ON THE EXISTENCE OF HOPF CYCLES IN OPTIMAL GROWTH MODELS WITH TIME DELAY

A Master's Thesis

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The Institute of Economic and Social Sciences
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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

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In this thesis, we analyzed the existence of cycles à la Poincaré-Andronov-Hopf in optimal growth models with time delay. The analysis builds upon a new method developed, which investigates the number of pure imaginary roots of the characteristic equation. The method was applied to the time-to-build models of Asea and Zak (1999) and Winkler (2004).

Keywords: Hopf Cycles, Optimal Growth Models, Delay.

ÖZET

ZAMAN GECİKMELİ OPTİMAL BÜYÜME MODELLERİNDE HOPF DÖNGÜLERİNİN VARLIĞI ÜZERİNE

Mustafa Kerem Yüksel

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Yrd. Doç. Dr. Hüseyin Çağrı Sağlam

Eylül 2008

Bu çalışmada zaman gecikmeli optimal büyüme modellerinde Poincaré-Andronov-Hopf tarzında döngülerin varlığı incelenmiştir. Burada kullanılan analiz karakteristik denklemlerinin saf sanal köklerinin sayısını irdeleyen yeni bir metod üzerine kurulmuştur. Bu metod Asea ve Zak (1999) ve Winkler (2004) tipi yatırım-üretim gecikmeli modellere uygulanmıştır.

Anahtar Kelimeler: Hopf Döngüleri, Optimal Büyüme Modelleri, Yatırım-Üretim Gecikmeli Modeller.

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CHAPTER 1

INTRODUCTION

1.1 Literature Survey

Just in the beginning of his monumental work *The Age of Revolution 1789-1848* (first publication 1962), which explores the world between this period, Eric J. Hobsbawn was wise to state that 'words are witnesses which often speak louder than documents' and only two sentences later he listed some words which had invented or gained meaning (in terms of their modern usage) within this period, words such as 'capitalism', 'industry', 'working class' etc. and more strikingly '(economic) crises' and 'statistics.'

Economic crises entered in economic literature as early as Jean-Baptiste Say (1803). By 1830, there were inquiries on early theories of cycles and crises and certainly there was some awareness of periodicity of times of prosperity and distress[1] (Besomi, 2008). According to Besomi (2008), one of the

[1]According to Besomi (2008) Wade (1833) supplied dates for some crises years (p. 150): 1763,1772, 1793, 1811, 1816, 1825–6. Jevons (1878) also gave years of crises: 1763, 1772–3, 1783, 1793, (1804–5?), 1815, 1825 (p. 231).

Wade, J. 1833. *History of the middle and working classes; with a popular exposition of the economical and political principles which have influenced the past and present condition of the industrious orders. Also an Appendix of prices, rates of wages, population, poor-rates, mortality, marriages, crimes, schools, education, occupations, and other statistical information, illustrative of the former and present state of society and of the agricultural, commercial, and manufactoring classes*, London: Effingham Wilson (reprinted: New York: Kelly, 1966). 2nd edition 1834, 3rd edition 1835.

Jevons, W.S. 1878 "Commercial crises and sun-spots", Pt. 1, *Nature*, vol. XIX, 14 November, pp. 33–37. Reprinted in *Investigations in Currency and Finance*, ed. by H. S. Foxwell, London: Macmillan, 1884, pp. 221–35.c(Besomi, 2008) first accounts of "waves" were by Thomas Tooke who in his 1823 publication *Thoughts and Details on the High and Low Prices of the Last Thirty Years*, attributed these crises mainly to exogeneous events such as bad seasons etc., and later incorporated some endogenous factors. Hyde Clarke (1838) was of interest with the idea that "cycles in nature and society are subject to an elementary mathematical law." (Besomi, 2008) Although Clarke was not specifically interested in economics, an enourmous literature built upon the crises and cycles in economics. Citing Besomi (2008); Coquelin[2] (1848) asserted that "commercial perturbations have become in certain countries in some degree periodical"; Lawson[3] (1848) declared these period would be five to seven years; Jevons (1878) claimed a strict periodicity of 11 years in his survey with reference to "most writers". One should note that early investigators were eager to identify the reasons to be exogeneous shocks to the system, such as wars, bad seasons, embargoes, oppressive duties, the dangers and difficulties of transportation, social unrest increasing uncertainty, arbitrary exactions, jobbing and speculations etc. The common point was that these shocks either distrupts the proper working of the system or the proper functioning of the exchange or production mechanisms (Besomi, 2008). These crises were assumed to be corrected in the course of the self-adjusting nature of the economy just after the exogenous determinant is removed.

A second group of analysts were then trying to model these cycles as a part of the natural course of the economy. These group views cycles as a resultant behaviour intrinsic to economic activity, not disjunct occurances. This approach forced them to identify the cyclic phenomenon and characterize it. Quoting Besomi (2008), the transition from the exogenous shock models to "proper theories of the cycle was a gradual process that took several decades, and was only completed at the eve of World War I with the

[2]Coquelin, C. 1848. "Les Crises Commerciales et la Liberté des Banques", *Revue des Deux Mondes XXVI*, 1 November, pp. 445–70. Abridged as Coquelin 1850. (Besomi, 2008)

[3]Lawson, J. A. 1848. *On commercial panics: a paper read before the Dublin Statistical Society*, Dublin. (Besomi, 2008)

theories of Tugan-Baranowsky, Spiethoff, Mitchell, Bouniatian, Aftalion and a few others." Once again, Wade was one of the first who "explicitly spoke of a commercial cycle intrinsic to a mercantile society" and "inseparable from mercantile pursuits." (Besomi 2008) Moreover, as the cause of fluctuations, Wade was one of the first to come up with the idea of "the lag between change in price, change in demand and change in production, on which the principal cyclical mechanism implicitly relied, becomes apparent." (Besomi, 2008)

In accordance with Besomi (2008), Persons (1926) also divides theorists into two groups (without giving exact references, but by just mentioning names) according to the their approach to cycles. We can replicate its taxonomy here. The first group consists of economist who emphasize on factors other than economic institutions:

- Periodic agricultural cycles generate economic cycles: W. S. Jevons, H. S. Jevons, H. L. Moore

- Uneven expansion in the output of organic and inorganic materials is the cause of the modern crisis: Werner Sombart

- A specific disturbance, such as an unusual harvest, the discovery of new mineral deposits, the outbreak of war, invention, or other "accident," may disturb economic equilibrium and set in motion a sequence which, however, will not repeat itself unless another specific disturbance occurs: Thornstein Veblen, Irving Fischer, A. B. Adams

- Variations in the mind of the business community (affected, of course, by specific economic disturbances) are the dominating cause of trade cycles: A. C. Pigou, Ellsworth Huntington, M. B. Hexter.

The second group economists are those who emphasize on factors related to economic institutions:

- Given our economic institutions (particularly capitalistic production and private property) it is their tendency to develop business fluctuations: Joseph Schumpeter, Gustav Cassel, E. H Vogel, R. E. May, C. F. Bickerdike.

- The capitalistic or roundabout system of production is the primary cause of business fluctuations: Arthur Spiethoff, D. H. Robertson, Albert Aftalion, T. E. Burton, G. H. Hull, L. H. Frank, T. W. Mitchell, J. M. Clark.

- Excessive accumulation of capital equipment, accompanied by maldistribution of income, is responsible for lapses from prosperity to depression: Mentor Bouniatian, Tugan-Baranowsky, John A. Hobson, M. T. England, W. H. Beveridge, N. Johannsen, E. J. Rich.

- The fluctuation of money profits is the center from which business cycles originate (eclectic theories): W. C. Mitchell, Jean Lescure, T. N. Carver.

- The nature of the flow of money and credit, under our present monetary system, is the element responsible for the interruption of business prosperity: R. G. Hawtrey, Major C. H. Douglas, W. T. Foster and Waddill Catchings, A. H. Hansen, W. C. Schluter, H. B. Hastings, H. Abbati, W. H. Wakinshaw, P. W. Martin, Bilgram and Levy.

Persons (1926) also gives the justification of this classification with reference to essential points of the theories thereafter.

One should also notice that the two groups are divided in their terminology, too, which is very apt with their theoretical background. Those who understood crises as disconnected events shaped their language accordingly with frequent use of "crises"; yet those who evaluate cycles as a part of the state of the economy exploits the use of the word "cycle". The crises theorist tried to identify to reasoning of each crisis with a particular exogenous shock which lies in the background of all the crisis. W. S. Jevons (1878), for example, thought that the sunspots with the exact periodicity of 10.45 years are the main cause of crop-failures of which he believed to be every 10.44 years and this results with an economic burst. H. S. Jevons considered heat emissions by the sun with the periodicity of 3.5 years to be prior reason of crop cycles and thus the economic cycles. Irving Fischer was the one who put forward most common causes of fluctuations as increase in the quantity of money, shock to business confidence, short crops and invention. Ellsworth Huntington, interestingly, makes a connection between business cycles and mental attitude of the community which depends on health. M. B. Hexter tried to find a link between fluctuations in birth-rate and in death-rate and fluctuations in business enterprise. (Persons, 1926) On the other hand, those who are tied with the cycles perspective tried to find a causality in the system where one state logically preceeds the other (Besomi, 2008). Joseph Schumpeter, for example, thought cycles to be "essentially a process of adapting the economic system to the gains or advances of the respective periods of expansion" (Persons, 1926). R. E. May blames increased productivity of labour; Albert Aftalion indicates the existence and the universality of the new industrial technique which has caused the appearance and repetition of economic cycles; L. H. Frank explains cycles with his theory of variations in the rates of production-consumption of consumers' goods; Mentor Bouniatian comes up with two ideas: (1) the idea that the modification of the social utility of wealth, resulting from changes in the relation between the production of goods and the need for them, is a cause of the general advance of prices in a period of prosperity [...] and of decline in a crisis, (2) the idea that the time-using capitalistic process [...] is at the basis of a period of advance." (Persons, 1926)[4]

As the theories of fluctuations improved from crises to cycles the question

[4]A more detailed list of theories and explanations can be found in Persons (1926).

"how" takes place of the question "why" (Besomi, 2008). Ragnar Frisch (1933) offered to define the dynamics in a theory within a mathematical setup[5]. Frisch and Holme (1935) tried to identify the roots a characteristic equation of a specific type of mixed difference and differential equation which occurs in economic dynamics of Michal Kalecki. (Kalecki will be discussed later.)

The crises of capitalist mode of production had also a particular place in marxist economic literature. Besomi (2008) references the "the young Friedrich Engels" who gives an elegant dialectical interpretation of the in his *Outlines of a Critique to Political Economy* (1844, pp. 433-4). Although neither Marx nor Engels put forward a complete theory of this cyclic crises, they assumed that this cycles are intrinsically embedded in the nature of capitalist production. Marx called these as "realization crises" which are based on the failure of the realization of the expected profits of the capitalist. Failure were assumed to be rooted in the overproduction of the economy due to insufficient planning, which Marx referred as the "anarchy of the capitalist production". It was Michal Kalecki who tried to find mathematical reasoning for the marxists approach in a series of papers during 1930s and later. In his one of the most influential articles, Kalecki introduced lag structure in the economy to explore the cyclic behaviour, which he showed rigorously for the first time that business cycles depends endogenously to production (investment) lags. (Kalecki, 1935) (A brief exposition of Kaleckian Model is still to be discussed with the literature that builds upon.)

Before discussing in detail the Kaleckian setup and other models, we should track the improvement of mathematical apparatus. Apparently, after a seminar by Kalecki at a meeting in the Econometric Society at Leyden, Frisch and Holme (1935) were first to analyze the roots of difference-

[5]Frisch (1933) was a model of persistent fluctuations as a result of the superposition of random exogenous schocks upon a damped system. (Besomi, 2006). These type of models will be revised later and finally evolve into real business cycle models.

differential equations of the form $\dot{y}(t) = ay(t) - cy(t-\theta)$ and characterize the main properties with respect to the roots according to the exogenous (emprical econometric) parameters a and c. It was James and Belz (1938) who contributed to the mathematics of the problem by further characterization. James and Belz (1938) suggested that "a solution of a difference-differential equation might be developed in terms of an infinite series of characteristic solutions" and investigates "the conditions under which such a development is possible." In addition to this, this paper gave methods "for determining the coefficients of the development, when it exists" and showed that the solutions of certain forms of integro-differential equations "can be given in the form of an infinite series derived from a consideration of related difference-differential equations." Hayes (1950) partially closed the literature on roots by giving the properties of the roots of transcendental equations of the form $\tau(s) = se^s - a_1e^s - a_2 = 0$ which is nothing but the resultant characteristic equation of a subset of difference-differential equations with constant coefficients, which frequently occur in dynamic economic systems with delays. As Zak (1999) points out, the first thorough analysis of a general class of Delay Differential Equations (DDEs) was by Bellman and Cooke (1963) with later fundamental work by Hale (1977).

