FREE ACTIONS ON CW-COMPLEXES AND VARIETIES OF SQUARE ZERO MATRICES

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Berrin Şentürk

September, 2011

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Özgün Ünlü(Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Ergün Yalçın

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Yıldıray Ozan

Approved for Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of Graduate School of Engineering and Science

ABSTRACT

FREE ACTIONS ON CW-COMPLEXES AND VARIETIES OF SQUARE ZERO MATRICES

Berrin Şentürk
M.S. in Mathematics
Supervisor: Assist. Prof. Dr. Özgün Ünlü
September, 2011

Gunnar Carlsson stated a conjecture which gives a lower bound on the rank of a differential graded module over a polynomial ring with coefficients in algebraically closed field k when it has a finite dimensional homology over k. Carlsson showed that this conjecture implies the rank conjecture about free actions on product of spheres. In this paper, to understand the Carlsson's conjecture about differential graded modules, we study the structure of the variety of upper triangular square zero matrices and the techniques which were investigated by Rothbach to determine its irreducible components . We hope these varieties could help prove Carlsson's conjecture.

Keywords: differential graded module, free action, variety.

ÖZET

CW-KOMPLEKSLERİ ÜZERİNE SERBEST ETKİLER VE KARESİ SIFIR OLAN MATRİSLERİN ÇEŞİTLEMELERİ

Berrin Şentürk
Matematik, Yüksek Lisans
Tez Yöneticisi: Yrd. Doç. Dr. Özgün Ünlü
Eylül, 2011

Gunnar Carlsson, katsayıları cebirsel kapalı cisim k'dan olan bir polinom halkası üzerinde bulunan ve homolojisi k üzerinde sonlu boyutlu olan, diferansiyel dereceli modülün mertebesine alt sınır veren bir sanı ortaya attı. Carlsson bu sanının kürelerin çarpımı üzerine serbest etkimeler ile ilgili olan mertebe sanısını gerektirdiğini gösterdi. Bu tezde, Carlsson'ın diferansiyel dereceli modüllerle ilgili olan sanısını anlamak için, üst üçgensel karesi sıfır olan matrislerin çeşitlemesinin yapısını ve bu çeşitlemenin indirgenemez parçalarını belirleyen, Rothbach tarafından bulunan teknikleri çalışmaktayız. Bu çeşitlemelerin Carlsson'ın sanısını kanıtlamada yardımcı olabileceğini umuyoruz.

Anahtar sözcükler: diferansiyel dereceli modül, serbest etki, çeşitleme.

Acknowledgement

I would like to express my sincere gratitude to my supervisor Asst. Prof. Dr. Özgün Ünlü for his excellent guidance, valuable suggestions, encouragement and patience.

I would also like to thank Assoc. Prof. Dr. Ergün Yalçın and Prof. Yıldıray Ozan for a careful reading of this thesis.

I am so grateful to have the chance to thank my family, especially my sister Berna Şentürk and my mother Emine Şentürk, for their encouragement, support, endless love and trust. This thesis would never be possible without their countenance.

I would like to thank my friend Çisem Güneş for encouragement and also for her help to solve all kinds of problem that I had. My thanks also goes to Mehmet Emin Aktaş who helped me about Latex with all his patience. Finally, I would like to thank all my friends in the department who increased my motivation, whenever I needed.

Contents

Chapter 1

Preliminaries

1.1 Introduction

In algebraic topology, there is a conjecture which states that if a finite group $(\mathbb{Z}/p)^r$ acts freely (see Section 1.2) on a finite complex $X \simeq S^{n_1} \times \ldots \times S^{n_l}$, then $r \leq l$. The equidimensional cases $(n = n_i)$ are verified with some additional conditions; with the induced action on integral homology is trivial by Carlsson [5], with the induced action on integral homology is unrestricted, but if $p = 2$, then $n \neq 1, 3, 7$ by Adem-Browder [1], with $p = 2$ and $n = 1$ by Yalçın [22]. Hanke [12] shows that the conjecture is true if p is large compared to the dimension of the product of spheres. Moreover, Carlsson generalizes the conjecture which asserts that if $(\mathbb{Z}/p)^r$ acts freely on a finite CW-complex X, then $\sum_i \mathrm{rk}_{\mathbb{Z}/p} H_i(X; \mathbb{Z}/p) \geq 2^r$. Carlsson proved it for $p = 2$ and $r \leq 3$ by using some facts about upper triangular square zero matrix corresponding to a differential graded module over a polynomial ring [4]. He achieved this by associating a chain complex with a differential graded module over a polynomial ring by a functor, he called it β (see Chapter 2). Carlsson showed that these differential graded modules can be taken solvable (see Definition 2.0.17) so they can be represented by upper triangular square zero matrices. Then he noticed that the rank of an upper triangular square zero matrices which represents module with nontrivial homology, over a polynomial ring must have submaximal rank. Earlier, Rothbach [19] studied the structure of the variety of upper triangular square zero matrices and its irreducible components.

In this paper, we examine Carlsson's conjecture and the algebraic analogue of the conjecture for $p = 2$ and how the algebraic analogue implies Carlsson's conjecture. We explain the method proposed by Karagueuzian, Oliver and Ventura [9] to understand Carlsson's conjecture. Karagueuzian, Oliver and Ventura relate this conjecture to square zero matrices. We use permutation matrices to calculate some varieties of matrices corresponding to differential graded module by following an idea of Rothbach.

The thesis is organized as follows:

In Chapter 1, we review some facts about free group action on spheres and introduce some definitions which we will use later.

In Chapter 2, we discuss Carlsson's conjecture about differential graded modules over a polynomial ring and its relation with conjectures mentioned in this section. We also give definitions and notations about varieties over algebraically closed field. Special attention is given to the particular irreducible component of variety of square zero matrices and its coordinate ring.

In Chapter 3, we examine the relation between Borel orbits in variety of square zero matrices and valid X^2 words. We also determine the order of valid X^2 words with moves which was investigated by Rothbach. Then we use bracket word to determine maximal valid X^2 words. Our ultimate goal, in this chapter is to show that the relation between permutation matrix and differential module corresponding to maximal valid X^2 word.

In Chapter 4, we show that Karagueuzian, Oliver, and Ventura's conjecture is true for $n = 2$ and prove a theorem related to their conjecture.

1.2 Free Group Actions on Spheres

In this section, we introduce the preliminary results about free group action on products of spheres after we give some basic definitions. We follow the notations of [2].

Definition 1.2.1. [2] An *action* of a topological group G on a topological space X is a map from $G \times X$ to X written as $\Theta(g, x)$ such that, $\Theta(g, \Theta(h, x)) = \Theta(gh, x)$ for all $g, h \in G$, $x \in X$ and $\Theta(e, x) = x$ for all $x \in X$, where e is the identity of G. The space X, together with a given action Θ of G, is called a G-space.

We often use gx for $\Theta(g, x)$ when Θ is understood from the context. The action Θ is called *free* if for all $x \in X$, $gx = x$ implies g is identity.

Definition 1.2.2. [2] If X is a G-space and $x \in X$, then the subspace

$$G(x) = \{gx \in X \mid g \in G\}$$

is called the *orbit* of x (under G). Note that if $gx = hy$ for some $g, h \in G$ and $x, y \in X$, then for any $f \in G$, $f(x) = fg^{-1}gx = fg^{-1}hy \in G(y)$ so that $G(x) \subset G(y)$; conversely $G(y) \subset G(x)$. Thus the orbits $G(x)$ and $G(y)$ of any two points $x, y \in X$ are either equal or disjoint.

The set $G_x = \{g \in G \mid gx = x\}$ is called the *isotropy* or *stability* subgroup of G at x.

Definition 1.2.3. Let G be a finite group. A *relative G-CW-complex* (X, A) is a pair of G-spaces together with a filtration $A = X_{-1} \subset X_0 \subset X_1 \subset \dots$ of $\bigcup_{n\geq -1} X_n = X$ such that A is a Hausdorff space, X carries the colimit topology with respect to this filtration and for each $n \geq 0$, the space X_n is obtained from X_{n-1} by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout diagram for each $n \geq 0$,

$$\begin{tikzcd}
\coprod_{i\in I} G/H_i \times \mathbb{S}^{n-1} \arrow[r] \arrow[d, hook] & X_{n-1} \arrow[d] \\
\coprod_{i\in I} G/H_i \times \mathbb{D}^{n} \arrow[r] & X_n
\end{tikzcd}$$

such that I_n is an index set and H_i is a subgroup of G for $i \in I_n$. If A is empty, then X is called a *G-CW-complex*. Elements of I_n are called equivariant n-cells and the map $G/H_i \times \mathbb{S}^{n-1} \to X_{n-1}$ is called the *attaching map* and the map $(G/H_i \times (\mathbb{D}^n, \mathbb{S}^{n-1})) \to (X_n, X_{n-1})$ is called the *characteristic map* of the corresponding n-cell. Here, we set $\mathbb{S}^{-1} = \emptyset$ and consider $\mathbb{D}^0$ as a point space. The spaces $\mathbb{S}^{n-1}$ and $\mathbb{D}^n$ have the trivial G-action. The G-subspace X_n is called the *n-skeleton* of (X, A).

The necessary and sufficient condition for a finite group G to act freely on a finite CW-complex the homotopy type of a sphere is that all abelian subgroups should be cyclic [20]. We know that $\mathbb{Z}_2$ is the only group that can acts freely on even dimensional spheres. It is more difficult to determine which finite groups act freely on an odd dimensional spheres.

