ESSENTIAL COHOMOLOGY AND RELATIVE COHOMOLOGY OF FINITE GROUPS

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ABSTRACT

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In this thesis, we study mod-p essential cohomology of finite p-groups. One of the most important problems on essential cohomology of finite p-groups is finding a group theoretic characterization of p-groups whose essential cohomology is non-zero. This is an open problem introduced in [22]. We relate this problem to relative cohomology. Using relative cohomology with respect to the collection of maximal subgroups of the group, we define relative essential cohomology. We prove that the relative essential cohomology lies in the ideal generated by the essential classes which are the inflations of the essential classes of an elementary abelian p-group.

To determine the relative essential cohomology, we calculate the essential cohomology of an elementary abelian p-group. We give a complete treatment of the module structure of it over a certain polynomial subalgebra. Moreover we determine the ideal structure completely. In [17], Carlson conjectures that the essential cohomology of a finite group is finitely generated and is free over a certain polynomial subalgebra. We also prove that Carlson's conjecture is true for elementary abelian p-groups.

Finally, we define inflated essential cohomology and in the case $p > 2$, we prove that for non-abelian p-groups of exponent p, inflated essential cohomology is zero. This also shows that for those groups, relative essential cohomology is zero. This result gives a partial answer to a particular case of the open problem in [22].

Keywords: Essential cohomology, inflated essential cohomology, relative cohomology, Mùi invariants, Steenrod algebra, Steenrod closedness .

ÖZET

SONLU GRUPLARIN ESAS KOHOMOLOJİSİ VE GÖRECELİ KOHOMOLOJİSİ

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Bu tezde, sonlu p-gruplarının mod-p esas kohomolojisini çalıştık. Sonlu p-gruplarının esas kohomolojisi ile ilgili en önemli problemlerden biri, esas kohomolojisi sıfır olmayan p-grupları için kuramsal bir nitelendirme bulmaktır. Bu problem, [22] nolu referansta tanıtılmış ve henüz tam sonucu bulunamamış bir problemdir. Bu problemi, sonlu grupların göreceli kohomolojisi ile ilişkilendirdik. Grubun bütün maksimal alt gruplarına göre göreceli kohomolojisini kullanarak göreceli esas kohomolojiyi tanımladık. Göreceli esas kohomolojinin, temel abel p-gruplarının esas sınıflarından yükseltilmiş sınıflar tarafından üretilmiş bir idealin içinde olduğunu ispatladık.

Göreceli esas kohomolojiyi belirleyebilmek için, bir temel abel p-grubunun esas kohomolojisini hesapladık. Bu esas kohomolojinin, belli bir polinom altcebiri üzerindeki modül yapısını tamamıyla verdik. Bunun yanı sıra, bu esas kohomolojinin ideal yapısını da tamamıyla belirledik. Carlson [17], sonlu bir grubun esas kohomolojisinin, belirli bir altcebir üzerinde sonlu ve serbest üreteçli olduğu sanısını ortaya koymuştur. Yukarıdakilere ek olarak, Carlson'nın bu sanısının temel abel p-grupları için de doğru olduğunu ispatladık.

Son olarak, yükseltilmiş esas kohomolojiyi tanımladık. p tek asal olduğu zaman, abel olmayan ve kuvveti p olan p-grupları için yükseltilmiş esas kohomolojinin sıfır olduğunu ispatladık. Böylece, bu gruplar için göreceli kohomolojinin de sıfır olduğunu gösterdik. Bu sonuçla, [22] nolu referanstaki açık problemin özel bir haline, kısmi bir yanıt verdik.

Anahtar sözcükler: Esas kohomoloji, Mùi değişmezleri, yükseltilmiş esas kohomoloji, göreceli kohomoloji sınıfları, Steenrod cebiri, Steenrod kapalılığı.

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ANNEME.

Contents

1 Introduction **1**

1.1 Essential cohomology of a finite group 2

1.2 Relative cohomology of a finite group 5

1.3 Statements of results 8

2 Preliminaries **14**

2.1 Complexes and homology 14

2.2 Projective resolutions and cohomology 19

2.3 The Künneth theorem 22

2.4 Group cohomology 24

2.4.1 The group algebra kG 24

2.5 Cohomology of groups and extensions 26

2.5.1 Low dimensional cohomology and group extensions 28

2.6 Minimal projective and injective resolutions 30

2.7 The ring structure of $H^*(G, k)$ 32

2.7.1 Restriction, inflation and transfer 34

3 Essential cohomology of a finite group **38**

3.1 Essential classes 38

3.2 Problems on essential cohomology 39

4 Essential cohomology of $(\mathbb{Z}/p\mathbb{Z})^n$ **45**

4.1 The generators of $\mathrm{Ess}^*((\mathbb{Z}/p\mathbb{Z})^n)$ 46

4.1.1 Mùi invariants 46

4.1.2 Relations between $\mathrm{Ess}^*((\mathbb{Z}/p\mathbb{Z})^n)$ and Mùi invariants . . . 49

4.1.3 The main theorem 50

4.2 The mod-p Steenrod algebra and $\mathrm{Ess}^*((\mathbb{Z}/p\mathbb{Z})^n)$ 54

4.2.1 Steenrod closedness 54

4.2.2 Action of the Steenrod algebra on Mùi invariants 57

4.3 $\mathrm{Ess}^*((\mathbb{Z}/p\mathbb{Z})^n)$ and the Steenrod closedness 60

5 Relative cohomology of finite groups **63**

5.1 Relative Cohomology of a finite group with respect to a collection of subgroups of the group 64

5.2 Relative cohomology with respect to a finite G-set X 67

5.3 Relations between X-relative cohomology and essential cohomology 69

6 Inflated essential cohomology **74**

6.1 Inflated essential cohomology when $p = 2$ 74

6.2 Inflated essential cohomology when $p > 2$ 76

Chapter 1

Introduction

Let G be a finite group and R be a commutative ring with identity. The cohomology of a group G with coefficients in a RG-module N, where RG is the group ring, is the cohomology of the cochain complex of the RG-modules:

$$0 \to \mathrm{Hom}_{RG}(P_0, N) \to \mathrm{Hom}_{RG}(P_1, N) \to \cdots$$

obtained by applying $\mathrm{Hom}_{RG}(-, N)$ to a projective resolution of the trivial RG-module R. We denote the cohomology of a group G with coefficients in N by $H^n(G, N)$. The most important cases for the ground ring R of the group ring RG are $R = \mathbb{Z}$ and a field, denoted by k, of characteristic p dividing the order of G. Note that, by Maschke's theorem [40], the group algebra kG is semisimple when the characteristic p of k does not divide the order of G. In this case, all kG-modules are projective and hence the cohomology $H^n(G, N)$ of G is zero for all $n > 0$. That is why we assume that the characteristic of k divides the order of G. When the coefficient N is the trivial kG-module k, there is a product

$$H^n(G, k) \otimes H^m(G, k) \to H^{n+m}(G, k)$$

which comes from the cup product. This gives a ring structure on

$$H^*(G, k) = \bigoplus_{n \geq 0} H^n(G, k).$$

Since the cup product is graded commutative, $H^*(G, k)$ is a graded commutative ring and it is a finitely generated k-algebra [29]. Moreover it is an unstable module over the mod-p Steenrod algebra $\mathcal{A}$. There are ring homomorphisms called restriction and inflation on cohomology rings which are also $\mathcal{A}$-module homomorphisms. We give the definitions of restriction and inflation homomorphisms in Chapter 2.

Throughout the thesis, G always denotes a finite group and k always denotes a field of characteristic p, unless otherwise stated.

1.1 Essential cohomology of a finite group

Let $\mathcal{H}$ be a collection of subgroups of the group G. We say that $\mathcal{H}$ detects the cohomology ring $H^*(G,k)$ if the product of the restriction maps

$$\prod_{H\in\mathcal{H}} \mathrm{res}^G_H : H^*(G,k) \to \prod_{H\in\mathcal{H}} H^*(H,k)$$

is an injection. In this case the collection $\mathcal{H}$ is called a detecting family. If the coefficient ring k is a general commutative ring, then the cohomology ring in positive degrees is detected on the Sylow p-subgroups of G for each prime p dividing the order of G. If k is a field of characteristic p, then the cohomology ring is detected on a Sylow p-subgroup of G. If we can find a detecting family, then we can obtain information about the cohomology ring of the group using restrictions to the members of the detecting family. The method for computing cohomology rings using detection is given by Adem, Carlson, Karagueuzian and Milgram in [3]. In [3] , the cohomology ring of the Sylow 2-group of the Higman-Sims group was computed by first finding a detecting family. Then the restrictions of the generators of the cohomology ring to each member in the detecting family were found. In the last step, the relations were calculated as the generators of the ideal that was the intersection of the kernels of the restrictions. So the existence of a detecting family is very important for calculating the cohomology ring of a group.

There are many examples of p-groups whose cohomology rings contain non-trivial cohomology classes that cannot be detected by any proper subgroups. These are the cohomology classes that restrict trivially on all proper subgroups. Such classes are called essential classes.

In this thesis, we study mod-p essential cohomology of a finite group.

Definition 1.1.1 *Let G be a finite group. We call an element $\zeta \in H^*(G,k)$ essential if $\mathrm{res}^G_H(\zeta) = 0$ for every proper subgroup H of G.*

These classes form a graded ideal in $H^*(G,k)$. This ideal is called the essential cohomology of G and it is denoted by $\mathrm{Ess}^*(G)$. Throughout the thesis, by essential cohomology, we mean the essential cohomology of the corresponding finite group in the text. If G is not a p-group, then $\mathrm{Ess}^*(G)$ is zero. It is difficult to obtain non-zero essential classes, but these classes have a very effective role in calculating of the cohomology rings of p-groups. A group theoretic characterization of groups with non-zero essential classes is very important in calculation methods. If all essential classes are zero, then the collection of the maximal subgroups is a detecting family. That is why it is important to classify p-groups having non-zero essential cohomology.

Problem 1.1.2 *For which p-groups is the essential cohomology non-zero?*

This problem is one of the most important problems in the cohomology of finite groups. The problem was first introduced in "J.F. Adams' Problem session for homotopy theory" which was held at the Arcata Topology Conference in 1986 [22]. The first attempt on the problem was made by M. Feshbach. He conjectured that $\mathrm{Ess}^*(G) \neq 0$ if and only if G satisfied the pC-condition i.e. every element of order p in G was central. In 1989, the conjecture was disproved by Rusin[48]. In [27], it was proved that if G satisfies the pC-condition, then the cohomology ring $H^*(G,k)$ is Cohen-Macaulay. Using this motivation, Adem and Karagueuzian [1] prove that for a finite p-group whose cohomology ring is Cohen-Macaulay, $\mathrm{Ess}^*(G)$ is non-zero if and only if G satisfies pC-condition.

Theorem 1.1.3 ([1]) *Let G be a finite group, then the following two conditions are equivalent:*
(1) $H^*(G,k)$ is Cohen-Macaulay and contains non-trivial essential elements.
(2) G is a p-group and every element of order p in G is central.

In [1], there is the following interesting consequence. For a group whose cohomology ring is Cohen-Macaulay and whose essential cohomology is non-zero, any subgroup has the same property. In general, there is no such relation between the structure of the group and the structure of its cohomology ring.

Essential cohomology also has an important role in the investigation of some ring theoretic invariants such as depth of the graded commutative ring $H^*(G, k)$. Recall that the depth of a graded commutative k-algebra is the length of the longest regular sequence of elements of the algebra. In Duflot's paper [27], it is proved that the depth of $H^*(G, k)$ is at least equal to the rank of the center of a Sylow p-subgroup of G. Our interest in depth is based on the fact that if d is the depth of $H^*(G, k)$, then the cohomology ring is detected on restriction to the centralizers of the elementary abelian p-groups of rank d. The relation between depth and essential cohomology follows from the fact that for a p-group G if the depth of $H^*(G, k)$ is strictly greater than the p-rank of the center of G, then $\mathrm{Ess}^*(G) = \{0\}$ (see [16]). In fact, this result together with Duflot's result in [27] mean that if $\mathrm{Ess}^*(G)$ is non-zero, then the depth of $H^*(G, k)$ is equal to the p-rank of the center of G. Because of that reason, getting non-zero essential classes is also very important to determine the depth of $H^*(G, k)$. In fact, this last result is also related to associated prime ideals in $H^*(G, k)$. If $\mathrm{Ess}^*(G) \neq 0$, then $\mathrm{Ess}^*(G)$ has an element ζ such that the annihilator of ζ is a prime ideal and it has dimension equal to the p-rank of the center of G which is equal to depth of $H^*(G, k)$ here . This is a particular case of Carlson's depth conjecture. Carlson [21] conjectures that if $H^*(G, k)$ has depth d, then there is always an associated prime of dimension d. This conjecture is stated for any finite group and David J. Green gives sufficient and necessary conditions for the problem in the case of p-groups (see [32]).

The essential cohomology has also a key role in Carlson's method for calculating the cohomology ring $H^*(G, k)$. In [17], Carlson describes a series of tests on a partial presentation for $H^*(G, k)$ and proves that the calculation is complete if it passes the tests. Carlson's tests depend on two conjectures about the structure of the cohomology ring. One of them is related to essential cohomology. In [17] he conjectures that the if the essential ideal is non-zero, then it is finitely generated and free over the polynomial subring $k[\zeta_1, ..., \zeta_d]$ where d is the depth and $\zeta_1, ..., \zeta_d$ is a regular sequence of maximal length in the cohomology ring.

David J. Green proves the conjecture for the groups which do not have an elementary abelian p-group of order p^2 as a direct factor.

Theorem 1.1.4 ([30]) *Let k be a field of characteristic p, and let G be a finite group which does not have the elementary abelian p-group of order p^2 as a direct factor. If the essential ideal $\mathrm{Ess}^*(G)$ in $H^*(G, k)$ is non-zero, then it is a Cohen-Macaulay module with Krull-dimension equal to the p-rank of the center of G.*

Notice that all these results are based on the fact that $\mathrm{Ess}^*(G)$ is non-zero.

Another problem related to essential cohomology is finding the nilpotency degree of $\mathrm{Ess}^*(G)$. The structure of the essential cohomology depends on whether G is an elementary abelian p-group or not. From Quillen's work in [46], we get that if G is not elementary abelian, then $\mathrm{Ess}^*(G)$ is nilpotent. Mùi [43] and T. Marx [39] independently conjectured that the nilpotency degree is 2. Later David J. Green gave a counterexample to the conjecture for 2-groups (see [31]).

Work to date on essential cohomology has concentrated on the problem of the nilpotency degree for the non-elementary abelian case. In this thesis, we study the essential cohomology of elementary abelian p-groups and give a complete treatment of elementary abelian case. We explain the results in Section 3. There are also some relations between essential cohomology and relative cohomology of finite groups. Before stating all of the results we need to define relative cohomology of a finite group G with respect to a collection of subgroups of G and with respect to a finite G-set.

1.2 Relative cohomology of a finite group

The definition of relative cohomology is based on the relative projectivity of an RG-module. The relative projectivity of an RG-module appears in many different forms in representation theory and category theory. One can study projectivity of an RG-module with respect to a subgroup of the group G (see [36], [33]) with respect to a G-set (with respect to a permutation of that group, see [10] and [54]), and with respect to a module (see [23]).

The first study about the relative cohomology is given by Higman [36]. Later in 1964 and 1965, Snapper defined the cohomology of a group relative to a permutation of the group [50, 51, 52, 53, 54]. In [50], he used relative cohomology to give a proof of the Frobenious theorem. In [35], Harris defined the cohomology of a group relative to a collection of subgroups of the group. These new definitions simplified many of Snapper's proofs. After Harris' paper [35], the cohomology of a group relative to a permutation is realized as the cohomology relative to the collection of stabilizer subgroups of the permutation representation.

In this thesis, we study the cohomology of p-groups relative to the collection of all maximal subgroups of the group. We notice that all extension classes relative to the collection of all maximal subgroups of the group, are essential classes and moreover we get that all extension classes relative to the collection of maximal subgroups lie in the set of essential classes inflated from the Frattini quotient which is isomorphic to an elementary abelian p-group. Before explaining these, we need to define relative cohomology with respect to the collection of all maximal subgroups.

The relative cohomology with respect to a collection of subgroups as well as relative cohomology with respect to an RG-module in modular representation theory is of fundamental importance. In [23], Carlson and Peng show the equivalence of the definition of group cohomology with respect to a collection $\mathcal{H} = \{H | \ H \leq G\}$ of subgroups of the group G and the definition of the relative cohomology with respect to a special module V where V is the direct sum $V = \bigoplus_{H \in \mathcal{H}} k \uparrow_H^G$. At the same time these two definitions are equivalent to the definition of the relative cohomology with respect to a finite G-set X where X is the set of all cosets of the subgroups in $\mathcal{H}$ (see [10], [23]). We use these equivalent definitions to prove some new results in Chapter 5 and below we explain one of these equivalent definitions, the relative cohomology with respect to a finite

G-set.

Let X be a finite G-set and let kX denote the permutation kG-module whose basis is given by the elements of X. To define relative cohomology, we need to give the following definitions:

Definition 1.2.1 *An exact sequence*

$$0 \to A \to B \to C \to 0$$

of kG-modules is said to be X-split if

$$0 \to A \otimes_k kX \to B \otimes_k kX \to C \otimes_k kX \to 0$$

splits.

Definition 1.2.2 *A kG-module M is said to be projective relative to X, or X-projective, if there exists a kG-module N such that M is a direct summand of $kX \otimes_k N$.*

Now, we can define an X-projective resolution of a kG-module M.

Definition 1.2.3 *A long exact sequence of kG-modules*

$$P_* : \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is said to be an X-projective resolution of M if each P_i is X-projective and for each i, the short exact sequence

$$0 \to \ker(\partial_n) \to P_n \to \operatorname{im}(\partial_n) \to 0$$

where $\partial_n : P_n \to P_{n-1}$ is the boundary map in the resolution, is X-split.

The usual comparison theorem for projective resolutions also holds for the relative projectivity and this enables us to define the relative cohomology. If

$$P_* : \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to k \to 0$$

is an X-projective resolution of k, then we have a cochain complex

$$0 \to \mathrm{Hom}_{kG}(P_0, k) \to \mathrm{Hom}_{kG}(P_1, k) \to \cdots \to \mathrm{Hom}_{kG}(P_n, k) \to \cdots .$$

We define $X\mathrm{Ext}^n_{kG}(k,k) = H^n(\mathrm{Hom}_{kG}(P_*, k), \delta^*)$. This definition is independent of the choice of an X-projective resolution P_*. Using this we can define the X-relative cohomology to be

$$XH^n(G,k) = X\mathrm{Ext}^n_{kG}(k,k).$$

As in usual group cohomology, we can consider the elements of $X\mathrm{Ext}^n_{kG}(k,k)$ as equivalence classes of X-split n-fold extensions

$$0 \to k \to M_{n-1} \to M_{n-2} \to \cdots \to M_0 \to k \to 0.$$

Two such extensions are equivalent if there is a map of X-split n-fold extensions taking one to the other. Note that two X-split n-fold extensions can be equivalent as n-fold extensions without being equivalent as X-split n-fold extensions.

There is a natural map

$$\varphi_{G,X} : X\mathrm{Ext}^n_{kG}(k,k) \to \mathrm{Ext}^n_{kG}(k,k)$$

which maps a X-split n-fold extension to itself in $\mathrm{Ext}^n_{kG}(k,k)$. This map is not necessarily injective (see [10]). By definition of the relative cohomology with respect to the collection of maximal subgroups equivalently the definition of the relative cohomology with respect to X where X is the set of cosets of all maximal subgroups of G, it is easy to see that

$$\mathrm{Im}\,\varphi_{G,X} \subseteq \mathrm{Ess}^*(G).$$

For that reason, finding a group theoretic characterization of finite p-groups having $\mathrm{Im}\,\varphi_{G,X} = 0$ is a solution for a particular form of the Problem 1.1.2. Because of that relation, it is interesting to study this homomorphism more closely.

1.3 Statements of results

Many results in group cohomology crucially differ on whether the group G is elementary abelian or not. One of these results is related to essential cohomology.

By Quillen's work [46], if G is not an elementary abelian p-group, then $\mathrm{Ess}^*(G)$ is nilpotent. If G is an elementary abelian p-group then there are non-nilpotent classes in the essential cohomology. Work to date on essential cohomology has concerned mostly with the non-elementary abelian case, but we find that the elementary abelian case is rather interesting and related to modular invariants and the action of Steenrod algebra $\mathcal{A}$ on $H^*(G,,k)$. It is well-known that the Steenrod algebra $\mathcal{A}$ has an action on the cohomology ring $H^*(G,k)$ and this action makes $H^*(G,k)$ an unstable $\mathcal{A}$-algebra. The Steenrod closure of a homogeneous subset $\mathcal{T}$ in $H^*(G,k)$ is the smallest homogeneous ideal which includes $\mathcal{T}$ and is closed under the action of Steenrod algebra $\mathcal{A}$.