Kalecki (1935)[6] introduced production lags, a time delay between the investment decisions and delivery of the capital goods, to show the generation of endogenous cycles. Kalecki employed a linear delay differential equation of the deviation of investment which he denoted as J.[7] The investment equation

[6] A brief exposition of the Kalecki (1935) model and its properties can be found in Zak (1999) and Szydłowski (2002). These texts reproduces Kalecki's results with contemporary techniques which are also employed in this thesis.

[7] Michal Kalecki studied the underlying forces of cycles in economy throughout his life and his bunch of theories vary from linear difference differential equation systems to exogenous factors. As Besomi (2006), in his study about Kalecki's business cycle theories, pointed out Kalecki "either failed to provide a rigorous proof of the stability of the cycle when the model was endogenous or failed to provide an explanation of the cycle relying on the properties of the economic system, resorting instead to exogenous shocks to explain the persistence of fluctuations." Kalecki interpreted cycles as the dynamic expression of the "intrinsic antagonism of capitalism" however he "acknowledged the existence of disturbing factors, from which he abstracted in order to isolate a *pure* cycle." Besomi (2006) also

was $\dot{J}(t) = AJ(t) - BJ(t-\theta)$[8]. Kalecki's models exhibit endogenous cycles by employing simple time lags in a linear DDE. Lags in the model serves two purposes: (1) Lag structure was emprically significant[9] and (2) linear ordinary differentials equations are known to be unable to give cyclic solutions while linear DDEs may exhibit endogenous cycles. Apart from showing that there can exist endogeously driven cycles in the economy rather than crises determined by exogeneous schocks, Kalecki developed the mathematical techniques to characterize the stabiliy properties in linear DDEs. Obviously, one should wait for Hayes (1950) for a full understanding of the stability properties in linear one delay DEs, although Kalecki (1935) presented a thorough stability analysis (Zak, 1999). Kaldor (1940) criticizes Kalecki (1935) by pointing out that the drawback of the model is that "the existence of an undamped cycle can be shown only as a result of a happy coincidence, of a particular constellation of the various time-lags and parameters assumed" and "the amplitude of the cycle depends on the size of the initial shock." Instead Kaldor (1940) proposed a nonlinear investment decision to obtain cycles of the economy. Inspired by Kaldor (1940), Ichimura (1954) explored the possibility of an economic system with a unique limit cycle; Chang and Smyth (1971) reexamined the model and stated the necessary and sufficient conditions of an existence of a limit cycle; Grasman and Wentzel (1994) considered the co-existence of a limit cycle and an equilibrium.The dynamics of Kaldor-Kalecki type of models have been extensively studied on a series of papers by Krawiec and Szydłowski (1999, 2000, 2001, 2005) and Krawiec, *et al.* (1999). Kaldor-Kalecki models has two mechanisms which would lead to

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reports that "Kalecki's models describes damped fluctuations around a line of stationary equilibrium and rely for the persistence o fluctuations on exogenous shocks" and moreover, all his models "crucially depend for cyclicality upon one or more reaction lags."

[8]The exact LDDE studied by M. Kalecki (1935, pp. 332) was $\dot{J}(t) = \frac{m}{\theta}J(t) - \frac{m+n\theta}{\theta}J(t-\theta)$ where m and n were assumed to constants.

[9]Kalecki (1935, pp. 337-338) estimates the lag between the curves of beginning and termination of building schemes (dwelling, industrial and public buildings) as 8 months and lags between orders and deliveries in the machinery-making industry as 6 months based on the data supplied by German *Institut fuer Konjunkturforschung*. He assumed "that the average duration of θ is 0.6 years."

cyclic behaviour, one being the nonlinearity of the investment function and the other being the time delay in investment (Krawiec and Szydłowski, 2001). Krawiec and Szydłowski (1999, 2001) proves that it is the time to build assumption rather that the nonlinear (s-shaped) investment function that leads to the generation of cycles.

The main tool in these papers for creating cycles is the Hopf bifurcation. "In 1942, Hopf published the ground-breaking work in which he presented the conditions necessary for the appearance of periodic solutions, represented in phase space by a limit cycle" (Szydłowski, 2002). With reference to the contributors of the study of the sufficient conditions under which periodic orbits occur from stationary states are called Poincaré–Andronov–Hopf theorems (These theorems are inserted just before their appropriate use in the thesis for the sake of completeness). As Kind (1999) points out generally it is easy to prove the Hopf bifurcation since it doesn't require any information on the nonlinear parts of the equation system. Moreover, in systems with the dimension higher than two, the Hopf bifurcation may be the only tool for the analysis of the cyclical equilibria, since the Poincaré-Bendixson theorem is not applicable. Furthermore, when the conditions of Hopf bifurcation is satisfied, it guarantees both the existence and uniqueness of periodic trajectories (Krawiec and Szydłowski, 1999). However, Hopf theorem gives no information on the number and the stability of closed orbits. On the other hand, nonlinear parts can be used in the calculation of a stability coefficient in order to determine the stability properties of the closed orbits (Kind, 1999). Guckenheimer and Holmes (1983, Thm 3.4.2, pp. 151-153) both gives the theory and an example in that direction. Feichtinger (1992) is an example of such a calculation in economic literature.

Zak (1999) summarized Kalecki's contribution and extended his results to a general equilibrium setup, which has been an open reseearch area until then[10]. Zak (1999) inserts a production lag into a basic one sector Solowian model and showed that the results also admits Hopf cycles under certain conditions. Later, Krawiec and Szydłowski (2004) reproduced the results and improved the analysis of the same model. Zak (1999) also copies the results of an important contribution to the literature which marked an important "false" attempt to extend the same analysis to the optimal growth models (OGM) with lags. Asea and Zak (1999) was the first to lay out the main tools and showed that there exists a cyclic behaviour in these type of model. However, this paper contains a little error on the dynamic equations which erroneously leads to Hopf cycles. The corrected characteristic equation[11] is not easy to analyze to find out whether the roots satisfy Hopf conditions, so studies afterwards turn to numerical analysis to reveal periodic behaviour. Winkler, *et al.* (2003), Winkler, *et al.* (2005), Collard, *et al.* (2006), Collard, *et al.* (2008), Brandt-Pollmann, *et al.* (2008) are among such studies.[12] Unlike Solowian systems which result with a characteristic equation of the form $h(\lambda) \stackrel{def}{=} \lambda - Ae^{-\lambda r} = 0$; in optimal growth models, one should deal with more complex characteristic equations. Apart from the nonlinearity of the utility and production functions, OGM is governed by a 2-by-2 system of equations (one for state and the other for control dynamics), so the degree of the polynomial is greater, if one can mention about degree of quasi polynomials. Collard, *et al.* (2006) numerically showed that the advanced terms in Euler equations governing the dynamic system dampens the fluctuation caused by the lags through a kind of smoothing effect (They call this phenamenon 'time-to-build echo'). Short run dynamics of time-to-build

[10]Zak (1999, pp. 325ff) also claimed that Kaleckian cycle in Kalecki (1935) was nothing but Hopf cycles.

[11]Winkler *et all.* (2003) gives the correct dynamics and characteristic equations for any utility and production function. In Collard *et all.* (2008) one can find the correct dynamics and characteristic equations for a specific concave production function ($f(k) = Ak^{\alpha}$) and in Collard *et all.* (2006) the case of CES utility function ($u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$) and the same production technology is studied.

[12]The mathematical background of the nonexistence of Hopf bifurcation will be main theme of this thesis and discussed later.

echoes was further studied by Collard, *et al.* (2008) in where one can find the associated numerical simulations. Winkler, *et al.* (2004) provides numerical solutions of models of time delay OGMs for a linear limitational production function, while Winkler, *et al.* (2005) gives a numerical analysis of a time-lagged capital accumulation OGM with Leontief type of production functions. Brandt-Pollmann, *et al.* (2008) extends the numerical solutions to objective functions with state externalities.

Dockner (1985) was of special interest since it directed a new research of Hopf cycles in economy. Dockner (1985) gave the root characteristics (local stability properties) of a 4-by-4 system of dynamic equations in a simple form, where these 4-by-4 is generally the resultant dynamics of nonlinear optimal control problems with one control and two state variables. These results have been exploited extensively by Wirl in a series of papers[13], with models of two states, one inducings an externality on the objective function. Note that the etiology of cycles in these models are the externality which is expressed with one of the state variables in objective function, rather than time delays in the evolution of states. The optimality of such cycles has been studied by Dockner and Feichtinger (1991). Optimality of cycles (in a similar two state approach) in more specific setups has also been studied. Wirl (1994) investigates cyclical optimality in a Ramsey model with wealth effects and Wirl (1995) repeats the same for renewable resource stocks can be exemplified. Wirl (1992) simplifies the findings of Dockner (1985) in economic framework of two-dimensional optimal control models and gives an economic interpretation to the necessary conditions for cyclic behaviour. Wirl (1994) repeats and extends Wirl (1992). Wirl (1997, 1999, 2002) further extend the results to optimal control problems with one state and an externality. Since the externality is not included in the Hamiltonian of the optimal control problem, the model has a 3-by-3 dynamics, yet the findings are in similar direction. Wirl (1999) constructs an

[13]See various papers in references.

environmental model and repeats the analysis. Wirl (2004) analyzes a model of optimal saving with optimal intertemporal renewable resources in terms of thresholds and cycles.

One should also mention the seminal work by Kydland and Prescott (1982). In their paper, Kydland and Prescott (1982) formulated a discrete time theoretical framework and showed that US post-war economy fitted well. This is one of the major studies that supports the idea that the time-to-build assumption contributes to the cyclical behaviour in the economy even when the simplest equilibrium growth model is employed.

In this thesis, the author tries to sharpen the analysis of one sector OGM with one control and one state variables and time delays. One of the by-products of this study is the proof of the nonexistence of Hopf bifurcation in a similar model of Asea and Zak (1999). Moreover, the nonexistence of Hopf bifurcations in OGM models of with time delays will be generalized. The main outcome of this study is the presentation of a new method for the analysis of the quasi-polynomials with a degree of two. With the employment of this method, the nonexistence of Hopf cycles in Ramsey type optimal growth models with delay was shown.

1.2 Characteristic Equation of Dynamic Systems and Its Roots

A dynamic system of differential equations induces a characteristic equation of which the placement of the roots of the equations in the complex plane gives clues about the behaviour (stability, indeterminacy etc.) of the system. The characteristic equation determines the behaviour of the system near its *steady state* (i.e. *equilibrium point*). Following Hale and Lunel (1993, pp. 17), a linear differential equation of the form $\dot{x}(t) = Ax(t) + Bx(t-r)$ has a nontrivial solution $ce^{\lambda t}$ (c, constant) if and only if $h(\lambda) \overset{def}{=} \lambda - A - Be^{-\lambda r} = 0$.

Because of the transcendental function of λ, this is not a polynomial but is the type of funtional form which is called *quasi-polynomials*. The analysis of quasi-polynomials in economics dates back to M. Kalecki (1935). In his paper, Kalecki (1935) introduced a *gestation period* to the model and ended up with a quasi-polynomial. Later, Frisch and Holme (1935) and James and Belz (1938) contributed to the literature on the characteristic solutions of mixed difference and differential equations. However, a major breakthrough in the analysis was by Hayes (1950). Hayes gave the properties of certain difference-differential equations, mainly the ones of the form $h(\lambda) \stackrel{def}{=} \lambda e^{\lambda r} - Ae^{\lambda r} - B = 0$[14]. Note that this equation is equivalent in roots with the equation above.

