PLANE SEXTICS WITH A TYPE E_7 SINGULAR POINT

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By
Mehmet Emin Aktaş
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Alexander Degtyarev(Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Özgün Ünlü

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Çetin Ürtiş

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of Graduate School of Engineering and Science

ABSTRACT

PLANE SEXTICS WITH A TYPE $\mathbf{E}_7$ SINGULAR POINT

Mehmet Emin Aktaş
M.S. in Mathematics
Supervisor: Assoc. Prof. Dr. Alexander Degtyarev
August, 2011

The computation of the fundamental grup of a plane sextic (i.e., curves $B \subset P^2$) still remain unanswered problem. There is an huge effort on this subject. In this thesis, we study plane sextic curves with a type $\mathbb{E}_7$ singular point, try to state a geometric approach to compute the fundamental groups of plane sextics with that type of singular points and develop a trick to find the commutant of these groups.

Keywords: trigonal curve, dessin, j-invariant, fundamental group.

ÖZET

$\mathbf{E}_7$ TİPİ TEKİL NOKTASI OLAN ALTINCI DERECE YÜZEY DENKLEMLERİ

Mehmet Emin Aktaş
Matematik, Yüksek Lisans
Tez Yöneticisi: Doç. Dr. Alexander Degtyarev
Ağustos, 2011

Altıncı dereceden denklemlerin temel gruplarının hesaplanması hala çözümü bulunamamış bir problemdir. Bu konuda büyük bir çaba sarfedilmektedir. Bu tezde $\mathbb{E}_7$ tipi tekil noktası olan altıncı dereceden denklemlerin temel gruplarının hesaplanması ile alakalı geometrik bir çözüm yolu ifade etmeye ve bu grupların komutantlarını bulmak için kullanılabilecek bir yöntem geliştirmeye çalıştık.

Anahtar sözcükler: trigonal eğriler, dessin, j-değişmezi, temel grup.

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Chapter 1

Introduction

The subject of this thesis is singular complex plane projective algebraic curves of degree six (sextics). We try to compute the fundamental group of the plane sextics with an $\mathbf{E}_7$ type singular point. By the fundamental group of a curve, we mean the group of its complement. Here, we consider a sextic B satisfying the following conditions:

1. B has simple (i.e. $\mathbf{A}-\mathbf{D}-\mathbf{E}$) singularities only,
2. B has a distinguished singular point P of type $\mathbf{E}_7$ and has no singular point of type $\mathbf{E}_6$ and $\mathbf{E}_8$.
3. B has a linear component through P.
4. B is a *maximizing* sextics, which means that the total Milnor number $\mu(B)$ of the singular points of B takes the maximal possible value, which is 19 (see U. Persson [11], where the term was introduced).

(Sextics with a type $\mathbf{E}_6$ point are considered in [5] and sextics with a type $\mathbf{E}_8$ point are considered in [4].) We confine ourself to maximizing sextics because they are most singular sextics and we show that maximal sextics correspond to maximal trigonal curves in Chapter 3.

When we have tried to compute the fundamental group of a plane sextic by using geometric approach and direct calculation, in most cases we have been inconclusive. It is really very big deal to compute it directly because it needs very huge calculations to do. Hence, an easy way was discovered by A. Degtyarev. Below, we outline the algorithm roughly that is used to compute the fundamental group.

1. We transform the sextics to a trigonal curve by elementary transformations.
2. We consider the (*functional*) *j-invariant* of the trigonal curve.
3. We reach the dessin of the trigonal curve by using its *j-invariant*.
4. The braid monodromy can easily be computed using the dessin.
5. Finally, by using the braid relations and van Kampen method, we compute the fundamental group.

1.1 Content of the thesis

In Chapter 2, we introduce trigonal curves in rational ruled surface, discuss their relations with the plane sextics and say something about (*functional*) *j-invariant* of the trigonal curve. In Chapter 3, we discuss the dessins of a trigonal curve. In Chapter 4, we speak about the braid monodromy and van Kampen method to find the fundamental group. Finally, we follow the outline and state a particular trick for a particular situation about the commutant of the fundamental group of plane sextics in Chapter 5.

Chapter 2

Trigonal Curves

In order to understand what a trigonal curve is, we need some terminology and definitions. Principal references are [3] and [2].

2.1 Hirzebruch surfaces

The Hirzebruch surface Σ_k, $k \geq 0$, is a rational geometrically ruled surface with a section E of self-intersection $-k$. If $k > 0$, the ruling is unique and there is a unique section E of self-intersection $-k$; it is called the *exceptional section*.

Let Σ_k, $k \geq 1$ be a Hirzebruch surface. Denote by $p\colon \Sigma_k \to \mathbb{P}^1$ the ruling. *Fibers* of a Σ are those of the projections p. Given a point b in the base $\mathbb{P}^1$, the fiber F_b is the fiber $p^{-1}(b)$. We can think of the fibers as "vertical" lines in Σ_k

2.2 Trigonal Curves

Definition 2.2.1. A *generalized trigonal curve* on a Hirzebruch surface is a reduced curve C not containing the exceptional section E and intersects each generic fiber at three points.

Definition 2.2.2. A *singular fiber* of a generalized trigonal curve $C \subset \Sigma_k$ is a fiber F of Σ_k intersecting $C + E$ geometrically at less than four points.

Thus F is singular if either C passes through $E \cap F$ (in Figure 2.1b), or C is tangent to F (in Figure 2.1a), or C has a singular point in F (in Figure 2.1c).

Definition 2.2.3. A singular fiber F is called *proper* if C does not pass through $E \cap F$ (in Figure 2.1a and Figure 2.1c).

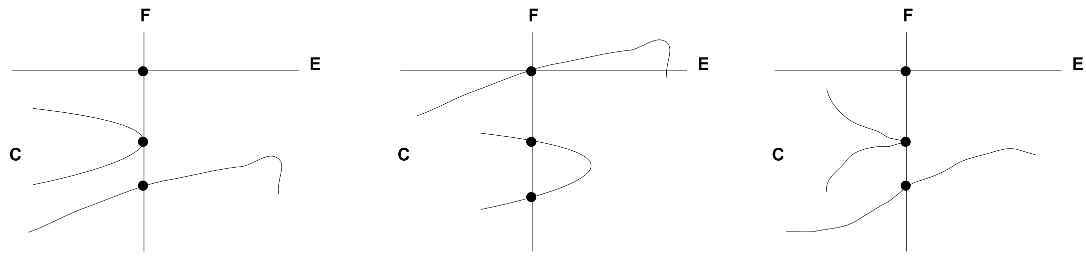

Figure 2.1: Three singular fibers

Definition 2.2.4. A *trigonal curve* is a generalized trigonal curve disjoint from the exceptional section.

For a trigonal curve $C \subset \Sigma_k$, we have $|C| = |3E + 3kF|$; conversely, any curve $C \in |3E + 3kF|$ not containing E as a component is a trigonal curve.