Let V be an elementary abelian p-group of rank $n > 0$. It is well-known that the cohomology ring of V is

$$H^*(V,\mathbb{F}_p) = \begin{cases} \mathbb{F}_p[x_1,x_2,...,x_n] & \text{if} \quad p=2, \quad \deg(x_i)=1 \\ \mathbb{F}_p[x_1,x_2,...,x_n] \otimes \bigwedge(a_1,...,a_n) & \text{if} \quad p>2. \end{cases}$$

When $p > 2$, we have $2\deg(a_i) = \deg(x_i) = 2$, $x_i = \beta(a_i)$ and $a_i^2 = 0$.

It is easy to see that $\mathrm{Ess}^*(V)$ is always non-zero and when $p = 2$, $\mathrm{Ess}^*(V)$ is a principal ideal generated by the product of all non-zero one dimensional classes. For $p > 2$, we show that $\mathrm{Ess}^*(V)$ is the Steenrod Closure of the product $a_1 \cdots a_n$. We also prove that $\mathrm{Ess}^*(V)$ is a Cohen-Macaulay module over the subalgebra $\mathbb{F}_p[x_1,...,x_n]$ for both cases. So Carlson's conjecture in [17] which says that the essential cohomology of an arbitrary p-group is free and finitely generated over a certain polynomial subalgebra in $H^*(G,k)$, holds for elementary abelian p-groups. Precisely we prove the following:

Definition 1.3.1 *Denote by L_n the polynomial*

$$L_n(X_1,...,X_n) = \begin{vmatrix} X_1 & X_2 & ... & X_n \\ {X_1}^p & {X_2}^p & ... & {X_n}^p \\ ... & ... & ... & ... \\ {X_1}^{p^{n-1}} & {X_2}^{p^{n-1}} & ... & {X_n}^{p^{n-1}} \end{vmatrix} \in \mathbb{F}_p[X_1,...,X_n].$$

Lemma 1.3.2 *Let V be an elementary abelian 2-group. The essential cohomology $\mathrm{Ess}^*(V)$ is the principal ideal in $H^*(V, \mathbb{F}_2)$ generated by $L_n(x_1, ..., x_n)$. Moreover $\mathrm{Ess}^*(V)$ is the free $\mathbb{F}_2[x_1, ..., x_n]$-module with the free generator $L_n(x_1, ..., x_n)$ and the Steenrod closure of this generator.*□

For $p > 2$, the ideal structure of $\mathrm{Ess}^*(V)$ is given by the theorem:

Theorem 1.3.3 *(See Theorem 4.1.10.) Let p be an odd prime and V a rank n elementary abelian p-group. Then the essential cohomology $\mathrm{Ess}^*(V)$ is the Steenrod closure of the product $a_1 \cdots a_n$ where $a_i \in H^1(V, \mathbb{F}_p)$. That is $\mathrm{Ess}^*(V)$ is the smallest ideal in $H^*(V, \mathbb{F}_p)$ which contains the one dimensional space generated by $a_1 \cdots a_n$ in $H^*(V, \mathbb{F}_p)$ and is closed under the action of the Steenrod algebra.*

The other result concerns the structure of $\mathrm{Ess}^*(V)$ as a module over the polynomial subalgebra $k[x_1, ..., x_n]$. We observe that the generators of $\mathrm{Ess}^*(V)$ are the Mùi invariants.

Let V be a finite dimensional vector space over the field k. Consider the natural action of $GL(V)$ over V^*. There is an induced action of $GL(V)$ over the polynomial algebra $S(V^*)$ and Dickson's invariants generate the invariants of the action of $GL(V)$ on $S(V^*)$. There is also an induced action on $S(V^*) \otimes_k \wedge(V^*)$. Mùi invariants are the $SL(V^*)$-invariants of this induced action. For the details of Mùi invariants see [45]. We proved that:

Theorem 1.3.4 *(See Theorem 4.3.1) Let p be an odd prime and V a rank n elementary abelian p-group. Then as a module over the polynomial subalgebra $k[x_1, ..., x_n]$ of the cohomology ring $H^*(V, \mathbb{F}_p)$, the essential cohomology $\mathrm{Ess}^*(V)$ is free on the set of Mùi invariants.*

The essential cohomology of elementary abelian p-groups has a crucial role for the essential classes that come from the relative cohomology of the group.

Let G be a p-group. Suppose that X is the set of all left cosets of maximal subgroups of G, as a finite G-set where the action is left multiplication. Consider the group homomorphism

$$\varphi_{G,X} : XH^n(G,k) \to H^n(G,k).$$

This homomorphism depends on G and finite G-set X. A quick look shows that the image of $\varphi_{G,X}$ lies in $\mathrm{Ess}^*(G)$. We define the relative essential ideal (which refers to the essential classes coming from the relative cohomology) as the ideal generated by $\operatorname{Im}\varphi_{G,X}$ and denote it by $\mathrm{RelEss}^*(G)$ for a finite p-group G. The problem of which p-groups $\mathrm{RelEss}^*(G)$ is non-zero is slightly different and is particular version of the problem of which groups $\mathrm{Ess}^*(G)$ is non-zero.

We proved that $\mathrm{RelEss}^*(G)$ of a p-group is closely related to the essential cohomology of elementary abelian p-groups and we defined the inflated essential cohomology. These inflated essential cohomology classes let us consider another problem which is also a particular form of the Problem 1.1.2.

Theorem 1.3.5 *(See Theorem 5.3.6) Let G be a p-group. Suppose that X is the set of all cosets of maximal subgroups, then*

$$\operatorname{Im}\varphi_{G,X} \subseteq \inf^G_{G/\Phi(G)}(\mathrm{Ess}^*(G/\Phi(G)))$$

where $\Phi(G)$ is the Frattini subgroup of G.

We define the inflated essential cohomology of G as the ideal generated by $\inf^G_{G/\Phi(G)}(\mathrm{Ess}^*(G/\Phi(G)))$ and denote it by $\mathrm{InfEss}^*(G)$. So under the given conditions of Theorem 5.3.6, we have $\mathrm{RelEss}^*(G) \subseteq \mathrm{InfEss}^*(G)$. It is clear that if $\mathrm{InfEss}^*(G) = 0$, then $\mathrm{RelEss}^*(G) = 0$. Now the problem is for which p-groups $\mathrm{InfEss}^*(G)$ is zero.

We know that the essential cohomology of an elementary abelian p-group is the Steenrod closure of the product of one dimensional classes in the cohomology ring (see Theorem 4.3.1). Using the above notation, we conclude that $\mathrm{InfEss}^*(G) = 0$ if and only if $\inf^G_{G/\Phi(G)}(\prod_{x\in H^1(G,\mathbb{F}_2)-\{0\}} x) = 0$ for $p = 2$ and $\inf^G_{G/\Phi(G)}(a_1 \cdots a_n) = 0$

for $p > 2$. So the classification problem (see Problem 1.1.2 turns out to be the classification of p-groups for which $\inf^G_{G/\Phi(G)}(\prod_{x\in H^1(G,\mathbb{F}_2)-\{0\}} x) = 0$ for $p = 2$ and $\inf^G_{G/\Phi(G)}(a_1 \cdots a_n) = 0$ for $p > 2$.

The classification of 2-groups whose inflated essential classes are zero, is complete.

Theorem 1.3.6 (Yalçın [59]) *If G is a non-abelian 2-group, then* $\mathrm{InfEss}^*(G) = 0$.

Now it follows easily that:

Corollary 1.3.7 *Suppose that X is the set of all cosets of maximal subgroups. If G is a non-abelian 2-group, then for any $n \geq 0$*

$$\mathrm{Im}(\varphi_{G,X} : XH^n(G,k) \to H^n(G,k)) = 0.$$

For $p > 2$, the classification is much more complicated. We prove that:

Theorem 1.3.8 *(See Theorem 6.2.10) If G is a non-abelian p-group of exponent p, then* $\mathrm{InfEss}^*(G) = 0$.

Corollary 1.3.9 *If G is non-abelian p-group of exponent p, then* $\mathrm{RelEss}^*(G) = 0$.

We also prove that the nilpotency degree of $\mathrm{InfEss}^*(G)$ is 2.

Theorem 1.3.10 *(See Theorem 6.2.17) Let G be a finite p-group such that* $\mathrm{InfEss}^*(G)$ *is non zero. Then the nilpotency degree of* $\mathrm{InfEss}^*(G)$ *is* 2.

The thesis is organized as follows:

In Chapter 2, we give some background material from homological algebra which contains definitions of cohomology, projective resolutions and some basic theorems of cohomology theory for an arbitrary commutative ring with identity. Also we study the group algebra kG, projective and injective kG-modules and resolutions, the relation between cohomology and extensions, first cohomology $H^1(G, N)$, minimal projective resolutions and finally the ring structure of $H^*(G, k)$.

Chapter 3 includes the definition of essential cohomology and its properties as well as the problems related to essential cohomology of finite groups.

In Chapter 4, we give a complete treatment of the essential cohomology of elementary abelian p-groups. This chapter is a detailed version of the paper [7].

In Chapter 5, we study relative cohomology of finite groups. We give the relations between the relative cohomology of finite groups with respect to a collection of subgroups and the essential cohomology.

In Chapter 6, we define the inflated essential cohomology and give the relations between the relative cohomology and the inflated essential cohomology. We also give some partial answers to Problem 1.1.2.

Chapter 2

Preliminaries

To define the cohomology of a finite group G, we need to consider projective resolutions of the trivial RG-module R, where RG is the group algebra and R is the ground ring which is commutative with identity. The most important cases for R is $R = \mathbb{Z}$ or R is a field, especially a field of characteristic p where p is a prime number. In this chapter, our main interest is the cohomology of a cochain complex of R-modules for any ring with identity. We give the general theory of the homology and the cohomology of a chain complex and a cochain complex of R-modules to obtain main applications to group algebra which are used in cohomology theory of groups. To get more details about the materials in this chapter, we refer the reader to [10], [15], [38].

2.1 Complexes and homology

Definition 2.1.1 *A chain complex C of R-modules is a family $C = \{C_n, \partial_n\}$, $n \in \mathbb{Z}$, where each C_n is an R-module and $\partial_n : C_n \to C_{n-1}$ is R-module homomorphism, satisfying $\partial_n \circ \partial_{n+1} = 0$. Here ∂_n is called the differential of the complex. Thus a complex C has the form*

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow C_{-1} \longrightarrow \cdots .$$

In this complex, instead of using lower indices, it is often convenient to write C^n for C_{-n} and $\delta^n : C^n \to C^{n+1}$ in place of $\partial_{-n} : C_{-n} \to C_{-n-1}$ for $n \geq 0$.

Definition 2.1.2 *A cochain complex C of R-modules is a family $C = \{C^n, \partial_n\}$, $n \in \mathbb{Z}$, where each C^n is an R-module and $\delta^n : C^n \to C^{n+1}$ is R-module homomorphism, satisfying $\delta^n \circ \delta^{n-1}$=0. Here δ^n is called the differential of the complex. Thus a cochain complex C has the form*

$$\cdots \longrightarrow C^{-1} \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^n \xrightarrow{\delta^n} C^{n+1} \longrightarrow \cdots .$$

The condition $\partial_n \circ \partial_{n+1}$=0 for all integers n gives that Im $\partial_{n+1} \subseteq \ker \partial_n$. The homology and similarly the cohomology measures the differences between Im ∂_{n+1} and ker ∂_n as follows.

Definition 2.1.3 *The homology of a chain complex C is defined as*

$$H_n(C) = H_n(C, \partial_*) = \ker\ (\partial_n : C_n \to C_{n-1}) / \operatorname{Im}\ (\partial_{n+1} : C_{n+1} \to C_n).$$

The cohomology of a cochain complex C is defined as

$$H^n(C) = H^n(C, \delta^*) = \ker\ (\delta^n : C^n \to C^{n+1}) / \operatorname{Im}\ (\delta^{n-1} : C^{n-1} \to C^n).$$

An n-cycle of C is an element of $Z_n(C) := \ker\ (\partial_n : C_n \to C_{n-1})$ and an n-boundary is an element of $B_n(C) := \operatorname{Im}\ (\partial_{n+1} : C_{n+1} \to C_n)$. Similarly an n-cocycle is an element of $Z^n(C) := \ker\ (\delta_n : C^n \to C^{n+1})$ and an n-coboundary is an element of $B^n(C) := \operatorname{Im}\ (\delta^{n-1} : C^{n-1} \to C^n)$. If $x \in C_n$ is such that $\partial_n(x) = 0$ then $x \in Z_n(C)$ and $[x]$ is the image of x in $H_n(C)$ and $[x]$ is called homology class. Two n-cycles x_1, x_2 are in the same homology class, that is $[x_1] = [x_2]$, if and only if $x_1 - x_2 \in \operatorname{Im}\ \partial_{n+1}$. And also if $x \in C_n$, then we say that x has dimension n.

Definition 2.1.4 *If C and D are chain complexes (respectively cochain complexes), a chain map (respectively cochain map) $f : C \to D$ is a family of module* *homomorphisms $f_n : C_n \to D_n$ (respectively $f^n : C^n \to D^n$), $n \in \mathbb{Z}$, such that the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & C_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r, "\partial'_{n-1}"] & D_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

That is $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$ for all n (Respectively

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n+1} \arrow[r, "\delta^{n+1}"] \arrow[d, "f^{n+1}"] & C^n \arrow[r, "\delta^n"] \arrow[d, "f^n"] & C^{n-1} \arrow[r, "\delta^{n-1}"] \arrow[d, "f^{n-1}"] & C^{n-2} \arrow[r] \arrow[d, "f^{n-2}"] & \cdots \\
\cdots \arrow[r] & D^{n+1} \arrow[r, "\delta^{n'}"] & D^n \arrow[r, "\delta^{n'}"] & D^{n-1} \arrow[r, "\delta^{n-1'}"] & D^{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

that is $\delta^{n'} \circ f^n = f^{n-1} \circ \delta^n$).

Lemma 2.1.5 *A chain map $f : C \to D$ induces a homomorphism $f_* : H_n(C) \to H_n(D)$ defined by $f_*([x]) = [f_n(x)]$ for $x \in Z_n(C)$ and similarly a cochain map $f : C \to D$ induces a homomorphism $f^* : H^n(C) \to H^n(D)$ defined by $f^*([x]) = [f^n(x)]$ for $x \in Z^n(C)$.*

Definition 2.1.6 *Let $f, f' : C \to D$ be chain maps. We say that f and f' are chain homotopic (written $f \simeq f'$), if there are module homomorphisms $h_n : C_n \to D_{n+1}$ such that $f_n - f'_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n$ holds for all $n \in \mathbb{Z}$ for the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "{f_{n+1}, f'_{n+1}}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "{f_n \,|\, f'_n}"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "{f_{n-1}, f'_{n-1}}"] \arrow[dl, "h_{n-1}"] & C_{n-2} \arrow[r] \arrow[d, "{f_{n-2}, f'_{n-2}}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial'_{n+1}"] & D_n \arrow[r, "\partial'_n"] & D_{n-1} \arrow[r, "\partial'_{n-1}"] & D_{n-2} \arrow[r] & \cdots .
\end{tikzcd}$$

Definition 2.1.7 *We say that C and D are chain homotopy equivalent (written $C \simeq D$), if there are chain maps $f : C \to D$ and $f' : D \to C$ such that $f \circ f' \simeq Id_D$ and $f' \circ f \simeq Id_C$. The chain maps f and f' are called chain equivalences.*

We have similar definitions for cochain complexes.

Proposition 2.1.8 *If $f, f^{'} : C \to D$ are chain homotopic, then*

$$f_* = f^{'}_* : H_n(C) \to H_n(D).$$

A homotopy equivalence $C \simeq D$ induces an isomorphism $H_n(C) \cong H_n(D)$ for all $n \in \mathbb{Z}$.

$\square$

The cohomological version of the above proposition is the following.

Proposition 2.1.9 *If $f, f^{'} : C \to D$ are cochain homotopic, then*

$$f^* = (f^{'})^* : H^n(C) \to H^n(D).$$

A homotopy equivalence $C \simeq D$ induces an isomorphism $H^n(C) \cong H^n(D)$ for all $n \in \mathbb{Z}$.

$\square$

Each R-module M may be thought as a trivial positive complex. That is $M_0 = M$ and $M_n = 0$ for $n \neq 0$ and $\partial = 0$.

Definition 2.1.10 *Let M be an R-module and C be a chain complex. A contracting homotopy for the chain map $\varepsilon : C \to M$ is a chain map $f : M \to C$ together with $\varepsilon \circ f = \mathrm{Id}_M$ and a homotopy $s : \mathrm{Id} \simeq f \circ \varepsilon$. That means a contracting homotopy consists of module homomorphisms $f : M \to C_0$ and $s_n : C_n \to C_{n+1}$, $n = 0, 1, 2 \ldots$ such that $\varepsilon \circ f = \mathrm{Id}$, $\partial_1 \circ s_0 + f \circ \varepsilon = \mathrm{Id}_{C_0}$ and $\partial_{n+1} \circ s_n + s_{n-1} \circ \partial_n = \mathrm{Id}$ for $n > 0$.*

Remark 2.1.11 *If $\varepsilon : C \to M$ has a contracting homotopy then we have $\varepsilon_* : H_0(C) \cong M$ for $n = 0$ and $H_n(C) = 0$ for $n > 0$. Contracting homotopy measures the exactness of the complex $\varepsilon : C \to M$.*

Definition 2.1.12 *A short exact sequence*

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

of chain complexes consists of chain maps $C' \to C$ and $C \to C''$ such that for each n,

$$0 \longrightarrow C'_n \xrightarrow{g_n} C_n \xrightarrow{f_n} C''_n \longrightarrow 0$$

is a short exact sequence.

Proposition 2.1.13 *Let*

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$

be a short exact sequence of chain complexes, then there is a long exact sequence

$$\dots \longrightarrow H_{n+1}(C'') \xrightarrow{\partial} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(C'') \xrightarrow{\partial} \dots$$

where ∂ is the connecting homomorphism.

□

The definition of the connecting homomorphism and the proof of this proposition can be found in [[10], Ch.2, pg. 27].

We have a similar exact sequence for cohomology:

Proposition 2.1.14 *Let*

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$

a short exact sequence of cochain complexes, then there is a long sequence

$$\dots \longrightarrow H^n(C') \xrightarrow{g^*} H^n(C) \xrightarrow{f^*} H^n(C'') \xrightarrow{\delta} H^{n+1}(C') \longrightarrow \dots$$

where δ is the connecting homomorphism.

□

2.2 Projective resolutions and cohomology

Definition 2.2.1 *An R-module P is called projective if for every homomorphism $f : P \to B$ and every epimorphism $g : A \to B$, there is a homomorphism $h : P \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "f"] & \\ A \arrow[r, "g"] & B \arrow[r] & 0 \end{tikzcd}$$

Definition 2.2.2 *An R-module I is called injective if for every homomorphism $\beta : A \to I$ and every monomorphism $\gamma : A \to B$, there is a homomorphism $\alpha : B \to I$ such that the following diagram commutes:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\gamma"] \arrow[d, "\beta"'] & B \arrow[dl, dashed, "\alpha"] \\ & I & \end{tikzcd}$$

Definition 2.2.3 *A projective resolution of an R-module M is a long exact sequence*

$$\cdots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

where each P_i is a projective R-module.

Remark 2.2.4 *Since every module is a homomorphic image of a free module and every free module is projective, projective resolution always exists.*

Theorem 2.2.5 *(Comparison Theorem) Any homomorphism of modules*

$$M \xrightarrow{f} N$$

can be extended to a chain map of projective resolutions with the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"'] & P_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] \arrow[dl, "h_n"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, "h_{n-1}"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_{n+1} \arrow[r, "\partial'_{n+1}"] & Q_n \arrow[r, "\partial'_n"] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

Given any two such chain maps f_n *and* f'_n*, there is a chain homotopy* $h_n : P_n \to Q_{n+1}$ *so that* $f_n - f'_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n$ *where* $\partial_n : P_n \to P_{n-1}$ *and* $\partial'_n : Q_n \to Q_{n-1}$ *are differentials of the resolutions.*

Proof: See [10].

□

Definition 2.2.6 *If* N *is right* R*-module and*

$$\cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \longrightarrow M \to 0$$

is a projective resolution of a left R*-module* M*, then we have a chain complex*

$$\ldots \longrightarrow N \otimes_R P_{n+1} \xrightarrow{Id \otimes \partial_{n+1}} N \otimes_R P_n \xrightarrow{Id \otimes \partial_n} N \otimes_R P_{n-1} \longrightarrow \ldots$$

$\mathrm{Tor}_n^R(N, M)$ *is defined as the homology of this complex:*

$$\mathrm{Tor}_n^R(N, M) := H_n(N \otimes P, Id \otimes \partial_*)$$

Definition 2.2.7 *If* N *is a left* R*-module and*

$$\cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \to M \to 0$$

is a projective resolution of a left R*-module* M*, then we have a cochain complex*

$$0 \longrightarrow \mathrm{Hom}_R(P_0, N) \xrightarrow{\delta^0} \mathrm{Hom}_R(P_1, N) \xrightarrow{\delta^1} \mathrm{Hom}_R(P_2, N) \to \cdots$$

Ext_R^n*(M,N) is defined as the cohomology of this complex:*

$$\mathrm{Ext}_R^n(M, N) := H^n(\mathrm{Hom}_R(P, N), \delta^*)$$

In these definitions, for $n = 0$, we have $\mathrm{Tor}_0^R(N, M) = N \otimes_R M$ and $\mathrm{Ext}_R^0(M, N) = \mathrm{Hom}_R(M, N)$.

