\overline{G}(\hat{s}^{-1}(Q^*_{2L})) &< p\ \overline{G}(\hat{Q}_{2L} + Q_{1H}) + (1-p)\,\overline{G}(\hat{s}^{-1}(\hat{Q}_{2L}))
\end{aligned}$$

which implies $Q^*_{1L} < \hat{Q}_{1L}$ and $Q^*_{2L} > \hat{Q}_{2L}$ (Remember that $\overline{G}$ is a decreasing function.). If we use this in equilibrium conditions,

$$\begin{aligned}
\overline{G}(s^{-1}(Q^*_{1L})) &< \overline{G}(\hat{Q}_{2L} + \hat{Q}_{1L}) \\
\overline{G}(Q^*_{2L} + Q^*_{1L}) &> \overline{G}(\hat{s}^{-1}(\hat{Q}_{2L}))
\end{aligned}$$

meaning that both $Q^*_{1L} + Q^*_{2L} > s^{-1}(Q^*_{1L}) > \hat{Q}_{2L} + \hat{Q}_{1L}$ and $\hat{Q}_{2L} + \hat{Q}_{1L} > \hat{s}^{-1}(\hat{Q}_{2L}) > Q^*_{2L} + Q^*_{1L}$ should be true, which is a contradiction. The proof for other cases are similar.

Thus, the solution given by Theorem 7.2 is unique. $\square$

## E.5 Proof of Theorem 7.4

Let $G_A$ and $G_B$ be the distribution functions of $D_A$ and $D_B$, respectively. $D_A$ stochastically dominates $D_B$. Thus, $G_A(x) \leq G_B(x)$ and $\overline{G}_A(x) \geq \overline{G}_B(x)$ for all $x$. Since $\overline{G}_A$ and $\overline{G}_B$ are decreasing functions, $\overline{G}_A^{-1}(y) \geq \overline{G}_B^{-1}(y)$ for all $y$. We define $(Q_{1L}^A, Q_{1H}^A, Q_{2L}^A, Q_{2H}^A)$ and $(Q_{1L}^B, Q_{1H}^B, Q_{2L}^B, Q_{2H}^B)$ as the equilibrium order quantities for $D_A$ and $D_B$, respectively.

Returning to the result of Theorem 7.2, we have three possible cases. Consider the equilibrium conditions in case $(i)$. Now, since $\hat{s}$ is an increasing function, there exists $\delta_{2H} = Q_{2H}^A - Q_{2H}^B = \hat{s}(\overline{G}_A^{-1}(c_{2H})) - \hat{s}(\overline{G}_B^{-1}(c_{2H})) \geq 0$. Note that, the stock–out probability of firm 2 under high type does not change.

Now, by $(i_2)$,

$$q\,\overline{G}_A(Q_{1L}^A+Q_{2H}^A)+(1-q)\,\overline{G}_A(s^{-1}(Q_{1L}^A)) = q\,\overline{G}_B(Q_{1L}^B+Q_{2H}^B)+(1-q)\,\overline{G}_B(s^{-1}(Q_{1L}^B)).$$

Since the stock–out probability of firm 2 under high type does not change and low type of firm 1 gets spillover only from high type of firm 2, the probability of firm 1's getting a spillover should not change.

Let $\delta_{1L} = Q_{1L}^A - Q_{1L}^B$. We can rewrite the equilibrium condition as,

$$\begin{aligned} &q\,\overline{G}_A(Q_{1L}^A + Q_{2H}^A) + (1-q)\,\overline{G}_A(s^{-1}(Q_{1L}^A)) \\ &\quad = q\,\overline{G}_B(Q_{1L}^A + Q_{2H}^A - \delta_{1L} - \delta_{2H}) + (1-q)\,\overline{G}_B(s^{-1}(Q_{1L}^A - \delta_{1L})). \end{aligned}$$

We know that for any $\{x_1, x_2\}$, if $\overline{G}_A(x_1) = \overline{G}_B(x_2)$ then $x_1 \geq x_2$. Moreover, since the spillover probability does not change, $\overline{G}_A(s^{-1}(Q_{1L}^A)) \geq \overline{G}_B(s^{-1}(Q_{1L}^A))$ should be satisfied. Thus, the difference between order quantities is positive, i.e., $\delta_{1L} \geq 0$ and $Q_{1L}^A \geq Q_{1L}^B$.

By a similar argument for $(i_2)$, $\delta_{1H} = Q_{1H}^A - Q_{1H}^B \geq 0$.

