N-TANGLE KANENOBU KNOTS WITH THE SAME JONES POLYNOMIALS

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By
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July, 2010

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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July, 2010

It is still an open question if there exists a non-trivial knot whose Jones polynomial is trivial. One way of attacking this problem is to develop a mutation on knots which keeps the Jones polynomial unchanged yet alters the knot itself. Using such a mutation; Eliahou, Kauffmann and Thistlethwaite answered, affirmatively, the analogous question for links with two or more components.

In a paper of Kanenobu, two types of families of knots are presented: a 2-parameter family and an n-parameter family for $n \geq 3$. Watson introduced braid actions for a generalized mutation and used it on the (general) 2-tangle version of the former family. We will use it on the n-tangle version of the latter. This will give rise to a new method of generating pairs of prime knots which share the same Jones polynomial but are distinguishable by their HOMFLY polynomials.

Keywords: braid action, Jones polynomial, Kanenobu knot, mutation, tangle.

ÖZET

ORTAK JONES ÇOKTERİMLİSİNE SAHİP N-DOLANIMLI KANENOBU DÜĞÜMLERİ

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Jones çokterimlisi 1 olan aşikâr düğüm haricinde bir düğüm olup olmadığı halen açık bir sorudur. Bu soruya saldırmanın bir yolu Jones çokterimlisini değiştirmeyip düğümün kendisini değiştiren bir düğüm dönüşümü bulmaktır. Eliahou, Kauffmann ve Thistlethwaite, bu şekilde, iki veya daha fazla parçaya sahip girişik halkalar için soruya olumlu yanıt vermişlerdir.

Kanenobu bir makalesinde, iki çeşit düğüm ailesini sunmuştur: 2-değişkenli aile ve n-değişkenli aile, $n \geq 3$. Watson daha genel bir dönüşüm için örgü etkilerini tanıtmış ve bunları ilk ailenin (genel) 2-dolanımlı sürümüne uygulamıştır. Biz de bu dönüşümü ikinci ailenin n-dolanımlı sürümüne uygulayacağız. Bu da aynı Jones çokterimlisine sahip fakat HOMFLY çokterimlisiyle ayrılabilen asal düğüm çiftleri üretmenin yeni bir yolunu verecektir.

Anahtar sözcükler: örgü etkisi, Jones çokterimlisi, Kanenobu Düğümü, dönüşüm, dolanım.

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Chapter 1

Introduction

Let us consider a piece of string tangled in two different ways and glued at both ends. It is a natural question to ask if one of the resulting objects can be deformed to the other without cutting the string. One can try to play with them and if lucky enough show that it is possible to do so. However, if that is not the case, we are left with the questions how to discriminate them or up to what extent the variety of these inequivalent objects can scale. This type of questions were, indeed, problems of early knot tabulators [T], [Lt] where a knot is represented mathematically by a homeomorphism from the unit circle to the 3-dimensional space $\mathbb{R}^3$ (or S^3).

Many knot invariants are developed to check inequivalence of knots. One of the most important of these is the Jones polynomial [J1], [J2] which is a Laurent polynomial with integer coefficients. In spite of its popularity, it is still unknown if there is a non-trivial knot whose Jones polynomial is 1. To get such a knot, one can try to use a "mutation" on unknot which keeps the Jones polynomial same but changes the unknot to a non-trivial knot. For this reason, study of mutations is closely related to the open problem. In this paper, we will give a new family of mutants which depend on a sequence of tangles.

In the first four chapters, basic concepts of the modern theory of knots will be introduced briefly. We start with clarifying what we mean by equivalence of knots and proceed with classical result of Reidemeister [R], which allows us to deal with diagrams of knots in two dimensions rather than knots themselves. Our next task is to define and examine the properties of both numerical and polynomial invariants. In the fourth chapter, Conway's very useful concept of tangle will be considered along with its relation to 2-bridge knots and we will shortly touch to the theory of braids.

In the fifth chapter, a review of some necessary background and tools for our construction takes place. In particular, it will be clear why we need a mutation distorting the HOMFLY polynomial in regard to the open question. A key part of underlying linear algebra machinery of the work of Eliahou, Kauffmann, Thistlethwaite [EKT] and Watson [W] is the idea of the skein module which was originally due to Conway and is later formalized by Rolfsen [Rf]. Watson formalized the mutation appeared in [EKT] by means of braid actions and applied it to 2-tangle Kanenobu knots. We will use it on n-tangle Kanenobu knots.

The last chapter constitutes the main part of this thesis. We define, study and generalize the original n-parameter family of Kanenobu knots for $n \geq 3$. Using an argument similar to [W], we show that the Jones polynomial is fixed by the mutation for our n-tangle family of knots. We then prove that the pairs obtained by the mutation do not have the same HOMFLY polynomial which in turn implies they are not Conway mutants. Remember that if we have any chance to solve the main problem we must avoid ending up with Conway mutants. Next, we give a simple condition to assure primeness of our family for arbitrary prime tangles inside. We close with some illustrations of our construction and further generalizations.

Chapter 2

Knots and Links

A *link* of n components is a homeomorphism from n copies of the unit circle into the 3-dimensional space $\mathbb{R}^3$ (or S^3). In particular, when $n = 1$ we have a *knot*. We will confuse this homeomorphism with its embedded image when we refer to a knot or link. We can also consider the projection of a knot to a 2-dimensional space, in our case, $\mathbb{R}^2$ is possible and enough. Nevertheless, we are to put some restrictions on this projection to prevent some pathologies. Firstly, to give the information of which string is above the other in the projection, the overpass-underpass diagrams will be used as in Fig. 2.1. We then impose the conditions

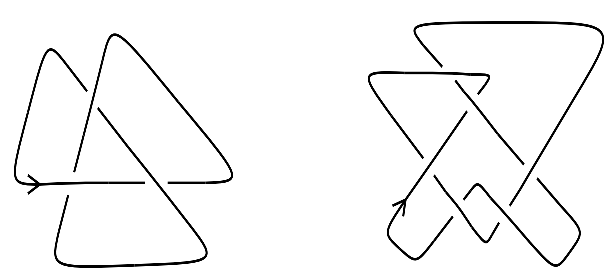


Figure 2.1: Knot Diagrams

on diagrams that no three strings intersect at one point and no string intersects another one non-transversely. In addition to these information, it is possible to assign an orientation to knots (or links). Depending on the context, we may sometimes omit the orientation.

2.1 Equivalence of knots

The natural idea of equivalence of knots comes essentially from deforming one knot to another. This can be paraphrased as if there exists an isotopic deformation h_t of $\mathbb{R}^3$ such that h_0 is identity and h_1 sends one knot to another then we want these two knots to be equivalent. Intuitive idea is close to the actual definition, *albeit* not exactly the same at first sight.

Definition 2.1.1. *Two knots (or links) K_1 and K_2 are said to be equivalent if there is an orientation preserving homeomorphism of $\mathbb{R}^3$ sending K_1 to K_2.*

Equivalent knots in the sense of deformation are also equivalent under this definition. Inverse question is, however, not trivial. Nevertheless, it is known [F] that every orientation preserving homeomorphism of $\mathbb{R}^3$ onto itself is realizable by an isotopic deformation. We shall also point out that there are knots whose mirror images are not equivalent to themselves yet they are not listed separately in the knot tables. Additionally, we say that a link is *tame* provided it is equivalent to a link which is a finite union of straight line segments. In particular, continuously differentiable links parametrized by arc length are tame [CF]. *Nota bene*, we will work with tame links only, from now on.

Having a relaxed condition to check equivalence, there is still room to reduce the process of comparison to just a few basic moves.

2.2 Reidemeister moves

Given two equivalent links L_1 and L_2, their diagrams are related by a sequence of the moves demonstrated in the Fig. 2.2, called *Reidemeister moves.* It is quite useful to reduce all types of other moves down to three moves. Particularly, when one candidate of knot invariant is being checked, it is enough to verify that it respects just these three moves.

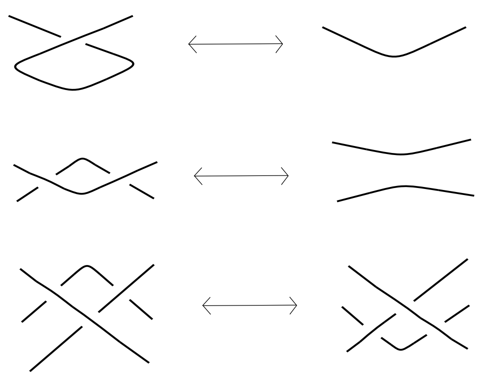


Figure 2.2: Reidemeister Moves

2.3 Knot decompositions

A *trivial knot* or *unknot* is defined to be a knot which bounds a copy of embedded disc in 3-dimensional space. Taking this as starting point, more complicated knots can be built using known ones. Let K be an oriented knot and S be an embedded sphere intersecting K at exactly two points transversely. Consider the part of K within S and take its union with an oriented simple curve $c \in S$ in a way that orientations match up. Let us call this new knot K_1 and we obtain K_2 by repeating the procedure for outside of S. In this case, K is said to be the *sum* (or *connected sum*) of K_1 and K_2, denoted $K_1 + K_2$. (See Fig. 2.3) Summation for links is defined exactly in the same fashion except sums of two links via different components may result in inequivalent links. If a knot has only

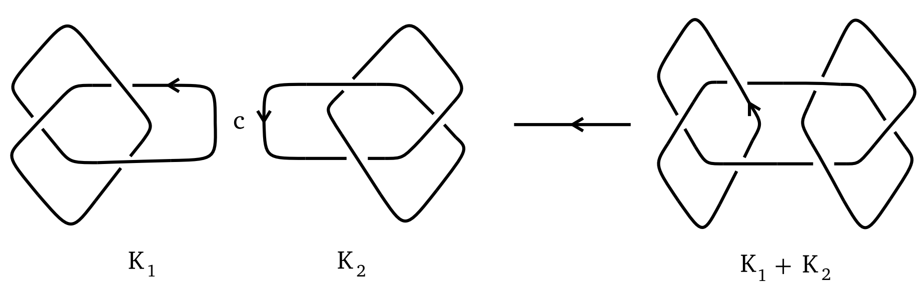


Figure 2.3: Decomposition of knots

trivial decomposition then we call it *prime*. In general, we can define a link to be prime if every sphere intersecting it at two points, transversely, bounds an *unknotted spanning arc* on one side of it. At this point, we can refer reader to

[S1] to see that such a decomposition to primes is always finite and unique up to reordering.

