PRODUCT ROLLOVER STRATEGY AND INVENTORY POLICY OF A MONOPOLY MANUFACTURING SUBSTITUTABLE PRODUCTS

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By

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July, 2010

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ABSTRACT

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July, 2010

In many industries, effective management of product rollovers is extremely important for being able to survive. In management of product rollovers, timing decision; i.e., time to introduce of a secondary product and time to phase out a primal product is critical. Inventory policy is another factor that affects management of rollovers.

In this study, we analyze primary rollover strategy of a monopoly manufacturing two substitute products together with its contingency strategies over a two period planning term. Specifically, we consider four different primary rollover strategies, namely Base Strategy, IS Strategy, ISES Strategy and IFES Strategy, derived with existence/non-existence of the products. Base Strategy is associated with the case where we decide to introduce and sell only the primary product. On the other hand, IS Strategy brings introduction of a newer (secondary) product in the second period. If monopoly chooses to make its move with IFES Strategy, it introduces both of the products simultaneously in the first period while phasing out the primary product in the beginning of the next period. Another alternative strategy, ISES Strategy, would be selling products in different periods, primary product first and secondary product next.

When a primary strategy is selected, there is a commitment to this strategy. In this study, to reflect market conditions, we consider two alternative demand forms; multiplicative and additive forms and there is an adjustment to market through inventory policy. Firm replenishes its stocks with an order-up-to policy in each period where demands for these substitute products are assumed to be correlated and these products assumed to be substitutable; i.e., there exists stock-out-induced substitution between the products.

In the analysis, we determine the optimal inventory levels when a specified rollover strategy is executed. Moreover, we explore the conditions, which play important role in making rollover strategies. Furthermore, factors that affect early and late introduction of a new product into the market are investigated. We also discuss the factors that motivate a monopoly to introduce a new product.

Keywords: New Product Introduction, Product Rollovers, Stock-out Induced Substitution, Substitute Products, Inventory Policy.

ÖZET

İKAME MALLAR ÜRETEN MONOPOL İÇİN ÜRÜN DEVİR STRATEJİLERİ VE ENVANTER POLİTİKASI

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Birçok endüstride ürün devrinin etkin yönetimi firmaların piyasada kalabilmeri için oldukça önemlidir. Ürün devri yönetiminde zamanlama kararı, ikincil ürünün piyasaya getirilme zamanı ve birincil ürünün piyasadan çekildiği zaman, oldukça kritik bir karardır. Envanter yönetimi ise ürün devri yönetiminde önemli olan diğer bir faktördür.

Bu çalışmada, ikame mallar üreten tekel firmanın iki dönemlik zaman dilimindeki birincil ve durumsal ürün devri stratejileri incelenmektedir. Özellikle, ürünlerin iki dönemlik zaman diliminde var olup olmamalarına göre türetilmiş, Temel Strateji, IS Stratejisi, IFES Stratejisi ve IFES Stratejisi olarak adlandırdığımız, dört ürün devir stratejisi degerlendirilmiştir. Temel Strateji sadece birincil ürünün pazara sürülmesi durumunu içeren stratejidir. Öte yandan, IS Stratejisi yeni/ikincil ürünün ikinci zaman diliminde piyasaya getirilmesini kapsamaktadır. Tekel firmanın IFES Straejisi ile hamle yaptığı durumda ise, her iki ürün de piyasaya ilk zaman diliminde getirilirken birincil ürün bir sonraki dönem başında piyasadan çekilir. Diğer bir strateji olan ISES Stratejisi ise her iki ürünün de pazarda farklı zaman dilimlerinde, birincil ürünün ilk zaman diliminde ve ikincil ürünün bir sonraki zaman diliminde, bulunmasını sağlar.

Birincil ürün stratejisine karar verildikten sonra, seçilen stratejiye tüm zaman aralığında bağımlılık söz konusudur. Bu çalışmada, piyasa şartlarını yansıtmak için iki farklı talep modeli, toplamsal ve çarpımsal talep modelleri kullanılmıştır. Tekel firma, piyasaya dönemlik envanter politikası ile tepki vermektedir. İkame malların taleplerinin bağımlı ve bu malların stokta bulunmama durumunda ikame edilebilir varsayıldığı bu problemde, firma stoklarını belirli bir seviyeye kadar ısmarlamalı envanter yönetimi ile yenilemektedir.

Bu çalışmada, belirli bir ürün devri stratejisi için en uygun envanter seviyeleri belirlenmektedir ve ürün devri stratejileri oluştrulurken göz önünde bulundurulması gereken durumlar incelenmektedir. Buna ek olarak, yeni ürünün erken veya geç olarak piyasa sürülmesi kararını etkileyen unsurlar incelenmektedir. Ayrıca tekel fimalarn piyasa yeni ürün getirmelerini teşvik edebilecek etmenler tartışılmaktadır.

Anahtar sözcükler: Yeni Ürün, Ürün Devri, Stokta Bulunmama Durumunda kame, İkame mallar, Envanter Politikası.

To My Parents Fatma & Memduh and My Husband Ahmet Melik

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Chapter 1

Introduction

Managing product rollovers, introducing a new product and phasing out an old one, is the challenge that several industries are frequently encountering (Lim and Tang, 2006). In many industries, to introduce new products is a necessity for being able to survive. Short product life cycles, changing customer preferences and technological innovations are only a few of several factors that push firms to develop new products. As a consequence of increasing product proliferation, products existing even for a short time become old and they are forced out of the market. As a result, phasing out an existing product becomes another issue in management of product introductions and as Billington et al. (1998) puts it, it is extremely important to coordinate the decisions regarding the introduction and displacement.

Lim and Tang (2006) explains that coordinating timing decisions for product rollovers and selecting appropriate rollover strategy is extremely significant because there is a risk attached to each decision. Too early introduction of a new product combined with too late elimination of the old product may cause demand of old product to be cannibalized by the new product whereas too late introduction of a new product may remove potential sales from the new product. If phasing out decision is too early it may bring firm to a financially risky position that it sells only the new product without support from the sales of old product. After selecting an appropriate strategy, firm may still suffer from problems such as excess or scarce inventories, technical problems with the new product and incorrect assessment of market and demand characteristics (Billington et al., 1998).

Despite the frequency of new product introductions, as Billington et al. (1998) states, there are a plenty of unsuccessful product introductions that companies experience. Being motivated by this, we present a formal model that incorporates several issues companies face when managing product rollovers. In general, we discuss primary and contingency strategies associated with new product introductions and older product eliminations.

In this thesis, we consider a monopoly market and by doing this, we omit the competition drive and its effect on product rollover strategy. We do this because we want to focus on the competition between own products of firm and its effect on our decisions. However, this study can be extended to the competitive markets and present a more realistic way of seeing new product introduction challenge today's business environment intensely experiencing.

We assume that the monopoly firm, decides over a two period time interval and lengths of these periods are not necessarily equal. Firm introduces a primary product in the beginning of the first period and it has not decided the time to introduce a secondary product, which is developed and ready. Moreover, the monopoly may also phase out its primary product in the end of first period. Hence by deciding whether to enter the secondary product or not in any period and whether to exit the primary product or not in the second period, it implicitly considers the timing issue as a part of its product rollover strategy. Thus, we study four different primary strategies associated with managing product rollovers.

Once monopoly decides which strategy to pursue in the long term(two periods in this study), it commits this strategy until the term ends. We think that this is a reasonable assumption since, primary strategies are long term plans and generally each of them is associated with big investments on issues such as production technology, supply chain activities or marketing activities. Related to this, we use different investment levels that include costs of production technology for each rollover strategy. When simultaneous existence of the products is the case, it may be advantageous to use a production line where postponement of differentiation is possible and as a result of this, investment to obtain such a system is needed. However, this investment could differ according to the existing production technology in the sense that there may be no production system currently and production line can be built from scratch or redesign the existing system for delayed product differentiation for the next period.

There may be however some control over primary strategy once committed through contingency strategies as Billington et al. (1998) discusses. Parallel to this, we include inventory policy that provides adjustment to market conditions in the short term. After selecting a rollover strategy, monopoly decides its order-up-to levels for each period in our problem setting. We assume that the decision maker can replenish its stock in the beginning of each period and replenishment lead time is zero. With replenishment of the stock, we mean ordering inputs from suppliers and producing end-products. There is no fix charge for ordering and total ordering cost, work in process (in-transit) inventory holding cost and processing cost are proportional to ordering quantity. Similarly, total holding cost is proportional to the end-of-period inventory. On the other hand, unsold finished items at the end of a period, can be sold in the next period at the price of those newly produced items. There is no penalty cost and the opportunity cost of not satisfying a customer is simply the foregone sale.

Market conditions are very significant in determining success or failure of a rollover strategy and as we put before, firm reacts market with inventory policy. Market conditions for our model are explained in the following arguments. Demand for each of the product is stochastic and total demand for a period is assumed to be the summation of independent and identically distributed unit time demands over the length of the period. We assume that there is a correlation between the demands of the products offered in a period. Moreover, there is a consumer driven substitution (Netessine & Rudi, 2003), or alternatively stock-out-induced substitution (Nagarajan & Rajagopalan, 2008), in the sense that when there are unsatisfied customers of a product, a portion of them can switch to the other product to satisfy their needs.

According to Lim and Tang (2006) a strategic decision about product rollovers should include three issues. One of them is time to introduce a new product and time to phase out an existing one. Another issue is pricing for old and new product before and after introduction. Finally, contingencies including competitor's actions and technical problems should be taken into account. Comprising this, in our analysis for a non-competitive environment, we focus on two decision streams; timing decision and contingency plans with replenishment decision when a strategy is committed. Timing decision is handled implicitly with different rollover strategies. Each strategy includes a decision whether to introduce a secondary product in one of the two periods and whether to phase out primary product or not in the second period. When a strategy is chosen, we control our stock according to demand conditions in each period. Pricing is not a decision in our model but it can easily be converted to a decision variable. Being aware of significance of price on rollover strategies, we compare different rollover strategies under different price levels through hypotheses of numerical analysis.

Having summarized the boundaries of our model, this thesis is organized as follows. In chapter 3, we introduce the profit model in detail and show the conditions where they are concave. Later, in Chapter 4, we focus on demand model and discuss two ways of considering randomness in demand. Moreover, with these models we incorporate price substitution and correlation between the demands when they are together in the market to our model. In Chapter 5, we compare dual and single (dual) rollover strategies, early and late introduction of a new product and explore incentives for a monopoly to introduce a new product under different settings with different price, demand and cost structures. We explore validity of hypotheses with numerical analysis. Finally, Chapter 6, gives concluding remarks and possible future research directions.

Chapter 2

Literature Review

New Product Introduction (NPI) is a popular subject which has been discussed in various aspects by engineering, marketing, strategy and economics literature. Economic literature generally focuses on contribution of new products to economy and competition. Nevo (2003) studies impacts of new products and quality changes of existing products over economic welfare using estimated demand systems and compare conclusions with literature. Segerstrom (1991) considers effects of improved products and their imitations on economic growth and concludes that if average level of innovation efforts over the long run is large enough, new products and their imitations effect economic growth positively. Petrin (2002) investigates new products in competitive minivan market and finds results supporting the idea that new products increase customer standards by promising even more new products because of firms seeking temporary market power after new products' cannibalization of existing products. Hausman and Leonard (2002) evaluates competitive effects of NPI with changes in price levels of existing products due to increase competition and high product variety in the market with data from bath tissue market. Kadiyali et al.(1999) discusses product line extensions in a competitive setting and provides effects of extension on prices, market power, sales and competition.