Proposition 1.2.4. *$\mathbb{Z}_n$ is the only finite group that can act freely on S^1.*

Proposition 1.2.5. *The group $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is prime, cannot act freely on any sphere.*

Proof. See [7], page 262. □

It is natural to look for analogous conditions for finite groups to act freely on finite CW complexes having the homotopy type $\prod_{i=1}^{r} S^n$. As we mention in Section 1, Carlsson generalizes the main conjecture as follows:

Conjecture 1.2.6 (Conjecture 1.3 in [6]). Suppose that $G = (\mathbb{Z}/p)^r$ acts freely on a finite CW-complex X. Then $\sum_i \mathrm{rk}_{\mathbb{Z}/p} H_i(X; \mathbb{Z}/p) \geq 2^r$.

Carlsson proved this conjecture for $p = 2$ and $r \leq 3$ by using facts about differential graded module over a polynomial ring [4].

1.3 Differential Modules

Let R be a commutative ring with identity. A *differential module* (F, ∂) is an R-module F with an endomorphism ∂ such that $\partial^2 = 0$. The endomorphism ∂ is called the *differential*.

A *differential graded module (DG-module)* is a graded module that has a differential compatible with the graded structure (that is, the differential is of degree

r for some r). Given a graded ring R, Carlsson defines a *differential graded R-module* F as a free, graded R-module with a graded R-module homomorphism $\partial : F \to F$ of degree -1 so that $\partial^2 = 0$. A graded module F is called *bounded above* when $F_i = 0$ for sufficiently large i. For a graded ring R, let $\mathcal{D}(R)$ denote the category of finitely generated DG-R-modules, and if R is bounded above, we let $\mathcal{D}_\infty(R)$ denote the category of bounded above DG-R-modules. Clearly, $\mathcal{D}(R)$ is a subcategory of $\mathcal{D}_\infty(R)$.

A *complex* of modules over a ring R is a sequence of R-modules and homomorphisms

$$F : \quad \ldots \longrightarrow F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \longrightarrow \ldots$$

where the degree-i component of complex F is F_i such that $\partial_i \partial_{i+1} = 0$ for each i. In other words, a complex F is a differential graded R-module in which the differential is of degree -1.

Definition 1.3.1. Let F be a differential module with differential ∂. Then the homology module of F is defined to be $H(F) = Ker\partial / Im\partial$. If F is differential graded module, then the i-th homology of F is

$$H_i(F) = Ker\partial_i / Im\partial_{i+1}.$$

We write $H(F)$ for the direct sum $\bigoplus_i H_i(F)$ of all the homology modules. Similarly, if F is a complex as above, then the homology module of F is $H(F) = Ker\partial / Im\partial$.

We say that the differential module (or a complex) F is *exact* if $H(F) = 0$.

Definition 1.3.2. [11] If (F, ∂) and (G, ∂) are differential modules, then a *map of differential modules* is a map of modules $\alpha : F \to G$ such that $\alpha\partial = \partial\alpha$. If F and G are complexes, then we insist that α preserve the grading as well. Explicitly, if

$$F : \quad \ldots \longrightarrow F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \longrightarrow \ldots$$

and

$$G : \quad \ldots \longrightarrow G_{i+1} \xrightarrow{\partial_{i+1}} G_i \xrightarrow{\partial_i} G_{i-1} \longrightarrow \ldots$$

are complexes of modules, then a map of complexes $\alpha : F \to G$ is a collection of maps

$$\alpha_i : F_i \to G_i$$

of modules making the diagrams

$$\begin{tikzcd} F_i \arrow[r, "\partial_i"] \arrow[d, "\alpha_i"'] & F_{i-1} \arrow[d, "\alpha_{i-1}"] \\ G_i \arrow[r, "\partial_i"'] & G_{i-1} \end{tikzcd}$$

commutative.

If $\alpha : (F, \partial) \to (G, \partial)$ is a map of differential modules, then α carries kernel of differential of F to kernel of differential of G and image of differential of F to image of differential of G. Thus α gives rise to an *induced map on homology*, which we also call α:

$$\alpha : H(F) \to H(G).$$

Definition 1.3.3. [11] If α, β: $(F, \partial) \to (G, \partial)$ are two maps of differential modules, then α is *homotopy equivalent* to β (or simply *homotopic* to β) if there is a map of modules $h : F \to G$ such that $\alpha - \beta = \partial h + h\partial$. In this case, F and G have the same homotopy type. If F and G are complexes (so that F and G are graded modules and ∂ have degree -1), then we insist that h have degree 1.

$$\begin{tikzcd} F_i \arrow[r, "\partial_i"] \arrow[d] & F_{i-1} \arrow[d] \arrow[dl, "h"'] \\ G_i \arrow[r, "\partial_i"'] & G_{i-1} \end{tikzcd}$$

Note that α is homotopy equivalent to β if and only if $\alpha - \beta$ is equivalent to 0.

The homotopy terminology comes from topology. If α and β are continuous maps from a space X to a space Y, then they induce maps of complexes from the singular chain complex (see definitions in [21], page $160 - 161$) of X to that of Y. A homotopy $H : X \times I \to Y$ from α to β determines a chain map $h(x) := H(x \times I)$ that raises dimensions by 1.

Let C be a chain complex. We mostly consider that $C_q = 0$ for $q < 0$. A *free chain complex* is a chain complex in which C_q is a free abelian group for every q. For an arbitrary discrete group G, there is a close connection between G-CW-complexes and ordinary CW-complexes.

Let X be a G-space and an ordinary CW-complex. We say that G *acts cellularly* on X if the following holds:
(i) For each $g \in G$ and each open cell E of X, the left translation gE is again an open cell of X.
(ii) If $gE = E$, then the induced map $E \to E$, $x \mapsto gx$ is the identity.

Proposition 1.3.4 (Proposition 1.15 in [8]). *Let X be a CW-complex with cellular action of G. Then X is a G-CW-complex.*

Proposition 1.3.5 (Proposition 1.16 in [8]). *Let G be a discrete group. Let X be a G-complex and H be a subgroup of G. Then X, considered as an H-space, is an H-CW-complex with the same skeleta.*

Hence, if G is discrete and X is a G-CW-complex, then X is a CW-complex with cellular G-action. Consequently, for discrete groups, we have two equivalent definitions of G-CW-complexes.

Let G be a finite group. Given a G-CW-complex, its cellular chain complex with R coefficients is defined by

$$\tilde{C}_* : \ldots \xrightarrow{\Delta_{n+1}} H_n(X_n, X_{n-1}; R) \xrightarrow{\Delta_n} H_{n-1}(X_{n-1}, X_{n-2}; R) \xrightarrow{\Delta_{n-1}} \ldots$$

where Δ is the connecting homomorphism of the triple (X_n, X_{n-1}, X_{n-2}). Then there is a functor which takes free G-CW-complex to free RG-chain complex and there is a natural isomorphism $H_*(X) \cong H_*(\tilde{C}_*(X))$ [15].

Let R be a unital commutative ring and M, N be R-modules. The tensor product (see [10], page 359) $M \otimes_R N$ is an R-module spanned by symbols $m \otimes n$ (simple tensor) satisfying distributive laws:

$$(m + m') \otimes n = m \otimes n + m' \otimes n, \; m \otimes (n + n') = m \otimes n + m \otimes n'.$$

Also multiplication by any $r \in R$ is associative with $\otimes$ on both sides:

$$r(m \otimes n) = (rm) \otimes n = m \otimes (rn).$$

Let $f : R \to S$ be a homomorphism of commutative rings with $f(1_R) = 1_S$. We use f to consider any S-module N as an R-module by $rn := f(r)n$. In particular, S itself is an R-module by $rs := f(r)s$. Passing from N as an S-module to N as an R-module in this way is called *restriction of scalars.*

We can also reverse the process of restriction of scalars.

Proposition 1.3.6. *The additive group $M \otimes_R S$ has a unique S-module structure satisfying $s'(m \otimes s) = m \otimes s's$, and this is compatible with the R-module structure in the sense that $rt = f(r)t$ for all $r \in R$ and $t \in M \otimes_R S$. Notice that any $t \in M \otimes_R S$ is a finite sum of elementary tensors, say*

$$t = m_1 \otimes s_1 + \ldots + m_k \otimes s_k.$$

We are mostly interested in DG-modules over a polynomial ring. Let $A = k[x_1, \ldots, x_n]$ where k is a field and each x_i assigned grading -1. Note that if F is DG-A-module and E is a graded A-module, then $H_*(F, E)$ denotes the homology of the DG-module $F \otimes_A E$.

Let B be a graded ring and augmented over a field k with augmentation $\epsilon : B \to k$.

Definition 1.3.7. Let F be finitely generated DG-B-module. We say that (F, ∂) is *minimal* if the map $\partial \otimes id : F \otimes_B k \to F \otimes_B k$ is the zero map.

Definition 1.3.8. [4] We say a DG-A-module F is totally finite if $\dim_k H_*(F)$ is positive integer where k is the ground field.

Chapter 2

Variety of Square Zero Matrices

In this section, we focus on the algebraic formulations of Carlsson's conjecture about DG modules over a polynomial ring and discuss the relation between conjectures mentioned in the introduction. For more details about conjectures, see [6]. We also give the reason why we study square zero matrices. Then we give definitions and notations about varieties corresponding to these matrices over an algebraically closed field.

