# VALID INEQUALITIES FOR THE PROBLEM OF OPTIMIZING A NONSEPARABLE PIECEWISE LINEAR FUNCTION ON 0-1 VARIABLES

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By

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July, 2011

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# ABSTRACT

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July, 2011

In many procurement and transportation applications, the unit prices depend on the amount purchased or transported. This results in piecewise linear cost functions. Our aim is to study the structure that arises due to a piecewise linear objective function and to propose valid inequalities that can be used to solve large procurement and transportation problems. We consider the problem of optimizing a nonseparable piecewise linear function on 0-1 variables. We linearize this problem using a multiple-choice model and investigate properties of facet defining inequalities. We propose valid inequalities and lifting results.

*Keywords:* Piecewise linear functions, valid inequalities, facet defining inequalities, sequential lifting, simultaneous lifting.

# ÖZET

## 0-1 DEĞİŞKENLİ AYRIŞMAYAN PARÇALI DOĞRUSAL BİR FONKSİYONUN ENİYİLENMESİ PROBLEMİ İÇİN GEÇERLİ EŞİTSİZLİKLER

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Birçok satın alma ve ulaşım uygulamasında, birim fiyatlar satın alınan ya da ulaşımı gerçekleştirilen ürün miktarına bağlıdır. Bu durum, parçalı doğrusal fonksiyonların kullanılmasına yol açmaktadır. Bizim bu çalışmadaki başlıca amacımız, parçalı doğrusal fonksiyonların kullanılmasıyla ortaya çıkan yapıyı incelemek ve büyük satın alma ve ulaşım problemlerinin çözümünde faydalı olabilecek geçerli eşitsizlikler üretmektir. Bu çalışmada, 0-1 değişkenli ayrışmayan parçalı doğrusal bir fonksiyonun eniyilenmesi problemi ele alınmaktadır. Bu problem, çok seçenekli model kullanarak doğrusallaştırılıp, yüzey tanımlayan eşitsizliklerin özellikleri incelenmektedir. Ayrıca, geçerli eşitsizlikler üretilip, kaldırma sonuçları sunulmaktadır.

*Anahtar sözcükler*: Parçalı doğrusal fonksiyonlar, geçerli eşitsizlikler, yüzey tanımlayan eşitsizlikler, sıralı kaldırma, eşzamanlı kaldırma.

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# Chapter 1

# Introduction

Piecewise linear functions are widely used in optimization problems in many applications, including transportation, telecommunications, and production planning. Any nonlinear function can be approximated to an arbitrary degree of accuracy as a piecewise linear function. The degree of accuracy is controlled by the number of the linear segments in the piecewise linear function.

Piecewise linear approximation is used in various specific applications, such as portfolio selection, network loading, merge-in-transit, minimum cost network flow with nonconvex piecewise linear costs, electronic circuit design, facility location with staircase costs, and optimization of gas network. Besides, procurement auctions with piecewise linear costs are common in industry. Cost functions in many supply chain problems are piecewise linear. Transportation costs in a network are mostly concave and piecewise linear, possibly with fixed costs.

In addition, piecewise linear optimization is an area of interest on its own. Piecewise linear functions that are convex can be minimized by linear programming since the slope of the segments are increasing. However, it is necessary to introduce nonlinearities in the model if the piecewise linear function is not convex.

This chapter consists of four sections. In the first section, the nonlinear problem with piecewise linear cost function that is studied throughout this thesis is defined. In the second section, research contributions are stated. Three formulations on linearization of the piecewise linear function are introduced in the third section. In the last section, outline of the thesis is provided.

## 1.1 Problem Definition

In this section, we first define the parameters of the problem and give the specifications of the piecewise linear cost function. After stating the objective of the problem, we introduce the nonlinear model.

We are given a set of items $N = \{1, \ldots, n\}$. For item $k \in N$, $p_k \geq 0$ is the revenue of selecting item $k$, and $w_k \geq 0$ is the weight of the item. We are also given a lower semicontinuous piecewise linear cost function $f$ with $t$ segments. Let $T = \{1, \ldots, t\}$. For segment $j \in T$, $b_j$ denotes the variable cost, $s_j$ denotes the fixed cost, and $a_j$ denotes the right breakpoint of the segment. Without loss of generality, we assume that $a_0 = 0$.

Our problem, called *Nonseparable Piecewise Linear Optimization (NPLO)*, is to select a subset of items in order to maximize the total profit. Since the cost function is a piecewise linear function, this problem is nonlinear.

For $k \in N$, $x_k$ is 1 if item $k$ is selected and 0 otherwise. Nonlinear model is the following formulation called $NLM$.

$$
\begin{aligned}
&\max \sum_{k \in N} p_k x_k - f\left(\sum_{k \in N} w_k x_k\right) \\
&\text{s.t. } x_k \in \{0, 1\} \qquad \forall k \in N
\end{aligned}
$$

The special case with a single piece with $a_1 < \sum_{k \in N} w_k$ and $f(y) = 0$ for

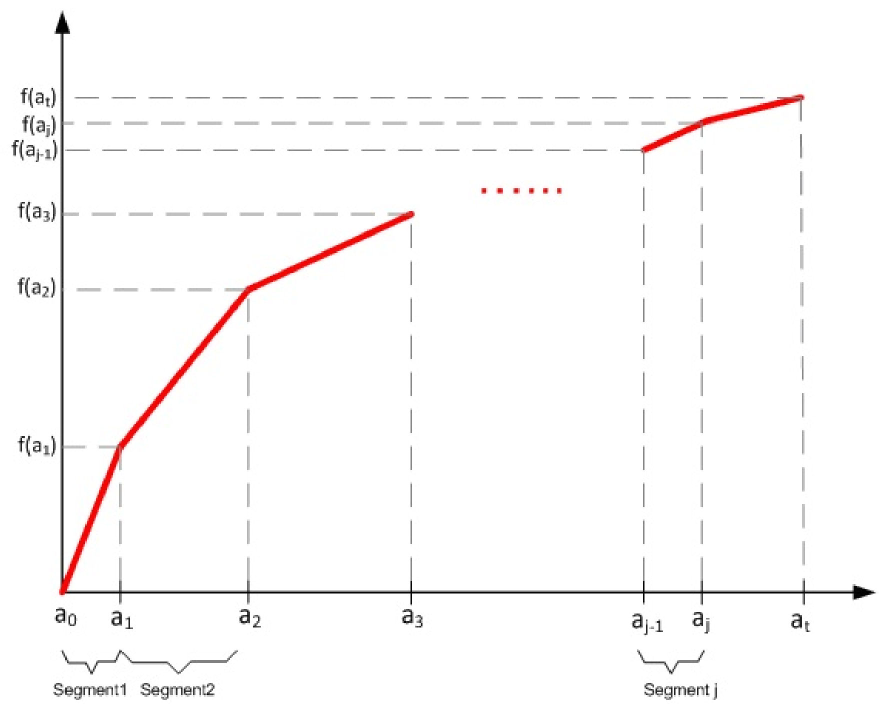


Figure 1.1: Piecewise linear cost function.

$y \in [0, a_1]$ is a knapsack problem.

## 1.2 Contribution

This thesis presents a nonlinear optimization problem with a piecewise linear cost function. Three formulations on linearization of the mathematical model, which are called multiple choice model, incremental cost formulation, and convex combination model, are provided for the given problem.

Since the knapsack problem is a special case of our problem, our problem is NP-hard. Therefore, deriving strong valid inequalities and facet defining inequalities is crucial for the solution of the problem. The aim of this research is to provide such inequalities for the proposed model.

In this study, a valid inequality in general form is presented and properties of facet defining inequalities are analyzed. Under some special conditions of the parameters in the valid inequality of the general form, strong valid inequalities are obtained.

In addition, some of the existing valid inequalities are lifted by sequential and simultaneous lifting techniques in order to achieve new strong valid inequalities. Lifting the existing inequalities are beneficial since they may lead to facet defining inequalities. Uplifting technique is used while determining the coefficients of the variables.

For each one of the two models, the multiple choice model with equality constraint and the relaxed version of it, an example with two segments and four items is analyzed. Using PORTA [21], all facet defining inequalities for the examples are obtained. All of the facets for the multiple choice model with equality constraint are explained either by the proposed valid inequalities and facet defining inequalities or lifting those inequalities. On the other hand, most of the facets for the relaxed version of the initial model are explained either by the proposed valid inequalities and facet defining inequalities or lifting those inequalities.

Ultimately, this research provides a problem with piecewise linear cost function and analysis on valid and facet defining inequalities. Lifting techniques are also used in order to obtain valid and facet defining inequalities. Lastly, strength of the proposed valid inequalities are shown on two examples in Appendix.

## 1.3 Linear Formulations

Piecewise linear functions that are convex can be minimized by linear programming since the slope of the segments are increasing. However, it is necessary to introduce nonlinearities in the model if the piecewise linear function is not convex. Since our piecewise linear cost function is not necessarily convex, we choose mixed integer programming approach (MIP) to solve our problem. In this MIP approach, the nonlinearities are formulated using binary variables. Besides, additional constraints are needed to relate those binary and continuous variables. Three fundamental MIP formulations are presented in this section.

This section consists of three subsections. Initially, we introduce the multiple choice model with the equality constraint, called $MCM_{=}$. Then, the relaxed version of it, $MCM_{\geq}$, is stated. In the second subsection, incremental cost formulation is presented. Finally, convex combination and alternative convex combination models are described in the last subsection.

### 1.3.1 Multiple Choice Model

Multiple choice model is first used by Balakrishnan and Graves [1]. In the multiple choice model, we use the following decision variables. For $k \in N$, $x_k$ is 1 if item $k$ is selected and 0 otherwise. For $j \in T$, $z_j$ is 1 if segment $j$ is selected and 0 otherwise, and $h_j$ is the amount of load on segment $j$.

The multiple choice model is the following formulation called $MCM_{=}$.

$$\max \sum_{k \in N} p_k x_k - \sum_{j \in T} (b_j h_j + s_j z_j) \tag{1.1}$$

$$\text{s.t.} \quad \sum_{j \in T} h_j = \sum_{k \in N} w_k x_k \tag{1.2}$$

$$a_{j-1} z_j \leq h_j \leq a_j z_j \qquad \forall j \in T \tag{1.3}$$

$$\sum_{j \in T} z_j \leq 1 \tag{1.4}$$

$$x_k \in \{0, 1\} \qquad \forall k \in N \tag{1.5}$$

$$z_j \in \{0, 1\} \qquad \forall j \in T \tag{1.6}$$

$$h_j \geq 0 \qquad \forall j \in T \tag{1.7}$$

The objective function (1.1) is the total revenue of selected items minus the total cost of the selected segment.

Constraints (1.2), (1.5), and (1.7) states that the total weight of the selected items is equal to the amount of load on the selected segment.

Constraints (1.3), (1.6), and (1.7) ensure that the amount of load on the selected segment is within the left and right breakpoints of that segment and that the amount of load on a segment that is not selected is zero.

Due to constraints (1.4) and (1.6) at most one segment is selected.

If the function $f$ is nondecreasing, then we can relax constraints (1.2) to

$$\sum_{j \in T} h_j \geq \sum_{k \in N} w_k x_k \tag{1.8}$$

without changing the optimal value. We call the resulting model $MCM_{\geq}$.

### 1.3.2 Incremental Cost Formulation

Incremental cost formulation is introduced by Manne and Markowitz [2]. Decision variables for this formulation are as follows. For $k \in N$, $x_k$ is 1 if item $k$ is selected and 0 otherwise. For $j \in T$, $r_j$ is 1 if segment $j$ is selected and 0 otherwise. For $j \in T$, $y_j$ is the amount of load on segment $j$.

The incremental cost formulation is the following formulation.

$$\max \sum_{k \in N} p_k x_k - \left( f(a_0) + \sum_{j \in T} (g_j y_j)/u_j \right) \tag{1.9}$$

$$\text{s.t. } a_0 + \sum_{j \in T} y_j = \sum_{k \in N} w_k x_k \tag{1.10}$$

$$u_1 r_1 \leq y_1 \leq u_1 \tag{1.11}$$

$$u_j r_j \leq y_j \leq u_j r_{j-1} \qquad \forall j = 2, 3, .., t-1 \tag{1.12}$$

$$0 \leq y_t \leq u_t r_{t-1} \tag{1.13}$$

$$x_k \in \{0, 1\} \qquad \forall k \in N \tag{1.14}$$

$$r_j \in \{0, 1\} \qquad \forall j = 1, 2, .., t-1 \tag{1.15}$$

The objective function (1.9) is the total profit of selected items minus the total cost of the selected segment.

Constraints (1.10), (1.13) and (1.14) ensure that total weight of the selected items cannot exceed the total amount of load on selected segments.

Constraints (1.11), (1.12), (1.13) and (1.15) ensure that the amount of load on a selected segment should be less than the difference between left and right breakpoint of that segment. If a segment is not selected, then the amount of load on that segment must be zero.

If the function $f$ is nondecreasing, then we can relax constraints (1.10) to

$$a_0 + \sum_{j \in T} y_j \geq \sum_{k \in N} w_k x_k \tag{1.16}$$

without changing the optimal value.

### 1.3.3 Convex Combination Model

Convex combination model is first used by Dantzig [3]. In this model, we use the following decision variables. For $k \in N$, $x_k$ is 1 if item $k$ is selected and 0 otherwise. For $j \in T$, $z_j$ is 1 if segment $j$ is selected and 0 otherwise, and $\lambda_{j-1}$ and $\lambda_j$ are the weights of the breakpoints on segment $j$.