The topological type of a proper singular fiber F is shown as $\tilde{\mathbf{T}}$, where $\mathbf{T}$ says the type of the singular point of C in F. When just $\mathbf{T}$ is not enough, a number of *'s, showing the extra tangency of the fiber and the curve is used. We use the following notation for the topological types of proper fibers:

- $\tilde{\mathbf{A}}_0$: a nonsingular fiber;
- $\tilde{\mathbf{A}}_0^*$: a simple vertical tangent;
- $\tilde{\mathbf{A}}_0^{**}$: a vertical inflection tangent;
- $\tilde{\mathbf{A}}_1^*$: a node of C with one of the branches vertical;
- $\tilde{\mathbf{A}}_2^*$: a cusp of C with vertical tangent;
- $\tilde{\mathbf{A}}_p$, $\tilde{\mathbf{D}}_q$, $\tilde{\mathbf{E}}_6$, $\tilde{\mathbf{E}}_7$, $\tilde{\mathbf{E}}_8$: a simple singular point of $\bar{C}$ of the same type with minimal possible local intersection index with the fiber.

2.3 Elementary transformations

Definition 2.3.1. An *elementary transformation* of Σ_k is a birational transformation $\Sigma_k \dashrightarrow \Sigma_{k+1}$ consisting in blowing up a point P in the exceptional section of Σ_k followed by blowing down the fiber F through P.

The inverse transformation $\Sigma_{k+1} \dashrightarrow \Sigma_k$ blows up a point P' *not* in the exceptional section of Σ_{k+1} and blows down the fiber F' through P'. The result of an elementary transformation is a ruled surface Σ' over the same base B and with an exceptional section E' (the proper transform of E) of self-intersection $E^2 \pm 1$.

Let $\bar{C} \subset \Sigma_k$ be a generalized triganol curve. Then, by a sequence of elementary transformations, one can resolve the points of intersection of $\bar{C}$ and E and obtain a true trigonal curve $\bar{C}' \subset \Sigma_{k'}$, $k' \geq k$, birationally equivalent to $\bar{C}$. Alternatively, given a trigonal curve $\bar{C} \subset \Sigma_k$ with triple singular points, one can apply a sequence of elementary transformations to obtain a trigonal curve $\bar{C}' \subset \Sigma_{k'}$, $k' \geq k$, birationally equivalent to $\bar{C}$ and with $\tilde{\mathbf{A}}$ type singular fibers only.

Let B be a sextic which satisfies the conditions (1)-(3) in Chapter 1. Then when we blow up the type $\mathbf{E}_7$ point of the sextic, we get a Hirzebruch surface Σ_1 with the exceptional section E (the exceptional divisor), a trigonal curve $C \subset \Sigma_1$ (the proper transform of B), and a fiber F of Σ_1 (the proper transform of the linear component of B). Later, we apply two elementary transformations i.e. we blow up the point of $C \cap E \cap F$ and blow down the fiber through this point. Finally, we get Σ_3 with a trigonal curve C'' and a fiber F'', which passes through the singular point of C''. The fiber F'' is called the *distinguished fiber*, it is of type $\tilde{\mathbf{A}}_p$, $p \geq 3$. Conversely, starting from a pair (C'', F''), where $C'' \subset \Sigma_3$ is a trigonal curve, we can apply inverse of the transformation as it was done above to get a plane sextic B satisfies (1)-(3) in Chapter 1.

Since B has a linear component through P, the birational transformation used establishes a diffeomorphism between $\mathbb{P}^2 \setminus B$ and $\Sigma_3 \setminus (C'' \cup F'' \cup E)$ hence, the fundamental group of $\mathbb{P}^2 \setminus B$ can be computed as the fundamental group of $\Sigma_3 \setminus (C'' \cup F'' \cup E)$ Thus, in the rest of the paper, instead of plane sextics as in Chapter 1, we speak about pairs $\Sigma_3 \setminus (C'' \cup F'' \cup E)$ as above and compute the fundamental group of $\Sigma_3 \setminus (C'' \cup F'' \cup E)$ instead of computing the fundamental group of $\mathbb{P}^2 \setminus B$.

2.4 The j-invariant of a trigonal curve

Definition 2.4.1. The (*functional*) j-invariant $j_C{:}P^1 \to P^1$ of a generalized trigonal curve $C \subset \Sigma_2$ is defined as the analytic continuation of the function sending a point b in the base P^1 of Σ_2 representing a nonsingular fiber F_b of C to the j-invariant (divided by 12^3) of the elliptic curve covering F_b and ramified at the four points of intersection of F_b and $C + E$.

The curve B is called *isotrivial* if j_C is constant. Since j_C is defined via affine charts and analytic continuation, it is obvious that it is invariant under elementary transformations.

The function $j{:}P^1 \to P^1$ has three special values: 0, 1 and ∞. The correspondence between the type of a fiber F_z and the value $j(z)$ is shown in Table 2.1.

Type of F_z	$j(z)$	Vertex	Valency
$\tilde{\mathbf{A}}_p$ ($\tilde{\mathbf{D}}_{p+5}$), $p \geq 1$	∞	$\times$	$p+1$
$\tilde{\mathbf{A}}_0^*$ ($\tilde{\mathbf{D}}_5$)	∞	$\times$	1
$\tilde{\mathbf{A}}_0^{**}$($\tilde{\mathbf{E}}_6$)	0	$\bullet$	1 mod 3
$\tilde{\mathbf{A}}_1^*$ ($\tilde{\mathbf{E}}_7$)	1	$\circ$	1 mod 2
$\tilde{\mathbf{A}}_2^*$ ($\tilde{\mathbf{E}}_8$)	0	$\bullet$	2 mod 3

Table 2.1: The values of $j(z)$ at singular fibers F_z

Chapter 3

Dessins and Skeletons

In mathematics, a dessin d'enfants, means children's drawing, is a type of graph drawing, which has first been introduced by Felix Klein [9]. After a century, it was rediscovered and named by A. Grothendieck [7] for studying rational maps with three critical points. Later, S. Orevkov [10] used it to study the case for the j-invariant of a trigonal curve or elliptic surfaces.

We have mentioned in the previous chapter that a trigonal curve is (almost) determined by its j-invariant, whereas the latter, is adequately described by its dessin.

3.1 Dessin of a trigonal curve

Definition 3.1.1. The *dessin* $\Gamma_{\bar{C}}$ of a non-isotrivial trigonal curve $\bar{C} \subset \Sigma_k$ is defined as the planar map ${j_{\bar{C}}}^{-1}(\mathbb{R}p^{-1}) \subset S^2 = \mathbb{P}^1$, enhanced with the following decorations: the pull-backs of 0, 1, and ∞ are called, respectively, $\bullet$-, $\circ$- and $\times$-vertices of $\Gamma_{\bar{C}}$, and the connected components of the pull-backs of $(0,1), (1,\infty)$ and $(-\infty, 0)$ are called, respectively, bold, dotted and solid edges of $\Gamma_{\bar{B}}$.

Clearly, the dessin is ivariant under elementary transformations of the curve.

Definition 3.1.2. The *skeleton* $Sk_{\bar{C}}$ of a trigonal curve $\bar{C}$ is the planar map obtained from the dessin $\Gamma_{\bar{C}}$ by removing all $\times$-vertices and solid and dotted edges.

Thus, $Sk_{\bar{C}}$ is Grothendieck's dessin d'enfant ${j_{\bar{C}}}^{-1}([0,1])$. The pull-backs of 0 are called $\bullet$- vertices, and the pull-backs of 1 are called $\circ$- vertices.

The $\bullet$-vertices of valency 1 mod 3 or 2 mod 3 and $\circ$- vertices of valency 1 mod 2 are called singular. All other $\bullet$- and $\circ$- vertices are called nonsingular.