Marketing and business literature discuss NPI in various aspects including new product development (NPD), business strategy , diffusion of new products, industry clockspeed and product rollovers. New Product Development literature focuses on the whole process from idea generation to product pricing to bring new products or services to market. Comprehensive reviews on NPD is presented by Krishnan and Ulrich (2001) and Ernst (2002) . Gatignon and Xuereb (1997) is a paper which evaluates NPIs from a strategical point of view and provides NPI strategies for different levels of competition for different market and demand structures. Diffusion models are used to examine the communication and adoption of innovation and new products in the market. Mahajan et al. (1990) provides a comprehensive literature review on this research area. Druehl et al (2009) investigates the relationship between the frequency of product upgrades in an industry with product development costs and diffusion rates. Fine (1998) suggests that industries operate at different clockspeeds and claims that technology clockspeed can be measured by rates of new product introduction. Souza et al. (2004) investigates the effects of industry clockspeed on optimal new-product introduction timing.

Product rollovers, introducing a new product and phasing out another product is the most relevant NPI literature for this study. Tang (2010) classifies product rollover as an operational component of new product development in their literature review for overlapping marketing and operations. Thus, one can come across with various marketing issues such as diffusion models or market segmentation and operational issues such as delayed inventory management in rollover literature.

According to Greenley et al. (1994) most of the time product launching and elimination end up with failure in the sense that company suffers from pure sales and unsatisfied. Motivated by empirical findings like this one and market practices, there has been research on product launch and product elimination (product rollovers). However, product rollover remains an understudied research area in comparison with its significance in NPI according to Lim and Tang (2006). Two strategical studies of product rollovers and new products are Billington et al. (1998) and Erhun et al. (2007). Billington et al.(1998) introduces market and product risk factors in managing product rollovers, conceptualize primary and contingency strategies to cope with risk factors and discusses two type of primary strategies: Single and Dual rolls. Erhun et al. (2007) provides a formal process for managing product transitions with their empirical study at Intel Corporation. In their analysis, they discuss product rollover risks, departmental factors to anticipate these risks and change of these factors over time. With these analyses, they provide a general process for mapping scenarios of demand and supply risks, effect of old product on new product and effected outcome of product transition with strategies to prevent risks and strategies to be able to manage product transition given risks.

Lim and Tang (2006) approaches managing product rollovers from an analytical point of view. They provide a model with deterministic demand when new product is ready to be introduced and old product can be eliminated any time and make pricing and timing decisions. Dual and single rolls are also discusses extensively in this paper and conditions when one of them is preferred over another is provided theoretically. Moreover, they also introduce a demand model, which deals with loyalty factors concerning the loyal customers that go on to buy the existing product in oppose to the unloyal customers with preferences shifted on behalf of new product. A recent paper by Koca et al. (2010) studies product rollover strategy of a firm using dynamic pricing. They correlate market risk and optimal rollover strategy: single versus dual rollover strategy. They also integrate inventory decisions to their model. Moreover, they provide optimal pricing path given reservation prices. Their study includes diffusion and preannouncements as well. Li and Gao (2008) discusses value of sharing upstream information in solo product rollovers. Arslan et al. (2009) is a comprehensive paper in the sense that it provides optimal timing and pricing strategies in both competitive and monopoly setting where prices of new products are dependent on existing products. Our study is different from them in the sense that we consider consumer-driven substitution but we do not discuss concepts such as diffusion, dynamic pricing or sharing information.

Most of the literature of NPI approaches the issue as product upgrades or product line extensions but not specify as rollover strategy. Moorthy and Png (1992) discusses product line extensions, a variant of an existing product, and identify the conditions under which solo or dual product rollover is optimal. Their demand is assumed to be dependent on quality and quality levels are also considered as a decision variable. Some papers incorporate market segmentation and price discrimination to either a solo or dual rollover. Bala and Carr (2005) and Bala and Carr (2009) are among the papers which discusses optimal pricing for a solo rollover under price discrimination and models demand using utility theory. They also consider market segmentation with different levels of product improvement. Wang and Li (2008), on the other hand, discusses similar settings under a dual rollover case. Wilhelm et al. (2003) is another paper which consider solo rollover strategy by providing operations side of new product introduction with manufacturing and supplying decisions according to different product design decisions. Klastorin and Tsai (2004) provides optimal dynamic strategy of a firm committed to a dual rollover under a competitive setting. They integrate product diffusion into their model as well. Kornish et al. (2008) discusses timing and pricing decisions when demand erodes in time and production is time consuming. Our study is different from this literature in the sense that we consider both solo and dual rollover strategies together with inventory/manufacturing policy. Moreover, we do not include marketing concepts such as product diffusion, market segmentation, utility theory or price discrimination in our model.

There are two papers that integrate consumer-driven substitution with product rollover strategy to the extent we are aware. One of them is Li and Shen (2008) which shows optimal timing of a new product when a firm decides to commit a dual rollover. They use diffusion model in their discussion. A more recent paper, Li et al. (2010) studies a similar setting with the decision of offering substitution looking at inventory levels of products. Our study is different than these two papers in the sense that we include two dual rollover strategies and a solo rollover. Figure 2 compares literature of product upgrades and rollovers with our study.

	Demand		Decisions			Rollover Strategy		Other Characteristics			
Papers	Stochastic Demand	Correlated Demand	Timing	Pricing	Inventory Planning	Solo Rolls	Dual Rolls	Consumer-Driven Substitution	Multiple Periods	Competition	Other Subjects
Our Study	X	X			X	X	X	X	X		Allows Extension for Timing and Pricing Decisions
Lim et al. (2006)			X	X		X	X		X		-
Arslan et al. (2009)			X	X		X	X		X	X	Rollback Product Introduction Generation Skipping Policy
Koca et al. (2010)	X		X	X	X	X	X		X		Product Diffusion Model Preannouncements
Li et al. (2010)	X		X		X		X	X	X		Dynamic Decision to Offer Stock-out Induced Substitution
Li and Gao (2008)	X				X	X			X		Product Obsolescence Contract Management Information Sharing
Billington et al. (1998)	Rollover Strategies										
Erhun et al. (2009)	Rollover Strategies										
Greenly et al. (1994)	Comparative Study of Rollovers										
Li et al. (2008)	X		X		X		X	X	X		Product Diffusion Model Early Substitution
Moorthy (1992)			X			X	X		X		Quality Dependent Demand
Bala et al. (2005)	X			X		X			X		Market Segmentation, Utility Theory, Price Discrimination
Bala et al. (2009)	X			X		X			X		Market Segmentation, Utility Theory, Price Discrimination
Klastorin et al. (2004)	X			X			X		X	X	Utility Theory, Dynamic Pricing Product Diffusion
Kornish et al. (2008)	X		X	X	X				X		Time-Consuming Production Product Obsolescence
Wang et al. (2008)	X		X	X			X		X		Market Segmentation, Price Discrimination
Wilhelm et al. (2003)			X		X	X			X		Product Design, Suppliers and Manufacturing

Figure 2.1: Literature for Rollovers and Upgrades

Another main research area of this study is inventory management with stock-out induced substitution which refers to the substitution due to stock outs broadly. Inventory literature with stock-out based substitution is extensively studied according to Nagarajan and Rajagopalan (2009). Substitution might be a result one of the two sources; consumer and decision maker as Nagarajan and Rajagopalan (2009) discusses. In case of decision maker driven substitution, decision maker offers solution such as transshipment of goods from one location to the other in case of stock-outs to prevent loss sales. Herer et al. (2002), Paterson et al. (2009) and Dong and Rudi (2004) includes comprehensive literature review on lateral transshipments, considers transshipment's effect on manufacturers and retailers and discusses role of transshipments in management of supply chain and designing both cost efficient and customer responsive supply chain system, respectively.

Consumer-driven substitution, on the other hand, occurs when customers are willing to consume an alternative product when one product is out of stock. There is an extensive research on stock-out induced substitution. However, they use different assumptions regarding consumer substitution behavior, demand structure, number of products and periods and dynamic versus static substitution. McGillivary and Silver (1978) is an early study, which considers two substitute products with partial substitution and stochastic demand. They use simulation and heuristics in their numerical analysis. Parlar and Goyal (1984) studies the same problem and show that expected profit functions are concave under a wide range of parameter settings. Parlar (1985) models new and old products with partial substitution over two periods using newsvendor problem structure. In a later study by Parlar (1988), an oligopoly market is analyzed and stock-out induced substitution across products of competitors is modeled using newsvendor structure. In finding optimal policy game theoretical framework is utilized. Pasternack and Drezner (1991) compares full substitution with no substitution in a single period. Drezner et al. (1995) considers an EOQ model with no, full and partial substitution and substitution is penalized with a cost and it is concluded that full substitution is always worse than no or partial substitution under their non-linear model. Rajaram and Tang (2001) studies inventory model with partial substitution and correlated demand. Netessine and Rudi (2003) studies an inventory policy of a multiple-product case with partial substitution and correlated demands. In oppose to the case of two periods, they claim that epected profit is not necessarily concave in multiple periods. Nagarajan and Rajagopalan (2009) develops a model to analyze multiple period inventory problem with partial substitution and stochastic demand. They conclude that under certain conditions, inventory policies of substitutable products are independent, partially decoupled. They also provide a numerical analysis using industry data. Mahajan and van Ryzin (2001) studies the case where customers dynamically decide which product to choose to maximize their utility according to inventory levels. Hopp and Xu (2008) brings static approximation to the case where there is dynamic substitution under competition. For a comprehensive literature review on stock-out induced substitution reader is referred to Mhajan and Van Ryzin (1998). In the following table, we locate our study among the closest consumer-driven substitution literature that we discuss as:

Papers	Correlated Demand	Partial Substitution	Competition in the Market	Multiple Periods
Our Study	X	X		
Nagarajan and Rajagopalan (2009)	X	X		X
Netessine and Rudi (2003)	X	X		X
Rajaram and Tang (2001)	X	X		
Parlar (1985)		X		X
Parlar and Goyal (1984)		X		
McGillivary and Silver (1978)		X		
Parlar (1988)		X	X	
Mahajan and van Ryzin (1998)	Literature Review			

Figure 2.2: Literature for Consumer Driven Substitution

In this study, product rollover strategies, dual and solo rolls are discussed as primary strategies. Moreover, we incorporate consumer-substitution concept as a significant issue in making contingency plans. Our study brings inventory policy and new product introduction issues together in a two-period monopoly setting. To our knowledge, we are the first to compare primary rollover strategies under consumer-driven substitution. Another contribution of our study is to present hypotheses examining to solo/dual rolls, early/late introduction, monopoly driven substitution under different market conditions. These hypotheses are verified with different literature including product rollovers, consumer-driven substitution and monopoly innovation.

Chapter 3

Model

In this chapter, we assume that the decision maker determines its primary rollover strategy before introducing primary product in the market and once this decision is made, it can not change its product portfolio. This assumption is reasonable since in the practice rollover strategies are long term decisions and often are associated with huge investments. In the short term, firm can determine the order-up-to-levels for its supplies at the beginning of each period and reacts to market conditions. Prices and period lengths of each period are assumed to be fixed.

As a consequence of timing decision with fixed period lengths, we analyze four cases, each of which are possibilities regarding the existence of secondary product and non-existence of primary product in each period. This chapter begins with the base case where only the primary product exists in both of the periods. In latter sections, introduction of secondary product and/or elimination of the primary product are integrated into the model. Assuming the introduction of primary product in all cases, four different scenarios except the base case are possible. We examine three of them in Sections 3.2, 3.3, and 3.4. The case where both of the products exist in both of the periods is omitted since it is not related to our discussion where the focus is given to introduction of a secondary product and the management of product rollover.

In Table 3.1, we show notation for Chapter 3 where primary product (secondary product) and product 1(product 2) are used interchangeably.

Table 3.1: Notation for Chapter 3

Notation	**Definition**
c_{0i}	Unit ordering/manufacturing cost of product i when it is alone in the market
c_i	Unit ordering/manufacturing cost of product i when it is not alone in the market
h_i	Inventory holding cost for product i
p_{ij}	Price of product i in period j
S_{ij}	Order-up-to level in for product i in period j
${S_{ij}}^*$	Optimal order-up-to level in for product i in period j
T	Investment for Base, IS and ISES Strategies the beginning of first period
U	Investment for IS Strategy at the beginning of second period
K	Investment for ISES Strategy at the beginning of second period
P	Investment for IFES Strategy the beginning of first period
R	Investment for IFES Strategy at the beginning of second period
f_j	Probability density function associated with primary product demand in period j
F_j	Cumulative distribution function associated with primary product demand in period j
g_j	Probability density function associated with secondary product demand in period j
G_j	Cumulative distribution function associated with secondary product demand in period j
$f_j(x_{1j}, x_{2j})$	Joint probability density function in period j

Continued on Next Page...