The convex combination model is the following formulation.

$$\max \sum_{k \in N} p_k x_k - \sum_{j=0}^{t} \lambda_j f(a_j) \tag{1.17}$$

$$\text{s.t.} \quad \sum_{j=0}^{T} a_j \lambda_j = \sum_{k \in N} w_k x_k \tag{1.18}$$

$$\sum_{j=0}^{T} \lambda_j = 1 \tag{1.19}$$

$$\lambda_0 \leq z_0 \tag{1.20}$$

$$\lambda_j \leq z_{j-1} + z_j \qquad \forall j = 1, 2, .., t-1 \tag{1.21}$$

$$\lambda_t \leq z_{t-1} \tag{1.22}$$

$$\sum_{j=0}^{t} z_j = 1 \tag{1.23}$$

$$x_k \in \{0, 1\} \qquad \forall k \in N \tag{1.24}$$

$$z_j \in \{0, 1\} \qquad \forall j = 0, 1, .., t \tag{1.25}$$

$$\lambda_j \geq 0 \qquad \forall j = 0, 1, .., t \tag{1.26}$$

The objective function (1.17) is the total profit of selected items minus the total cost of the selected segment.

Constraints (1.18), (1.24), and (1.26) ensure that total weight of the selected items cannot exceed the total amount of load on selected segments.

Constraints (1.19) and (1.26) ensure that the total weight of the breakpoints should be equal to 1.

Constraints (1.20), (1.21), (1.22), (1.25), and (1.26) ensure that the weight of a breakpoint should be zero, unless the corresponding segment or segments to that breakpoint is selected.

Constraints (1.23) and (1.25) ensure that at most one segment can be selected.

If the function $f$ is nondecreasing, then we can relax constraints (1.18) to

$$\sum_{j=0}^{T} a_j \lambda_j \geq \sum_{k \in N} w_k x_k \tag{1.27}$$

without changing the optimal value.

Sherali [4] introduced an alternative formulation for the convex combination model. When we adjust that formulation to our problem, we get the alternative convex combination model. In this new formulation, following decision variables are used. For $k \in N$, $x_k$ is 1 if item $k$ is selected and 0 otherwise. For $j \in T$, $z_j$ is 1 if segment $j$ is selected and 0 otherwise, and $\lambda_j^L$ and $\lambda_j^R$ are the weights of the left and right breakpoints on segment $j$, respectively.

Alternative convex combination formulation is the following formulation.

$$\max \sum_{k \in N} p_k x_k - \sum_{j=1}^{T} (\lambda_j^L f(a_{j-1}) + \lambda_j^R f(a_j)) \tag{1.28}$$

$$\text{s.t.} \quad \sum_{j=1}^{T} (a_{j-1} \lambda_j^L + a_j \lambda_j^R) = \sum_{k \in N} w_k x_k \tag{1.29}$$

$$\sum_{j=0}^{T} \left(\lambda_j^L + \lambda_j^R\right) = z_j \tag{1.30}$$

$$\sum_{j=0}^{T} z_j = 1 \tag{1.31}$$

$$x_k \in \{0,1\} \qquad \forall k \in N \tag{1.32}$$

$$z_j \in \{0,1\} \qquad \forall j = 0,1,..,T \tag{1.33}$$

$$\lambda_j^L, \lambda_j^R \geq 0 \qquad \forall j = 0,1,..,T \tag{1.34}$$

The objective function (1.28) is the total profit of selected items minus the total cost of the selected segment.

Constraints (1.29), (1.32), and (1.34) ensure that total weight of the selected items cannot exceed the total amount of load on selected segments.

Constraints (1.30), (1.33), and (1.34) ensure that the weights of both left and right breakpoints of a segment should be zero, unless that segment is selected.

Constraints (1.31) and (1.33) ensure that at most one segment can be selected.

If the function $f$ is nondecreasing, then we can relax constraints (1.29) to

$$\sum_{j=1}^{T}(a_{j-1}\lambda_j^L + a_j\lambda_j^R) \geq \sum_{k\in N} w_k x_k \tag{1.35}$$

without changing the optimal value.

Croxton, Gendron, and Magnanti [5] propose that any feasible solution of the LP relaxation of multiple choice model, incremental cost formulation, and alternative convex combination model corresponds to a feasible solution to the other two with the same cost. Consequently, the LP relaxations of the three formulations are equivalent.

## 1.4 Contents

The rest of this thesis is organized as follows. A review of the piecewise linear functions in the literature is presented in Chapter 2. Chapter 3 introduces a set of valid inequalities in general form and analyzes properties of facet defining inequalities. Besides, valid inequalities for the problem and information about lifting valid inequalities in order to strengthen them are provided in Chapter 3. Chapter 4 contains any conclusions drawn from this research and possible discussions about possible future work. Finally, examples for both $MCM_{\geq}$ and $MCM_{=}$, and facet defining inequalities obtained by PORTA are presented in Appendix. In addition, those facet defining inequalities are explained by the valid inequalities proposed in Chapter 3 and lifting.

# Chapter 2

# Literature Review

In this chapter, a review of literature related to our study is presented. In our study, we have conducted a literature research on two distinct areas. Hence, this chapter consists of two sections; piecewise linear functions, and lifting, respectively. In the first section, a literature review about piecewise linear functions, their characteristics and linearization methods is presented. Second section consists of a review of the literature that is related to sequential, simultaneous, and sequence independent lifting techniques for valid inequalities.

## 2.1 Piecewise Linear Functions

Piecewise linear functions are commonly used in optimization problems in many real life applications such as transportation, telecommunication, and production planning problems. It is possible to approximate any nonlinear function to a piecewise linear function. The accuracy of the approximation is simply based on the size of the linear segments in the corresponding piecewise linear function.

There exist three fundamental formulations on linearization of the piecewise linear functions in the literature; namely the multiple choice model, the incremental cost formulation, and the convex combination model. Optimization of these three basic formulations is an area of interest in itself. There exist numerous studies on linearization and optimization of piecewise linear functions in the literature. In this section, we provide a review of the literature that is related with piecewise linear functions, their specifications and linearizations.

Croxton et al. [5] study a generic minimization problem with separable nonconvex piecewise linear costs. They state three basic formulations; incremental model, multiple choice model, and convex combination model. Then, they prove that the LP relaxations of the incremental, multiple choice, and convex combination formulations are equivalent. Moreover, they prove that the LP relaxation of any one of these three formulations approximates the cost function with its lower convex envelope.

In the work of de Farias et al. [6] special ordered set of type 2 (SOS2) approach is proposed for the optimization of a discontinuous separable piecewise linear function. A set of variables are said to be SOS2, when at most two variables are nonzero, and two nonzero variables are adjacent. They show that the given SOS2 approach can be used even when a mixed integer programming (MIP) model is not available for the given problem. They also prove that the LP relaxation bound of their SOS2 formulation is as good as the MIPs, when a MIP formulation is available. Besides, they state the advantages of SOS2 approach over the MIP model.

Keha et al. [7] study incremental model and convex combination model with and without additional binary variables. They show that both formulations without additional binary variables have the same LP bounds with the corresponding formulations with additional binary variables. When there are no additional binary variables, they enforce the nonlinearities algorithmically, by branching on

SOS2 variables in the branch and bound algorithm. They prove that SOS2 formulation, as well as incremental and convex combination models, is also locally ideal, which means that the binary variables are integer in every vertex of the set. They conclude that the formulations without binary variables should be better models for piecewise linear functions than the MIP models, since they are more compact.

In the study of Vielma et al. [8] the branch-and-cut algorithm for LPs with piecewise linear continuous costs are extended to the lower semicontinuous case. They also extend the SOS2 formulation for LPs with piecewise linear continuous costs to the lower semicontinuous case. Then, they adapt valid inequalities developed by Kehaet al. [9] to the new model. The discontinuous case caused by a fixed charge is also studied and two new valid inequalities are developed. According to the computational results, adding SOS2 based cuts can significantly increase the performance of the branch-and-cut procedure for one class of problems. For the other class of problems, adding SOS2 based cuts can significantly increase the performance regarding the number of nodes and best gaps obtained.

Vielma et al. [10] study the modeling of piecewise linear functions as MIPs. They review several existing and new formulations for continuous functions and study on their extension to the multivariate nonseparable case. Then, they compare these formulations both with respect to their theoretical properties and their computational performance. In addition, extensions of these formulations considering lower semicontinuous functions are studied.

Keha [11] studies the polyhedral structure of piecewise linear optimization problems and derived strong valid inequalities. Then, a polyhedral study of the one row relaxation of a separable piecewise linear optimization problem is presented. In addition, several classes of valid inequalities are derived for this problem, and a branch-and-cut algorithm is presented. He also derives some valid inequalities via lifting procedure. He shows that most of the valid inequalities that he derived are facet defining for a lower dimensional polytope.

## 2.2 Lifting

One of the most commonly used techniques to obtain strong valid inequalities is lifting. Lifting procedure aims at generating a strong valid inequality from an existing valid inequality by adjusting coefficients of one or more variables in the initial inequality. Lifting of a valid inequality may result in a facet defining inequality in some cases. Lifting techniques can be discussed in three categories; sequential or simultaneous, exact or approximate, and up or down lifting.

Since lifting is an important tool for generating strong valid inequalities, it is very useful for optimizing mixed integer programming problems. Therefore, there are plenty of studies on lifting techniques in the literature. In this section, we provide a review of the literature that is related with lifting valid inequalities.

Gu et al. [12] study lifting flow cover inequalities for mixed 0-1 integer programs. They discuss sequential and sequence independent lifting, and compare them. Since sequential lifting of flow cover inequalities is computationally challenging, they present a computationally efficient way to lift them using sequence independent lifting technique which can be classified under simultaneous lifting. Finally, they show the effectiveness of their lifting techniques by giving computational results.

In the study of Gu et al. [13] sequence independent lifting techniques are presented for both flow cover and knapsack cover inequalities. They discuss the relation between superadditive functions and sequence independent lifting techniques. Then, they show that the lifting coefficients are sequence independent if that lifting function is superadditive. In addition, they obtain good approximations to maximum lifting by introducing the idea of valid superadditive lifting functions.

Easton and Hooker [14] study simultaneously lifting sets of binary variables into cover inequalities for knapsack polytopes. They introduce a polynomial time algorithm that finds valid and facet defining inequalities using simultaneously lifting techniques. They show that the resulting simultaneously lifted cover inequality cannot be derived by lifting any cover inequality sequentially in many instances.

Atamturk [15] studies sequence independent lifting for mixed integer programming. He shows that superadditive lifting functions lead to sequence independent lifting of inequalities for general mixed integer programming. He also discusses that mixed integer rounding can be viewed as an application of sequence independent lifting. Besides, he analyzes facet defining conditions for mixed integer rounding inequalities for mixed integer knapsacks.

In the work of Shebalov and Klabjan [16] convex hull of a set defined by a single inequality with continuous and binary variables is analyzed. Flow cover inequality is extended and shown that it is valid when the set is restricted by fixing some variables. In addition, conditions under which that inequality is facet defining are presented. Furthermore, the way to lift that inequality in order to obtain valid inequalities for the original set using sequence independent lifting techniques is shown.

Sharma [17] presents a new algorithm called the Maximal Simultaneous Lifting Algorithm which basicly produces a cover inequality using sequence independent uplifting techniques over a set of binary variables. Then, he shows that under some assumptions, this algorithm results in strong inequalities that are facet defining. Their computational studies show that this algorithm can find numerous strong valid inequalities for knapsack problems in negligible time.

Bolton [18] introduces a new lifting method called synchronized simultaneous lifting. Then, he shows that some of the inequalities obtained by this method cannot be produced by any previous lifting methods. He also presents an algorithm called the Synchronized Simultaneous Lifting Algorithm which runs in quadratic time. This algorithm produces synchronized simultaneous lifting inequalities. His computational study shows that the proposed algorithm is significantly helpful in solving integer programs.

The contribution of this thesis to the literature is as follows: A problem of optimizing a nonseparable piecewise linear function is introduced. Out of three most commonly used linearization models in the literature, multiple choice model is studied. The task is to find valid and facet defining inequalities for the given problem. First, some properties of facet defining inequalities are presented. Besides, several strong valid inequalities are proposed for the problem. Moreover, lifting techniques are used to obtain strong valid inequalities. Exact methods are applied while determining the coefficients of the lifting variables. In addition, both sequential and simultaneous lifting techniques are applied. Furthermore, since we find out that there is no study on finding valid inequalities and providing properties of facet defining inequalities for this problem, our study is innovative for the piecewise linear optimization literature.

# Chapter 3

# Valid and Facet Defining Inequalities

*NPLO* is NP-hard. Therefore, the computational time for an instance may be too long, or even it may not be solved by a solver, especially if that instance is a big one. In order to reduce the computational time, one may think of several techniques. The most two common techniques are decomposition methods and the cutting plane algorithm. In this paper, we are interested in the latter one. Aim of the cutting plane technique is to generate valid inequalities that eliminate some portion of feasible region of the linear relaxation without cutting any feasible solutions of the original problem. The most useful cutting planes are the facet defining inequalities.

One way to create valid and facet defining inequalities is the lifting technique. Lifting technique aims at strengthening an existing valid inequality by adjusting coefficients of some of the variables in the inequality.

This chapter consists of three sections. In the first section, a valid inequality in general form for our problem is introduced. Then, four properties of facet defining inequalities are presented. In the second section, first, a valid inequality for the general problem is proposed. Then, the valid inequality in general form is analyzed in four specific cases when the piecewise linear function consists of two segments. For each case, at least one valid inequality is proposed. In the last section, sequential and simultaneous lifting techniques that are used to obtain valid and facet defining inequalities are discussed.

Before we proceed to the first section, we discuss the dimension of the multiple choice model given in the first chapter.