For $(i_3)$, we have

$$\begin{aligned} &p\;\overline{G}_A(Q_{2L}^A + Q_{1H}^A) + (1-p)\,\overline{G}_A(Q_{2L}^A + Q_{1L}^*) \\ &\quad = p\;\overline{G}_B(Q_{2L}^B + Q_{1H}^A - \delta_{1H}) + (1-p)\,\overline{G}_B(Q_{2L}^B + Q_{1L}^A - \delta_{1L}). \end{aligned}$$

From previous argument, we know that the stock–out probability of firm 1 does not change with a stochastic increase in demand distribution. (Equilibrium order quantities increase to compensate the change in demand distribution.) Using a similar argument for $(i_3)$, $\delta_{2L} = Q_{2L}^A - Q_{2L}^B \geq 0$. Thus all the equilibrium order quantities increase.

Similar proof for cases $(ii)$ and $(iii)$. $\square$

## E.6 Proof of Theorem 7.5

As $s$ increases uniformly, $\hat{s}^{-1}$ increase, $\hat{s}$ and $s^{-1}$ decreases. From Theorem 7.2, as $s$ increases, $Q_{2H}^*$ decreases.

From $(i_1)$,

$$q\,\overline{G}^A(Q_{1L}^* + Q_{2H}^*) + (1-q)\,\overline{G}^A(s^{-1}(Q_{1L}^*)) = c_{1L}.$$

If $s$ increases uniformly, $s^{-1}$ decreases. Hence, $Q_{1L}$ should increase to satisfy the equilibrium condition. Similarly, $Q_{1H}^*$ increases as $s$ increases.

From $(i_3)$,

$$p\;\overline{G}^A(Q_{2L}^* + Q_{1H}^*) + (1-p)\,\overline{G}^A(Q_{2L}^* + Q_{1L}^*) = c_{2L}.$$

Since $Q_{1L}^*$ and $Q_{1H}^*$ increase, $Q_{2L}^*$ should decrease to compensate. Similar argument applies for cases $(ii)$ and $(iii)$. $\square$

## E.7 Comparative Statics

This section summarizes the comparative statics results for general demand distributions. But we need the following results.

First note that $s' = \partial s(D)/\partial D > 0$ and $\hat{s}' = \partial \hat{s}(D)/\partial D > 0$ since we assume both $s$ and $\hat{s}$ are increasing and deterministic functions. Then the derivative of the inverses of the split functions can be found by

$$
\begin{aligned}
(s^{-1})' &= \frac{\partial s^{-1}(Q)}{\partial Q} = \frac{1}{s'(s^{-1}())} > 0 \\
(\hat{s}^{-1})' &= \frac{\partial \hat{s}^{-1}(Q)}{\partial Q} = \frac{1}{\hat{s}'(\hat{s}^{-1}())} > 0
\end{aligned}
$$

We use these results to find the signs of derivatives of order quantities with respect to each parameter in the model.

Table E.1: Derivatives of equilibrium order quantities w.r.t. $c_{1L}$

| | Q | Conditions | $\mathbf{c_{1L}}$ | Sign |
|---|---|---|---|---|
| | $Q_{2H}$ | | $0$ | |
| **(i)** | $Q_{1L}$ | $\overline{G}(s^{-1}(Q_{1L})) > 0$ | $-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1L}))}$ | $< 0$ |
| | | $\overline{G}(s^{-1}(Q_{1L})) = 0$ | $-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$ | $< 0$ |
| | $Q_{1H}$ | | $0$ | |
| | $Q_{2L}$ | $\overline{G}(Q_{1L}+Q_{2L}) > 0$ | $-\frac{(1-p)g(Q_{1L}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}(\frac{\partial Q_{1L}}{\partial c_{1L}})$ | $> 0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L}) = 0$ | $0$ | |
| **(ii)** | $Q_{1L}$ | $\overline{G}(Q_{1L}+Q_{2L}) > 0$ | $-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}$ | $< 0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L}) = 0$ | $-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$ | $< 0$ |
| | $Q_{1H}$ | | $0$ | |
| | $Q_{2L}$ | | $0$ | |
| **(iii)** | $Q_{1L}$ | $\overline{G}(Q_{1L}+Q_{2L}) > 0$ | $-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$ | $< 0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L}) = 0$ | $-\frac{1}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$ | $< 0$ |
| | $Q_{1H}$ | | $0$ | |
| | $Q_{2L}$ | | $0$ | |

.