Proposition 2.2.8 *If M is projective R-module and N is any R-module, then $\mathrm{Ext}_R^n(M, N) = 0 = \mathrm{Tor}_n^R(M, N)$ for all $n > 0$.*

$\mathrm{Tor}_n^R(-, -)$ and $\mathrm{Ext}_R^n(-, -)$ preserve direct sums.

Proposition 2.2.9 *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of left R-modules.*
i) If N is a right R-module, then there is a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Tor}_n^R(N, M_1) \to \mathrm{Tor}_n^R(N, M_2) \to \mathrm{Tor}_n^R(N, M_3) \to \dots \\ \to N \otimes_R M_1 \to N \otimes_R M_2 \to N \otimes_R M_3 \to 0 \end{aligned}$$

ii) If N is a left R-module, there is a long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(N, M_1) \to \mathrm{Hom}_R(N, M_2) \to \mathrm{Hom}_R(N, M_3) \to \\ \cdots \to \mathrm{Ext}_R^n(N, M_1) \to \mathrm{Ext}_R^n(N, M_2) \to \mathrm{Ext}_R^n(N, M_3) \to \dots . \end{aligned}$$

$\square$

$N \otimes_R -$ or $- \otimes_R N$ are covariant functors. $\mathrm{Hom}_R(N, -)$ is a covariant functor, but $\mathrm{Hom}_R(-, N)$ is a contravariant functor.

Proposition 2.2.10 *Let*

$$0 \to M_0 \to M_1 \to M_2 \to 0$$

be a short exact sequence of right R-modules.
i) N is a left R-module. Then there is a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Tor}_n^R(M_0, N) \to \mathrm{Tor}_n^R(M_1, N) \to \mathrm{Tor}_n^R(M_2, N) \to \dots \\ \to M_0 \otimes_R N \to M_1 \otimes_R N \to M_2 \otimes_R N \to 0 \end{aligned}$$

ii) Let

$$0 \to M_0 \to M_1 \to M_2 \to 0$$

be a short exact sequence of left R-modules and N is a left R-modules. Then there is a long exact sequence

$$0 \to \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N) \to \mathrm{Hom}_R(M_0, N) \to ..$$
$$\cdots \to \mathrm{Ext}^n_R(M_2, N) \to \mathrm{Ext}^n_R(M_1, N) \to \mathrm{Ext}^n_R(M_0, N) \to \dots$$

□

2.3 The Künneth theorem

Let C and D be chain complexes of right, respectively left, R-modules. We can construct a new complex in the following form

$$(C \otimes_R D)_n = \bigoplus_{i+j=n} (C_i \otimes_R C_j)$$

The differential $\partial_n : (C \otimes_R D)_n \to (C \otimes_R D)_{n-1}$ is given by

$$\partial_n(x \otimes y) = \partial_i(x) \otimes y + (-1)^i x \otimes \partial_j(y)$$

for $x \in C_i$ and $y \in D_j$ and we have $\partial_n \circ \partial_{n+1} = 0$. This formula shows that the tensor product $x_1 \otimes x_2$ of cycles is a cycle in $C \otimes D$ and the tensor product of a cycle and a boundary is a boundary. Thus if x_1 and x_2 are cycles in C and D respectively then we have a well defined group homomorphism

$$\rho : H_i(C) \otimes_R H_j(D) \to H_{i+j}(C \otimes_R D)$$

such that $\rho : [x_1] \otimes [x_2] \mapsto [x_1 \otimes x_2]$.

Definition 2.3.1 *A left R-module N is called flat if for any long exact sequence of right R-modules*

$$\cdots \to M_n \to M_{n-1} \to M_{n-2} \to \dots$$

the sequence

$$\cdots \to M_n \otimes_R N \to M_{n-1} \otimes_R N \to M_{n-2} \otimes_R N \to \dots$$

is also exact.

Theorem 2.3.2 *(The Künneth Theorem) Let C be a chain complex of right R-modules and D be a chain complex of left R-modules. If the cycles $Z_n(C)$ and the boundaries $B_n(C)$ are flat modules for all n, then there is a short exact sequence of R-modules*

$$0 \to \bigoplus_{i+j=n} H_i(C) \otimes_R H_j(D) \to H_n(C \otimes_R D) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H_i(C), H_j(D)) \to 0.$$

Proof: See [[10], Ch.2, pg. 39].

□

Let C be a chain complex such that $Z_n(C)$ and $H_n(C)$ are projective. Then the exact sequence

$$0 \to B_n(C) \to Z_n(C) \to H_n(C) \to 0$$

splits, and hence $B_n(C)$ is projective. Since projective modules are also flat, $Z_n(C)$, $H_n(C)$ and $B_n(C)$ are flat and by the definition of a flat module $\mathrm{Tor}_1^R(H_i(C), H_j(D)) = 0$. Using the Künneth Theorem, we obtain the following corollaries.

Corollary 2.3.3 *If $Z_n(C)$ and $H_n(C)$ are projective R-modules for all n, then*

$$H_n(C \otimes_R D) \cong \bigoplus_{i+j=n} H_i(C) \otimes_R H_j(D).$$

Corollary 2.3.4 *If $Z_n(C)$ and $H_n(C)$ are projective R-modules and either C or D exact, then so is $C \otimes_R D$.*

After giving definition of group cohomology, we state cohomological version of Künneth formula in the case R is a field of characteristic p.

2.4 Group cohomology

Let G be a finite group and k be a field of characteristic p. In this section, we give some properties of the projective and the injective kG-modules. We give the definition of the group cohomology and study the relation between the cohomology and extensions, in particular, we study the first cohomology $H^1(G, -)$. Using the existence of the projective cover of a kG-module M, we give the existence of the minimal projective resolution.

2.4.1 The group algebra kG

Definition 2.4.1 *Let G be a finite group with elements $\{g_1, \ldots, g_n\}$ and k be a field of characteristic p. The group ring kG is the set of all formal finite sums*

$$\{\sum_{i=1}^{n} a_i g_i, a_i \in k\}$$

with addition and multiplication defined by

$$\sum_{i=1}^{n} a_i g_i + \sum_{i=1}^{n} b_i g_i = \sum_{i=1}^{n} (a_i + b_i) g_i$$

$$(\sum_{g \in G} a_g g)(\sum_{h \in G} b_h h) = \sum_{g,h \in G} a_g b_h (gh).$$

Since k is a field, kG is a vector space with basis $g_1, \ldots, g_n$. The scalar multiplication is defined $\lambda u = \sum_{i=1}^{n} (\lambda a_i) g_i$ for $\lambda \in k$ and $u = \sum_{i=1}^{n} a_i g_i$ in kG. So kG is an algebra which we call the group algebra kG. The group algebra kG has a multiplicative identity $1 = 1_k 1_G$. For any kG-module M, we define the k-dual $M^* = \mathrm{Hom}(M, k)$ as the kG-module of the k linear homomorphisms from M to the trivial module k. M^* is a kG-module with G-action $(gf)(m) = f(g^{-1}m)$ for $g \in G, f \in M^*, m \in M$.

We now list some of the basic properties of kG.

Proposition 2.4.2 *$kG \cong kG^*$ as kG-modules, that is, kG is a Frobenius algebra.*

Proof: For proof see [[20], pg. 8].

□

Proposition 2.4.3 ([20]) *kG is an injective kG-module, that is, kG is self-injective.*

Corollary 2.4.4 *Every finitely generated injective kG-module is projective, and every finitely generated projective kG-module is injective.*

Proposition 2.4.5 ([16]) *A kG-module M is projective if and only if M is a direct summand of a free module.*

Proposition 2.4.6 *If P is a projective kG-module and M is any kG-module, then $P \otimes M$ is a projective kG-module.*

Proof: See [[20], pg. 11].

□

Definition 2.4.7 *Let M be a kG-module, H a subgroup of G, and L be a kH-module. We denote the restriction of M to H as $M\downarrow_H$. The induced module $L\uparrow^G$ as a kG-module is defined as $L\uparrow^G := kG \otimes_{kH} L$ and here kG acts by left multiplication.*

Proposition 2.4.8 *If P is a projective kG-module and H is a subgroup of G, then $P\downarrow_H$ is a projective kH-module.*

Proof: See [[4], Ch.2, pg. 33].

□

Proposition 2.4.9 *If H is a subgroup of G and L is a projective kH-module, then $L\uparrow^G$ is a projective kG-module.*

Proof: See [[4], Ch.3, pg. 57].

$\square$

2.5 Cohomology of groups and extensions

Definition 2.5.1 *Let M and N be finitely generated kG-modules. Let*

$$P_* \xrightarrow{\varepsilon} M$$

be any projective resolution of M. Applying $\mathrm{Hom}_{kG}(-, N)$ we get the complex

$$0 \to \mathrm{Hom}_{kG}(P_0, N) \to \mathrm{Hom}_{kG}(P_1, N) \to \cdots$$

Then $\mathrm{Ext}^n_{kG}(M, N)$ is defined as the cohomology of the complex in the following way.

$$\mathrm{Ext}^n_{kG}(M, N) := H^n(\mathrm{Hom}_{kG}(P_*, N)).$$

If $M = k$ is the trivial kG-module then we have a special notation $H^n(G, N) := \mathrm{Ext}^n_{kG}(k, N)$ and it is called " the n-th cohomology of G with coefficients in N".

If we have $N = k$, then $H^*(G, k) = \mathrm{Ext}^*_{kG}(k, k)$.

Note that $\mathrm{Ext}^n_{kG}(-, -)$ does not depend on the choice of the projective resolution (see [[20], Ch.2, pg. 29]).

Let $U^n(M, N)$ be the set of all exact sequences of finitely generated kG-modules of the form

$$E : 0 \to N \to B_{n-1} \to \cdots \to B_0 \to M \to 0.$$

Such sequences are called n-fold extensions of M by N.

Define a relation $\equiv$ on $U^n(M,N)$ by $E_1 \equiv E_2$ if there is a commuting diagram

$$\begin{array}{ccccccccccccc}
E_1 : 0 & \longrightarrow & N & \longrightarrow & B_{n-1} & \longrightarrow & \dots & \longrightarrow & B_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \theta_{n-1}} & & & & \downarrow{\scriptstyle \theta_0} & & \| & & \\
E_2 : 0 & \longrightarrow & N & \longrightarrow & C_{n-1} & \longrightarrow & \dots & \longrightarrow & C_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

The relation $\equiv$ is not an equivalence relation, because it is not symmetric. To have an equivalence relation define $\sim$ as follows. $E_1 \sim E_2$ provided there exists a chain $F_0, ..., F_m \in U^n(M,N)$ with $E_1 = F_0$, $E_2 = F_m$ and for each $i = 1, ..., m$ either $F_{i-1} \equiv F_i$ or $F_i \equiv F_{i-1}$. We can denote the equivalence classes of an exact sequence E by $class(E)$. There is an addition which makes $U^n(M,N)/\sim$ an abelian group. We have the following:

Theorem 2.5.2 *Let M and N be kG-modules. Then there is an isomorphism*

$$\mathrm{Ext}^n_{kG}(M,N) \cong U^n(M,N)/\sim .$$

Proof: (See [20]) Let

$$P_* \xrightarrow{\epsilon} M$$

be a projective resolution. For a given $E \in U^n(M,N)$, we get a chain map μ_*.

$$\begin{array}{ccccccccccccc}
\longrightarrow & P_{n+1} & \xrightarrow{\partial_{n+1}} & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \dots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow 0 \\
 & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \mu_{n-1}} & & & & \downarrow{\scriptstyle \mu_0} & & \| & \\
 & 0 & \longrightarrow & N & \longrightarrow & B_{n-1} & \longrightarrow & \dots & \longrightarrow & B_0 & \longrightarrow & M & \longrightarrow 0
\end{array}$$

From the diagram one gets $\mu_n \circ \partial_{n+1} = 0$ which means $\mu_n : P_n \to N$ is a cocycle. The assignment $class(E) \mapsto [\mu_n]$ gives a well defined homomorphism θ from $U^n(M,N)/\sim$ to $\mathrm{Ext}^n_{kG}(M,N)$. Conversely given $\zeta \in \mathrm{Ext}^n_{kG}(M,N)$, choose a cocycle $\hat{\zeta} : P_n \to N$ representing ζ. We have a commutative diagram

$$\begin{array}{cccccccccccc}
\longrightarrow & P_{n+1} & \longrightarrow & P_n & \xrightarrow{\partial_n} & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \dots & \longrightarrow & M \longrightarrow 0 \\
 & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \hat{\zeta}} & & \downarrow{\scriptstyle g} & & \| & & & & \| \\
 & 0 & \longrightarrow & N & \xrightarrow{f} & B & \xrightarrow{h} & P_{n-2} & \longrightarrow & \dots & \longrightarrow & M \longrightarrow 0
\end{array}$$

where B is the pushout of the first square. This gives a well defined map ϕ on the opposite direction. It is easy to see that θ and ϕ are inverses to each other.

$\square$

2.5.1 Low dimensional cohomology and group extensions

Definition 2.5.3 *An extension of a group G by a group N is a short exact sequence of groups*

$$1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1. \tag{2.1}$$

Another extension

$$1 \longrightarrow N \longrightarrow E' \longrightarrow G \longrightarrow 1 \tag{2.2}$$

of G by N is said to be equivalent to (3.1) if there is a map $E \to E'$ making the diagram

$$\begin{tikzcd} & & E \arrow[dd] \arrow[dr] & & \\ 1 \arrow[r] & N \arrow[ur] \arrow[dr] & & G \arrow[r] & 1 \\ & & E' \arrow[ur] & & \end{tikzcd}$$

commute. Such a map is necessarily an isomorphism. The main problem in the theory of group extensions is to classify the extensions of G by N up to equivalence. In fact, we are looking for all possible ways of building a group E with N as a normal subgroup and G as the quotient. This problem is closely related to the cohomology $H^i(G, -)$ for $i = 1, 2, 3$. For this section, we consider only the case where N is an abelian group written additively. In this case, G has an action on N, that is N is a G-module.

Definition 2.5.4 *A function $d : G \to N$ is called derivation if it satisfies $d(gh) = d(g) + g \cdot d(h)$ for all $g, h \in G$.*

A function $p : G \to N$ of the form $p : g \mapsto g \cdot a - a$ is called principal derivation for $g \in G$ and for some fixed $a \in N$.

There is an isomorphism between the first cohomology and the quotient group

$$H^1(G, N) \cong \mathrm{Der}(G, N)/P(G, N)$$

where $\mathrm{Der}(G,N)$ is the abelian group of derivations and $P(G,N)$ is the group of principal derivations.

In Chapter 3, we calculate the essential cohomology of an elementary abelian p-groups. In the last chapter we define inflated essential classes. For that reasons we need the followings:

Definition 2.5.5 *If G is a group, Frattini subgroup $\Phi(G)$ is defined as the intersection of all the maximal subgroups of G.*

Lemma 2.5.6 ([47]) *If G is a finite p-group, then $G/\Phi(G)$ is a vector space over $\mathbb{Z}/p\mathbb{Z}$.*

Proposition 2.5.7 ([10], Ch.3, pg. 86) *Let G be a p-group. There is a natural isomorphism*

$$H^1(G,k) = \mathrm{Ext}^1_{kG}(k,k) \cong \mathrm{Hom}(G/\Phi(G), k^+)$$

where k^+ denotes the additive group of k. Thus if $G/\Phi(G)$ is elementary abelian of rank n, then $\mathrm{Ext}^1_{kG}(k,k)$ is an n-dimensional vector space over k.

Proof: A representation of G over k is a group homomorphism $\phi : G \to GL_n(k)$ where $GL_n(k)$ is the group of non-singular $n \times n$ matrices over k, for some n. The vector space k^n is a kG-module with G-action $(\sum_i r_i g_i)x = \sum_i r_i \phi(g_i)(x)$ where $x \in k^n$. This gives a one to one correspondence between the representations and finitely generated kG-modules.

Consider the representation $\phi : G \to GL_2(k)$. An extension $0 \to k \to M \to k \to 0$ of kG-modules has a matrix representation of the form

$$\begin{pmatrix} 1 & \alpha(g) \\ 0 & 1 \end{pmatrix}$$

where $\alpha : G \to k^+$ is a homomorphism of groups from G to the additive group of k. By the help of this matrix representations, we have a one to one correspondence between $\mathrm{Ext}^1_{kG}(k,k)$ and $\mathrm{Hom}(G, k^+)$. The desired result follows from the fact that the kernel of α must contain $\Phi(G)$, since k^+ is abelian of exponent p and ker α is a maximal subgroup.

$\square$

2.6 Minimal projective and injective resolutions

Definition 2.6.1 *A projective cover of a kG-module M is a projective module P_M together with a surjective homomorphism $\varepsilon : P_M \to M$ satisfying the following property:*

If $\theta : Q \to M$ is a surjective homomorphism from a projective kG-module Q onto M, then there is an injective homomorphism $\sigma : P_M \to Q$ such that the diagram commutes:

$$\begin{tikzcd} & P_M \arrow[dl, dashed, "\sigma"'] \arrow[d, "\varepsilon"] \\ Q \arrow[r, "\theta"] & M \end{tikzcd}$$

By definition, if

$$P_M \xrightarrow{\varepsilon} M$$

is a projective cover of M, then no proper projective submodule of P_M is mapped onto M. And projective cover, if they exist, are unique up to isomorphism.

Theorem 2.6.2 *Let M be a finitely generated kG-module. Then M has projective cover.*

Proof: (See [20]) Choose P_M to be a projective kG-module of smallest k-vector space such that there exist $P_M \twoheadrightarrow M$. Suppose we are given Q and θ as in the definition above. P_M and Q are projective there is a commutative diagram

$$\begin{tikzcd} P_M \arrow[d, shift right, "\sigma"'] \arrow[dr, "\varepsilon"] & \\ Q \arrow[u, shift right, "\tau"'] \arrow[r, "\theta"] & M \end{tikzcd} .$$

Let $\varphi := \tau \circ \sigma : P_M \to P_M$.

To complete the proof it is enough to prove that φ is an automorphism. Since P_M is finite dimensional by Fitting's Lemma (see [[10], Ch.1, pg. 7]), $P_M = \ker \varphi^n \oplus \operatorname{Im} \varphi^n$ for sufficiently large n. Since P_M is projective $\ker \varphi^n$ and $\operatorname{Im} \varphi^n$ are projective. By the commutativity of the diagram, we have $\varepsilon \circ \varphi^n = \varepsilon$. By minimality, we have $\ker \varphi^n = 0$. That is φ is an automorphism. So, σ is injective as desired, and P_M is a projective cover by the definition.

$\square$

Definition 2.6.3 *A projective resolution*

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

or in short writing

$$P_* \xrightarrow{\varepsilon} M$$

is called minimal projective resolution if there is another projective resolution

$$Q_* \xrightarrow{\theta} M$$

of M*, then there is an injective chain map* $\mu_* : (P_* \twoheadrightarrow M) \hookrightarrow (Q_* \twoheadrightarrow M)$ *and a surjective chain map* $\mu_*^{'} : (Q_* \twoheadrightarrow M) \twoheadrightarrow (P_* \twoheadrightarrow M)$ *such that both* μ_* *and* $\mu_*^{'}$ *lift the identity on* M*.*

Minimal projective resolutions always exists. Let

$$P_0 \xrightarrow{\varepsilon} M$$

be a projective cover of M, $P_1 \twoheadrightarrow \ker \varepsilon$ a projective cover of $\ker \varepsilon$ and repeating the same procedure, we get the minimal projective resolution. The advantage of using a minimal projective resolution is that if W is any simple module, then the differentials in the complexes $\mathrm{Hom}_{kG}(P_*, W)$ and $P_* \otimes_{kG} W$ are trivial. For this reason

$$\mathrm{Tor}_n^{kG}(M, W) = P_n \otimes_{kG} W$$

$$\mathrm{Ext}_{kG}^n(M, W) = \mathrm{Hom}_{kG}(P_n, W)$$

for any kG-module M. In particular, $\dim_k H^n(G, k) = \dim_k \mathrm{Hom}_{kG}(P_n, k)$.