Let $P_{\geq}$ be the set of points that satisfy all the constraints of the model $MCM_{\geq}$. Define $e^x_k$, $e^h_j$ and $e^z_j$ such that $e^x_k$ is the unit vector of size $n$ such that the $k^{th}$ entry is 1 and others are zero, $e^h_j$ is the unit vector of size $t$ such that the $j^{th}$ entry is 1 and others are zero, $e^z_j$ is the unit vector of size $t$ such that the $j^{th}$ entry is 1 and others are zero.

Define $P_{\geq} = \{(x,z,h) : (1.3) - (1.8)\}$ and $P^{conv}_{\geq}$ is the convex hull of $P_{\geq}$. Define also, $P_{=} = \{(x,z,h) : (1.2) - (1.7)\}$ and $P^{conv}_{=}$ is the convex hull of $P_{=}$. We assume that $w_k \leq a_t$ for all $k \in N$.

**Proposition 1** *$dim(P^{conv}_{\geq}) = n + 2t$.*

*Proof.* Suppose that every solution $(x,z,h)$ in $P^{conv}_{\geq}$ satisfies $\sum_{j\in T} \delta_j h_j + \sum_{j=1}^{t-1} \mu_j z_j + \sum_{k\in N} \sigma_k x_k = \rho$. Since $(0,0,0) \in P^{conv}_{\geq}$, $\rho = 0$. As $(0, e^z_1, 0) \in P^{conv}_{\geq}$, $\mu_1 = 0$. Let $1 > \epsilon > 0$ be a small number. Since $(0, e^z_1, \epsilon e^h_1) \in P^{conv}_{\geq}$, $\delta_1 = 0$. Both solutions $(0, e^z_2, a_1 e^h_2)$, $(0, e^z_2, a_2 e^h_2)$ are in $P^{conv}_{\geq}$. Therefore, $\delta_2 = 0$, $\mu_2 = 0$. Similarly, $(0, e^z_m, a_{m-1} e^h_m)$, $(0, e^z_m, a_m e^h_m) \in P^{conv}_{\geq}$. Hence, $\delta_m = 0$, $\mu_m = 0$ for every $m \in \{2, 3, .., t\}$. Since $(e^x_1, e^z_t, a_t e^h_t) \in P^{conv}_{\geq}$, $\sigma_1 = 0$. $\sigma_2$ is also 0 since $(e^x_2, e^z_t, a_t e^h_t) \in P^{conv}_{\geq}$. Similarly, $\sigma_k = 0$ for every $k \in N$. Hence, there is no equation satisfied by all points $(x,z,h) \in P^{conv}_{\geq}$ and $dim(P^{conv}_{\geq}) = n + 2t$. $\square$

## 3.1 Properties of Facet Defining Inequalities

In this section, we first present a valid inequality in general form. Then, we propose four propositions on the properties of some of the coefficients in the given inequality.

Suppose that inequality

$$\rho + \sum_{k\in N} \alpha_k x_k + \sum_{j\in T} \beta_j z_j \leq \sum_{j\in T} \gamma_j h_j \tag{3.1}$$

is a valid inequality for $P_{\geq}^{conv}$.

**Proposition 2** *If (3.1) is facet defining for $P_{\geq}^{conv}$ and is different from $\sum_{j\in T} z_j \leq 1$, then $\rho = 0$.*

*Proof.* Let $F_{\geq} = \left\{(x,z,h) \in P_{\geq}^{conv} : \rho + \sum_{k\in N} \alpha_k x_k + \sum_{j\in T} \beta_j z_j = \sum_{j\in T} \gamma_j h_j\right\}$. There exists $(x,z,h) \in F_{\geq}$ such that $\sum_{j\in T} z_j = 0$. Since otherwise, all $(x,z,h) \in F_{\geq}$ satisfy $\sum_{j\in T} z_j = 1$. When $\sum_{j\in T} z_j = 0$, $x = 0$ and $h = 0$. Hence, $\rho$ must be equal to zero. □

**Proposition 3** *Let $l \in T$. If (3.1) is facet defining for $P_{\geq}^{conv}$ and is different from $h_l \leq a_l z_l$, then $\gamma_l \geq 0$.*

*Proof.* Let $F_{\geq} = \left\{(x,z,h) \in P^{conv} : \rho + \sum_{k\in N} \alpha_k x_k + \sum_{j\in T} \beta_j z_j = \sum_{j\in T} \gamma_j h_j\right\}$ and $l \in T$. There exists $(x,z,h) \in F_{\geq}$ such that $h_l < a_l z_l$. Since otherwise, all $(x,z,h) \in F_{\geq}$ satisfy $h_l = a_l z_l$. Consider solution $(x,z,\bar{h})$ where $\bar{h}_j = h_j$ for every $j \in T \setminus \{l\}$ and $\bar{h}_l = h_l + \epsilon$ for very small $\epsilon > 0$. As $(x,z,\bar{h})$ is a feasible solution, we need $\rho + \sum_{k\in N} \alpha_k x_k + \sum_{j\in T} \beta_j z_j \leq \sum_{j\in T} \gamma_j \bar{h}_j$ as inequality (3.1) is valid. Hence, $\gamma_l$ must be greater than or equal to zero. □

**Proposition 4** *Let $m \in N$. If (3.1) is facet defining for $P_{\geq}^{conv}$ and is different from $x_m \geq 0$, then $\alpha_m \geq 0$.*

*Proof.* Let $F_{\geq} = \left\{(x,z,h) \in P^{conv} : \rho + \sum_{k \in N} \alpha_k x_k + \sum_{j \in T} \beta_j z_j = \sum_{j \in T} \gamma_j h_j\right\}$ and $m \in N$. There exists $(x,z,h) \in F_{\geq}$ such that $x_m > 0$. Consider solution $(\bar{x}, z, h)$ where $\bar{x}_k = x_k$ for every $k \in N \setminus \{m\}$ and $\bar{x}_m = 0$. As this solution is feasible, we need $\alpha_m \geq 0$. □

**Proposition 5** *If (3.1) is facet defining for $P_{\geq}^{conv}$ and is different from $\sum_{j \in T} z_j \leq 1$ and $h_l \leq a_l z_l$ for all $l \in T$, then define*

$$
\begin{aligned}
\hat{\beta}_i = \min & \sum_{k \in N} (\gamma_i w_k - \alpha_k) x_k \\
s.t.\ & a_{i-1} \leq \sum_{k \in N} w_k x_k \leq a_i \\
& x_k \in \{0,1\} \qquad \forall k \in N
\end{aligned}
$$

*and*

$$
\begin{aligned}
\tilde{\beta}_i = \min & \ \gamma_i a_{i-1} - \sum_{k \in N} \alpha_k x_k \\
s.t.\ & \sum_{k \in N} w_k x_k \leq a_{i-1} \\
& x_k \in \{0,1\} \qquad \forall k \in N
\end{aligned}
$$

*for all $i \in T$. Then, $\beta_i = min\left\{\hat{\beta}_i, \tilde{\beta}_i\right\}$.*

*Proof.* Suppose that (3.1) is facet defining for $P_{\geq}^{conv}$ and is different from $\sum_{j \in T} z_j \leq 1$ and $h_l \leq a_l z_l$ for all $l \in T$. Let $i \in T$. By *Proposition 2*, we know that $\rho = 0$ and by *Proposition 3*, we know that $\gamma_i \geq 0$. If $z_i = 1$, then the inequality becomes $\sum_{k \in N} \alpha_k x_k + \beta_i \leq \gamma_i h_i$. Hence, $\beta_i \leq \gamma_i h_i - \sum_{k \in N} \alpha_k x_k$ should be satisfied by all feasible points $(x,z,h)$ with $z_i = 1$. Given $x$, if $\sum_{k \in N} w_k x_k \geq a_{i-1}$, then best value for $h_i$ is $\sum_{k \in N} w_k x_k$, since $\gamma_i \geq 0$. Otherwise, best value for $h_i$ is $a_{i-1}$. For these $h_i$ values, $\hat{\beta}_i$ and $\tilde{\beta}_i$ are the upper bounds for $\beta_i$, respectively. Then, the best value for $\beta_i$ is the minimum of these two bounds. □

## 3.2 Valid Inequalities

In this section, we first present a valid inequality for arbitrary number of segments. Then, we analyze four cases on an instance when the piecewise linear function has two segments and introduce strong valid inequalities for each case. Besides, an example for each valid inequality that defines a facet for the given problem is presented.

For $S \subseteq N$, let $w(S) = \sum_{k \in S} w_k$. Let $j \in T$. We define

$$\begin{aligned} \bar{w}_j(S) = \max & \sum_{k \in S} w_k x_k \\ \text{s.t.} & \sum_{k \in S} w_k x_k \leq a_j \\ & x_k \in \{0,1\} \, \forall k \in S \end{aligned}$$

and $\lambda = w(S) - \bar{w}_j(S)$.

**Proposition 6** *Let $j \in T$, $S \subseteq N$, and $C \subseteq \{k \in S : w_k > \lambda\}$. The inequality*

$$\sum_{k \in C} \lambda x_k + \sum_{k \in S \setminus C} w_k x_k + \beta_j z_j \leq \sum_{i \in T \setminus \{j\}} h_i \tag{3.2}$$

*is a valid inequality for $P_{\geq}^{conv}$.*

*Proof.* If $z_j = 0$, then $h_j = 0$ and inequality (3.2) becomes $\sum_{k \in C} \lambda x_k + \sum_{k \in S \setminus C} w_k x_k \leq \sum_{i \in T \setminus \{j\}} h_i$. This is satisfied as $\lambda \leq w_k$ for all $k \in C$ and $w_k$ and $x_k$ are nonnegative for all $k \in N$.
If $z_j = 1$, then as $\sum_{i \in T \setminus \{j\}} h_i = 0$, inequality (3.2) becomes

$$\sum_{k \in C} \lambda x_k + \sum_{k \in S \setminus C} w_k x_k + \beta_j \leq 0$$

By definitons of $\lambda$ and $\beta_j$, (3.2) is again satisfied. $\square$

### 3.2.1 The case of two segments

Now, we will focus on the piecewise linear cost function with two segments. When there are two segments in the piecewise linear cost function, inequality (3.1) becomes;

$$\sum_{k\in N} \alpha_k x_k + \beta_1 z_1 + \beta_2 z_2 \leq \gamma_1 h_1 + \gamma_2 h_2 \tag{3.3}$$

Define $\beta_1$ and $\beta_2$ as *Proposition 5* proposes.

Now, we will analyze four cases and propose five valid inequalities for them. First two inequalities are special cases of (3.2). Last three inequalities are basicly influenced from *Proposition 5*. Each of them contains the $\beta_i$ structure. Besides, $\gamma_j$'s are either zero or one in these three inequalities. $\alpha_k$ values are formed as the facet defining inequalities that are obtained by PORTA are analyzed thoroughly. After proving the validity of every inequality, an example of the corresponding inequality that defines a facet for the following instance will be shown.

$$\begin{aligned}
&h_1 + h_2 \geq 17x_1 + 17x_2 + 18x_3 + 2x_4\\
&0 \leq h_1 \leq 24z_1\\
&24z_2 \leq h_2 \leq 53z_2\\
&z_1 + z_2 \leq 1\\
&x_1, x_2, x_3, x_4, z_1, z_2 \in \{0, 1\}
\end{aligned}$$

First case is for inequalities on $z_2$ and $h_1$ as follows.

**Case 1** $\beta_1 = 0, \gamma_2 = 0, \gamma_1 = 1$

*For this case, we have inequality (3.2) in the following form;*

$$\sum_{k\in C} \lambda x_k + \sum_{k\in S\setminus C} w_k x_k + \beta_2 z_2 \leq h_1 \tag{3.4}$$

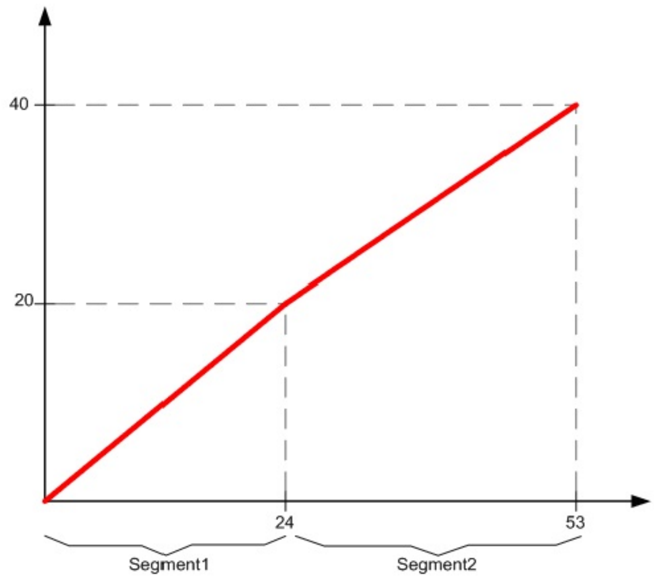


Figure 3.1: An example with two segments.

As an example, let $S = \{1,2,3,4\}$ and $C = \{1,3\}$. Then, as $w(S) = 54$ and $\bar{w}_2(S) = 52$, $\lambda$ is 2. Hence, $\alpha_1 = 2$, $\alpha_2 = 17$, $\alpha_3 = 2$, and $\alpha_4 = 2$. Since $\beta_2 = -21$, inequality (3.4) becomes $2x_1 + 17x_2 + 2x_3 + 2x_4 - 21z_2 \leq h_1$ which defines a facet for the problem as PORTA points out.

Next case is for inequalities on $z_1$ and $h_2$ as follows.

**Case 2** $\beta_2 = 0, \gamma_1 = 0, \gamma_2 = 1$;
*For this case, we have inequality (3.2) in the following form;*

$$\sum_{k \in C} \lambda x_k + \sum_{k \in S \setminus C} w_k x_k + \beta_1 z_1 \leq h_2 \tag{3.5}$$

As an example, let $S = \{1,2,3\}$ and $C = \emptyset$. Then, as $w(S) = 52$ and $\bar{w}_1(S) = 18$, $\lambda$ is 34. Besides, $\beta_1 = -18$. Hence, inequality (3.5) becomes $17x_1 + 17x_2 + 18x_3 - 18z_1 \leq h_2$. This inequality is also facet defining for the problem according to PORTA.