Table E.2: Derivatives of equilibrium order quantities w.r.t. $c_{1H}$

| | Q | Conditions | $\mathbf{c_{1H}}$ | Sign |
|---|---|---|---|---|
| | $Q_{2H}$ | | $0$ | |
| **(i)** | $Q_{1L}$ | | $0$ | |
| | $Q_{1H}$ | | $-\frac{1}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$ | $< 0$ |
| | $Q_{2L}$ | $\overline{G}(Q_{1L}+Q_{2L}) > 0$ | $-\frac{pg(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}(\frac{\partial Q_{1H}}{\partial c_{1H}})$ | $> 0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L}) = 0$ | $-\frac{\partial Q_{1H}}{\partial c_{1H}}$ | $> 0$ |
| **(ii)** | $Q_{1L}$ | $\overline{G}(Q_{1L}+Q_{2L}) > 0$ | $-\frac{(1-q)g(Q_{1L}+Q_{2L})}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}(\frac{\partial Q_{2L}}{\partial c_{1H}})$ | $< 0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L}) = 0$ | $0$ | |
| | $Q_{1H}$ | | $-\frac{1}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$ | $< 0$ |
| | $Q_{2L}$ | $\overline{G}(\hat{s}^{-1}(Q_{2L})) > 0$ | $-\frac{pg(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)(\hat{s}^{-1})'g(\hat{s}^{-1}(Q_{2L}))}(\frac{\partial Q_{1H}}{\partial c_{1H}})$ | $> 0$ |
| | | $\overline{G}(\hat{s}^{-1}(Q_{2L})) = 0$ | $-\frac{\partial Q_{1H}}{\partial c_{1H}}$ | $> 0$ |
| **(iii)** | $Q_{1L}$ | | $0$ | |
| | $Q_{1H}$ | $\overline{G}(Q_{1H}+Q_{2L}) > 0$ | $-\frac{1}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$ | $< 0$ |
| | | $\overline{G}(Q_{1H}+Q_{2L}) = 0$ | $-\frac{1}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$ | $< 0$ |
| | $Q_{2L}$ | | $0$ | |

.

Table E.3: Derivatives of equilibrium order quantities w.r.t. $c_{2L}$

| | Q | Conditions | $\mathbf{c_{2L}}$ | Sign |
|---|---|---|---|---|
| | $Q_{2H}$ | | $0$ | |
| **(i)** | $Q_{1L}$ | | $0$ | |
| | $Q_{1H}$ | | $0$ | |
| | $Q_{2L}$ | $\overline{G}(Q_{1L}+Q_{2L})>0$ | $-\frac{1}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}$ | $<0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L})=0$ | $-\frac{1}{pg(Q_{1H}+Q_{2L})}$ | $<0$ |
| **(ii)** | $Q_{1L}$ | $\overline{G}(Q_{1L}+Q_{2L})>0$ | $-\frac{(1-q)g(Q_{1L}+Q_{2L})}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}(\frac{\partial Q_{2L}}{\partial c_{2L}})$ | $>0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L})=0$ | $0$ | |
| | $Q_{1H}$ | | $0$ | |
| | $Q_{2L}$ | $\overline{G}(\hat{s}^{-1}(Q_{2L}))>0$ | $-\frac{1}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$ | $<0$ |
| | | $\overline{G}(\hat{s}^{-1}(Q_{2L}))=0$ | $-\frac{1}{pg(Q_{1H}+Q_{2L})}$ | $<0$ |
| **(iii)** | $Q_{1L}$ | $\overline{G}(Q_{1L}+Q_{2L})>0$ | $\frac{(1-q)\hat{s}'g(Q_{1L}+Q_{2L})/g(\overline{G}^{-1}(c_{2L}))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$ | $>0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L})=0$ | $0$ | |
| | $Q_{1H}$ | $\overline{G}(Q_{1H}+Q_{2L})>0$ | $\frac{(1-q)\hat{s}'g(Q_{1H}+Q_{2L})/g(\overline{G}^{-1}(c_{2L}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$ | $>0$ |
| | | $\overline{G}(Q_{1H}+Q_{2L})=0$ | $0$ | |
| | $Q_{2L}$ | | $-\frac{\hat{s}'}{g(\overline{G}^{-1}(c_{2L}))}$ | $<0$ |

.