A skeleton is a bipartite graph. For this reason, in the drawings, we omit bivalent $\circ$- vertices, assuming that such a vertex is to be inserted in the middle of each edge connecting two $\bullet$- vertices. For example, in Figure 3.1., there are 6 $\circ$- vertices where they are omitted. In particular, for a skeleton, only singular monovalent $\circ$- vertices are drawn.

A *region* of a skeleton Sk $\subset \mathbb{P}^1$ is a connected component of the complement $\mathbb{P}^1 \backslash Sk$. Closed regions are connected components of the manifold theoretical cut of $\mathbb{P}^1$ along Sk. We say that a region R is an m-gon (or an m-gonal region) if the boundary of the corresponding closed region $\bar{R}$ contains m $\bullet$-vertices. For example, in Figure 3.1., the dessin has one *monogone*, one *bigone*, one $3-gon$, one $4-gon$ and one $8-gon$.

We use skeletons especially in the study of *maximal* trigonal curves. Let us firstly define what a *maximal* trigonal curve is:

Definition 3.1.3. A non-isotrivial trigonal curve $\bar{C}$ is called *maximal* if it has the following properties:

1. $\bar{C}$ has no singular fibers of type $\tilde{\mathbf{D}}_4$;
2. $j = j_{\bar{C}}$ has no critical values other than 0, 1, and ∞;
3. each point in the pull-back $j^{-1}(0)$ has ramification index at most 3;
4. each point in the pull-back $j^{-1}(1)$ has ramification index at most 2.

The skeleton Sk_C of any *maximal* curve C has the following properties:

1. Sk_C is connected.
2. each $\bullet$-vertex of Sk_C has valency 1,2, or 3; each $\circ$-vertex has valency 1 or 2 and is connected to a $\bullet$-vertex.

We can think vice versa; any $Sk_C \in S^2$ satisfying (1) and (2) above extends to a unique, up to orientation preserving diffeomorphism of S^2, dessin of *maximal* trigonal curve. We insert a $\circ$-vertex in the middle of each edge connecting two $\bullet$-vertices, place a $\times$-vertex inside each region R of Sk_C and connect the $\times$-vertex by disjoint solid (dotted) edges to all $\bullet$- (respectively, $\circ$-) vertices in the boundary of R.

Normally, if we have a sextic with simple singularities, we transform it to a trigonal curve, consider the j-invariant of the corresponding trigonal curve, get the dessin of the trigonal curve by using its j-inavriant. However, from the previous paragraph, if the curve is maximal, we can move in the opposite direction; the trigonal curve can be computed by using its skeleton. It can be effectively studied using skeletons. In the rest of the paper, we will try to use this strategy.

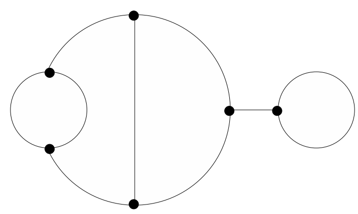

Figure 3.1: A dessin

3.2 The case of maximal trigonal curves with A type singularities

The type $\tilde{\mathbf{A}}_0^{**}$, $\tilde{\mathbf{A}}_1^{*}$, $\tilde{\mathbf{A}}_2^{*}$ singular fibers of a trigonal curve are called *unstable*, and all other singular fibers are called *stable*. A trigonal curve is called *stable* if all its singular fibers are stable. In the dessins of stable maximal trigonal curves with **A** type singularities, all ○- vertices are bivalent and all •- vertices are trivalent. For example, in Figure 3.2., the dessin of the trigonal curve that has $\mathbf{A}_8 \oplus \mathbf{A}_5$ singular points has no monovalent ○- vertex and six trivalent •- vertices.

In order to be able to get the trigonal curve with **A** type singularities by using its skeleton, the trigonal curve should be maximal. A maximal trigonal curve C with **A** type singularities only can be characterized in terms of its *total Milnor number* $\mu(C)$ which is the sum of the Milnor numbers of all singular points of B. The following theorem is proved in [3].

Theorem 3.2.1. *For a non-isotrivial genuine trigonal curve $C \in \Sigma_k$ with simple singularities only one has*

$$\mu(C) \leq 5k - 2 - \#\{\textit{unstable fibers of } C\},$$

the equality holds if and only if C is maximal.

Proposition 3.2.2. *Under the construction of the section 2.3., a maximal trigonal curve C corresponds to a maximizing sextic.*

Proof: We will use the Theorem 3.2.1. In our situation, C has no unstable singularities. Moreover, since the trigonal curve is in the Hirzebruch surface Σ_3, k is equal to 3. Hence, $\mu(C) = 13$. In the maximal case, the distinguished fiber of C boils down to a *distinguished* region in the skeleton Sk_C, which has to have at least four vertices. To obtain the sextic B, we apply two inverse elementary transformations to the singularity of the related distinguished region to get the corresponding singular point of the sextic B. By doing this, we add 6 to $\mu(C)$ and we get $\mu(B) = 19$, which means that the sextic B is maximizing.

Indeed, if the singular point corresponding to the distinguished region is $\mathbf{A}_3$, we get $\mathbf{E}_7 \oplus 2\mathbf{A}_1$ after the inverse elementary transformations. If it is $\mathbf{A}_4$, the sextic has $\mathbf{E}_7 \oplus \mathbf{A}_3$ and if it is $\mathbf{A}_p$ where $p > 4$, the sextic has $\mathbf{E}_7 \oplus \mathbf{D}_{p-1}$. Hence, in three cases, we add to the Milnor number 6, and we get $\mu(B) = 13 + 6 = 19$. This means a maximal trigonal curves with $\mathbf{A}$ type singularities corresponds a maximizing sextic and vice versa.

$\square$

For example, in Figure 3.2., the dessin has five regions whose dimensions are 9,6,1,1,1. Hence it has type of $\mathbf{A}_8$ and $\mathbf{A}_5$ singular points. If we choose the distinguished region as the region that has nine edges, the sextic has $\mathbf{E}_7 \oplus \mathbf{D}_7$ singularity, i.e. the sextic is $\mathbf{E}_7 \oplus \mathbf{D}_7 \oplus \mathbf{A}_5$ The sum of corresponding indices is $7 + 7 + 5 = 19$ which is equal to the maximum value of the $Milnor$ number and the sextic is maximizing. We can also choose the distinguished region as the region that has six edges. Hence the sextic has $\mathbf{E}_7 \oplus \mathbf{D}_4$ singularity, i.e. the sextic is $\mathbf{E}_7 \oplus \mathbf{D}_4 \oplus \mathbf{A}_8$. The sum of corresponding indices is $7 + 4 + 8 = 19$ again.

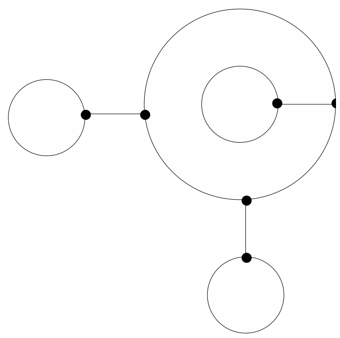

Figure 3.2: Dessin of the trigonal curve that has $\mathbf{A}_8 \oplus \mathbf{A}_5$ singular points

Chapter 4

The Braid Monodromy

After we get the skeletons of a trigonal curve, in order to compute the fundamental group, we will use the *braid monodromy*. Hence, let us try to understand how to use the braid monodromy to find the fundamental group. Firstly, we should know what the braid group is.