First we recall Carlsson's conjecture;

Conjecture 2.0.9. Let $G = (\mathbb{Z}/p)^r$ and X be a finite free G-CW-complex. Then $\sum_i \operatorname{rk}_{\mathbb{Z}/p} H_i(X;\mathbb{Z}/p) \geq 2^r$.

Carlsson constructs the algebraic analogue of this conjecture by letting $\Lambda_r = k[G]$ where $G = (\mathbb{Z}/2)^r$ and k be a field of characteristic two. As an algebra Λ_r is isomorphic to the exterior algebra $E(y_1, \ldots, y_r)$, by considering $y_i = T_i + 1$, where $\{T_1, \ldots, T_r\}$ is a basis for $(\mathbb{Z}/2)^r$. Λ_r can be considered as a graded ring by assuming the grading 0 to all elements of Λ_r. Then $\mathcal{D}(\Lambda_r)$ denotes the category of finitely generated DG-Λ_r-modules. As we mentioned in Section 1.3, DG-Λ_r-module is a free, graded Λ_r-modules with a differential ∂. Then an algebraic analogue of Conjecture 2.0.9 is the following conjecture;

Conjecture 2.0.10. [6, Conjecture 2.2] Let $F \in \mathcal{D}(\Lambda_r)$ and $H_*(F) \neq 0$. Then $\operatorname{rk}_k H_*(F) \geq 2^r$.

Let X be a free, finite G-CW-complex. The cellular chain complex $\tilde{C}_*(X;k)$ is a finitely generated chain complex of free Λ_r-modules. Hence it is in $ob\mathcal{D}(\Lambda_r)$ and $H_*(X;k) = H_*(\tilde{C}_*(X;k))$. Therefore, Conjecture 2.0.10 implies Conjecture 2.0.9 for $p = 2$.

Let A_r denote the polynomial ring $k[x_1, \ldots, x_r]$ which we grade by assigning each variable the grading (-1). Carlsson shows in [3] that there is a functor $\beta : \mathcal{D}(\Lambda_r) \to \mathcal{D}(A_r)$ defined as follows. The functor β takes a DG-Λ_r-module (F, ∂) to $F \otimes_k A_r$ and the differential ∂ on $\beta(F)$ is defined by

$$\partial(f \otimes h) = \partial f \otimes h + \sum_{i=1}^{r} y_i f \otimes x_i h.$$

Proposition 2.0.11 (Propositions 2.1 and 2.2 in [3]). *There are natural isomorphisms $H_*(F) \to H_*(\beta F, k)$ and $H_*(F, k) = H_*(F \otimes_{k[G]} k) \to H_*(\beta F)$*

Corollary 2.0.12 (Corollary 2.3 in [3]). *For any $F \in ob\mathcal{D}(\Lambda_r)$, $H_*(\beta F)$ is finitely generated as a k-vector space. In other words, if F is finitely generated DG Λ_r-module, then βF is totally finite (see Definition 1.3.8).*

We also have following proposition:

Proposition 2.0.13 (Proposition 2.6 in [6]). *For every $(F, \partial) \in ob\mathcal{D}_\infty(A)$, there exists $(\bar{F}, \bar{\partial}) \in ob\mathcal{D}_\infty(A)$, where $(\bar{F}, \bar{\partial})$ is minimal and is chain equivalent to (F, ∂).*

Let $\mathcal{D}^0(A_r)$ and $\mathcal{D}^0_\infty(A_r)$ denote the full subcategories of $\mathcal{D}(A_r)$ and $\mathcal{D}_\infty(A_r)$, respectively, whose objects are the DG-A_r-modules (F, ∂) for which $H_*(F)$ is nontrivial and finite dimensional k-vector space. Let $\mathcal{D}^0_\infty(\Lambda_r)$ denote the full subcategories of $\mathcal{D}_\infty(\Lambda_r)$ whose objects are chain equivalent to object in $\mathcal{D}^0(\Lambda_r)$. We also let $h\mathcal{D}(\Lambda_r)$ and $h\mathcal{D}(A_r)$ denote the homotopy categories of $\mathcal{D}(\Lambda_r)$, $\mathcal{D}(A_r)$ and $h\beta : h\mathcal{D}(\Lambda_r) \to h\mathcal{D}(A_r)$ be the induced map on homotopy categories. By Corollary 2.0.12, $h\beta$ can be extended to a functor $H : h\mathcal{D}^0_\infty(\Lambda_r) \to h\mathcal{D}^0_\infty(A_r)$.

Theorem 2.0.14 (Theorem 2.7 in [6]). *$H : h\mathcal{D}^0_\infty(\Lambda_r) \to h\mathcal{D}^0_\infty(A_r)$ is an equivalence of categories.*

Conjecture 2.0.15 (Conjecture 2.8 in [6]). Let $F \in ob\mathcal{D}^0(A_r)$. Then $\operatorname{rk}_{A_r} F \geq 2^r$.

This conjecture is equivalent to the following conjecture.

Conjecture 2.0.16. Let $\bar{F} \in ob\mathcal{D}^0(A_r)$ and $\bar{F}$ is minimal. Then $\operatorname{rk}_{A_r} \bar{F} \geq 2^r$.

Finally, we are ready to show that Conjecture 2.0.15 is equivalent to Conjecture 2.0.10. By Theorem 2.0.14 and Proposition 2.0.11, Conjecture 2.0.10 is equivalent to the conjecture that for all $F \in ob\mathcal{D}^0(A_r)$, $\operatorname{rk}_k H_*(F,k) \geq 2^r$. By Proposition 2.0.13, F is equivalent to a minimal DG-A_r-module $\bar{F}$ with augmentation $\epsilon : A_r \to k$ given by $\epsilon(f(x_1,\ldots,x_r)) = a_0$ where f is a polynomial with constant term a_0. Then $\operatorname{rk}_{A_r} F \geq \operatorname{rk}_{A_r} \bar{F} = \operatorname{rk}_k(\bar{F} \otimes_{A_r} k) = \operatorname{rk}_k H_*(\bar{F},k) \geq 2^r$.

As a summary, we start with a finite G-CW-complex X. Then we get a finitely generated chain complex of free Λ_r-modules, say C. By functor β, let $\beta C = F$, we obtain a totally finite DG-A_r-module F. Finally, there exists $\bar{F}$ which is minimal and chain equivalent to F. Note that $\operatorname{rk}_{A_r} \bar{F} = \operatorname{rk}_k(\bar{F} \otimes_{A_r} k) = \operatorname{rk}_k H_*(\bar{F},k) = \operatorname{rk}_k H_*(F,k) = \operatorname{rk}_k H_*(C) = \operatorname{rk}_k H_*(X,k)$. Therefore, $\operatorname{rk}_{A_r} \bar{F} \geq 2^r$ implies $\operatorname{rk}_k H_*(X,k) \geq 2^r$.

Definition 2.0.17. Let F be a finitely generated DG-A_r-module. A *composition series* for a DG-A_r-module F is a sequence $0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_q = F$ of DG-submodules so that each quotient F_j/F_{j-1} is a free, finitely generated DG-A_r-module whose differential is identically zero. F is said to be *solvable* if there is a chain equivalence $f : \bar{F} \to F$ where $\bar{F}$ is a free, finitely generated DG-A_r-module admitting a composition series.

Carlsson shows that every free, finitely generated DG-A_r-module is solvable [3]. When we choose a basis for F compatible with the composition series, the differential ∂ can be represented by the special matrix. Since $\partial^2 = 0$ and the degree of ∂ is (-1) in composition series, we have strictly upper triangular square zero matrix. Indeed, the dimension of this matrix is even by the following propositions.

Proposition 2.0.18 (Proposition 1.8 in [4]). *Suppose that F is a totally finite, free, finitely generated DG-A_r-module and m is any maximal ideal of A_r other than $(x_1, \ldots, x_n)$. Then, $H_*(F, A_r/m) = 0$.*

Proposition 2.0.19 (Proposition 1.9 in [4]). *Let F' be any finite dimensional differential module with differential ∂ over a field k'. Then $H_*(F') = 0$ if and only if $2 \operatorname{rk} \partial = \dim_{k'} F'$.*

Proof. Assume $H_*(F') = 0$. Then $Ker\partial_n = Im\partial_{n+1}$. Consider the homomorphism $\partial : F' \to F'$. By 1-st Isomorphism Theorem, $F'/Ker\partial \cong Im\partial$, so $\dim F' - \dim (Ker\partial) = \dim (Im\partial)$ over a field k'. Then $\dim_{k'} F' = 2\dim(Im\partial) = 2 \operatorname{rk} \partial$. Conversely, assume $2 \operatorname{rk} \partial = \dim_{k'} F'$, to the contrary $H_*(F') \neq 0$. By again 1-st Isomorphism Theorem, we have $\dim_{k'} F' = \dim(Ker\partial) + \dim(Im\partial)$. Since, $H_*(F') \neq 0$, $\dim (Ker\partial) > \dim (Im\partial)$. Then we have $\dim_{k'} F' > 2(Im\partial)$, that is $\dim_{k'} F' > 2 \operatorname{rk} \partial$ which is a contradiction. □

Let $\bar{F'} = \bar{F} \otimes_{A_r} (A_r/m)$ where $\bar{F}$ is minimal and m is any maximal ideal of A_r other than $(x_1, \ldots, x_r)$. Then $\dim_{k'} \bar{F'}$ is even for a field k' by Proposition 2.0.19. We also let $\bar{\partial}'$ denote the differential of DG-A_r-module $\bar{F'}$ and M denote the matrix which represents $\bar{\partial}'$. Then $\dim_{k'} M$ is also even and $\operatorname{rk} M = (\dim_{k'} M)/2 = (\dim_{k'} \bar{F'})/2$.