Periodic solutions to dynamic systems are also analyzed extensively in control theory. One way to detect limit cycles is Hopf bifurcation. Hopf bifurcation discards tedious calculations and provides a powerful and easy tool to detect limit cycles. Kind (1999) comfirms this by stating "in most cases the proof of a Hopf bifurcation is not difficult because it does not require any information on the nonlinear parts of the equation system. Moreover, in systems whose dimensions are higher than two, the Hopf bifurcation theorem may constitute the only tool for the analysis of cyclical equilibria, since the Poincaré–Bendixson theorem is not applicable in these cases". Hopf cycles appear when a fixed point loses or gains stability due to a change in a parameter and meanwhile a cycle either emerges from or collapses in to the fixed point (Asea and Zak, 1999). Under the circumstances the system can either have a stable fixed point sorrounded by an unstable cycle (called a *subcritical* Hopf bifurcation); or a stable cycle loses its stability and a stable cycle appears (called a *supercritical* Hopf bifurcation) as the parameter(s) approaches to a critical value (Asea and Zak, 1999). Both cases can be economically significantly meaningful. Supercritical case which implies a stable cycle can be considered as a stylized business cycle or growth cycles and the subcritical

[14]For a summary of the roots of certain types of quasi-polynomials, see Özbay (2000, pp. 110-113).

case can correspond to the corridor stability. (Kind, 1999)

Let us state the Poincaré-Andronov-Hopf Theorem (Hale and Koçak, 1991, Thm. 11.12, pp. 344) here for the sake of completeness:

Theorem 1 *(Poincaré-Andronov-Hopf, Hale and Koçak, 1991) Let* $\dot{\mathbf{x}} = A(\mu)\mathbf{x} + \mathbf{F}(\mu, \mathbf{x})$ *be a* C^k, *with* $k \geq 3$, *planar vector field depending on a scalar parameter* μ *such that.* $\mathbf{F}(\mu, \mathbf{0}) = \mathbf{0}$ *and* $D_x\mathbf{F}(\mu, \mathbf{0}) = \mathbf{0}$ *for all sufficienty small* $|\mu|$. *Assume that the linear part* $A(\mu)$ *at the origin has the eigenvalues* $\alpha(\mu) \pm i\beta(\mu)$ *with* $\alpha(0) = 0$ *and* $\beta(0) \neq 0$. *Furthermore, suppose that the eigenvalues cross the imaginary axis with nonzero speed, that is,* $\frac{d\alpha}{d\mu}(0) \neq 0$. *Then, in any neighborhood* U *of the origin in* $\mathbb{R}^2$ *and any given* $\mu_0 > 0$ *there is a* $\overline{\mu}$ *with* $|\overline{\mu}| < \mu_0$ *such that the differential equation* $\dot{\mathbf{x}} = A(\bar{\mu})\mathbf{x} + \mathbf{F}(\bar{\mu}, \mathbf{x})$ *has a nontrivial periodic orbit in* U.

According to the above theorem, one can summarize the sufficient conditions for Hopf Bifurcation as follows:

- (H1) $A(\mu)$ has *only* one pair of pure imaginary eigenvalues[15]. *(Pre-Hopf Condition)*[16]

- (H2) These eigenvalues cross the imaginary axis with nonzero speed, i.e., $\frac{d\alpha}{d\mu}(0) \neq 0$. *(Transverse Crossing)*

The pre-Hopf condition is necessary for Hopf Bifurcation. Therefore, if this condition is not met Hopf Bifurcation doesn't exist for the system which implies that limit cycles do not occur via Hopf Bifurcation, if not via any other way[17].

[15] Note that $A(\mu)$ is nothing but the Jacobian matrix that results from *linearization* of the system, if the system is nonlinear. If $\bar{\mathbf{x}}$ is the equilibrium point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then the linear differential equation $\dot{\mathbf{x}} = D\mathbf{f}(\bar{\mathbf{x}})\mathbf{x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) \end{pmatrix}$ is the *linear variational equation* or the *linearization* of the vector field $\mathbf{f}$ at the equilibrium point $\bar{\mathbf{x}}$. (Hale and Koçak, 1991, Defn. 9.4, pp. 267)

[16] The name is given by the author of the thesis.

[17] Asea and Zak (1999, pp. 1164ff) mentions other ways in which periodic orbits may arise. Heteroclinic orbits are given as an option, yet there are stated to be "rare".

The scope of this thesis is limited to 2-by-2 systems, if not the results can be generalized to larger dimensional systems. In a 2-by-2 dynamic system of differential equations, the characteristic equation is generally a *quadratic* one, if not a *quasi polynomial*. Below, we presented a method to determine one pair of pure imaginary eigenvalues from the characteristic equation. Define $h_1(\lambda)$ be the characteristic equation of a 2-by-2 system of differential equations of $\dot{x}$ and $\dot{u}$, which is of the form:

$$\left(\frac{\partial \dot{u}\,(t)}{\partial u\,(t)}|_{(x,u)} - \lambda\right)\left(\frac{\partial \dot{x}\,(t)}{\partial x\,(t)}|_{(x,u)} - \lambda\right) - \frac{\partial \dot{u}\,(t)}{\partial x\,(t)}|_{(x,u)}\frac{\partial \dot{x}\,(t)}{\partial u\,(t)}|_{(x,u)} = 0. \quad (1.1)$$

Define $h_2(\lambda, m)$ where $m \in \mathbb{C}$ as follows:

$$\left(\frac{\partial \dot{u}\,(t)}{\partial u\,(t)}|_{(x,u)} - \lambda - m\right)\left(\frac{\partial \dot{x}\,(t)}{\partial x\,(t)}|_{(x,u)} - \lambda + m\right) = 0. \quad (1.2)$$

Proposition 1 $\{\lambda \in \mathbb{C}|h_2(\lambda, m) = 0 \wedge h_2(\lambda, m) = h_1(\lambda)\} = \{\lambda \in \mathbb{C}|h_1(\lambda) = 0\}$

Proof *Suppose* $\lambda \in \{\mu \in \mathbb{C}|h_2(\mu, m) = 0 \wedge h_2(\mu, m) = h_1(\mu)\}$ *.Then there exists* $m \in \mathbb{C}$ *such that* $h_2(\lambda, m) = 0 \wedge h_2(\lambda, m) = h_1(\lambda)$*. But then* $h_1(\lambda) = h_2(\lambda, m) = 0$*, that is* $\lambda \in \{\mu \in \mathbb{C}|h_1(\mu) = 0\}$*, i.e.*

$$\{\lambda \in \mathbb{C}|h_2(\lambda, m) = 0 \wedge h_2(\lambda, m) = h_1(\lambda)\} \subseteq \{\lambda \in \mathbb{C}|h_1(\lambda) = 0\}$$

On the contrary, suppose $\lambda \in \{\mu \in \mathbb{C}|h_1(\mu) = 0\}$*. Now let* m *be such that*

$$m(\frac{\partial \dot{u}\,(t)}{\partial u\,(t)}|_{(x,u)} - \frac{\partial \dot{x}\,(t)}{\partial x\,(t)}|_{(x,u)}) - m^2 = -\frac{\partial \dot{u}\,(t)}{\partial x\,(t)}|_{(x,u)}\frac{\partial \dot{x}\,(t)}{\partial u\,(t)}|_{(x,u)}$$

Then we have:

$$h_1(\lambda) = \left(\frac{\partial \dot{u}\,(t)}{\partial u\,(t)}|_{(x,u)} - \lambda\right)\left(\frac{\partial \dot{x}\,(t)}{\partial x\,(t)}|_{(x,u)} - \lambda\right) - \frac{\partial \dot{u}\,(t)}{\partial x\,(t)}|_{(x,u)}\frac{\partial \dot{x}\,(t)}{\partial u\,(t)}|_{(x,u)} = 0$$

$$= \left(\frac{\partial \dot{u}\,(t)}{\partial u\,(t)}|_{(x,u)} - \lambda \right) \left(\frac{\partial \dot{x}\,(t)}{\partial x\,(t)}|_{(x,u)} - \lambda \right) + m(\frac{\partial \dot{u}\,(t)}{\partial u\,(t)}|_{(x,u)} - \frac{\partial \dot{x}\,(t)}{\partial x\,(t)}|_{(x,u)}) - m^2$$

$$= \left(\frac{\partial \dot{u}\,(t)}{\partial u\,(t)}|_{(x,u)} - \lambda - m \right) \left(\frac{\partial \dot{x}\,(t)}{\partial x\,(t)}|_{(x,u)} - \lambda + m \right) = h_2(\lambda, m).$$

Thus, $\lambda \in \{\mu \in \mathbb{C} | h_2(\mu, m) = 0 \wedge h_2(\mu, m) = h_1(\mu)\}$,

i.e.

$$\{\lambda \in \mathbb{C} | h_2(\lambda, m) = 0 \wedge h_2(\lambda, m) = h_1(\lambda)\} \supseteq \{\lambda \in \mathbb{C} | h_1(\lambda) = 0\}.$$

Therefore,

$$\{\lambda \in \mathbb{C} | h_2(\lambda, m) = 0 \wedge h_2(\lambda, m) = h_1(\lambda)\} = \{\lambda \in \mathbb{C} | h_1(\lambda) = 0\}$$

□

The proposition above declares that roots of the $h_1(\lambda) = 0$ is also the roots of $h_2(\lambda, m) = 0$ for some $m \in \mathbb{C}$, and vice versa. That is, no roots of the characteristic equation is discarded with the transformation. The point in this transformation of $h_1(\lambda)$ to $h_2(\lambda, m)$ is that now $h_2(\lambda, m)$ is a product of two polynomials (possibly quasi-polynomials if delay is incorporated in the model) which is easy to study. One can show the nonexistence of the Hopf Bifurcation by showing that there are more than one pair of pure imaginary roots to any of the polynomials of which their product constitutes the characteristic equation, so by contradicting the pre-Hopf condition. On the contrary, one can also show that pre-Hopf condition is met by simply showing that one of the polynomials admit one pair of pure imaginary roots and the other admits none.

CHAPTER 2

A GENERAL ONE SECTOR MODEL WITH DELAY

Consider the following model which will be base for the analysis of models in the thesis:

$$\max \qquad \int_0^\infty e^{-rt} f(x(t), u(t)) dt$$

subject to

$$\dot{x}(t) = g_1(x(t-d)) + g_2(u(t-\tau)) + g_3(x(t)),$$
$$x(0) = x_0 \text{ and } (x(t), \dot{x}(t)) \subset \mathbb{R}^2,$$

For all the models used, the following assumptions on parameters were made throughout the text, unless otherwise stated. The discount factor is positive ($r > 0$); the delay parameters are nonnegative if they are employed ($\tau, d \geq 0$); depreciation is nonnegative if it is used ($\delta \geq 0$). The results holds for any assumption on the utility and production functions provided that the solution exists, given their differentiability. So, suppose $f(x, u) \in C^3(\mathbb{R}^2, \mathbb{R})$ and $g_1(x) \in C^3(\mathbb{R}, \mathbb{R})$.

The corresponding Hamiltonian of the system will be:

$$H(x(t), u(t), \lambda(t), t) = \\ e^{-rt} f(x(t), u(t)) + \lambda(t) \left[g_1(x(t-d)) + g_2(u(t-\tau)) + g_3(x(t)) \right], \quad (2.1)$$

Given the standard notation for partial derivatives, i.e. $f_x \equiv \frac{\partial f}{\partial x}$, the first order conditions (FOCs) will be as follows:

$$
\begin{aligned}
H_u = 0: \quad & f_u e^{-rt} + \lambda(t+\tau) g_{2u} = 0, \\
H_x = -\dot{\lambda}(t): \quad & -\dot{\lambda}(t) = e^{-rt} f_x + \lambda(t+d) g_{1x} + \lambda(t) g_{3x}, \\
H_\lambda = \dot{x}(t): \quad & \dot{x}(t) = g_1(x(t-d)) + g_2(u(t-d)) + g_3(x(t)),
\end{aligned}
$$

After some tedious calculations which is given in appendix A

$$
\begin{aligned}
&\dot{u}(t)\left[f_{\mathrm{uu}} - \frac{f_u g_{2\mathrm{uu}}}{g_{2u}}\right] + f_{ux}\dot{x} = \\
&(r - g_{3x}(t+d)) f_u + g_{2u}\left(e^{-r\tau} f_x(t+\tau) - e^{-rd}\frac{f_u(t+d)}{g_{2u}(t+d)} g_{1x}(t+\tau)\right), \quad (2.2)
\end{aligned}
$$

Consistent with the standart assumptions of economic theory, let us concentrate on the case that

$$
g_{1u} = g_{2x} = g_{3u} = g_{2\mathrm{uu}} = g_{2ux} = 0
$$

Then, from the first order conditions, the dynamics of the DDE system will be as follows:

$$
\begin{aligned}
&f_{\mathrm{uu}}\dot{u}(t) + f_{ux}\dot{x} = \\
&\quad\left[(r - g_{3x}(t+d)) f_u + g_{2u}\left(e^{-r\tau} f_x(t+\tau) - e^{-rd}\frac{f_u(t+d)}{g_{2u}(t+d)} g_{1x}(t+\tau)\right)\right], \quad (2.3)
\end{aligned}
$$

$$
\dot{x}(t) = g_1(x(t-d)) + g_2(u(t-d)) + g_3(x(t)). \quad (2.4)
$$

Given $f_{\mathrm{uu}} \neq 0$, the steady state equations will be as follows:

$$(r - g_{3x}(x))f_u(x,u) + g_{2u}(u)\left(e^{-r\tau}f_x(x,u) - e^{-rd\frac{f_u(x,u)}{g_{2u}(u)}}g_{1x}(x)\right) = 0,$$

$$g_1(x) + g_2(u) + g_3(x) = 0.$$

In order to determine the characteristic equation of the system, we should first obtain the characteristic matrix. The elements of the characteristic matrix are as follows yet their derivation is given in the Appendix B.