4.1 The Braid Group

A *geometric braid* on n strands, where $n \in \mathbb{N}$, is an injective map $\beta : I \times \{1, ..., n\} \to \mathbb{R}^3$ with the following properties:

- the x-coordinate of $\beta(x, k)$ equals x for all $x \in I$ and $k \in \{1, ..., n\}$;
- one has $\beta(0, k) = (0, k)$ and for each $k \in \{1, ..., n\}$ there is a $k' \in \{1, ..., n\}$ such that $\beta(1, k) = (1, k')$.

Two geometric braids are said to be *equivalent* if they are isotopic in the class of geometric braids. An equivalence class of geometric braids is called *braid*. The *product* $\beta_1 \cdot \beta_2$ of two geometric braids β_1, β_2 is defined as:

$$\beta_1 \cdot \beta_2(x,k) = \begin{cases} \beta_1(2x,k), & \text{if } x \leq 1/2 \\ \beta_2(2x-1,k), & \text{if } x \geq 1/2 \end{cases}$$

and the *inverse* β^{-1} of a geometric braid is defined as:

$$\beta^{-1}(x,k) = \beta(1-x,k).$$

An intuitive description of the *geometric braid* on n strands, where n $\in$ N is as follows; assume there are two sets of n items lying on a table, where the items are ordered on two vertical and parallel lines and the sets are sitting together. We connect each items of the first set with an item of the second set having one-to-one correspondence. This connection is called a *braid*. We can see three examples of braids for $n = 4$ in Figure 4.1. The first braid in Figure 4.1. is different from the second braid and equivalent with the third braid.

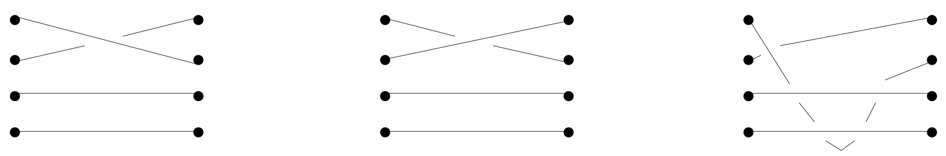

Figure 4.1: Three braids

We can compose two braids by drawing the first braid next to second and erasing items in the middle as in Figure 4.2.

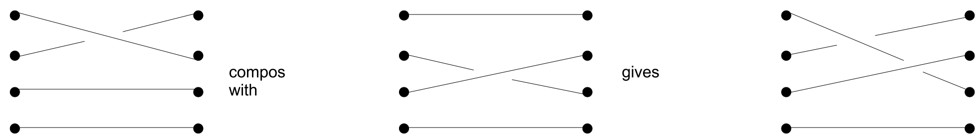

Figure 4.2: Composition of braids

Moreover, the composition of braids is a group operation. The inverse element is the mirror image of the first braid. We call this group as *Braid* group on n strands and denote this group as $\mathbb{B}_n$.

4.1.1 Generators and Relations of $\mathbb{B}_n$

Every braid in $\mathbb{B}_n$ can be written as a composition of the so called *Artin generators* σ_i , $i = 1, ..., n-1$, where σ_i twists the i-th and i+1-st strands through an angle of π in the counterclockwise direction while leaving the other strands intact. In this basis, the defining relations for $\mathbb{B}_n$ are

$$[\sigma_i, \sigma_j] = 1 \text{ if } |i-j| > 1, \text{ and } \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

4.1.2 Relation between $\mathbb{B}_3$ and the modular group

From the previous section one has

$$\mathbb{B}_3 = \langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

Let $a = \sigma_1\sigma_2\sigma_1$ and $b = \sigma_1\sigma_2$. If we apply the braid relations, we get $a^2 = b^3$ and let $c = a^2 = b^3$. If we again apply the relations, we see that

$$\sigma_1 c {\sigma_1}^{-1} = \sigma_2 c {\sigma_2}^{-1} = c.$$

Hence we find that c is in the center of $\mathbb{B}_3$. The subgroup generated by c is normal subgroup of $\mathbb{B}_3$. Therefore, take the quotient group $\mathbb{B}_3 \, / \langle c \rangle$ and get the quotient group is isomorphic to the modular group, (the modular group $\Gamma = \langle x, y | x^2 = y^3 = 1 \rangle$.) i.e. $\Gamma \simeq \mathbb{B}_3 \, / \langle c \rangle$.

4.1.3 $\mathbb{B}_3$ as the group of automorphisms

We have also that the braid group $\mathbb{B}_3$ can be defined as the group of automorphisms of the free group $G = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$. There is a theorem about this:

Theorem 4.1.1 (**Artin**[1]). *An automorphism φ of the free group $\mathbb{F}_n = \langle \alpha_1, ..., \alpha_n \rangle$ is a braid if and only if each image ${\alpha_i}' := \varphi(\alpha_i)$ is a conjugate of one of the generators and one has ${\alpha_1}'...{\alpha_n}' = \alpha_1...\alpha_n$.*

This automorphism sends each generator of G to the conjugate of another generator and finally, the product $\alpha_1\alpha_2\alpha_3$ stays same. We define the generators σ_1, σ_2 as

$$\sigma_1 : (\alpha_1, \alpha_2, \alpha_3) \to (\alpha_1\alpha_2{\alpha_1}^{-1}, \alpha_1, \alpha_3), \ \sigma_2 : (\alpha_1, \alpha_2, \alpha_3) \to (\alpha_1, \alpha_2\alpha_3{\alpha_2}^{-1}, \alpha_2).$$

We introduce also $\sigma_3 = {\sigma_1}^{-1}\sigma_2\sigma_1$ and $\tau = \sigma_2\sigma_1 = \sigma_3\sigma_2 = \sigma_1\sigma_3$. The center of $\mathbb{B}_3$ is the infinite cyclic group generated by τ^3

4.2 The Braid Monodromy

We will use van Kampen's method to find the fundamental group. Let, firstly state van Kampen theorem;

Theorem 4.2.1 (Van Kampen). *Let U_1 and U_2 be open subsets of a space X such that:*

- $U_1 \cup U_2 = X$ *and*
- $U_1 \cap U_2$ *is path connected.*

Then,

$$\pi_1(X) = \pi_1(U_1) *_{\pi(U_1 \cap U_2)} \pi_1(U_2).$$

Let C be a generalized trigonal curve. Let $F_1, F_2, ..., F_r$ be the singular fibers of C and E be the exceptional section. Pick a nonsingular fiber F and let $F^\sharp = F \setminus (C \cup E)$. Clearly, $F^\sharp$ is equal to $F \setminus E$ with three punctures. (As in Figure 4.3) Let $U_1 = \Sigma_d \setminus (E \cup C \cup \sum F_i)$ as the same notation in the van Kampen theorem. Let $B^\sharp = \mathbb{P}_1 \setminus \{p_1, p_2, ..., p_r\}$ where p_i is the image under the ruling of the corresponding singular fiber F_i. Since the singular fibers have been removed, we have a locally trivial fibration, $F^\sharp \hookrightarrow U_1 \to B^\sharp$. Hence, from the Serre theorem, we get a Serre exact sequence [8] as follows;

$$... \to \pi_2(B^\sharp) \to \pi_1(F^\sharp) \to \pi_1(U_1) \to \pi_1(B^\sharp) \to \pi_0(F^\sharp) \to ...$$