Table 3.1 – Continued

Notation	**Definition**
r_j	Discount rate for finding net present value of a cash stream j period
$\Pi_1(S_{11})$	Expected profit in period 1 for an order-up-to level of S_{11} for Base, IS and ISES Strategy
$\Pi_1(S_{11}, S_{21})$	Expected profit in period 1 for an order-up-to level of S_{11} of primary and S_{21} of secondary product for IFES Strategy
$\Pi_2(I_{11})$	Expected profit in period 2 for an initial primary product inventory of I_{11} for Base and IS Strategy
$\Pi_2(I_{21})$	Expected profit in period 2 for an initial secondary product inventory I_{21} of IFES Strategy
$\Pi(S_{11})$	Expected total profit for an order-up-to level of $S_{11}$for Base and IS Strategies
$\Pi(S_{21})$	Expected total profit for an order-up-to level of S_{21} for IFES Strategy
$\Pi(S_{11}, S_{21})$	Expected total profit for an order-up-to level of S_{11} and S_{21} for ISES Strategy
$L_2(S_{12})$	Expected profit in period 2 for an order-up-to level of S_{12} without initial inventory (Base Case)
$L_2(S_{12}, S_{22})$	Expected profit in period 2 for an order-up-to level of S_{12} and S_{22} without initial inventory (IS Strategy)
$L_1(S_{11}, S_{21})$	Expected profit in period 1 for an order-up-to level of S_{11} and S_{21} without initial inventory (IFES Strategy)
l_j	Length of period j

3.1 Never Introduce Secondary Product (Base Case)

When there is only the primary product in both of the periods, this problem resembles to the two period newsboy problem. Before introducing the profit functions, we make a few notes on the assumptions. It is assumed that $p_{1j} > c_{01}$ where $j = \{1, 2\}$. As a result, there is a chance to make profit and selling a product makes sense. Since we assume same production related costs, c_{01}, in both of the periods, there would not be any tendency to hold inventory and to sell it in next period. We assume that residual inventory from the previous period has the same quality with newly produced items and can be sold at the same price with them. Another important assumption is that probability demand distributions reflect total demand distributions over given and fixed period lengths. We assume that cash flows occur at the end of each period. For a fixed first period inventory level, say S_{11}, the expected profit function is given as:

$$\Pi_1(S_{11}) = p_{11}\mu_{11} - h_1(S_{11} - \mu_{11}) - c_{01}S_{11} - \int_{S_{11}}^{\infty}(p_{11} + h_1)(x_{11} - S_{11})f_1(x_{11})dx_{11} \tag{3.1}$$

For an order-up-to level of S_{12} with no initial inventory, second period expected profit function is given as:

$$L(S_{12}) = p_{12}\mu_{12} - h_1(S_{12} - \mu_{12}) - c_{01}S_{12} - \int_{S_{12}}^{\infty}(p_{12} + h_1)(x_{12} - S_{12})f_2(x_{12})dx_{12} \tag{3.2}$$

The second derivative of $L(S_{12})$ with respect to S_{12} is as follows:

$$\frac{\partial^2 L}{\partial {S_{12}}^2} = -(p_{12} + h_1)f_2(S_{12}) \tag{3.3}$$

Since $f(x_{12})$, p_{12} and h_1 are positive, we conclude that profit function of the second period, i.e., $L(S_{12})$ is a strictly concave function of S_{12} and there is a unique order-up to level which optimizes the profit function of the second period. To find this optimum point, we equate the first derivative, with respect to S_{12}, to zero and solve the resulting equation for S_{12} which gives following equation:

$$\hat{S_{12}} = F_2^{-1}\left(\frac{p_{12} - c_{01}}{p_{12} + h_1}\right) \tag{3.4}$$

When order-up-to level is $\hat{S_{12}}$ with no initial inventory, second period profit, i.e. $L(\hat{S_{12}})$, is obtained by plugging Equation 3.4 into Equation 3.2 and is shown as in the following;

$$L(\hat{S_{12}}) = (p_{12} + h_1)\int_0^{\hat{S_{12}}} x_{12} f_2(x_{12}) dx_{12} \tag{3.5}$$

Let I_{11} be residual inventory from the first period or equivalently initial inventory of the second period, i.e. $I_{11} = max\left\{0, (S_{11} - x_{11})\right\}$.If initial inventory level is less than $\hat{S_{12}}$, it is optimal to order such that inventory level after ordering is $\hat{S_{12}}$. On the other hand, if inventory level before ordering exceeds $\hat{S_{12}}$, it is optimal not to order and produce anything. Thus, optimum order-up-to-level, ${S_{12}}^*$, can be expressed as:

$$ {S_{12}}^* = \begin{cases} \hat{S_{12}} & \text{for} \quad I_{11} \leq \hat{S_{12}} \\ I_{11} & \text{for} \quad I_{11} > \hat{S_{12}} \end{cases} $$

or, equivalently, as ${S_{12}}^* = max\left\{\hat{S_{12}}, I_{11}\right\}$.

We can write expected profit function of second period with a fixed second period inventory level before ordering of I_{11}, as:

$$ \Pi_2(I_{11}) = \begin{cases} L(\hat{S_{12}}) + c_{01} I_{11} & \text{for} \quad I_{11} \leq \hat{S_{12}} \\ L(I_{11}) + c_{01} I_{11} & \text{for} \quad I_{11} > \hat{S_{12}} \end{cases} $$

Next, we write total profit function, $\hat{\Pi}(S_{11})$, at a fixed inventory level after ordering of S_{11} as:

$$
\begin{aligned}
\Pi(S_{11}) &= -T + r_1\Pi_1(S_{11}) + r_2\Bigg(\int_0^{(S_{11}-\hat{S_{12}})^+} (L(S_{11}-x_{11}) + c_{01}(S_{11}-x_{11}))f_1(x_{11})dx_{11} \\
&+ \int_{(S_{11}-\hat{S_{12}})^+}^{S_{11}} (L(\hat{S_{12}}) + c_{01}(S_{11}-x_{11}))f_1(x_{11})dx_{11} \\
&+ \int_{S_{11}}^{\infty} L(\hat{S_{12}})f_1(x_{11})dx_{11}\Bigg)
\end{aligned} \tag{3.6}
$$

In 3.6, first term, $r_1\Pi_1(S_{11})$ is the net present value of the first period expected profit. If S_{11} is smaller than or equal to $\hat{S_{12}}$, second term vanishes and lower limit of the next term becomes 0. In other words, when S_{11} is smaller than or equal to $\hat{S_{12}}$, it is optimal to order up to the level of $\hat{S_{12}}$ no matter the demand of the first period. However, when S_{11} is larger than $\hat{S_{12}}$, there is the possibility of beginning second period with a an inventory exceeding $\hat{S_{12}}$. When this is the case, it is optimal not to order and begin the second period with the left-over items from the first period. The expression $\int_0^{(S_{11}-\hat{S_{12}})^+}(L(S_{11}-x_{11}) + c_{01}(S_{11}-x_{11}))f_1(x_{11})dx_{11}$ shows the expected profit of the second period when this is the situation. On the other hand, if first period demand is larger than the difference between S_{11} and $\hat{S_{12}}$, it is optimal to order up to the level of $\hat{S_{12}}$ in second period. The third and fourth terms in equation 3.6 show this situation.

Lemma 3.1.1 *The expected total profit function is concave in S_{11} in the regions where $S_{11} \leq \hat{S_{12}}$ and in the region where $S_{11} > \hat{S_{12}}$*

Proof. Second derivative of $\Pi(S_{11})$ with respect to S_{11} when $S_{11} \leq \hat{S_{12}}$ and when $S_{11} > \hat{S_{12}}$ are shown in the following functions, respectively as:

$$
\frac{\partial^2\Pi}{\partial {S_{11}}^2}|S_{11} \leq \hat{S_{12}} \quad = \quad f_1(S_{11})\Big(r_2c_{01} - r_1(h_1 + p_{11})\Big) \tag{3.7}
$$

$$\frac{\partial^2\Pi}{\partial {S_{11}}^2}|S_{11} > \hat{S_{12}} = f_1(S_{11})\Big(r_2c_{01} - r_1(h_1 + p_{11})\Big) - r_2(h_1 + p_{12}) \int_0^{S_{11}-\hat{S_{12}}} f_1(x_{11})f_2(S_{11} - x_{11})dx_{11} \tag{3.8}$$

Since we are able to find second derivatives, we conclude that total profit function is continuously differentiable in the region where $S_{11} \leq \hat{S_{12}}$ and in the region where $S_{11} > \hat{S_{12}}$. Equation 3.7 is negative because of two reasons. First, $r_1(h_1 + p_{11})$ is greater than r_2c_{01} because we have made the usual assumption in the sense that p_{11} is greater than c_{01} and because of the fact that discount factor associated with the first period, i.e. r_1, is greater than the discount factor of total time period, i.e. r_2. Second, we have assumed that we have probability distributions with positive pdf's, i.e. $f_1(x_{11}) > 0$. For Equation 3.8, we notice that it is the summation of Equation 3.7 and a term. We claim that this term is negative. This is true because of the following. $\int_0^{S_{11}-\hat{S_{12}}} f_1(x_{11})f_2(S_{11} - x_{11})dx_{11}$ is convolution of f_1 and f_2 up to a point and since f_1 and f_2 are assumed to be positive, this expression is positive. Thus, the term is negative and this proves the negativity of Equation 3.8. Hence second order conditions for total profit function holds in each of the region and we conclude that $\hat{\Pi}(S_{11})$ is strictly concave in each of the region.■

Next, we investigate first order conditions by equating first derivative of the total profit function to zero. Let Y_1 be the value of S_{11} which makes $\frac{\partial\Pi}{\partial S_{11}}|S_{11} \leq \hat{S_{12}} = 0$. Similarly, let us denote the value of S_{11} which makes $\frac{\partial\Pi}{\partial S_{11}}|S_{11} > \hat{S_{12}} = 0$ as Y_2. Then, first order conditions for the regions of $S_{11} \leq \hat{S_{12}}$ and $S_{11} > \hat{S_{12}}$ are shown as in the following, respectively.

$$\frac{\partial\Pi}{\partial S_{11}}|S_{11} \leq \hat{S_{12}} = r_1(p_{11} - c_{01}) - \{r_1(p_{11} + h_1) - r_2c_{01}\} F_1(Y_1) = 0 \tag{3.9}$$

$$\frac{\partial \Pi}{\partial S_{11}} | S_{11} > \hat{S_{12}} = r_1(p_{11} - c_{01}) - \{r_1(p_{11} + h_1) - r_2 c_{01}\} F_1(Y_2) + r_2\left(\int_0^{Y_2 - \hat{S_{12}}} \{(p_{12} - c_{01}) - (p_{12} + h_1)F_2(Y_2 - x_{11})\} f_1(x_{11})dx_{11}\right) = 0 \tag{3.10}$$

In the region where $S_{11} \leq \hat{S_{12}}$, optimum order level is found as:

$$Y_1 = F_1^{-1}\left(\frac{r_1(p_{11} - c_{01})}{r_1(p_{11} + h_1) - r_2 c_{01}}\right) \tag{3.11}$$

Cost of underage is $r_1(p_{11} - c_{01})$ which is same with one period newsboy problem and cost of overage is $(r_1 h_1 - r_2 c_{01})$, different from one period newsboy problem. This is reasonable because residual inventory from first period is used in the second period.

For remaining region, finding a close form expression for optimum level is not possible without the knowledge of probability distributions, because optimum level is dependent on both first and second period demand. In the following theorem, complete discussion on optimum value for first period order-up-to-level exists and the discussion of this theorem is similar to the discussion in Linh and Hong (2009).