2.7 The ring structure of $H^*(G, k)$

For this section, k also denote the trivial kG-module. The existence of the projective resolution of k lets us to calculate the cohomology groups $H^n(G, k)$ which we define previously. These are abelian groups, moreover these are vector spaces over the field k. There is a product structure over the infinite direct sum $\bigoplus_{n \geq 0} H^n(G, k)$ which is called cup product (for details about cup product see [16]). This product structure gives a ring structure on the infinite direct sum. We denote this infinite sum by $H^*(G, k)$ and it is called cohomology ring of G. This is a k-algebra and it is graded. That is we have

$$H^m(G, k) \cdot H^n(G, k) \subseteq H^{m+n}(G, k).$$

We say $H^*(G, k)$ is graded commutative, because the elements of odd degree anticommute. That is, if $x \in H^m(G, k)$ and $y \in H^n(G, k)$, then $x \cdot y = (-1)^{mn} y \cdot x$.

One of the fundamental theorems in group cohomology is about finite generation of cohomology ring.

Theorem 2.7.1 (Evens [29], Venkov [56]) *Let k be any commutative Noetherian ring. The cohomology ring is finitely generated as a k-algebra, and if M is any finitely generated kG-module, then $H^*(G, M)$ is a finitely generated module over $H^*(G, k)$.*

The theorem says that $H^*(G,k)$ is Noetherian ring. Moreover, if $p = 2$, then $H^*(G,k) \cong k[x_1,...,x_n]/I$, where $x_1,...,x_n$ are homogeneous generators, and ideal I is homogeneous which means generated by homogeneous elements.

For $p > 2$, the elements of odd degree anticommute, we have nilpotent elements with nilpotency degree 2. That is $H^*(G,k) \cong k[x_1,...,x_n] \otimes \bigwedge(a_1,...,a_n)/I$ where $x_1,...,x_n$ have even degree and $a_1,...,a_n$ have odd degree. The ideal I is again homogeneous.

Example 2.7.2 Let $G = \langle g|\ g^{p^n} = 1\rangle$ and let k be a field of characteristic $p > 0$. Then

$$H^*(G,k) = \begin{cases} k[x_1] & \text{if} \quad p^n = 2 \\ k[x_1,x_2]/(x_1^2) & \text{if} \quad p^n > 2 \end{cases}$$

Here $x_i \in H^i(G,k)$.

We can compute the cohomology rings of direct product of finite groups using the Künneth theorem.

Theorem 2.7.3 ([16]) *The cohomology ring of the direct product $G_1 \times G_2$ is isomorphic to $H^*(G_1,k) \otimes_k H^*(G_2,k)$.*

Example 2.7.4 Let G be an elementary abelian p-group of rank n. That is $G = (\mathbb{Z}/p\mathbb{Z})^n$.

$$H^*(G,k) = \begin{cases} k[x_1,x_2,...,x_n] & \text{if} \quad p = 2 \\ k[y_1,y_2,...,y_n] \otimes \wedge(x_1,...,x_n) & \text{if} \quad p > 2 \end{cases}$$

Here $x_i \in H^1(G,k)$, $y_i \in H^2(G,k)$ and $x_i^2 = 0$, $\beta(x_i) = y_i$ where $\beta : H^1(G,\mathbb{F}_p) \to H^2(G,\mathbb{F}_p)$ is the connecting homomorphism in the long exact sequence associated to the short exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

It is called the Bockstein homomorphism.

For $p = 2$, the cohomology ring of a finite group is commutative and for $p > 2$, $H^*(G,k)/\mathrm{Rad}H^*(G,k) = k[x_1, x_2, ..., x_n]$ is commutative. For that reason one can study commutative algebra on $H^*(G,k)$.

As $H^*(G,k)$ is finitely generated, the aim is to find the generators and relations between these generators (see [17]) in calculations. In Chapter 3, we explain some open problems related to essential cohomology which are very important in computation methods. To explain these problems precisely, we need some basic definitions from commutative algebra such as Krull dimension, depth, regular sequence, etc. Before giving these definitions we need definitions and properties of some basic operations on group cohomology.

2.7.1 Restriction, inflation and transfer

Definition 2.7.5 *Let H be a subgroup of G. The group algebra kG is a free kH-module with basis given by any set of representatives of the left cosets of H in G. It follows that projective kG-modules are projective as kH-modules. Let (P_*, ϵ) be a projective resolution of a kG-module M. Then the restriction of this projective resolution to H is a projective resolution of M as a kH-module. For a kG-module N, applying $\mathrm{Hom}_{kG}(-, N)$ to this resolutions we get an inclusion of complexes*

$$\mathrm{Hom}_{kG}(P_*, N) \hookrightarrow \mathrm{Hom}_{kH}(P_*, N_H).$$

This inclusion induces a map on cohomology which is denoted by

$$\mathrm{res}^G_H : \mathrm{Ext}^n_{kG}(M, N) \to \mathrm{Ext}^n_{kH}(M_H, N_H)$$

for any n. It is called restriction homomorphism.

Definition 2.7.6 *Let H be a normal subgroup of G and let M be a $k(G/H)$-module. We can consider M as a kG-module on which H acts trivially. If (Q_*, ϵ) is a projective $k(G/H)$-resolution of M and if (P_*, ϵ') is a projective kG-resolution of M, then there is a chain map $\theta : (P_*, \epsilon') \to (Q_*, \epsilon)$ that lifts the identity on M. The inflation map on cohomology is induced from the map of complexes*

$\theta^* : \mathrm{Hom}_{k(G/H)}(Q_*, N) \to \mathrm{Hom}_{kG}(P_*, N)$ *and denoted by*

$$\mathrm{inf}^G_{G/H} : \mathrm{Ext}^n_{kG/H}(M, N) \to \mathrm{Ext}^n_{kG}(M, N)$$

for any $k(G/H)$-modules M and N and any $n \geq 0$.

These two maps on cohomology are ring homomorphisms (see [16]).

The transfer map on cohomology is similar to induction on modules, but needs more explanation. Let H be a subgroup of G and M, N are kG-modules. If $\alpha \in \mathrm{Hom}_{kH}(M, N)$, then $\beta = \sum_{gH} g \cdot \alpha$ where the sum is over any complete set of representatives of the left cosets of H, is an element in $\mathrm{Hom}_{kG}(M, N)$.

Definition 2.7.7 *Let (P_*, ϵ) be a kG-projective resolution of M. Let $\zeta \in \mathrm{Ext}^n_{kH}(M, N)$ for some n. The transfer of ζ, denoted by $\mathrm{tr}^G_H(\zeta)$ is the cohomology class cls($\sum_{gH} g \cdot f$) where $f : P_n \to N$ is any cocyle representing ζ.*

Transfer is not a ring homomorphism, it is a k-linear homomorphism.

Proposition 2.7.8 *Let H be a subgroup of G. Then for any $\zeta \in \mathrm{Ext}^n_{kG}(M, N)$, we have $\mathrm{tr}^G_H \mathrm{res}^G_H(\zeta) = |G : H|\zeta$.*

Proof: See [16].

$\square$

The following corollary is an important fact for the essential cohomology of a finite group G.

Corollary 2.7.9 *If P is a Sylow p-subgroup of a finite group G, then res^G_P is injective.*

As I stated before we need some definitions of ring theoretic invariants such as depth, associated primes, regular sequences and Krull dimension in Chapter 3.

Definition 2.7.10 *A set of homogeneous elements $x_1, ..., x_d$ is called a homogeneous system of parameters for $H^*(G,k)$ if they are algebraically independent and $H^*(G,k)$ is a finitely generated module over the polynomial subring $k[x_1, ..., x_d]$.*

Note that the length d above is unique.

Definition 2.7.11 *The Krull dimension of $H^*(G,k)$ is the length of any homogeneous system of parameters of $H^*(G,k)$.*

A celebrated theorem of Quillen states that the Krull dimension of $H^*(G,k)$ is equal to the p-rank of G.

Definition 2.7.12 *A sequence $x_1, ..., x_t$ of elements of $\bigoplus_{n>0} H^n(G,k)$ is a regular sequence if the multiplication by x_1 on $H^*(G,k)$ is injective and for every $i = 2, ..., t$, the multiplication by x_i on the quotient $H^*(G,k)/(x_1, ..., x_{i-1})$ is injective.*

Using this definition, we can define depth of $H^*(G,k)$.

Definition 2.7.13 *The depth of $H^*(G,k)$ is defined to be the length of the longest regular sequence of $H^*(G,k)$.*

Theorem 2.7.14 (Duflot [27]) *The depth of $H^*(G,k)$ is at least equal to the p-rank of the center of a Sylow p-subgroup of G.*

We need one more definition in Chapter 3.

Definition 2.7.15 *An associated prime for $H^*(G,k)$ is a prime ideal $\mathfrak{p} \subseteq H^*(G,k)$ such that $\mathfrak{p} = \mathrm{Ann}_{H^*(G,k)}(x)$ for some element $x \in H^*(G,k)$.*

The most important relation between essential cohomology and associated prime in group cohomology is given by the following theorem: Let $\mathcal{A}_s$ be the set of all elementary abelian p-subgroups of G of p-rank s. Let $\mathcal{H}_s = \{C_G(E) |\ E \in \mathcal{A}_s\}$.

Theorem 2.7.16 *Suppose that for some $n > 0$, there is a non-zero element $\zeta \in H^n(G,k)$, satisfying $\mathrm{res}^G_H(\zeta) = 0$ for all $H \in \mathcal{H}_s$. Then $H^*(G,k)$ has an associated prime $\mathfrak{p}$ with the property $\mathrm{dim V_G}(\mathfrak{p}) < \mathrm{s}$. Moreover, in this case the depth of $H^*(G,k)$ is less than s.*

Proof: See [21].

□

In this theorem, $V_G(\mathfrak{p})$ is the set of maximal ideals in $H^*(G,k)$ containing $\mathfrak{p}$.

Chapter 3

Essential cohomology of a finite group

Let k be a field of characteristic p. There are computer programs to calculate generators and relations for the cohomology ring $H^*(G, k)$ of a p-group G to get much more information about the algebraic structure of the cohomology ring (for details see [17], [19]). If $\mathcal{H}$ is a detecting family and if we know the cohomology ring of each member of $\mathcal{H}$, then we can get much more information about the generators of $H^*(G, k)$. But there are many examples where we do not have a detecting family. That is, there are many examples of non-trivial cohomology classes which restrict trivially on all proper subgroups. Such classes are called essential. These classes are important for many reasons which are explained in the next section.

3.1 Essential classes

Definition 3.1.1 *An element $x \in H^*(G, k)$ is called essential if for every proper subgroup $H \leq G$, we have $\mathrm{res}^G_H(x) = 0$.*

These classes form an ideal in $H^*(G,k)$. It is a graded ideal and it is denoted by $\mathrm{Ess}^*(G)$. In the literature, it is called the essential cohomology of G or essential ideal. Throughout the thesis, when we write essential cohomology we mean that essential cohomology of the corresponding finite group in the text.

It is well-known that restriction to a Sylow p-subgroup is injective (see Corollary 2.7.9). This means that when G is not a p-group, $\mathrm{Ess}^*(G)$ is zero. For that reason we concentrate on essential cohomology of p-groups. The structure of essential cohomology depends on whether G is elementary abelian or not. We can conclude from the following theorem that if G is not an elementary abelian p-group, then $\mathrm{Ess}^*(G)$ is nilpotent.

Theorem 3.1.2 (Quillen [46]) *Let M be a kG-module and let I be the ideal in $\mathrm{Ext}^*_{kG}(M,M)$ consisting of all elements ζ having the property that the restriction $\mathrm{res}^G_E(\zeta)=0$ for all elementary abelian p-subgroups E of G. Then $I^n=0$ for some positive integer n.*

In the next section, including the nilpotency degree of the essential cohomology, we explain some problems related to essential cohomology which have not been solved completely.

3.2 Problems on essential cohomology

Throughout this section, G denotes a p-group. There are many interesting p-groups for which $\mathrm{Ess}^*(G)=0$. This vanishing is an important key point in most known calculations. Because if $\mathrm{Ess}^*(G)=0$, then the cohomology ring $H^*(G,k)$ is detected on maximal subgroups [19] and using this detection, $H^*(G,k)$ can be determined completely.

There are also many examples with non-zero essential classes such as Q_8. These non-trivial classes cause difficulties in calculations because in this case cohomolog ring can not be detected on any proper subgroups, but such groups are universal detectors in the cohomology of finite groups. For that reasons it is natural to search for group theoretic characterization of such groups.

Problem 3.2.1 *For which p-groups G, is* $\mathrm{Ess}^*(G)$ *non-zero?*

This problem was originally stated as a problem in "J.F.Adams' Problem Session for Homotopy theory" which held at the Arcata Topology Conference in 1986 [22] as follows:

Can one give a useful alternative description of p-groups with non-zero essential cohomology?

There were some attempts to solve the problem. M. Feshbach's original conjecture was that $\mathrm{Ess}^*(G) \neq 0$ if and only if G satisfies the pC condition, i.e., every element of order p in G is central. The conjecture was disproved by Rusin [48]. He gave an example of a group of order 32 such that $\mathrm{Ess}^*(G) \neq 0$, but it did not satisfy 2C condition. The extra-special p-group of order p^3 and exponent p where p is an odd prime greater than 3, is the counterexample for p odd [37]. In both examples, the cohomology ring of G is not Cohen-Macaulay. Recall that the cohomology ring of G is Cohen-Macaulay if it is free and finitely generated over a polynomial subalgebra.

The latest result on the problem due to Adem and Karagueuzian. They proved that:

Theorem 3.2.2 ([1]) *Let G be a finite group, then the following two conditions are equivalent:*
(1) $H^*(G,k)$ is Cohen-Macaulay and contains non-trivial essential elements.
(2) G is a p-group and every element of order p in G is central.

The structure of a group is far away from its cohomology ring in general. Thus this kind of group theoretic characterization is not common in the cohomology of finite groups. The above theorem has an interesting consequence. If we have a finite group with Cohen-Macaulay cohomology ring and non-trivial essential classes, the same property holds for any subgroup. That is existence of non-zero essential classes gives us much more information about the algebraic structure of the group.

There are also some partial results using some ring theoretic invariants such as depth.

Proposition 3.2.3 *Suppose that G is a p-group and the depth of $H^*(G,k)$ is strictly greater than the p-rank of the center of G. Then* $\mathrm{Ess}^*(G) = 0$.

Proof: See [16].

□

Another result about this problem is given by Minh for extra special p-groups.

Theorem 3.2.4 (Minh [41]) *Let p be an odd prime. If G is an extraspecial p-group, then* $\mathrm{Ess}^*(G) = 0$ *if and only if $exp(G) = 3$ and $|G| = 3^3$.*

Note that it is still an open problem to classify all p-groups with non-zero essential classes.

Another problem is a conjecture due to Carlson. In calculations, Carlson's completion criteria depends on a couple of conjectures about the structure of the cohomology rings. One of these conjectures is related to the structure of the essential cohomology as a module over a certain subalgebra of the cohomology ring. He conjectures that:

Conjecture 3.2.5 (Carlson [19]) *If* $\mathrm{Ess}^*(G)$ *is non-zero, then it is finitely generated and free over a certain polynomial subalgebra of $H^*(G,k)$.*

The problem about calculations is the fact that $H^*(G,k)$ is an infinite object but any calculation of $H^*(G,k)$ is finite. So the problem is calculating the degree bound of generators and relations between the generators. The above conjecture is checked for calculating degree bound of relations between the generators in $H^*(G,k)$. We can state the conjecture as a problem in an explicit form:

Problem 3.2.6 (Carlson [19]) *Assume that* $\mathrm{Ess}^*(G) \neq 0$ *and the dimension of the annihilator of* $\mathrm{Ess}^*(G)$ *is* d*, the* p*-rank of the center of* G*. Let* $\zeta_1, ..., \zeta_d$ *be a regular sequence of maximal length, then is* $\mathrm{Ess}^*(G)$ *a free module over the polynomial subring* $k[\zeta_1, ..., \zeta_d]$*?*

If one can give an affirmative answer for this question, then it allows us to find an upper bound on the maximum degrees of a minimal set of relations among the generators of the cohomology ring.

David J. Green proved that for some certain p-groups, the essential cohomology is Cohen-Macaulay. This gives a partial answer for Carlson's question.

Theorem 3.2.7 (David J. Green [30]) *Let* k *be a field of characteristic* p*, and let* G *be a finite* p*-group which does not have the elementary abelian* p*-group of the order* p^2 *as a direct factor. If the essential ideal* $\mathrm{Ess}^*(G)$ *in* $H^*(G,k)$ *is non-zero, then it is a Cohen-Macaulay module with Krull dimension equal to the* p*-rank of the centre of* G*.*

For the remaining groups, Carlson's conjecture is still open.

By Proposition 3.2.3, we see that $\mathrm{Ess}^*(G) = 0$ unless depth of $H^*(G,k)$ is equal to the p-rank of $Z(G)$. This also gives relations between essential cohomology and depth of $H^*(G,k)$. The existence of non-zero essential classes determines the depth of $H^*(G,k)$. In addition, there is another problem related to depth and associated primes in $H^*(G,k)$. This problem is Carlson's Depth Conjecture.

Problem 3.2.8 (Carlson [21]) *Does the cohomology ring always have an associated prime* $\mathfrak{p}$ *whose dimension equal to the depth of the cohomology ring* $H^*(G,k)$*?*

The relation between the Carlson's Depth Conjecture and the essential cohomology is as follows:

Proposition 3.2.9 ([17]) *If the essential cohomology is non-zero, then the dimension of its annihilator is equal to the p-rank of the center.*

The dimension of the annihilator A of the essential cohomology of G is the same as the dimension of its variety (for varieties one can see [11]) $V_G(A)$ or the Krull dimesion of the ring $H^*(G,k)/A$. In fact, proposition concerns with the existence of an associated prime $\mathfrak{p}$ whose dimension is equal to the depth of the cohomology ring. In fact, if $\mathrm{Ess}^*(G) \neq 0$, then it has an element whose annihilator is a prime ideal and having dimension equal to the depth of the cohomology ring.

The conjecture is stated for any finite group and David J. Green gives an answer to the conjecture in the case where G is a p-group.

Theorem 3.2.10 (Green [32]) *Suppose that G is a p-group whose center has p-rank z. Then the following statements are equivalent:*

1. The mod p-cohomology ring $H^*(G,k)$ is not detected on the centralizers of its rank $z+1$ elementary abelian subgroups.

2. There is an associated prime $\mathfrak{p}$ such that $H^*(G,k)/\mathfrak{p}$ has dimension z.

3.The depth of $H^*(G,k)$ equals to z.

Finding a group theoretic characterization of p-groups with non-zero essential classes also gives an affirmative answer to the Carlson's Depth conjecture and determines the depth as well.

As we stated previously, if G is elementary abelian then there is a non-nilpotent class which is equal to the product of Bocksteins of all non-zero one dimensional classes. If G is not an elementary abelian p-group then $\mathrm{Ess}^*(G)$ is nilpotent. One of the most important problems is what the nilpotency degree of

$\mathrm{Ess}^*(G)$ is. H.Mùi [44] and T.Marx [39] conjecture that if G is a finite p-group which is not elementary abelian, then $\mathrm{Ess}^*(G)^2 = 0$. This is known as essential conjecture.

In [42], Minh gives an upper bound for the nilpotency degree of $\mathrm{Ess}^*(G)$.

Theorem 3.2.11 *Let G be a non-elementary abelian p-group. If x is essential, then $x^p = 0$.*

David J. Green gives a counterexample to the essential conjecture.

Theorem 3.2.12 (Green [31]) *Let G be a Sylow 2-subgroup of finite unitary group $SU_3(4)$. Then $\mathrm{Ess}^*(G)^2 \neq 0$. To be more precise, there are essential classes in $H^4(G, \mathbb{F}_2)$ and $H^{10}(G, \mathbb{F}_2)$ whose product is non-zero. This non-zero element of $H^{14}(G, \mathbb{F}_2)$ is the last survivor in the sense of Benson and Carlson [12]. Moreover there are essential classes in degree six and eight whose product is the last survivor.*

In [42], Minh predicted that nilpotency degree was 2, because each essential class was a sum of transfer form proper subgroups. The question is whether essential classes are always sums of transfers. The answer is no.

Corollary 3.2.13 (Green, [31]) *Let G be a Sylow 2-subgroup of $SU_3(4)$. Then $\mathrm{Ess}^*(G)$ is not contained in the ideal $\mathrm{Tr(G)}$ in $H^*(G, \mathbb{F}_2)$ which is generated by all transfers from proper subgroups.*

Chapter 4

Essential cohomology of $(\mathbb{Z}/p\mathbb{Z})^n$

For a finite group G, elementary abelian p-subgroups of G play an important role in cohomology and modular representation theory of G. There are many remarkable papers which show the importance of these subgroups. Quillen's Dimension Theorem on varieties [46], Chouinard's theorem on projective modules [25], and the other applications such as [5], [6], [8], [9], [18], [46] are some of these papers. The module theory for group algebras of elementary abelian p-groups is different from that other p-groups (see [49]).