We know that $\pi_2(B^\sharp) = 0$ and $\pi_0(F^\sharp) = 0$, hence we get a short exact sequence. Moreover, pick a generic section S which is disjoint from E and intersecting all fibers $F, F_1, ..., F_r$ outside of C. We know that $\pi_1(F^\sharp) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ where α_i is the loop which covers i-th intersection point of fiber $F^\sharp$ and the generalized trigonal curve C and $\pi_1(B^\sharp) = \langle \gamma_1, ..., \gamma_r \rangle$ where γ_i is the loop which covers p_i. For each $j = 1, ..., r$ dragging the fiber F along γ_j and keeping the base point in S results in a certain automorphism $m_j : \pi_1(F^\sharp) \to \pi_1(F^\sharp)$, called the *braid monodromy* along γ_j. It has the property that the image $m_j(\alpha_i)$ of each generator α_i, $i = 1, 2, 3$, is a conjugate of another generator $\alpha_{i'}$. According to van Kampen, we get:

$$\pi_1(U_1) = \langle \alpha_1, \alpha_2, \alpha_3, \gamma_1, ..., \gamma_r | {\gamma_j}^{-1} \alpha_i \gamma_j = m_j(\alpha_i), i = 1, 2, 3, j = 1, ..., r \rangle.$$

Moreover, let U_2, in the notation of the Van Kampen theorem, be an open cylinder around a singular fiber F_j. Patching back a fiber F_j makes γ_j contractible and this gives an additional relation $\gamma_j = 1$. Hence, by using Van Kampen theorem, in this notation, we reach van Kampen method which is actually equivalent to the Zariski van Kampen theorem:

Theorem 4.2.2 (Zariski – van Kampen)**.** *One has,*

$$\pi_1(\Sigma_d \setminus (E \cup C)), F \cap S) = \langle \alpha_1, \alpha_2, \alpha_3 | \alpha_i = m_j(\alpha_i), i = 1, 2, 3, j = 1, ..., r \rangle.$$

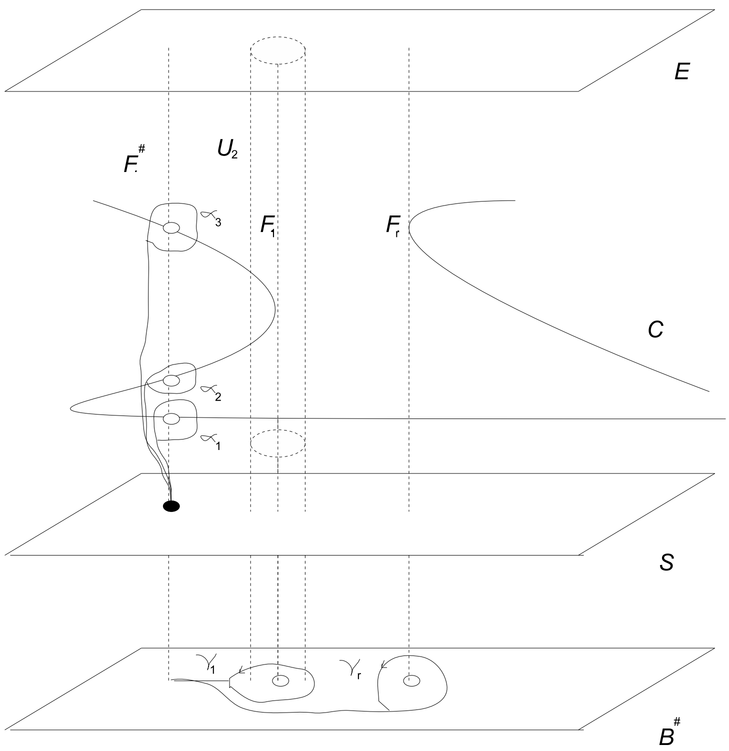

Figure 4.3: Van Kampen Method

Chapter 5

The Fundamental Group

After we get dessins of sextics, we can use the strategy mentioned in Chapter 4 (Van Kampen method) and find the fundamental group. In order to be able to use this strategy, we should find the braid monodromy.

5.1 Computation of the braid monodromy

We know that the skeleton of a trigonal curves with **A** type singular point has five regions and one of the region, the distinguished region, does not provide relations. We can use the other four regions to find the braid monodromies.

Given a base point in an edge e, there is a standard basis $\{\alpha_1, \alpha_2, \alpha_3\}$, which is well defined up to simultaneous conjugation of the generators by a power of $\alpha_1\alpha_2\alpha_3$ for $\mathbb{F}_3$, related to this edge. The two regions adjacent to a given edge are distinguishable: one is to the right, and the other one is to the left. The braid relations arising from the right region adjacent to e are very simple. They are calculated in the coming paragraph. On the other hand, in order to compute the group, we should fix one common base point c and write all relations in one common basis, $\{\alpha_1, \alpha_2, \alpha_3\}$. Hence, we need a way to relate the standard basis over the common point and the standard basis over an auxiliary point a_i next to a given i-th region. For this sake, we choose a path γ_i in the skeleton from c to a_i and obtain the relations by conjugating the 'standard' ones by the monodromy along γ_i.

While we move from the common base point c to an auxiliary point a_i, if we pass over a •- vertex and our direction is counterclockwise, we apply the braid $\sigma_1\sigma_2$ to the standard base $\alpha_1, \alpha_2, \alpha_3$. This braid projects to the x in the modular group Γ. If we pass over a •- vertex and our direction is clockwise, we apply the inverse of the first braid. This braid projects to the x^2 in Γ. If we pass over a ∘-vertex, we apply the braid $\sigma_1\sigma_2\sigma_1$. This braid projects to the y in Γ. Hence we get a series of x and y as defining the path. In order to make some simplification, we write '1,-1 and 0' instead of 'x,x^2 and y'. Hence, we get a sequence of number which only includes $-1, 0$ and 1. There are four regions in the dessin other than the distinguished region. This means, we get four paths and by using these paths, we can find the braid monodromies m_j as follows:

$$m_j = \text{(i-th path)}{\sigma_1}^m\text{(i-th path)}^{-1},\ i = 1, 2, 3, 4,$$

where m is the size of the region where the corresponding path ends. By using van Kampen method, we get the fundamental group as follows:

$$\pi_1(U_1) = \langle \alpha_1, \alpha_2, \alpha_3 | \alpha_i = m_j(\alpha_i), i = 1, 2, 3, j = 1, ..., r\rangle.$$

For example, in Figure 5.1, we study on $\mathbf{E}_7 \oplus \mathbf{A}_8 \oplus \mathbf{D}_4$. We choose the outher region as the distinguished region. It has eight vertices hence it has more than four vertices. We place the base point on the inner monogone. We move from the base point to the four regions. Hence, we get four paths corresponding to that four regions as follows; $[]$, $[1]$, $[1, 0, 1, 0, -1, 0, -1]$ and $[1, 0, 1, 0, 1, 0, -1, 0, -1]$. Consequently, we get the braid monodromies as follows; σ_1, $[1]{\sigma_1}^6[1]^{-1}$, $[1, 0, 1, 0, -1, 0, -1]\sigma_1[1, 0, 1, 0, -1, 0, -1]^{-1}$ and $[1, 0, 1, 0, 1, 0, -1, 0, -1]\sigma_1[1, 0, 1, 0, 1, 0, -1, 0, -1]^{-1}$.