The concept of genus of a knot turns out to be useful for producing information about primeness.

Definition 2.3.1. *A connected, compact, orientable surface assuming a link L to be its boundary is called a Seifert surface of L.*

Theorem 2.3.2 (Seifert Algorithm)**.** *Any oriented link has a Seifert surface.*

Proof. In the diagram of L, crossings can be distorted as in Fig. 2.4, so that we end up with non-intersecting oriented *circuits*. We can assign each circuit a disc and connect these discs by half twisted strips. If the remaining object is not connected, small discs on disconnected components are removed and their boundaries are replaced with boundaries of tubes. It can be seen at once that this surface satisfies all desired properties. □

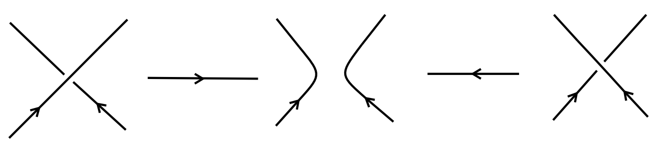


Figure 2.4: Distortion of crossings

Now we can define *genus* $g(K)$ of a knot K as

$$g(K) = \min \{ \text{ genus}(S) : S \text{ is a Seifert surface of } K \}$$

In particular, $g(unknot) = 0$. Moreover, it is well-known that genus is additive (*cf.* [L1], pg. 17-18) which implies that only unknot has additive inverse, prime decomposition is finite, a knot of genus 1 is prime and there are infinitely many distinct knots. This suggests why knot tables are designed to include prime knots only.

Chapter 3

Invariants

As we have mentioned earlier, classifying knots requires *invariants* to distinguish them, that is some functions from the set of all knots to an abstract domain such that values on the equivalence classes are the same. Equivalently, they respect Reidemeister moves. Historically, numerical knot invariants are introduced first. Most of them are easy to define but difficult to compute. Thus, more calculable invariants appeared as the theory progressed. Alexander polynomial is first such effective polynomial invariant. It was followed by the Jones polynomial and the HOMFLY polynomial. (Sometimes called the 2-variable Jones polynomial or the HOMFLYPT polynomial.)

3.1 Numerical invariants

The *crossing number* $c(K)$ of a knot K is defined to be the minimum possible number of crossing over all diagrams of the knot. Knot tabulations and enumerations are made in an increasing order of crossing number, yet there is a big difficulty working with this type of data which depends on arbitrary number of diagrams. For example, additivity of crossing number is another open question. Nonetheless, one of the applications of the Jones polynomial proved this to hold [M1] for alternating knots, *i.e.* those whose crossings alternate, or for even wider family of links [L1], chapter 5.

We can always consider a knot in $\mathbb{R}^3$ living partially in the xy-plane and partially in the upper half space. Let us call the pieces of strings above the plane, *bridges*, then we can define the *bridge number* $br(K)$ of a knot K to be the minimum number of bridges over all diagrams of K. For example, knots with bridge number 2 are completely classified. (See section 4.2)

Similarly, the *unknotting number* $u(K)$ of a knot K is the minimum number of overpass-underpass violations in the diagram of K to make it into an unknot. We can assign each crossing a sign as in Fig. 3.1 and define the *linking number*

Figure 3.1: Signs of crossings

$lk(K_1, K_2)$ of two oriented knots K_1 and K_2 to be the half of the sum of signs of all crossings where one string is from K_1 and other one is from K_2. A simple check of the effect of Reidemeister moves on $lk(K_1, K_2)$ shows that linking number is an invariant of oriented links of two components. The *writhe* $w(K)$ of an oriented knot K is defined, in a similar fashion, to be the sum of signs of all self-intersections of K. The writhe itself is not really a knot invariant but it plays a crucial rôle to make the Kauffmann bracket into the Jones polynomial invariant. (See section 3.3)

3.2 The Alexander polynomial

Given a link L of n components, let F be its Seifert surface with genus g. Consider the first homology group of F with integer coefficients, which can be given by ([M2], [L1])

$$H_1(F; \mathbb{Z}) = \oplus_{2g+n-1}\mathbb{Z}$$

where generators are equivalence classes of oriented simple closed curves $f_i \in F$. In general; for a connected, compact, orientable surface S with boundary, there corresponds [L1] a unique bilinear form

$$\beta : H_1(S;\mathbb{Z}) \times H_1(S^3 \backslash S;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

such that $H_1(S^3\backslash S;\mathbb{Z})$ is isomorphic to $H_1(S;\mathbb{Z})$ and $\beta([c],[d]) = lk(c,d)$ for oriented simple closed curves $c \in S^3\backslash S$, $d \in S$. Now, let N be a tubular neighborhood of F and X be the closure of $S^3\backslash N$. We update F as $X \cap F$ and take a neighborhood $F \times [-1,1]$ of $F = F \times 0$ in X. Letting $i^{\pm}(x) = x \times \pm 1 = x^{\pm}$ for $x \in F$, the *Seifert form* of F

$$\alpha : H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

is defined by $\alpha(f,h) = \beta(f, i_*^+(h)) = lk(f,h^+)$. The matrix A of this bilinear form is called the *Seifert matrix* for F. Let X_i be a copy of the closure of $X\backslash(F\times(-1,1))$ and X_∞ be the space obtained by gluing $F \times -1$ of ∂X_i to $F \times 1$ of ∂X_{i+1}. Defining $t : X_\infty \longrightarrow X_\infty$ by $t(X_i) = X_{i+1}$ canonically, $\langle T \rangle$ acts on X_∞ as a group of homeomorphisms. Thus, the group ring $\mathbb{Z}\langle t\rangle = \mathbb{Z}[t^{\pm}]$ has an induced action on $H_1(X_\infty;\mathbb{Z})$, consequently $H_1(X_\infty;\mathbb{Z})$ is a $\mathbb{Z}[t^{\pm}]$-module, called the *Alexander module*. Now, $B = tA - A^\tau$ is a square presentation matrix for $H_1(X_\infty;\mathbb{Z})$ (*cf.* [L1]), *i.e.* B is a transition matrix from the basis f_i of F to the basis e_i of E for the exact sequence

$$F \longrightarrow E \longrightarrow H_1(X_\infty;\mathbb{Z}) \longrightarrow 0.$$

Finally, the *Alexander polynomial* $\Delta_L(t)$ of L is defined, up to a power of $\pm t^{\pm}$, to be $\det(B)$. There is a way to calculate Alexander polynomial via Fox calculus [CF] as follows. Let G be a finite group and $\mathbb{Z}G$ be its group ring.

Definition 3.2.1. *A map $D : \mathbb{Z}G \to \mathbb{Z}G$ is called a derivative if*

(i) $D(v_1 + v_2) = Dv_1 + Dv_2$

(ii) $D(v_1v_2) = Dv_1\tau v_2 + v_1Dv_2$

where $\tau : \mathbb{Z}G \to \mathbb{Z}$ *is defined by* $\tau(g) = 1$ *for* $g \in G$ *and* $v_1, v_2 \in \mathbb{Z}G$*.*

Indeed, a derivative is the unique linear extension of any mapping $D : G \to \mathbb{Z}G$ to $\mathbb{Z}G$ satisfying

$$D(g_1g_2) = Dg_1 + g_1Dg_2$$

Clearly, any derivative is uniquely determined by its values on any generating subset of G. We are interested in derivatives in the group ring of a free group which is the reason Fox calculus is sometimes called free calculus.

Let $F = F(\vec{x})$ be a free group with basis $\vec{x} = (x_1, x_2, \ldots)$ so that $\mathbb{Z}F$ becomes the ring of finite sums of finite products of powers of x_i's, in other words *free polynomials* in x_i's, in defiance of negative powers and non-commutativity of x_i's.