$$\frac{\partial \dot{u}(t)}{\partial u(t)}|_{(x,u)} = \left(r - e^{-rd}e^{d\lambda}g_{1x} - g_{3x}\right) + \frac{f_{xu}}{f_{\mathrm{uu}}}\left(e^{-r\tau}e^{\lambda\tau} - e^{-\lambda\tau}\right)g_{2u}, \qquad (2.5)$$

$$\frac{\partial \dot{u}(t)}{\partial x(t)}|_{(x,u)} = \frac{1}{f_{\mathrm{uu}}}[(r - 2g_{3x} - e^{-rd}e^{d\lambda}g_{1x} - g_{1x}e^{-\lambda d})f_{ux}$$
$$- f_u\left(e^{-rd}e^{\lambda\tau}g_{1xx} + g_{3xx}e^{\lambda d}\right) + g_{2u}e^{-r\tau}f_{xx}e^{\lambda\tau}], \quad (2.6)$$

$$\frac{\partial \dot{x}(t)}{\partial u(t)}|_{(x,u)} = g_{2u}e^{-\lambda\tau}, \qquad (2.7)$$

$$\frac{\partial \dot{x}(t)}{\partial x(t)}|_{(x,u)} = g_{1x}e^{-\lambda d} + g_{3x}. \qquad (2.8)$$

Accordingly, the general form of the characteristic equation can be recast as:

$$\left(\frac{\partial \dot{u}(t)}{\partial u(t)}|_{(x,u)} - \lambda\right)\left(\frac{\partial \dot{x}(t)}{\partial x(t)}|_{(x,u)} - \lambda\right) - \frac{\partial \dot{u}(t)}{\partial x(t)}|_{(x,u)}\frac{\partial \dot{x}(t)}{\partial u(t)}|_{(x,u)} = 0. \qquad (2.9)$$

2.1 Extended Ramsey Setup: Standart Ramsey with Wealth Externalities

Suppose $f(x(t), u(t))$ be some utility function, $d = 0 = \tau$, $g_1(x(t)) = p(x(t)) - \delta x(t)$ for some production function and $g_2(u(t)) = -u(t)$ for the control (consumption) and state (capital) variables $u(t)$ and $x(t)$.

$$\begin{aligned}
&\max && \int_0^\infty e^{-rt} f(x(t), u(t))dt \\
&\text{subject to} && \\
& && \dot{x}(t) = p(x(t)) - u(t) - \delta x(t), \\
& && x(0) = x_0 \text{ and } (x(t), \dot{x}(t)) \subset \mathbb{R}^2,
\end{aligned}$$

This model is a simple Ramsey type optimal growth model with wealth externalities in the objective function. The corresponding Hamiltonian of the system will be:

$$H(x(t), c(t), \lambda(t), t) = e^{-rt} u(x(t), c(t)) + \lambda(t)\left[p(x(t)) - u(t) - \delta x(t)\right]. \tag{2.10}$$

The FOC will be as follows:

$$\begin{aligned}
H_c = 0: & \quad f_u e^{-rt} - \lambda(t) = 0, \\
H_x = -\dot{\lambda}(t): & \quad -\dot{\lambda}(t) = e^{-rt} f_x + \lambda(t)(p_x - \delta), \\
H_\lambda = \dot{x}(t): & \quad \dot{x}(t) = p(x(t)) - u(t) - \delta x(t)).
\end{aligned}$$

Then, from the first order conditions, the dynamics of the DE system will be as follows:

$$\dot{u}(t) = \frac{1}{f_{uu}}\left((r + \delta - p_x) f_u - f_x - f_{ux}\dot{x}\right), \tag{2.11}$$

$$\dot{x}(t) = p(x(t)) - u(t) - \delta x(t)). \tag{2.12}$$

Given $f_{\mathrm{uu}} \neq 0$, the steady state equations will be as follows:

$$\begin{aligned} (r + \delta - p_x) f_u &= f_x, \\ p(x) - \delta x &= u. \end{aligned}$$

The corresponding characteristic equation of the system will be obtained from the following elements of the characteristic matrix.

$$\frac{\partial \dot{u}(t)}{\partial u(t)}|_{(x,u)} = (r + \delta - p_x), \tag{2.13}$$

$$\frac{\partial \dot{u}(t)}{\partial x(t)}|_{(x,u)} = \frac{1}{f_{\mathrm{uu}}} \left[(r + 2(\delta - p_x))f_{ux} + f_u p_{xx} - f_{xx}\right], \tag{2.14}$$

$$\frac{\partial \dot{x}(t)}{\partial u(t)}|_{(x,u)} = -1, \tag{2.15}$$

$$\frac{\partial \dot{x}(t)}{\partial x(t)}|_{(x,u)} = p_x - \delta. \tag{2.16}$$

As mentioned before, the general form of the characteristic equation will be

$$(r + \delta - p_x - \lambda)(p_x - \delta - \lambda) + \frac{1}{f_{\mathrm{uu}}} \left[(r + 2(\delta - p_x))f_{ux} + f_u p_{xx} - f_{xx}\right] = 0. \tag{2.17}$$

This is a quadratic equation where the roots are

$$\lambda_{1,2} = \frac{r}{2} \pm \frac{\sqrt{(r + 2(\delta - p_x))^2 - \frac{4}{f_{\mathrm{uu}}}\left[(r + 2(\delta - p_x))f_{ux} + f_u p_{xx} - f_{xx}\right]}}{2}. \tag{2.18}$$

Since $r \neq 0$, there is no Hopf Bifurcation in the model because the pre-Hopf condition is not satisfied.

2.2 The Model with $\dot{x}(t) = p(x(t-d)) - \delta x(t-d) - u(t)$

Let $f(x(t), u(t))$ be an utility function and $\tau = 0$, $g_1(x(t)) = p(x(t)) - \delta x(t)$ for some production function $p(.)$ and $g_2(u(t)) = -u(t)$ for the control (consumption) and state (capital) variables $u(t)$ and $x(t)$.

$$
\begin{aligned}
&\max && \int_0^\infty e^{-rt} f(x(t), u(t)) dt \\
&\text{subject to} && \\
& && \dot{x}(t) = p(x(t-d)) - \delta x(t-d) - u(t), \\
& && x(0) = x_0 \text{ and } (x(t), \dot{x}(t)) \subset \mathbb{R}^2,
\end{aligned}
$$

This model is an extended version of the Ramsey model with time-to-build delay. Asea and Zak (1999) analyzes a simpler version where the wealth externality is omitted, which will also be the main interest here. This model is optimized here to obtain the fisrt order conditions of the most general form at hand. The corresponding Hamiltonian of the system will be:

$$
\begin{aligned}
&H(x(t), u(t), \lambda(t), t) = \\
&\qquad e^{-rt} f(x(t), u(t)) + \lambda(t)\left[p(x(t-d)) - \delta x(t-d) - u(t)\right]. \qquad (2.19)
\end{aligned}
$$

The FOCs are as follows:

$$
\begin{aligned}
H_u = 0 : &\quad f_u e^{-rt} = \lambda(t), \\
H_x = -\dot{\lambda}(t) : &\quad -\dot{\lambda}(t) = e^{-rt} f_x + \lambda(t+d)(p_x - \delta), \\
H_\lambda = \dot{x}(t) : &\quad \dot{x}(t) = p(x(t-d)) - \delta x(t-d) - u(t).
\end{aligned}
$$

Then, from the first order conditions, the dynamics of the DE system will be as follows:

$$\dot{u}(t) = \frac{1}{f_{\mathrm{uu}}}\left[rf_u - f_x - e^{-rd}f_u(t+d)(p_x - \delta) - f_{ux}\dot{x}\right], \tag{2.20}$$

$$\dot{x}(t) = p(x(t-d)) - \delta x(t-d) - u(t). \tag{2.21}$$

Given $f_{\mathrm{uu}} \neq 0$, the steady state equations are as follows:

$$\begin{aligned} &\left(r - e^{-rd}(p_x - \delta)\right) f_u = f_x, \\ &p(x) - \delta x = u. \end{aligned}$$

The corresponding characteristic equation of the system is obtained from the following elements of the characteristic matrix.

$$\frac{\partial \dot{u}(t)}{\partial u(t)}|_{(x,u)} = \left(r - e^{-rd}e^{\lambda d}(p_x - \delta)\right), \tag{2.22}$$

$$\begin{aligned} &\frac{\partial \dot{u}(t)}{\partial x(t)}|_{(x,u)} = \\ &\qquad \frac{1}{f_{\mathrm{uu}}}\left[(rf_{ux} - f_{xx} + (e^{-rd}e^{\lambda d} - e^{-\lambda d})(p_x - \delta)f_{ux} - f_u e^{-rd}p_{xx}\right], \end{aligned} \tag{2.23}$$

$$\frac{\partial \dot{x}(t)}{\partial u(t)}|_{(x,u)} = -1, \tag{2.24}$$

$$\frac{\partial \dot{x}(t)}{\partial x(t)}|_{(x,u)} = e^{-\lambda d}(p_x - \delta). \tag{2.25}$$

Characteristic equation of the dynamic system is obtained as follows

$$\begin{aligned} &\left(r - e^{-rd}e^{\lambda d}(p_x - \delta) - \lambda\right)\left(e^{-\lambda d}(p_x - \delta) - \lambda\right) + \\ &\quad \frac{1}{f_{\mathrm{uu}}}\left[(rf_{ux} - f_{xx} + (e^{-rd}e^{\lambda d} - e^{-\lambda d})(p_x - \delta)f_{ux} - f_u e^{-rd}p_{xx}\right] = 0. \end{aligned} \tag{2.26}$$

Lets switch back to Asea and Zak (1999) and find out the reasons of the nonexistence of Hopf cycles. The only difference of this model and Asea and Zak (1999) is the wealth externality in the objective function, which is absent in the beforementioned paper. To achieve the same model, we can simply assume $f_x = 0$. Then, we obtain the following characteristic equation:

$$\left(r - e^{-rd}e^{\lambda d}(p_x - \delta) - \lambda\right)\left(e^{-\lambda d}(p_x - \delta) - \lambda\right) - \frac{f_u}{f_{uu}}e^{-rd}p_{xx} = 0, \qquad (2.27)$$

The important point in this analysis is the existence of one pair of imaginary root to the characteristic equation that would lead to Hopf bifurcation. Asea and Zak (1999) unfortunately obtained an erroneous characteristic equation and showed the existence of Hopf cycles. Collard, *et al.* (2008) couldn't show the existence of such roots for the corrected equation. Actually, their conjucture was the cycles are smoothened by the advanced terms in dynamic equations of the system. This is numerically verified.

The main contribution of the thesis is that it presents a coinsize method to show whether there are pure imaginary roots to the characteristic equation or not, and whether there are one piar or more given their existence. Before applying our method it must be noted that the steady state equation reduces to $r = e^{-rd}(p_x - \delta)$. For the ease of notation, let us define $A \equiv -\frac{f_u}{f_{uu}}e^{-rd}p_{xx} \in \mathbb{R}$. Suppose that there exists

$$\alpha + i\beta = m \in \mathbb{C}$$

such that

$$\begin{aligned}\left(r - e^{-rd}e^{\lambda d}(p_x - \delta) - \lambda - m\right)\left(e^{-\lambda d}(p_x - \delta) - \lambda + m\right) = 0 \\ = \left(r - e^{-rd}e^{\lambda d}(p_x - \delta) - \lambda\right)\left(e^{-\lambda d}(p_x - \delta) - \lambda\right) + A. \qquad (2.28)\end{aligned}$$

We are interested in pure imaginary roots to the equation, so suppose there exists $\lambda = i\omega$ where $\omega \in \mathbb{R}$:

$$\left(r - e^{-rd}e^{i\omega d}(p_x - \delta) - i\omega - m\right)\left(e^{-i\omega d}(p_x - \delta) - i\omega + m\right) = 0, \tag{2.29}$$

$$m(r - e^{-rd}e^{i\omega d}(p_x - \delta) - e^{-i\omega d}(p_x - \delta)) - m^2 = A. \tag{2.30}$$

Now the equation (**??**) leads to two equations since the real and imaginary parts of the left and right hand sides should be equal. Recalling the Euler's formula which states $e^{i\theta} = \cos\theta + i\sin\theta$, the extension of the equation (**??**) is as follows:

$$\begin{aligned}(\alpha + i\beta)\left[r - (p_x - \delta)(e^{-rd} + 1)\cos d\omega - i(p_x - \delta)(e^{-rd} - 1)\sin d\omega\right]& \\ - (\alpha + i\beta)^2 = A,&\end{aligned} \tag{2.31}$$