Table E.4: Derivatives of equilibrium order quantities w.r.t. $c_{2H}$

| | Q | Conditions | $\mathbf{c_{2H}}$ | Sign |
|---|---|---|---|---|
| | $Q_{2H}$ | | $-\frac{\hat{s}'}{g(\overline{G}^{-1}(c_{2H}))}$ | $< 0$ |
| (i) | $Q_{1L}$ | $\overline{G}(s^{-1}(Q_{1L})) > 0$ | $\frac{\hat{s}' g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))/g(\overline{G}^{-1}(c_{2H}))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})' g(s^{-1}(Q_{1L}))}$ | $> 0$ |
| | | $\overline{G}(s^{-1}(Q_{1L})) = 0$ | $\frac{\hat{s}'}{qg(\overline{G}^{-1}(c_{2H}))}$ | $> 0$ |
| | $Q_{1H}$ | | $\frac{\hat{s}' g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))/g(\overline{G}^{-1}(c_{2H}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})' g(s^{-1}(Q_{1H}))}$ | $> 0$ |
| | $Q_{2L}$ | $\overline{G}(Q_{1L} + Q_{2L}) > 0$ | $-\frac{pg(Q_{1H}+Q_{2L})\partial Q_{1H}/\partial c_{2H}+(1-p)g(Q_{1L}+Q_{2L})\partial Q_{1L}/\partial c_{2H}}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}$ | $< 0$ |
| | | $\overline{G}(Q_{1L} + Q_{2L}) = 0$ | $-\frac{\partial Q_{1H}}{\partial c_{1H}}$ | $< 0$ |
| (ii) | $Q_{1L}$ | $\overline{G}(Q_{1L} + Q_{2L}) > 0$ | $\frac{q\hat{s}' g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))/g(\overline{G}^{-1}(c_{2H}))+(1-q)g(Q_{1L}+Q_{2L})(\partial Q_{2L}/\partial c_{2H})}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}$ | $> 0$ |
| | | $\overline{G}(Q_{1L} + Q_{2L}) = 0$ | $\frac{\hat{s}'}{g(\overline{G}^{-1}(c_{2H}))}$ | $> 0$ |
| | $Q_{1H}$ | | $\frac{\hat{s}' g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))/g(\overline{G}^{-1}(c_{2H}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})' g(s^{-1}(Q_{1H}))}$ | $> 0$ |
| | $Q_{2L}$ | $\overline{G}(\hat{s}^{-1}(Q_{2L})) > 0$ | $-\frac{pg(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)(\hat{s}^{-1})' g(\hat{s}^{-1}(Q_{2L}))}(\frac{\partial Q_{1H}}{\partial c_{2H}})$ | $< 0$ |
| | | $\overline{G}(\hat{s}^{-1}(Q_{2L})) = 0$ | $-\frac{\partial Q_{1H}}{\partial c_{2H}}$ | $< 0$ |
| (iii) | $Q_{1L}$ | $\overline{G}(Q_{1L} + Q_{2L}) > 0$ | $\frac{q\hat{s}' g(Q_{1L}+Q_{2H})/g(\overline{G}^{-1}(c_{2H}))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$ | $> 0$ |
| | | $\overline{G}(Q_{1L} + Q_{2L}) = 0$ | $\frac{\hat{s}'}{g(\overline{G}^{-1}(c_{2H}))}$ | $> 0$ |
| | $Q_{1H}$ | $\overline{G}(Q_{1H} + Q_{2L}) > 0$ | $\frac{q\hat{s}' g(Q_{1H}+Q_{2H})/g(\overline{G}^{-1}(c_{2H}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$ | $> 0$ |
| | | $\overline{G}(Q_{1H} + Q_{2L}) = 0$ | $\frac{\hat{s}'}{g(\overline{G}^{-1}(c_{2H}))}$ | $> 0$ |
| | $Q_{2L}$ | | $0$ | |

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Table E.5: Derivatives of equilibrium order quantities w.r.t. $p$

| | Q | Conditions | **p** | Sign |
|---|---|---|---|---|
| | $Q_{2H}$ | | 0 | |
| **(i)** | $Q_{1L}$ | | 0 | |
| | $Q_{1H}$ | | 0 | |
| | $Q_{2L}$ | $\overline{G}(Q_{1L}+Q_{2L})>0$ | $\frac{\overline{G}(Q_{1H}+Q_{2L})-\overline{G}(Q_{1L}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}$ | $>0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L})=0$ | $\frac{\overline{G}(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})}$ | $>0$ |
| **(ii)** | $Q_{1L}$ | $\overline{G}(Q_{1L}+Q_{2L})>0$ | $-\frac{(1-q)g(Q_{1L}+Q_{2L})}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}(\frac{\partial Q_{2L}}{\partial p})$ | $<0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L})=0$ | 0 | |
| | $Q_{1H}$ | | 0 | |
| | $Q_{2L}$ | $\overline{G}(\hat{s}^{-1}(Q_{2L}))>0$ | $\frac{\overline{G}(Q_{1H}+Q_{2L})-\overline{G}(\hat{s}^{-1}(Q_{2L}))}{pg(Q_{1H}+Q_{2L})+(1-p)(\hat{s}^{-1})'g(\hat{s}^{-1}(Q_{2L}))}$ | $>0$ |
| | | $\overline{G}(\hat{s}^{-1}(Q_{2L}))=0$ | $\frac{\overline{G}(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})}$ | $>0$ |
| **(iii)** | $Q_{1L}$ | | 0 | |
| | $Q_{1H}$ | | 0 | |
| | $Q_{2L}$ | | 0 | |