Theorem 3.2.2. *To each free generator* x_j*, there corresponds a unique derivative* $D_j = \dfrac{\partial}{\partial x_j}$ *in* $\mathbb{Z}F$ *satisfying* $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$*.*

A detailed proof can be found in [CF]. Now, suppose that

$$F \xrightarrow{\gamma} F/R = |\vec{x} : \vec{r}| = G$$

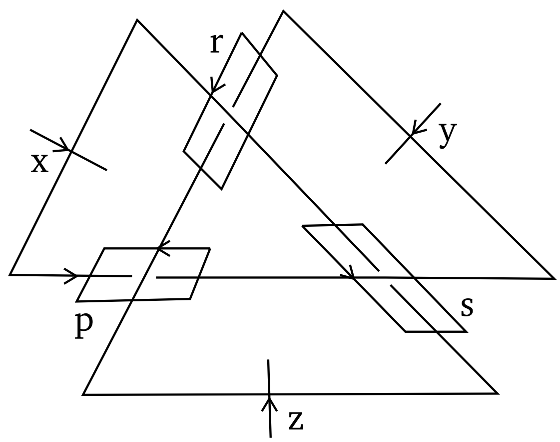
where R is the smallest free normal subgroup containing all *relators* $\vec{r} = (r_1, r_1, \ldots)$. Also, taking $H := G/[G,G]$ and $\alpha : \mathbb{Z}G \to \mathbb{Z}(G/[G,G])$ to be the extension of $G \to G/[G,G]$, we have a composition

$$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x_j}} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}|\vec{x} : \vec{r}| \xrightarrow{\alpha} \mathbb{Z}H$$

Here, Alexander matrix $[a_{ij}]$ of $|\vec{x} : \vec{r}|$ is given by

$$a_{ij} := \alpha\gamma \frac{\partial r_i}{\partial x_j}$$

The reason we are dealing with presentations of groups is because there is a constructive way ([CF], chapter 6) to deduce a finite presentation $|\vec{x} : \vec{r}|$ of the *knot group*, $\pi(\mathbb{R}^3 \backslash K)$, for a prescribed knot K. This is achieved by assigning each overpass strand a generator vector and then obtaining relators by taking tours around the boundaries of thin neighborhoods of underpass strands. (See Fig. 3.2) Now, the Alexander polynomial $\Delta(K)$ of a knot K can be computed



$p = z^{-1}x^{-1}zy$, $r = xzx^{-1}y^{-1}$; $s = y^{-1}z^{-1}yx$

Figure 3.2: $\pi(S^3 \backslash K) = |x, y, z : p, r, s|$

as the greatest common divisor of the determinants of all $(n-1) \times (n-1)$ sub matrices of the Alexander matrix (a.k.a. generators of the *first elementary ideal* of the Alexander matrix) of the finite presentation of the knot group of K. There are some technicalities about this construction which are examined at length in [CF]. For example, it is shown that all generators can be reduced to one generator which becomes the variable of the polynomial and the g.c.d. always exists.

This approach provides a computational algorithm. For example, results [K] of Kanenobu about the Alexander polynomial of the n-parameter family of knots are obtained by making use of Fox calculus. The advantage of the topological approach, on the other hand, is its practical use to derive the following skein relation for normalized Alexander polynomials. (*cf.* [Kw], pg.105)

(i) $\Delta(\bigcirc) = 1$

(ii) $\Delta(\nwarrow\!\!\!\!\nearrow) - \Delta(\nearrow\!\!\!\!\nwarrow) = (t^{1/2} - t^{-1/2})\Delta()()$

Here, symbols are used to denote the same links except for the part drawn inside. This theorem also shows that Alexander polynomial is symmetric.

3.3 The Kauffmann bracket and the Jones polynomial

Even though Jones polynomial was discovered out of certain algebras [J1], we will define it in a different and simpler way due to Kauffmann.

Definition 3.3.1. *The Kauffmann bracket $\langle D\rangle$ of an unoriented diagram D is the polynomial in $\mathbb{Z}[A^{-1}, A]$ defined by the relations:*

(i) $\langle \bigcirc \rangle = 1$

(ii) $\langle D \amalg \bigcirc \rangle = (-A^{-2} - A^{2})\langle D\rangle$

(iii) $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$

where bracket symbols represent almost same links except the parts drawn inside the brackets and $\langle \bigcirc \rangle$ represents the unknot. One can easily verify the following properties of Kauffmann bracket.

(i) $\langle \bigcirc_k \rangle = (-A^{-2} - A^{2})^{k-1}$

(ii) $\langle \overline{D} \rangle = \overline{\langle D\rangle}$

(iii) $\langle \circlearrowleft \rangle = -A^{3}\langle \frown \rangle$

where $\langle \bigcirc_k \rangle$ is *unlink* of k components, overline represents the mirror image and conjugate, respectively. Kauffmann bracket respects the first two Reidemeister moves but is not an invariant under the third move. Still, assigning minus 1 and plus 1, we can make it respect the third Reidemeister move too.

Theorem 3.3.2. *Let D be a diagram of a link L, then $(-A)^{-3w(D)}\langle D\rangle$ is an invariant of the oriented link where $w(D)$ is the writhe of D.*

Now, we define the *Jones polynomial* $V_L(t)$ of an oriented link L as

$$V_L(t) := (-A)^{-3w(D)}\langle D\rangle$$

where D is any oriented diagram for L and the indeterminate t is identified with A^{-4}. Employing induction on the number of crossings and using bracket relations, we have $V_L(t) \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$. Second bracket property and corresponding writhe change gives the following skein relation.

(i) $V(\bigcirc) = 1$

(ii) $t^{-1}V(\quad) - tV(\quad) = (t^{1/2} - t^{-1/2})V(\quad)$

3.4 The HOMFLY polynomial

The skein relations for Alexander and Jones polynomials suggests a general polynomial satisfying a skein relation with variable coefficients. Indeed, shortly after the discovery of Jones polynomial, following theorem is proved [HOMFLY].

Theorem 3.4.1. *There is a unique function P from the set of isotopy classes of tame oriented links to the set of homogeneous Laurent polynomials of degree 0 in x, y, z such that*

(i) $xP_{L_+}(x,y,z) + yP_{L_-}(x,y,z) + zP_{L_0}(x,y,z) = 0,$

(ii) $P_L(x,y,z) = 1$ *if L consists of a single unknotted component.*

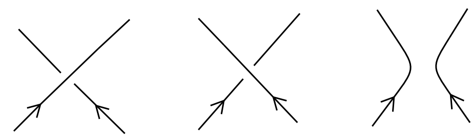


Figure 3.3: L_+, L_-, L_0

where L_+, L_-, L_0 are as in Fig. 3.3. This relation can be used to compute polynomials by forming a *skein tree* with trivial end points, since every link can be converted to unlink by changing crossings.

Letting $x = l$, $y = l^{-1}$ and $z = m$; we have the following skein relation for two variable non-homogeneous polynomial $\mathcal{P}(l, m)$

$$l\mathcal{P}_{L_+}(l,m) + l^{-1}\mathcal{P}_{L_-}(l,m) + m\mathcal{P}_{L_0}(l.m) = 0$$

Since we can recover $P(x, y, z)$ as $\mathcal{P}\left(\frac{\sqrt{x}}{\sqrt{y}}, \frac{z}{\sqrt{xy}}\right)$, $\mathcal{P}$ is the same polynomial invariant, denoted with the same symbol P from now on and called *HOMFLY polynomial.* Some of the immediate observations are:

(i) $\Delta_L(t) = P(i, -i(t^{1/2} - t^{-1/2}))$ and $V_L(t) = P(it^{-1}, -i(t^{1/2} - t^{-1/2}))$

(ii) $P_{\bigcirc_k} = \left(-\frac{l^2+1}{lm}\right)^{k-1}$

(iii) $P_L = P_{L_1}P_{L_2}$ for $L = L_1 + L_2$

(iv) Reversing the orientation of all components of a links does not change HOMFLY polynomial.

Chapter 4

Tangles and Braids

4.1 Prime Tangles

A (2-string) *tangle* is a pair (B, t) where B is a 3-ball, t is a union of 2 strings inside B whose ends points are attached to the boundary ∂B and possibly a number of closed strings in B. We mark the four end points *a priori* and define the *numerator* $N(T)$, *denominator* $D(T)$ of a tangle T and *horizontal*, *vertical summation* of two tangles; $T + U$, $T \oplus U$ as shown in Fig. 4.1. *Union* $T \cup U$ of

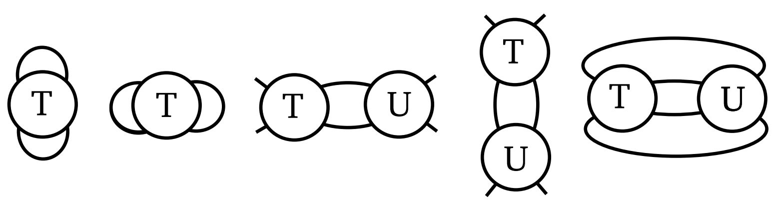


Figure 4.1: $N(T)$, $D(T)$, $T + U$, $T \oplus U$, $T \cup U$

two tangles is given by $N(T+U)$. Two tangles (B_1, t_1) and (B_2, t_2) are *equivalent* if there is a homeomorphism of pairs from one to the other which is fixed on the boundary of the sphere. We will denote the tangle  by 0 and  by ∞. Let $\phi : 0 \to (D, v)$ be a homeomorphism of pairs which is not necessarily identity on the boundary but sends four fixed points to themselves, then the tangle (D, v) is called *trivial*. Also, a tangle (B, t) will be called *locally trivial* if every sub

3-ball $A \subset B$ meeting t at exactly two points transversely, bounds an unknotted spanning arc, that is, an arc resulting in unknot when its endpoints are connected along a string lying on ∂A. With these conventions, we can say a tangle is *prime* if it is locally trivial but non-trivial. Note that, these two conditions imply that there is no embedded disc in B which separates two arcs of t. Prime tangles can be used to build prime knots or to check primeness, as we shall do in section 6.5. For this reason, we state some results due to Lickorish [L2].