A quick analysis of this equation states that $\beta \neq 0$ (i.e. $m \in \mathbb{C} \setminus \mathbb{R}$). If $m \in \mathbb{R}$ i.e. $\beta = 0$, then the equation becomes

$$\alpha\left[r - (p_x - \delta)(e^{-rd} + 1)\cos d\omega - i(p_x - \delta)(e^{-rd} - 1)\sin d\omega\right] - \alpha^2 = A$$

which implies that $(p_x - \delta)(e^{-rd} - 1) = r(1 - e^{rd}) = 0$ which contradicts with $rd \neq 0$. Thus $\beta \neq 0$. The two equations that are derived from the real and complex parts of the equation (**??**) are as follows:

$$\alpha\left[r-(p_x-\delta)(e^{-rd}+1)\cos d\omega\right]+\beta\left[(p_x-\delta)(e^{-rd}-1)\sin d\omega\right]$$
$$-\left(\alpha^2-\beta^2\right)=A, \quad (2.32)$$

$$-\alpha\left[(p_x-\delta)(e^{-rd}-1)\sin d\omega\right]+\beta\left[r-(p_x-\delta)(e^{-rd}+1)\cos d\omega\right]$$
$$-2\alpha\beta=0, \quad (2.33)$$

The characteristic equation was cast as

$$\underbrace{\left(r-e^{-rd}e^{\lambda d}(p_x-\delta)-\lambda-m\right)}_{\text{Polynomial 1}}\ \underbrace{\left(e^{-\lambda d}(p_x-\delta)-\lambda+m\right)}_{\text{Polynomial 2}}=0.$$

i. Let us first suppose

$$\left(e^{-i\omega d}(p_x-\delta)-i\omega+m\right)=0 \tag{2.34}$$

and ignore polynomial 1 of the equation (**??**). This will lead to two equations from the real and imaginary parts of the equality, which are;

$$\cos d\omega=\frac{-\alpha}{p_x-\delta}, \tag{2.35}$$

$$\sin d\omega=\frac{\beta-\omega}{p_x-\delta}. \tag{2.36}$$

By means of these equations:

$$|\alpha|\leq|p_x-\delta|, \tag{2.37}$$

$$|\beta-\omega|\leq|p_x-\delta|. \tag{2.38}$$

Now, substituting (**??**) and (**??**) into equations (**??**) and (**??**), we obtain that

$$\alpha\left[r+\alpha(e^{-rd}+1)\right]+\beta\left[(e^{-rd}-1)(\beta-\omega)\right]-\left(\alpha^2-\beta^2\right)=A, \tag{2.39}$$

$$-\alpha\left[(e^{-rd}-1)(\beta-\omega)\right]+\beta\left[r+\alpha(e^{-rd}+1)\right]-2\alpha\beta=0. \tag{2.40}$$

A brief analysis states that $\alpha \neq 0$. If $\alpha = 0$, then $\beta r = 0$, i.e. $r = 0$, a contradiction. With the earlier finding of $\beta \neq 0$, we found that m is neither of the form $m = \alpha \in \mathbb{R}$ nor $m = i\beta \in \mathbb{C}$, but $m = \alpha + i\beta \in \mathbb{C}$.

If we rearrange the terms of the equation (**??**):

$$\beta=\frac{-\omega(e^{-rd}-1)}{r}\alpha. \tag{2.41}$$

If we substitute β from equation (**??**) into equation (**??**), we obtain the following quadratic equation:

$$\alpha^2\left[1+\left(\frac{\omega(e^{-rd}-1)}{r}\right)^2\right]e^{-rd}+\alpha\left[r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r}\right]-A=0. \tag{2.42}$$

With every solution α to the equation (**??**), we have a corresponding β that will constitute a solution $\alpha + i\beta = m$ to the equation (**??**).

Note that if $\left(r - e^{-rd}e^{i\omega d}(p_x-\delta) - i\omega - m\right) = 0$ has no solution and this quadratic equation has only one root , then pre-Hopf condition is verified. However, if there exists two different α's to the quadratic equation, then there will definitely be more than one pure imaginary roots for the characteristic equation irrespective of the number of solutions to

$\left(r - e^{-rd}e^{i\omega d}(p_x - \delta) - i\omega - m\right) = 0$. Without any effort, this will imply that the pre-Hopf condition is not justified. This is one of the vital elements of this thesis that it provides a clear cut method for the analysis of the verification of the pre-Hopf condition.

In order to eliminate the imaginary roots and justify the existence of two distinct solutions the following relation should be justified:

$$\left(r + \frac{\left(\omega(e^{-rd} - 1)\right)^2}{r}\right)^2 + 4A\left[1 + \left(\frac{\omega(e^{-rd} - 1)}{r}\right)^2\right]e^{-rd} > 0. \tag{2.43}$$

Note that this relation holds for any parameter combination if $A > 0$. However, under the neoclassical assumptions, which are $f_u > 0$, $f_{\mathrm{uu}} < 0$ and $p_{xx} < 0$, $A = -\frac{f_u}{f_{\mathrm{uu}}}e^{-rd}p_{xx} < 0$. Therefore a further analysis should be made to determine the root characteristics of equation (**??**).

ii. We also have to concentrate on the other polynomial of the characteristic equation (**??**). So, suppose

$$\left(r - e^{-rd}e^{i\omega d}(p_x - \delta) - i\omega - m\right) = 0. \tag{2.44}$$

This implies that:

$$\cos d\omega = \frac{(r - \alpha)}{p_x - \delta}e^{rd} = \frac{(r - \alpha)}{r}, \tag{2.45}$$

$$\sin d\omega = -\frac{(\omega + \beta)}{p_x - \delta}e^{rd} = \frac{-(\omega + \beta)}{r}. \tag{2.46}$$

First of all, equations (**??**) and (**??**) insert two inequalities:

$$|r - \alpha| \leq e^{-rd}\,|p_x - \delta| = |r|\,, \tag{2.47}$$

$$|\omega + \beta| \leq e^{-rd} |p_x - \delta| = |r|. \tag{2.48}$$

Now, substituting (**??**) and (**??**) into (**??**) and (**??**), we achieve that

$$\alpha \left[r - (r - \alpha)(e^{rd} + 1) \right] - \beta \left[(\omega + \beta)(1 - e^{rd}) \right] - \left(\alpha^2 - \beta^2 \right) = A, \tag{2.49}$$

$$\alpha \left[(\omega + \beta)(1 - e^{rd}) \right] + \beta \left[r - (r - \alpha)(e^{rd} + 1) \right] - 2\alpha\beta = 0. \tag{2.50}$$

If we rearrange the terms of the equation (**??**):

$$\beta = \frac{\omega(e^{rd} - 1)}{r} \alpha. \tag{2.51}$$

If we insert β of the equation (**??**) into the equation (**??**), we obtain the following quadratic equation:

$$e^{rd}(1 + \left(\frac{\omega(e^{rd} - 1)}{r} \right)^2)\alpha^2 - \left(re^{rd} + \frac{\left(\omega(e^{rd} - 1)\right)^2}{r} \right) \alpha - A = 0. \tag{2.52}$$

In order to eliminate the imaginary roots and justify the existence of two distinct solutions the following relation should be justified:

$$\left(re^{rd} + \frac{\left(\omega(e^{rd} - 1)\right)^2}{r} \right)^2 + 4Ae^{rd} \left(1 + \left(\frac{\omega(e^{rd} - 1)}{r} \right)^2 \right) > 0. \tag{2.53}$$

Similarly we should further our studies about the roots of this equation under neoclassical assumptions $A = -\frac{f_u}{f_{uu}} e^{-rd} p_{xx} < 0$.

Obviously, number of roots to the characteristic equation (**??**) will be determined by the signs of the relations (**??**) and (**??**). Consider the following table (Let $\#m$ denote the number of roots to equations (**??**) or (**??**)):

$\#m$	eqn. (**??**)>0	eqn. (**??**)$=0$	eqn. (**??**)<0
eqn. (**??**)>0	≥ 2	≥ 2	≥ 2
eqn. (**??**)$=0$	≥ 2	≤ 2	≤ 1
eqn. (**??**)<0	≥ 2	≤ 1	$=0$

If we rewrite this table in terms of pre-Hopf condition we arrive the following:

$\#m$	eqn. (**??**)>0	eqn. (**??**)$=0$	eqn. (**??**)<0
eqn. (**??**)>0	No Hopf	No Hopf	No Hopf
eqn. (**??**)$=0$	No Hopf	≤ 2	≤ 1
eqn. (**??**)<0	No Hopf	≤ 1	No Hopf

Note that the elements of the first row and first column don't meet the pre-Hopf condition already since in these conditions there exists more than one pure imaginary roots. However relations (**??**)<0 and (**??**)<0 doesn't meet the pre-Hopf condition because there is no pure imaginary root. For the sake of completeness, we will show that the rest of the cases are not possible simultaneously.

Before all, consider the functional form in equation (**??**):

$$\left(r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r}\right)^2+4A\left[1+\left(\frac{\omega(e^{-rd}-1)}{r}\right)^2\right]e^{-rd}$$

Rearranging the terms,

$$\left(r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r}\right)^2+4A\left[1+\left(\frac{\omega(e^{-rd}-1)}{r}\right)^2\right]e^{-rd}=$$

$$\left(r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r}\right)^2+\frac{4A}{r}\left[r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r}\right]e^{-rd}=$$

$$\left(r+\frac{\big(\omega(e^{-rd}-1)\big)^2}{r}+\frac{2A}{r}e^{-rd}\right)^2-\frac{4A^2}{r^2}e^{-2rd}=$$

$$\frac{1}{r}\left(r+\frac{\big(\omega(e^{-rd}-1)\big)^2}{r}\right)\left(r^2+\big(\omega(e^{-rd}-1)\big)^2+4Ae^{-rd}\right)=$$

$$\underbrace{\frac{1}{r}}_{>0}\quad\underbrace{\left(r+\frac{\big(\omega(e^{-rd}-1)\big)^2}{r}\right)}_{>0}\quad\left(r^2+\big(\omega(e^{-rd}-1)\big)^2+4Ae^{-rd}\right). \tag{2.54}$$

It is then clear that $\left(r+\frac{\big(\omega(e^{-rd}-1)\big)^2}{r}\right)^2+4A\left[1+\left(\frac{\omega(e^{-rd}-1)}{r}\right)^2\right]e^{-rd}\gtreqless 0$ if and only if $r^2+\big(\omega(e^{-rd}-1)\big)^2+4Ae^{-rd}\gtreqless 0$.

Case 1: Suppose relations (**??**)<0 and (**??**)$=0$ hold simultaneously. These imply that

$$r^2+\big(\omega(e^{-rd}-1)\big)^2+4Ae^{-rd}<0, \tag{2.55}$$

and

$$A=-\frac{\left(r^2e^{rd}+\big(\omega(e^{rd}-1)\big)^2\right)^2}{4e^{rd}\left(r^2+\big(\omega(e^{rd}-1)\big)^2\right)}. \tag{2.56}$$

Substituting (**??**) into (**??**):

$$r^2+\big(\omega(e^{-rd}-1)\big)^2-\frac{\left(r^2e^{rd}+\big(\omega(e^{rd}-1)\big)^2\right)^2}{e^{rd}\left(r^2+\big(\omega(e^{rd}-1)\big)^2\right)}e^{-rd}=$$

$$r^2+\big(\omega(e^{-rd}-1)\big)^2-\frac{\left(r^2+e^{-rd}\big(\omega(e^{rd}-1)\big)^2\right)^2}{r^2+\big(\omega(e^{rd}-1)\big)^2}=$$

$$\frac{r^2\big(\omega(e^{rd}-1)\big)^2+r^2e^{-2rd}\big(\omega(e^{rd}-1)\big)^2-2r^2e^{-rd}\big(\omega(e^{rd}-1)\big)^2}{r^2+\big(\omega(e^{rd}-1)\big)^2}=$$

$$\frac{r^2\left(\omega(e^{rd}-1)\right)^2}{r^2+\left(\omega(e^{rd}-1)\right)^2}\left(1+e^{-2rd}-2e^{-rd}\right)=$$

$$\frac{r^2\left(\omega(e^{rd}-1)\right)^2}{r^2+\left(\omega(e^{rd}-1)\right)^2}\left(1-e^{-rd}\right)^2>0 \tag{2.57}$$

which leads to a conradiction with (**??**). Thus this case is not possible.