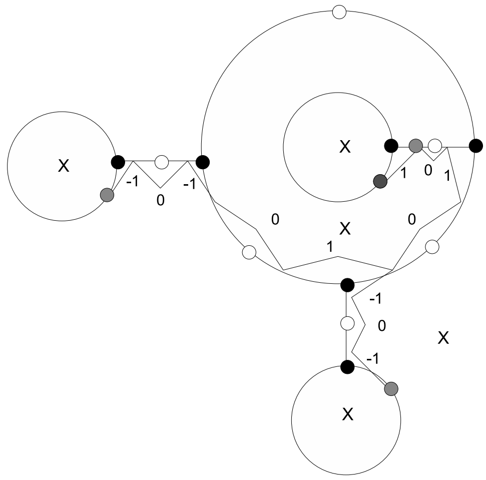

Figure 5.1: Paths of the dessin of the trigonal curve with $\mathbf{A}_8 \oplus \mathbf{A}_5$ singular points

5.2 The commutant of the fundamental group

Even if we find the braid monodromies, the fundamental groups are infinite and it is difficult to say something about its structure. Hence, we should find alternative ways to say something about the fundamental group. There exist a computer programme `GAP` (Groups, Algorithm, Programming) which can inform us about finite groups. Therefore, if we relate the fundamental groups with finite groups, we can use `GAP` and say something about the fundamental groups. There is a nice trick which uses this strategy which uses the following lemma:

Lemma 5.2.1 ([5]). *Let H be a group, and let $a \in H$ be a central element whose projection to the abelianization $H/[H,H]$ has infinite order. Then the projection $H \to H/a$ restricts to an isomorphism $[H,H] = [H/a, H/a]$.*

Therefore, if we can find central elements of the fundamental group and take the quotient of the group with these elements, both of the groups have same commutants. We know from algebra that if the commutant of a group is trivial, the group is abelian. Hence, if the quotient group has trivial commutant, it is abelian and from the lemma, the fundamental group is also abelian.

There is a particular trick to find a central element for a special type of skeletons. If we can place the base point on a monogone of the skeleton and this monogone is not adjacent to the distinguished region, we can find a central element as follows; From the first braid monodromy, we have found that $m_1(\alpha_1) = \sigma_1(\alpha_2) = \alpha_2$. This implies $\alpha_1 = \alpha_2$. For the second braid monodromy, we move from base point to the nearest region to the monogone. Hence, we only pass over a $\bullet$- vertex and our direction is counterclockwise and our path is [1]. Let the number of the $\bullet$- vertices on the boundary of the corresponding region be n. Therefore, the braid monodromy is

$$m_2 = [1]{\sigma_1}^n[1]^{-1}$$

i.e. we have $\sigma_1\sigma_2{\sigma_1}^n{\sigma_2}^{-1}{\sigma_1}^{-1}$. After some easy calculations by using braid relations, we get ${\sigma_2}^n(\alpha_2) = \alpha_2$. By using these two relations, we get $[(\alpha_2\alpha_3)^n, \alpha_2] = 1$, and $[(\alpha_2\alpha_3)^n, \alpha_3] = 1$ (no need to check α_1, since $\alpha_1 = \alpha_2$). Namely, $(\alpha_2\alpha_3)^n$ is a central element and we can apply the previous lemma.

We use the programming language `GAP` to do the calculations. In the corresponding `GAP` code, there is free group $G = \mathbb{F}_3$ and $g = \pi_1/a$ where $\pi_1 = \mathbb{F}_3/\cup(RelsA, ...)$ from the van Kampen theorem and a is the central element. Then, the code is as follows:

g := G / Union ([$(G.2 * G.3)^n$], RelsA ([1st path],m_1), RelsA ([2nd path],m_2), RelsA ([3rd path],m_3), RelsA ([4th path],m_4));

where m_i is the number of the $\bullet$ vertices on the boundary of the region where the i-th path ends, n is the number of the $\bullet$ vertices on the boundary of the region that the monogone containing the base point is adjacent and `RelsA` is the function which returns the relators in G and it is defined in `pi1.txt`.

For example, in figure below, we study on $\mathbf{E}_7 \oplus \mathbf{A}_8 \oplus \mathbf{D}_4$. If we apply the trick, we get the following code and result:

```
gap> Read (”braid.txt”)
gap> Read (”common.txt”)
gap> Read (”pi1.txt”)
gap> g := G / Union ([(G.2 G.3)^n, G.1/G.2], RelsA ([1],6), RelsA
([1,0,1,0,-1,0,-1],1), RelsA ([1,0,1,0,1,0,-1,0,-1],1));
<f_p group on the generators [f1, f2, f3]>
gap> Size(g);
12
gap> AbelianInvariants(g);
[ 3 , 4 ]
```

Abelian invariants [3 , 4] means $g \setminus [g, g]= \mathbb{Z}_3 \oplus \mathbb{Z}_4$ and the size of the group g is 12, which means $g \setminus [g, g] = g$. This implies that g has a trivial commutant and finally we get that the fundamental group of the sextic with $\mathbf{E}_7 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4$ type singular points has also trivial commutant, i.e. it is abelian.

5.3 Computation

We need dessins having six •- vertices. When we look at the dessins coming from the Miranda-Persson table in [6], we reach lists of all dessins which have eight trivalent •- vertices. In order to get dessins having six •- vertices, we omit a monogone and the edge which is adjacent to the monogone. By doing this to the dessins in the Miranda-Persson table, we reduce number of the vertices by two and we get lists of all dessins which have six vertices. Secondly, in order to apply our trick, dessins should satisfy two conditions; firstly, the distinguished region, which has to have at least four edges, can be chosen. Secondly, there has to exist a monogone which is not adjacent to the distinguished region. We put the suitable dessins in the figures below. If we apply the same strategy for these dessins, write the corresponding `GAP` codes and finally, we get the following table.