Consequently, following conjecture implies Conjecture 2.0.9 for $p = 2$.

Conjecture 2.0.20. Let M be a upper triangular matrix in $Mat_{2n}(A_r)$. If (A_r^{2n}, M) is a totally finite, minimal DG-A_r-module, then $2n \geq 2^r$.

Therefore, in this paper, we focus on the varieties of upper triangular square zero $2n \times 2n$ matrices over a field k.

2.1 Notations

In algebraic geometry, the n-dimensional affine space A^n (or $A^n(k)$) over the field k is the set of n-tuples of elements of k. An element $p = (p_1, \ldots, p_n) \in A^n$ is called a *point*, p_i's are *(affine) coordinates* of p. Indeed, a subset $V \subseteq A^n$ is an *affine algebraic variety*, if it is a zero set of a finite set of polynomials in $k[x_1, \ldots, x_n]$ and let $f_1, \ldots, f_k \in k[x_1, \ldots, x_n]$, then

$$V = Z(f_1, \ldots, f_k) = \{p \in A^n \mid f_i(p) = 0 \ \forall\ i\}.$$

Given a variety V in the n-dimensional space k^n, the *coordinate ring* of V is the quotient ring

$$k[V] = k[x_1, \ldots, x_n]/I(V)$$

where $I(V) = \{f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \ \ for \ \ all \ \ x \in V\}$.

Definition 2.1.1. A variety $V \subset k^n$ is irreducible if it is nonempty and not the union of two proper subvarieties; that is, if $V = V_1 \cup V_2$ for varieties V_1, V_2 then $V = V_1$ or V_2.

As we mentioned at the beginning of this chapter, we are concerned with the varieties of the even dimensional matrices (see Proposition 2.0.19).

In this section, we give definitions and notations about varieties over k which is an algebraically closed field and we follow the terminology in [9]. All the varieties we consider are over k.

Notation 2.1.1. [9] We will use the following notations in the rest of the thesis:

- Let U_{2n} be the variety of strictly upper-triangular $2n \times 2n$ matrices over k. In other words $U_{2n} = k^l$ where $l = 2n^2 - n$. Then we can define U_{2n} by the following matrix;

$$U_{2n} = \left\{ \begin{pmatrix} 0 & P_{12} & P_{13} & P_{14} & \dots & P_{1(2n)} \\ 0 & 0 & P_{23} & P_{24} & \dots & P_{2(2n)} \\ 0 & 0 & 0 & P_{34} & \dots & P_{3(2n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & 0 & P_{(2n-1)(2n)} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \textit{where } P_{ij} \in k \textit{ and } i < j \right\}.$$

Note that $i < j$ makes the matrix strictly upper triangular.

- Let V_{2n} be the variety of square zero matrices in U_{2n}. In other words; $V_{2n} = \{X \in U_{2n} | X^2 = 0\}$.

- The coordinate ring of U_{2n} is $R(U_{2n}) = k[x_{ij} \mid i < j]$. For instance $R(U_4) = k[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]$ for $n = 2$.

- For V_{2n}, $R(V_{2n}) = R(U_{2n})/I(V(J))$ where J is corresponding ideal for V_{2n}. By Hilbert's Nullstellensatz [18], $I(V(J)) = \sqrt{J}$. For instance, for $n = 2$,

$$R(V_4) = k[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]/\sqrt{(x_{12}x_{23}, x_{12}x_{24} + x_{13}x_{34}, x_{23}x_{34})}.$$

- Let Z be a particular irreducible component of V_{2n}. So, we have $Z \subseteq V_{2n} \subseteq U_{2n}$. By recalling Definition 2.1.1, Z cannot be written as the union of two proper varieties.

- Let R be the coordinate ring of Z.

- Let M be a $2n \times 2n$ upper triangular matrix over ring R such that the (i, j)-entry of M is the image in R of x_{ij}. Note that the coordinate ring of U_{2n} is $R(U_{2n}) = k[x_{ij} \mid i < j]$. There are surjections of coordinate rings

$$\varphi : R(U_{2n}) \to R(V_{2n}) \to R$$

corresponding to the inclusions $Z \hookrightarrow V_{2n} \hookrightarrow U_{2n}$. Using these surjections, we can regard the images of the x_{ij} as elements of R. Then we also define the (i, j)-entry of M; $M_{ij} = \varphi(x_{ij})$. Moreover, since $R(V_{2n})$ is a ring of variety of square zero matrices of U_{2n} and M follow the map $R(U_{2n}) \to R(V_{2n}) \to R$, $M^2 = 0$ and we regard M as a differential on the R-module R^{2n}. The differential M can be represented by the special matrix when we choose a proper basis for differential R-module.

$$M = \begin{pmatrix} 0 & M_{12} & M_{13} & M_{14} & \dots & M_{1(2n)} \\ 0 & 0 & M_{23} & M_{24} & \dots & M_{2(2n)} \\ 0 & 0 & 0 & M_{34} & \dots & M_{3(2n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & 0 & M_{(2n-1)(2n)} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \textit{with } M^2 = 0.$$

In the rest of the paper, $R/m = k$ where m is maximal ideal of R.

- Let Y be the subvariety of matrices of rank less than n.

- Let I denote the ideal of R corresponding to Y. Note that I is the ideal generated by all $n \times n$ minors of the universal matrix M. So I is generated by the determinant of all $n \times n$ matrix obtained from M by deleting n rows and n columns.

We have a differential R-module with a differential $M : R^{2n} \to R^{2n}$ where R is same as above and $M^2 = 0$.

Notice that, the homology of differential module F is generally denoted by $H(F)$, but in the rest of paper, by following the idea of Karagueuzian, Oliver and Ventura, we take the homology of matrix M corresponding to differential of module F. It is more reasonable since we concern with the changes of the rank of the matrix corresponding to differential M. Then, we can set the homology of R^{2n} is the quotient module $H(M) = Im(M)/Ker(M)$.

Chapter 3

Valid X^2 Words and Bracket Words

The main purpose of this chapter is to examine the relation between Borel orbits in V_{2n} and valid X^2 words. In this chapter, most of definitions are taken from [9] and [19]. We clarify these definitions with an example. In Section 3.2, we show that how some part of M can be calculated by using a permutation matrix.

3.1 Valid X^2 Words

Let V_{2n} be same as in previous chapter and B_{2n} be the $2n \times 2n$ Borel group of invertible upper triangular matrices. B_{2n} acts on V_{2n} by $b(X) = b^{-1}Xb$ for all $b \in B$, $X \in V$.

Definition 3.1.1. The Borel orbits in V_{2n} are the orbits of the conjugation action of the Borel group of all invertible upper-triangular matrices on V_{2n}.

There is a one-to-one correspondence between the Borel orbits and *valid X^2 words* (see Definition 3.1.3); an ordering of valid X^2 words can be described by *moves.* A move is a function from the set of valid X^2 words to itself; the ordering is describes by moves in the sense that if $w < w'$ then there is a sequence of moves which transforms w' into w.

Definition 3.1.2. A matrix P is said to be a *partial permutation matrix* if it has at most one nonzero entry in each row and column, and these nonzero entries (if any) are all 1.

To an upper-triangular partial permutation matrix we can associate a sequence of nonnegative integers $(a_1, a_2, \ldots, a_{2n})$ by setting

$$a_i = \begin{cases} j & \text{if } Pe_i = e_j \\ 0 & \text{if } Pe_i = 0 \end{cases}$$

Definition 3.1.3. Given a partial permutation matrix P, define the word $w = a_1 \ldots a_{2n}$ where a_i's satisfy the condition above. Then a word is said to be *valid X^2 word* if it is of the form w for some $P \in V_{2n}$ with $P^2 = 0$. In other words, there are three conditions to be a valid X^2 word w where a_i are letters of w;

- $a_j < j$ for all j (which makes the matrix strictly upper triangular)
- $a_j \neq a_k$ for all $j \neq k$ (which makes the matrix a partial permutation matrix)
- $a_j = i$ implies $a_i = 0$ (else $P^2 e_j = Pe_i$ is nonzero.)

Rothbach proved that each Borel orbit contains a unique partial permutation matrix [19]. Therefore, each Borel orbit is associated a unique valid X^2 word.

If w is a valid X^2 word, $\mathrm{rk}(w)$ is the number of nonzero integers a_i in w, i.e., the rank of the partial permutation matrix associated to w.

The closure of a Borel orbit is the closure of an image of the Borel group, which is an irreducible variety, so these closures are themselves irreducible varieties [17]. Then, the closure of a Borel orbit is itself a union of Borel orbits. Rothbach defined *moves* to give an order relation on the valid X^2 words. This order relation help us to determine which Borel orbits are contained in the closure of a given Borel orbit, in terms of the corresponding valid X^2 words. Then we need to introduce the following terminology to explain this.

Definition 3.1.4. A letter a_i of a valid X^2 word $a_1 \dots a_{2n}$ is a *bound zero* (or simply bound) if $a_i = 0$ and there exists a j such that $a_j = i$. A letter a_i is *free* if it is not bound.