Theorem 3.1.1 *Optimum order-up-to-level or the first period, S_{11}^*, is found as in the following;*

$$S_{11}{}^* = \begin{cases} Y_2 & \text{if} \quad Y_1 > \hat{S_{12}} \\ Y_1 & \text{if} \quad Y_1 \leq \hat{S_{12}} \end{cases}$$

Proof. Proof consists of two parts. If $Y_1 \leq \hat{S_{12}}$, we claim that $\frac{\partial \Pi}{\partial S_{11}} | S_{11} > \hat{S_{12}}$ is negative for value of $\left\{S_{11} : S_{11} > \hat{S_{12}}\right\}$. This claim is true because $r_1(p_{11} -$

$c_{01}) + \{r_1(p_{11} + h_1) - r_2c_{01}\} F_1(S_{11}) < 0$ when $S_{11} > Y_1$. Moreover, when $S_{11} > \hat{S_{12}}$, the expression $\int_0^{S_{11}-\hat{S_{12}}} \{(p_{12} - c_{01}) - (p_{12} + h_1)F_2(S_{11} - x_{11})\} f_1(x_{11})dx_{11}$ is negative, because it is summation of negative values and a zero coming from $(p_{12} - c_{01}) - (p_{12} + h_1)F_2(\hat{S_{12}}) = 0$ as an upper limit. Thus, when $Y_1 \leq \hat{S_{12}}$, we have proved that there is no S_{11} that makes 3.10 valid. Thus, the optimum is Y_1 in this region. On the other hand, if $Y_1 > \hat{S_{12}}$, we claim that Y_2 exists and it is feasible. When $\{S_{11} : S_{11} > Y_1\}$, $\frac{\partial \Pi}{\partial S_{11}}|S_{11} > \hat{S_{12}}$ is negative and when $\left\{S_{11} : S_{11} = \hat{S_{12}}\right\}$, $\frac{\partial \Pi}{\partial S_{11}}|S_{11} > \hat{S_{12}}$ is positive and thus there is a value which makes $\frac{\partial \Pi}{\partial S_{11}}|S_{11} > \hat{S_{12}}$ equal to zero in between because of Lemma 3.1.1. This proves the existence and feasibility of Y_2.■

Consequently, when $Y_1 > \hat{S_{12}}$, secondary period order-up-to-level is found to be $max\left\{\hat{S_{12}}, I_{11}\right\}$. This is true because for an order-up-to-level of Y_2 which is greater than ideal secondary period level of $\hat{S_{12}}$ in the first period, either a level greater than $\hat{S_{12}}$ is carried to the next period where we do not order in the second period. Another possibility is carrying a level less than $\hat{S_{12}}$ and the optimal thing to do is replenishing up to ideal amount of $\hat{S_{12}}$. On the other hand, when $Y_1 \leq \hat{S_{12}}$, optimal secondary period order-up-to level is $\hat{S_{12}}$ because initial inventory for the second period is always less than $\hat{S_{12}}$.

Based on the discussion for the optimal levels of first and second period, the total profit function is given as:

$$\hat{\Pi}({S_{11}}^*) = \begin{cases} -T + \{r_1(p_{11} + h_1) - r_2c_{01}\} \int_0^{Y_1} x_{11}f_1(x_{11})dx_{11} + \\ r_2(p_{12} + h_1) \int_0^{\hat{S_{12}}} x_{12}f_2(x_{12})dx_{12} & \text{if} \quad Y_1 \leq \hat{S_{12}} \\ \\ -T + \{r_1(p_{11} + h_1) - r_2c_{01}\} \int_0^{Y_2} x_{11}f_1(x_{11})dx_{11} + \\ r_2(p_{12} - c_{01}) \int_0^{Y_2-\hat{S_{12}}} x_{11}f_1(x_{11})dx_{11} + \\ r_2(p_{12} + h_1) \int_0^{\hat{S_{12}}} x_{12}f_2(x_{12})dx_{12} + \\ r_2(p_{12} + h_1) \int_0^{Y_2-\hat{S_{12}}} x_{11}F_2(Y_2 - x_{11})f_1(x_{11})dx_{11} + \\ r_2(p_{12} + h_1) \int_{\hat{S_{12}}}^{Y_2} x_{12}F_1(Y_2 - x_{12})f_2(x_{12})dx_{12} & \text{if} \quad Y_1 > \hat{S_{12}} \end{cases}$$

We finish our discussion for the base case. To summarize, we provide close form expression for the second period order-up-to-level. Next, we state optimal solution for the first period inventory level after ordering. Finally, we provided total profit function. In the next subsections, from 3.2 to 3.4, we examine other cases where secondary product exist in one or more periods.

3.2 Secondary Product Introduction in the Second Period (IS)

In the second period, we introduce a new product, the secondary product. We call this strategy as IS strategy which is the abbreviation of "Introduce in the Second". According to Billington et al (1998), in a dual product roll both new and old products exist simultaneously for a period of time. Therefore, IS strategy is a dual product roll.

We follow the research stream which considers two types substitution to model inventory for substitutable products. These are consumer-driven substitution and demands are negatively correlated. Consumer-driven substitution or stock-out-induced substitution exists when customers of a product may switch to the substitute if the product is out of stock. We assume that a fixed proportion of unsatisfied customers of a product may switch to the other product as in Parlar (1988).

In addition to an investment in the first period as in base case, there is an extra investment for redesigning production line for delayed product differentiation. We denote this investment with U.

Decision maker has the option to replenish its stock for both of the products in the beginning of the second period in addition to its option to determine the amount to produce for primary product in the first period.

Notation is a little different than the base case. Parameters and variables are assumed to be not necessarily same for the products: c_i, h_i $p_{i,j}$ and $S_{i,j}$ take a subscript i denoting the associated product where $i = \{1, 2\}$ in addition to the subscript, j, denoting the time. This case has S_{11}, S_{12}, S_{22} as decision variables. Similarly, we denote optimal order-up-to levels by ${S_{i,j}}^*$ where $i, j \in \{1, 2\}$. Moreover, we denote the joint probability density function and joint cumulative distribution function of the demand for secondary product in period j by $f_j(x_{1j}, x_{2j})$ and $F_j(x_{1j}, x_{2j})$, respectively. In addition to the notation of the base case, we have also individual probability distribution function and cumulative distribution function for the secondary product and they are denoted as $g_j(x_{2j})$ and $G_j(x_{2j})$ in the period j, respectively. Regarding the proportions of the unsatisfied customers switching to the other product, we use α and β. α is the proportion of customers switching to the secondary product when primary product is out of stock. Similarly, β denotes the proportion of secondary product customers preferring to use primary product as a second choice demand because of primary product shortages.

There is the chance to make profit on both of the products; thus, $p_{i,j} > c_i$, for each $i = \{1, 2\}$. Moreover, there is a low tendency to hold a product and sell it next period because production and ordering related cost parameter, c_i, is not dependent on time. To guarantee concavity, we make other assumptions regarding the relations of some parameters and they are shown through the discussion of this section.

First period expected profit is same with base case given in 3.1 because there is only the primary product. In the second period, monopoly sells both of the products: primary and the secondary product. There may be primary product inventory from the first period. Thus, residual inventory, I_{11}, may be consumed by customers of both product.

For a fixed second period inventory levels, say S_{12} and S_{22}, if there is no initial inventory of primary product, the expected profit function is given as:

$$
\begin{aligned}
L(S_{12}, S_{22}) &= -U - c_1 S_{12} - c_2 S_{22} + \int_0^{S_{22}} \int_0^{S_{12}} \Big(p_{12}x_{12} + p_{22}x_{22} \\
&- h_1(S_{12} - x_{12}) - h_2(S_{22} - x_{22}) \Big) f_2(x_{12}, x_{22}) dx_{12} dx_{22} \\
&+ \int_0^{S_{22}} \int_{S_{12}}^{\infty} \Big(p_{12}S_{12} + p_{22}(x_{22} + min\{\alpha(x_{12} - S_{12}), (S_{22} - x_{22})\}) \\
&- h_2 \left[(S_{22} - x_{22} - \alpha(x_{12} - S_{12}) \right]^+ \Big) f_2(x_{12}, x_{22}) dx_{12} dx_{22} \\
&+ \int_{S_{22}}^{\infty} \int_0^{S_{12}} \Big(p_{12}(x_{12} + min\{\beta(x_{22} - S_{22}), (S_{12} - x_{12})\}) + p_{22}S_{22} \\
&- h_1 \left[(S_{12} - x_{12} - \beta(x_{22} - S_{22}) \right]^+ \Big) f_2(x_{12}, x_{22}) dx_{12} dx_{22} \\
&+ \int_{S_{22}}^{\infty} \int_{S_{12}}^{\infty} \Big(p_{12}S_{12} + p_{22}S_{22} \Big) f_2(x_{12}, x_{22}) dx_{12} dx_{22}
\end{aligned}
\tag{3.12}
$$

U represents the investment made to modify the system such that some operations of the existing product line become common for both of the products and made to build secondary product specific operations. Following two terms represents the production and ordering related costs. When demand is less than the order-up-to level, amount of sales is equal to the demand and remaining amount is held. Thus, the expression $\int_0^{S_{22}} \int_0^{S_{12}} \Big(p_{12}x12 + p_{22}x22 - h_1(S_{12} - x_{12}) - h_2(S_{22} - x_{22}) \Big) f_2(x_{12}, x_{22}) dx_{12} dx_{22}$ represents the expected profit when demand is less than the initial inventory for both of the products. On the other hand, if demand is larger than the amount on hand, two things can happen. First, if inventory level of substitute is larger than its demand, some of the unsatisfied demand can be met with substitute. After both demand groups are satisfied, there may be still some inventory of the substitute product. The expression $\int_0^{S_{22}} \int_{S_{12}}^{\infty} \Big(p_{12}S_{12} + p_{22}(x_{22} + min\{\alpha(x_{12} - S_{12}), (S_{22} - x_{22})\}) - h_2 \left[(S_{22} - x_{22} - \alpha(x_{12} - S_{12}) \right]^+ \Big) f_2(x_{12}, x_{22}) dx_{12}$ shows the expected profit when primary product is out of stock and secondary product is used to satisfy primary product customers. Similarly, the next expression represents the expected profit when there is unsatisfied demand of secondary product demand. Second, when both of the products are out of stock, amount of sales is equal to the amount on hand. Thus, last expression shows the expected profit of such a situation.