## E.8 Equilibrium under Uniform Demand and Linear Market Shares

The equilibrium conditions under the assumption $D \sim \text{Uniform}(0,1)$ are as follows:

$$q(1-\min\{1,Q_{1H}+Q_{2H}\})+(1-q)(1-\min\{1,\max\{Q_{2L}/(1-s),Q_{1H}+Q_{2L}\}\})$$
$$+(1-q)(\max\{\min\{1,Q_{2L}/(1-s)\}-\min\{1,Q_{1H}/s\},0\})=c_{1H}$$

$$q(1-\min\{1,Q_{1L}+Q_{2H}\})+(1-q)(1-\min\{1,\max\{Q_{2L}/(1-s),Q_{1L}+Q_{2L}\}\})$$
$$+(1-q)(\max\{\min\{1,Q_{2L}/(1-s)\}-\min\{1,Q_{1L}/s\},0\})=c_{1L}$$

$$Q_{2H}/(1-s)=1-c_{2H}$$

$$p(1-\min\{1,\max\{Q_{1H}/s,Q_{1H}+Q_{2L}\}\}+\max\{\min\{1,Q_{1H}/s\}-\min\{1,Q_{2L}/(1-s)\},0\})$$
$$+(1-p)(1-\min\{1,\max\{Q_{1L}/s,Q_{1L}+Q_{2L}\}\})$$
$$+(1-p)(\max\{\min\{1,Q_{1L}/s\}-\min\{1,Q_{2L}/(1-s)\},0\})=c_{2L}$$

Solution for $Q_{2H}=(1-s)(1-c_{2H})$ is straight forward. However in order to obtain the solutions for $Q_{1L}$,$Q_{1H}$ and $Q_{2L}$ we have to know the ordering for $Q_{1L}/s$, $Q_{1H}/s$, $Q_{2L}/(1-s)$, 1 and whether $Q_{1L}+Q_{2L}$, $Q_{1H}+Q_{2L}$,$Q_{1L}+Q_{2H}$ and $Q_{1H}+Q_{2H}$ are greater than 1 or not. We can summarize all the possibilities as:

Table E.6: Derivatives of equilibrium order quantities w.r.t. $q$

| | Q | Conditions | **q** | Sign |
|---|---|---|---|---|
| | $Q_{2H}$ | | $0$ | |
| **(i)** | $Q_{1L}$ | $\overline{G}(s^{-1}(Q_{1L})) > 0$ | $\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))-\overline{G}(s^{-1}(Q_{1L}))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1L}))}$ | $> 0$ |
| | | $\overline{G}(s^{-1}(Q_{1L})) = 0$ | $\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$ | $> 0$ |
| | $Q_{1H}$ | | $\frac{\overline{G}(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))-\overline{G}(s^{-1}(Q_{1H}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$ | $> 0$ |
| | $Q_{2L}$ | $\overline{G}(Q_{1L}+Q_{2L}) > 0$ | $-\frac{pg(Q_{1H}+Q_{2L})\partial Q_{1H}/\partial q+(1-p)g(Q_{1L}+Q_{2L})\partial Q_{1L}/\partial q}{pg(Q_{1H}+Q_{2L})+(1-p)g(Q_{1L}+Q_{2L})}$ | $< 0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L}) = 0$ | $-\frac{\partial Q_{1H}}{\partial q}$ | $< 0$ |
| **(ii)** | $Q_{1L}$ | $\overline{G}(Q_{1L}+Q_{2L}) > 0$ | $\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}(c_{2H})))-\overline{G}(Q_{1L}+Q_{2L})-(1-q)g(Q_{1L}+Q_{2L})(\partial Q_{2L}/\partial q)}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+Q_{2L})}$ | $> 0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L}) = 0$ | $\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}(c_{2H})))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$ | $> 0$ |
| | $Q_{1H}$ | | $\frac{\overline{G}(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))-\overline{G}(s^{-1}(Q_{1H}))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)(s^{-1})'g(s^{-1}(Q_{1H}))}$ | $> 0$ |
| | $Q_{2L}$ | $\overline{G}(\hat{s}^{-1}(Q_{2L})) > 0$ | $-\frac{pg(Q_{1H}+Q_{2L})}{pg(Q_{1H}+Q_{2L})+(1-p)(\hat{s}^{-1})'g(\hat{s}^{-1}(Q_{2L}))}(\frac{\partial Q_{1H}}{\partial q})$ | $< 0$ |
| | | $\overline{G}(\hat{s}^{-1}(Q_{2L})) = 0$ | $-\frac{\partial Q_{1H}}{\partial q}$ | $< 0$ |
| **(iii)** | $Q_{1L}$ | $\overline{G}(Q_{1L}+Q_{2L}) > 0$ | $\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}(c_{2H})))-\overline{G}(Q_{1L}+\hat{s}(\overline{G}(c_{2L})))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$ | $> 0$ |
| | | $\overline{G}(Q_{1L}+Q_{2L}) = 0$ | $\frac{\overline{G}(Q_{1L}+\hat{s}(\overline{G}(c_{2H})))}{qg(Q_{1L}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$ | $> 0$ |
| | $Q_{1H}$ | $\overline{G}(Q_{1H}+Q_{2L}) > 0$ | $\frac{\overline{G}(Q_{1H}+\hat{s}(\overline{G}(c_{2H})))-\overline{G}(Q_{1H}+\hat{s}(\overline{G}(c_{2L})))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))+(1-q)g(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2L})))}$ | $> 0$ |
| | | $\overline{G}(Q_{1H}+Q_{2L}) = 0$ | $\frac{\overline{G}(Q_{1H}+\hat{s}(\overline{G}(c_{2H})))}{qg(Q_{1H}+\hat{s}(\overline{G}^{-1}(c_{2H})))}$ | $> 0$ |
| | $Q_{2L}$ | | $0$ | |