As we state in the previous chapter, the elementary abelian p-groups also determine the structure of essential cohomology. It is very well-known that if G is not an elementary abelian p-group, then $\mathrm{Ess}^*(G)$ is nilpotent (see Theorem 3.1.2). On the contrary, the essential cohomology of an elementary abelian p-group has a non-nilpotent class which is equal to product of Bocksteins of all one dimensional classes of $H^*(G, k)$. Work to date on the essential cohomology concentrate on non-elementary abelian case [31], [42]. In this study, we give a complete treatment for the elementary abelian case. In this chapter, we calculate the essential cohomology of an elementary abelian p-group. This chapter is a detailed version of the paper [7].

4.1 The generators of $\mathrm{Ess}^*((\mathbb{Z}/p\mathbb{Z})^n)$

Let V be an elementary abelian group of rank n. We prove that $\mathrm{Ess}^*(V)$ is a free module over the polynomial part of the cohomology ring $H^*(V,k)$ and the free generators are the $SL(V)$-invariants of the action of $GL(V)$ over the cohomology ring $H^*(V,k)$. These invariants which we define in the next section are called Mùi invariants.

4.1.1 Mùi invariants

Let k be a field and V be a n-dimensional k-vector space. Note that in this case we have the isomorphism $GL(n,k) \cong GL(V)$. Consider the natural action of $GL(V)$ on V^*. This action induces an action on symmetric algebra $S(V^*)$. The Dickson invariants (see [26], [58]) generate the invariants for the induced action of $GL(V)$ on the polynomial algebra $S(V^*)$. There is also an induced action of $GL(V)$ on the polynomial tensor exterior algebra $S(V^*)\bigotimes_k \bigwedge(V^*)$, and the Mùi invariants are the $SL(V)$-invariants of this action. For more information see Mùi's original paper [45] or Crabb's paper [26]. To see all these in an explicit way, consider an elementary abelian p-group $G = (\mathbb{Z}/p\mathbb{Z})^n$. This group is a vector space over $\mathbb{F}_p$. So let us denote this elementary abelian p-group as V. The cohomology ring of an elementary abelian p-group is well-known.

If $p = 2$, then $H^*(V,\mathbb{F}_2) = \mathbb{F}_2[x_1,...,x_n]$ where $x_i \in H^1(V,\mathbb{F}_2)$. If $p > 2$, then $H^*(V,\mathbb{F}_p) = \mathbb{F}_p[x_1,...,x_n]\bigotimes_{\mathbb{F}_p} \bigwedge(a_1,...,a_n)$ where $a_i \in H^1(V,\mathbb{F}_p)$ and $x_i \in H^2(V,\mathbb{F}_p)$ and $\beta(a_i) = x_i$ where β is Bockstein connecting homomorphism.

Let $f(a_1,...,a_n,x_1,...,x_n)$ be an element in $H^*(V,\mathbb{F}_p)$. The action of $GL(n,\mathbb{F}_p) \cong GL(V)$ on f is the following: Let $w = (w_{ij})$ be an element in $GL(V)$, then $(wf)(a_1,...,a_n,x_1,...,x_n) = f(wa_1,...wa_n,wx_n,...,wx_n)$ where $wa_j = \sum_{i=1}^n w_{ij}a_i$ and $wx_j = \sum_{i=1}^n w_{ij}x_i$ for $1 \le j \le n$.

If $w \cdot f = f$ for all $w \in GL(V)$, then f is called an invariant of $GL(V)$. Now we can define the Mùi invariants as follows:

Definition 4.1.1 *Denote by L_n the polynomial*

$$L_n(X_1, ..., X_n) = \begin{vmatrix} X_1 & X_2 & ... & X_n \\ {X_1}^p & {X_2}^p & ... & {X_n}^p \\ ... & ... & ... & ... \\ {X_1}^{p^{n-1}} & {X_2}^{p^{n-1}} & ... & {X_n}^{p^{n-1}} \end{vmatrix} \in \mathbb{F}_p[X_1, ..., X_n].$$

Consider the linear combination $a_1X_1 + a_2X_2 + \ldots a_nX_n$. If $a_i \neq 0$ but $a_j = 0$ for all $1 \leq j < i$ then a_i is called the leading coefficient. We have a well-known alternative description of L_n as given in the following way:

Lemma 4.1.2 *1. L_n is the product of all monic linear forms in $X_1, ..., X_n$.*

2. For an n-dimensional $\mathbb{F}_p$-vector space V we may define $L_n(V) \in S(V^*)$ up to a non-zero scalar multiple by

$$L_n(V) = \lambda \prod_{[x] \in \mathbb{P}V^*} x.$$

Proof: A linear form in $X_1, ..., X_n$ is monic if the leading coefficient is one. For the first part, observe that $L_n(X_1, ...X_n)$ and the product of all monic linear forms in $X_1, ..., X_n$ have the same total degree and the right side divides the left side. The coefficient of $X_1{X_2}^p{X_3}^{p^2} \cdots {X_n}^{p^{n-1}}$ is $+1$ for $L_n(X_1, ...X_n)$ and for the product of all monic linear forms in $X_1, ..., X_n$. All these imply that they are equal.

The second part follows from the first part.

$\square$

Example 4.1.3 *Let $n = 2$ and $p = 3$. Then $L_2(X_1, X_2) = X_1X_2(X_1 + X_2)(X_1 + 2X_2)$.*

Now consider the polynomials in $H^*(V, \mathbb{F}_p) = \mathbb{F}_p[x_1, ..., x_n] \bigotimes_{\mathbb{F}_p} \bigwedge(a_1, ..., a_n)$ defined as follows: Let $(s_1, ..., s_k)$ be a sequence of integers with $0 \leq s_1 < s_2 < ... < s_k < n$. For $0 \leq s \leq n-1$, define

$$M_{n,s} = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ {x_1}^{p^{s-1}} & {x_2}^{p^{s-1}} & \dots & {x_n}^{p^{s-1}} \\ {x_1}^{p^{s+1}} & {x_2}^{p^{s+1}} & \dots & {x_n}^{p^{s+1}} \\ \dots & \dots & \dots & \dots \\ {x_1}^{p^{n-1}} & {x_2}^{p^{n-1}} & \dots & {x_n}^{p^{n-1}} \end{vmatrix}.$$

And let $L_n(x_1, ..., x_n)$ be as defined in previous part. The product

$$M_{n,s_1} \cdot M_{n,s_2} \cdots M_{n,s_k}$$

has the factor ${L_n}^{k-1}$ (see [45]). We have Mùi invariants as follows:

$$M_{n,s_1,\dots s_k} = (-1)^{k(k-1)/2} M_{n,s_1} \cdots M_{n,s_k} / {L_n}^{k-1}.$$

For a short writing, let $S = \{s_1, ..., s_k\}$, $M_{n,S} = (-1)^{k(k-1)/2} M_{n,s_1} \cdots M_{n,s_k} / {L_n}^{k-1}$. Note in particular that $L_n = M_{n,\emptyset}$.

These are $SL(V)$-invariants, because for $w \in GL(V)$ we have $w \cdot M_{n,s} = \det w \cdot M_{n,s}$ and $w \cdot L_n = \det w \cdot L_n$ where $\det : GL(V) \to (\mathbb{Z}/p\mathbb{Z})^*$ is the determinant function.

Example 4.1.4 *Let $n = 2$ and $p = 3$. Then there are 2^2 Mùi invariants:*

$$M_{2,1} = \begin{vmatrix} a_1 & a_2 \\ x_1 & x_2 \end{vmatrix}, M_{2,0} = \begin{vmatrix} a_1 & a_2 \\ x_1^3 & x_2^3 \end{vmatrix}, L_2 = \begin{vmatrix} x_1 & x_2 \\ x_1^3 & x_2^3 \end{vmatrix}, M_{2,0,1} = -a_1 \cdot a_2$$

All these polynomials are essential classes in $H^*(V, \mathbb{F}_p)$.

These invariants are very important in describing the cohomology of symmetric groups (see [2, 45]).

4.1.2 Relations between $\mathrm{Ess}^*((\mathbb{Z}/p\mathbb{Z})^n)$ and Mùi invariants

For this section we need to consider the direct sum decomposition of the cohomology ring

$$H^*(V,\mathbb{F}_p) = \bigoplus_{k=0}^{n} N_k(V)$$

where n is the rank of V and

$$N_k(V) = \mathbb{F}_p[x_1, ..., x_n] \otimes_{\mathbb{F}_p} \wedge^k(a_1, ..., a_n).$$

If we consider the restriction to each subgroups we see that the essential cohomology is well-behaved with respect to this decomposition. That is $\mathrm{Ess}^*(V)$ can be decomposed as follows:

$$\mathrm{Ess}^*(\mathrm{V}) = \bigoplus_{k=0}^{n} N_k(V) \cap \mathrm{Ess}^*(V).$$

For example let $V = (\mathbb{Z}/p\mathbb{Z})^2$ and $H^*(V,\mathbb{F}_p) = \mathbb{F}_p[x,y] \otimes \bigwedge(a,b)$. Then for $\zeta \in H^*(V,\mathbb{F}_p)$, we have $\zeta = a\cdot b\cdot f_1(x,y)+a\cdot f_2(x,y)+b\cdot f_3(x,y)+f_4(x,y)$. Then $\mathrm{res}^G_H(\zeta)=0$ if and only if $\mathrm{res}^G_H(a\cdot b\cdot f_1(x,y))=0$, $\mathrm{res}^G_H(a\cdot f_2(x,y)+b\cdot f_3(x,y))=0$ and $\mathrm{res}^G_H(f_4(x,y))=0$ for any proper subgroup H of G.

Lemma 4.1.5 *$M_{n,s} \in N_1(V) \cap \mathrm{Ess}^*(V)$.*

Proof: It is clear that $M_{n,s} \in N_1(V)$ by definition. The cohomology group $H^1(G,\mathbb{F}_p)$ is isomorphic to $\mathrm{Hom}(G,\mathbb{F}_p)$. So for a p-group G, any non-zero cohomology class $\zeta \in H^1(G,\mathbb{F}_p)$ corresponds to a homomorphism $\tilde{\zeta}$ whose kernel is a maximal subgroup $\ker(\tilde{\zeta}) = H_\zeta \subseteq G$. It is clear that the restriction of ζ to the kernel H_ζ is zero. For each maximal subgroup H there is a corresponding one dimensional class which is a linear combination of $a_1, ..., a_n$. If we restrict to H we kill the corresponding linear combination. But this gives a linear dependence on the a_i's and the same linear dependence on the x_i's. So this is the linear dependence between the columns of $\mathrm{res}^G_H(M_{n,s})$ which means $\mathrm{res}^G_H(M_{n,s})=0$.

□

Lemma 4.1.6 $\mathrm{Ess}^*(V)^2 = L_n(V) \cdot \mathrm{Ess}^*(V)$

Proof: Each factor of $L_n(V)$ is a Bockstein of a class in $H^1(V)$ and each one dimensional class corresponds a maximal subgroup of V, which means that $L_n(V)$ is essential. As $L_n(V)$ is essential we have $L_n(V) \cdot \mathrm{Ess}^*(V) \subseteq \mathrm{Ess}^*(V)^2$. For the converse, let H be the maximal subgroup of V and ζ be the corresponding non-zero one dimensional class and $x = \beta(\zeta)$. The kernel of the restriction to H is an ideal I_H which is generated by ζ and x. If $f, g \in I_H$ then we may write $f = \zeta \cdot f' + x \cdot f''$ and $g = \zeta \cdot g' + x \cdot g''$ and $f \cdot g = \zeta \cdot x \cdot (f'' \cdot g' \pm f' \cdot g'') + f'' \cdot g'' \cdot x^2$, that is $f \cdot g = x \cdot h$ where $h = (f''g' \pm f'g'') \cdot \zeta + f'' \cdot g'' \cdot x$. This means $h \in I_H$.

On the other hand, $H^*(V, \mathbb{F}_p)$ is a free module over the unique factorization ring $k[x_1, ..., x_n]$. This gives that $f \cdot g = L_n(V) \cdot y$ for some $y \in H^*(V, \mathbb{F}_p)$ and $h = \frac{L_n(V)}{x} \cdot y$ since $f \cdot g = L_n(V) \cdot y = x \cdot h$. As $h \in I_H$ that is $\mathrm{res}^G_H(h) = 0$ and $\mathrm{res}^G_H(\frac{L_n(V)}{x})$ is a non-zero divisor, we deduce that $\mathrm{res}^G_H(y) = 0$ which means $y \in \mathrm{Ess}^*(V)$.

$\square$

Corollary 4.1.7 $M_{n,s_1,\ldots,s_r} \in N_r(V) \cap \mathrm{Ess}^*(V)$.

Remark 4.1.8 *Observe that*

$$M_{n,S} \cdot M_{n,T} = \left\{ \begin{array}{lll} (-1)^{|S||T|} L_n(V) M_{n,S\cup T} & if & S \cap T = \emptyset \\ 0 & & \text{otherwise} \end{array} \right\}.$$

where $S = \{s_1, ..., s_r\}$ *with* $s_1 < ... < s_r$ *and* $T = \{t_1, ..., t_k\}$ *with* $t_1 < ... < t_k$ *are subsets of* $\{0, ..., n-1\}$

4.1.3 The main theorem

In this part, we investigate the structure of $\mathrm{Ess}^*(V)$ as a module over the polynomial subalgebra $k[x_1, ..., x_n]$ of the cohomology ring $H^*(V, \mathbb{F}_p)$ and calculate the generators of $\mathrm{Ess}^*(V)$. The case $p = 2$ is well-known: Recall that for an elementary abelian 2-group G, the cohomology ring is $H^*(G, \mathbb{F}_2) = \mathbb{F}_2[x_1, ..., x_n]$

where $\{x_1, ..., x_n\}$ is a basis for $H^1(V, \mathbb{F}_2)$. The essential cohomology of G is given as follows:

Lemma 4.1.9 *Let V be an elementary abelian 2-group. The essential cohomology* $\mathrm{Ess}^*(V)$ *is the principal ideal in $H^*(V, \mathbb{F}_2)$ generated by $L_n(x_1, ..., x_n)$. Moreover* $\mathrm{Ess}^*(V)$ *is the free $\mathbb{F}_2[x_1, ..., x_n]$-module with free generator $L_n(V)$.*

Proof: Recall that every maximal subgroup is the kernel of a non-zero linear form. The restriction to a maximal subgroup kills the corresponding linear form which is a factor of $L_n(V)$. This means $L_n(V)$ is essential. For the converse part, assume that y is essential, and let $x \in V^*$ be a non-zero linear form. Now consider the subspace W spanned by x and let U be a complement of W. Now we can write $y = y_1 \cdot x + y_2$ where $y_1 \in \mathbb{F}_2[x_1, ..., x_n]$ and $y_2 \in S(U)$. For $H = \ker(x)$, $\mathrm{res}^V_H(y_2) = 0$ since y is essential. The restriction $\mathrm{res}^V_H : V^* \to H^*$ satisfies $\ker(\mathrm{res}^V_H) \cap U = 0$, thus res^V_H is injective on $S(U)$ which means $y_2 = 0$. We get $y = y_1 \cdot x$. Repeating this procedure for all maximal subgroups we get that $L_n(V)$ divides y since $S(V^*)$ is a unique factorization ring. So $\mathrm{Ess}^*(V)$ is the principal ideal generated by $L_n(V)$. It is clearly free on this one generator.

□

The main result of this section is the following:

Theorem 4.1.10 *Let p be an odd prime and V a rank n elementary abelian p-group. Then as a module over the polynomial subalgebra $k[x_1, ..., x_n]$ of the cohomology ring $H^*(V, \mathbb{F}_p)$, the essential cohomology* $\mathrm{Ess}^*(V)$ *is free on the set of Mùi invariants.*

For the proof of this theorem we need to calculate the joint annihilators of the $M_{n,S}$ with $|S| = r$.

Lemma 4.1.11 *The joint annihilator of $M_{n,0}, M_{n,1}, ..., M_{n,n-1}$ is $N_n(V)$.*

Proof: The product $a_1 \cdots a_n$ is a basis for $\wedge^n(a_1, ..., a_n)$ and is annihilated by each $M_{n,s}$ since ${a_i}^2 = 0$. For the converse, suppose that $y \neq 0$ is annihilated by each $M_{n,s}$. Observing that $M_{n,s} \cdot N_r(V) \subseteq N_{r+1}(V)$, we may assume without loss of generality that $y \in N_r(V)$ for some r. Multiplying once or more by suitably chosen elements a_i, we can reduce to the case $y \in N_{n-1}$.

Consider the field of fractions of $k[x_1, ..., x_n]$ and denote it by K. Let $W = K \otimes_k \wedge^{n-1}(a_1, ..., a_n)$. Consider the linear form $\phi_s : W \to K$ given by $\phi_s(w)a_1 \cdots a_n = M_{n,s} \cdot w$ for each $M_{n,s}$. Since $M_{n,s} \cdot y = 0$ it follows that y is in the kernel of ϕ_s. For W, the elements $a_1 \cdots \hat{a_r} \cdots a_n$ for $1 \leq r \leq n$ form a basis and then we have

$$M_{n,s} \cdot a_1 \cdots \hat{a_r} \cdots a_n = (-1)^{r+1} \gamma_{s,r} a_r \cdot a_1 \cdots \hat{a_r} \cdots a_n,$$

and thus $\phi_s(a_1 \cdots \hat{a_r} \cdots a_n) = \gamma_{r,s}$.

Now consider the matrix $\Gamma \in M_n(K)$ given by $\Gamma_{r,s} = \gamma_{r,s}$. Let C be the matrix with entries $C_{s,i} = x_i^{s-1}$ for $1 \leq s \leq n$. If one transposes Γ and then multiplies the i-th row by $(-1)^i$ and the j-th column by $(-1)^j$, then one obtains the adjugate matrix of C. As the determinant of C is $L_n(V)$ and in particular non-zero, it follows that $\det\Gamma \neq 0$. The construction of Γ gives that ϕ_s form a basis of W^*. In this case their common kernel should be zero, which means $y = 0$. Contradiction.

□

Corollary 4.1.12 *The joint annihilator of* $\{M_{n,S} : |S| = r\}$ *is* $\bigoplus_{s \geq n-r+1} N_s(V)$.

Proof: Proof follows by induction on r. The case $r = 1$ is the previous lemma. The annihilator is as large as claimed because $M_{n,S} \in N_{|S|}(V)$ and $N_r(V)N_{|S|}(V) \subseteq N_{r+|S|}$. Suppose that there exists an element $y \in H^*(V, \mathbb{F}_p)$ which annihilates all $\{M_{n,S} : |S| = r\}$ but is not in $\bigoplus_{s \geq n-r+1} N_s(V)$. So one can write $y = \sum_{s=0}^n y_s$ where $y_s \in N_s(V)$. Let $s_0 = \min\{s| \ \ y_s \neq 0\}$. Since y is not in $\bigoplus_{s \geq n-r+1} N_s(V)$ we have $s_0 \leq n - r$. And y_{s_0} is not in $N_n(V)$ which means that it is not an annihilator for any $M_{n,t}$ for $0 \leq t \leq n$ by Lemma 4.1.11. $y_{s_0} \cdot M_{n,t} \in N_{s_0+1}(V)$ we conclude that $y \cdot M_{n,t}$ lies outside $\bigoplus_{s \geq n-r+2} N_s(V)$. Inductive hypothesis means that there is some T with $|T| = r - 1$ such that the annihilator of $M_{n,T}$ is $\bigoplus_{s \geq n-r+2} N_s(V)$. Since $y \cdot M_{n,t}$ lies outside $\bigoplus_{s \geq n-r+2} N_s(V)$

we have $y \cdot M_{n,t} \cdot M_{n,T} \neq 0$. But this means that $y \cdot M_{n,S} \neq 0$ for $S = T \cup \{t\}$ and $|S| = r$. Contradiction (note that since $M_{n,t} \cdot M_{n,t} = 0$, $t \in T$ is not possible).

□

Corollary 4.1.13 *Every $M_{n,S}$ is non-zero. For $S = \underline{n} = 0, ..., n-1$ we have $M_{n,\underline{n}}$ is a non-zero scalar multiple of $a_1 \cdot a_2 \cdots a_n$.*

Proof: Checking the degree of $M_{n,\underline{n}}$ and the degree of the product $a_1 \cdot a_2 \cdots a_n$ we see that $M_{n,\underline{n}}$ is a scalar multiple of $a_1 \cdot a_2 \cdots a_n$. Letting $r = n$ in Corollary 4.1.12, we see that $1 \in N_0$ does not annihilate $M_{n,\underline{n}}$, that is $1 \cdot M_{n,\underline{n}} \neq 0$. By Remark 4.1.8, every $M_{n,S}$ divides $L_n(V) \cdot M_{n,\underline{n}} \neq 0$, so as a divisor $M_{n,S}$ is non-zero.

□

Proof: (Proof of Theorem 4.1.10) As we said before, we can decompose $\mathrm{Ess}^*(V)$ as

$$\mathrm{Ess}^*(\mathrm{V}) = \bigoplus_{r=0}^{n} N_r(V) \cap \mathrm{Ess}^*(V).$$

So it is enough if we show that the Mùi invariants

$$\{M_{n,S} \mid |S| = r\}$$

form a basis for $k[x_1, ..., x_n]$-module $N_r(V) \cap \mathrm{Ess}^*(V)$ for each r. We know that by Corollary 4.1.7, each $M_{n,S}$ lies in this module.