Parlar (1988) shows the expected profit function of a player competing with another player through stock-out-induced substitution and simplifies the function by getting rid of maximum and minimum functions. In our profit function, we do similar simplifications with following analysis:

$$min\,\{\alpha(x_{12}-S_{12}),(S_{22}-x_{22})\} = \begin{cases} \alpha(x_{12}-S_{12}) & \text{for} \quad x_{12} \le (S_{22}-x_{22})/\alpha + S_{12} \\ (S_{22}-x_{22}) & \text{for} \quad otherwise \end{cases}$$

$$[(S_{22}-x_{22})-\alpha(x_{12}-S_{12})]^{+} = \begin{cases} (S_{22}-x_{22})-\alpha(x_{12}-S_{12}) & \text{for} \quad x_{12} \le (S_{22}-x_{22})/\alpha + S_{12} \\ 0 & \text{for} \quad otherwise \end{cases}$$

$$min\,\{\beta(x_{22}-S_{22}),(S_{12}-x_{12})\} = \begin{cases} \beta(x_{22}-S_{22}) & \text{for} \quad x_{22} \le (S_{12}-x_{12})/\beta + S_{22} \\ (S_{12}-x_{12}) & \text{for} \quad otherwise \end{cases}$$

$$[(S_{12}-x_{12})-\beta(x_{22}-S_{12})]^{+} = \begin{cases} (S_{12}-x_{12})-\beta(x_{22}-S_{22}) & \text{for} \quad x_{22} \le (S_{12}-x_{12})/\beta + S_{22} \\ 0 & \text{for} \quad otherwise \end{cases}$$

Using the substitutions of $A = (S_{22}-x_{22})/\alpha+S_{12}$ and $B = (S_{12}-x_{12})/\beta+S_{22}$ in our profit function, we obtain the following simplified second period profit function:

$$
\begin{aligned}
L(S_{12},S_{22}) &= p_{12}S_{12}+p_{22}S_{22}-c_1S_{12}-c_2S_{22}+ \\
&+ \int_0^{S_{22}}\int_{S_{12}}^{A}\Big((p_{22}+h_2)(\alpha(x_{12}-S_{12})-(S_{22}-x_{22})\Big)f_2(x_{12},x_{22})dx_{12}dx_{22} \\
&+ \int_0^{S_{12}}\int_{S_{22}}^{B}\Big(p_{12}+h_1)(\beta(x_{22}-S_{22})-(S_{12}-x_{12})\Big)f_2(x_{12},x_{22})dx_{22}dx_{12} \\
&- \int_0^{S_{22}}\int_0^{S_{12}}\Big((p_{12}+h_1)(S_{12}-x_{12})+(p_{22}+h_2)(S_{22}-x_{22})\Big)f_2(x_{12},x_{22})dx_{12}dx_{22}
\end{aligned}
\tag{3.13}
$$

Theorem 3.2.1 *The expected second period total profit function, i.e. $L(S_{12},S_{22})$ is jointly concave in S_{12} and S_{22} if $(p_{12}+h_1) > \alpha(p_{22}+h_2)$ and $(p_{22}+h_2) > \beta(p_{12}+h_1)$ where $\alpha \neq 0$ and $\beta \neq 0$*

Proof.Second derivatives of $L(S_{12},S_{22})$ with respect to S_{12} and S_{22} are as follows, respectively:

$$
\begin{aligned}
\frac{\partial^2 L}{{S_{12}}^2} &= (\alpha(p_{22}+h_2)-(p_{12}+h_1))\int_0^{S_{22}} f_2(S_{12},x_{22})dx_{22} \\
&- \alpha(p_{22}+h_2)\int_0^{S_{22}} f_2(A,x_{22})dx_{22} \\
&- (p_{12}+h_1))/\beta\int_0^{S_{12}} f_2(x_{12},B)dx_{12}
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
\frac{\partial^2 L}{{S_{22}}^2} &= (\beta(p_{12}+h_1)-(p_{22}+h_2))\int_0^{S_{12}} f_2(x_{12},S_{22})dx_{12} \\
&- \beta(p_{12}+h_1)\int_0^{S_{12}} f_2(x_{12},B)dx_{12} \\
&- \frac{(p_{22}+h_2)}{\alpha}\int_0^{S_{22}} f_2(A,x_{22})dx_{22}
\end{aligned}
\tag{3.15}
$$

$$\begin{aligned}\frac{\partial^2 L}{\partial S_{12}\partial S_{22}} &= -(p_{12}+h_1)\int_0^{S_{12}} f_2(x_{12},B)dx_{12} \\ &- \quad (p_{22}+h_2))\int_0^{S_{22}} f_2(A,x_{22})dx_{22}\end{aligned} \tag{3.16}$$

$$\begin{aligned}\frac{\partial^2 L}{\partial S_{22}\partial S_{12}} &= -(p_{12}+h_1)\int_0^{S_{12}} f_2(x_{12},B)dx_{12} \\ &- \quad (p_{22}+h_2))\int_0^{S_{22}} f_2(A,x_{22})dx_{22}\end{aligned} \tag{3.17}$$

Then it is clearly seen that if $(p_{12}+h_1) > \alpha(p_{22}+h_2)$, 3.14,which is first leading principal of Hessian matrix is negative. Moreover, if $(p_{22}+h_2) > \beta(p_{12}+h_1)$, in addition to previous condition, determinant of Hessian Matrix is positive. Then, Hessian is negative definite. This proves Theorem 3.2.1.■

Thus, parallel to Nagarajan and Rajagopalan (2009), we need conditions of $(p_{12}+h_1) > \alpha(p_{22}+h_2)$ and $(p_{22}+h_2) > \beta(p_{12}+h_1)$ in addition to positive substitution rates to guarantee that profit function is jointly concave in S_{12} and S_{22}. If we assume that $h_1 \leq h_2$, these conditions make stocking a product worthwhile by eliminating the possibility of earning higher revenue by not stocking the original product but increasing the stock of substitute product.

Next, for first order necessary optimality conditions, we equate first partial derivative of second period profit function with respect to S_{12} to zero, i.e. $\frac{\partial L}{S_{12}}$. Similarly, we find first partial derivative of second period profit function with respect to S_{22} and equate it to zero. First order conditions are shown in the following equations as:

$$\begin{aligned}\frac{\partial L}{\partial S_{12}} &= -c_1 + p_{12} - (p_{12}+h_1)\int_0^{S_{12}}\int_0^{B} f_2(x_{12},x_{22})dx_{22}dx_{12} \\ &- \alpha(p_{22}+h_2)\int_0^{S_{22}}\int_{S_{12}}^{A} f_2(x_{12},x_{22})dx_{12}dx_{22} = 0\end{aligned} \tag{3.18}$$

$$\begin{aligned}\frac{\partial L}{\partial S_{22}} &= -c_2 + p_{22} - (p_{22}+h_2)\int_0^{S_{22}}\int_0^{A} f_2(x_{12},x_{22})dx_{12}dx_{22} \\ &- \beta(p_{12}+h_1)\int_0^{S_{12}}\int_{S_{22}}^{B} f_2(x_{12},x_{22})dx_{22}dx_{12} = 0\end{aligned} \tag{3.19}$$

In Netessine and Rudi (2003), a formula is given to express first order conditions by using an alternative technique other than Leibniz's formula. According to the related proposition, first order conditions for S_{12} and S_{22} could be expressed by the following equations, respectively:

$$\begin{aligned}\frac{p_{12}-c_1}{p_{12}+h_1} &= Pr(x_{12}<S_{12}) - Pr(x_{12}<S_{12}<x_{12}+\beta(x_{22}-S_{22})) + \\ &+ \frac{\alpha(p_{22}+h_2)}{p_{12}+h_1}Pr((x_{22}+\alpha(x_{12}-S_{12})<S_{22}),(x_{22}<S_{22}))\end{aligned} \tag{3.20}$$

$$\begin{aligned}\frac{p_{22}-c_2}{p_{22}+h_2} &= Pr(x_{22}<S_{22}) - Pr(x_{22}<S_{22}<x_{22}+\alpha(x_{12}-S_{12})) + \\ &+ \frac{\beta(p_{12}+h_1)}{p_{22}+h_2}Pr((x_{12}+\beta(x_{22}-S_{22})<S_{12}),(x_{12}<S_{12}))\end{aligned} \tag{3.21}$$

First order conditions found with Leibniz rule of differentiation under the integral sign, 3.18 and 3.19 are parallel with 3.20 and 3.21. This way of expressing makes one to compare optimum order-up-to levels with one period newsboy problem. These equations have intuitive interpretations, as in Nagarajan and Rajagopalan (2009), and are explained in following discussion. In general, equations turn out to be newsboy first order conditions without the second term which adjust optimal order-up-to-level upwards due to customer switches from other product and third term which adjust the optimal value downwards due to decrease in opportunity cost of not stocking with switches to the other product. In particular, as substitution rate from the product increases, optimal order-up-to-level for that product decreases because of the third term. On the other hand as switching rate to product increases, order-up-to level of that product increases because of the second term. As a summary, in addition to the probability of using the product for both demand groups the probability of using the substitute for the product in case of stock-out is considered and the sum is equated to the newsboy ratio. However, when considering the possibility of eliminating a portion of lost sales through the substitute, a discount factor is used. In particular, discount factor when considering the lost sales of S_{12} is $\frac{\alpha(p_{22}+h_2)}{p_{12}+h_1}$. Similarly, $\frac{\beta(p_{12}+h_1)}{p_{22}+h_2}$ is used as a discount factor for the case of S_{22}. These discount factors are assumed to be less than 1, because of our assumption to guarantee concavity. Therefore, the possibility of eliminating lost sales with the substitute and downward pressure on the amount of the product is limited. Hence, we say that our model is relatively conservative in decreasing the amount of a product by considering stock-out-induced substitution to the other product.

As a result of Theorem 3.2.1, there exists unique optimum solutions, $\tilde{S_{12}}$ and $\tilde{S_{22}}$. First order conditions are found to be curves in (S_{12}, S_{22}) plane. Therefore, optimum order-up-to levels are found by solving them simultaneously.

Nagarajan and Rajagopalan (2009) provides close form expressions under certain parameters and distributions. On the other hand, here solving such a system without any knowledge of distributions or parameters is quite tedious. In the next chapters, Chapter 4 with explicit demand functions and Chapter 5 with assumed demand distributions, we provide solutions to optimum levels for replenishment amounts.

If the left over inventory from the first period is less than $\tilde{S_{12}}$, it is optimal to replenish primary product inventory up to this level. On the other hand, if the amount left over is larger than $\tilde{S_{12}}$, we do not replenish because of concavity. Thus, it is optimal to order-up-to, ${S_{12}}^*$, where ${S_{12}}^* = max\left\{\tilde{S_{12}}, I_{11}\right\}$. It is important to note that, value of ${S_{22}}^*$ is dependent on the value of ${S_{12}}^*$ because the left over inventory comes from only the primary product. To be more specific, ${S_{22}}^*$ is equal to $\tilde{S_{22}}$ if ${S_{12}}^*$ takes value of $\tilde{S_{12}}$. On the other hand, when ${S_{12}}^*$ is equal to I_{11}, ${S_{22}}^*$ takes the value that makes $\frac{\partial L}{\partial S_{12}} = 0$. We call this value as ${S_{22}}^*(I_{11})$. Next, second period expected profit function for an initial inventory level of I_{11} is shown as:

$$\Pi_2(I_{11}) = \begin{cases} L(\tilde{S_{12}}, \tilde{S_{22}}) + c_1(I_{11}) & \text{for} \quad I_{11} \leq \tilde{S_{12}} \\ L(I_{11}, {S_{22}}^*(I_{11})) + c_1(I_{11}) & \text{for} \quad I_{11} > \tilde{S_{12}} \end{cases}$$

Inspired by further analysis on properties of first order conditions of Parlar (1998), we provide following lemmas:

Lemma 3.2.1 $\frac{\partial L}{\partial S_{12}} = 0$ *is a strictly decreasing curve in the* (S_{12}, S_{22}) *plane, given that* $(p_{12} + h_1) > \alpha(p_{22} + h_2)$ *and* $\beta \neq 0$

Proof. Being unable to write S_{22} as a function of S_{12} from $\frac{\partial L}{S_{12}} = 0$, we use implicit differentiation. Let du/dS_{12} be the derivative of $\frac{\partial L}{S_{12}} = 0$ at (S_{12}, S_{22}). Then following holds:

$$\begin{aligned} \frac{du}{dS_{12}} &= \frac{(\alpha(p_{22}+h_2) - (p_{12}+h_1)) \int_0^{S_{22}} f_2(S_{12}, x_{22}) dx_{22}}{(p_{12}+h_1) \int_0^{S_{12}} f_2(x_{12}, B) dx_{12} + (p_{22}+h_2) \int_0^{S_{22}} f_2(A, x_{22}) dx_{22}} \\ &- \frac{(p_{12}+h_1) \int_0^{S_{12}} f_2(x_{12}, B) dx_{12}}{\beta \left\{ (p_{12}+h_1) \int_0^{S_{12}} f_2(x_{12}, B) dx_{12} + (p_{22}+h_2) \int_0^{S_{22}} f_2(A, x_{22}) dx_{22} \right\}} \\ &- \frac{\alpha(p_{22}+h_2) \int_0^{S_{22}} f_2(A, x_{22}) dx_{22}}{(p_{12}+h_1) \int_0^{S_{12}} f_2(x_{12}, B) dx_{12} + (p_{22}+h_2) \int_0^{S_{22}} f_2(A, x_{22}) dx_{22}} \end{aligned} \tag{3.22}$$