Theorem 4.1.1. *Let L be a link of one or two components in S^3 and a 2-sphere intersect with L at four points transversely. If the 2-sphere separates (S^3, L) to two prime tangles then L is prime.*

Theorem 4.1.2. *Let (B, t) be a prime or trivial tangle, a ball $A \subset B$ intersect both components of t in single intervals and (A, u) be a prime tangle. Supposing $\partial u = \partial A \cap t$, $(B, (t \backslash (t \cap A)) \cup u)$ is prime.*

4.2 2-bridge knots

If we start classifying links with respect to their bridge numbers, the first non-trivial family is 2-*bridge* knots. *Caveat lector*, 2-bridge knots can have more than one components, contrary to what the name suggests. A fast observation is that every 2-bridge knot is prime and contains at most two components since we have the relation

$$br(K_1 + K_2) = br(K_1) + br(K_2) - 1$$

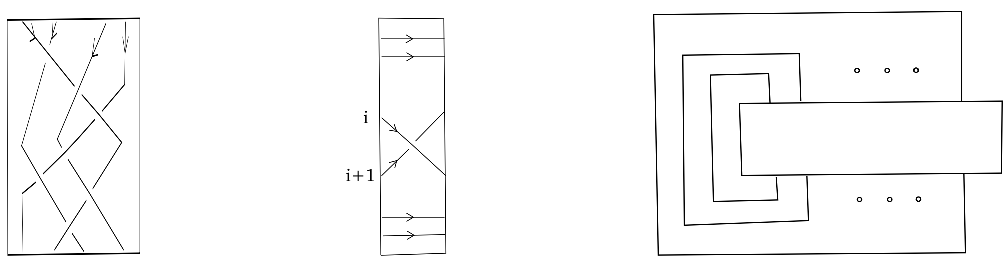
due to Schubert [S2]. This family is completely classified with the help of trivial tangles. (*cf.* [M2], pg. 183)

Theorem 4.2.1. *A 2-bridge knot is the denominator of some trivial tangle and denominator of a trivial tangle is a 2-bridge knot.*

This theorem completely classifies 2-brigde knots since every trivial tangle can be characterized by an alternating sequence of horizontal and vertical twists on 0 or ∞. Classification problem of 3-bridge knots, however, is still open.

4.3 The braid group

Consider a unit 3-dimensional cube in $\mathbb{R}^3$, mark n points B_i on the base and n points C_i on the ceiling of the cube, each of which is aligned on the plane $x = 1/2$. In addition, we choose them in a way that vertical projection of C_i gives B_i for $i = 1, \cdots, n$. Let $s_1, \cdots, s_n$ be mutually disjoint finite-segmented polygonal arcs and $\beta = s_1 \cup \cdots \cup s_n$. β is said to be an *n-braid* provided $\partial\beta = \cup_i (B_i \cup C_i)$ and every plane parallel to the base intersects each *string* s_i at exactly one point. (See Fig.4.2(a)) Two braids β_1 and β_2 are *equivalent* if there is an isotopy h_t of



(a) A braid (b) σ_k (c) Closure of a braid

Figure 4.2: Braids and braid closure

the unit cube such that h_t is identity on the boundary for all $t \in [0, 1]$, $h_0 = id$ and $h_1(\beta_1) = \beta_2$. Let B_n denote the set of all equivalence classes of braids. We can define *product* of two braids by putting one on top of other. This makes B_n a group, called the *n-braid group*. We specify the element shown in Fig. 4.2(b) as σ_k. Since we can divide a braid horizontally to sub-braids each of which contains only one twist, B_n is generated by σ_k for $k = 1, 2, \cdots, n-1$. Also, the relations

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| > 1$

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \cdots, n-1$

are seen to hold from geometric pictures at once. Besides, it was shown that no further relation is possible (due to Artin, *cf.* [B] for a proof). Hence, above generators and relators give a presentation of B_n.

The *closure* of a braid is obtained by connecting base points to corresponding ceiling points as in the Fig. 4.2(c).Now, it is natural to ask about the relationship between braids and knots.

Theorem 4.3.1 (Alexander)**.** *Every (oriented) link is a closure of some braid.*

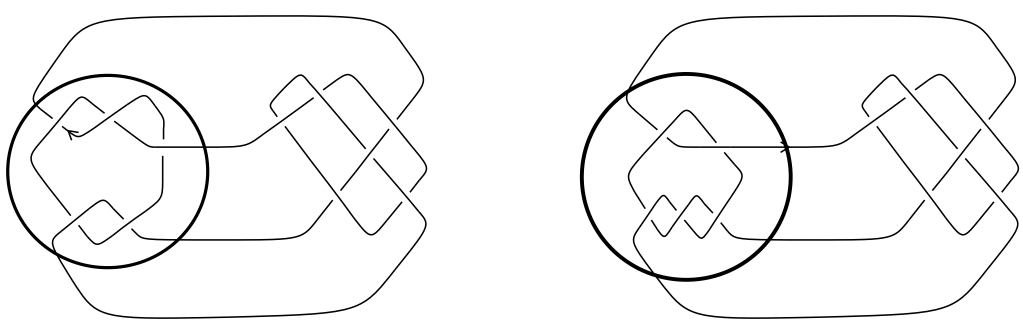
Taking closure of equivalent braids yields equivalent knots but not *vice versa*. Indeed, closures of two braids are equivalent if and only if the braids are related by certain braid moves *a.k.a. Markov moves.*

Chapter 5

Mutations

5.1 The Conway mutation

Let D be a knot diagram and T be a tangle as shown in Fig. 5.1(a). Now, we rotate T in one of the x, y, z-axis by π, provided four end points are sent to themselves and obtain a new diagram D', Fig. 5.1(b). This procedure is called *Conway mutation*. We observe that this mutation does not change Jones



(a) Kinoshita-Terasaka knot

(b) Conway knot

Figure 5.1: Conway mutation

polynomial of knots. Indeed, we can form the same skein tree by deforming the crossings inside the tangles for both the original diagram and its mutant. This process yields one of symmetric tangles in Fig. 5.2. Therefore, the end points of the skein trees are equivalent, so D and D' have the same Jones polynomial.

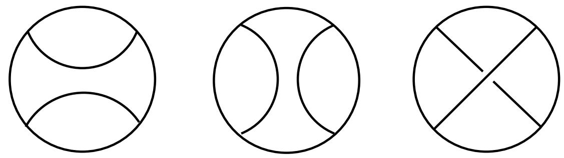


Figure 5.2: Symmetric tangles

Same argument is also valid for their HOMFLY polynomials. As a matter of fact, this mutation changes the knot itself sometimes, as in the Fig. 5.1. On the other hand, applying this mutation to an unknot gives another unknot [Kw], [Rf]. This fact is the motivation to find other mutations which keep the Jones polynomial same but change the HOMFLY polynomial so that resulting knots are not Conway mutants.

5.2 Skein module

The idea of forming a vector space with tangles is due to Conway and is formalized by Rolfsen [Rf]. We will follow the conventions used in [EKT] and [W]. Let $\mathcal{M}$ denote the free $\mathbb{Z}[A, A^{-1}]$-module generated by all equivalence classes of diagrams of (2-string) tangles, and $\mathcal{I}$ the 2-sided ideal generated by the elements

(i) $T \amalg \bigcirc - \delta T$

(ii) $(\times) - A()() - A^{-1}(\asymp)$

where $\delta = (-A^{-2} - A^2)$, $T \in \mathcal{M}$ and symbols inside the parentheses denote an arbitrary tangle except the part drawn. The *skein module* $\mathcal{S}$ is defined as the quotient $\mathcal{M}/\mathcal{I}$. Note that condition (i) ensures that there are no free components of tangles in $\mathcal{S}$ and $\mathcal{S}$ is generated by the tangles 0 and ∞. Let us write $T \in \mathcal{S}$ as

$$T = T_0 \cdot 0 + T_\infty \cdot \infty = \begin{bmatrix} T_0 & T_\infty \end{bmatrix} \begin{bmatrix} 0 \\ \infty \end{bmatrix} := br(T) \begin{bmatrix} 0 \\ \infty \end{bmatrix}$$

where T_0, T_∞ lie in $\mathbb{Z}[A, A^{-1}]$ and the *bracket vector* $br(T)$ of a tangle T is given as above. Some immediate observations [EKT] are as follows.

Proposition 5.2.1.

$$br(T+U) = br(T)\begin{bmatrix} U_0 & U_\infty \\ 0 & U_0 + \delta U_\infty \end{bmatrix}$$

$$br(T \oplus U) = br(T)\begin{bmatrix} \delta U_0 + U_\infty & 0 \\ U_0 & U_\infty \end{bmatrix}$$

Proof. Following equalities hold plainly.

$$br(0+0) = br(0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad br(0+\infty) = br(0)\begin{bmatrix} 0 & 1 \\ 0 & \delta \end{bmatrix}$$

$$br(\infty+0) = br(\infty)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad br(\infty+\infty) = br(\infty)\begin{bmatrix} 0 & 1 \\ 0 & \delta \end{bmatrix}$$

As $br(\infty + U) = U_0\, br(\infty) + \delta U_\infty\, br(\infty)$, we have

$$br(0+U) = br(0)\begin{bmatrix} U_0 & U_\infty \\ 0 & U_0 + \delta U_\infty \end{bmatrix}, \quad br(\infty+U) = br(\infty)\begin{bmatrix} U_0 & U_\infty \\ 0 & U_0 + \delta U_\infty \end{bmatrix}$$

Similarly, $br(T+U) = T_0\, br(0+U) + T_\infty\, br(\infty+U)$ yields the required result. The argument for $T \oplus U$ is the same. $\square$

5.3 Braid actions

One particular way of manipulating tangles is given in [EKT] (see Fig. 5.3). This

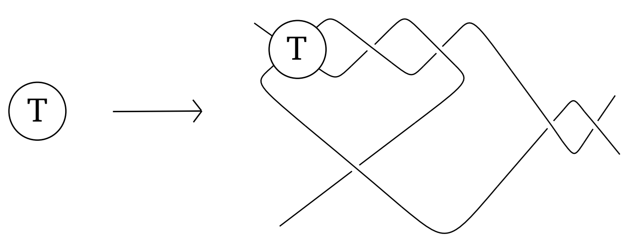


Figure 5.3: $T \longrightarrow T^\omega$

picture suggests the use of alternative sequence of horizontal and vertical twists to tangles. The same procedure used in [M2] pg. 183-187, can be applied to this picture to obtain a braid instead of these twists. In fact, Watson used this idea to generalize the tangle operation via braid actions [W]. Consider the group action

$$\begin{aligned} \mathcal{S} \times B_3 &\longrightarrow \mathcal{S} \\ (T, \beta) &\longmapsto T^\beta \end{aligned}$$

where T^β is defined by Fig. 5.4. Taking generators σ_1, σ_2 of the 3-braid group