For the above two cases, all facet defining inequalities for the given instance can be described by (3.4) and (3.5) (See Appendix A).

Next case is for inequalities on $z_2$, $h_1$, and $h_2$ as follows.

**Case 3** $\beta_1 = 0, \gamma_1 = 1, \gamma_2 = 1$;

***Proposition 7*** *Let $S \subseteq N$ and $C \subseteq S$ such that $w(C) \geq a_1$. Define*

$$w'(S) = \min_{k \in S : w(S \setminus \{k\}) > a1} w(S \setminus \{k\}) - a_1$$

*Let $\alpha_k = w(C) - a_1$ if $k \in C$, and $\alpha_k = w'(S)$ if $k \in S \setminus C$. The inequality*

$$\sum_{k \in S} min\{\alpha_k, w_k\} x_k + \sum_{k \in N \setminus S} w_k x_k + \beta_2 z_2 \leq h_1 + h_2 \tag{3.6}$$

*is a valid inequality.*

*Proof.* *If $z_2 = 0$, then $h_2 = 0$ and inequality (3.6) becomes $\sum_{k \in S} min\{\alpha_k, w_k\} x_k + \sum_{k \in N \setminus S} w_k x_k \leq h_1$. This is satisfied as $min\{\alpha_k, w_k\} \leq w_k$ for all $k \in S$ and $h_1 \geq \sum_{k \in N} w_k x_k$.*
*If $z_2 = 1$, then $h_1 = 0$ and inequality (3.6) becomes $\sum_{k \in S} min\{\alpha_k, w_k\} x_k + \sum_{k \in N \setminus S} w_k x_k + \beta_2 \leq h_2$. By definition of $\beta_2$ from Proposition 5, $\sum_{k \in S} min\{\alpha_k, w_k\} x_k + \beta_2 \leq \sum_{k \in S} w_k x_k$. Since $h_2 \geq \sum_{k \in N} w_k x_k$, inequality (3.6) is again satisfied.* □

*As an example, let $S = \{1, 2, 3, 4\}$ and $C = \{1, 3, 4\}$. Then, as $w(C) = 37$ and $w'(S) = 12$; $\alpha_1 = 13$, $\alpha_2 = 12$, $\alpha_3 = 13$, and $\alpha_4 = 2$. Besides, $\beta_2$ is 9. Hence, inequality (3.6) becomes $13x_1 + 12x_2 + 13x_3 + 2x_4 + 9z_2 \leq h_1 + h_2$ which* *is also a facet defining inequality for the problem as PORTA shows.*

***Proposition 8*** *For the same case define* $\alpha' = a_1 - \bar{w}_1(S)$. *For* $k \in S$; *let* $\alpha_k = w_k - \alpha'$ *if* $w_k > \alpha'$, *and* $\alpha_k = w_k$, *otherwise. Then;*

$$\sum_{k \in S} \alpha_k x_k + \beta_2 z_2 \leq h_1 + h_2 \tag{3.7}$$

*is a valid inequality.*

*Proof.* *The validity follows from the definition of* $\beta_2$ *and the fact that* $\alpha_k \leq w_k$ *for all* $k \in N$. $\Box$

As an example, let $S = \{1, 2, 3, 4\}$. Then, as $\bar{w}_1(S) = 20$; $\alpha' = 4$. Accordingly, $\alpha_1 = 13$, $\alpha_2 = 13$, $\alpha_3 = 14$, and $\alpha_4 = 2$. Besides, $\beta_2$ is 8. Hence, inequality (3.7) becomes $13x_1 + 13x_2 + 14x_3 + 2x_4 + 8z_2 \leq h_1 + h_2$ which also defines a facet for the problem as PORTA indicates.

For the above case, four out of ten facet defining inequalities for the given instance can be described by (3.6) and (3.7). Remaining six facets are described by sequential lifting procedure.

Next case is for inequalities on $z_1$, $z_2$, and $h_2$ as follows.

**Case 4** $\gamma_1 = 0, \gamma_2 = 1, \beta_2 \neq 0$;

***Proposition 9*** *Let* $S \subseteq N$ *and* $C \subseteq S$ *such that* $w(C) \geq a_1$. *Let also* $\alpha_k = w(C) - a_1$ *if* $k \in C$, *and* $\alpha_k = w'(S)$ *if* $k \in S \setminus C$. *Then*

$$\sum_{k \in S} min\left\{\alpha_k, w_k\right\} x_k + \beta_1 z_1 + \beta_2 z_2 \leq h_2 \tag{3.8}$$

*is a valid inequality.*

*Proof.* *If* $z_1 = 1$*, then* $h_2 = 0$ *and inequality (3.8) becomes* $\sum_{k \in S} min\{\alpha_k, w_k\} x_k + \beta_1 \leq 0$*. By definition of* $\beta_1$ *this is satisfied.*
*If* $z_2 = 1$*, then* $h_1 = 0$ *and inequality (3.8) becomes* $\sum_{k \in S} min\{\alpha_k, w_k\} x_k + \beta_2 \leq h_2$*. By definition of* $\beta_2$*,* $\sum_{k \in S} min\{\alpha_k, w_k\} x_k + \beta_2 \leq \sum_{k \in S} w_k x_k$*. Since* $h_2 \geq \sum_{k \in N} w_k x_k$*, inequality (3.8) is again satisfied.* □

As an example, let $S = \{1, 2, 3\}$ and $C = \{1, 2\}$. Then, since $w(C) = 34$ and $w'(S) = 10$, $\alpha_1 = 10, \alpha_2 = 10$, and $\alpha_3 = 10$. Besides, $\beta_1$ is -10 and $\beta_2$ is 14. Hence, inequality (3.8) becomes $10x_1 + 10x_2 + 10x_3 - 10z_1 + 14z_2 \leq h_2$ which is also a facet defining inequality for the problem as PORTA shows.

For the above case, seven out of fifteen facet defining inequalities for the given instance can be described by (3.8). Besides, seven facets are described by sequential lifting procedure.

## 3.3 Lifting

Lifting was first proposed by Gomory [19] and is one of the most widely used techniques to generate strong valid inequalities. Main purpose of lifting is to start with a valid inequality and obtain a strong valid inequality by changing the coefficients of one or more variables in the initial inequality. In some cases, lifting of a valid inequality results in a facet defining inequality. Lifting techniques can be categorized in three parts; sequential or simultaneous, exact or approximate, and up or down lifting. In our study, we applied both sequential and simultaneous lifting in order to obtain valid and facet defining inequalities. We used uplifting technique while determining the exact coefficients of the variables to be lifted.

This section consists of two subsections; sequential lifting, and simultaneous lifting, respectively. In each one, corresponding lifting technique is briefly explained. Then, strength of the lifting techniques are shown by providing examples from the problem that is given in the previous section.

### 3.3.1 Sequential Lifting

Sequential lifting is the most commonly used lifting technique in the literature. In most cases, it is very helpful on deriving facet defining inequalities. There are two types of sequential lifting; uplifting and downlifting.

Sequential lifting adjusts the coefficients of the variables one at a time. At each time a variable is to be lifted, an optimization problem is required to be solved. Typically, the lifting process is done iteratively until all of the possible variables are lifted. Besides, the order of the variables that are to be lifted can vary. The coefficients of the lifted variables can also vary, since they depend on the lifting order.

The sequential lifting algorithm for an $x$ variable, $x_l$, assumes that $\sum_{k\in N\setminus\{l\}} \alpha_k x_k + \sum_{j\in T} \beta_j z_j \leq \sum_{j\in T} \gamma_j h_j$ is a valid inequality for $P_{\geq}^{conv}$ when $x_l = 0$, and seeks to generate a valid inequality $\alpha_l x_l + \sum_{k\in N\setminus\{l\}} \alpha_k x_k + \sum_{j\in T} \beta_j z_j \leq \sum_{j\in T} \gamma_j h_j$ for $P_{\geq}^{conv}$.

The methodology to perform sequential lifting of an $x$ variable, $x_l$, starts with setting $x_l$ to 1. Then, inequality $\alpha_l x_l + \sum_{k\in N\setminus\{l\}} \alpha_k x_k + \sum_{j\in T} \beta_j z_j \leq \sum_{j\in T} \gamma_j h_j$ is obtained. This inequality, together with our original constraints constitute the constraint set. Our aim is to solve the optimization problem which tries to maximize $\alpha_l$ while satisfying the constraint set. In addition, one or more $z_j$'s may have to be equal to zero after setting $x_l$ to 1. Setting those variables can help solving the problem.

The sequential lifting algorithm for a $z$ variable, $z_r$, assumes that $\sum_{k \in N} \alpha_k x_k + \sum_{j \in T \setminus \{r\}} \beta_j z_j \leq \sum_{j \in T} \gamma_j h_j$ is a valid inequality for $P_{\geq}^{conv}$ when $z_r = 0$, and seeks to generate a valid inequality $\sum_{k \in N} \alpha_k x_k + \sum_{j \in T \setminus \{r\}} \beta_j z_j + \beta_r z_r \leq \sum_{j \in T} \gamma_j h_j$ for $P_{\geq}^{conv}$.

The methodology to perform sequential lifting of a $z$ variable, $z_r$, starts with setting $z_r$ to 1. Then, inequality $\sum_{k \in N} \alpha_k x_k + \beta_r \leq \gamma_r h_r$ is obtained. In addition, one or more $x_k$'s may have to be equal to zero after setting $z_r$ to 1. Adjusting values of those variables can help solving the optimization problem which is to be solved in order to find the maximum value for $\beta_r$.

As an example, let $12x_2 + 13x_3 + 2x_4 \leq h_1 + h_2$ be the inequality to be lifted for $MCM_{\geq}$ model. Obviously, this inequality is valid when $x_1 = 0$ and $z_2 = 0$, since $17x_1 + 17x_2 + 18x_3 + 2x_4 \leq h_1 + h_2$. Assume that $x_1$ is the variable to be uplifted. Let $x_1 = 1$, then $z_1 = 1$. Thus, we get the following inequality:
$\alpha_1 + 12x_2 + 13x_3 + 2x_4 \leq h_1$. Maximum value that $\alpha_1$ can take is 17, since otherwise the inequality $17x_1 + 17x_2 + 18x_3 + 2x_4 \leq h_1$ would not be valid. Now, we lift the inequality with $z_2$. We set $z_2$ to 1, and get the following inequality:
$17x_1 + 12x_2 + 13x_3 + 2x_4 + \beta_2 \leq h_2$. $\hat{\beta}_2 = 5$ and $\tilde{\beta}_2 = 5$, so maximum value that $\beta_2$ can take is 5, which are obtained as the following optimization problems are solved:

$$\begin{aligned} \hat{\beta}_2 = & \min\,(5x_2 + 5x_3) \\ & \text{s.t. } 24 \leq 17x_1 + 17x_2 + 18x_3 + 2x_4 \leq 53 \\ & \quad x_k \in \{0,1\} \qquad \forall k \in N \end{aligned}$$

and

$$\begin{aligned} \tilde{\beta}_2 = & \min\,\{24 - (17x_1 + 12x_2 + 13x_3 + 2x_4)\} \\ & \text{s.t. } 17x_1 + 17x_2 + 18x_3 + 2x_4 \leq 24 \\ & \quad x_k \in \{0,1\} \qquad \forall k \in N \end{aligned}$$

Then, $\beta_2 = min\left\{\hat{\beta}_2, \tilde{\beta}_2\right\}$.

After finding $\beta_2$, we get the valid inequality $17x_1 + 12x_2 + 13x_3 + 2x_4 + 5z_2 \leq h_1 + h_2$ which is a facet defining inequality for this example.

### 3.3.2 Simultaneous Lifting

Simultaneous lifting is another lifting method for generating valid inequalities. Its main difference from sequential lifting is that it allows for more than one variable to be lifted at the same time. Simultaneous lifting technique for binary variables is first proposed by Zemel [20]. He used an exact method to lift multiple variables simultaneously. Although his method is technically precise, it is very demanding, computationally.

Main advantage of simultaneous lifting is that it requires less number of optimization problems to be solved by lifting a set of variables at once. Moreover, the inequalities generated by simultaneous lifting tend to be stronger [17].

The simultaneous lifting algorithm for a pair of $z$ and $h$ variables, $z_r$ and $h_r$, assumes that $\sum_{k\in N}\alpha_k x_k + \sum_{j\in T\setminus\{r\}}\beta_j z_j \leq \sum_{j\in T\setminus\{r\}}\gamma_j h_j$ is a valid inequality for $P^{conv}_{\geq}$ when $z_r = 0$, and seeks to create a valid inequality $\sum_{k\in N}\alpha_k x_k + \sum_{j\in T\setminus\{r\}}\beta_j z_j + \beta_r z_r \leq \sum_{j\in T\setminus\{r\}}\gamma_j h_j + \gamma_r h_r$ for $P^{conv}_{\geq}$.

The methodology to perform simultaneous lifting of a pair of $z$ and $h$ variables, $z_r$ and $h_r$, starts with setting $z_r$ to 1, and consequently $\sum_{k\in N} w_k x_k \leq h_r$ and $a_{r-1} \leq h_r \leq a_r$. Then, inequality $\sum_{k\in N}\alpha_k x_k + \beta_r \leq \gamma_r h_r$ is obtained. In addition, one or more $x_k$'s may have to be equal to zero or one after setting $z_r$ to 1. Adjusting values of those variables can be useful while finding the proper coefficients of $z_r$ and $h_r$. In order to find these coefficients, the set of inequalities

$a_{r-1} \leq \sum_{k\in N} w_k x_k \leq a_r$ and $\sum_{k\in N} \alpha_k x_k + \beta_r \leq \gamma_r \sum_{k\in N} w_k x_k \leq \gamma_r h_r$ are analyzed. Out of the feasible pairs, the pair of $\beta_r^*$ and $\gamma_r^*$ that satisfy two of these inequalities as equality are the proper lifting coefficients.