Case 2: Suppose relations (**??**)$=0$ and (**??**)$=0$ hold simultaneously. These imply that

$$r^2+\left(\omega(e^{-rd}-1)\right)^2+4Ae^{-rd}=0, \tag{2.58}$$

and

$$A=-\frac{\left(r^2e^{rd}+\left(\omega(e^{rd}-1)\right)^2\right)^2}{4e^{rd}\left(r^2+\left(\omega(e^{rd}-1)\right)^2\right)}. \tag{2.59}$$

Substituting (**??**) into (**??**):

$$r^2+\left(\omega(e^{-rd}-1)\right)^2-\frac{\left(r^2e^{rd}+\left(\omega(e^{rd}-1)\right)^2\right)^2}{e^{rd}\left(r^2+\left(\omega(e^{rd}-1)\right)^2\right)}e^{-rd}=$$

$$r^2+\left(\omega(e^{-rd}-1)\right)^2-\frac{\left(r^2+e^{-rd}\left(\omega(e^{rd}-1)\right)^2\right)^2}{r^2+\left(\omega(e^{rd}-1)\right)^2}=$$

$$\frac{r^2\left(\omega(e^{rd}-1)\right)^2+r^2e^{-2rd}\left(\omega(e^{rd}-1)\right)^2-2r^2e^{-rd}\left(\omega(e^{rd}-1)\right)^2}{r^2+\left(\omega(e^{rd}-1)\right)^2}=$$

$$\frac{r^2\left(\omega(e^{rd}-1)\right)^2}{r^2+\left(\omega(e^{rd}-1)\right)^2}\left(1+e^{-2rd}-2e^{-rd}\right)=$$

$$\frac{r^2\left(\omega(e^{rd}-1)\right)^2}{r^2+\left(\omega(e^{rd}-1)\right)^2}\left(1-e^{-rd}\right)^2>0. \tag{2.60}$$

which leads to a conradiction with (**??**). Thus this case is not possible.

Case 3: Suppose relations (**??**)$= 0$ and (**??**)< 0 hold simultaneously. These imply that

$$r^2 + \left(\omega(e^{-rd} - 1)\right)^2 + 4Ae^{-rd} = 0 \tag{2.61}$$

i.e., $A = -\frac{r^2+\left(\omega(e^{-rd}-1)\right)^2}{4e^{-rd}}$ and

$$\left(re^{rd} + \frac{\left(\omega(e^{rd} - 1)\right)^2}{r}\right)^2 + 4Ae^{rd}\left(1 + \left(\frac{\omega(e^{rd} - 1)}{r}\right)^2\right) < 0. \tag{2.62}$$

Substituting (**??**) into (**??**):

$$\left(re^{rd} + \frac{\left(\omega(e^{rd} - 1)\right)^2}{r}\right)^2 - \frac{r^2 + \left(\omega(e^{-rd} - 1)\right)^2}{e^{-rd}} e^{rd}\left(\frac{r^2 + \left(\omega(e^{rd} - 1)\right)^2}{r^2}\right) =$$

$$\frac{1}{r^2}[\left(r^2e^{rd} + \left(\omega(e^{rd} - 1)\right)^2\right)^2 - \left(r^2 + \left(\omega(e^{-rd} - 1)\right)^2\right) e^{2rd}\left(r^2 + \left(\omega(e^{rd} - 1)\right)^2\right)] <$$

$$\frac{1}{r^2}\left(r^2e^{rd} + \left(\omega(e^{rd} - 1)\right)^2\right)\left[r^2e^{rd} - r^2e^{2rd}\right] =$$

$$e^{rd}\left(r^2e^{rd} + \left(\omega(e^{rd} - 1)\right)^2\right)\left[1 - e^{rd}\right] < 0, \tag{2.63}$$

Even if at first glance (**??**)$= 0$ and (**??**)< 0 seems to be consistent. However, further investigation on the roots wil lead a contradiction. $r^2 +$

$\left(\omega(e^{-rd}-1)\right)^2 + 4Ae^{-rd} = 0$ implies that equation (**??**), i.e.,

$$\alpha^2 \left[1 + \left(\frac{\omega(e^{-rd}-1)}{r}\right)^2\right] e^{-rd} + \alpha \left[r + \frac{\left(\omega(e^{-rd}-1)\right)^2}{r}\right] - A = 0$$

has only one (double) root. Then;

$$\alpha = -\frac{r}{2}e^{rd}, \tag{2.64}$$

By equation (**??**):

$$\beta = \frac{\omega(1-e^{rd})}{2}, \tag{2.65}$$

Then by equations (**??**) and (**??**),

$$\cos d\omega = \frac{1}{2}\frac{re^{rd}}{p_x - \delta} = \frac{1}{2}, \tag{2.66}$$

$$\sin d\omega = \frac{\omega(\frac{1-e^{rd}}{2} - 1)}{p_x - \delta} = -\frac{1}{2}\frac{\omega(e^{rd}+1)}{p_x - \delta}, \tag{2.67}$$

Then $d\omega = \pm\frac{\pi}{3}$. Suppose $d\omega = \frac{\pi}{3}$, then $\sin d\omega = \frac{\sqrt{3}}{2} = -\frac{1}{2}\frac{\omega(e^{rd}+1)}{p_x-\delta}$, i.e. $\omega = -\frac{\sqrt{3}(p_x-\delta)}{(e^{rd}+1)} = -\frac{\sqrt{3}re^{rd}}{(e^{rd}+1)}$. That is, $0 > \frac{-\pi}{3\sqrt{3}} = \frac{rde^{rd}}{(e^{rd}+1)} > 0$, a contradiction. Now, suppose $d\omega = -\frac{\pi}{3}$, then $\sin d\omega = \frac{-\sqrt{3}}{2} = -\frac{1}{2}\frac{\omega(e^{rd}+1)}{p_x-\delta}$, i.e. $\omega = \frac{\sqrt{3}(p_x-\delta)}{(e^{rd}+1)} = \frac{\sqrt{3}re^{rd}}{(e^{rd}+1)}$. That is, $0 > \frac{-\pi}{3\sqrt{3}} = \frac{rde^{rd}}{(e^{rd}+1)} > 0$, another contradiction. Thus, even if there is only one pure imaginary root, this root is not consistent with the rest of the system. Therefore, this case is not also possible.

Finally, after showing that the three cases are also not possible we can conclude that no matter what the sign of the relations (**??**) and (**??**) pre-Hopf condition is not verified. Therefore Hopf cycles for this type of optimal growth models with time-to-build delay is not analytically possible.

2.3 The Model with $\dot{x}(t) = p(x(t-d)) - u(t-d) - \delta x(t)$

Suppose $f(x(t), u(t))$ be some utility function, $\tau = d$, $g_1(x(t)) = p(x(t))$ and $g_3(x(t)) = -\delta x(t)$ for some production function and $g_2(u(t)) = -u(t)$ for the control (consumption) and state (capital) variables $u(t)$ and $x(t)$.

$$
\begin{aligned}
&\max && \int_0^\infty e^{-rt} f(x(t), u(t))dt \\
&\text{subject to} && \\
& && \dot{x}(t) = p(x(t-d)) - u(t-d) - \delta x(t), \\
& && x(0) = x_0 \text{ and } (x(t), \dot{x}(t)) \subset \mathbb{R}^2,
\end{aligned}
$$

This is another type of delay structure in the literature.Winkler (2004) analyzes a simpler version where the wealth externality is omitted, which will also be the main interest here. The corresponding Hamiltonian of the system is:

$$
\begin{aligned}
&H(x(t), u(t), \lambda(t), t) = \\
&\qquad e^{-rt} f(x(t), u(t)) + \lambda(t)\left[p(x(t-d)) - u(t-d) - \delta x(t)\right]. \qquad (2.68)
\end{aligned}
$$

The FOCs are as follows:

$$
\begin{aligned}
H_u &= 0: & f_u e^{-rt} &= \lambda(t+d), \\
H_x &= -\dot{\lambda}(t): & -\dot{\lambda}(t) &= e^{-rt} f_x + \lambda(t+d) p_x - \lambda(t)\delta, \qquad (2.69) \\
H_\lambda &= \dot{x}(t): & \dot{x}(t) &= p(x(t-d)) - u(t-d) - \delta x(t).
\end{aligned}
$$

After some tedious calculations:

$$\dot{u}\,(t)\, f_{\mathrm{uu}} = (r-\delta) f_u - \left(e^{-rd} f_x(t+d) + e^{-rd} f_u(t+d) p_x(t+d)\right) - f_{ux}\dot{x}. \quad (2.70)$$

Now in correlation with the assumptions of the economic theory, assume

$$g_{1u} = g_{2x} = g_{3u} = g_{2\mathrm{uu}} = g_{2ux} = 0 \quad (2.71)$$

Then, from the first order conditions, the dynamics of the DE system are as follows:

$$\begin{aligned} &\dot{u}\,(t) + \frac{f_{ux}}{f_{\mathrm{uu}}}\dot{x} = \\ &\qquad \frac{1}{f_{\mathrm{uu}}}\left[(r-\delta) f_u - \left(e^{-rd} f_x(t+d) + e^{-rd} f_u(t+d) p_x(t+d)\right)\right], \end{aligned} \quad (2.72)$$

$$\dot{x}\,(t) = p\,(x(t-d)) - u(t-d) - \delta x(t). \quad (2.73)$$

Given $f_{\mathrm{uu}} \neq 0$, the steady state equations are as follows:

$$\begin{aligned} &(r + \delta - e^{-rd} p_x(x)) f_u = e^{-rd} f_x, \\ &p(x) - \delta x = u. \end{aligned}$$

The corresponding characteristic equation of the system is obtained from the following elements of the characteristic matrix.

$$\frac{\partial \dot{u}\,(t)}{\partial u\,(t)}|_{(x,u)} = \left(r + \delta - e^{-rd} e^{d\lambda} p_x\right) - \frac{f_{xu}}{f_{\mathrm{uu}}}\left(e^{-rd} e^{\lambda d} - e^{-\lambda d}\right), \quad (2.74)$$

$$\frac{\partial \dot{u}(t)}{\partial x(t)}|_{(x,u)} = \frac{1}{f_{\mathrm{uu}}}[(r+2\delta - e^{-rd}e^{d\lambda}p_x - p_x e^{-\lambda d})f_{ux} - f_u e^{-rd}e^{\lambda d}p_{xx} - e^{-rd}f_{xx}e^{\lambda d}], \quad (2.75)$$

$$\frac{\partial \dot{x}(t)}{\partial u(t)}|_{(x,u)} = -e^{-\lambda d}, \quad (2.76)$$

$$\frac{\partial \dot{x}(t)}{\partial x(t)}|_{(x,u)} = p_x e^{-\lambda d} - \delta. \quad (2.77)$$

Consider the model without wealth externality in the model, i.e. $f_x = 0$. Then the characteristic equation reduces:

$$\left(r+\delta - e^{-rd}e^{d\lambda}p_x - \lambda\right)\left(p_x e^{-\lambda d} - \delta - \lambda\right) - \frac{f_u}{f_{\mathrm{uu}}}e^{-rd}p_{xx} = 0. \quad (2.78)$$

Note that the steady state condition turns into $r+\delta = e^{-rd}p_x$. Moreover, for the ease of notation assume $A \equiv -\frac{f_u}{f_{\mathrm{uu}}}e^{-rd}p_{xx} \in \mathbb{R}$. Suppose there exists

$$\alpha + i\beta = m \in \mathbb{C}$$

such that

$$\left(r+\delta - e^{-rd}e^{d\lambda}p_x - \lambda - m\right)\left(p_x e^{-\lambda d} - \delta + m\right) = 0$$
$$= \left(r+\delta - e^{-rd}e^{d\lambda}p_x - \lambda\right)\left(p_x e^{-\lambda d} - \delta - \lambda\right) + A. \quad (2.79)$$

We are interested in pure imaginary roots to the equation, so suppose there exists $\lambda = i\omega$ where $\omega \in \mathbb{R}$. So the equations become;

$$\left(r + \delta - e^{-rd} e^{d\lambda} p_x - \lambda - m\right) \left(p_x e^{-\lambda d} - \delta - \lambda + m\right) = 0, \tag{2.80}$$

$$m(r + 2\delta - \left(e^{-rd} e^{d\lambda} + e^{-\lambda d}\right) p_x) - m^2 = A. \tag{2.81}$$

Substituting m in (**??**):

$$(\alpha + i\beta) \left[r + \delta - p_x(e^{-rd} + 1) \cos d\omega - i(e^{-rd} - 1) p_x \sin d\omega\right] \\ - (\alpha + i\beta)^2 = A. \tag{2.82}$$

A quick analysis of this equation states that $\beta \neq 0$ (i.e. $m \in \mathbb{C} \setminus \mathbb{R}$). If $m \in \mathbb{R}$ i.e. $\beta = 0$, then this will imply that $(e^{-rd} - 1)p_x = 0$ which contradicts with $rd \neq 0$. Thus $\beta \neq 0$. Now the equation (**??**) leads to two equations from the real and imaginary parts.