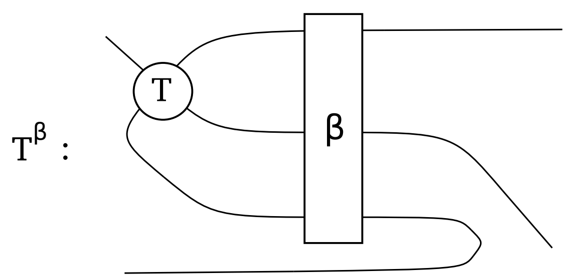


Figure 5.4: Braid action on tangles

B_3 as in Fig. 4.2(b) and applying proposition 5.2.1, one can compute that

$$br(T) = br(T^{\sigma_1}) \begin{bmatrix} A & A^{-1} \\ 0 & -A^{-3} \end{bmatrix}, \quad br(T) = br(T^{\sigma_2}) \begin{bmatrix} -A^{-3} & 0 \\ A^{-1} & A \end{bmatrix}$$

These two matrices, M_1 and M_2, define a group homomorphism

$$\Phi : B_3 \longrightarrow GL_2(\mathbb{Z}[A, A^{-1}])$$

such that $\Phi(\sigma_i) = M_i$ as one can check the braid relation $M_1 M_2 M_1 = M_2 M_1 M_2$.

5.4 The Watson mutation

We will follow the formalism used in [W]. Let $K = K(T, U)$ be a knot (or link) such that strings of both tangles T and U are included in K. The item (ii) of $\mathcal{I}$ assures the following equation

$$\langle K(T,U)\rangle = br(T)\,\mathcal{K}\,br^t(U)$$

where

$$\mathcal{K} = \begin{bmatrix} \langle K(0,0)\rangle & \langle K(0,\infty)\rangle \\ \langle K(\infty,0)\rangle & \langle K(\infty,\infty)\rangle \end{bmatrix}$$

is the *evaluation matrix* of K. Defining $K^\beta = K(T^\beta, U^{\beta^{-1}})$, it ensues that

$$\langle K^\beta\rangle = br(T)\Phi(\beta)\mathcal{K}\Phi^t(\beta^{-1})br^t(U)$$

Now, consider the B_3 action on $GL_2(\mathbb{Z}[A,A^{-1}])$ given by $\mathcal{K}^\beta = \Phi(\beta)\mathcal{K}\Phi^t(\beta^{-1})$. It is our content that $\mathcal{K}^\beta = \mathcal{K}$, *i.e.* $\beta \in B_{\mathcal{K}}$, the stabilizer of $\mathcal{K}$ in B_3.

Proposition 5.4.1 (Watson)**.** *The invertible matrix*

$$\mathcal{X} = \begin{bmatrix} x & \delta \\ \delta & \delta^2 \end{bmatrix}$$

is fixed by σ_1, i.e. σ_1 lies in $B_{\mathcal{X}}$, where $x \in \mathbb{Z}[A,A^{-1}]$.

For a knot (or link) K whose matrix is in this form, we say that K and K^{σ_1} are *Watson mutants*. A similar construction is given in [EKT] with $\beta = \sigma_1^2\sigma_2^{-1}\sigma_1^2$, $\mathcal{X} = \left(\begin{smallmatrix} x & \delta^2 \\ \delta^2 & \delta \end{smallmatrix}\right)$ and $K(T,U)$ as in Fig. 5.5.

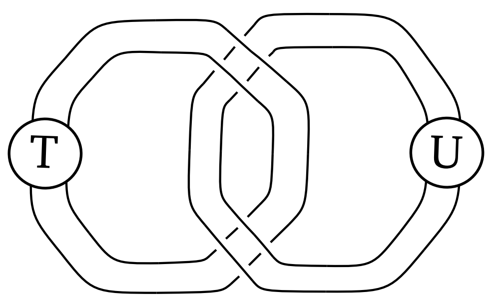


Figure 5.5: 2-parallel of Hopf link with two tangles

5.5 2-tangle Kanenobu knots

The 2-*parameter Kanenobu knot* was originally defined [K] as in Fig.5.6 except that T and U were taken to be horizontal twists only. For this general family,

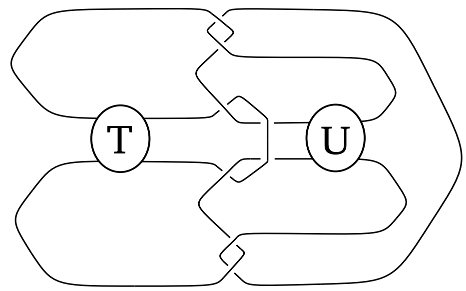


Figure 5.6: 2-tangle Kanenobu knots

we will use the term 2-*tangle Kanenobu knots* and denote them with the same notation $K(T, U)$. By direct computation, $\mathcal{K}$ is presented in the form described in proposition 5.4.1. It is also shown [W] that Watson mutants obtained in this way are prime and do not have a common HOMFLY polynomial provided the tangle T is prime and U is the mirror image of T. In the next chapter, we will show that the *n-tangle Kanenobu knots* shown in Fig. 6.2 share the same Jones polynomial, have different HOMFLY polynomials and are prime under natural conditions.

Chapter 6

n-tangle Kanenobu knots

6.1 n-parameter Kanenobu knots

The *n-parameter Kanenobu knots* $K(p_1, p_2, \cdots, p_n)$ defined again in [K] are as in Fig. 6.1 such that each band consists of p_i positive horizontal (half) twists, where positive (half) twist is as in Fig. 3.1. Next theorem [K] gives information

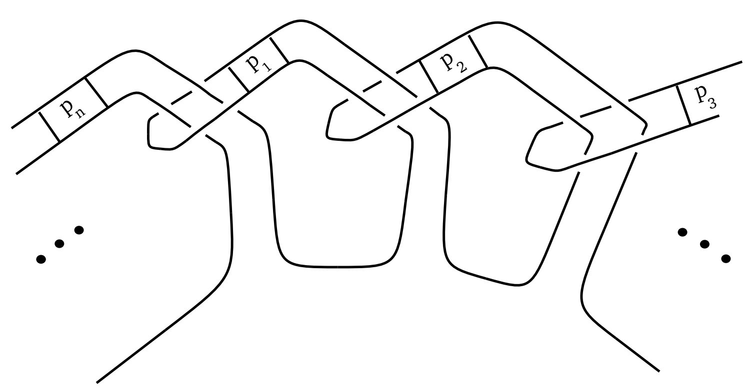


Figure 6.1: n-parameter Kanenobu knots for $n \geq 3$

about the polynomial invariants of this original family.

Theorem 6.1.1 (Kanenobu)**.** *Suppose $P(p_1, p_2, \cdots, p_n)$, $V(p_1, p_2, \cdots, p_n)$ and $\Delta(p_1, p_2, \cdots, p_n)$ are given as the HOMFLY, Jones and Alexander polynomials of $K(p_1, p_2, \cdots, p_n)$. Let ε_i be 0 if p_i is even and 1 if p_i is odd. Let e be the number* *of* 0*'s in* $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$. *Then*

(i) $P(p_1, p_2, \cdots, p_n) = (-l^2)^{\sum_{i=1}^{n}(p_i-\varepsilon_i)/2}(P(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) - 1) + 1$

(ii) $V(p_1, p_2, \cdots, p_n) = (-t)^{\sum_{i=1}^{n} p_i}(V(0, 0, \cdots, 0) - 1) + 1$

(iii) $\Delta(p_1, p_2, \cdots, p_n) = \Delta(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = f(t)f(t^{-1})$
where $f(t) = (-t)^e - (1-t)^n$

6.2 Generalization

Definition 6.2.1. *Let* $K_{\mathcal{V}}(T, U)$ *denote an* $(n+2)$*-tangle Kanenobu knot where* $\mathcal{V}$ *denotes a sequence of tangles* $(V_1, V_2, \cdots, V_n)$ *and* T, U *are chosen arbitrarily from the* $n+2$ *arms,* $n \geq 1$. *(See Fig.6.2)*

Caveat lector, if the case $n = 0$ was not excluded, it would not coincide with the 2-tangle knots defined in section 5.5. In particular, $K(0,0) = 4_1 + 4_1$ [K] but $K_{\varnothing}(0,0) = 6_1$ [1] with Rolfsen numbering. Their Jones polynomials are also seen to be different directly.[2] Another warning is that, depending on the tangles, $K_{\mathcal{V}}(T, U)$ can be a link of several components, contrary to what its name suggests. We now apply the braid actions, described in section 5.3, to T and U. Let

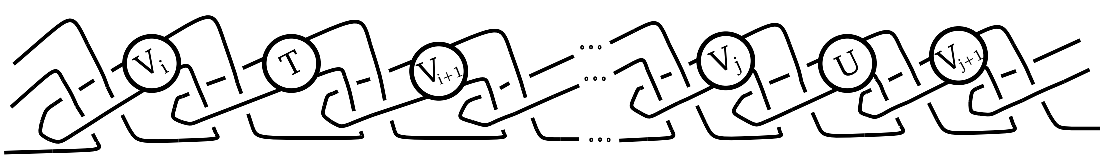


Figure 6.2: $K_{\mathcal{V}}(T, U)$

$K^{\beta}_{\mathcal{V}}(T, U)$ denote the family $K_{\mathcal{V}}(T^{\beta}, U^{\beta^{-1}})$. In particular, whenever $K_{\mathcal{V}}(T, U)$ is a knot, so is $K^{\beta}_{\mathcal{V}}(T, U)$. For orientations; firstly, each string whose both endpoints are attached to the same tangle V_i changes its orientation if and only if the orientations of the other two strings attached to V_i are changed. Secondly, T

[1]computed by the software Knotscape [Kns].
[2]computed by the software KNOT [Kn].

impose its orientation to T^β and U to $U^{\beta^{-1}}$ by definition. In principle, this may lead to a conflict of orientations on strings of $K_{\mathcal{V}}^{\beta}(T,U)$ outside the tangles. Nonetheless, this is not the case for our particular $\beta \in B_3$ in the next section.