$\{\frac{Q_{1L}}{s} > 1, \frac{Q_{1L}}{s} \leq 1\}$ $\quad$ $\{\frac{Q_{1H}}{s} > 1, \frac{Q_{1H}}{s} \leq 1\}$ $\quad$ $\{\frac{Q_{2L}}{(1-s)} > 1, \frac{Q_{2L}}{(1-s)} \leq 1\}$

$\{\frac{Q_{1L}}{s} > \frac{Q_{2L}}{(1-s)}, \frac{Q_{1L}}{s} \leq \frac{Q_{2L}}{(1-s)}\}$ $\quad$ $\{\frac{Q_{1H}}{s} > \frac{Q_{2L}}{(1-s)}, \frac{Q_{1H}}{s} \leq \frac{Q_{2L}}{(1-s)}\}$

$\{Q_{1L}+Q_{2L} > 1, Q_{1L}+Q_{2L} \leq 1\}$ $\quad$ $\{Q_{1H}+Q_{2L} > 1, Q_{1H}+Q_{2L} \leq 1\}$

$\{Q_{1L}+Q_{2H} > 1, Q_{1L}+Q_{2H} \leq 1\}$ $\quad$ $\{Q_{1H}+Q_{2H} > 1, Q_{1H}+Q_{2H} \leq 1.\}$

We have 512 different possibilities for $Q_{1L}$,$Q_{1H}$ and $Q_{2L}$ each leading to a different region in the 7 dimensional space. However, the number of regions can be reduced to 8 regions as shown below.

First, if both of the players have a high type, then the total inventory cannot exceed 1 and if second firm has high type since he does not expect any spillover. This is simply due to the suboptimality of all values greater than 1. Second, some of the conditions imply the others. For example, if $Q_{1L}/s > 1$ and $Q_{2L}/(1-s) > 1$ then $Q_{1L}+Q_{2L} > 1$. Third, $Q_{2L}/(1-s) > Q_{1L}/s$ implies $Q_{2L}/(1-s) > Q_{1H}/s$ since low type of a firm orders as much as high type of the firm due to submodularity. Similarly, $Q_{2L}/(1-s) \leq Q_{1H}/s$ implies $Q_{2L}/(1-s) \leq Q_{1L}/s$.

Using these kind of arguments we reduce the conditions to form 8 different regions. It can be shown that it is not possible to reduce the conditions further without making additional assumptions on the parameters.

| **Region** | **Conditions** |
|---|---|
| **1** | $\frac{Q_{1L}}{s} > 1$ , $\frac{Q_{2L}}{(1-s)} > 1$ |
| **2** | $Q_{1L} + Q_{2L} > 1$ , $\frac{Q_{1L}}{s} \leq 1$ |
| **3** | $Q_{1L} + Q_{2L} \leq 1$ , $\frac{Q_{1L}}{s} \leq \frac{Q_{2L}}{(1-s)}$ |
| **4** | $Q_{1L} + Q_{2L} > 1$ , $\frac{Q_{2L}}{(1-s)} \leq 1$ , $\frac{Q_{1H}}{s} \leq \frac{Q_{2L}}{(1-s)}$ |
| **5** | $Q_{1L} + Q_{2L} \leq 1$ , $\frac{Q_{1L}}{s} > \frac{Q_{2L}}{(1-s)}$ , $\frac{Q_{1H}}{s} \leq \frac{Q_{2L}}{(1-s)}$ |
| **6** | $Q_{1H} + Q_{2L} > 1$ , $\frac{Q_{1H}}{s} > \frac{Q_{2L}}{(1-s)}$ |
| **7** | $Q_{1L} + Q_{2L} > 1$ , $Q_{1H} + Q_{2L} \leq 1$ , $\frac{Q_{1H}}{s} > \frac{Q_{2L}}{(1-s)}$ |
| **8** | $Q_{1L} + Q_{2L} \leq 1$ , $\frac{Q_{1H}}{s} > \frac{Q_{2L}}{(1-s)}$ |

In each of the regions, the given inequalities simplify the equilibrium conditions leading to an easy computation of the equilibrium order quantities.