Because of each probability distribution function being assumed to be positive and β given positive, above function is continuous. Moreover, because it is assumed that $(p_{12} + h_1) > \alpha(p_{22} + h_2)$, all of the terms in 3.22 are negative.■

Lemma 3.2.2 $\frac{\partial L}{\partial S_{22}} = 0$ *is a strictly decreasing curve in the* (S_{12}, S_{22}) *plane, given that* $(p_{22} + h_2) > \beta(p_{12} + h_1)$ *and* $\alpha \neq 0$

Proof Similar to Lemma 3.2.1, we use implicit differentiation. Let dv/dS_{22} be the derivative of $\frac{\partial L}{\partial S_{22}} = 0$ at (S_{12}, S_{22}):

$$\frac{dv}{dS_{12}} = -\frac{(p_{12} + h_1)\int_0^{S_{12}} f_2(x_{12}, B)dx_{12} + (p_{22} + h_2)\int_0^{S_{22}} f_2(A, x_{22})dx_{22}}{\eta} \tag{3.23}$$

where η is equal to $(p_{22} + h_2 - \beta(p_{12} + h_1))\int_0^{S_{12}} f_2(x_{12}, S_{22})dx_{12} + \beta(p_{12} + h_1)\int_0^{S_{12}} f_2(x_{12}, B)dx_{12} + (\frac{(p_{22}+h_2)}{\beta})\int_0^{S_{22}} f_2(A, x_{22})dx_{22}$

Given that $(p_{22} + h_2) > \beta(p_{12} + h_1)$, it turns out that 3.23 is negative.■

To summarize, conditions for concavity of second profit function in S_{12} and S_{22} guarantee optimum level of S_{22} be a decreasing function of S_{12} and optimum level S_{12} being a decreasing function of S_{22}, respectively. As a result of these lemmas, we can present upper and lower bounds on optimum levels of S_{12} and S_{22} as the discussion in Parlar (1988). These bounds are important for numerical analysis of Chapter 5. For finding upper and lower bounds on optimum S_{12}, we equate S_{22} to 0 and ∞, respectively, in 3.18. By doing so, we obtain following equation for upper bound, say $\overline{S}_{12}$ as:

$$\int_0^{\overline{S}_{12}}\int_0^{\frac{\overline{S}_{12}-x_{12}}{\beta}} f_2(x_{12}, x_{22})dx_{22}dx_{12} = \frac{p_{12} - c_1}{p_{12} + h_1} \tag{3.24}$$

Without knowledge on joint probability distribution function for the demands, it is not possible to express $\overline{S}_{12}$ explicitly. We can rewrite 3.24 as

$Pr(\beta x_{22} + x_{12} < \overline{S}_{12}) = \frac{p_{12}-c_1}{p_{12}+h_1}$. Therefore, by defining $\beta x_{22} + x_{12}$ as a random variable, inverse c.d.f of this random variable at the newsboy ratio is equal to $\overline{S}_{12}$. This is reasonable because if we were not to stock secondary product, demand faced would be primary product demand plus the unsatisfied secondary product demand ready to use primary product, i.e. $\beta x_{22} + x_{12}$.

For the lower bound on S_{12}, say $\underline{S}_{12}$, the following equation is obtained from 3.18 by equating S_{22} to ∞ as:

$$\int_0^{\underline{S}_{12}} \int_0^{\infty} f_2(x_{12}, x_{22}) dx_{22} dx_{12} = \frac{p_{12} - c_1 - \alpha(p_{22} + h_2)}{p_{12} + h_1 - \alpha(p_{22} + h_2)} \tag{3.25}$$

Therefore, lower bound for S_{12} is equal to the following:

$$\underline{S}_{12} = F_2^{-1}\left(\frac{p_{12} - c_1 - \alpha(p_{22} + h_2)}{p_{12} + h_1 - \alpha(p_{22} + h_2)}\right) \tag{3.26}$$

If $p_{12} - c_1 - \alpha(p_{22} + h_2) > 0$ following consideration holds. When we stock infinitely many of secondary product, cost of underage for primary product is $p_{12} - c_1 - \alpha(p_{22} + h_2)$ and cost of overage is $h_1 - c_1$. Cost of overage is same with one period newsboy problem whereas cost of underage is adjusted downwards with stock-out-induced substitution to secondary product. Hence, $\hat{S_{12}}$ will be somewhere in between, $\overline{S}_{12}$ and $\underline{S}_{12}$.

Similar arguments apply to secondary product order-up to level as well and upper and lower value are found from the equations $Pr(\alpha x_{12} + x_{22} < \overline{S}_{22}) = \frac{p_{22}-c_2}{p_{22}+h_2}$ and $Pr(x_{22} < \underline{S}_{22}) = \frac{p_{22}-c_2-\beta(p_{12}+h_1)}{p_{22}+h_2-\beta(p_{12}+h_1)}$, respectively. Not being able to find an expression for upper bound, lower bound for the secondary product optimal order-up-to-level in the second period is found as provided that $p_{22} - c_2 - \beta(p_{12} + h_1) > 0$:

$$\underline{S}_{22} = G_2^{-1}\left(\frac{p_{22} - c_2 - \beta(p_{12} + h_1)}{p_{12} + h_1 - \beta(p_{12} + h_1)}\right) \tag{3.27}$$

Similar to the base case, expectedtotal profit function with an order-up-to level of S_{11} is given as:

$$\begin{aligned}\Pi(S_{11}) &= -T - r_1 U + r_1 \Pi(S_{11}) + r_2 \Bigg(\int_0^{(S_{11}-\tilde{S_{12}})^+} (L(S_{11}-x_{11}, {S_{22}}^*(S_{11}-x_{11})) \\ &+ c_1(S_{11}-x_{11})) f_1(x_{11}) dx_{11} + \int_{(S_{11}-\tilde{S_{12}})^+}^{S_{11}} (L(\tilde{S_{12}}, \tilde{S_{22}}) + c_1(S_{11}-x_{11})) f_1(x_{11}) dx_{11} \\ &+ \int_{S_{11}}^{\infty} L(\tilde{S_{12}}, \tilde{S_{22}}) f_1(x_{11}) dx_{11} \Bigg)\end{aligned} \tag{3.28}$$

First term of $\Pi(S_{11})$ is net present value of expected first period profit where as the second term shows the discounted second period profit. If $S_{11} > \tilde{S_{12}}$ following discussion holds. The last term is composed of three parts. First part denotes the expected second period profit when primary product demand is such that the residual inventory of the first period, i.e. I_{11}, is bigger than the optimum order-up-to level for the primary product, $\tilde{S_{12}}$. Therefore, we do not produce extra amount for primary product and replenish secondary product such that it is equal the best possible value given the primary product order-up-to level of I_{11}. On the other hand, in the second and third parts of the second term, order-up-to levels for both of the product are in the optimum levels because first period primary product demand is such that amount left over for primary product, I_{11}, is less than the optimal level, $\tilde{S_{12}}$. In the second part, production in the amount of the difference between the optimal level and the left over amount occurs. On the other hand, in the third part, there is a production in the full amount of optimal order-up-to level for the primary product. On the other hand, if $S_{11} \leq \tilde{S_{12}}$, first part of the second term vanishes and the lower limit for the second part changes to 0.

Next, we discuss concavity of the total profit function. Second order condition for the expected profit function when $S_{11} \leq \tilde{S_{12}}$ is given as:

$$\frac{\partial^2 \Pi}{\partial {S_{11}}^2} = -f_1(S_{11}) \left\{ r_1(h_1 + p_{11}) - r_2 c_1 \right\} \tag{3.29}$$

We conclude that the function, Π_2 is strictly concave since $\{r_1(h_1 + p_{11}) - r_2 c_1\}$

is positive due to our assumptions regarding the discount rates and the relationship of p_{11} and c_1. As a result, total profit function is concave when $S_{11} \leq \tilde{S_{12}}$ with first order condition given as:

$$\frac{\partial \Pi}{\partial S_{11}} = r_1(p_{11} - c_{01}) - F_1(S_{11})\{r_1(h_1 + p_{11}) - r_2 c_1\} = 0 \tag{3.30}$$

Therefore, we conclude that optimal order-up-to level for the primary product in the first period is as follows:

$$F_1^{-1}(\frac{r_1(p_{11} - c_{01})}{r_1(h_1 + p_{11}) - r_2 c_1}) \tag{3.31}$$

if this value is feasible, i.e. it is less than $\tilde{S_{12}}$. We recall that this value is same with the Y_1 of base case.

On the other hand, when $S_{11} > \tilde{S_{12}}$, first order condition of total profit function with respect to S_{11} is found as in the following:

$$\begin{aligned}\frac{\partial \Pi}{\partial S_{11}} &= r_1(p_{11} - c_{01}) - F_1(S_{11})\{r_1(h_1 + p_{11}) - r_2 c_1\} \\ &+ \int_0^{(S_{11} - \tilde{S_{12}})} \frac{\partial L(S_{11} - x_{11}, S_{22}^*(S_{11} - x_{11}))}{\partial S_{11}} f_1(x_{11}) dx_{11}\end{aligned} \tag{3.32}$$

In the appendix it is shown that total profit function is strictly concave in this region if Equation 3.22 is smaller than Equation 3.23 at the point $(S_{11} - x_{11}, S_{22}^*(S_{11} - x_{11}))$. This condition is dependent on several factors including price levels, demands and substitution rates. Equation 3.22 denotes the required increase (decrease) in order-up-to level for secondary product to push optimal order-up-to level for primary product to decrease (increase) whereas Equation 3.23 shows the optimal order-up-to for secondary product given a change in primary product order-up-to level.

In the Appendix A, it is shown that when $(\frac{\alpha - \beta}{\beta} + \alpha)(p_{22} + h_2) > (p_{12} + h_1)$ and $\alpha > \beta$, total profit function is concave and in the Appendix B three examples are given. These instances can be explained by the following arguments. For concavity, we need total opportunity costs for underage and overage of second choice demand for primary product to be less than the costs of first choice demand. However, this difference is limited because to guarantee concavity we also need that substitution rate of primary product customers is large enough compared to secondary product to ensure $(\frac{\alpha-\beta}{\beta}+\alpha)(p_{22}+h_2) > (p_{12}+h_1)$. Primary product customers should be such eager to use second choice product that change in the optimal secondary product as a result of change in secondary product is larger than the required change in secondary product level to push optimal level for primary product at the same level. This happens because changes in optimal secondary product order-up-to level as a result of a change in the primary product are more sensitive to primary product substitution rates. On the other hand, the associated change for pushing optimal primary product level to change is more sensitive to substitution rates of secondary product.

When the total profit function is concave, we claim that 3.32 is negative for $S_{11} > Y_1$ and positive for $S_{11} = \tilde{S_{12}}$. This claim is true because of the following arguments. For $S_{11} > Y_1$, the expression $r_1(p_{11}-c_{01})-F_1(S_{11})\left\{r_1(h_1+p_{11})-r_2c_1\right\}$ is negative because $F_1(S_{11})$ is greater than $\frac{r_1(p_{11}-c_1)}{r_1(h_1+p_{11})-r_2c_1}$. Rest of 3.32 is negative because $\frac{\partial L(S_{11}-x_{11},S_{22}^*(S_{11}-x_{11}))}{\partial S_{11}}$ takes negative values when $x_{11} > S_{11}-\tilde{S_{12}}$ as a conclusion of Lemma 3.2.1 and is zero when $x_{11} = S_{11}-\tilde{S_{12}}$. When $S_{11} = \tilde{S_{12}}$, $F_1(S_{11})$ is less than $\frac{r_1(p_{11}-c_{01})}{r_1(h_1+p_{11})-r_2c_1}$ and hence first part of 3.32 is positive. Rest of the equation is negative because $\frac{\partial L(\tilde{S_{12}},\tilde{S_{22}})}{\partial S_{11}}$ is zero as a conclusion of Lemma 3.2.1. Thus, 3.32 takes a zero value in between Y_1 and $\tilde{S_{12}}$ and we denote this as Y_3.