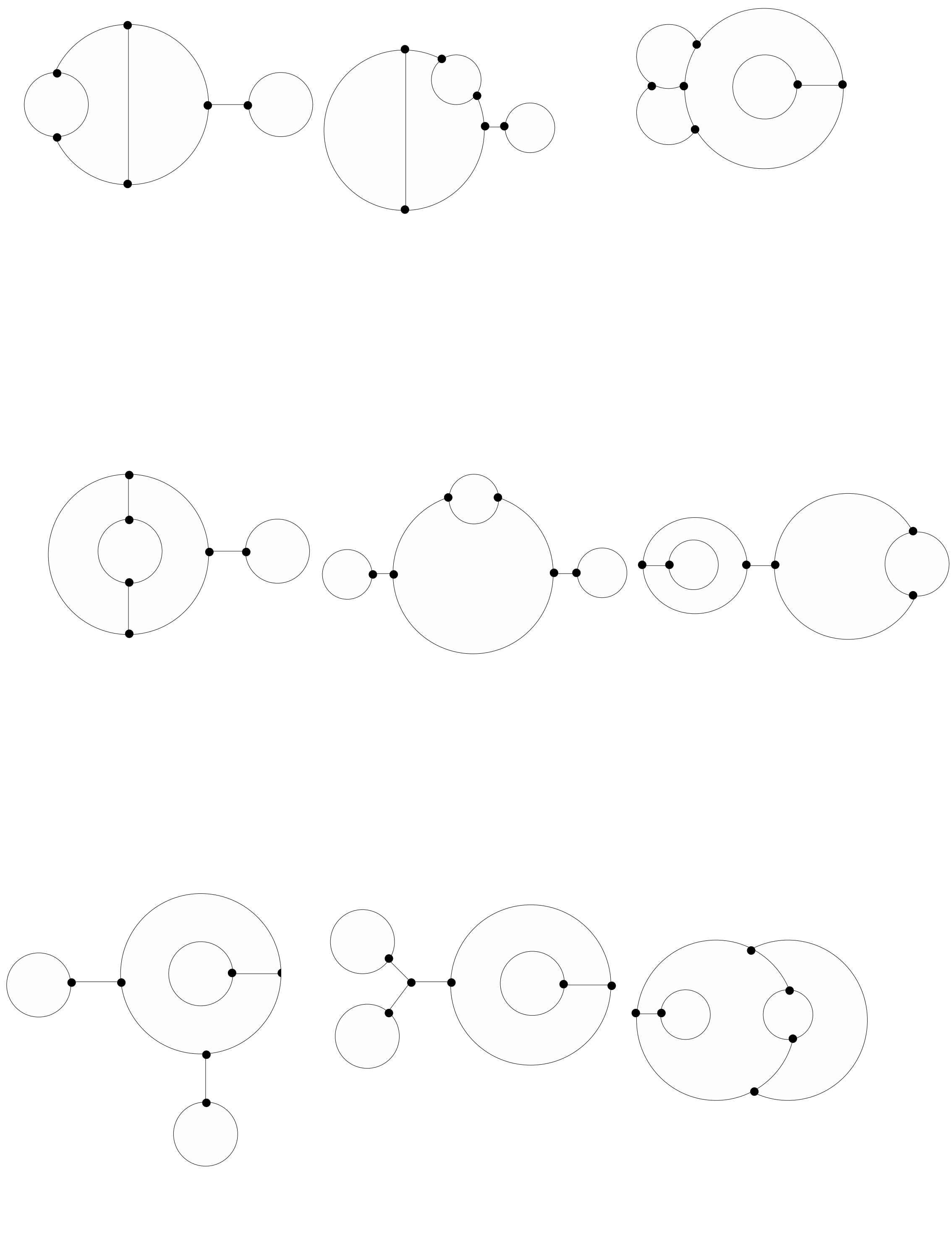

Figure 5.2: Dessins of $\mathbf{E}_7$ singularities

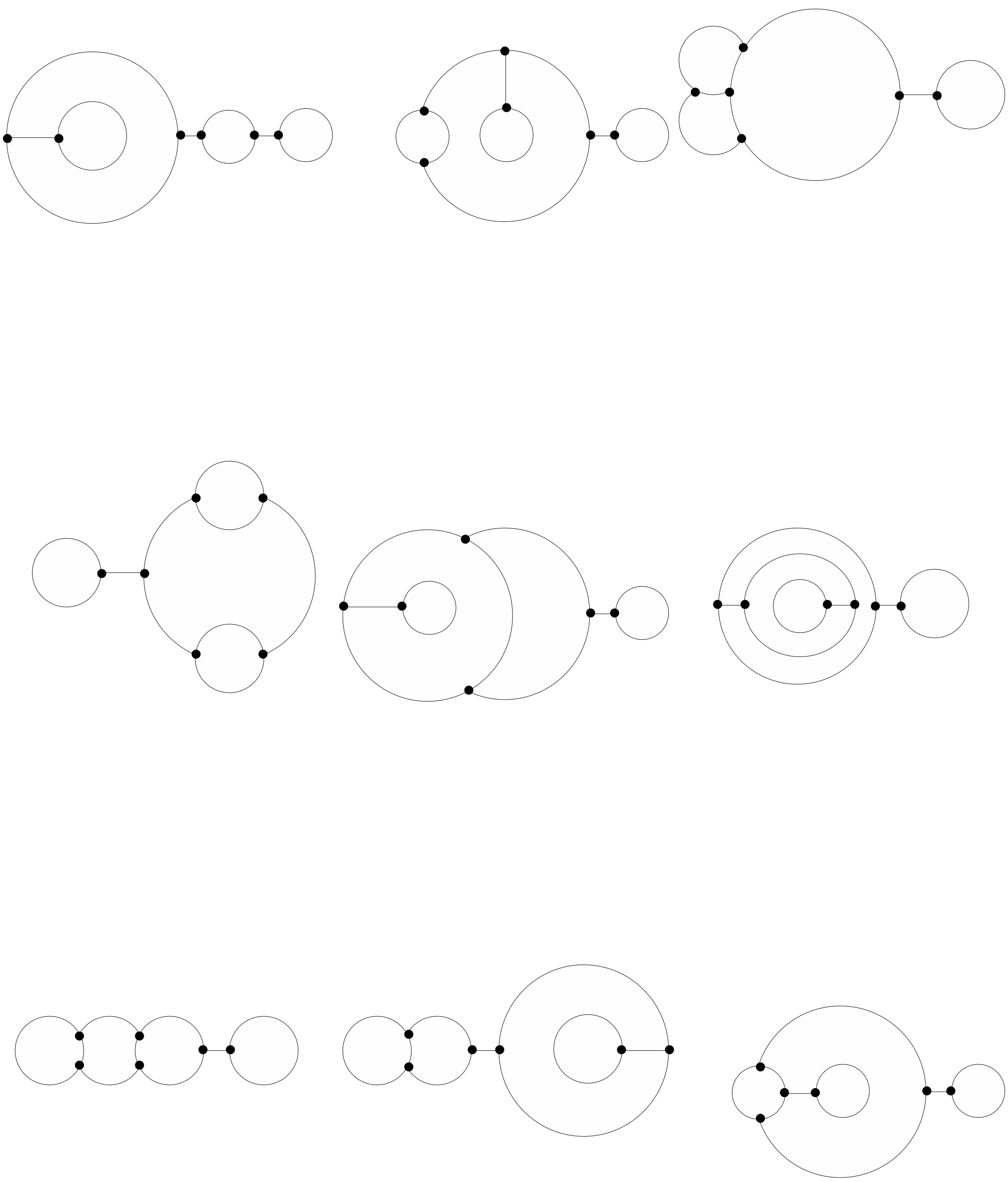

Figure 5.3: Dessins of $\mathbf{E}_7$ singularities

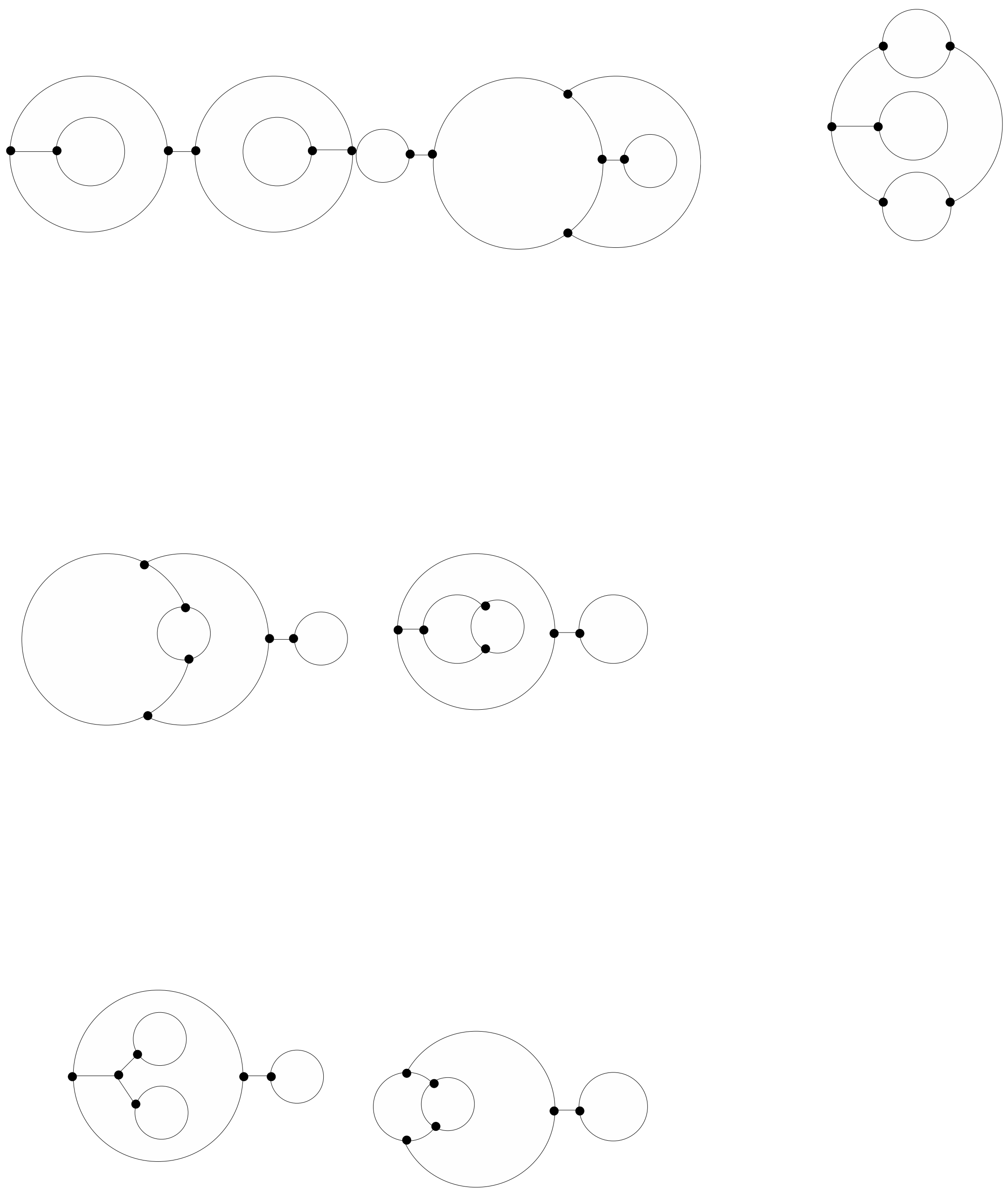

Figure 5.4: Dessins of $\mathbf{E}_7$ singularities

Num	Set of singularities	Dessins	Size of g	Abelian Invariants
1	$\mathbf{E}_7 \oplus \mathbf{A}_7 \oplus \mathbf{A}_2 \oplus 3\mathbf{A}_1$	5.2.a	16	[16]
2	$\mathbf{E}_7 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_1$	5.2.b	∞	[0,8]
3	$\mathbf{E}_7 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	5.2.c	14	[2,7]
4	$\mathbf{E}_7 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus 3\mathbf{A}_1$	5.2.d	∞	[0,2,3]
5	$\mathbf{E}_7 \oplus \mathbf{A}_9 \oplus 3\mathbf{A}_1$	5.2.e	20	[4,5]
6	$\mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	5.2.f	10	[2,5]
7	$\mathbf{E}_7 \oplus \mathbf{D}_7 \oplus \mathbf{A}_5$	5.4.f	12	[3,4]
8	$\mathbf{E}_7 \oplus \mathbf{D}_8 \oplus \mathbf{A}_4$	5.2.h	10	[2,5]
9	$\mathbf{E}_7 \oplus \mathbf{A}_7 \oplus \mathbf{A}_2 \oplus 3A_1$	5.2.i	∞	[0,8]
10	$\mathbf{E}_7 \oplus \mathbf{D}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	5.3.a	10	[2,5]
11	$\mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	5.3.b	14	[2,7]
12	$\mathbf{E}_7 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	5.3.c	14	[2,7]
13	$\mathbf{E}_7 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_1$	5.3.d	∞	[0,8]
14	$\mathbf{E}_7 \oplus \mathbf{A}_6 \oplus \mathbf{D}_4 \oplus \mathbf{A}_2$	5.3.e	12	[3,4]
15	$\mathbf{E}_7 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	5.3.f	10	[2,5]
16	$\mathbf{E}_7 \oplus \mathbf{A}_7 \oplus \mathbf{A}_2 \oplus 3A_1$	5.3.g	∞	[0,8]
17	$\mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	5.3.h	10	[2,5]
18	$\mathbf{E}_7 \oplus \mathbf{D}_4 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	5.3.i	14	[2,7]
19	$\mathbf{E}_7 \oplus \mathbf{D}_4 \oplus 2\mathbf{A}_4$	5.4.a	1200	[2,5]
20	$\mathbf{E}_7 \oplus \mathbf{D}_4 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$	5.4.b	∞	[0,2,3]
21	$\mathbf{E}_7 \oplus \mathbf{A}_7 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	5.4.c	∞	[0,8]
22	$\mathbf{E}_7 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4 \oplus 3\mathbf{A}_1$	5.4.d	∞	[0,2,3]