For Region 1, we reduce the equilibrium conditions to the following form:

$$\begin{aligned} q(1 - Q_{1H} - Q_{2H}) + (1 - q)(1 - Q_{1H}/s) &= c_{1H}, \\ q(1 - Q_{1L} - Q_{2H}) &= c_{1L}, \\ Q_{2H}/(1 - s) &= 1 - c_{2H}, \\ p(1 - Q_{1H} - Q_{2L}) &= c_{2L}. \end{aligned}$$

It is straightforward to find the order quantities for this region:

$$\begin{array}{ll} Q_{1H} = \frac{(1-c_{1H}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)} & Q_{1L} = 1 - \frac{c_{1L}}{q} - (1-s)(1-c_{2H}), \\ Q_{2H} = (1-s)(1-c_{2H}) & Q_{2L} = 1 - \frac{c_{2L}}{p} - \frac{(1-c_{1H}-q(1-s)(1-c_{2H}))}{(q+(1-q)/s)}. \end{array}$$

Now, by plugging these quantities into necessary inequalities, we obtain:

$$\begin{aligned} &\frac{Q_{1L}}{s} > 1 \qquad \Rightarrow \qquad 1 - \frac{c_{1L}}{q} - (1-s)(1-c_{2H}) > s \\ &\Rightarrow \frac{c_{1L}}{q} - (1-s)(1-c_{2H}) < 1 - s \qquad \Rightarrow \qquad \frac{c_{1L}}{q} - (1-s)c_{2H} < 0 \\ &\Rightarrow c_{1L} < q(1-s)c_{2H} \end{aligned}$$

$$
\begin{aligned}
&\frac{Q_{2L}}{(1-s)} > 1 \qquad \Rightarrow \qquad 1 - \frac{c_{2L}}{p} - \frac{(1 - c_{1H} - q(1-s)(1-c_{2H}))}{(q + (1-q)/s)} > 1 - s \\
&\Rightarrow \frac{c_{2L}}{p} + \frac{s(1 - c_{1H} - q(1-s)(1-c_{2H}))}{(1-(1-s)q)} < s \qquad \Rightarrow \qquad \frac{c_{2L}}{p} - \frac{s(c_{1H} - q(1-s)(c_{2H}))}{(1-(1-s)q)} < 0 \\
&\Rightarrow c_{2L} < \frac{sp(c_{1H} - q(1-s)c_{2H})}{1-(1-s)q}
\end{aligned}
$$

Thus, Region 1 can be characterized by two inequalities:

$$
\begin{aligned}
c_{1L} &< q(1-s)c_{2H}, \\
c_{2L} &< \frac{sp(c_{1H} - q(1-s)c_{2H})}{1-(1-s)q}.
\end{aligned}
$$

These conditions are necessary and sufficient, i.e., if these inequalities are satisfied, then equilibrium order quantities take the values in Region 1.

In a similar fashion, we can obtain the conditions for all 8 regions. This is summarized in Figure 1.

## E.9 Comparative Statics under Uniform Demand and Linear Market Shares

$Q_{1L} \rightarrow$

| | $\mathbf{Q_{1L}^{\alpha}}$ | $\mathbf{Q_{1L}^{\beta}}$ | $\mathbf{Q_{1L}^{\gamma}}$ | $\mathbf{Q_{1L}^{\delta}}$ |
|---|---|---|---|---|
| $\mathbf{c_{1L}}$ | $-\frac{1}{q}$ | $-\frac{1}{(q+(1-q)/s)}$ | $-1$ | $-1$ |
| $\mathbf{c_{1H}}$ | $0$ | $0$ | $-\frac{s(1-s)(1-q)p}{(1-(1-s)q)(1-sp)}$ | $0$ |
| $\mathbf{p}$ | $0$ | $0$ | $\frac{s(1-s)(1-q)(c_{2L}-c_{1H}+q(1-s)(c_{2H}-c_{2L}))}{(1-(1-s)q)(1-sp)^2}$ | $0$ |
| $\mathbf{c_{2L}}$ | $0$ | $0$ | $\frac{(1-s)(1-q)}{1-sp}$ | $(1-q)(1-s)$ |
| $\mathbf{c_{2H}}$ | $(1-s)$ | $\frac{s(1-s)q}{(1-sp)}$ | $\frac{q(1-s)(1-q(1-s)-ps^2)}{(1-(1-s)q)(1-sp)}$ | $q(1-s)$ |
| $\mathbf{q}$ | $\frac{c_{1L}}{q^2}$ | $\frac{s(1-s)(c_{2H}-c_{1L})}{(1-(1-s)q)^2}$ | $\frac{(1-s)((1-(1-s)q)^2(c_{2H}-c_{2L})-ps^2(c_{2H}-c_{1H}))}{(1-(1-s)q)^2(1-ps)}$ | $(1-s)(c_{2H}-c_{2L})$ |
| $\mathbf{s}$ | $(1-c_{2H})$ | $1-c_{2H}+\frac{(1-q)(c_{2H}-c_{1L})}{(1-(1-s)q)^2}$ | $q(1-c_{2H})+(1-q)(\frac{(1-s)(1-q)(1-c_{2L})}{(1-sp)^2}+\frac{s(1-s)pq(1-c_{2H})}{(1-(1-s)q)(1-sp)}$ $+\frac{p(1-2s+s^2p-(1-s)^2q)(1-c_{1H}-q(1-s)(1-c_{2H}))}{(1-(1-s)q)^2(1-sp)^2})$ | $q(1-c_{2H})+(1-q)(1-c_{2L})$ |