6.3 Equivalence of Jones polynomials

Proposition 6.3.1. *The Jones polynomial of the family $K_{\mathcal{V}}^{\sigma_1}(T,U)$ is the same with that of $K_{\mathcal{V}}(T,U)$.*

Proof. Lets compute the evaluation matrix $\mathcal{K}_{\mathcal{V}}$ of $K_{\mathcal{V}}(T,U)$.

$$\mathcal{K}_{\mathcal{V}}(T,U) = \begin{bmatrix} \langle K_{\mathcal{V}}(0,0)\rangle & \langle K_{\mathcal{V}}(0,\infty)\rangle \\ \langle K_{\mathcal{V}}(\infty,0)\rangle & \langle K_{\mathcal{V}}(\infty,\infty)\rangle \end{bmatrix}$$

One can observe from Fig. 6.2 that $K_{\mathcal{V}}(0,\infty)$ and $K_{\mathcal{V}}(\infty,0)$ are equivalent to a sum of denominators of the tangles, $D(V_1) + D(V_2) + \cdots + D(V_n)$ and an unlinked copy of unknot. Similarly, $K_{\mathcal{V}}(\infty,\infty)$ is equivalent to the same link with an additional trivial unlinked component. Writing the Kauffmann bracket of $K_{\mathcal{V}}(0,0)$ as x and that of $K_{\mathcal{V}}(0,\infty)$ as u, we have

$$\mathcal{K}_{\mathcal{V}}(T,U) = \begin{bmatrix} x & u \\ u & u\delta \end{bmatrix}$$

Now, we check if $\mathcal{K}_{\mathcal{V}}(T,U)$ is fixed by σ_1:

$$
\begin{aligned}
\mathcal{K}_{\mathcal{V}}^{\sigma_1}(T,U) &= \Phi(\sigma_1)\mathcal{K}_{\mathcal{V}}(T,U)\Phi^t({\sigma_1}^{-1}) \\
&= \begin{bmatrix} A & A^{-1} \\ 0 & -A^{-3} \end{bmatrix} \begin{bmatrix} x & u \\ u & u\delta \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ A & -A^{3} \end{bmatrix} \\
&= \begin{bmatrix} Ax + A^{-1}u & Au + A^{-1}u\delta \\ -A^{-3}u & -A^{-3}u\delta \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ A & -A^{3} \end{bmatrix} \\
&= \begin{bmatrix} x + A^{-2}u + A^{2}u + u\delta & -A^{4}u - A^{2}u\delta \\ -A^{-4}u - A^{-2}u\delta & u\delta \end{bmatrix} \\
&= \begin{bmatrix} x + u(A^{-2} + A^{2} + \delta) & -A^{2}u(A^{2} + \delta) \\ -A^{-2}u(A^{-2} + \delta) & u\delta \end{bmatrix} \\
&= \begin{bmatrix} x & u \\ u & u\delta \end{bmatrix} \\
&= \mathcal{K}_{\mathcal{V}}(T,U)
\end{aligned}
$$

So we have, $\sigma_1 \in B_{\mathcal{K}_{\mathcal{V}}(T,U)}$ which implies the equivalence of Kauffmann brackets of $K_{\mathcal{V}}^{\sigma_1}(T,U)$ and $K_{\mathcal{V}}(T,U)$. (See section 5.4). Note that (1) the contribution of all crossings, outside the tangles, to writhe is zero. (2) All the strings of the tangles other than T^{σ_1} and U^{σ_1} change their orientation, therefore preserve their contribution to writhe. (3) The total contribution, to writhe, coming from the crossings of T^{σ_1}, U^{σ_1} is equivalent to that coming from the crossings of T, U. Hence, we conclude that $V_{K_{\mathcal{V}}^{\sigma_1}(T,U)} = V_{K_{\mathcal{V}}(T,U)}$. $\square$

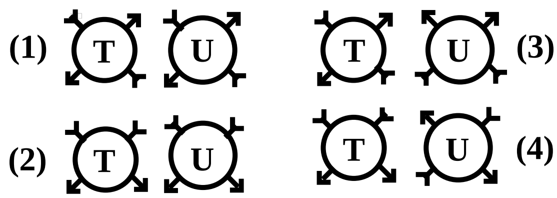
The same computation with σ_2 yields that $\sigma_2 \in B_{\mathcal{K}_{\mathcal{V}}(T,U)}$ only if $x = u\delta$, which means that $B_{\mathcal{K}_{\mathcal{V}}(T,U)}$ is, in general, a proper subset of B_3. We end this section by stating the problem in a general form.

Question 6.3.2. *Given an abstract knot K which depends on at least two tangles and suppose B_n has an action on tangles, then what is the group $B_{\mathcal{K}}$?*

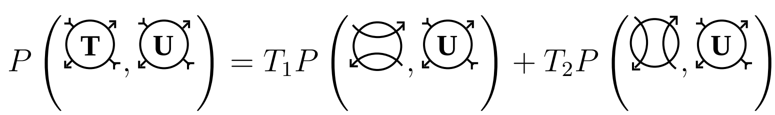
6.4 Inequivalence of HOMFLY polynomials

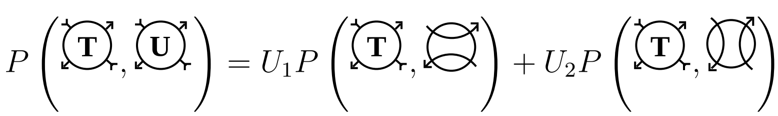
Proposition 6.4.1. *The HOMFLY polynomial of the family $K^{\sigma_1}_{\mathcal{V}}(T,U)$ is different from that of $K_{\mathcal{V}}(T,U)$.*

Proof. An inspection of Fig. 6.2 shows that there are eight possible orientations for the tangles T, U but half of these possibilities is eliminated due to the fact that inversion of orientations of all components of a link preserves the HOMFLY polynomial. One instance of remaining possibilities are as follows:

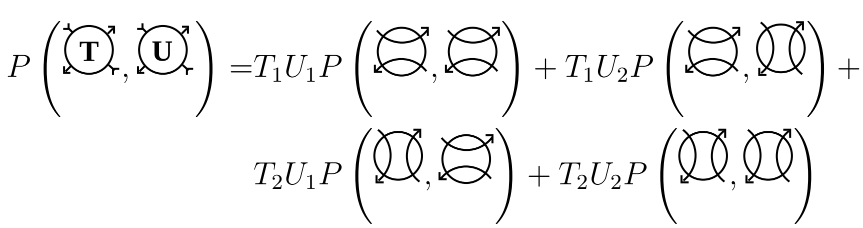


Temporarily, we switch the notation $P_{K_{\mathcal{V}}(T,U)}$ to $P(T,U)$ to ease reading.
Case(1) Using the skein relations for HOMFLY polynomial (see section 3.4), suppose we have

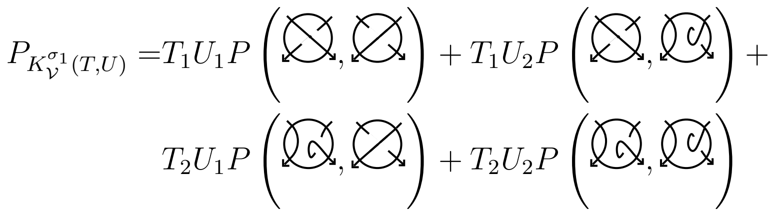




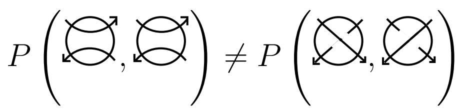
where $T_i, U_j \in \mathbb{Z}[l^{\pm}, m^{\pm}]$. These two equations imply that

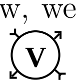
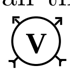
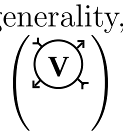
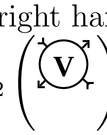


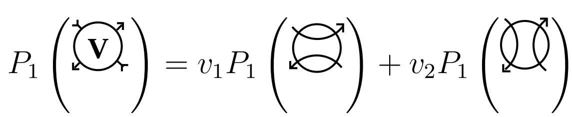
Applying σ_1 to $P_{K_{\mathcal{V}}(T,U)}$, we get



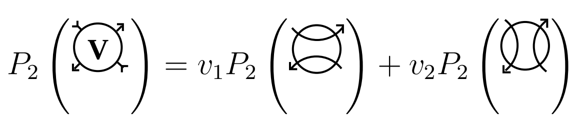
Inspecting Fig. 6.2 and remembering the fact that reversal of all components of a link preserves the HOMFLY polynomial, last three terms of last two equations are seen to be the same. Thus, it suffices to show that