As an example, let $204x_1 + 204x_2 + 204x_3 + 34x_4 + 170z_2 \leq 17h_2$ be the inequality to be lifted for $MCM_{\geq}$ model. This inequality is valid, since $170 \leq 17h_2 - 204x_1 - 204x_2 - 204x_3 - 34x_4$ is satisfied by all feasible points $(x, z, h)$ when $z_1 = 0$. Also, 170 is the maximum value for $\beta_2$ for these $\alpha_k$ and $\gamma_j$ values as *Proposition 5* proposes. Assume that $z_1$ and $h_1$ are the variables to be lifted simultaneously. Let $z_1 = 1$, then $z_2 = 0$ and $h_1 \leq 24$. Consequently, we get the following inequality:
$204x_1 + 204x_2 + 204x_3 + 34x_4 + \beta_1 \leq \gamma_1 h_1$.

Find the set of feasible solutions that satisfy following constraints:
$17x_1 + 17x_2 + 18x_3 + 2x_4 \leq 24$
$204x_1 + 204x_2 + 204x_3 + 34x_4 + \beta_1 \leq \gamma_1 \sum_{k\in N} w_k x_k \leq \gamma_1 h_1$

Feasible solutions and corresponding inequalities can be listed as follows:
If $x_1 = 1$ and $x_4 = 0$, then: $204 + \beta_1 \leq 17\gamma_1 \leq \gamma_1 h_1$ (i)
If $x_1 = 1$ and $x_4 = 1$, then: $238 + \beta_1 \leq 19\gamma_1 \leq \gamma_1 h_1$ (ii)
If $x_1 = 0$, $x_3 = 0$, and $x_4 = 1$, then: $34 + \beta_1 \leq 2\gamma_1 \leq \gamma_1 h_1$ (iii)
If $x_3 = 1$ and $x_4 = 0$, then: $204 + \beta_1 \leq 18\gamma_1 \leq \gamma_1 h_1$ (iv)
If $x_3 = 1$ and $x_4 = 1$, then: $238 + \beta_1 \leq 20\gamma_1 \leq \gamma_1 h_1$ (v)

When the above five cases are analyzed, it can be seen that (iv) is dominated by (i), and (v) is dominated by (ii). Besides, at most two of (i), (ii), and (iii) can be binding. Either if (i) and (ii) or (i) and (iii) are binding, the remaining one would not be valid. If (ii) and (iii) are binding, then $\beta_1 = -10$ and $\gamma_1 = 12$. Then, we get the inequality $204x_1 + 204x_2 + 204x_3 + 34x_4 - 10z_1 + 170z_2 \leq 12h_1 + 17h_2$ which is also a facet defining inequality for this example.

# Chapter 4

# Conclusion and Future Research

In this thesis, we first presented a nonlinear optimization problem with a piecewise linear cost function. After introducing four linearizations of the mathematical model, we showed that we can relax a set of constraints if the piecewise linear function $f$ is nondecreasing. Relaxing that set of constraints lead us to other versions of the models. We picked multiple choice model, $MCM_{\geq}$ and $MCM_{=}$, as the scope of our study.

In Chapter 2, we provided a review of the literature that contains specifications of piecewise linear functions, their linearization methods and formulations, and piecewise linear optimization.

We introduced a valid inequality in general form in Chapter 3. Then, we analyzed that inequality and discussed properties of facet defining inequalities for our formulation. After deriving valid inequalities from the valid inequality in general form, we studied on how to strengthen them. Moreover, we used both sequential and simultaneous lifting in order to obtain new strong valid inequalities and facet defining inequalities. While determining the coefficients of the variables, we used uplifting technique.

In addition, we analyzed two examples with two segments and four items, one for $MCM_{\geq}$ and one for $MCM_{=}$. We obtained all facet defining inequalities for both problems, using PORTA. For $MCM_{=}$ formulation, we managed to explain all the facet defining inequalities either by the valid and facet defining inequalities that we proposed, or lifting those inequalities. For $MCM_{\geq}$ formulation, we explained most of the facet defining inequalities either by the valid and facet defining inequalities that we proposed, or lifting those inequalities. Analysis on those examples can be found in Appendix.

This study can be extended in many ways. In this study, we picked multiple choice model out of three fundamental formulations on linearization of piecewise linear cost functions. One can study remaining two basic formulations, which are incremental cost formulation and convex combination model. An alternative way could be to convert valid and facet defining inequalities that we obtained to these two formulations. Besides, new valid and facet defining inequalities can be derived as a future research. We studied a relaxed version of the initial model, $MCM_{\geq}$, for which the piecewise linear cost function needs to be nondecreasing. One may be interested in dealing with different special cases of the piecewise linear cost function. In addition, we clarified strengths of our valid and facet defining inequalities by explaining the facets of our small sized instances with them. One can write an algorithm to produce and introduce valid inequalities to large sized instances and test their strength as a future work.

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# Appendix A

# An example for $MCM_{\geq}$

Using PORTA, we generated all facet defining inequalities for the convex hull of the set $Q$ defined by:

$$h_1 + h_2 \geq 17x_1 + 17x_2 + 18x_3 + 2x_4$$
$$0 \leq h_1 \leq 24z_1$$
$$24z_2 \leq h_2 \leq 53z_2$$
$$z_1 + z_2 \leq 1$$
$$x_1, x_2, x_3, x_4, z_1, z_2 \in \{0, 1\}$$

Trivial inequalities:

$$h_1 \leq 24z_1 \tag{15}$$
$$h_2 \leq 53z_2 \tag{16}$$
$$0 \leq x_1 \tag{17}$$
$$0 \leq x_2 \tag{18}$$
$$0 \leq x_3 \tag{19}$$
$$0 \leq x_4 \tag{20}$$
$$0 \leq h_1 \tag{21}$$
$$24z_2 \leq h_2 \tag{22}$$
$$z_1 + z_2 \leq 1 \tag{83}$$

$$17x_1 + 17x_2 + 18x_3 + 2x_4 \leq h_1 + h_2 \qquad (58)$$

Upper bounds on $x_k$'s ($x_k \leq \sum_{j \in T: w_k \leq a_j} z_j$ dominates $x_k \leq 1$):

$$x_4 - z_1 - z_2 \leq 0 \qquad (64)$$
$$x_3 - z_1 - z_2 \leq 0 \qquad (65)$$
$$x_2 - z_1 - z_2 \leq 0 \qquad (66)$$
$$x_1 - z_1 - z_2 \leq 0 \qquad (67)$$

Inequalities on $z_1$ and $z_2$ (cover inequalities):

$$x_2 + x_3 - z_1 - 2z_2 \leq 0 \qquad (71)$$
$$x_1 + x_3 - z_1 - 2z_2 \leq 0 \qquad (72)$$
$$x_1 + x_2 - z_1 - 2z_2 \leq 0 \qquad (73)$$
$$x_1 + x_2 + x_3 - z_1 - 3z_2 \leq 0 \qquad (74)$$
$$x_1 + x_2 + x_3 + x_4 - 2z_1 - 3z_2 \leq 0 \qquad (77)$$

Inequalities on $h_1$ and $z_2$ (these are inequalities (3.4): from (23) to (69), $\lambda = 0$ and $C = \emptyset$, from (44) to (51), $S = \{1, 2, 3, 4\}$ and $\lambda = 2$.):

$$18x_3 - 18z_2 \leq h_1 \qquad S = \{3\} \qquad (23)$$
$$17x_2 - 17z_2 \leq h_1 \qquad S = \{2\} \qquad (24)$$
$$17x_2 + 18x_3 - 35z_2 \leq h_1 \qquad S = \{2, 3\} \qquad (25)$$
$$17x_1 - 17z_2 \leq h_1 \qquad S = \{1\} \qquad (30)$$
$$17x_1 + 18x_3 - 35z_2 \leq h_1 \qquad S = \{1, 3\} \qquad (31)$$
$$17x_1 + 17x_2 - 34z_2 \leq h_1 \qquad S = \{1, 2\} \qquad (32)$$
$$18x_3 + 2x_4 - 20z_2 \leq h_1 \qquad S = \{3, 4\} \qquad (36)$$
$$17x_2 + 2x_4 - 19z_2 \leq h_1 \qquad S = \{2, 4\} \qquad (37)$$
$$17x_1 + 2x_4 - 19z_2 \leq h_1 \qquad S = \{1, 4\} \qquad (42)$$
$$2x_4 - 2z_2 \leq h_1 \qquad S = \{4\} \qquad (69)$$

$$17x_1 + 17x_2 + 18x_3 + 2x_4 - 52z_2 \leq h_1 \qquad C = \emptyset \qquad (44)$$
$$2x_1 + 17x_2 + 18x_3 + 2x_4 - 37z_2 \leq h_1 \qquad C = \{1\} \qquad (45)$$
$$17x_1 + 2x_2 + 18x_3 + 2x_4 - 37z_2 \leq h_1 \qquad C = \{2\} \qquad (46)$$
$$17x_1 + 17x_2 + 2x_3 + 2x_4 - 36z_2 \leq h_1 \qquad C = \{3\} \qquad (47)$$
$$2x_1 + 2x_2 + 18x_3 + 2x_4 - 22z_2 \leq h_1 \qquad C = \{1,2\} \qquad (48)$$
$$2x_1 + 17x_2 + 2x_3 + 2x_4 - 21z_2 \leq h_1 \qquad C = \{1,3\} \qquad (49)$$
$$17x_1 + 2x_2 + 2x_3 + 2x_4 - 21z_2 \leq h_1 \qquad C = \{2,3\} \qquad (50)$$
$$2x_1 + 2x_2 + 2x_3 + 2x_4 - 6z_2 \leq h_1 \qquad C = \{1,2,3\} \qquad (51)$$

Inequalities on $h_2$ and $z_1$ (these are inequalities (3.5)):

$$17x_1 + 17x_2 + 18x_3 - 18z_1 \leq h_2 \qquad S = \{1,2,3\}, \lambda = 34, C = \emptyset \qquad (34)$$
$$17x_1 + 17x_2 + 18x_3 + 2x_4 - 20z_1 \leq h_2 \quad S = \{1,2,3,4\}, \lambda = 34, C = \emptyset \qquad (43)$$

Inequalities on $h_1$, $h_2$ and $z_2$:

Following three inequalities can be described with sequential lifting procedure.

$170x_1 + 170x_2 + 170x_3 + 238z_2 \leq 10h_1 + 17h_2$ (1)
$170x_1 + 170x_2 + 170x_3 + 238z_2 \leq 17h_2$ is a valid inequality for $Q^{conv}$ when $z_1 = 0$. Assume that $h_1$ is the variable to be lifted. Let $z_1 = 1$ and above inequality becomes $170x_1 + 170x_2 + 170x_3 \leq \gamma_1 h_1$. Then, minimum value for $\gamma_1$ is 10. Then, we get
$170x_1 + 170x_2 + 170x_3 + 238z_2 \leq 10h_1 + 17h_2$ (1)

$170x_1 + 187x_2 + 187x_3 + 221z_2 \leq 11h_1 + 17h_2$ (2)
$170x_1 + 187x_2 + 187x_3 + 221z_2 \leq 17h_2$ is a valid inequality for $Q^{conv}$ when $z_1 = 0$. Assume that $h_1$ is the variable to be lifted. Let $z_1 = 1$ and above inequality becomes $170x_1 + 187x_2 + 187x_3 \leq \gamma_1 h_1$. Then, minimum value for $\gamma_1$ is 11. Then, we get
$170x_1 + 187x_2 + 187x_3 + 221z_2 \leq 11h_1 + 17h_2$ (2)

$187x_1 + 170x_2 + 187x_3 + 221z_2 \leq 11h_1 + 17h_2$ (6)

$187x_1 + 170x_2 + 187x_3 + 221z_2 \leq 17h_2$ is a valid inequality for $Q^{conv}$ when $z_1 = 0$. Assume that $h_1$ is the variable to be lifted. Let $z_1 = 1$ and above inequality becomes $187x_1 + 170x_2 + 187x_3 \leq \gamma_1 h_1$. Then, minimum value for $\gamma_1$ is 11. Then, we get

$187x_1 + 170x_2 + 187x_3 + 221z_2 \leq 11h_1 + 17h_2$ (6)

Following three inequalities can be described with the valid inequality (3.6).

$12x_1 + 12x_2 + 12x_3 + 2x_4 + 10z_2 \leq h_1 + h_2$ (54)

$S = \{1, 2, 3, 4\}, C = \{1, 2, 4\}, w'(S) = 12, \beta_2 = 10$

$12x_1 + 13x_2 + 13x_3 + 2x_4 + 9z_2 \leq h_1 + h_2$ (55)

$S = \{1, 2, 3, 4\}, C = \{2, 3, 4\}, w'(S) = 12, \beta_2 = 9$

$13x_1 + 12x_2 + 13x_3 + 2x_4 + 9z_2 \leq h_1 + h_2$ (56)

$S = \{1, 2, 3, 4\}, C = \{1, 3, 4\}, w'(S) = 12, \beta_2 = 9$

Following inequality can be described with the valid inequality (3.7).

$13x_1 + 13x_2 + 14x_3 + 2x_4 + 8z_2 \leq h_1 + h_2$ (57)

$S = \{1, 2, 3, 4\}, \bar{w}_1(S) = 4, \alpha' = 20, \beta_2 = 8$

Following three inequalities can be described with sequential lifting procedure.