Let $y \in N_r(V) \cap \mathrm{Ess}^*(V)$. We need to write y as

$$y = \sum_{|S|=r} f_S M_{n,S}$$

for some $f_S \in k[x_1, ..., x_n]$. Let $T = \underline{n} - S$. Then we have $M_{n,S} \cdot M_{n,T} = \pm L_n(V) \cdot M_{n,\underline{n}}$ as $T \cap S = \emptyset$. Define $\varepsilon_s \in \{+1, -1\}$ by $M_{n,S} \cdot M_{n,T} = \varepsilon_s L_n(V) \cdot M_{n,\underline{n}}$. Let's define f_s by $f_s \cdot M_{n,\underline{n}} = \frac{1}{L_n(V)} \varepsilon_s y \cdot M_{n,T}$ so that $M_{n,S'} \cdot M_{n,T} = 0$ for all $S' \neq S$ with $|S| = r$. This definition of f_s makes sense because $y M_{n,T}$ lies in both $N_r(V)N_{n-r} = N_n(V)$ and $L_n(V) \cdot \mathrm{Ess}^*(V)$ as $y \cdot M_{n,T} \in \mathrm{Ess}^*(V)^2$. Consider the annihilator of $M_{n,T}$ for every $|T| = n - r$. By Corollary 4.1.12, the joint annihilator of it is $\sum_{s \geq r+1} N_s(V)$. On the other hand by definition of f_s we can conclude that

$$(y - \sum_{|S|=r} f_s M_{n,S}) \cdot M_{n,T} = 0$$

for every $|T| = n - r$. But $(y - \sum_{|S|=r} f_s \cdot M_{n,S})$ can not be annihilator as it lies in $N_r(V)$. So $y = \sum_{|S|=r} f_s \cdot M_{n,S}$.

As a last step we need to show that $M_{n,S}$ are linearly independent. Suppose that $g_S \in k[x_1, ..., x_n]$ are such that $\sum_{|S|=r} g_S \cdot M_{n,S} = 0$. Take one S and let $T = \underline{n} - S$. Multiplying by $M_{n,T}$ we conclude that

$$g_S \cdot M_{n,S} \cdot M_{n,T} = g_S.\varepsilon_S L_n(V) \cdot M_{n,\underline{n}} = 0.$$

This follows that $g_S = 0$.

$\square$

4.2 The mod-p Steenrod algebra and $\mathrm{Ess}^*((\mathbb{Z}/p\mathbb{Z})^n)$

The Steenrod operations were first conceived as operations on the cohomology of topological spaces with coefficients $\mathbb{Z}/p\mathbb{Z}$ for some prime p. The cohomology of a finite group G can be viewed as the cohomology of its classifying space BG, that is why Steenrod operations can be applied to the mod p-cohomology of G. In this chapter, we prove that $\mathrm{Ess}^*(V)$ is in fact the Steenrod closure of the product $a_1 \cdots a_n$. To prove this we need to study the action of Steenrod Algebra on cohomology ring for any finite group and also on Mùi invariants.

4.2.1 Steenrod closedness

The action of Steenrod Algebra $\mathcal{A}$ is given in the following theorems in the case where $p = 2$ and p is odd. We see that the cohomology ring of a finite group is a module over the Steenrod Algebra.

Theorem 4.2.1 *Let G be a finite group and $p = 2$. The cohomology ring $H^*(G, \mathbb{F}_2)$ is a module over the mod-2 Steenrod Algebra $\mathcal{A}$. Moreover the action of the operations has the following properties:*

1. The additive homomorphisms $Sq^i : H^r(G, \mathbb{F}_2) \to H^{r+i}(G, \mathbb{F}_2)$ are natural transformations of functors for all i and all r.

2. Suppose we have $\zeta \in H^r(G, \mathbb{F}_2)$. If $r < i$, then $Sq^i(\zeta) = 0$ and $Sq^r(\zeta) = \zeta^2$.

3. The Bockstein homomorphism is equal to the first Steenrod square,

$$\beta = Sq^1 : H^r(G, \mathbb{F}_2) \to H^{r+1}(G, \mathbb{F}_2)$$

4. (Cartan Formula)

$$Sq^i(\eta \cdot \zeta) = \sum_{j=0}^{i} Sq^j(\eta) Sq^{i-j}(\zeta).$$

In the p odd case, we have very similar action:

Theorem 4.2.2 *Let G be a finite group and p be odd prime. The cohomology ring $H^*(G, \mathbb{F}_p)$ is a module over the mod-p Steenrod Algebra $\mathcal{A}$. The action of the operations has the following properties:*

1. The additive homomorphisms $\mathcal{P}^i : H^r(G, \mathbb{F}_p) \to H^{r+2(p-1)i}(G, \mathbb{F}_p)$ are natural transformations of functors for all i and all r.

2. Suppose we have $\zeta \in H^r(G, \mathbb{F}_p)$. If $r < 2i$, then $\mathcal{P}^i(\zeta) = 0$.

3. If $\zeta \in H^2 r(G, \mathbb{F}_p)$, then $P^r(\zeta) = \zeta^p$.

4. The Cartan formula,

$$\mathcal{P}^i(\eta \cdot \zeta) = \sum_{j=0}^{i} \mathcal{P}^j(\eta) \mathcal{P}^{i-j}(\zeta).$$

Note that the naturality of these operations give that restriction on group cohomology rings is an $\mathcal{A}$-module homomorphism. This means that $\mathrm{Ess}^*(G)$ is invariant under the action of Steenrod algebra.

The cohomology ring is not an arbitrary module over $\mathcal{A}$. It is an unstable module over $\mathcal{A}$.

Definition 4.2.3 *An unstable $\mathcal{A}$-module is a graded $\mathcal{A}$-module $M = \sum_{i\geq 0} M^i$ which satisfies the instability conditions:*

- *For $p = 2$, M is unstable if for any homogeneous $m \in M$, then we have $Sq^j(m) = 0$ for $\deg(m) < j$.*

- *For the case p is odd, M is unstable if for every homogeneous $m \in M$, $i \in \{0, 1\}$ then $\beta^i \mathcal{P}^j(m) = 0$ whenever $\deg(m) < 2j + i$.*

Definition 4.2.4 *In addition to the above conditions, suppose that M is an algebra. We say that M is an unstable $\mathcal{A}$-algebra if the multiplication is compatible with the Steenrod operations as it is expressed in Cartan Formula, and if it satisfies the following conditions:*

- *For $p = 2$, for every $m \in M^r$, $Sq^r(m) = m^2$*

- *For $p > 2$, for every $m \in M^{2r}$, $\mathcal{P}^r(m) = m^p$.*

Now by Theorem 4.2.1 and Theorem 4.2.2 it is easy to see that the cohomology ring $H^*(G, \mathbb{F}_p)$ is an unstable $\mathcal{A}$-algebra(For details about Steenrod Algebra one can see [16]).

We are ready to define Steenrod closedness:

Definition 4.2.5 *Let $\mathcal{K}$ be an unstable $\mathcal{A}$-algebra and $\mathcal{T}$ is a homogeneous subset of $\mathcal{K}$. The Steenrod closure of $\mathcal{T}$ is the smallest homogeneous ideal which contains $\mathcal{T}$ and is closed under the Steenrod algebra action.*

Note that since multiplication is compatible with Steenrod operations, the Steenrod closure of $\mathcal{T}$ is the ideal generated by the set $\{\alpha(t) | \alpha \in \mathcal{A}, t \in \mathcal{T}\}$.

The reason why we need this definition is that we show that the essential cohomology of an elementary abelian p-group is the Steenrod closure of the one dimensional subspace generated by $a_1 \cdots a_n$ in $H^*(V,k) = k[x_1,...,x_n] \otimes \wedge(a_1,...,a_n)$ in the case where p is odd. It is clear that the product $a_1 \cdots a_n$ is essential. If one wants to have more essential classes, she may apply Steenrod operations to the product. This is the motivation to consider the Steenrod closures.

4.2.2 Action of the Steenrod algebra on Mùi invariants

The second main result of this chapter is that $\mathrm{Ess}^*(V)$ is the Steenrod closure of the product $a_1 \cdots a_n$. To prove this we need the action of Steenrod algebra on Mùi invariants.

Lemma 4.2.6

$$\beta(M_{n,s}) = \begin{cases} L_n(V) & \text{if} \quad s = 0 \\ 0 & \quad\text{otherwise} \end{cases} \tag{4.1}$$

For $0 \leq s \leq n-2$ *we have:*

$$\mathcal{P}^{p^s}(M_{n,r}) = \begin{cases} M_{n,r-1} & \text{if} \quad r = s+1 \\ 0 & \quad\text{otherwise} \end{cases} \tag{4.2}$$

In particular $\beta(L_n(V)) = 0$ *and* $\mathcal{P}^{p^s}(L_n(V)) = 0$.

Proof: For equation (4.1) consider the definition of $M_{n,s}$:

$$M_{n,s} = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ {x_1}^{p^{s-1}} & {x_2}^{p^{s-1}} & \dots & {x_n}^{p^{s-1}} \\ {x_1}^{p^{s+1}} & {x_2}^{p^{s+1}} & \dots & {x_n}^{p^{s+1}} \\ \dots & \dots & \dots & \dots \\ {x_1}^{p^{n-1}} & {x_2}^{p^{n-1}} & \dots & {x_n}^{p^{n-1}} \end{vmatrix}.$$

Applying Bockstein to $M_{n,s}$ is just applying it to the first row because $\beta(x_i) = 0$. So $\beta(M_{n,0}) = L_n(V)$ and for $s > 0$, $\beta(M_{n,s}) = 0$ because $\beta(M_{n,s})$ is equal to the determinant of a matrix with repeated rows. $\beta(L_n(V)) = 0$ because

$$L_n(V) = \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ {x_1}^{p} & {x_2}^{p} & \dots & {x_n}^{p} \\ {x_1}^{p^2} & {x_2}^{p^2} & \dots & {x_n}^{p^2} \\ \dots & \dots & \dots & \dots \\ {x_1}^{p^{n-1}} & {x_2}^{p^{n-1}} & \dots & {x_n}^{p^{n-1}} \end{vmatrix}.$$

where each $x_i = \beta(a_i)$ and $\beta(x_i) = 0$. For equation (4.2) we again consider the determinants and some properties of Steenrod powers. Note that for every $m > 0$, $\mathcal{P}^m(a_i) = 0$ and $\mathcal{P}^m(x_i^{p^s}) = x_i^{p^{s+1}}$ if $m = p^s$ and zero otherwise. So we just concerns with m which is a p-th power. If we consider the Cartan formula

$$\mathcal{P}^m(xy) = \sum_{i+j=m} \mathcal{P}^i(x)\mathcal{P}^j(y)$$

we only need to apply $\mathcal{P}^{p^s}$ to one row and other rows stay unchanged. That is we have

$$\mathcal{P}^{p^{s-1}}(M_{n,s}) = \mathcal{P}^{p^{s-1}}\left(\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ {x_1}^{p^{s-1}} & {x_2}^{p^{s-1}} & \dots & {x_n}^{p^{s-1}} \\ {x_1}^{p^{s+1}} & {x_2}^{p^{s+1}} & \dots & {x_n}^{p^{s+1}} \\ \dots & \dots & \dots & \dots \\ {x_1}^{p^{n-1}} & {x_2}^{p^{n-1}} & \dots & {x_n}^{p^{n-1}} \end{vmatrix} \right) = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ {x_1}^{p^{s-2}} & {x_2}^{p^{s-2}} & \dots & {x_n}^{p^{s-2}} \\ {x_1}^{p^{s}} & {x_2}^{p^{s}} & \dots & {x_n}^{p^{s}} \\ \dots & \dots & \dots & \dots \\ {x_1}^{p^{n-1}} & {x_2}^{p^{n-1}} & \dots & {x_n}^{p^{n-1}} \end{vmatrix}$$

as we just apply $\mathcal{P}^{p^{s-1}}$ to the row $({x_1}^{p^{s-1}}, {x_2}^{p^{s-1}}, ..., {x_n}^{p^{s-1}})$. But second determinant is just $M_{n,s-1}$ nothing else. For $\mathcal{P}^{p^s}(L_n(V))$ consider

$$\mathcal{P}^{p^s}(L_n(V)) = \mathcal{P}^{p^s}(\begin{vmatrix} x_1 & x_2 & ... & x_n \\ ... & ... & ... & ... \\ {x_1}^{p^{s-1}} & {x_2}^{p^{s-1}} & ... & {x_n}^{p^{s-1}} \\ {x_1}^{p^s} & {x_2}^{p^s} & ... & {x_n}^{p^s} \\ {x_1}^{p^{s+1}} & {x_2}^{p^{s+1}} & ... & {x_n}^{p^{s+1}} \\ ... & ... & ... & ... \\ {x_1}^{p^{n-1}} & {x_2}^{p^{n-1}} & ... & {x_n}^{p^{n-1}} \end{vmatrix}) = \begin{vmatrix} x_1 & x_2 & ... & x_n \\ ... & ... & ... & ... \\ {x_1}^{p^{s-1}} & {x_2}^{p^{s-1}} & ... & {x_n}^{p^{s-1}} \\ {x_1}^{p^{s+1}} & {x_2}^{p^{s+1}} & ... & {x_n}^{p^{s+1}} \\ {x_1}^{p^{s+1}} & {x_2}^{p^{s+1}} & ... & {x_n}^{p^{s+1}} \\ ... & ... & ... & ... \\ {x_1}^{p^{n-1}} & {x_2}^{p^{n-1}} & ... & {x_n}^{p^{n-1}} \end{vmatrix}$$

Since we have repeated rows in the right side of the equality we get zero.

□

Lemma 4.2.7 *Let* $S = \{s_1, ..., s_r\}$ *with* $0 \leq s_1 < s_2 < ... < s_r \leq n-1$.

1. *Suppose that* 0 *is not in* S*. Then* $M_{n,S} = \beta(M_{n,S\cup\{0\}})$.

2. $L_n(V)^{r-1}\mathcal{P}^m(M_{n,S}) = \mathcal{P}^m(M_{n,s_1} \cdots M_{n,s_r})$ *for each* $m < p^{n-1}$.

3. $1 \leq u \leq n-1$ *set* $X = \{s \in S | s \leq u\}$ *and* $Y = \{s \in S | s > u\}$*. Then* $L_n(V)\mathcal{P}^{p^{u-1}}(M_{n,S}) = \mathcal{P}^{p^{u-1}}(M_{n,X}) \cdot M_{n,Y}$.

4. *For* $0 \leq r \leq n-1$ *and* $0 < m < p^{n-1}$*, one has* $\mathcal{P}^m(M_{n,0,\dots,r}) = 0$

5. *For* $1 \leq u \leq n-1$ *one has* $\mathcal{P}^{p^{u-1}}(M_{n,0,\dots,u-2,u}) = M_{n,1,\dots,u-1}$.

Proof: (1)By definition, $M_{n,S}$ satisfies the following:

$$L_n(V)^{r+1} \cdot M_{n,S\cup\{0\}} = L_n(V) \cdot M_{n,0} \cdot M_{n,s_1} \cdots M_{n,s_r}. \tag{4.3}$$

Apply Bockstein to the equality (4.3) above. Then we have

$$L_n(V)^{r+1} \cdot \beta(M_{n,S\cup\{0\}}) = L_n(V) \cdot \beta(M_{n,0}) \cdot M_{n,s_1} \cdots M_{n,s_r}$$

by equation (4.1). As $\beta(M_{n,0}) = L_n(V)$ and $M_{n,S} = \frac{1}{L_n(V)^{r-1}} \cdot M_{n,s_1} \cdots M_{n,s_r}$ we get that $M_{n,S} = \beta(M_{n,S\cup 0})$.

(2) Apply $\mathcal{P}^m$ to $L_n(V)^r \cdot M_{n,S} = L_n(V) \cdot M_{n,s_1} \cdots M_{n,s_r}$. Then we get $L_n(V)^{r-1}\mathcal{P}^m(M_{n,S}) = \mathcal{P}^m(M_{n,s_1} \cdots M_{n,s_r})$ by equation (4.2).

(3) Recall that by the Adem relations we can express each $\mathcal{P}^m$ in terms of the $\mathcal{P}^{p^s}$ with $p^s \leq m$. Since $S = X \cup Y$ and $X \cap Y = \emptyset$, we have $L_n(V) \cdot M_{n,S} = M_{n,X} \cdot M_{n,Y}$ and apply $\mathcal{P}^m$ to this. Third part follows since $\mathcal{P}^m(M_{n,s}) = 0$ if $0 < m \leq p^{u-1}$ and $s > u$.

(4) Consider induction on r. Case $r = 0$ follows from the Adem relations and Equation (4.2). Assume that $\mathcal{P}^m(M_{n,0,1,\dots,r_1}) = 0$. We know that it is enough to consider just only $\mathcal{P}^{p^s}$ for $0 \leq s \leq n-2$. Applying $\mathcal{P}^{p^s}$ to $L_n(V) \cdot M_{n,0,\dots,r} = M_{n,0,\dots,r-1} \cdot M_{n,r}$ we deduce that

$$L_n(V) \cdot \mathcal{P}^{p^s}(M_{n,0,\dots,r}) = M_{n,0,\dots,r-1} \cdot \mathcal{P}^{p^s}(M_{n,r})$$

by inductive step. But this is zero because by Equation (6.2) we have
$\mathcal{P}^{p^s}(M_{n,r}) = M_{n,r-1}$ and $M_{n,0,\dots,r-1} \cdot M_{n,r-1} = 0$.

(5) Using the fourth part and a similar argument to the third part we deduce that

$$L_n(V) \cdot \mathcal{P}^{p^{u-1}}(M_{n,0,\dots,u-2,u}) = M_{n,\{1,\dots,u-2\}} \cdot \mathcal{P}^{p^{u-1}}(M_{n,u}) = M_{n,\{0,\dots,u-2\}} \cdot M_{n,u-1}$$

but this is just $L_n(V) \cdot M_{n,0,\dots,u-1}$.

□

4.3 $\mathrm{Ess}^*((\mathbb{Z}/p\mathbb{Z})^n)$ and the Steenrod closedness

Using the action of Steenrod algebra on Mùi invariants, we can prove the following theorem:

Theorem 4.3.1 *Let p be an odd prime and V a rank elementary abelian p-group. Then the essential cohomology* $\mathrm{Ess}^*(V)$ *is the Steenrod closure of the product*

$a_1 \cdots a_n$. *That is* $\mathrm{Ess}^*(V)$ *is the smallest ideal in* $H^*(V, \mathbb{F}_p)$ *which contains the one dimensional space generated by* $a_1 \cdots a_n$ *in* $H^*(V, \mathbb{F}_p)$ *and is closed under the action of the Steenrod algebra.*

Proof: We know that $M_{n,\underline{n}}$ is a non-zero scalar multiple of $a_1 \cdots a_n$ and the Mùi invariants generates $\mathrm{Ess}^*(V)$. So it is sufficient to show that for every $M_{n,S}$ there is an element θ in the Steenrod Algebra $\mathcal{A}$ such that $M_{n,S} = \theta(M_{n,\underline{n}})$. For this part proof follows on decreasing induction on $r = |S|$. $r = n$ is the trivial case since in this case we can consider θ as identity in $\mathcal{A}$. Now assume that $r < n$. Consider all S with $|S| = r$ and let u be the smallest element of $\underline{n} - S$. Then we can write $S = \{0, ..., u-1\} \cup Y$ with $s > u$ for every $s \in S$. Now we can apply induction on u. The case $u = 0$ follows from the first part of the Lemma 4.2.6. So assume $u \geq 2$. Set $T = \{0, ..., u-2, u\}$. We can complete the induction by showing that $M_{n,S} = P^{p^{u-1}}(M_{n,T\cup Y})$. Part three of Lemma 4.2.7 gives that

$$L_n(V) \cdot P^{p^{u-1}}(M_{n,T\cup Y}) = P^{p^{u-1}}(M_{n,T}) \cdot M_{n,Y}.$$

Part 5 of the Lemma 4.2.6 says that $P^{p^{u-1}}(M_{n,T}) = M_{n,\{1,\ldots,u-1\}}$. So we get $P^{p^{u-1}}(M_{n,T\cup Y}) = M_{n,S}$ as we claimed.

$\square$

One can calculate $\mathrm{Ess}^*(V)$ for small ranks by direct calculations.