With above argument we have proven the existence and feasibility of the value found from 3.32, Y_3. Moreover, we have also proven that 3.32 is negative at Y_1 and thus, optimal level is Y_3 if $\tilde{S_{12}} < Y_1$. On the other hand, if $\tilde{S_{12}} \geq Y_1$, the optimal value is Y_1 because 3.32 is negative for $\tilde{S_{12}} < S_{11}$ and thus, there is no candidate feasible point from the second region for optimality.

As a conclusion of above discussions, we write optimal points for a concave total profit function as in the following:

$$S_{11}{}^{*} = \begin{cases} Y_3 & \text{if} \quad Y_1 > \tilde{S_{12}} \\ Y_1 & \text{if} \quad Y_1 \leq \tilde{S_{12}} \end{cases}$$

$$S_{12}{}^{*} = \begin{cases} max\left\{\tilde{S_{12}}, I_{11}\right\} & \text{if} \quad Y_1 > \tilde{S_{12}} \\ \tilde{S_{12}} & \text{if} \quad Y_1 \leq \tilde{S_{12}} \end{cases}$$

$$S_{22}{}^{*} = \begin{cases} min\left\{\tilde{S}_{22}, S_{22}^{*}(I_{11})\right\} & \text{if} \quad Y_1 > \tilde{S_{12}} \\ \tilde{S_{22}} & \text{if} \quad Y_1 \leq \tilde{S_{12}} \end{cases}$$

Based on the discussion for the optimal levels of first and second period, the total profit function, when it is concave, is given as:

$$\Pi({S_{11}}^{*}) = \begin{cases} -r_1U - T + \{r_1(p_{11}+h_1) - r_2c_1\}\int_0^{Y_1} x_{11}f_1(x_{11})dx_{11} + \\ r_2(p_{22}+h_2)\int_0^{\tilde{S}_{22}}\int_{\tilde{S}_{12}}^{A}(\alpha x_{12}+x_{22})f_2(x_{12},x_{22})dx_{12}dx_{22} + \\ r_2(p_{12}+h_1)\int_0^{\tilde{S}_{12}}\int_{\tilde{S}_{22}}^{B}(\beta x_{22}+x_{12})f_2(x_{12},x_{22})dx_{22}dx_{12} + \\ r_2\int_0^{\tilde{S}_{12}}\int_0^{\tilde{S}_{22}}((p_{12}+h_1)x_{12}+(p_{22}+h_2)x_{22})f_2(x_{12},x_{22})dx_{22}dx_{12} & \text{if} \quad Y_1 \leq \tilde{S_{12}} \\ \\ -r_1U - T + \{r_1(p_{11}+h_1) - r_2c_1\}\int_0^{Y_3} x_{11}f_1(x_{11})dx_{11} + \\ r_2(1-F_1(Y_3-\tilde{S}_{12}))(p_{22}+h_2)\int_0^{\tilde{S}_{22}}\int_{\tilde{S}_{12}}^{A}(\alpha x_{12}+x_{22}) \\ f_2(x_{12},x_{22})dx_{12}dx_{22} + \\ r_2(1-F_1(Y_3-\tilde{S}_{12}))(p_{12}+h_1)\int_0^{\tilde{S}_{12}}\int_{\tilde{S}_{22}}^{B}(\beta x_{22}+x_{12}) \\ f_2(x_{12},x_{22})dx_{22}dx_{12} + \\ r_2(1-F_1(Y_3-\tilde{S}_{12}))\int_0^{\tilde{S}_{12}}\int_0^{\tilde{S}_{22}}((p_{12}+h_1)x_{12}+(p_{22}+h_2)x_{22}) \\ f_2(x_{12},x_{22})dx_{22}dx_{12} + \\ r_2\int_0^{Y_3-\tilde{S}_{12}}\Bigg(-(p_{12}-c_1)x_{11} + \\ (p_{12}+h_1)\int_0^{S_{22}(Y_3)}\int_{S_{11}-x_{11}}^{A(Y_3)}((\alpha x_{12}+x_{11})+x_{22}) \\ f_2(x_{12},x_{22})dx_{22}dx_{12} + \\ (p_{12}+h_1)\int_0^{S_{11}-x_{11}}\int_{S_{22}(Y_3)}^{B(Y_3)}(\beta x_{22}+x_{12}+x_{11}) \\ f_2(x_{12},x_{22})dx_{22}dx_{12} - \\ \int_0^{S_{22}(Y_3)}\int_0^{Y_3}((p_{12}+h_1)(x_{12}+x_{11})+(p_{22}+h_2)x_{22}) \\ f_2(x_{12},x_{22})dx_{22}dx_{12}\Bigg)f_1(x_{11})dx_{11} & \text{if} \quad Y_1 > \tilde{S_{12}} \end{cases}$$

We note that previous discussion is not valid unless $(\frac{\alpha-\beta}{\beta}+\alpha)(p_{22}+h_2) > (p_{12}+h_1)$ and $\alpha > \beta$ under identical demand and price conditions. When that is not the case, we can not guarantee concavity of total profit function. In that case, we search optimum levels in both of the regions and compare the feasible optimum points of each region. According to the analysis, we select the best point which maximizes total profit function.

The findings of this section form the basis of our discussion of Chapter 5 where numerical studies are made.

3.3 Secondary Product Introduction while Phasing out the Primary Product in the Second Period (ISES)

ISES strategy is a single roll strategy since we assume that there is only one product at a time. In other words, if there is any left over primary product from the first period, they are assumed to be thrown away. As an advantage of such a strategy, substitution due to demand correlation is out of consideration. On the other hand, as a disadvantage, with ISES strategy, we give up the opportunity to slow down customer losses with substitute product. Therefore, we investigate net advantage of this strategy over the others.

Different than the notation for the IS strategy, we express expected total profit as a function of S_{11} andS_{22} because of the structure of this case. In this case, S_{11} andS_{22} are decision variables. We assume that the production line designed for the primary product is modified such that it is appropriate for the secondary product and the best way to achieve this is to design the system according to the postponement case. Thus, as production line investments, we use the same costs with the IS strategy.

There is an investment of T at the beginning of first period for single production of primary product and an amount of K to redesign the production line for single production of secondary product.

Total profit function with first period order-up-to level of S_{11} and second period order-up-to level of S_{22} is shown as below:

$$\begin{aligned}
\Pi(S_{11}, S_{22}) &= -T - r_1 K + r_1\Big(p_{11}\mu_{11} - h_1(S_{11} - \mu_{11}) - c_{01}S_{11} \\
&- \int_{S_{11}}^{\infty}(p_{11} + h_1)(x_{11} - S_{11})f_1(x_{11})dx_{11}\Big) \\
&+ r_2\Big(- S + p_{22}\mu_{22} - h_2(S_{22} - \mu_{22}) - c_{02}S_{22} \\
&- \int_{S_{22}}^{\infty}(p_{22} + h_2)(x_{22} - S_{22})g_2(x_{22})dx_{22}\Big)
\end{aligned} \tag{3.33}$$

It follows that $\Pi(S_{11}, S_{22})$ is jointly concave in S_{11} and S_{22} with second order conditions of $-r_1(p_{11} + h_1)f_1(S_{11})$ and $-r_2(p_{22} + h_2)g_2(S_{22})$, respectively. Therefore, optimal levels for first and second period are found as:

$$S_{11}{}^* = F_1^{-1}(\frac{p_{11} - c_{01}}{p_{11} + h_1}) \tag{3.34}$$

$$S_{22}{}^* = G_2^{-1}(\frac{p_{22} - c_{02}}{p_{22} + h_2}) \tag{3.35}$$

As seen by our analysis of optimal levels, this strategy provides replenishment levels not dependent on each other and they are simply equal to one period newsboy ratios. There are two reasons behind this. One of them is our assumption regarding the independence of demands between the periods. As the other reason, left over items from the previous period are not used and hence decision variables are not linked to each other.

As a last point, we provide optimal expected profit as in the following:

$$\begin{aligned}
\Pi(S_{11}{}^*, S_{22}{}^*) &= -T - r_2 K + r_1\Big((p_{11} + h_1)\int_0^{S_{11}{}^*} x_{11}f_1(x_{11})dx_{11}\Big) \\
&+ r_2\Big(- S + (p_{22} + h_2)\int_0^{S_{22}{}^*} x_{22}g_2(x_{22})dx_{22}\Big)
\end{aligned} \tag{3.36}$$

3.4 Secondary Product Introduction in the First Period While Phasing out the Primary in the Second Period (IFES)

This strategy is a dual product roll because there is a time period, first period, in which both of the products exist. However, different from the IS strategy, these two products are introduced together in the beginning of the first period. Therefore, primary product has a short life compared to secondary product. As a real life situation, firm introduces arty professional software into the market together with a limited or primal version which is cheaper or easier to access. After the first period where customers are attracted through the primal model, only the advanced model is available in the market.

When comparing this strategy with the others, the advantage of early introduction through stock out-induced substitution and the advantage of early elimination because of negative correlation between the demands are weighed up against the disadvantage of early introduction due to price substitution between the products and the disadvantage of early elimination because of not being able to use stock out-induced substitution.

We assume that P is invested for building the production line and this amount is assumed to be less than $T + U$, which is the summation of individual primary production investment and modification for delayed product differentiation. This assumption is realistic because transforming an existing system is harder and costly than building a system from scratch. In the second period, we redesign the system for single production of secondary product and invest an amount of R for this.

With order-up-to levels of S_{11} and S_{21}, the first period expected profit function is shown as below where $A = (S_{21} - x_{21})/\alpha + S_{11}$ and $B = (S_{11} - x_{11})/\beta + S_{21}$:

$$\begin{aligned}
\Pi_1(S_{11}, S_{21}) &= -P + p_{11}S_{11} + p_{21}S_{21} - c_1S_{11} - c_2S_{21} + \\
&+ \int_0^{S_{21}} \int_{S_{11}}^{A} \Big((p_{21} + h_2)(\alpha(x_{11} - S_{11}) - (S_{21} - x_{21}) \Big) f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&+ \int_0^{S_{11}} \int_{S_{21}}^{B} \Big((p_{11} + h_1)(\beta(x_{21} - S_{21}) - (S_{11} - x_{11}) \Big) f_1(x_{11}, x_{21}) dx_{21} dx_{11} \\
&- \int_0^{S_{21}} \int_0^{S_{11}} \Big((p_{11} + h_1)(S_{11} - x_{11}) + (p_{21} + h_2)(S_{21} - x_{21}) \Big) f_1(x_{11}, x_{21}) dx_{11} dx_{21}
\end{aligned} \tag{3.37}$$

In the next period, we sell only the secondary product. As a result, left over secondary product inventory is used whereas any primary product left is eliminated. We denote the inventory left over by I_{21} and $I_{21} = max\{0, S_{21} - x_{21}\}$.