$12x_1 + 17x_2 + 13x_3 + 2x_4 + 5z_2 \leq h_1 + h_2$ (61)

$12x_1 + 13x_3 + 2x_4 \leq h_1 + h_2$ is a valid inequality when $x_2 = 0$ and $z_2 = 0$, since $17x_1 + 17x_2 + 18x_3 + 2x_4 \leq h_1 + h_2$. Assume that $x_2$ is the variable to be uplifted. Let $x_2 = 1$, then $z_1 = 1$. Thus, we get the following inequality:

$12x_1 + \alpha_2 x_2 + 13x_3 + 2x_4 \leq h_1$. Maximum value that $\alpha_2$ can take is 17, since

otherwise the inequality $17x_1 + 17x_2 + 18x_3 + 2x_4 \leq h_1$ would not be valid. Now, we lift the inequality with $z_2$. We set $z_2$ to 1, and get the following inequality: $12x_1 + 17x_2 + 13x_3 + 2x_4 + \beta_2 \leq h_2$. Maximum value that $\beta_2$ can take is 5, as the methodology for sequential lifting of a $z$ variable is applied. Then, we get
$12x_1 + 17x_2 + 13x_3 + 2x_4 + 5z_2 \leq h_1 + h_2$ (61)

$17x_1 + 12x_2 + 13x_3 + 2x_4 + 5z_2 \leq h_1 + h_2$ (62)
$12x_2 + 13x_3 + 2x_4 \leq h_1 + h_2$ is a valid inequality when $x_1 = 0$ and $z_2 = 0$, since $17x_1 + 17x_2 + 18x_3 + 2x_4 \leq h_1 + h_2$ Assume that $x_1$ is the variable to be uplifted. Let $x_1 = 1$, then $z_1 = 1$. Thus, we get the following inequality:
$\alpha_1 x_1 + 12x_2 + 13x_3 + 2x_4 \leq h_1$. Maximum value that $\alpha_1$ can take is 17, since otherwise the inequality $17x_1 + 17x_2 + 18x_3 + 2x_4 \leq h_1$ would not be valid. Now, we lift the inequality with $z_2$. We set $z_2$ to 1, and get the following inequality: $17x_1 + 12x_2 + 13x_3 + 2x_4 + \beta_2 \leq h_2$. Maximum value that $\beta_2$ can take is 5, as the methodology for sequential lifting of a $z$ variable is applied. Then, we get
$17x_1 + 12x_2 + 13x_3 + 2x_4 + 5z_2 \leq h_1 + h_2$ (62)

$13x_1 + 13x_2 + 18x_3 + 2x_4 + 4z_2 \leq h_1 + h_2$ (63)
$13x_1 + 13x_2 + 2x_4 \leq h_1 + h_2$ is a valid inequality when $x_3 = 0$ and $z_2 = 0$, since $17x_1 + 17x_2 + 18x_3 + 2x_4 \leq h_1 + h_2$ Assume that $x_3$ is the variable to be uplifted. Let $x_3 = 1$, then $z_1 = 1$. Thus, we get the following inequality:
$13x_1 + 13x_2 + \alpha_3 x_3 + 2x_4 \leq h_1$. Maximum value that $\alpha_3$ can take is 18, since otherwise the inequality $17x_1 + 17x_2 + 18x_3 + 2x_4 \leq h_1$ would not be valid. Now, we lift the inequality with $z_2$. We set $z_2$ to 1, and get the following inequality: $13x_1 + 13x_2 + 18x_3 + 2x_4 + \beta_2 \leq h_2$. Maximum value that $\beta_2$ can take is 4, as the methodology for sequential lifting of a $z$ variable is applied. Then, we get
$13x_1 + 13x_2 + 18x_3 + 2x_4 + 4z_2 \leq h_1 + h_2$ (63)

Inequalities on $h_2$, $z_1$ and $z_2$:

Following seven inequalities can be described with the valid inequality (3.8).

$10x_1 + 10x_2 + 10x_3 - 10z_1 + 14z_2 \leq h_2$ (26)
$S = \{1, 2, 3\}, C = \{1, 2\}, w(C) = 34, w'(S) = 10, \beta_1 = -10, \beta_2 = 14$

$10x_1 + 11x_2 + 11x_3 - 11z_1 + 13z_2 \leq h_2$ (27)
$S = \{1, 2, 3\}, C = \{2, 3\}, w(C) = 35, w'(S) = 10, \beta_1 = -11, \beta_2 = 13$

$11x_1 + 10x_2 + 11x_3 - 11z_1 + 13z_2 \leq h_2$ (28)
$S = \{1, 2, 3\}, C = \{1, 3\}, w(C) = 35, w'(S) = 10, \beta_1 = -11, \beta_2 = 13$

$12x_1 + 12x_2 + 12x_3 + 2x_4 - 14z_1 + 10z_2 \leq h_2$ (38)
$S = \{1, 2, 3, 4\}, C = \{1, 2, 4\}, w(C) = 36, w'(S) = 12, \beta_1 = -14, \beta_2 = 10$

$12x_1 + 13x_2 + 13x_3 + 2x_4 - 15z_1 + 9z_2 \leq h_2$ (39)
$S = \{1, 2, 3, 4\}, C = \{2, 3, 4\}, w(C) = 37, w'(S) = 12, \beta_1 = -15, \beta_2 = 9$

$13x_1 + 12x_2 + 13x_3 + 2x_4 - 15z_1 + 9z_2 \leq h_2$ (40)
$S = \{1, 2, 3, 4\}, C = \{1, 3, 4\}, w(C) = 37, w'(S) = 12, \beta_1 = -15, \beta_2 = 9$

Following seven inequalities can be described with sequential lifting procedure.

$13x_1 + 13x_2 + 14x_3 + 2x_4 - 16z_1 + 8z_2 \leq h_2$ (41)
$13x_1 + 13x_2 + 14x_3 + 2x_4 + 8z_2 \leq h_2$ is a valid inequality when $z_1 = 0$. Assume that $z_1$ is the variable to be uplifted. Let $z_1 = 1$, then $z_2 = 0$. Thus, we get the following inequality:
$13x_1 + 13x_2 + 14x_3 + 2x_4 + \beta_1 \leq 0$. Maximum value that $\beta_1$ can take is -16, as the methodology for sequential lifting of a $z$ variable is applied. Then, we get
$13x_1 + 13x_2 + 14x_3 + 2x_4 - 16z_1 + 8z_2 \leq h_2$ (41)

$17x_1 + 18x_2 + 18x_3 - 18z_1 - z_2 \leq h_2$ (52)
$17x_1 + 18x_3 - 18z_1 \leq h_2$ is a valid inequality when $x_2 = 0$ and $z_2 = 0$. Assume

that $x_2$ is the variable to be uplifted. Let $x_2 = 1$, then $z_1 = 1$. Thus, we get the following inequality:
$17x_1 + \alpha_2 x_2 + 18x_3 \leq 18$. Maximum value that $\alpha_2$ can take is 18, trivially. Now, we lift the inequality with $z_2$. We set $z_2$ to 1, and get the following inequality:
$17x_1 + 18x_2 + 18x_3 + \beta_2 z_2 \leq h_2$. Maximum value that $\beta_2$ can take is 1, as the methodology for sequential lifting of a $z$ variable is applied. Then, we get
$17x_1 + 18x_2 + 18x_3 - 18z_1 - z_2 \leq h_2$ (52)

$18x_1 + 17x_2 + 18x_3 - 18z_1 - z_2 \leq h_2$ (53)
$18x_1 + 17x_2 + 18x_3 - z_2 \leq h_2$ is a valid inequality when $z_1 = 0$. Assume that $z_1$ is the variable to be uplifted. Let $z_1 = 1$, then $z_2 = 0$. Thus, we get the following inequality:
$18x_1 + 17x_2 + 18x_3 + \beta_1 \leq 0$. Maximum value that $\beta_1$ can take is -18, trivially. Then, we get
$18x_1 + 17x_2 + 18x_3 - 18z_1 - z_2 \leq h_2$ (53)

$17x_1 + 18x_2 + 18x_3 + 2x_4 - 20z_1 - z_2 \leq h_2$ (59)
$17x_1 + 18x_2 + 18x_3 - z_2 \leq h_2$ is a valid inequality when $x_4 = 0$ and $z_1 = 0$. Assume that $x_4$ is the variable to be uplifted. Let $x_4 = 1$. Then, we get the following inequality:
$17x_1 + 18x_2 + 18x_3 + \alpha_4 x_4 - z_2 \leq h_2$. Maximum value that $\alpha_4$ can take is 2. Now, we lift the inequality with $z_1$. We set $z_1$ to 1, and get the following inequality:
$17x_1 + 18x_2 + 18x_3 + 2x_4 + \beta_1 \leq 0$. Maximum value that $\beta_1$ can take is -20, as the methodology for sequential lifting of a $z$ variable is applied. Then, we get
$17x_1 + 18x_2 + 18x_3 + 2x_4 - 20z_1 - z_2 \leq h_2$ (59)

$18x_1 + 17x_2 + 18x_3 + 2x_4 - 20z_1 - z_2 \leq h_2$ (60)
$18x_1 + 17x_2 + 18x_3 - z_2 \leq h_2$ is a valid inequality when $x_4 = 0$ and $z_1 = 0$. Assume that $x_4$ is the variable to be uplifted. Let $x_4 = 1$. Then, we get the following inequality:
$18x_1 + 17x_2 + 18x_3 + \alpha_4 x_4 - z_2 \leq h_2$. Maximum value that $\alpha_4$ can take is 2. Now,

we lift the inequality with $z_1$. We set $z_1$ to 1, and get the following inequality: $18x_1 + 17x_2 + 18x_3 + 2x_4 + \beta_1 \leq 0$. Maximum value that $\beta_1$ can take is -20, as the methodology for sequential lifting of a $z$ variable is applied. Then, we get
$18x_1 + 17x_2 + 18x_3 + 2x_4 - 20z_1 - z_2 \leq h_2$ (60)

$18x_1 + 18x_2 + 18x_3 - 18z_1 - 2z_2 \leq h_2$ (68)
$18x_1 + 18x_3 - 2z_2 \leq h_2$ is a valid inequality when $x_2 = 0$ and $z_1 = 0$. Assume that $x_2$ is the variable to be uplifted. Let $x_2 = 1$. Then, we get the following inequality:
$18x_1 + \alpha_2 x_2 + 18x_3 - 2z_2 \leq h_2$. Maximum value that $\alpha_2$ can take is 18. Now, we lift the inequality with $z_1$. We set $z_1$ to 1, and get the following inequality:
$18x_1 + 18x_2 + 18x_3 + \beta_1 \leq 0$. Maximum value that $\beta_1$ can take is -18, as the methodology for sequential lifting of a $z$ variable is applied. Then, we get
$18x_1 + 18x_2 + 18x_3 - 18z_1 - 2z_2 \leq h_2$ (68)

$18x_1 + 18x_2 + 18x_3 + 2x_4 - 20z_1 - 2z_2 \leq h_2$ (70)
$18x_1 + 18x_3 + 2x_4 - 2z_2 \leq h_2$ is a valid inequality when $x_2 = 0$ and $z_1 = 0$. Assume that $x_2$ is the variable to be uplifted. Let $x_2 = 1$. Then, we get the following inequality:
$18x_1 + \alpha_2 x_2 + 18x_3 + 2x_4 - 2z_2 \leq h_2$. Maximum value that $\alpha_2$ can take is 18. Now, we lift the inequality with $z_1$. We set $z_1$ to 1, and get the following inequality:
$18x_1 + 18x_2 + 18x_3 + 2x_4 + \beta_1 \leq 0$. Maximum value that $\beta_1$ can take is -20, as the methodology for sequential lifting of a $z$ variable is applied. Then, we get
$18x_1 + 18x_2 + 18x_3 + 2x_4 - 20z_1 - 2z_2 \leq h_2$ (70)

Inequalities on $h_1$, $z_1$ and $z_2$:

$$17x_1 + 17x_2 + 17x_3 + 17x_4 - 15z_1 - 51z_2 - h_1 \leq 0 \quad (33)$$

$$17x_1 + 17x_2 + 18x_3 + 17x_4 - 15z_1 - 52z_2 - h_1 \leq 0 \quad (35)$$

Above two inequalities can be described with sequential lifting procedure.

$17x_1 + 17x_2 + 17x_3 + 17x_4 - 15z_1 - 51z_2 \leq h_1$ (33)
$17x_1 + 17x_2 + 17x_3 + 17x_4 - 15z_1 \leq h_1$ is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable to be uplifted. Let $z_2 = 1$, then $z_1 = 0$. Thus, we get the following inequality:
$17x_1 + 17x_2 + 17x_3 + 17x_4 + \beta_2 \leq 0$. Maximum value that $\beta_2$ can take is -51, trivially. Then, we get
$17x_1 + 17x_2 + 17x_3 + 17x_4 - 15z_1 - 51z_2 \leq h_1$ (33)

$17x_1 + 17x_2 + 18x_3 + 17x_4 - 15z_1 - 52z_2 \leq h_1$ (35)
$17x_1 + 17x_2 + 18x_3 + 17x_4 - 15z_1 \leq h_1$ is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable to be uplifted. Let $z_2 = 1$, then $z_1 = 0$. Thus, we get the following inequality:
$17x_1 + 17x_2 + 18x_3 + 17x_4 + \beta_2 \leq 0$. Maximum value that $\beta_2$ can take is -52, as the methodology for sequential lifting of a $z$ variable is applied. Then, we get
$17x_1 + 17x_2 + 18x_3 + 17x_4 - 15z_1 - 52z_2 \leq h_1$ (35)

Inequalities on $h_1$, $h_2$, $z_1$ and $z_2$:

Following four inequalities can be described with simultaneous lifting procedure.