23	$\mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	5.4.e	10	[2,5]
24	$\mathbf{E}_7 \oplus \mathbf{A}_9 \oplus \mathbf{A}_3$	5.4.g	20	[4,5]
25	$\mathbf{E}_7 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_1$	5.4.h	∞	[0,2,3]
26	$\mathbf{E}_7 \oplus \mathbf{D}_4 \oplus \mathbf{A}_8$	5.2.g	16	[16]
27	$\mathbf{E}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2$	5.3.e	10	[2,5]
28	$\mathbf{E}_7 \oplus 2\mathbf{A}_5 \oplus 2\mathbf{A}_1$	5.4.b	∞	[0,2,3]

Table 5.1: Maximal sets of singularities with a type $\mathbf{E}_7$ point

Normally, a dessin may correspond to more than one sextic. It depends on the choice of the distinguished region. As it was mentioned before, the distinguished region has to have at least four edges. Moreover, in order to use our trick, there should exist a monogone which is not adjacent to the distinguished region. If we apply both two conditions, in our figures, there is only one possible sextic that each skeleton corresponds and we wrote them on the table.

If we look at the table, we can say that the fundamental group of sextics which of size of g is equal to the product of the components of its abelian invariants are abelian. i.e. fundamental groups of the sextics with singularities in Table 5.1 line 1,3,5,6,7,8,10,11,12,14,15,16,17,18,23,24,26 and 27 are all abelian. If the size of the corresponding group g is infinite, `GAP` fails and we cannot say anything about the commutant of the fundamental groups of the sextics. Hence, we do not say anything about the commutant of the fundamental groups of the sextics with singularities in Table 5.1 line 2,4,9,13,20,21,22,25 and 28. Finally, the size of the group g corresponding to the fundamental group of the sextic with $\mathbf{E}_7 \oplus \mathbf{D}_4 \oplus 2\mathbf{A}_4$ (in Table 5.1 line 19) is 1200 and it has abelian invariants as $[2, 5]$. Hence, its commutant q has order $120 = 1200/(2 * 5)$. Let $h = [q, q]$, then by using `GAP`, we have $h = [h, h]$, hence h is a *perfect* group. The only perfect group of order 120 is $SL(2, \mathbb{F}_5)$. Hence, the commutant is $SL(2, \mathbb{F}_5)$. Again, we can say that the fundamental group is non-abelian.

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