We regard valid X^2 words as partial permutations of the set $\{1, \dots, 2n\}$. Then, the word $(a_1, \dots, a_{2n})$ is regarded as the partial permutation with domain $\{i \mid a_i \neq 0\}$, which sends i to a_i. We also know that $X^2 = 0$. Therefore, the domain and range of the permutation are disjoint. These can be illustrated by diagrams with arrows. For instance, the word 002105 is illustrated by the following diagram and arrows:

$$(\ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6\ ).$$

Then the three moves are the following:

- A *move of type 1* takes a nonzero letter a_k and replaces it with $a_k{}^*$ which is the largest integer less than a_k so that the replacement yields a new valid X^2 word. Since replacement with 0 always yields a valid X^2 word, $a_k{}^*$ always exists. If we set $j = a_k$ and $i = a_k{}^*$ that makes $i < j < k$, then this move sends

  $$(\ i \quad j \quad k\ ) \text{ to } (\ i \quad j \quad k\ ).$$

  or

  $$(\ j \quad k\ ) \text{ to } (\ j \quad k\ )$$

  when $i \neq 0$ or $i = 0$, respectively.

- A *move of type 2* takes two free letters a_k, a_l where $k < l$ and $a_k > a_l$, and swaps their locations. In other words, if we set $a_k = j$ and $a_l = i$ where i is nonzero so $i < j < k < l$, then this move sends

  $$(\ i \quad j \quad k \quad l\ ) \text{ to } (\ i \quad j \quad k \quad l\ )$$

  or if we set $a_k = j$ and $a_l = 0$ that gives $j < k < l$, then it sends

  $$(\ j \quad k \quad l\ ) \text{ to } (\ j \quad k \quad l\ ).$$

- A *move of type 3* is defined if $i < j < k < l$ such that $i = a_j$ and $k = a_l$ (so $a_i = a_k = 0$) and replaces a_l by j, a_k by i and a_j by 0. Schematically, it sends

$$(i \quad j \qquad k \quad l) \text{ to } (i \qquad j \qquad k \qquad l).$$

Notice that a move of type 2 or 3 preserves the rank of words. Actually, the only way of getting a word of smaller rank is to replace a letter by zero which can be done by applying move 1 one or more times. A sequence of moves of type 1 which results in a letter being replaced by zero will be called a *move of type* $1'$.

For two valid X^2 words w, w', the order relation is defined by letting $w \geq w'$ if and only if w can be transformed into w' by a finite sequence of moves. Therefore, the maximal valid X^2 words are not the result of any of the three types of moves.

Example 3.1.5. The word $(0, 1, 0, 3)$ is transformed to $(0, 1, 0, 0)$ by a move of type 1, so $(0, 1, 0, 0) < (0, 1, 0, 3)$. The word $(0, 1, 0, 0)$ is transformed to $(0, 0, 1, 0)$ by a move of type 2, so $(0, 0, 1, 0) < (0, 1, 0, 0)$. In addition, the word $(0, 1, 0, 3)$ is transformed to $(0, 0, 1, 2)$ by a move of type 3, hence we have $(0, 0, 1, 2) < (0, 1, 0, 3)$. We are determined three words smaller than $(0, 1, 0, 3)$. Actually, it is a maximal valid X^2 word.

Theorem 3.1.6 (Rothbach, [19])**.** *For any pair of valid X^2 words v, w,the Borel orbit $\mathcal{O}_v$ associated to v is contained in the closure of the Borel orbit $\mathcal{O}_w$ associated to w if and only if $v \leq w$. The irreducible components of V_{2n} are thus the closures of the Borel orbits associated to the maximal valid X^2 words; and the irreducible component of V_{2n} associated to a maximal valid X^2 word w is the union of the Borel orbits associated to the valid X^2 words which are less than or equal to w.*

A brief sketch of Rothbach's techniques is as follows: Let $k^i \subseteq k^{2n}$ be the subspace of elements $(x_1, \ldots, x_i, 0, \ldots, 0)$ for $x_1, \ldots, x_i \in k$. These are the subspaces of k^{2n} which are invariant under the action of all elements in the Borel group. For any $P \in V_{2n}$ and $0 \leq j < i$, define $r(i, j, P) = dim_k(P(k^i) + k^j)$, i.e., these dimensions are invariants of the Borel orbits. For any valid X^2 word v, associated to a partial permutation matrix P, set $v_{ij} = r(i, j, P)$. Using these, Rothbach shows:

- Two matrices $P, Q \in V_{2n}$ are in the same Borel orbit if and only if $r(i,j,P) = r(i,j,Q)$ for all i,j. The Borel orbit associated to v is thus the set $\{P \in V_{2n} \mid r(i,j,P) = v_{ij} \ \forall i,j\}$.
- For any two valid X^2 words v, w, we have $v \leq w$ if and only if $v_{ij} \leq w_{ij}$ for all i,j.
- If v is obtained from w by a move of one of the the above types, then the Borel orbit $\mathcal{O}_v$ is in the closure of the Borel orbit $\mathcal{O}_w$.

For any valid X^2 word w, the union of the Borel orbits associated to words $v \leq w$ is the set

$$\{P \in V_{2n} \mid r(i,j,P) \leq w_{ij} \ \forall i,j\}.$$

The maximal valid X^2 words are also called *bracket words* because there is a one-to-one correspondence between maximal valid X^2 words and sequences of left and right parentheses of length $2n$. A bracket word corresponds to the valid X^2 word $(a_1, \ldots, a_{2n})$ where $a_i = 0$ if the i-th parenthesis in the bracket word is a left parenthesis, and $a_i = j$ if the i-th parenthesis is a right parenthesis which closes the j-th parenthesis.

Remark 3.1.7. For a bracket word w of length $2n$, we have $rank(w) = n$.

Note that the number of bracket words is C_n well known Catalan numbers are given by

$$C_n = \frac{(2n)!}{(n+1)!n!} \ .$$

Definition 3.1.8. We say that a bracket word is *irreducible* if it cannot be expressed as the concatenation of bracket words of smaller length.

Example 3.1.9. For $n = 3$, we have $C_3 = 5$, then ((())) and (()()) are irreducible bracket words, but ()(()), (())() and ()()() are reducible bracket words.

3.1.1 Some calculations for $n = 2$

Example 3.1.10. For $n = 2$, we have 4×4 matrices over k. Then

- U_4 is the variety of strictly upper-triangular 4×4 matrices over k,

$$U_4 = k^6 = \left\{ \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix} \;\middle|\; a, b, c, d, e, f \in k \right\}.$$

- Then V_4 must satisfy the following;

$$\begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & ad & ae + bf \\ 0 & 0 & 0 & df \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0.$$

  So $V_4 = \{(a, b, c, d, e, f) \mid ad = 0, ae + bf = 0, df = 0\}$.

- The coordinate ring of U_4 is $R(U_4) = k[x_a, x_b, x_c, x_d, x_e, x_f]$. Then the coordinate ring of V_4 is

$$R(V_4) = k[x_a, x_b, x_c, x_d, x_e, x_f]/I(V(x_a x_d, x_a x_e + x_b x_f, x_d x_f)).$$

  By Hilbert's Nullstellensatz [18], $I(V(J)) = \sqrt{J}$. Then, $R(V_4) = k[x_a, x_b, x_c, x_d, x_e, x_f]/\sqrt{(\langle x_a x_d, x_a x_e + x_b x_f, x_d x_f \rangle)}$.

Using bracket words, we have two maximal valid X^2 words; $(()) = 0021$ which is irreducible and $()() = 0103$ which is reducible.

Case(1) For $w_1 = 0021$, the corresponding diagram is

$$\begin{tikzcd}[column sep=small] (\,1 & 2 & 3 \arrow[l, bend right] & 4\,). \arrow[lll, bend right] \end{tikzcd}$$

We have $P.e_1 = 0, P.e_2 = 0, P.e_3 = e_2, P.e_4 = e_1$, then

$$P_{w_1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We defined $w_{ij} = r(i, j, P_w) = dim_k(P(k^i) + k^j)$ at the beginning of section. Then we have $w_{10} = dim(P_{w_1}(k \oplus 0 \oplus 0 \oplus 0)) = 0$ and $w_{20} = dim(P_{w_1}(k \oplus k \oplus 0 \oplus 0)) = 0$. In addition, $w_{21} = dim(P_{w_1}(k \oplus k \oplus 0 \oplus 0) + (k \oplus 0 \oplus 0 \oplus 0)) = 1$ and $w_{42} = dim(P_{w_1}(k \oplus k \oplus k \oplus k) + (k \oplus k \oplus 0 \oplus 0)) = 2$. Similarly, $w_{30} = 1$, $w_{31} = 2$, $w_{32} = 2$, $w_{40} = 2$, $w_{41} = 2$ and $w_{43} = 3$. We can define w as

$$w = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 a + x_3 b + x_4 c \\ x_3 d + x_4 e \\ x_4 f \\ 0 \end{bmatrix} \text{where } x_1, x_2, x_3, x_4 \in k.$$

We need to find all P such that $r(i, j, P) \leq w_{ij}$. We have $r_{10} = 0$ which does not give any result but $r_{20} = 0$ certainly implies $a = 0$. We also have $r_{30} \leq 1$ which implies $a + b = 0$ or $d = 0$ and $r_{40} \leq 2$ implies $a + b + c = 0$ or $d + e = 0$ or $f = 0$. The conditions $r_{21} \leq 1$, $r_{31} \leq 2$, $r_{32} \leq 2$ and $r_{43} \leq 3$ are not enough to get any results, but $r_{41} \leq 2$ implies $d + e = 0$ or $f = 0$. Finally, $r_{42} \leq 2$ implies $f = 0$. Then we definitely know that $a = f = 0$. Now, we are ready to find Z, R, Y and I for $n = 2$.