Example 4.3.2 Let $V = (\mathbb{Z}/p\mathbb{Z})^2$. Then the cohomology ring is $H^*(V, k) = \mathbb{F}_2[x, y] \otimes_{\mathbb{F}_2} \wedge(a, b)$ where $a, b \in H^1(V, k)$ and $x, y \in H^2(V, k)$ and $x = \beta(a)$ and $y = \beta(b)$. We are claiming that there are 2^2 generators for the essential cohomology of G and the generators are:

$$M_{2,1} = \begin{vmatrix} a & b \\ x & y \end{vmatrix}, M_{2,0} = \begin{vmatrix} a & b \\ x^3 & y^3 \end{vmatrix}, L_2 = \begin{vmatrix} x & y \\ x^3 & y^3 \end{vmatrix}, M_{2,0,1} = -a \cdot b.$$

In fact, here we have $M_{2,1} = -\beta(a \cdot b)$, $M_{2,0} = -P\beta(a \cdot b)$, $M_{2,0,1} = -\beta P\beta(a \cdot b)$. Let $I = \langle a \cdot b, \beta(a \cdot b), P\beta(a \cdot b), \beta P\beta(a \cdot b)\rangle$. We are claiming that $\mathrm{Ess}^*(V) = I$.

It is clear that $I \subset \mathrm{Ess}^*(V)$.

For the converse let $\zeta \in \mathrm{Ess}^*(V)$. As an element of $H^*(V, \mathbb{F}_2)$

$$\zeta = a \cdot b \cdot f_1(x, y) + a \cdot f_2(x, y) + b \cdot f_3(x, y) + f_4(x, y).$$

Since ζ is essential $f_4(x, y) = L_2(x, y) \cdot f(x, y)$ and know it is enough to consider $\zeta = a \cdot f_2(x, y) + b \cdot f_3(x, y)$ because $a \cdot b$ is essential. Restriction of ζ to the kernel of a gives that $f_3(x, y) = x \cdot h_(x, y)$. Restriction to the kernel of b gives that $f_2(x, y) = y \cdot g(x, y)$. Then $\zeta = a \cdot y \cdot g(x, y) + b \cdot x \cdot h(x, y)$. Consider the sum:

$$\hat{\zeta} = \zeta + g(x, y) \cdot \beta(a \cdot b).$$

The sum is also essential and we have

$$\zeta \equiv \hat{\zeta} \text{ mod I.}$$

Since $\hat{\zeta} = b \cdot H(x, y)$ where $H(x, y) = g(x, y) + h(x, y)$ is essential, by restrictions to the kernels of $x+y, x+2y, ..., x+(p-1)y$ we get $\hat{\zeta} = (b \cdot x^p - b \cdot x \cdot y^{p-1}) \cdot \hat{H}(x, y)$. On the other hand we have $b \cdot x^p - b \cdot x \cdot y^{p-1} = P\beta(a \cdot b) - \beta(a \cdot b) \cdot y^{p-1}$. This means that $\hat{\zeta} \equiv 0$ mod I. So $\mathrm{Ess}^*(V) \subset I$.

Chapter 5

Relative cohomology of finite groups

Let G be a finite group and let R be a commutative ring with unity. To define relative cohomology, we consider definition of relative projectivity of an RG-module. Relative projectivity can be studied in different forms both in representation theory and category theory. One can study relative projectivity of an RG-module with respect a subgroup of G [10], [36], [33]; with respect to a collection of subgroups [14, 35], with respect to a G-set [10], with respect to a module [23]. Using definitions of relative projectivity with respect to those objects above, we can define relative cohomology with respect a subgroup of G , with respect to a collection of subgroups of the group, with respect to a G-set and with respect to an RG-module.

In this chapter, we consider the relative cohomology of G with respect to a collection of subgroups of G. We find some relations between the relative cohomology and essential cohomology of G.

5.1 Relative Cohomology of a finite group with respect to a collection of subgroups of the group

To define relative cohomology of G with respect to a collection of subgroups of G, we define relatively H-projectivity where H is a subgroup of G.

Definition 5.1.1 *Let H be a subgroup of G. An RG-module M is said to be projective relative to H or relatively H-projective if whenever we are given RG-modules M_1 and M_2, a map $\lambda : M \to M_1$ and an epimorphism $\mu : M_2 \to M_1$ such that there exists a map of RH-modules $\nu : M \downarrow_H \to M_2 \downarrow_H$ with $\lambda = \mu \circ \nu$, then there exists a map of RG-modules $\nu' : M \to M_2$ with $\lambda = \mu \circ \nu'$.*

Note that if H is the trivial subgroup and R is a field, then this is the definition of projective RG-module.

Definition 5.1.2 *A short exact sequence of RG-modules is H-split if it splits when it is restricted to H.*

In the rest of the chapter, we use some other definitions of relatively H-projectivity. The following proposition gives these equivalent definitions of relatively H-projectivity.

Proposition 5.1.3 (D.G.Higman, [36]) *Let M be an RG-module and H a subgroup of G. Then the following are equivalent*

i) M is projective relative to H

ii) Every H-split epimorphism of RG-modules $\lambda : M' \to M$ (i.e. one which splits as a map of RH-modules) splits.

iii) M is a direct summand of $M \downarrow_H \uparrow^G$

iv) M is a direct summand of some module induced from H.

Proof: See [10].

□

We can also define relative projectivity with respect to a collection of subgroups of G.

Definition 5.1.4 *Let $\mathcal{H}$ be a collection of subgroups of G. An RG-module M is said to be relatively $\mathcal{H}$-projective if each indecomposable direct summand of M is relatively H-projective for some $H \in \mathcal{H}$.*

Definition 5.1.5 *A short exact sequence is said to be $\mathcal{H}$-split if it splits when it is restricted to each member $H \in \mathcal{H}$.*

Definition 5.1.6 *A relatively $\mathcal{H}$-projective resolution of an RG-module M is a long exact sequence*

$$\to X_n \to \cdots \to X_2 \to X_1 \to X_0$$

of RG-modules such that

i) $X_0/\operatorname{Im}\partial_1 \cong M$ where $\partial_1 : X_1 \to X_0$ is the boundary map.

ii) Each X_i is relatively $\mathcal{H}$-projective.

iii) Each short exact sequence $0 \to \ker(\partial_n) \to X_n \to \operatorname{Im}\partial_n \to 0$ splits when it is restricted to each member $H \in \mathcal{H}$. Here $\partial_n : X_n \to X_{n-1}$ for $n \geq 1$ is the boundary map.

Note that the map $\bigoplus_{H\in\mathcal{H}} M \downarrow_H\uparrow^G \to M$ is always $\mathcal{H}$-split. That means relatively $\mathcal{H}$-projective resolution always exist.

Also note that, by definition an RG-module N is relatively $\mathcal{H}$-projective if its indecomposable summands are relatively H-projective for some $H \in \mathcal{H}$, but there is not necessarily a summand for each $H \in \mathcal{H}$. That is relatively $\mathcal{H}$-projective resolution depends only on the maximal non-conjugate subgroups H contained in $\mathcal{H}$ and not all subgroups.

Theorem 5.1.7 (The relative comparison theorem, [10]) *Given a map of modules $M \to M'$ and relatively $\mathcal{H}$-projective resolution $\mathcal{P}$ and $\mathcal{P}'$ of M and M' respectively, we can extend to a map of chain complexes $\{f_n\} : X_n \to X_n^{'}$ and given any two such maps, f_n, $f_n^{'}$ there is a contracting chain homotopy $h_n : X_n \to X'_{n+1}$ such that $f_n - f'_n = \partial_{n+1} \circ h_n + h_{n+1} \circ \partial_n$.*

Now we can define the relative $\mathcal{H}$-cohomology as follows:

Definition 5.1.8 *Let M' be an RG-module and*

$$\mathcal{P} : \cdots \to X_n \to \cdots \to X_2 \to X_1 \to X_0 \to M \to 0$$

be a relatively $\mathcal{H}$-projective resolution of M. Define

$$\mathrm{Ext}^n_{G,\mathcal{H}}(M, M') = H^n(\mathrm{Hom}_{RG}(\mathcal{P}, M'), \delta^*)$$

and the relative $\mathcal{H}$-cohomology of G is

$$H^n(\mathcal{H}, G, M) = \mathrm{Ext}^n_{G,\mathcal{H}}(R, M).$$

We may view elements of $\mathrm{Ext}^n_{G,\mathcal{H}}(M, M')$ as equivalence classes of $\mathcal{H}$-split n-fold extensions

$$0 \to M' \to M_{n-1} \to \cdots \to M_0 \to M \to 0.$$

Equivalence relation is given by a map of $\mathcal{H}$-split n-fold extensions taking one to the other.

For a special case, the definition of relative cohomology with respect to a collection of subgroups is equivalent to the definition of relative cohomology with respect a finite G-set which we define in the next section.

5.2 Relative cohomology with respect to a finite G-set X

Let G be a finite group and k be a field of characteristic p. Let X be a finite G-set and kX denote the permutation module whose basis is given by the elements of X.

Definition 5.2.1 *An exact sequence*

$$0 \to A \to B \to C \to 0$$

of kG-modules is said to be X-split if

$$0 \to A \otimes_k kX \to B \otimes_k kX \to C \otimes_k kX \to 0$$

splits.

Definition 5.2.2 *A kG-module M is said to be projective relative to X, or X-projective, if there exists a kG-module N such that M is a direct summand of $kX \otimes N$.*

For example, kX itself is an X-projective kG-module.

Now we can define an X-projective resolution of a kG-module M, using these two definitions.

Definition 5.2.3 *A long exact sequence*

$$P_* : \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is said to be an X-projective resolution of M if each P_i is X-projective and for each i, the short exact sequence

$$0 \to \ker(\partial_n) \to P_n \to \operatorname{im}(\partial_n) \to 0$$

where $\partial_n : P_n \to P_{n-1}$ is the boundary map in the resolution, is X-split.

We give the definition of an X-projective cover of a kG-module M to define a minimal X-projective resolution.

Definition 5.2.4 *For any kG-module M, (P, ε) is said to be an X-projective cover of M if P is X-projective and $\varepsilon : P \to M$ is a right X-split surjection and has no X-projective summands in its kernel.*

A minimal X-projective resolution of M is an X-projective resolution in which each P_n is the minimal X-projective cover of $\ker(\partial_n)$. Every kG-module has a minimal X-projective resolution and it is unique up to chain homotopy equivalence (see [23]). The usual comparison theorem for projective resolutions also holds for the relative projectivity and this enables us to define relative cohomology. If

$$P_* : \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to k \to 0$$

is an X-projective resolution of k, then we have a cochain complex

$$0 \to \mathrm{Hom}_{kG}(P_0, k) \to \mathrm{Hom}_{kG}(P_1, k) \to \cdots \to \mathrm{Hom}_{kG}(P_n, k) \to \cdots .$$

The cohomology groups of this chain complex are independent of choice of X-projective resolution and we define $X\mathrm{Ext}^n_{kG}(k, k) = H^n(\mathrm{Hom}_{kG}(P_*, k), \delta^*)$. Using this we can define the X-relative cohomology to be

$$XH^n(G, k) = X\mathrm{Ext}^n_{kG}(k, k).$$

As it is in usual group cohomology, we can consider the elements of $X\mathrm{Ext}^n_{kG}(k, k)$ as equivalence classes of X-split n-fold extensions

$$0 \to k \to M_{n-1} \to M_{n-2} \to \cdots \to M_0 \to k \to 0.$$

Two such extensions are equivalent if there is a map of X-split n-fold extensions taking one to the other. Note that two X-split n-fold extensions can be equivalent as n-fold extensions without being equivalent as X-split n-fold extensions (see [10]).

There is a relation between the H-relative cohomology and X- relative cohomology when X is transitive permutation representation of G with point stabilizer H. This is given in the following lemma not only for field k but also for ring R.

Lemma 5.2.5 *Suppose X is a transitive permutation representation of G with point stabilizer H. Then a short exact sequence of RG-modules is X-split if and only if it is H-split. An RG-module M is relatively X-projective if and only if it is relatively H-projective.*

Proof: See [10].

□

Corollary 5.2.6 *If X is the set of coset representatives of a collection $\mathcal{H}$ of subgroups of G, then X-relative cohomology is isomorphic to $\mathcal{H}$-relative cohomology.*

Lemma 5.2.7 *Let G be a finite group and X be a finite G-set. If G acts on X without fixed point (that is if $g \cdot x = x$ for some $x \in X$, then $g = 1$.), then $XH^n(G, k)$ is agree with usual cohomology group $H^n(G, k)$.*

Proof: Recall that any finite G-set X can be written as a disjoint union of left cosets, $X = \sqcup_{H \leq G} G/H$ where H is point stabilizer of some point $x \in X$. Then the kG-module kX is the direct sum of the induced modules $\bigoplus_{H \leq G} k \uparrow_H^G$. Since X is fixed point free $H = 1$ which means $kX = \bigoplus kG$. In this case the X-projective resolution is just the usual free (projective) resolution of G.

□

The above lemma means that the relative cohomology is interesting for the case X is not fixed point-free.

5.3 Relations between X-relative cohomology and essential cohomology

For this section, let G be a p-group and k be a field of characteristic p. Let $\mathcal{H}$ be the collection of all maximal subgroups of G and let X be the set of all cosets of maximal subgroups of G, and let kX denote the permutation module associated to X. As a kG-module, kX is the direct sum of permutation modules of type $k\uparrow_H^G$ where the sum is over all maximal subgroups H of G.

For any finite G-set X, each X-split n-fold extension can be considered as a n-fold extension of kG-modules. This means that we have a map

$$\varphi_{G,X} : XH^n(G,k) \to H^n(G,k).$$

This is a group homomorphism. Two X-split n-fold extension may be equivalent as n-fold extension of kG-modules without being equivalent as X-split n-fold extensions which means that $\varphi_{G,X}$ is not injective in general (see [10]).

This homomorphism gives a relation between the relative cohomology with respect to a particular G-set and essential cohomology of G. We are interested in the image of $\varphi_{G,X}$.

Proposition 5.3.1 *Suppose X is the set of all cosets of maximal subgroups of G. Then the image of $\varphi_{G,X}$ lies in* $\mathrm{Ess}^*(G)$.

Proof: Let $\mathcal{H}$ be the collection of maximal subgroups of G. By Corollary 5.2.6, X-relative cohomology and $\mathcal{H}$-relative cohomology of G is equivalent. Thus restriction of any X-split n-fold extension to any maximal subgroup $H \in \mathcal{H}$ is zero by definition of $\mathcal{H}$-relative cohomology of G. This means that any X-split n-fold extension is essential.

□

Because of that relation, it is interesting to study this homomorphism more closely.

Let X be a G-set. Consider the normal subgroup N of G which fixes all elements of X. The quotient group G/N also acts on X and the action is given by $(gN)x = gx$ where $g \in G$ and $x \in X$.

Lemma 5.3.2 *Let N be a normal subgroup of G which fixes all elements of X. Then*

$$XH^n(G,k) \cong XH^n(G/N,k)$$

for all $n \geq 0$

Proof: Consider the X-projective resolution of k where the action is G-action:

$$\cdots \to kX^{\otimes^n} \to kX^{\otimes^{n-1}} \to \cdots \to kX \otimes kX \to kX \to k \to 0.$$

Since X is a G/N-set we can view this resolution as X-projective resolution of k with G/N-action. This gives an isomorphism from $\mathrm{Hom}_{kG/N}(kX^{\otimes^n}, k)$ onto $\mathrm{Hom}_{kG}(kX^{\otimes^n}, k)$ for each $n \geq 1$. Thus we have a cochain complex which has isomorphic groups at each dimension $n \geq 0$.

□

Corollary 5.3.3 *Let X be a transitive G-set and N be a normal subgroup that fixes an element $x \in X$. Then $XH^n(G,k) \cong H^n(G/N,k)$.*

Proof: Since X is a transitive G-set all point stabilizers equal to N and X is fixed point free as G/N-set. Then corollary follows from the previous lemma.

□

Proposition 5.3.4 *All X-split extensions of length ≥ 2 are equivalent to an extension of the form*

$$0 \to k \to \cdots \to kX \to k \to 0.$$

Proof: Consider the X-projective resolution of k

$$\cdots \to kX^{\otimes^n} \to kX^{\otimes^{n-1}} \to \cdots \to kX \otimes kX \to kX \to k \to 0.$$

As it is in the proof of Theorem 2.5.2, we have a commutative diagram

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & kX^{\otimes^n} & \xrightarrow{\partial_n} & kX^{\otimes^{n-1}} & \longrightarrow & kX^{\otimes^{n-2}}\cdots & \longrightarrow & kX & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \| & & \| & & \\
0 & \longrightarrow & k & \longrightarrow & B & \longrightarrow & kX^{\otimes^{n-2}}\cdots & \longrightarrow & kX & \longrightarrow & k & \longrightarrow & 0
\end{array}$$

where B is the pushout of the diagram. The proposition follows from the fact that pushout of X-split sequence is also X-split (see [23]).

□

Lemma 5.3.5 *Let X be a G-set and N be a normal subgroup which acts on X trivially. Then* $\operatorname{Im}\varphi_{G,X} \subseteq \operatorname{Im}\inf^G_{G/N}$.

Proof: We have a commutative diagram

$$\begin{array}{ccc} XH^n(G/N,k) & \xrightarrow{\varphi_{G/N,X}} & H^n(G/N,k) \\ \downarrow & & \downarrow{\scriptstyle \inf^G_{G/N}} \\ XH^n(G,k) & \xrightarrow{\varphi_{G,X}} & H^n(G,k) \end{array}$$

where the left vertical map is the isomorphism in Lemma 5.3.2. The statement follows from the commutativity.

□

Theorem 5.3.6 *Let G be a finite p-group. Suppose that X is the set of all cosets of maximal subgroups. Then*

$$\operatorname{Im}\varphi_{G,X} \subseteq \inf^G_{G/\Phi(G)}(\operatorname{Ess}^*(G/\Phi(G))).$$

Proof: We know that $\operatorname{Im}\varphi_{G/\Phi(G),X}$ lies in $\operatorname{Ess}^*(G/\Phi(G))$ by Proposition 5.3.1. On the other hand $\Phi(G)$ acts on X trivially because by definition it is the intersection of all maximal subgroups in G. By Lemma 5.3.5, $\operatorname{Im}\varphi_{G,X} \subseteq \inf^G_{G/\Phi(G)}(\operatorname{Im}\varphi_{G/\Phi(G),X}) \subseteq \inf^G_{G/\Phi(G)}(\operatorname{Ess}^*(G/\Phi(G)))$.

□

Definition 5.3.7 *We call the ideal generated by* $\operatorname{Im}\varphi_{G,X}$ *the relative essential cohomology of G and denote it by* $\operatorname{RelEss}^*(G)$.

It is clear that for a composite group $\operatorname{RelEss}^*(G)$ is zero, because $\operatorname{Ess}^*(G)$ is zero. It is natural to ask for which p-groups $\operatorname{RelEss}^*(G)$ is zero.

Problem 5.3.8 *For which p-groups is* $\mathrm{RelEss}^*(G)$ *zero?*

If we classify p-groups so that $\mathrm{RelEss}^*(G)$ is non-zero, we give a partial answer for the open problem about classification of groups with non-zero essential classes. In some sense, this problem is a particular form of the open problem. One approach to the problem is to consider for which p-groups $\mathrm{inf}^G_{G/\Phi(G)}(\mathrm{Ess}^*(G/\Phi(G))) = 0$. By Theorem 5.3.6, $\mathrm{RelEss}^*(G)$ lies in the ideal generated by $\mathrm{inf}^G_{G/\Phi(G)}(\mathrm{Ess}^*(G/\Phi(G)))$ which we define as inflated essential cohomology and denote by $\mathrm{InfEss}^*(G)$ in the next chapter. We know that $\mathrm{RelEss}^*(G) \subseteq \mathrm{InfEss}^*(G)$, but we do not know whether the converse is true or not.

If we find p-groups with $\mathrm{InfEss}^*(G) = 0$, then we find a partial solution to the Problem 5.3.8. It is partial because there may be p-groups with $\mathrm{inf}^G_{G/\Phi(G)}(\mathrm{Ess}^*(G/\Phi(G))) \neq 0$, but $\mathrm{RelEss}^*(G) = 0$. We study the Problem 5.3.8 in the next chapter.

Chapter 6

Inflated essential cohomology

In Chapter 3, we calculated essential cohomology of elementary abelian p-groups. Recall that for any finite p-group, the Frattini quotient $G/\Phi(G)$ is an elementary abelian p-group. For this chapter we consider the the image $\inf^G_{G/\Phi(G)}(\mathrm{Ess}^*(G/\Phi(G)))$. It is well known that $\inf^G_{G/\Phi(G)}$ is a ring homomorphism, but the image $\inf^G_{G/\Phi(G)}(\mathrm{Ess}^*(G/\Phi(G)))$ is not an ideal unless $\inf^G_{G/\Phi(G)}$ is surjective. In this chapter, we consider the ideal generated by $\inf^G_{G/\Phi(G)}(\mathrm{Ess}^*(G/\Phi(G)))$ which we call inflated essential cohomology. We give some partial results on classifying p-groups G with $\inf^G_{G/\Phi(G)}(\mathrm{Ess}^*(G/\Phi(G))) = 0$.