$204x_1 + 204x_2 + 204x_3 + 34x_4 - 10z_1 + 170z_2 \leq 12h_1 + 17h_2$ (10)
$204x_1 + 204x_2 + 204x_3 + 34x_4 + 170z_2 \leq 17h_2$ is valid, since $170 \leq 17h_2 - 204x_1 - 204x_2 - 204x_3 - 34x_4$ is satisfied by all feasible points $(x, z, h)$ when $z_1 = 0$. Also, 170 is the maximum value for $\beta_2$ for these $\alpha_k$ and $\gamma_j$ values as *Proposition 5* proposes. Assume that $z_1$ and $h_1$ are the variables to be lifted simultaneously. Let $z_1 = 1$, then $z_2 = 0$ and $h_1 \leq 24$. Consequently, we get the following inequality:
$204x_1 + 204x_2 + 204x_3 + 34x_4 + \beta_1 \leq \gamma_1 h_1$.

Find the set of feasible solutions that satisfy following constraints:
$17x_1 + 17x_2 + 18x_3 + 2x_4 \leq 24$
$204x_1 + 204x_2 + 204x_3 + 34x_4 + \beta_1 \leq \gamma_1 \sum_{k \in N} w_k x_k \leq \gamma_1 h_1$

Feasible solutions and corresponding inequalities can be listed as follows:
If $x_1 = 1$ and $x_4 = 0$, then: $204 + \beta_1 \leq 17\gamma_1 \leq \gamma_1 h_1$ (i)
If $x_1 = 1$ and $x_4 = 1$, then: $238 + \beta_1 \leq 19\gamma_1 \leq \gamma_1 h_1$ (ii)
If $x_1 = 0$, $x_3 = 0$, and $x_4 = 1$, then: $34 + \beta_1 \leq 2\gamma_1 \leq \gamma_1 h_1$ (iii)
If $x_3 = 1$ and $x_4 = 0$, then: $204 + \beta_1 \leq 18\gamma_1 \leq \gamma_1 h_1$ (iv)
If $x_3 = 1$ and $x_4 = 1$, then: $238 + \beta_1 \leq 20\gamma_1 \leq \gamma_1 h_1$ (v)

When the above five cases are analyzed, it can be seen that (iv) is dominated by (i), and (v) is dominated by (ii). Besides, at most two of (i), (ii), and (iii) can be binding. Either if (i) and (ii) or (i) and (iii) are binding, the remaining one would not be valid. If (ii) and (iii) are binding, then $\beta_1 = -10$ and $\gamma_1 = 12$. Then, we get
$204x_1 + 204x_2 + 204x_3 + 34x_4 - 10z_1 + 170z_2 \leq 12h_1 + 17h_2$

$204x_1 + 221x_2 + 221x_3 + 34x_4 - 8z_1 + 153z_2 \leq 13h_1 + 17h_2$ (11)
$204x_1 + 221x_2 + 221x_3 + 34x_4 + 153z_2 \leq 17h_2$ is valid, since $153 \leq 17h_2 - 204x_1 - 221x_2 - 221x_3 - 34x_4$ is satisfied by all feasible points $(x, z, h)$ when $z_1 = 0$. Also, 153 is the maximum value for $\beta_2$ for these $\alpha_k$ and $\gamma_j$ values as *Proposition 5* proposes. Assume that $z_1$ and $h_1$ are the variables to be lifted simultaneously. Let $z_1 = 1$, then $z_2 = 0$ and $h_1 \leq 24$. Consequently, we get the following inequality:
$204x_1 + 221x_2 + 221x_3 + 34x_4 + \beta_1 \leq \gamma_1 h_1$.

Find the set of feasible solutions that satisfy following constraints:
$17x_1 + 17x_2 + 18x_3 + 2x_4 \leq 24$
$204x_1 + 221x_2 + 221x_3 + 34x_4 + \beta_1 \leq \gamma_1 \sum_{k \in N} w_k x_k \leq \gamma_1 h_1$

Feasible solutions and corresponding inequalities can be listed as follows:
If $x_1 = 1$ and $x_4 = 0$, then: $204 + \beta_1 \leq 17\gamma_1 \leq \gamma_1 h_1$ (i)
If $x_1 = 1$ and $x_4 = 1$, then: $238 + \beta_1 \leq 19\gamma_1 \leq \gamma_1 h_1$ (ii)
If $x_1 = 0$, $x_3 = 0$, and $x_4 = 1$, then: $34 + \beta_1 \leq 2\gamma_1 \leq \gamma_1 h_1$ (iii)
If $x_2 = 1$ and $x_4 = 0$, then: $221 + \beta_1 \leq 17\gamma_1 \leq \gamma_1 h_1$ (iv)

If $x_2 = 1$ and $x_4 = 1$, then: $255 + \beta_1 \leq 19\gamma_1 \leq \gamma_1 h_1$ (v)

When the above five cases are analyzed, it can be seen that (i) is dominated by (iv), and (ii) is dominated by (iv). Besides, at most two of (iii), (iv), and (v) can be binding. Either if (iii) and (iv) or (iv) and (v) are binding, the remaining one would not be valid. If (iii) and (v) are binding, then $\beta_1 = -8$ and $\gamma_1 = 13$. Then, we get
$204x_1 + 221x_2 + 221x_3 + 34x_4 - 8z_1 + 153z_2 \leq 13h_1 + 17h_2$ (11)

$221x_1 + 204x_2 + 221x_3 + 34x_4 - 8z_1 + 153z_2 \leq 13h_1 + 17h_2$ (13)
$221x_1 + 204x_2 + 221x_3 + 34x_4 + 153z_2 \leq 17h_2$ is valid, since $153 \leq 17h_2 - 221x_1 - 204x_2 - 221x_3 - 34x_4$ is satisfied by all feasible points $(x, z, h)$ when $z_1 = 0$. Also, 153 is the maximum value for $\beta_2$ for these $\alpha_k$ and $\gamma_j$ values as *Proposition 5* proposes. Assume that $z_1$ and $h_1$ are the variables to be lifted simultaneously. Let $z_1 = 1$, then $z_2 = 0$ and $h_1 \leq 24$. Consequently, we get the following inequality:
$221x_1 + 204x_2 + 221x_3 + 34x_4 + \beta_1 \leq \gamma_1 h_1$.

Find the set of feasible solutions that satisfy following constraints:
$17x_1 + 17x_2 + 18x_3 + 2x_4 \leq 24$
$221x_1 + 204x_2 + 221x_3 + 34x_4 + \beta_1 \leq \gamma_1 \sum_{k \in N} w_k x_k \leq \gamma_1 h_1$

Feasible solutions and corresponding inequalities can be listed as follows:
If $x_1 = 1$ and $x_4 = 0$, then: $221 + \beta_1 \leq 17\gamma_1 \leq \gamma_1 h_1$ (i)
If $x_1 = 1$ and $x_4 = 1$, then: $255 + \beta_1 \leq 19\gamma_1 \leq \gamma_1 h_1$ (ii)
If $x_1 = 0$, $x_3 = 0$, and $x_4 = 1$, then: $34 + \beta_1 \leq 2\gamma_1 \leq \gamma_1 h_1$ (iii)
If $x_3 = 1$ and $x_4 = 0$, then: $221 + \beta_1 \leq 18\gamma_1 \leq \gamma_1 h_1$ (iv)
If $x_3 = 1$ and $x_4 = 1$, then: $255 + \beta_1 \leq 20\gamma_1 \leq \gamma_1 h_1$ (v)

When the above five cases are analyzed, it can be seen that (iv) is dominated by (i), and (v) is dominated by (ii). Besides, at most two of (i), (ii), and (iii) can be binding. Either if (i) and (ii) or (i) and (iii) are binding, the remaining one would not be valid. If (ii) and (iii) are binding, then $\beta_1 = -8$ and $\gamma_1 = 13$. Then, we get

$221x_1 + 204x_2 + 221x_3 + 34x_4 + 153z_2 \leq 8z_1 + 13h_1 + 17h_2$ (13)

$117x_1 + 117x_2 + 126x_3 + 18x_4 - 4z_1 + 72z_2 \leq 7h_1 + 9h_2$ (78)

$117x_1 + 117x_2 + 126x_3 + 18x_4 + 72z_2 \leq 9h_2$ is valid, since $72 \leq 9h_2 - 117x_1 - 117x_2 - 126x_3 - 18x_4$ is satisfied by all feasible points $(x, z, h)$ when $z_1 = 0$. Also, 72 is the maximum value for $\beta_2$ for these $\alpha_k$ and $\gamma_j$ values as *Proposition 5* proposes. Assume that $z_1$ and $h_1$ are the variables to be lifted simultaneously. Let $z_1 = 1$, then $z_2 = 0$ and $h_1 \leq 24$. Consequently, we get the following inequality:

$117x_1 + 117x_2 + 126x_3 + 18x_4 + \beta_1 \leq \gamma_1 h_1$.

Find the set of feasible solutions that satisfy following constraints:

$17x_1 + 17x_2 + 18x_3 + 2x_4 \leq 24$

$117x_1 + 117x_2 + 126x_3 + 18x_4 + \beta_1 \leq \gamma_1 \sum_{k \in N} w_k x_k \leq \gamma_1 h_1$

Feasible solutions and corresponding inequalities can be listed as follows:

If $x_1 = 1$ and $x_4 = 0$, then: $117 + \beta_1 \leq 17\gamma_1 \leq \gamma_1 h_1$ (i)

If $x_1 = 1$ and $x_4 = 1$, then: $135 + \beta_1 \leq 19\gamma_1 \leq \gamma_1 h_1$ (ii)

If $x_1 = 0$, $x_3 = 0$, and $x_4 = 1$, then: $18 + \beta_1 \leq 2\gamma_1 \leq \gamma_1 h_1$ (iii)

If $x_3 = 1$ and $x_4 = 0$, then: $126 + \beta_1 \leq 18\gamma_1 \leq \gamma_1 h_1$ (iv)

If $x_3 = 1$ and $x_4 = 1$, then: $144 + \beta_1 \leq 20\gamma_1 \leq \gamma_1 h_1$ (v)

When the above five cases are analyzed, it can be seen that (i) is dominated by (iv), (iv) is dominated by (ii), and (ii) is dominated by (v). If (iii) and (v) are binding, then $\beta_1 = -4$ and $\gamma_1 = 7$. Then, we get

$117x_1 + 117x_2 + 126x_3 + 18x_4 - 4z_1 + 72z_2 \leq 7h_1 + 9h_2$ (78)

Finally, following is the set of facets that we could not explain with our proposed valid inequalities or lifting techniques.

$51x_1 + 51x_2 + 54x_3 + 8x_4 - 2z_1 + 34z_2 \leq 3h_1 + 4h_2$ (80)
$172x_1 + 187x_2 + 187x_3 + 22x_4 + 151z_2 \leq 11h_1 + 15h_2$ (3)
$181x_1 + 187x_2 + 198x_3 + 210z_2 \leq 11h_1 + 17h_2$ (4)
$183x_1 + 187x_2 + 198x_3 + 22x_4 + 140z_2 \leq 11h_1 + 15h_2$ (5)
$187x_1 + 172x_2 + 187x_3 + 22x_4 + 151z_2 \leq 11h_1 + 15h_2$ (7)
$187x_1 + 181x_2 + 198x_3 + 210z_2 \leq 11h_1 + 17h_2$ (8)
$187x_1 + 183x_2 + 198x_3 + 22x_4 + 140z_2 \leq 11h_1 + 15h_2$ (9)
$33x_1 + 33x_2 + 36x_3 + 36z_2 \leq 2h_1 + 3h_2$ (75)
$34x_1 + 34x_2 + 34x_3 + 4x_4 + 34z_2 \leq 2h_1 + 3h_2$ (76)
$50x_1 + 50x_2 + 54x_3 + 6x_4 + 36z_2 \leq 3h_1 + 4h_2$ (79)
$85x_1 + 85x_2 + 90x_3 + 102z_2 \leq 5h_1 + 8h_2$ (81)
$85x_1 + 85x_2 + 90x_3 + 10x_4 + 68z_2 \leq 5h_1 + 7h_2$ (82)
$11x_1 + 11x_2 + 12x_3 - 12z_1 + 12z_2 \leq h_2$ (29)
$217x_1 + 221x_2 + 234x_3 + 34x_4 - 8z_1 + 140z_2 \leq 13h_1 + 17h_2$ (12)
$221x_1 + 217x_2 + 234x_3 + 34x_4 - 8z_1 + 140z_2 \leq 13h_1 + 17h_2$ (14)

# Appendix B

# An example for $MCM_=$

Using PORTA, we generated all facet defining inequalities for the convex hull of the set $Q'$ defined by:

$$
\begin{aligned}
&h_1 + h_2 = 17x_1 + 17x_2 + 18x_3 + 2x_4 \\
&0 \leq h_1 \leq 24z_1 \\
&24z_2 \leq h_2 \leq 53z_2 \\
&z_1 + z_2 \leq 1 \\
&x_1, x_2, x_3, x_4, z_1, z_2 \in \{0, 1\}
\end{aligned}
$$

Following inequalities are trivial inequalities.

$z_1 + z_2 \leq 1$ (50)
$x_2 \geq 0$ (11)
$x_3 \geq 0$ (12)
$x_4 \geq 0$ (13)
$h_1 \geq 0$ (14)
$h_2 \geq 0$ (15)
$x_1 \geq 0$ (28)

Upper bounds on $x_k$'s:

$x_4 \leq z_1 + z_2$ (30)
$x_3 \leq z_1 + z_2$ (31)
$x_2 \leq z_1 + z_2$ (32)
$x_1 \leq z_1 + z_2$ (43)

Following inequalities are cover inequalities.