- $Z = \{(a, b, c, d, e, f) | a = 0, f = 0\}$.
- $R = k[x_a, \ldots, x_f]/\sqrt{(x_a, x_f)}$.

- $$X_{w_1} = \begin{bmatrix} 0 & 0 & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Since Y is the subvariety of matrices of rank less than 2,

$$Y = \{(a,b,c,d,e,f) | a = 0, f = 0, be - cd = 0\}.$$

- $I = \sqrt{(x_a, x_f, x_b x_e - x_c x_d)}$.

Case(2) For $w_2 = 0103$, we have the corresponding diagram

$$(\, 1 \leftarrow 2 \qquad 3 \leftarrow 4 \,).$$

Then, $P.e_1 = 0, P.e_2 = e_1, P.e_3 = 0, P.e_4 = e_3$, so

$$P_{w_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $w_{ij} = r(i,j,P_w) = dim_k(P(k^i) + k^j)$, $w_{10} = dim(P_{w_2}(k \oplus 0 \oplus 0 \oplus 0)) = 0$. Similarly, $w_{20} = 1$, $w_{21} = 1$ and $w_{30} = 1$. In addition, $w_{31} = dim(P_{w_2}(k \oplus k \oplus k \oplus 0) + (k \oplus 0 \oplus 0 \oplus 0)) = dim(k \oplus 0 \oplus 0 \oplus 0) = 1$. Similarly, $w_{32} = 2$, $w_{40} = 2$, $w_{41} = 2$, $w_{42} = 3$ and $w_{43} = 3$. Then

$$w = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 a + x_3 b + x_4 c \\ x_3 d + x_4 e \\ x_4 f \\ 0 \end{bmatrix}.$$

Similarly, we have $r_{10} = 0$, $r_{21} \leq 1$, $r_{32} \leq 2$ and $r_{43} \leq 3$ which do not give any results but $r_{20} \leq 1$ implies $a \neq 0$. We also have $r_{30} \leq 1$ which gives $a + b = 0$ or $d = 0$ and $r_{40} \leq 2$ gives $a + b + c = 0$ or $d + e = 0$ or $f = 0$. Since $r_{31} \leq 1$, we have $d = 0$. The results $r_{41} \leq 2$ implies $d + e = 0$ or $f = 0$ and $r_{42} \leq 3$ implies

$f \neq 0$. Therefore we definitely know $d = 0$, but $a \neq 0$ and $f \neq 0$.

Then we have

$$X_{w_2} = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since ${X_{w_2}}^2 = 0$, we also have $ae + bf = 0$.

- So $Z = \{(a, b, c, d, e, f) \mid d = 0, ae + bf = 0\}$.
- $R = k[x_a, \ldots, x_f]/\sqrt{(x_d, x_a x_e + x_b x_f)}$.
- Since Y is the subvariety of matrices of $rank < 2$,

$$Y = \{(a, b, c, d, e, f) \mid d = 0, ae = 0, bf = 0, af = 0, be = 0\}$$

- $I = \sqrt{(x_d, x_a x_e, x_b x_f, x_a x_f, x_b x_e)}$.

3.2 Bracket Words

In Section 3.1, we already defined that bracket word corresponds to the valid X^2 word $(a_1, \ldots, a_{2n})$ where $a_i = 0$ if the i-th parenthesis in the bracket word is a left parenthesis, and $a_i = j$ if the i-th parenthesis which closes the j-th parenthesis. In this section, P denotes the permutation matrix corresponding to a maximal valid X^2 word. So every bracket represented by 1 in P.

Remark 3.2.1. Let $p_{ij} = 1$ where $i < j$ in P. Then $j - i$ is odd. In other words, for any 1 in P, the distance between 1 and diagonal of P is odd.

By using the definition of bracket word above, we can say the distance between left and corresponding right parenthesis is odd. So, if $a_j = i$, then $j - i$ is odd. We have already defined a_j as

$$a_j = \begin{cases} i & \text{if } Pe_j = e_i \\ 0 & \text{if } Pe_j = 0 \end{cases}$$

Therefore, if $P.e_i = 0$ and $P.e_j = e_i$, then i-th row of P, $P_i = e_j$ and j-th row of P, $P_j = 0$. So $a_j = i$ implies $p_{ij} = 1$. Thus $j - i$ is odd when $p_{ij} = 1$.

Remark 3.2.2. Let p_{ij} be coordinates of P. Then at least one $p_{i(i+1)} = 1$ where $1 \leq i \leq 2n$. In other words, P has at least one 1 on the line $j = i + 1$.

The remark is obvious since the maximal valid X^2 word is represented by a bracket word and every bracket word contains consecutive right, left parenthesis closes each other which corresponds a 1.

Remark 3.2.3. Let p_{ij} be any coordinate of P. If $p_{ij} = 1$ for some $i < j$, then there exists a permutation matrix such that its coordinates are p_{ab} where $i + 1 \leq a, b \leq j - 1$ so it has same diagonal with P.

$$\begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \begin{bmatrix} & & \\ & \mathrm{P} & \\ & & \end{bmatrix} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

By Remark 3.2.1, when $p_{ij} = 1$, we know that there exists a bracket word such that j-th parenthesis closes i-th parenthesis with $j - i$ is odd. If $j - i > 1$, then there is a bracket word in between i-th parenthesis and j-th parenthesis corresponds P such that p_{ab} are coordinates of P where $i + 1 \leq a, b \leq j - 1$.

$$\begin{array}{ccccc} \ldots & (& \text{a bracket word} &) & \ldots \\ & i & & j & \end{array}$$

If $j - i = 1$, then we have empty bracket word.

Claim 3.2.1. *If $p_{a(a+1)} = 1$, $p_{b(b+1)} = 1$ and $p_{i(i+1)} = 0$ where $a + 1 \leq i \leq b - 1$ in P, then $\forall\ p_{ij} = 0$ where $a + 1 \leq i \leq b - 1$ and $a + 2 \leq j \leq b$.*

$$\begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & 1 \quad 0 \quad \cdots & 0 & & \\ & & & \begin{bmatrix} & & \\ & 0 & \\ & & \end{bmatrix} & \begin{matrix} \vdots \\ 0 \\ 1 \end{matrix} & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

Proof. Assume $p_{a(a+1)} = 1$, $p_{b(b+1)} = 1$ and $p_{i(i+1)} = 0$ where $a+1 \leq i \leq b-1$ in P, to the contrary $p_{ij} = 1$ for some i, j such that $a+1 \leq i \leq b-1$ and $a+2 \leq j \leq b$. Then recall Remark 3.2.3, we have another permutation matrix such that coordinates are p_{cd} where $i+1 \leq c, d \leq j-1$. However, by Remark 3.2.2, we know that there must be at least one $p_{i(i+1)} = 1$ where $a+1 \leq i \leq b-1$. Since all $p_{i(i+1)} = 0$ for $a+1 \leq i \leq b-1$ in our assumption, it is a contradiction. □

Claim 3.2.2. *Let $a < b$ and p_{ab} be coordinates of $2n \times 2n$ matrix P. Assume ($a = 0$ or $a \geq 1$ and $p_{a(a+1)} = 1$), ($b = 2n$ or $b \leq 2n-1$ and $p_{b(b+1)} = 1$), ($p_{(j+1)(j+2)} = 0$ where $a \leq j \leq b-2$). Then $p_{(j+1)i} = 0$ and $w_{ij} = j$ where $a \leq j \leq b-2$, $a+2 \leq i \leq b$.*

Proof. Suppose $a < b$ and $a = 0$ or $a \geq 1$ and $p_{a(a+1)} = 1$, $b = 2n$ or $b \leq 2n-1$ and $p_{b(b+1)} = 1$, $p_{(j+1)(j+2)} = 0$ where $a \leq j \leq b-2$. Then by Claim 3.2.1, $p_{(j+1)i} = 0$ where $a \leq j \leq b-2$, $a+2 \leq i \leq b$. Recall definition of w_{ij};

$$w_{ij} = dim(P(k^i) + k^j).$$

Note that in the first i-th columns there is no 1 below a-th row so there is no 1 at the j-th row or below j-th row. Therefore, $P(k^i) \leq k^j$ and then $dim(P(k^i)+k^j) = j$ that is $w_{ij} = j$. □

Remark 3.2.4. $w_{ij} = j \Rightarrow m_{(j+1)i} = 0$ where $j - i \geq 2$.

Proposition 3.2.5. *$\forall P$, $p_{(j+1)i}$ which satisfy conditions in Claim 3.2.2 imply $m_{(j+1)i} = 0$ where $a \leq j \leq b-2$, $a+2 \leq i \leq b$.*

Example 3.2.6. Let $w = (0,0,2,0,4,1)$ be our valid X^2 word. Then $n = 3$ and corresponding bracket word is (( ) ( )). Then corresponding permutation matrix is

$$P_w = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Proposition 3.2.5, the coordinates of M_w; m_{11}, m_{12}, m_{22}, m_{33}, m_{34}, m_{44}, m_{55}, m_{56} and m_{66} are all equal to zero. So,

$$M_w = \begin{bmatrix} 0 & 0 & m_{13} & m_{14} & m_{15} & m_{16} \\ 0 & 0 & m_{23} & m_{24} & m_{25} & m_{26} \\ 0 & 0 & 0 & 0 & m_{35} & m_{36} \\ 0 & 0 & 0 & 0 & m_{45} & m_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Chapter 4

Conjecture

In this chapter, we will prove Conjecture 4.0.8 for $n = 2$. Remember that in Chapter 2, we define the coordinate ring of U_{2n} as $R(U_{2n}) = k[x_{ij} \mid i < j]$ where $1 \leq i < j \leq 2n$. Since k is a field, so that it is a Unique Factorization Domain, a polynomial ring in an arbitrary number of variables with coefficient in k is also a Unique Factorization Domain. Note that because R is a commutative ring, R/I is commutative for any ideal I in R.