6.1 Inflated essential cohomology when $p = 2$

Lemma 6.1.1 *Let G be a p-group and N be a normal subgroup of G which is contained in all maximal subgroups. Then*

$$\inf^G_{G/N}(\mathrm{Ess}^*(G/N)) \subseteq \mathrm{Ess}^*(G).$$

Proof: Let $x \in \mathrm{Ess}^*(G/N)$. We have

$$\mathrm{res}^G_H(\inf^G_{G/N}(x)) = \inf^H_{H/N}(\mathrm{res}^{G/N}_{H/N}(x))$$

and $\mathrm{res}^{G/N}_{H/N}(x) = 0$ because x is essential and any maximal subgroup of G/N is in the form H/N.

$\square$

Corollary 6.1.2 *Let G be a p-group. Then*

$$\inf_{G/\Phi(G)}^{G}(\mathrm{Ess}^*(G/\Phi(G))) \subseteq \mathrm{Ess}^*(G).$$

Definition 6.1.3 *We call the ideal generated by $\inf_{G/\Phi(G)}^{G}(\mathrm{Ess}^*(G/\Phi(G)))$ the inflated essential cohomology of G.*

We denote this ideal by $\mathrm{InfEss}^*(G)$ for a finite group G. Since inflated essential cohomology is a subset of $\mathrm{Ess}^*(G)$, it is important to get non-zero essential classes for G, and because of that it is natural to ask for which p-groups $\mathrm{InfEss}^*(G)$ is non-zero. In fact, this is a particular form of the original open problem on essential cohomology. Using this inflated essential classes we can not classify all p-groups with non-zero essential classes, but we can find a subset of p-groups with non-zero essential classes.

The classification has different results for odd primes and even prime 2. For $p = 2$ the classification is determined completely. For odd primes, the classification is much more difficult.

For $p = 2$, the classification of 2-groups with non-zero inflated essential cohomology follows from a result of Yalçın.

Theorem 6.1.4 (Yalçın [59]) *Let G be a 2-group and $\sigma_G = \prod_{x \in H^1(G,\mathbb{F}_2)-\{0\}} x$. If G is non-abelian, then $\sigma_G = 0$.*

Corollary 6.1.5 *If G is a non-abelian 2-group, then* $\mathrm{InfEss}^*(G) = 0$.

Proof: Recall that the cohomology of elementary abelian 2-group is $H^*(G/\Phi(G), \mathbb{F}_2) = \mathbb{F}_2[x_1, ..., x_n]$ where $x_i \in H^1(G/\Phi(G), \mathbb{F}_2)$. It is well-known that $\inf_{G/\Phi(G)}^{G} : H^1(G/\Phi(G), \mathbb{F}_2) \to H^1(G, \mathbb{F}_2)$ is bijective. Recall that $\mathrm{Ess}^*(G/\Phi(G))$ is generated by $L_n(x_1, ..., x_n)$ (see Lemma 4.1.9). We have $\inf^G_{G/\Phi(G)}(L_n(x_1, ..., x_n)) = \sigma_G$. Then $\mathrm{InfEss}^*(G) = 0$ by Theorem 6.1.4.

□

Corollary 6.1.6 *If G is non-abelian 2-group, then $\mathrm{RelEss}^*(G) = 0$.*

Proof: We know that $\mathrm{RelEss}^*(G) \subseteq \mathrm{InfEss}^*(G)$. Then $\mathrm{RelEss}^*(G) = 0$ by Theorem 6.1.5.

□

There are abelian 2-groups with $\sigma_G = 0$. For example $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. The following theorem is a characterization of 2-groups with non-zero σ_G.

Theorem 6.1.7 (Yalçın [59]) *Let G be a 2-group. Then $\sigma_G \neq 0$ if and only if $G \cong \mathbb{Z}/2^k \times (\mathbb{Z}/2)^n$ for some $n \geq 0$ and $k \geq 1$.*

Now we can easily conclude that the characterization of 2-groups with non-zero inflated essential cohomology is the following.

Corollary 6.1.8 *Let G be a 2-group. Then $\mathrm{InfEss}^*(G) \neq 0$ if and only if $G \cong \mathbb{Z}/2^k \times (\mathbb{Z}/2)^n$ for some $n \geq 0$ and $k \geq 1$.*

6.2 Inflated essential cohomology when $p > 2$

The classification is much more complicated for $p > 2$. We know that $G/\Phi(G)$ is an elementary abelian p-group. The cohomology of $G/\Phi(G)$,

$$H^*(G/\Phi(G), k) = k[x_1, ..., x_n] \otimes \wedge(a_1, ..., a_n)$$

where $a_i \in H^1(G/\Phi(G), k)$ and $x_i = \beta(a_i)$.

Corollary 6.2.1 (Corollary of Theorem 4.3.1) *Let G be a p-group. The inflated essential cohomology $\mathrm{InfEss}^*(G)$ is zero if and only if*

$$\inf^G_{G/\Phi(G)}(a_1 \cdots a_n) = 0.$$

Proof: This follows from the fact that essential cohomology of elementary abelian p-group is the Steenrod closure of $a_1 \cdots a_n$ (see Theorem 4.3.1) and Steenrod operations commute with inflation by naturality.

$\square$

Now the problem about finding p-groups with non-zero inflated essential classes (so that non-zero essential classes) is reduced to the problem of finding the finite p-groups such that $\inf^G_{G/\Phi(G)}(a_1 \cdots a_n) \neq 0$. To get non-zero inflated essential classes, hence non-zero essential classes, it is enough to find p-groups such that $\inf^G_{G/\Phi(G)}(a_1 \cdots a_n) \neq 0$. One of the motivations is considering the extraspecial p-groups.

Definition 6.2.2 *A p-group is called extraspecial if the center $Z(G)$ is cyclic of order p and $Z(G) = G' = \Phi(G)$.*

It is well-known that the extraspecial p-groups have a distinctive role in the cohomology of finite groups. These are the minimal non-abelian p-groups in the sense that any non-trivial factor group is elementary abelian. Recall that if G is an extraspecial p-group, there is a central extension

$$0 \longrightarrow C_p \longrightarrow G \longrightarrow V \longrightarrow 0$$

where V is an elementary abelian p-group. There are two types of extraspecial p-groups of order p^3.

$$\mathbb{E} \cong \langle x, y, z \mid x^p = y^p = z^p = [x, z] = [y, z] = 1, [x, y] = z \rangle$$

and

$$\mathbb{M} \cong \langle x, y \mid x^p = y^{p^2} = 1, xyx^{-1} = y^{p+1} \rangle.$$

It is seen from the presentation that $\exp(\mathbb{E}) = p$ and $\exp(\mathbb{M}) = p^2$. An extraspecial p-group is of order p^{2n+1} and is isomorphic to one of the following central products:

$$\mathbb{E}_n = \mathbb{E} * \cdots * \mathbb{E} \text{ (n times)}$$

$$\mathbb{M}_n = \mathbb{M} * \mathbb{E}_{n-1}$$

If we consider the central product $C_{p^2} * \mathbb{M}_n$ or $C_{p^2} * \mathbb{E}_n$, we get almost extraspecial group of order p^{2n+2}. In fact these groups also fit into an extension of the form:

$$1 \to C_p \to G \to V \to 1$$

where V is an elementary abelian p-group which is isomorphic to $G/\Phi(G)$.

When G is an extraspecial p-group the dimension of V is even and when G is an almost extraspecial p-group the dimension of V is odd.

We consider the inflated essential classes of extraspecial p-groups. For notation, let $H^*(V, \mathbb{F}_p) = \mathbb{F}_p[x_1, ..., x_t] \otimes \wedge(a_1, ..., a_t)$ where $t = 2n$ if G is extraspecial and $t = 2n + 1$ if G is almost extraspecial.

Lemma 6.2.3 *Let p be an odd prime. The cohomology class of the extension of V by C_p is the class $\alpha \in H^2(V, \mathbb{F}_p)$ given as follows: If G is $\mathbb{E}_n$,*

$$\alpha = a_1a_2 + ... + a_{2n-1}a_{2n},$$

if G is $\mathbb{M}_n$, then

$$\alpha = a_1a_2 + ... + a_{2n-1}a_{2n} + x_{2n}.$$

If G is almost extraspecial group of order p^{2n+2} then

$$\alpha = a_1a_2 + ... + a_{2n-1}a_{2n} + x_{2n+1},$$

and α is in the kernel of inflation inf_V^G.

Proof: See [16].

$\square$

Lemma 6.2.4 *Let G be an extraspecial p-group of exponent p. Then* $\mathrm{InfEss}^*(G) = 0$.

Proof: We know that if $\inf^G_{G/\Phi(G)}(a_1 \cdots a_{2n}) = 0$ then $\mathrm{InfEss}^*(G) = 0$. By previous lemma we have

$$\alpha = a_1 a_2 + ... + a_{2n-1} a_{2n},$$

in the kernel of $\inf^G_{G/\Phi(G)}$. Then $\alpha a_3 \cdot a_4 \cdots a_{2n} = a_1 \cdots a_{2n}$ is in the kernel of $\inf^G_{G/\Phi(G)}$ which implies $\mathrm{InfEss}^*(G) = 0$.

□

In [41], it is proved that if G is an extraspecial p-group, then $\mathrm{Ess}^*(G) = 0$ if and only if $\exp(G) = 3$ and $|G| = 3^3$. Now it is clear that for this extraspecial p-group, $\mathrm{InfEss}^*(G) = 0$. It is also proved that $\mathrm{InfEss}^*(\mathbb{M}_n) \neq 0$ (see Proposition 4 in [41]) and $\mathrm{InfEss}^*(G) \neq 0$ (see Proposition 5 in [41]) where G is an almost extraspecial p-group. There is a question in [41]:

Question 6.2.5 *Let G be an extraspecial p-group. For $G \ncong \mathbb{E}$, is it true that* $\mathrm{Ess}^*(G) \cap \mathrm{Im}\ \inf^G_V \neq \{0\}$?

It is clear that $\mathrm{InfEss}^*(G) \subseteq \mathrm{Ess}^*(G) \cap \mathrm{Im}\ \inf^G_V$. So, we can consider the following question.

Question 6.2.6 *Let G be an extraspecial p-group. For $G \ncong \mathbb{E}$, is it true that* $\mathrm{InfEss}^*(G) \neq \{0\}$?

We prove that the Question 6.2.6 is not true for extraspecial p-group of exponent p. So about Question 6.2.5, we can say that $\mathrm{Ess}^*(G) \cap \inf^G_V(\mathrm{Ess}^*(\mathrm{V})) = \{0\}$ for extraspecial p-groups of exponent p.

Now, we wonder for which finite p-groups we have $\mathrm{InfEss}^*(G) = \{0\}$.

The motivation for considering extraspecial p-groups comes from the following. Many of the theorems such as Serre's theorem [49] can be proved by reducing them to the extraspecial case and then using induction. In fact, this is because of the following lemma:

Lemma 6.2.7 *Let G be a non-abelian p-group and let H be a maximal element in the collection of normal subgroups of G that do not contain the Frattini subgroup of G. Then the quotient $Q = G/H$ is an extraspecial or almost extraspecial p-group.*

Proof: There are two cases. In the first case, H may lie in $\Phi(G)$. Since H is maximal $\Phi(G)/H \cong C_p$ and $\Phi(G/H) = \Phi(G)/H$. For p-groups, we have $G' \leq \Phi(G)$ and since G is non abelian we have $(G/H)' = \Phi(G/H)$. The corresponding extension $0 \to \Phi(G/H) \to G/H \to G/\Phi(G) \to 0$ is a non-trivial extension.

In the second case, H may not lie in $\Phi(G)$ but by maximality of H, the intersection $H \cap \Phi(G)$ is maximal in $\Phi(G)$ and the required result follows (for details see [16], page 154).

$\square$

Lemma 6.2.8 *Let G be a p-group. If* $\mathrm{InfEss}^*(G) \neq 0$, *then* $\mathrm{InfEss}^*(G/N) \neq 0$ *for any proper quotient G/N.*

Proof: We prove that $\mathrm{InfEss}^*(G/N) = 0$, then $\mathrm{InfEss}^*(G) = 0$. We have a commutative diagram:

$$\begin{array}{ccc} H^1(G/N/\Phi(G/N), k) & \xrightarrow{\text{inf}} & H^1(G/\Phi(G), k) \\ {\scriptstyle\text{inf}}\downarrow & & {\scriptstyle\text{inf}}\downarrow \\ H^1(G/N, k) & \xrightarrow{\text{inf}} & H^1(G, k) \end{array}$$

$\mathrm{InfEss}^*(G/N) = 0$ if and only if $\mathrm{inf}^{G/N}_{G/N/\Phi(G/N)}(a_1 \cdots a_t) = 0$ where a_i's are the generators of $H^1(G/N/\Phi(G/N), k)$. Thus $\mathrm{inf}^{G}_{G/N}\mathrm{inf}^{G/N}_{G/N/\Phi(G/N)}(a_1 \cdots a_t) = 0$. On the other hand let $e_1, ..., e_n$ be the generators of $H^1(G/\Phi(G), k)$. It is clear that $t \leq n$, so we can view $e_i = \mathrm{inf}^{G/\Phi(G)}_{G/N/\Phi(G/N)}(a_i)$ for $1 \leq i \leq k$. By commutativity of the diagram $\mathrm{inf}^{G}_{G/\Phi(G)}(e_1 \cdots e_k) = 0$. Then $\mathrm{inf}^{G}_{G/\Phi(G)}(e_1 \cdots e_n) = 0$ which means $\mathrm{InfEss}^*(G) = 0$.

$\square$

With the same notation in Lemma 6.2.7 we have:

Lemma 6.2.9 *Let G be a non-abelian p-group such that the quotient $Q = G/H$ is extraspecial of exponent p. Then* $\mathrm{InfEss}^*(G) = 0$.

Proof: By Lemma 6.2.4 $\mathrm{InfEss}^*(Q) = 0$. Since Q is a proper quotient of G, by Lemma 6.2.8 $\mathrm{InfEss}^*(G) = 0$.

□

Theorem 6.2.10 *Let G be a non-abelian p-group of exponent p. Then* $\mathrm{InfEss}^*(G) = 0$.

Proof: If G is of exponent p any proper quotient is also exponent p. So the extraspecial quotient Q is also exponent p. So by Lemma 6.2.9, $\mathrm{InfEss}^*(G) = 0$ since $\mathrm{InfEss}^*(Q) = 0$.

□

Corollary 6.2.11 *If G is a p-group such that* $\mathrm{InfEss}^*(G) \neq 0$. *Then* $[G, G] \leqslant G^p$.

Proof: If G is an abelian p-group then there is nothing to do. Assume G is a non-abelian p-group. $[G, G] \leqslant G^p$ if and only if $\Phi(G) = G^p$. Now assume that $G^p < \Phi(G)$. Then the quotient G/G^p is a non abelian p-group of exponent p. Thus $\mathrm{InfEss}^*(G/G^p) = 0$ and then $\mathrm{InfEss}^*(G) = 0$.

□

For non-abelian p-groups having an extraspecial p-group of exponent p as a quotient, inflated essential classes are zero.

If G is an abelian p-group, then $\mathrm{InfEss}^*(G)$ is non-zero. This follows from the fact that the cohomology ring of an abelian p-group is the tensor product of the cohomology rings of the cyclic p-groups. And the cohomology ring of a cyclic p-group is $k[a, x]/(a^2)$ where $\deg a = 1$ and $\deg x = 2$.

Proposition 6.2.12 *Let G and H be p-groups such that* $\mathrm{InfEss}^*(G) \neq 0$ *and* $\mathrm{InfEss}^*(H) \neq 0$. *Then* $\mathrm{InfEss}^*(G \times H)$ *is non-zero.*

Proof: Since $\mathrm{InfEss}^*(G) \neq 0$ and $\mathrm{InfEss}^*(H) \neq 0$ we have $\inf^G_{G/\Phi(G)}(a_1 \cdots a_k) \neq 0$ where $a_i \in H^1(G/\Phi(G), k)$ and $\inf^H_{H/\Phi(H)}(e_1 \cdots e_l) \neq 0$ where $e_i \in H^1(H/\Phi(H), k)$. Now consider $\tilde{a_i} = \inf^G_{G/\Phi(G)}(a_i)$ and $\tilde{e_i} = \inf^H_{H/\Phi(H)}(e_i)$. $\mathrm{InfEss}^*(G \times H)$ is non-zero if and only if $\tilde{a_1} \cdots \tilde{a_k} \cdot \tilde{e_1} \cdots \tilde{e_l} \neq 0$. Second follows from the fact that $H^*(G \times H, k) \cong H^*(G, k) \otimes H^*(H, k)$.

Corollary 6.2.13 *Let G be a p-group such that* $\mathrm{InfEss}^*(G) \neq 0$. *If H is an abelian p-group, then* $\mathrm{InfEss}^*(G \times H)$ *is non-zero.*

If G is an extraspecial of exponent p^2 or an almost extraspecial p-group, then $\mathrm{InfEss}^*(G)$ is non-zero (see [41], Proposition 4 and Proposition 5). So any direct product of G with an abelian p-group has non-zero inflated essential classes.

With the same notation in Lemma 6.2.7, we have a question:

Question 6.2.14 *If $Q = G/H$ is extraspecial p-group of exponent of p^2 or almost extraspecial p-group then is it true that* $\mathrm{InfEss}^*(G) \neq \{0\}$*?*

Unfortunately, the answer is no.

Example 6.2.15 *By definition of central product, we can consider $\mathbb{M}_n$ in the extension*

$$0 \to C_p \to \mathbb{E}_{n-1} \times \mathbb{M} \to \mathbb{M}_n \to 0.$$

$Q = \mathbb{M}_n$ *and we know that* $\mathrm{InfEss}^*(\mathbb{M}_n) \neq 0$, *but* $\mathrm{InfEss}^*(\mathbb{E}_{n-1} \times \mathbb{M}) = 0$ *as* $\mathrm{InfEss}^*(\mathbb{E}_{n-1}) = 0$.

Example 6.2.16 *Since an almost extraspecial p-group Γ_n of order p^{2n+2} is the central product $C_{p^2} * \mathbb{E}_n$, we can consider the extension*

$$0 \to C_p \to \mathbb{E}_n \times C_{p^2} \to \Gamma_n \to 0.$$

We know that $\mathrm{InfEss}^*(\Gamma_n) \neq 0$, *but* $\mathrm{InfEss}^*(\mathbb{E}_n \times C_{p^2}) = 0$.

We also get some information about the nilpotency degree of the inflated essential cohomology of a p-group.

Theorem 6.2.17 *Let G be a finite p-group such that* $\mathrm{InfEss}^*(G)$ *is non-zero. Then the nilpotency degree of* $\mathrm{InfEss}^*(G)$ *is* 2.

Proof: Let V be the Frattini quotient of G of rank n. By definition $\mathrm{InfEss}^*(G)$ is generated by $\mathrm{inf}_V^G(\mathrm{Ess}^*(V))$, so it is enough to show $\mathrm{inf}_V^G(\mathrm{Ess}^*(V))^2 = 0$.

We know that the essential cohomology of V satisfies $\mathrm{Ess}^*(V)^2 = L_n(V) \cdot \mathrm{Ess}^*(V)$ by Lemma 4.1.6. Applying inflation to the equality we get $\mathrm{inf}_V^G(\mathrm{Ess}^*(V))^2 = \mathrm{inf}_V^G(L_n(V)) \cdot \mathrm{inf}_V^G(\mathrm{Ess}^*(V))$. By Lemma 4.1.2, we have

$$L_n(V) = \lambda \prod_{[x]\in \mathbb{P}H^1(V,k)} \beta(x).$$

Inflation is an $\mathcal{A}$-module homomorphism, so we have

$$\mathrm{inf}_V^G(L_n(V)) = \lambda \prod_{[x]\in \mathbb{P}H^1(V,k)} \beta(\mathrm{inf}_V^G(x)) = \lambda \prod_{[\bar{x}]\in \mathbb{P}H^1(G,k)} \beta(\bar{x}).$$

where $\bar{x} = \mathrm{inf}_V^G(x)$. By the following celebrated theorem of Serre, we get $\mathrm{inf}_V^G(L_n(V)) = 0$.

$\square$

Theorem 6.2.18 (Theorem 1.3 in [49]) *Let S be a subset of $H^1(G,k)$ which does not contain* 0 *and contains exactly one point from each line in $H^1(G,k)$. If G is not elementary abelian then*

$$\prod_{x\in S} \beta(x) = 0 \quad \text{in} \quad H^{even}(G,k)$$

Notice that $S = \mathbb{P}H^1(G, k)$.

As a conclusion in this thesis, we try to contribute to Problem 1.1.2. We give a complete treatment of the module structure and the ideal structure of an elementary abelian p-group. We introduce the relative essential cohomology and the inflated essential cohomology which lie in the essential cohomology of a finite group. We try to classify finite p-groups whose relative essential cohomology and inflated essential cohomology are zero. We prove that for non-abelian p-groups having an extraspecial p-group of exponent p as a quotient, inflated essential classes are zero. We give examples showing that Question 6.2.14 is not true. We also determine the nilpotency degree of inflated essential classes.

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