Now, we fix all the tangles but $V = V_1$. There are two possible orientations for V:  and .
Without loss of generality, consider the first case. We denote the right hand side polynomial as P_1  and the left hand side polynomial as P_2 , then write



and



As before, the second terms of the last two equations are the same so we only need to compare the first terms. We continue the same procedure for $V_2, V_3, \cdots, V_n$. At each step, we have two choices but the surviving terms are the same tangles for both sides. Thus, it suffices to show that

$$P(\cdots, \hat{0}, \cdots, \hat{0}, \cdots) \neq P(\cdots, \hat{1}, \cdots, \widehat{-1}, \cdots)$$

where hats signify different terms. (See section 6.1 for notation.) We compute the Alexander polynomials of $K(\cdots,\hat{0},\cdots,\hat{0},\cdots)$ and $K(\cdots,\hat{1},\cdots,\hat{-1},\cdots)$ by using Theorem 6.1.1.

$$\Delta(\cdots,\hat{0},\cdots,\hat{0},\cdots) = \tag{6.1}$$
$$= f(t)f(t^{-1}) \tag{6.2}$$
$$= [(-t)^{e+2} - (1-t)^n][(-t^{-1})^{e+2} - (1-t^{-1})^n] \tag{6.3}$$
$$= 1 - (-t)^{e+2}(1-t^{-1})^n - (1-t^n)(-t^{-1})^e + (2-t-t^{-1})^n \tag{6.4}$$

where $e+2$ is the number of zeros in the reduced form of $K(\cdots,\hat{0},\cdots,\hat{0},\cdots)$ so that

$$\Delta(\cdots,\hat{1},\cdots,\hat{-1},\cdots) = \tag{6.5}$$
$$= f(t)f(t^{-1}) \tag{6.6}$$
$$= [(-t)^{e} - (1-t)^n][(-t^{-1})^{e} - (1-t^{-1})^n] \tag{6.7}$$
$$= 1 - (-t)^{e}(1-t^{-1})^n - (1-t^n)(-t^{-1})^e + (2-t-t^{-1})^n \tag{6.8}$$

The degrees of the two middle terms are $n-e-2$ and $n-e$, respectively. Therefore, Alexander polynomials are distinct, hence so are HOMFLY polynomials, as required. Other three cases can be argued similarly. □

Corollary 6.4.2. *$K^{\sigma_1}_{\mathcal{V}}(T,U)$ and $K_{\mathcal{V}}(T,U)$ are not Conway mutants.*

6.5 Primeness of $K_{\mathcal{V}}(T,U)$

Proposition 6.5.1. *Suppose $K = K_{\mathcal{V}}(T,U)$ is a link of one or two components such that $K_{(0,\cdots,0)}(0,0)$ is not 2-bridged and T, U, $\mathcal{V}$ are prime. Then K is prime.*

Proof. Consider the following tangle W, shown in Fig. 6.3. We claim that W is a prime tangle. (1) W is locally trivial. Suppose not then there exists a sub

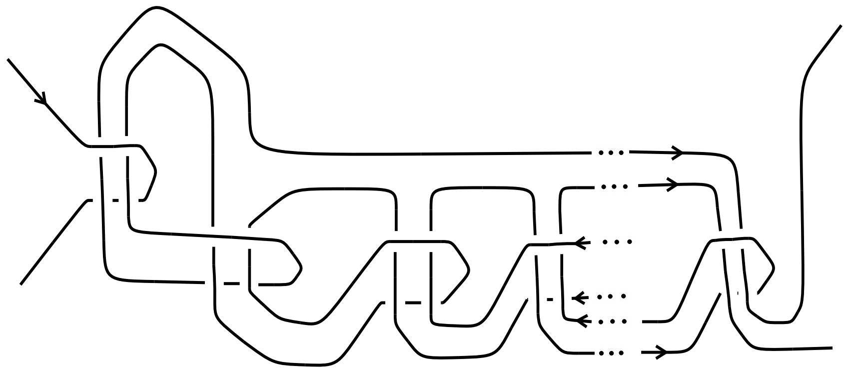


Figure 6.3: Tangle W

3-ball A, inside W, bounding a knotted arc, let us say connecting two end points of this arc along ∂A gives a non-trivial knot H. Now, $W \cup 0$ gives an unlink of two components, which means H is a summand of a trivial component $\bigcirc$. This contradicts to the additivity of genus.

(2) Suppose that W is trivial, then the denominator $D(W)$ of the tangle W must be a 2-bridge knot by theorem 4.2.1. However, $D(W) = K$. This establishes the claim.

Primeness of U and the sequence $\mathcal{V}$ implies that the tangle $\overline{W}$, shown in Fig. 6.4, is prime by consecutive applications of Theorem 4.1.2. The black ellipses represent the tangle sequence $\mathcal{V}$ and the tangle U which is somewhere in between the elements of $\mathcal{V}$. Since $K = \overline{W} \cup T$, K is prime by Theorem 4.1.1. $\square$

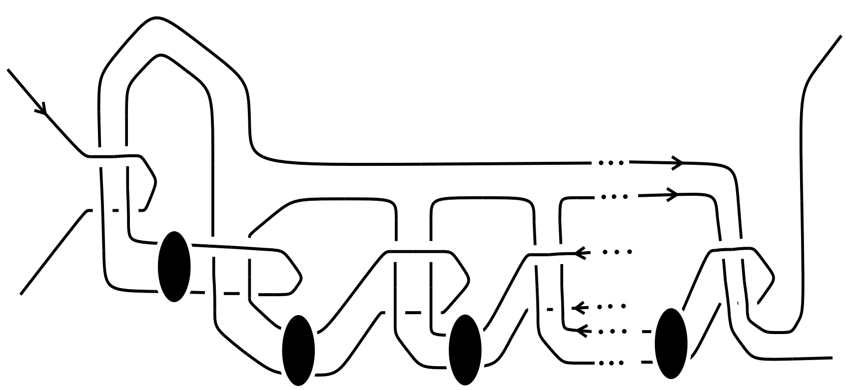


Figure 6.4: Tangle $\overline{W}$

Any partial choice of the tangles T, U and $\mathcal{V}$ as 0, instead of prime tangles, would still imply the primeness of K.

6.6 Examples

In this section, we demonstrate pairs of prime links with their Jones and HOMFLY polynomials. [3] The tangles in the examples are chosen to be either prime or 0 to insure primeness. Also, an odd number of applications of σ_1 are made to obtain the mutants with the desired properties. The reason for this is explained in item (1) of section 6.7.

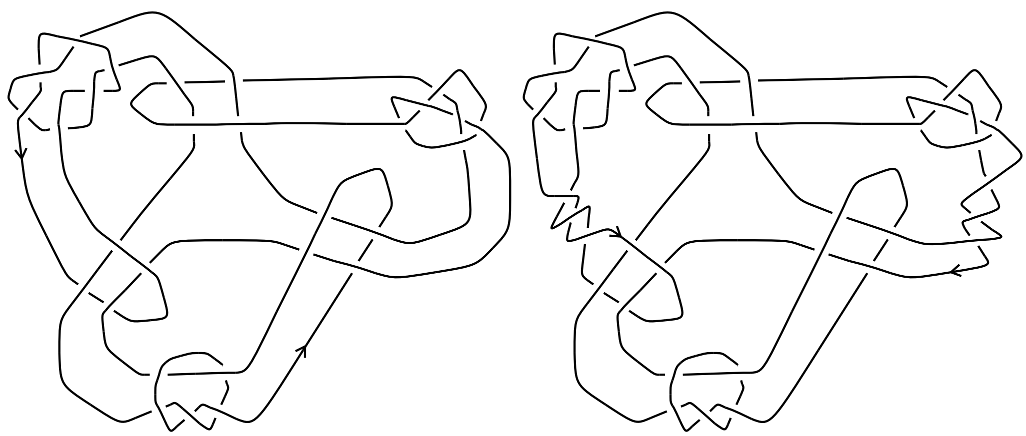


Figure 6.5: L_1 and L_2

Example 1. $V_{L_1}(t) = V_{L_2}(t) = -t^{-23/2}+8t^{-21/2}-31t^{-19/2}+79t^{-17/2}-150t^{-15/2}+223t^{-13/2}-261t^{-11/2}+231t^{-9/2}-123t^{-7/2}-38t^{-5/2}+203t^{-3/2}-323t^{-1/2}+357t^{1/2}-304t^{3/2}+186t^{5/2}-52t^{7/2}-51t^{9/2}+99t^{11/2}-96t^{13/2}+65t^{15/2}-32t^{17/2}+11t^{19/2}-2t^{21/2}$

$P_{L_1}(z,v) = z^{-1}(v^3-4v+4v^{-1}-v^{-3})+z(-v^3+8v-24v^{-1}+31v^{-3}-18v^{-5}+4v^{-7})+z^3(-2v^5+5v^3+11v-55v^{-1}+74v^{-3}-41v^{-5}+8v^{-7})+z^5(-9v^5+43v^3-65v+17v^{-1}+37v^{-3}-28v^{-5}+5v^{-7})+z^7(-16v^5+86v^3-164v+129v^{-1}-34v^{-3}-2v^{-5}+v^{-7})+z^9(-14v^5+81v^3-161v+133v^{-1}-43v^{-3}+4v^{-5})+z^{11}(-6v^5+40v^3-80v+61v^{-1}-16v^{-3}+v^{-5})+z^{13}(-v^5+10v^3-20v+13v^{-1}-2v^{-3})+z^{15}(v^3-2v+v^{-1})$

$P_{L_2}(z,v) = z^{-1}(v^3-4v+4v^{-1}-v^{-3})+z(v-4v^{-1}+v^{-3}+7v^{-5}-7v^{-7}+2v^{-9})+$

[3] Computations and graphics are done by the software KNOT [Kn].