Let R be a ring and M be a left R-module. The ideal

$$AnnM = \{r \in R \mid rm = 0 \text{ for all } m \in M\}$$

is called the *annihilator* of M. The *support* of M is the subset

$$SuppM = \{P \in SpecR \mid M_P \neq 0\}$$

where Spec of ring R is the set of prime ideals of R and $M_P = S^{-1}M$ is the module of fractions with respect to $S = R \backslash P$.

Proposition 4.0.7 (Proposition 3.1 in [9]). *$I \subseteq AnnH(M)$ and $Y \supseteq suppH(M)$.*

Conjecture 4.0.8 (Conjecture 3.2 in [9]). *$I = AnnH(M)$, or equivalently, $Y = suppH(M)$.*

For the following proof, we modify the notation of M. Remember that every M consists of coordinates $M_{ij} = \varphi(x_{ij})$ where $\varphi : R(U_{2n}) \to R$ such that $R(U_{2n}) = k[x_a, \ldots]$ and $R = k[\bar{x_a}, \ldots]$. But we take $\bar{x_a} = x_a$ in the following proof.

Proof. (for $n = 2$) Let $n = 2$ and suppose $x \in AnnH(M)$. Then $x.h = 0$ for $h \in H(M) = Ker(M)/Im(M)$. We can denote h as

$$h = \begin{bmatrix} k \\ l \\ m \\ n \end{bmatrix} \text{ where } h \in KerM.$$

Using the fact that $h \in KerM$,

$$Mh = \begin{bmatrix} 0 & x_a & x_b & x_c \\ 0 & 0 & x_d & x_e \\ 0 & 0 & 0 & x_f \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k \\ l \\ m \\ n \end{bmatrix} = \begin{bmatrix} lx_a + mx_b + nx_c \\ mx_d + nx_e \\ nx_f \\ 0 \end{bmatrix} = 0.$$

We have

$$lx_a + mx_b + nx_c = 0 \tag{0.1}$$

$$mx_d + nx_e = 0 \tag{0.2}$$

$$nx_f = 0. \tag{0.3}$$

Then for all k, l, m, n which satisfy Equations (0.1), (0.2) and (0.3), since $x \in AnnH(M)$ there exists l', m', n' such that

$$x \begin{bmatrix} k \\ l \\ m \\ n \end{bmatrix} = \begin{bmatrix} l'x_a + m'x_b + n'x_c \\ m'x_d + n'x_e \\ n'x_f \\ 0 \end{bmatrix}.$$

Firstly, recall Example 3.1.10 **Case**(2). We need to show $x \in I = \sqrt{(x_d, x_a x_e, x_b x_f, x_a x_f, x_b x_e)}$. We have four equations which are obtained from above equation;

$$kx = l'x_a + m'x_b + n'x_c \tag{0.4}$$

$$lx = m'x_d + n'x_e \tag{0.5}$$

$$mx = n'x_f \tag{0.6}$$

$$nx = 0. \tag{0.7}$$

We know that $x_d = 0$ in R. Thus, we can choose a representative of x which does not contain the term x_d in $R(U_{2n})$. Let us take $n = x_d$, $l = x_b$ and $m = -x_a$. Then Equations $(0.1), (0.2), (0.3)$ and (0.7) hold.

Consider Equation 0.6 in the polynomial ring $R(U_{2n})/x_d$ which is a UFD.

$$(-x_a)x = n'x_f + c(x_a x_e + x_b x_f) \text{ where } c \text{ is coefficient in } R(U_{2n}).$$

Terms which do not contain x_a in the right side of equation above must cancel each other and rest of the terms are divided by x_a. So $x = x_f k_1 + x_e k_2$ in $R(U_{2n})/x_d$ and similarly, $n' = x_a k_3 + x_b k_4$ in $R(U_{2n})/x_d$ where k_1, k_2, k_3 and k_4 are coefficients in $R(U_{2n})/x_d$.

Note that k can be chosen as equal to 1 because k does not affect image of h under M. Then

$$x = l'x_a + m'x_b + n'x_c.$$

Then x can be written as

$$x^2 = (x_f k_1 + x_e k_2)(l'x_a + m'x_b + (x_a k_3 + x_b k_4)x_c).$$

Notice that $x_a x_f$, $x_b x_f$, $x_a x_e$, $x_b x_e$ are minors thus $x^2 \in I$. Since I is radical ideal $x \in I$.

Similarly, recall Example 3.1.10 **Case**(1), let us take h' as

$$h' = \begin{bmatrix} p \\ r \\ s \\ t \end{bmatrix} \text{ where } h' \in KerM.$$

Using the fact that $h' \in KerM$, $Mh' = 0$. Then we have

$$rx_a + sx_b + tx_c = 0, \tag{0.8}$$

$$sx_d + tx_e = 0, \tag{0.9}$$

$$tx_f = 0 \text{ in } R. \tag{0.10}$$

Then for all p, r, s, t which satisfy Equations (0.8), (0.9) and (0.10), since $x \in AnnH(M)$ there exists r', s', t' in R such that

$$x \begin{bmatrix} p \\ r \\ s \\ t \end{bmatrix} = \begin{bmatrix} r'x_a + s'x_b + t'x_c \\ s'x_d + t'x_e \\ t'x_f \\ 0 \end{bmatrix}.$$

In this case, R is UFD and we need to show $x \in I = \sqrt{(x_a, x_f, x_bx_e - x_cx_d)}$. We have four equations which are obtained from above equation

$$px = r'x_a + s'x_b + t'x_c \tag{0.11}$$

$$rx = s'x_d + t'x_e \tag{0.12}$$

$$sx = t'x_f \tag{0.13}$$

$$tx = 0 \text{ in } R. \tag{0.14}$$

We know that $x_a = 0$, $x_f = 0$ in R and $x_bx_e - x_cx_d \in I$. Multiply Equation (0.8) with x_d and Equation (0.9) with x_b. Then we have

$$sx_dx_b + tx_dx_c = 0 \quad \text{and} \quad sx_dx_b + tx_ex_b = 0.$$

After subtraction, we obtain $t(x_c x_d - x_e x_b) = 0$. Since $x_c x_d - x_e x_b \neq 0$ in R, $t = 0$. Similarly, multiply Equation (0.8) with x_e and Equation (0.9) with x_c. Then we have

$$sx_b x_e + tx_e x_c = 0 \text{ and } sx_c x_d + tx_e x_c = 0.$$

After subtraction we have $s(x_b x_e - x_c x_d) = 0$. Again, since $x_c x_d - x_e x_b \neq 0$ in R, $s = 0$.

Notice that p and r can be chosen 1 since they do not affect image of h' under M. Then we obtain

$$x = s' x_b + t' x_c \text{ and } x = s' x_d + t' x_e.$$

After subtraction, we get

$$s'(x_b - x_d) + t'(x_c - x_e) = 0.$$

Because x_b, x_d, x_c, x_e are nonzero and $x_b \neq x_d$, $x_c \neq x_e$ in R, $t' = k(x_b - x_d)$ and $s' = -k(x_c - x_e)$. If we put t' and s' in Equation (0.11) or (0.12), then

$$x = k(x_b x_e - x_c x_d) \text{ where } k \in R.$$

Since x is some power of $(x_b x_e - x_c x_d)$, $x \in I$.

□

As related to Proposition 4.0.7, we will prove the following theorem.

Theorem 4.0.9. *Let M be a matrix as above. Then*

$$AnnH(M) \subseteq m \Rightarrow H(M \otimes_R R/m) \neq O$$

where m is maximal ideal of ring R.

Proof of Theorem 4.0.9. Suppose that $H(M \otimes_R R/m) = 0$. Then to show $AnnH(M) \nsubseteq m$, we need to find $r \in AnnH(M)$, but $r \notin m$. Note that we have $I \subseteq AnnH(M)$ where I is the ideal generated by all $n \times n$ minors of the universal matrix M, proved in [9]. So, the idea is taking r as nonzero $n \times n$ minor of M which is not in m. Since $I \subseteq AnnH(M)$, r must be in $AnnH(M)$.

Notice that if all $n \times n$ minors of M are in m then $\mathrm{rk}(M \otimes_R R/m) \leq n-1$. So $\dim_{R/m}(Im(M \otimes_R R/m)) \leq n-1$ and $\dim_{R/m}(Ker(M \otimes_R R/m)) \geq n+1$. Then $\dim_{R/m}(H(M \otimes_R R/m)) \geq 2$ which contradicts with $H(M \otimes_R R/m) = 0$. Therefore there exists $r \in AnnH(M)$, but $r \notin m$. □

Before giving an example of Theorem 4.0.9, we should define M as different from previous one. Now, $M_{ij} = \varphi_{ij}$ where $\varphi : R \to R/m$ such that x_i goes to $\bar{x}_i$. Notice that in following example the minor r is explicitly determined.