$Q_{2L} \rightarrow$

| | $\mathbf{Q_{2L}^{\alpha}}$ | $\mathbf{Q_{2L}^{\beta}}$ | $\mathbf{Q_{2L}^{\gamma}}$ | $\mathbf{Q_{2L}^{\delta}}$ |
|---|---|---|---|---|
| $\mathbf{c_{1L}}$ | $0$ | $\frac{s(1-p)}{(1-(1-s)q)}$ | $0$ | $0$ |
| $\mathbf{c_{1H}}$ | $\frac{s}{(1-(1-s)q)}$ | $\frac{sp}{(1-(1-s)q)}$ | $\frac{s(1-s)p}{(1-(1-s)q)(1-sp)}$ | $0$ |
| $\mathbf{p}$ | $\frac{c_{2L}}{p^2}$ | $\frac{s(c_{1H}-c_{1L})}{(1-(1-s)q)}$ | $\frac{s(1-s)(c_{1H}-c_{2L}-q(1-s)(c_{2H}-c_{2L}))}{(1-(1-s)q)(1-sp)^2}$ | $0$ |
| $\mathbf{c_{2L}}$ | $-\frac{1}{p}$ | $-1$ | $-\frac{(1-s)}{(1-sp)}$ | $-(1-s)$ |
| $\mathbf{c_{2H}}$ | $-\frac{qs(1-s)}{(1-(1-s)q)}$ | $-\frac{qs(1-s)}{(1-(1-s)q)}$ | $-\frac{pqs(1-s)^2}{(1-(1-s)q)(1-sp)}$ | $0$ |
| $\mathbf{q}$ | $-\frac{s(1-s)(c_{2H}-c_{1H})}{(1-(1-s)q)^2}$ | $-\frac{s(1-s)(c_{2H}-c_{1L}-p(c_{1H}-c_{1L}))}{(1-(1-s)q)^2}$ | $-\frac{ps(1-s)^2(c_{2H}-c_{1H})}{(1-(1-s)q)^2(1-sp)}$ | $0$ |
| $\mathbf{s}$ | $-\frac{(1-q)(1-c_{1H})}{(1-(1-s)q)^2}$ $-\frac{q^2(1-s)^2(1-c_{2H})-q(1-2s)(1-c_{2H})}{(1-(1-s)q)^2}$ | $-\frac{(1-q)(1-pc_{1H}-(1-p)c_{1L})}{(1-(1-s)q)^2}$ $+\frac{q^2(1-s)^2(1-c_{2H})-q(1-2s)(1-c_{2H})}{(1-(1-s)q)^2}$ | $-\frac{(1-s)(1-q)(1-c_{2L})}{(1-sp)^2} - \frac{s(1-s)pq(1-c_{2H})}{(1-(1-s)q)(1-sp)}$ $-\frac{p(1-2s+s^2p-(1-s)^2q)(1-c_{1H}-q(1-s)(1-c_{2H}))}{(1-(1-s)q)^2(1-sp)^2}$ | $-(1-c_{2L})$ |

$Q_{1H}$ and $Q_{2H} \rightarrow$

| | $\mathbf{Q_{1L}^{\alpha}}$ | $\mathbf{Q_{1L}^{\beta}}$ | $\mathbf{Q_{1L}^{\gamma}}$ | $\mathbf{Q_{2H}}$ |
|---|---|---|---|---|
| $\mathbf{c_{1L}}$ | $0$ | $0$ | $0$ | $0$ |
| $\mathbf{c_{1H}}$ | $-\frac{s}{(1-(1-s)q)}$ | $-\frac{1}{q}$ | $-1$ | $0$ |
| $\mathbf{p}$ | $0$ | $0$ | $0$ | $0$ |
| $\mathbf{c_{2L}}$ | $0$ | $0$ | $(1-q)(1-s)$ | $0$ |
| $\mathbf{c_{2H}}$ | $\frac{qs(1-s)}{(1-(1-s)q)}$ | $(1-s)$ | $q(1-s)$ | $-(1-s)$ |
| $\mathbf{q}$ | $\frac{s(1-s)(c_{2H}-c_{1H})}{(1-(1-s)q)^2}$ | $\frac{c_{1H}}{q^2}$ | $(1-s)(c_{2H}-c_{2L})$ | $0$ |
| $\mathbf{s}$ | $\frac{sq(1-c_{2H})}{(1-(1-s)q)} + \frac{(1-q)(1-c_{1H}-q(1-s)(1-c_{2H}))}{(1-(1-s)q)^2}$ | $(1-c_{2H})$ | $q(1-c_{2H})+(1-q)(1-c_{2L})$ | $-(1-c_{2H})$ |

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