$z^3(-2v^5+6v^3-15v^{-1}+9v^{-3}+9v^{-5}-8v^{-7}+v^{-9})+z^5(-v^7-v^5+16v^3-21v-9v^{-1}+22v^{-3}-6v^{-5})+z^7(-2v^7+3v^5+15v^3-30v-2v^{-1}+26v^{-3}-11v^{-5}+v^{-7})+z^9(-v^7+3v^5+6v^3-16v-2v^{-1}+13v^{-3}-3v^{-5})+z^{11}(v^5+v^3-3v-v^{-1}+2v^{-3})$

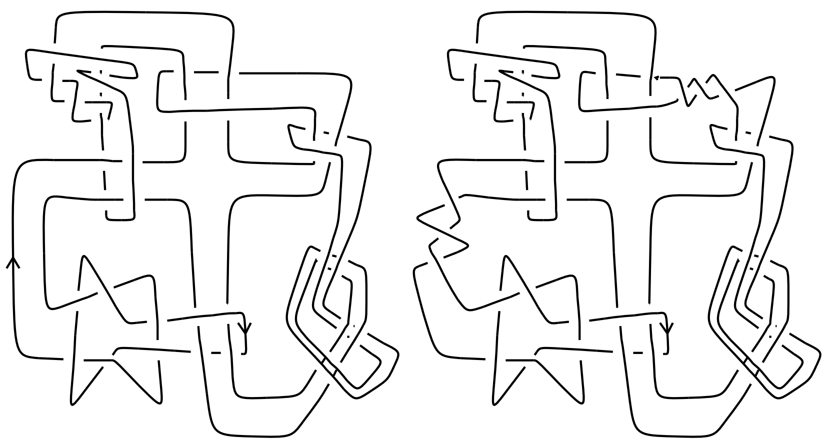


Figure 6.6: L_3 and L_4

Example 2. $V_{L_3}(t) = V_{L_4}(t) = -2t^{-39/2}+14t^{-37/2}-53t^{-35/2}+143t^{-33/2}-294t^{-31/2}+476t^{-29/2}-610t^{-27/2}+582t^{-25/2}-312t^{-23/2}-183t^{-21/2}+777t^{-19/2}-1267t^{-17/2}+1484t^{-15/2}-1371t^{-13/2}+989t^{-11/2}-504t^{-9/2}+83t^{-7/2}+163t^{-5/2}-221t^{-3/2}+172t^{-1/2}-127t^{1/2}+160t^{3/2}-266t^{5/2}+375t^{7/2}-420t^{9/2}+376t^{11/2}-272t^{13/2}+160t^{15/2}-75t^{17/2}+27t^{19/2}-7t^{21/2}+t^{23/2}$

$P_{L_3}(z,v) = z^{-1}(-v^7+v^5)+z(v^{15}-12v^{13}+58v^{11}-148v^9+214v^7-178v^5+84v^3-16v)+z^3(v^{17}-2v^{15}-50v^{13}+309v^{11}-800v^9+1,126v^7-893v^5+328v^3+85v-168v^{-1}+84v^{-3}-16v^{-5})+z^5(4v^{17}-21v^{15}-76v^{13}+687v^{11}-1,874v^9+2,629v^7-1,959v^5+421v^3+643v-691v^{-1}+292v^{-3}-48v^{-5})+z^7(3v^{17}-34v^{15}-42v^{13}+832v^{11}-2,518v^9+3,611v^7-2,497v^5+19v^3+1,478v-1,210v^{-1}+425v^{-3}-56v^{-5})+z^9(v^{17}-23v^{15}+v^{13}+608v^{11}-2,133v^9+3,208v^7-2,038v^5-521v^3+1,777v-1,167v^{-1}+328v^{-3}-32v^{-5})+z^{11}(-8v^{15}+12v^{13}+283v^{11}-1,187v^9+1,907v^7-1,089v^5-647v^3+1,267v-665v^{-1}+141v^{-3}-9v^{-5})+z^{13}(-v^{15}+6v^{13}+83v^{11}-439v^9+760v^7-370v^5-398v^3+551v-222v^{-1}+32v^{-3}-v^{-5})+z^{15}(v^{13}+14v^{11}-105v^9+197v^7-72v^5-139v^3+141v-40v^{-1}+3v^{-3})+z^{17}(v^{11}-15v^9+30v^7-6v^5-26v^3+19v-3v^{-1})+z^{19}(-v^9+2v^7-2v^3+v)$

$P_{L_4}(z,v) = z^{-1}(-v^7+v^5)+z(v^{17}-12v^{15}+58v^{13}-148v^{11}+217v^9-187v^7+90v^5-$

$16v^3)+z^3(v^{19}-5v^{17}-20v^{15}+201v^{13}-638v^{11}+1,082v^9-1,043v^7+489v^5+37v^3-168v+84v^{-1}-16v^{-3})+z^5(v^{19}-12v^{17}+5v^{15}+261v^{13}-1,151v^{11}+2,344v^9-2,572v^7+1,268v^5+298v^3-787v+448v^{-1}-96v^{-3})+z^7(-8v^{17}+28v^{15}+154v^{13}-1,117v^{11}+2,846v^9-3,628v^7+1,985v^5+601v^3-1,615v+965v^{-1}-200v^{-3})+z^9(-v^{17}+18v^{15}+31v^{13}-623v^{11}+2,111v^9-3,211v^7+2,019v^5+625v^3-1,871v+1,127v^{-1}-216v^{-3})+z^{11}(3v^{15}-5v^{13}-196v^{11}+975v^9-1,840v^7+1,366v^5+380v^3-1,338v+797v^{-1}-137v^{-3})+z^{13}(-2v^{13}-32v^{11}+273v^9-678v^7+609v^5+136v^3-607v+354v^{-1}-52v^{-3})+z^{15}(-2v^{11}+43v^9-153v^7+172v^5+26v^3-172v+97v^{-1}-11v^{-3})+z^{17}(3v^9-19v^7+28v^5+2v^3-28v+15v^{-1}-v^{-3})+z^{19}(-v^7+2v^5-2v+v^{-1})$

6.7 Final Remarks

(1) Several applications of σ_1 to $K_{\mathcal{V}}(T,U)$ gives a sequence of links with the same Jones polynomial and it can be shown by the same argument as in the proof of Theorem 6.4 that HOMFLY polynomials of $K_{\mathcal{V}}^{\sigma_1^n}(T,U) \neq K_{\mathcal{V}}^{\sigma_1^m}(T,U)$ if $n \neq m$ mod (2). Moreover, it is possible to get the same HOMFLY polynomials if $n = m$ mod (2). For example, the HOMFLY polynomials of the links shown in Fig. 6.7 are are the same. [4]

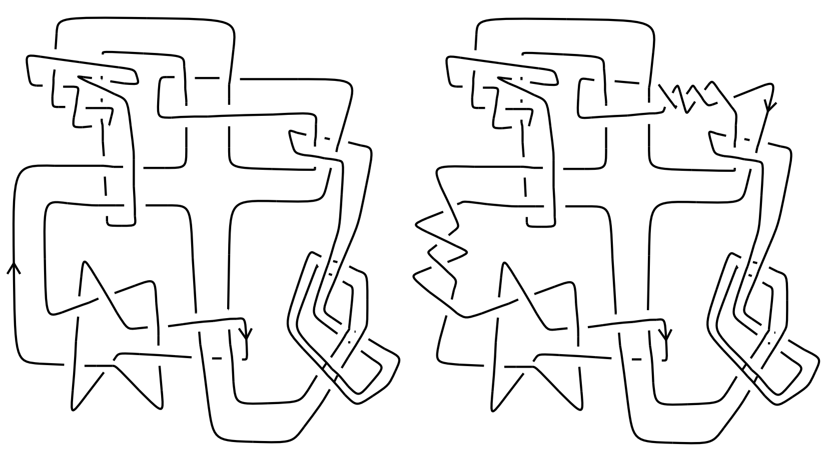


Figure 6.7: Links with the same HOMFLY polynomials

(2) A further generalization of the 2-tangle Kanenobu knots (see section 5.5) can be given as in Fig. 6.8. Similar results can be obtained for this family as well.

[4]computed by the software KNOT [Kn].

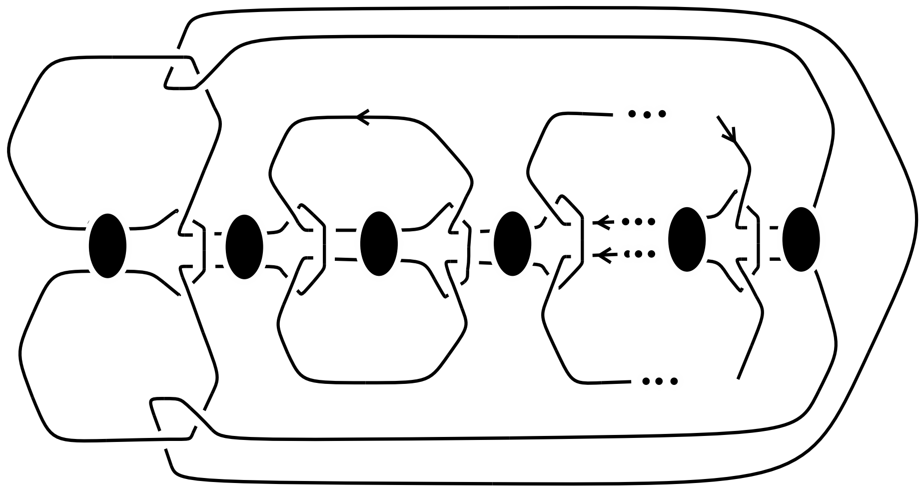


Figure 6.8: Generalization of 2-tangle Kanenobu knots

(3) The question if the Jones polynomial detects knottedness would be answered in the negative if one could arrange a diagram of unknot which would fit into a diagram of $K_{\mathcal{V}}(T, U)$.

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