Example 4.0.10. In previous chapter Example 3.1.10, we calculate M for $n = 2$. Now, we show that Theorem 4.0.9 holds for $n = 2$.
Case(1) For $w_1 = 0021$, take $\bar{M} = M \otimes_R R/m$ and consider the minor

$$\bar{M}_{23} = \left[\begin{array}{c|cc} 0 & \bar{x}_b & \bar{x}_c \\ 0 & \bar{x}_d & \bar{x}_e \\ \hline 0 & 0 & 0 \end{array}\right].$$

Assume $H(\bar{M}) = 0$ and by Proposition 2.0.19, $\mathrm{rk}\,\bar{M} = 2$, then $\bar{x}_b\bar{x}_e - \bar{x}_c\bar{x}_d \neq 0$. Take an element $r = x_b x_e - x_c x_d$, i.e., $r \notin m$. Since $I \subseteq AnnH(M)$, $r \in AnnH(M)$. So $AnnH(M) \nsubseteq m$.
Case(2) For $w_2 = 0103$, consider minors in $\bar{M}_{w_2}$ such that;

$$\bar{M}_{w_2} = \begin{bmatrix} 0 & \bar{x}_a & \bar{x}_b & \bar{x}_c \\ 0 & 0 & 0 & \bar{x}_e \\ 0 & 0 & 0 & \bar{x}_f \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To prove the theorem, it is enough to find r which is in $AnnH(M)$, but not in m. Since $r \in AnnH(M)$,

$$r \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in Im(M) \text{ for } \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in Ker(M).$$

Therefore

$$\begin{aligned} rx &= x_a y' + x_b z' + x_c w', \\ ry &= x_e w', \\ rz &= x_f w', \\ rw &= 0 \end{aligned}$$

where $x_a y + x_b z + x_c w = 0$, $x_e w = 0$ and $x_f w = 0$.

We assume $H(\bar{M}) = 0$ and by Proposition 2.0.19, $\mathrm{rk}(\bar{M}) = 2$. Then ($\bar{x}_e \neq 0$ or $\bar{x}_f \neq 0$) and ($\bar{x}_a \neq 0$ or $\bar{x}_b \neq 0$). We also have $\bar{x}_a\bar{x}_e + \bar{x}_b\bar{x}_f = 0$ since $(\bar{M}_{w_2})^2 = 0$. Notice that if $\bar{x}_b = \bar{x}_f = 0$ then $\bar{x}_a \neq 0$ and $\bar{x}_e \neq 0$. It contradicts with $\bar{x}_a\bar{x}_e + \bar{x}_b\bar{x}_f = 0$. Hence ($\bar{x}_b \neq 0$ or $\bar{x}_f \neq 0$) and ($\bar{x}_a \neq 0$ or $\bar{x}_e \neq 0$).

Case(2.1) Assume $\bar{x}_a = 0$. Then $\bar{x}_e \neq 0$ and $\bar{x}_b \neq 0$. So we can take $r = x_b x_e$ which is in I so in $AnnH(M)$ but not in m. Hence $AnnH(M) \nsubseteq m$.

Case(2.2) Assume $\bar{x}_a \neq 0$. Then we must consider the cases $\bar{x}_b = 0$ and $\bar{x}_b \neq 0$.

Case(2.2.1) Assume $\bar{x}_a \neq 0$ and $\bar{x}_b = 0$. Since $\bar{x}_b = 0$, $\bar{x}_f \neq 0$. Moreover, by equation $\bar{x}_a\bar{x}_e + \bar{x}_b\bar{x}_f = 0$, $\bar{x}_e = 0$. Then take $r = x_a x_f$. Since r is in I, by Proposition 4.0.7 $r \in AnnH(M)$. Notice that it is also possible to show $r \in AnnH(M)$ without using this proposition. Firstly, r has to satisfy the following equations;

$$x_a x_f x = x_a y' + x_c w' \tag{0.15}$$

$$x_a x_f y = x_e w' \tag{0.16}$$

$$x_a x_f z = x_f w' \tag{0.17}$$

$$x_a x_f w = 0 \tag{0.18}$$

Since $x_f w = 0$, Equation (0.18) holds. Choose $w' = x_a z$ to satisfy Equation (0.17). Then Equation (0.16) holds. Finally, Equation (0.15) holds when y' is chosen properly that is $y' = x_f x - x_c z$. Therefore, there exists $r \in AnnH(M)$ which proves $AnnH(M) \nsubseteq m$.

Case(2.2.2)Assume $\bar{x}_a \neq 0$ and $\bar{x}_b \neq 0$. Then $\bar{x}_e \neq 0$ and $\bar{x}_f \neq 0$. Take $r = x_a x_b x_e x_f$. Then r is consist of products of two minors. Then by Proposition 4.0.7, $r \in AnnH(M)$ which proves that $AnnH(M) \nsubseteq m$ when $H(M \otimes R/m) = 0$.

Remark 4.0.11. Converse of Theorem 4.0.9 is true if Conjecture 4.0.8 is true.

Proof. To show that $H(M \otimes R/m) \neq 0 \Rightarrow AnnH(M) \subseteq m$, assume $H(M \otimes R/m) \neq 0$, to the contrary $AnnH(M) \nsubseteq m$. Then there exists $r \in AnnH(M)$, but $r \notin m$. Suppose Conjecture 4.0.8 is true. Then $r \in I$ and there exists $n \times n$ minor in $M \otimes_R R/m$ with nonzero determinant. Therefore the rank of this minor is n and by Proposition 2.0.19, $H(M \otimes R/m) = 0$. This is a contradiction. □

Remark 4.0.12. Consider the following commutative diagram

$$\begin{tikzcd} R^{2n} \arrow[r, "M"] \arrow[d, "\psi"'] & R^{2n} \arrow[d, "\psi"] \\ (R/m)^{2n} \arrow[r, "\bar{M}"'] & (R/m)^{2n} \end{tikzcd}$$

Assume we have $v \in Ker(M)$, $\bar{v} \notin Im(\bar{M})$ where $\bar{v} = \psi(v)$. Then $v \notin Im(M)$. So $r \in AnnH(M)$ implies $rv \in Im(M)$. Moreover $r\bar{v} \in Im(\bar{M})$. Then $r \in m$. Thus, $AnnH(M) \subseteq m$.

Bibliography

[1] ADEM, A. and BROWDER, B., *The free rank of symmetry of* $(S^n)^k$, Invent. Math. **92** (1988), $431 - 440$.

[2] BREDON, GLEN E., *Introduction to Compact Transformation Groups*, Academic Press, 1972.

[3] CARLSSON, G., *On the Homology of Finite Free* $(Z/2)^n$*-complexes*, Invent. Math. **74** (1983), no 1, $139 - 147$.

[4] CARLSSON, G., *Free* $(Z/2)^3$*-actions on finite complexes*,Algebraic Topology and Algebraic K-Theory(Princeton N.J.,1983), Ann. of Math. Stud. **113**, (1987), $332 - 344$.

[5] CARLSSON, G., *On the rank of Abelian groups acting freely on* $(S^n)^k$, Invent. Math. **69**, $393 - 400$(1982).

[6] CARLSSON, G., *Free* $(Z/2)^r$*-actions and a problem in commutative algebra*,Lecture Notes in Math, **1217**, (1985), $79 - 83$.

[7] CARTAN, H. and EILENBERG, S., *Homological Algebra*, Princeton University Press, 1956.

[8] TOM DIECK, T., *Transformation Groups*, Studies in Mathematics **8**, de Gruyter, 1987.

[9] D. KARAGUEUZIAN, B. OLIVER and J. VENTURA, *The Components of a Variety of Matrices with Square Zero and Submaximal Rank*, **2** (2002), $1 - 9$.

[10] Dummit, D. and R.M., Foote, *Abstract Algebra*, John Wiley and Sons, 2004.

[11] Eisenbud, D., *Commutative Algebra with a View Toward Algebraic Geometry*, Springer, New York, 1994.

[12] Hanke, B., *The Stable Free Rank of Symmetry of Products of Spheres*, Invent. Math. **178**, $265 - 298$(2009).

[13] Keenan, A., *Which Finite Groups Act Freely on Spheres*, 2003.

[14] Kenneth, S. Brown, *Cohomolgy of Groups*, Springer, 1982.

[15] Lück, W., *Transformation Groups and Algebraic K-Theory*, Lecture Notes in Mathematics **1408**, 1989.

[16] Milnor, J., *Groups Which Act on S^n without Fixed Points*, J. Math. **79** (1957), pp. 623-630.

[17] Mumford, D., *The Red Book of Varieties and Schemes, Second Expanded Edition*, Lecture Notes in Mathematics **1358**, Springer-Verlag, 1999.

[18] Reid, M., *Undergraduate Commutative Algebra*, Cambridge, 1995.

[19] Rothbach, B., $X^2 = 0$ *and the strictly upper triangular matrices*, preprint 2002.

[20] Smith, P. A., *Permutable Periodic Transformations*, Proc. Nat. Acad. Sci. **30** (1944), $105 - 108$.

[21] Spanier, E.H., *Algebraic Topology*, McGraw-Hill, 1966.

[22] Yalçin, E. *Group actions and group extensions*, Trans. Amer. Math. Soc. **352** $(2000), 2689 - 2700$.