$x_1 + x_2 + x_3 + x_4 \leq 2z_1 + 3z_2$ (24)
$z_2 \leq x_1 + x_2$ (27)
$z_2 \leq x_2 + x_3$ (29)
$2z_2 \leq x_1 + x_2 + x_3$ (33)
$z_2 \leq x_1 + x_3$ (34)

Following inequalities can be described with the valid inequality (3.2).

$2x_1 + 2x_2 + 18x_3 + 2x_4 - 22z_2 \leq h_1$ (1)
$S = N$, $C = \{1, 2\}$, $\lambda = 2$, $\beta_2 = -22$

$2x_1 + 17x_2 + 18x_3 + 2x_4 - 37z_2 \leq h_1$ (2)
$S = N$, $C = \{1\}$, $\lambda = 2$, $\beta_2 = -37$

$17x_1 + 2x_2 + 2x_3 + 2x_4 - 21z_2 \leq h_1$ (4)
$S = N$, $C = \{2, 3\}$, $\lambda = 2$, $\beta_2 = -21$

$17x_1 + 2x_2 + 18x_3 + 2x_4 - 37z_2 \leq h_1$ (5)
$S = N$, $C = \{2\}$, $\lambda = 2$, $\beta_2 = -37$

$17x_1 + 17x_2 + 2x_3 + 2x_4 - 36z_2 \leq h_1$ (7)
$S = N$, $C = \{3\}$, $\lambda = 2$, $\beta_2 = -36$

$17x_1 + 17x_2 + 18x_3 + 2x_4 - 20z_1 \leq h_2$ (8)
$S = N$, $C = \{\}$, $\lambda = -34$, $\beta_1 = -20$

$17x_1 + 17x_2 + 18x_3 + 2x_4 - 52z_2 \leq h_1$ (9)
$S = N$, $C = \{\}$, $\lambda = 2$, $\beta_2 = -52$

$17x_2 - 17z_2 \leq h_1$ (18)
$S = \{2, 3, 4\}$, $C = \{3, 4\}$, $\lambda = 0$, $\beta_2 = -17$

$17x_2 + 18x_3 - 35z_2 \leq h_1$ (19)
$S = \{2, 3, 4\}$, $C = \{4\}$, $\lambda = 0$, $\beta_2 = -35$

$2x_1 + 2x_2 + 2x_3 + 2x_4 - 6z_2 \leq h_1$ (35)
$S = N$, $C = \{1, 2, 3\}$, $\lambda = 2$, $\beta_2 = -6$

$2x_1 + 2x_3 + 2x_4 - 21z_2 \leq h_1$ (36)
$S = N$, $C = \{1, 3\}$, $\lambda = 2$, $\beta_2 = -21$

$17x_1 - 17z_2 \leq h_1$ (38)
$S = \{1, 3, 4\}$, $C = \{3, 4\}$, $\lambda = 0$, $\beta_2 = -17$

$17x_1 + 18x_3 - 35z_2 \leq h_1$ (39)
$S = \{1, 3, 4\}$, $C = \{4\}$, $\lambda = 0$, $\beta_2 = -35$

$17x_1 + 17x_2 - 34z_2 \leq h_1$ (41)
$S = \{1, 2, 4\}$, $C = \{4\}$, $\lambda = 0$, $\beta_2 = -34$

$17x_1 + 17x_2 + 18x_3 - 18z_1 \leq h_2$ (42)

$S = \{1, 2, 3\}$, $C = \{\}$, $\lambda = 34$, $\beta_1 = -18$

Following inequalities can be described with sequential lifting.

$h_1 + x_1 \leq 18z_1 + z_2 + 2x_4$ (3)
$h_1 \leq 2x_4 + 18z_1$ is a valid inequality when $x_1 = 0$ and $z_2 = 0$. Assume that $x_1$ is the variable to be uplifted. Let $x_1 = 1$. Then, we get the following inequality:
$h_1 + \alpha_1 x_1 \leq 2x_4 + 18z_1$. Maximum value that $\alpha_1$ can take is 1. Now, we lift the inequality with $z_2$. We set $z_2$ to 1, and get the following inequality:
$x_1 \leq 2x_4 + \beta_2 z_2$. Minimum value that $\beta_2$ can take is 1. Then, we get
$h_1 + x_1 \leq 2x_4 + 18z_1 + z_2$ (3)

$h_1 + x_1 + x_2 \leq 18z_1 + 2z_2 + 2x_4$ (6)
$h_1 + x_1 \leq 18z_1 + 2x_4$ is a valid inequality when $x_2 = 0$ and $z_2 = 0$. Assume that $x_2$ is the variable to be uplifted. Let $x_2 = 1$. Then, we get the following inequality:
$h_1 + x_1 + \alpha_2 x_2 \leq 2x_4 + 18z_1$. Maximum value that $\alpha_2$ can take is 1. Now, we lift the inequality with $z_2$. We set $z_2$ to 1, and get the following inequality:
$h_1 + x_1 + x_2 \leq 2x_4 + 18z_1 + \beta_2 z_2$. Minimum value that $\beta_2$ can take is 1. Then, we get
$h_1 + x_1 + x_2 \leq 18z_1 + 2z_2 + 2x_4$ (6)

$h_2 + 15x_4 \leq 15z_1 + 52z_2$ (10)
$h_2 \leq 52z_2$ is a valid inequality when $x_4 = 0$ and $z_1 = 0$. Assume that $x_4$ is the variable to be uplifted. Let $x_4 = 1$. Then, we get the following inequality:
$h_2 + \alpha_4 x_4 \leq 52z_2$. Maximum value that $\alpha_4$ can take is 15. Now, we lift the inequality with $z_1$. We set $z_1$ to 1, and get the following inequality:
$15x_4 \leq \beta_1 z_1$. Minimum value that $\beta_1$ can take is 15. Then, we get
$h_2 + 15x_4 \leq 15z_1 + 52z_2$ (10)

$18x_3 \leq 18z_2 + h_1$ (16)
$18x_3 \leq h_1$ (14) is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable

to be uplifted. Let $z_2 = 1$. Then, we get the following inequality:
$18x_3 \leq \beta_2 z_2$. Minimum value that $\beta_2$ can take is 18. Then, we get
$18x_3 \leq 18z_2 + h_1$ (16)

$2x_4 + 36z_2 \leq 2z_1 + h_2 + x_1 + x_2$ (17)
$2x_4 + 36z_2 \leq h_2 + x_1 + x_2$ is a valid inequality when $z_1 = 0$. Assume that $z_1$ is the variable to be uplifted. Let $z_1 = 1$. Then, we get the following inequality:
$2x_4 \leq x_1 + x_2 + \beta_1 z_1$. Minimum value that $\beta_1$ can take is 2. Then, we get
$2x_4 + 36z_2 \leq 2z_1 + h_2 + x_1 + x_2$ (17)

$2x_4 + 35z_2 \leq 2z_1 + h_2 + x_1$ (20)
$2x_4 + 35z_2 \leq x_1 + h_2$ is a valid inequality when $z_1 = 0$. Assume that $z_1$ is the variable to be uplifted. Let $z_1 = 1$. Then, we get the following inequality:
$2x_4 \leq x_1 + \beta_1 z_1$. Minimum value that $\beta_1$ can take is 2. Then, we get
$2x_4 + 35z_2 \leq 2z_1 + h_2 + x_1$ (20)

$h_1 + 36z_2 \leq 18x_1 + 18x_2 + 18x_3 + 2x_4$ (21)
$h_1 \leq 18x_1 + 18x_2 + 18x_3 + 2x_4$ is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable to be uplifted. Let $z_2 = 1$. Then, we get the following inequality:
$\beta_2 z_2 \leq 18x_1 + 18x_2 + 18x_3 + 2x_4$. Maximum value that $\beta_2$ can take is 36. Then, we get
$h_1 + 36z_2 \leq 18x_1 + 18x_2 + 18x_3 + 2x_4$ (21)

$h_1 + 35z_2 \leq 18x_1 + 17x_2 + 18x_3 + 2x_4$ (22)
$h_1 \leq 18x_1 + 17x_2 + 18x_3 + 2x_4$ is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable to be uplifted. Let $z_2 = 1$. Then, we get the following inequality:
$\beta_2 z_2 \leq 18x_1 + 17x_2 + 18x_3 + 2x_4$. Maximum value that $\beta_2$ can take is 35. Then, we get
$h_1 + 35z_2 \leq 18x_1 + 17x_2 + 18x_3 + 2x_4$ (22)

$h_2 + 15x_4 \leq 15z_1 + 51z_2 + x_3$ (23)
$h_2 + 15x_4 \leq x_3 + 52z_2$ is a valid inequality when $z_1 = 0$. Assume that $z_1$ is the variable to be uplifted. Let $z_1 = 1$. Then, we get the following inequality:
$15x_4 \leq x_3 + \beta_1 z_1$. Minimum value that $\beta_1$ can take is 15. Then, we get
$h_2 + 15x_4 \leq 15z_1 + 51z_2 + x_3$ (23)

$h_1 + x_2 \leq 20z_1 + z_2$ (25)
$h_1 \leq 20z_1$ is a valid inequality when $x_2 = 0$ and $z_2 = 0$. Assume that $x_2$ is the variable to be uplifted. Let $x_2 = 1$. Then, we get the following inequality:
$h_1 + \alpha_2 x_2 \leq 20z_2$. Maximum value that $\alpha_2$ can take is 1. Now, we lift the inequality with $z_2$. We set $z_2$ to 1, and get the following inequality:
$x_2 \leq \beta_2 z_2$. Minimum value that $\beta_2$ can take is 1. Then, we get
$h_1 + x_2 \leq 20z_1 + z_2$ (25)

$h_1 + 35z_2 \leq 17x_1 + 18x_2 + 18x_3 + 2x_4$ (26)
$h_1 \leq 17x_1 + 18x_2 + 18x_3 + 2x_4$ is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable to be uplifted. Let $z_2 = 1$. Then, we get the following inequality:
$\beta_2 z_2 \leq 17x_1 + 18x_2 + 18x_3 + 2x_4$. Maximum value that $\beta_2$ can take is 35. Then, we get
$h_1 + 35z_2 \leq 17x_1 + 18x_2 + 18x_3 + 2x_4$ (26)

$h_1 + x_1 \leq 20z_1 + z_2$ (37)
$h_1 + x_1 \leq 20z_1$ is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable to be uplifted. Let $z_2 = 1$. Then, we get the following inequality:
$x_1 \leq \beta_2 z_2$. Minimum value that $\beta_2$ can take is 1. Then, we get
$h_1 + x_1 \leq 20z_1 + z_2$ (37)

$h_1 + x_1 + x_2 \leq 20z_1 + 2z_2$ (40)
$h_1 + x_1 + x_2 \leq 20z_1$ is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable to be uplifted. Let $z_2 = 1$. Then, we get the following inequality:
$x_1 + x_2 \leq \beta_2 z_2$. Minimum value that $\beta_2$ can take is 2. Then, we get

$h_1 + x_1 + x_2 \leq 20z_1 + 2z_2$ (40)

$x_1 + x_2 \leq z_1 + 2z_2$ (44)
$x_1 + x_2 \leq z_1$ is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable to be uplifted. Let $z_2 = 1$. Then, we get the following inequality:
$x_1 + x_2 \leq \beta_2 z_2$. Minimum value that $\beta_2$ can take is 2. Then, we get
$x_1 + x_2 \leq z_1 + 2z_2$ (44)

$2x_4 + 34z_2 \leq h_2 + 2z_1$ (45)
$2x_4 + 34z_2 \leq h_2$ is a valid inequality when $z_1 = 0$. Assume that $z_1$ is the variable to be uplifted. Let $z_1 = 1$. Then, we get the following inequality:
$2x_4 \leq \beta_1 z_1$. Minimum value that $\beta_1$ can take is 2. Then, we get
$2x_4 + 34z_2 \leq h_2 + 2z_1$ (45)

$x_1 + x_3 \leq z_1 + 2z_2$ (46)
$x_1 + x_3 \leq z_1$ is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable to be uplifted. Let $z_2 = 1$. Then, we get the following inequality:
$x_1 + x_3 \leq \beta_2 z_2$. Minimum value that $\beta_2$ can take is 2. Then, we get
$x_1 + x_3 \leq z_1 + 2z_2$ (46)

$h_1 + x_2 \leq 18z_1 + z_2 + 2x_4$ (47)
$h_1 + x_2 \leq 18z_1 + 2x_4$ is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable to be uplifted. Let $z_2 = 1$. Then, we get the following inequality:
$x_2 \leq \beta_2 z_2 + 2x_4$. Minimum value that $\beta_2$ can take is 1. Then, we get
$h_1 + x_2 \leq 18z_1 + z_2 + 2x_4$ (47)

$x_2 + x_3 \leq z_1 + 2z_2$ (48)
$x_2 + x_3 \leq z_1$ is a valid inequality when $z_2 = 0$. Assume that $z_2$ is the variable to be uplifted. Let $z_2 = 1$. Then, we get the following inequality:
$x_2 + x_3 \leq \beta_2 z_2$. Minimum value that $\beta_2$ can take is 2. Then, we get

$x_2 + x_3 \leq z_1 + 2z_2$ (48)

$2x_4 + 35z_2 \leq 2z_1 + h_2 + x_2$ (49)
$2x_4 + 35z_2 \leq h_2 + x_2$ is a valid inequality when $z_1 = 0$. Assume that $z_1$ is the variable to be uplifted. Let $z_1 = 1$. Then, we get the following inequality:
$2x_4 \leq \beta_1 z_1$. Minimum value that $\beta_1$ can take is 2. Then, we get
$2x_4 + 35z_2 \leq 2z_1 + h_2 + x_2$ (49)