$$\alpha \left[r + \delta - p_x(e^{-rd} + 1) \cos d\omega\right] + \beta(e^{-rd} - 1) p_x \sin d\omega \\ - \left(\alpha^2 - \beta^2\right) = A, \tag{2.83}$$

$$-\alpha(e^{-rd} - 1) p_x \sin d\omega + \beta \left[r + \delta - p_x(e^{-rd} + 1) \cos d\omega\right] - 2\alpha\beta = 0. \tag{2.84}$$

The characteristic equation was cast as:

$$\underbrace{\left(r + \delta - e^{-rd} e^{d\lambda} p_x - \lambda - m\right)}_{\text{Polynomial 1}} \underbrace{\left(p_x e^{-\lambda d} - \delta - \lambda + m\right)}_{\text{Polynomial 2}} = 0.$$

i. Let us first suppose

$$p_x e^{-\lambda d} - \delta - \lambda + m = 0. \tag{2.85}$$

and ignore polynomial 1 of (**??**). From the real and imaginary parts;

$$\cos d\omega = \frac{\delta - \alpha}{p_x}, \tag{2.86}$$

$$\sin d\omega = \frac{\beta - \omega}{p_x}. \tag{2.87}$$

By means of these equations:

$$|\delta - \alpha| \leq p_x, \tag{2.88}$$

$$|\beta - \omega| \leq p_x. \tag{2.89}$$

Now, inserting (**??**) and (**??**) into (**??**) and (**??**), we obtain

$$\alpha\left[r - \delta e^{-rd} + \alpha(1 + e^{-rd})\right] + \beta(e^{-rd} - 1)(\beta - \omega) - \left(\alpha^2 - \beta^2\right) = A, \tag{2.90}$$

$$-\alpha(e^{-rd} - 1)(\beta - \omega) + \beta\left[r - \delta e^{-rd} + \alpha(1 + e^{-rd})\right] - 2\alpha\beta = 0. \tag{2.91}$$

If we rearrange the terms of the equation (**??**),

$$\beta = \frac{-\omega(e^{-rd} - 1)}{r - \delta e^{-rd}}\alpha. \tag{2.92}$$

If we insert β into equation (**??**), we obtain the following quadratic equation:

$$\alpha^2\left[1+\left(\frac{\omega(e^{-rd}-1)}{r-\delta e^{-rd}}\right)^2\right]e^{-rd}+\alpha\left[r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right]-A=0. \quad (2.93)$$

Note that this is similiar with equation (**??**). The condition for the existence of real roots is as follows,

$$\left(r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right)^2+4A\left[1+\left(\frac{\omega(e^{-rd}-1)}{r-\delta e^{-rd}}\right)^2\right]e^{-rd}>0. \quad (2.94)$$

Under the neoclassical assumptions, $f_u>0$, $f_{\mathrm{uu}}<0$ and $p_{xx}<0$. Thus, $A=-\frac{f_u}{f_{\mathrm{uu}}}e^{-rd}p_{xx}<0$. We should further our studies about the roots of this equation under neoclassical assumptions.

ii. We should also have to concentrate on the other part of the characteristic equation (**??**). Now suppose

$$\left(r+\delta-e^{-rd}e^{d\lambda}p_x-\lambda-m\right)=0.$$

Then

$$\cos d\omega=\frac{(r+\delta-\alpha)}{p_x}e^{rd}, \quad (2.95)$$

$$\sin d\omega=\frac{-(\omega+\beta)}{p_x}e^{rd}. \quad (2.96)$$

First of all, these equations insert two inequalities:

$$|r+\delta-\alpha|\leq e^{-rd}p_x, \quad (2.97)$$

$$|\omega+\beta| \leq e^{-rd}p. \tag{2.98}$$

Now, inserting (**??**) and (**??**) into (**??**) and (**??**), we achieve that

$$\alpha\left[\alpha(1+e^{rd})-(r+\delta)\,e^{rd}\right]-\beta(1-e^{rd})(\omega+\beta) \\ -\left(\alpha^2-\beta^2\right)=A, \tag{2.99}$$

$$\alpha(1-e^{rd})(\omega+\beta)+\beta\left[\alpha(1+e^{rd})-(r+\delta)\,e^{rd}\right]-2\alpha\beta=0. \tag{2.100}$$

If we rearrange the terms of the equation (**??**),

$$\beta=\frac{\omega(1-e^{rd})}{e^{rd}\,(r+\delta)}\alpha. \tag{2.101}$$

If we insert β into (**??**), we obtain the following quadratic equation:

$$e^{rd}(1+\left(\frac{\omega(1-e^{rd})}{e^{rd}\,(r+\delta)}\right)^2)\alpha^2-\left((r+\delta)\,e^{rd}+\frac{\left(\omega(1-e^{rd})\right)^2}{e^{rd}\,(r+\delta)}\right)\alpha-A=0. \tag{2.102}$$

For the elimination of the imaginary solutions and the existence of two distinct solutions we should the following relation should hold,

$$\left((r+\delta)\,e^{rd}+\frac{\left(\omega(1-e^{rd})\right)^2}{e^{rd}\,(r+\delta)}\right)^2+4Ae^{rd}\left(1+\left(\frac{\omega(1-e^{rd})}{e^{rd}\,(r+\delta)}\right)^2\right)>0. \tag{2.103}$$

Now, we have to analyze equations (**??**) and (**??**) so that we can determine the number of roots the characteristic equation (**??**). Consider the following table (Let $\#m$ denote the number of roots to equations (**??**) or (**??**)) :

#m	eqn. (**??**)>0	eqn. (**??**)$=0$	eqn. (**??**)<0
eqn. (**??**)>0	≥ 2	≥ 2	≥ 2
eqn. (**??**)$=0$	≥ 2	≤ 2	≤ 1
eqn. (**??**)<0	≥ 2	≤ 1	$=0$

If we rewrite this table in terms of pre-Hopf condition we arrive the following:

#m	eqn. (**??**)>0	eqn. (**??**)$=0$	eqn. (**??**)<0
eqn. (**??**)>0	No Hopf	No Hopf	No Hopf
eqn. (**??**)$=0$	No Hopf	≤ 2	≤ 1
eqn. (**??**)<0	No Hopf	≤ 1	No Hopf

Note that the elements of the first row and first column don't meet the pre-Hopf condition already since in these conditions there exists more than one pure imaginary roots. However relations (**??**)<0 and (**??**)<0 doesn't meet the pre-Hopf condition because there is no pure imaginary root. For the sake of completeness, we will show that the rest of the cases are not possible simultaneously.

Before all, consider the functional form of equation (**??**), i.e.,

$$\left((r+\delta)\,e^{rd}+\frac{\left(\omega(1-e^{rd})\right)^2}{e^{rd}\,(r+\delta)}\right)^2+4Ae^{rd}\left(1+\left(\frac{\omega(1-e^{rd})}{e^{rd}\,(r+\delta)}\right)^2\right).$$

Rearranging the terms

$$\left((r+\delta)\,e^{rd}+\frac{\left(\omega(1-e^{rd})\right)^2}{e^{rd}\,(r+\delta)}\right)^2+4Ae^{rd}\left(1+\left(\frac{\omega(1-e^{rd})}{e^{rd}\,(r+\delta)}\right)^2\right)=$$

$$\left(\left(r+\delta\right)e^{rd}+\frac{\left(\omega(1-e^{rd})\right)^2}{e^{rd}\left(r+\delta\right)}+\frac{2A}{\left(r+\delta\right)}\right)^2-\frac{4A^2}{\left(r+\delta\right)^2}=$$

$$\left(\left(r+\delta\right)e^{rd}+\frac{\left(\omega(1-e^{rd})\right)^2}{e^{rd}\left(r+\delta\right)}\right)\left(\left(r+\delta\right)e^{rd}+\frac{\left(\omega(1-e^{rd})\right)^2}{e^{rd}\left(r+\delta\right)}+\frac{4A}{\left(r+\delta\right)}\right)=$$

$$\frac{e^{rd}}{\left(r+\delta\right)}\left(\left(r+\delta\right)e^{rd}+\frac{\left(\omega(1-e^{rd})\right)^2}{e^{rd}\left(r+\delta\right)}\right)\left(\left(r+\delta\right)^2+\left(\omega(e^{-rd}-1)\right)^2+4Ae^{-rd}\right)=$$

$$\underbrace{\left(e^{2rd}+\left(\frac{\omega(1-e^{rd})}{\left(r+\delta\right)}\right)^2\right)}_{>0}\left(\left(r+\delta\right)^2+\left(\omega(e^{-rd}-1)\right)^2+4Ae^{-rd}\right). \tag{2.104}$$

So, we conclude that $\left(\left(r+\delta\right)e^{rd}+\frac{\left(\omega(1-e^{rd})\right)^2}{e^{rd}(r+\delta)}\right)^2+4Ae^{rd}\left(1+\left(\frac{\omega(1-e^{rd})}{e^{rd}(r+\delta)}\right)^2\right)\gtreqless 0$ if and only if $\left(r+\delta\right)^2+\left(\omega(e^{-rd}-1)\right)^2+4Ae^{-rd}\gtreqless 0$.

Case 1: Suppose (**??**)<0 and (**??**)$=0$ simultaneously. These imply that

$$\left(r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right)^2+4A\left[1+\left(\frac{\omega(e^{-rd}-1)}{r-\delta e^{-rd}}\right)^2\right]e^{-rd}<0 \tag{2.105}$$

and

$$A=-\frac{\left(\left(r+\delta\right)^2+\left(\omega(e^{-rd}-1)\right)^2\right)}{4e^{-rd}}. \tag{2.106}$$

Substituting A;

$$\left(r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right)^2 - \left((r+\delta)^2+\left(\omega(e^{-rd}-1)\right)^2\right)\left[1+\left(\frac{\omega(e^{-rd}-1)}{r-\delta e^{-rd}}\right)^2\right]=$$

$$\frac{1}{(r-\delta e^{-rd})^2}[\left(r\left(r-\delta e^{-rd}\right)+\left(\omega(e^{-rd}-1)\right)^2\right)^2 - \left((r+\delta)^2+\left(\omega(e^{-rd}-1)\right)^2\right)\left(\left(r-\delta e^{-rd}\right)^2+\left(\omega(e^{-rd}-1)\right)^2\right)]=$$

$$\frac{-\delta}{(r-\delta e^{-rd})^2}\left[(2r+\delta)\left(r-\delta e^{-rd}\right)^2+\left(2r+\delta(1+e^{-rd})\right)\right]<0. \quad (2.107)$$

At first glance (**??**)<0 and (**??**)$=0$ seems consistent. However, further investigation wil lead a contradiction. (**??**)$=0$ implies that

$$e^{rd}(1+\left(\frac{\omega(1-e^{rd})}{e^{rd}\left(r+\delta\right)}\right)^2)\alpha^2-\left(\left(r+\delta\right)e^{rd}+\frac{\left(\omega(1-e^{rd})\right)^2}{e^{rd}\left(r+\delta\right)}\right)\alpha-A=0. \quad (2.108)$$

has only one (double) root. Then;

$$\alpha=\frac{r+\delta}{2}, \quad (2.109)$$

$$\beta=\frac{\omega(1-e^{rd})}{2e^{rd}}. \quad (2.110)$$

Then,

$$\cos d\omega=\frac{(r+\delta)}{2p_x}e^{rd}=\frac{1}{2},$$

$$\sin d\omega=-\frac{1}{2}\frac{\omega(e^{rd}+1)}{p_x}. \quad (2.111)$$

Then $d\omega=\pm\frac{\pi}{3}$. Suppose $d\omega=\frac{\pi}{3}$, then $\sin d\omega=\frac{\sqrt{3}}{2}=-\frac{1}{2}\frac{\omega(e^{rd}+1)}{p_x}$, i.e. $\omega=-\frac{\sqrt{3}p_x}{(e^{rd}+1)}=-\frac{\sqrt{3}(r+\delta)e^{rd}}{(e^{rd}+1)}$. That is, $0>\frac{-\pi}{3\sqrt{3}}=\frac{\sqrt{3}(r+\delta)de^{rd}}{(e^{rd}+1)}>0$, a contradiction. Now, suppose $d\omega=-\frac{\pi}{3}$, then $\sin d\omega=\frac{-\sqrt{3}}{2}=-\frac{1}{2}\frac{\omega(e^{rd}+1)}{p_x}$, i.e. $\omega = \frac{\sqrt{3}p_x}{(e^{rd}+1)} = \frac{\sqrt{3}(r+\delta)e^{rd}}{(e^{rd}+1)}$. That is, $0 > \frac{-\pi}{3\sqrt{3}} = \frac{\sqrt{3}(r+\delta)de^{rd}}{(e^{rd}+1)} > 0$, another contradiction. Thus, even if there is only one pure imaginary root, this root is not consistent with the rest of the system. Therefore, pre-Hopf condition is not met for this case.

Case 2: Suppose (??)= 0 and (??)= 0 simultaneously. These imply that

$$\left(r + \frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right)^2 + 4A\left[1 + \left(\frac{\omega(e^{-rd}-1)}{r-\delta e^{-rd}}\right)^2\right]e^{-rd} = 0 \quad (2.112)$$

and

$$A = -\frac{\left((r+\delta)^2 + \left(\omega(e^{-rd}-1)\right)^2\right)}{4e^{-rd}}. \quad (2.113)$$

Substituting A;

$$\left(r + \frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right)^2 - \frac{\left((r+\delta)^2 + \left(\omega(e^{-rd}-1)\right)^2\right)}{e^{-rd}}\left[1 + \left(\frac{\omega(e^{-rd}-1)}{r-\delta e^{-rd}}\right)^2\right]e^{-rd} =$$

$$\left(r + \frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right)^2 - \left((r+\delta)^2 + \left(\omega(e^{-rd}-1)\right)^2\right)\left[1 + \left(\frac{\omega(e^{-rd}-1)}{r-\delta e^{-rd}}\right)^2\right] =$$

$$\frac{1}{(r-\delta e^{-rd})^2}[\left(r\left(r-\delta e^{-rd}\right) + \left(\omega(e^{-rd}-1)\right)^2\right)^2 - \left((r+\delta)^2 + \left(\omega(e^{-rd}-1)\right)^2\right)\left(\left(r-\delta e^{-rd}\right)^2 + \left(\omega(e^{-rd}-1)\right)^2\right)] =$$

$$\frac{-\delta}{(r-\delta e^{-rd})^2}\left[(2r+\delta)\left(r-\delta e^{-rd}\right)^2+\left(2r+\delta(1+e^{-rd})\right)\right]<0. \quad (2.114)$$

This contradicts with equation (**??**). Therefore this case is also not possible.

Case 3: Suppose (**??**)$=0$ and (**??**)<0 simultaneously. These imply that

$$\left(r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right)^2+4A\left[1+\left(\frac{\omega(e^{-rd}-1)}{r-\delta e^{-rd}}\right)^2\right]e^{-rd}=0, \quad (2.115)$$

and

$$(r+\delta)^2+\left(\omega(e^{-rd}-1)\right)^2+4Ae^{-rd}<0. \quad (2.116)$$

Substituting A;

$$\frac{1}{r-\delta e^{-rd}}\left[r-\delta e^{-rd}+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right]\left[(r+\delta)^2+\left(\omega(e^{-rd}-1)\right)^2\right] - \left(r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right)^2>$$

$$\frac{1}{r-\delta e^{-rd}}\left[r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right]\left[(r+\delta)^2+\left(\omega(e^{-rd}-1)\right)^2\right] - \left(r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right)^2=$$

$$\left(r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right)\left(\frac{(r+\delta)^2}{r-\delta e^{-rd}}-r\right)=$$

$$\delta\left(r+\frac{\left(\omega(e^{-rd}-1)\right)^2}{r-\delta e^{-rd}}\right)\left(\frac{2r+\delta+e^{-rd}}{r-\delta e^{-rd}}\right)>0. \quad (2.117)$$

However, this is contradicting to equation (??). Therefore this case is also not possible. Note that these steps are correct irrespespective of the sign of $r - \delta e^{-rd}$.

Finally, after showing that the three cases are also not possible we can conclude that no matter what the sign of the relations (??) and (??) pre-Hopf condition is not verified. Therefore Hopf cycles for this type of optimal growth models with this time-to-build delay structure is not analytically possible.

CHAPTER 3

CONCLUSION

The main outcome of this study is the presentation of a new method for the analysis of the quasi-polynomials with a degree of two[1]. With the employment of this method, the nonexistence of Hopf cycles in Ramsey type optimal growth models with delay was shown. The existence of Hopf cycles was of interest especially after Asea and Zak (1999), yet their cycle was a result of an unfortunate erroneous characteristic equation. Collard, *et al.* (2008) conjectures a damping oscillation in the resultant dynamics due to the advanced term in governing dynamic equations. Collard, *et al.* (2008) employed a numerical simulation to expose the conjecture. The thesis proves the nonexistence of the periodic Hopf cycles in OGM with delay analytically for the first time.

The analysis relies upon the number of pure imaginary roots to the characteristic equation. The Hopf cycles can only be obtained under the existence of only one pair of pure imaginary roots. With the employment of the technique presented here, the study shows that the characteristic equation of Ramsey type optimal growth models leads to zero or two pure imaginary roots. This, in turn, guarantees the nonexistence of Hopf cycles, since the

[1]Degree of quasi-polynomials is an abuse of language. Here the quasi-polynomials of degree two implies a functional form

$$(e^{\pm\lambda\tau} + c_1 - \lambda)(e^{\pm\lambda\tau} + c_2 - \lambda) = 0.$$

pre-Hopf condition is not satisfied.

The study can be extended in many ways. The case with wealth externalities which is incorporated in the objective function is one extension that comes to mind, naturally. The complexity of the resultant equations are one of the obstacles that prevents such an analysis in a brief way. The study can also be extended so that it covers endogenous discounting, embodied and disembodied technical change etc.

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APPENDIX

Finding $\dot{u}$ in the general model:

From the first order conditions

$$H_u = 0: \quad f_u e^{-rt} + \lambda\,(t+\tau)\,g_{2u} = 0, \tag{3.1}$$

$$H_x = -\dot{\lambda}\,(t): \quad -\dot{\lambda}\,(t) = e^{-rt} f_x + \lambda(t+d) g_{1x} + \lambda(t) g_{3x}, \tag{3.2}$$

$$H_\lambda = \dot{x}\,(t): \quad \dot{x}\,(t) = g_1\,(x(t-d)) + g_2\,(u(t-d)) + g_3\,(x(t)). \tag{3.3}$$

Now taking the time derivative of equation (**??**), we obtain:

$$-r f_u e^{-rt} + e^{-rt}(f_{ux}\dot{x} + f_{\mathrm{uu}}\dot{u}) + \dot{\lambda}\,(t+\tau)\,g_{2u} + \lambda\,(t+\tau)\,g_{2\,\mathrm{uu}}\dot{u} = 0, \tag{3.4}$$

From equation (**??**), we also obtain the following equalities:

$$\lambda\,(t+\tau) = -e^{-rt}\frac{f_u}{g_{2u}}, \tag{3.5}$$

$$\lambda\,(t) = -e^{-r(t-\tau)}\frac{f_u(t-\tau)}{g_{2u}(t-\tau)}, \tag{3.6}$$

$$\lambda\,(t+d) = -e^{-r(t+d-\tau)}\frac{f_u(t+d-\tau)}{g_{2u}(t+d-\tau)}. \tag{3.7}$$

Substituting equation (**??**) and (**??**) into (**??**), we obtain:

$$\begin{aligned}\dot{\lambda}\,(t) = -e^{-rt} f_x + e^{-r(t+d-\tau)}\frac{f_u(t+d-\tau)}{g_{2u}(t+d-\tau)} g_{1x} \\ + e^{-r(t-\tau)}\frac{f_u(t-\tau)}{g_{2u}(t-\tau)} g_{3x},\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\dot{\lambda}\,(t+\tau) &= -e^{-r(t+\tau)} f_x\,(t+\tau) \\ &+ e^{-r(t+d)}\frac{f_u(t+d)}{g_{2u}(t+d)} g_{1x}\,(t+\tau) + e^{-rt}\frac{f_u}{g_{2u}} g_{3x}\,(t+\tau)\,.\end{aligned} \tag{3.9}$$

Using equation (**??**), (**??**) and (**??**), we find the following equation

$$\left[-e^{-r\tau}f_x\left(t+\tau\right)+e^{-rd}\frac{f_u(t+d)}{g_{2u}(t+d)}g_{1x}\left(t+\tau\right)+\frac{f_u}{g_{2u}}g_{3x}\left(t+\tau\right)\right]g_{2u}$$
$$-rf_u+\left(f_{ux}\dot{x}+f_{\mathrm{uu}}\dot{u}\right)-\frac{f_u}{g_{2u}}g_{2\,\mathrm{uu}}\dot{u}=0, \quad (3.10)$$

and finally by rearranging terms

$$\dot{u}\left(t\right)\left[f_{\mathrm{uu}}-\frac{f_u g_{2\,\mathrm{uu}}}{g_{2u}}\right]+f_{ux}\dot{x}=$$
$$\left(r-g_{3x}(t+d)\right)f_u+g_{2u}\left(e^{-r\tau}f_x(t+\tau)-e^{-rd}\frac{f_u(t+d)}{g_{2u}(t+d)}g_{1x}(t+\tau)\right), \quad (3.11)$$

Derivation of the Elements of the Characteristic Matrix:

From the equation (3.11),

$$\frac{\partial\dot{u}\left(t\right)}{\partial u\left(t\right)}|_{(x,u)}=\frac{1}{f_{\mathrm{uu}}}[rf_{\mathrm{uu}}-f_{ux}\left(g_{1u}e^{-\lambda d}+g_{2u}e^{-\lambda\tau}+g_{3u}\right)$$
$$+g_{2\,\mathrm{uu}}\left(e^{-rd}f_x-e^{-r\tau}\frac{f_u}{g_{2u}}g_{1x}\right)-g_{3x}f_{\mathrm{uu}}-g_{3xu}f_u e^{\lambda d}$$
$$+g_{2u}\left(e^{-r\tau}f_{xu}e^{\lambda\tau}-e^{-rd}\frac{\left(f_{\mathrm{uu}}e^{d\lambda}g_{1x}+f_u e^{\lambda\tau}g_{1xu}\right)-\frac{g_{2\,\mathrm{uu}}}{g_{2u}}f_u g_{1x}e^{\lambda\tau}}{g_{2u}}\right)], \quad (3.12)$$

Now since $g_{1u}=g_{2x}=g_{3u}=g_{2\,\mathrm{uu}}=g_{2ux}=0$;

$$\frac{\partial\dot{u}\left(t\right)}{\partial u\left(t\right)}|_{(x,u)}=\frac{1}{f_{\mathrm{uu}}}[rf_{\mathrm{uu}}-f_{ux}g_{2u}e^{-\lambda\tau}$$
$$+g_{2u}\left(e^{-r\tau}f_{xu}e^{\lambda\tau}-e^{-rd}\frac{f_{\mathrm{uu}}e^{\lambda d}g_{1x}}{g_{2u}}\right)-g_{3x}f_{\mathrm{uu}}] \quad (3.13)$$

$$=\frac{1}{f_{\mathrm{uu}}}[\left(r-e^{-rd}e^{d\lambda}g_{1x}-g_{3x}\right)f_{\mathrm{uu}}-f_{ux}g_{2u}e^{-\lambda\tau}+g_{2u}e^{-r\tau}f_{xu}e^{\lambda\tau}] \quad (3.14)$$

$$=\left(r-e^{-rd}e^{d\lambda}g_{1x}-g_{3x}\right)+\frac{f_{xu}}{f_{\mathrm{uu}}}\left(e^{-r\tau}e^{\lambda\tau}-e^{-\lambda\tau}\right)g_{2u}. \quad (3.15)$$

Similarly,

$$
\begin{aligned}
\frac{\partial \dot{u}(t)}{\partial x(t)}|_{(x,u)} = \frac{1}{f_{\mathrm{uu}}}[&(r-g_{3x})f_{ux} - g_{3xx}f_u e^{\lambda d} \\
&+ g_{2ux}\left(e^{-r\tau}f_x - e^{-rd}\frac{f_u}{g_{2u}}g_{1x}\right) - f_{ux}\left(g_{1x}e^{-\lambda d} + g_{3x}\right) \\
&+ g_{2u}\left(e^{-r\tau}f_{xx}e^{\lambda\tau} - e^{-rd}\frac{(f_{ux}e^{d\lambda}g_{1x} + f_u e^{\lambda\tau}g_{1xx}) - \frac{g_{2ux}}{g_{2u}}f_u g_{1x}e^{\lambda\tau}}{g_{2u}}\right)],
\end{aligned}
\tag{3.16}
$$

With the same assumptions, i.e., $g_{1u} = g_{2x} = g_{3u} = g_{2\,\mathrm{u}\,\mathrm{u}} = g_{2ux} = 0$;

$$
\begin{aligned}
\frac{\partial \dot{u}(t)}{\partial x(t)}|_{(x,u)} = \frac{1}{f_{\mathrm{uu}}}[&(r - 2g_{3x} - e^{-rd}e^{d\lambda}g_{1x} - g_{1x}e^{-\lambda d})f_{ux} \\
&- f_u\left(e^{-rd}e^{\lambda\tau}g_{1xx} + g_{3xx}e^{\lambda d}\right) + g_{2u}e^{-r\tau}f_{xx}e^{\lambda\tau}],
\end{aligned}
\tag{3.17}
$$