Without any initial inventory, second period expected profit when order-up-to level is fixed at S_{22} is as follows:

$$L(S_{22}) = -R + p_{22}\mu_{22} - h_2(S_{22} - \mu_{22}) - c_{02}S_{22} - \int_{S_{22}}^{\infty} (p_{22} + h_2)(x_{22} - S_{22}) g_2(x_{22}) dx_{22} \tag{3.38}$$

We claim that $L(S_{22})$ is concave in S_{22} because its second partial derivative with respect to S_{22} is $-(p_{22} + h_2)g_2(x_{22})$ and it is negative. Therefore we are able to find global maximum of the function $L(S_{22})$ from the first order condition and it is as follows:

$$\check{S_{22}} = G_2^{-1}\left(\frac{p_{22} - c_{02}}{p_{22} + h_2}\right) \tag{3.39}$$

If initial inventory is less than $\check{S_{12}}$, we replenish up-to $\check{S_{12}}$ level. On the other hand we begin the second period with an inventory level larger than $\check{S_{12}}$, we do not order. Hence, optimal second period order-up-to level is found as:

$$S_{22}^{*} = \begin{cases} \check{S_{22}} & \text{for} \quad I_{21} \leq \check{S_{22}} \\ I_{21} & \text{for} \quad I_{21} > \check{S_{22}} \end{cases}$$

As a result of the previous discussion, we show second period expected profit function with an initial inventory of I_{21} as:

$$\Pi_2(I_{21}) = \begin{cases} L(\check{S_{22}}) + c_{02}I_{21} & \text{for} \quad I_{21} \leq \check{S_{22}} \\ L(I_{21}) + c_{02}I_{21} & \text{for} \quad I_{21} > \check{S_{22}} \end{cases}$$

Therefore, expected total profit function with order-up-to levels of S_{11} and S_{21} for the first period and S_{22} for the second period is expressed by:

$$\begin{aligned}
\Pi(S_{11}, S_{21}) &= -P - r_2 R + r_1 \Pi(S_{11}, S_{21}) \\
&+ r_2 \Bigg(\int_0^{(S_{21}-\check{S_{22}})^+} \int_0^{S_{11}} [L(S_{21} - x_{21}) + c_{02}(S_{21} - x_{21})] f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&+ \int_0^{(S_{21}-\check{S_{22}})^+} \int_{S_{11}}^{\frac{(S_{21}-\check{S_{22}}-x_{21})}{\alpha}+S_{11}} [L(S_{21} - x_{21} - \alpha(x_{11} - S_{11})) \\
&+ c_{02}(S_{21} - x_{21} - \alpha(x_{11} - S_{11}))] f_1(x_{11}, x_{21}) dx_{21} dx_{11} \\
&+ \int_0^{(S_{21}-\check{S_{22}})^+} \int_{\frac{(S_{21}-\check{S_{22}}-x_{21})}{\alpha}+S_{11}}^{A} [L(\check{S_{22}}) + c_{02}(S_{21} - x_{21} - \alpha(x_{11} - S_{11}))] \\
&\quad f_1(x_{11}, x_{21}) dx_{21} dx_{11} + \int_0^{(S_{21}-\check{S_{22}})^+} \int_A^{\infty} L(\check{S_{22}}) f_1(x_{11}, x_{21}) dx_{21} dx_{11} \Bigg) \\
&+ r_2 \Bigg(\int_{(S_{21}-\check{S_{22}})^+}^{S_{21}} \int_0^{S_{11}} [L(\check{S_{22}}) + c_{02}(S_{21} - x_{21}) f_1(x_{11}, x_{21}) dx_{21} dx_{11} \\
&+ \int_{(S_{21}-\check{S_{22}})^+}^{S_{21}} \int_{S_{11}}^{A} [L(\check{S_{22}}) + c_{02}(S_{21} - x_{21} - \alpha(x_{11} - S_{11}))] f_1(x_{11}, x_{21}) dx_{21} dx_{11} \\
&+ \int_{(S_{21}-\check{S_{22}})^+}^{S_{21}} \int_A^{\infty} L(\check{S_{22}}) f_1(x_{11}, x_{21}) dx_{21} dx_{11} \Bigg) \\
&+ r_2 \Bigg(\int_{S_{21}}^{\infty} L(\check{S_{22}}) f_1(x_{11}, x_{21}) dx_{21} dx_{11} \Bigg)
\end{aligned} \tag{3.40}$$

Total profit function is composed of four terms. First term of 3.40 is the expected profit coming from the first period whereas the rest is related with the second term.

Second term is valid if secondary product order-up-to level of the first period turns out to be larger than $\check{S_{22}}$. If not, second term vanishes. Inside the second term, there are four parts each of which is expressed with integrals of different limits. First part shows the case where there is still inventory left and it is larger than $\check{S_{22}}$ after the demand is satisfied. As a result, the best thing to do is not to order but use this left over amount. A similar situation arises when there is inventory larger than $\check{S_{22}}$ after first choice demand and second choice demand is satisfied and this is shown in the second part of the second term. On the other hand, when left over inventory is less than $\check{S_{22}}$ after first choice and second choice demand is satisfied, optimal level to order-up-to is exactly $\check{S_{22}}$. In the third part and fourth part, second choice demand is larger than the difference between $\check{S_{22}}$ and left inventory after first choice demand is satisfied.

The third term of 3.40 has three parts and in each of them we order exactly $\check{S_{22}}$. If order-up-to level for secondary product in the first period exceeds $\check{S_{22}}$, each part has outer integral with lower limit of $S_{21} - \check{S_{22}}$. It says that when secondary demand exceeds $S_{21} - \check{S_{22}}$, inventory left is less than $\check{S_{22}}$ no matter the primary product demand and it is optimal to order $\check{S_{22}}$. On the other hand, if S_{21} is less than $\check{S_{22}}$, lower limit of outer integrals turn out to be 0. In such a situation, there is no way for left over inventory to exceed $\check{S_{22}}$ and as a result of this, optimal level to order up to is $\check{S_{22}}$.

Fourth term shows the case where secondary product faces a demand bigger than its stock. Thus, there is no inventory left and it is optimal to order up to $\check{S_{22}}$.

Next, we propose the following theorem.

Theorem 3.4.1 *For the region where $S_{21} \leq \check{S_{22}}$, the expected total profit function, i.e. $\Pi(S_{11}, S_{21})$ is strictly concave in S_{11} and S_{21} if $(p_{11}+h_1) > \alpha(p_{21}+h_2)$ and $r_1(p_{21}+h_2) > r_1\beta(p_{11}+h_1) + r_2c_{02}$ where $\alpha \neq 0$ and $\beta \neq 0$. First order* *condition of S_{11} and of S_{21} is shown in the following equations as:*

$$
\begin{aligned}
\frac{\partial \Pi}{\partial S_{11}} | S_{21} \leq \check{S_{22}} &= r_1(p_{11}-c_1)+\alpha\left\{r_2c_{02}-r_1(p_{21}+h_2)\right\}\int_0^{S_{21}}\int_{S_{11}}^{A} f_1(x_{11},x_{21})dx_{11}dx_{21} \\
&- r_1(p_{11}+h_1)\int_0^{S_{11}}\int_0^{B} f_1(x_{11},x_{21})dx_{21}dx_{11}=0
\end{aligned} \tag{3.41}
$$

$$
\begin{aligned}
\frac{\partial \Pi}{\partial S_{21}} | S_{21} \leq \check{S_{22}} &= r_1(p_{21}-c_2)+\left\{r_2c_{02}-r_1(p_{21}+h_2)\right\}\int_0^{S_{21}}\int_0^{A} f_1(x_{11},x_{21})dx_{11}dx_{21} \\
&- \beta r_1(p_{11}+h_1)\int_0^{S_{11}}\int_{S_{21}}^{B} f_1(x_{11},x_{21})dx_{21}dx_{11}=0
\end{aligned} \tag{3.42}
$$

Proof. Second partial derivatives of $\hat{\Pi}(S_{11},S_{21})$ is given in the following equations when $S_{21} \leq \check{S_{22}}$:

$$
\begin{aligned}
\frac{\partial^2 \Pi}{\partial {S_{11}}^2} &= -r_2\alpha c_{02}\int_0^{S_{21}} f_1(S_{11},x_{21})dx_{21} - r_1(p_{11}+h_1-\alpha(p_{21}+h_2))\int_0^{S_{21}} f_1(S_{11},x_{21})dx_{21} \\
&- r_1(p_{11}+h_1)\int_0^{S_{11}} \frac{f_1(x_{11},B)dx_{11}}{\beta} + \alpha(r_2c_{02}-r_1(p_{21}+h_2))\int_0^{S_{21}} f_1(A,x_{21})dx_{21}
\end{aligned} \tag{3.43}
$$

$$
\begin{aligned}
\frac{\partial^2 \Pi}{\partial {S_{21}}^2} &= (r_2c_{02}-r_1(p_{21}+h_2))\int_0^{S_{21}} \frac{f_1(A,x_{21})}{\alpha}dx_{11} \\
&+ \left\{r_2c_{02}+r_1\beta(p_{11}+h_1)-r_1(p_{21}+h_2)\right\}\int_0^{S_{11}} f_1(x_{11},S_{21})dx_{11} \\
&- \beta r_1(p_{11}+h_1)\int_0^{S_{11}} f_1(x_{11},B)dx_{11}
\end{aligned} \tag{3.44}
$$

$$\begin{aligned}\frac{\partial^2\Pi}{\partial S_{11}\partial S_{21}} &= (r_2c_{02} - r_1(p_{21}+h_2))\int_0^{S_{21}} f_1(A, x_{21})dx_{11} \\ &- r_1(p_{11}+h_1)\int_0^{S_{11}} f_1(x_{11}, B)dx_{11}\end{aligned} \tag{3.45}$$

$$\begin{aligned}\frac{\partial^2\Pi}{\partial S_{21}\partial S_{11}} &= (r_2c_{02} - r_1(p_{21}+h_2))\int_0^{S_{21}} f_1(A, x_{21})dx_{11} \\ &- r_1(p_{11}+h_1)\int_0^{S_{11}} f_1(x_{11}, B)dx_{11}\end{aligned} \tag{3.46}$$

When $(p_{11}+h_1) > \alpha(p_{21}+h_2)$, Equation 3.43 is negative and when $r_1(p_{21}+h_2) > r_1\beta(p_{11}+h_1)+r_2c_{02}$ determinant of Hessian matrix is positive and thus we show that expected total profit function has a negative definite Hessian. Thus, we prove strict joint of expected total profit function in this region.■

Let us denote the values found by solving the following system as Y_4 and Y_5 respectively for primary product and secondary product.

$$\begin{aligned}\frac{\partial\Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}} = 0 \\ \frac{\partial\Pi}{\partial S_{21}}|S_{21} \leq \check{S_{22}} = 0\end{aligned} \tag{3.47}$$

Moreover, we denote the value that makes $\frac{\partial\Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}} = 0$ zero, with an order-up-to level of (S_{21}) for secondary product, as ${S_{11}}^*(S_{21})$. Similarly, the value that makes $\frac{\partial\Pi}{\partial S_{21}}|S_{21} \leq \check{S_{22}} = 0$ zero, with an order-up-to level of (S_{11}) for primary product is denoted as ${S_{21}}^*(S_{11})$.

Since first order condition for primary product and secondary product turns out to be a curve, we question if it is an increasing or a decreasing curve with the following lemmas.

Lemma 3.4.1 $\frac{\partial \Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}} = 0$ *is a strictly decreasing curve in the* (S_{11}, S_{21}) *plane, given that* $(p_{11} + h_1) > \alpha(p_{21} + h_2)$ *and* $\beta \neq 0$

Proof. It is not possible to write S_{21} as a function of S_{11} from $\frac{\partial \Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}} = 0$. So, we use implicit differentiation. Let du/dS_{11} be the derivative of $Z\frac{\partial \Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}}1 = 0$ at (S_{11}, S_{21}). Then following holds:

$$\begin{aligned}
\frac{du}{dS_{11}} &= -\frac{r_2 \alpha c_{02} \int_0^{S_{21}} f_1(S_{11}, x_{22}) dx_{21}}{\chi} \\
&- \frac{r_1(-\alpha(p_{21} + h_2) + p_{11} - h_1) \int_0^{S_{21}} f_1(S_{11}, x_{21}) dx_{21}}{\chi} \\
&- \frac{r_1(p_{11} + h_1) \int_0^{S_{11}} f_1(x_{11}, B) dx_{11}}{\beta \chi} \\
&- \frac{\alpha(p_{21} + h_2) \int_0^{S_{21}} f_1(A, x_{21}) dx_{21}}{\chi}
\end{aligned} \tag{3.48}$$

where χ expresses $r_2 \alpha c_{02} \int_{S_{11}}^{\infty} f_1(x_{11}, S_{22}) dx_{11} + r_1 \Big((p_{11}+h_1) \int_0^{S_{11}} f_1(x_{11}, B) dx_{11} + (p_{21} + h_2) \int_0^{S_{21}} f_1(A, x_{21}) dx_{21} \Big)$ which is a positive term. Because of each probability distribution function being assumed to be positive and β given positive, above function is continuous. Moreover, because it is assumed that $(p_{11} + h_1) > \alpha(p_{21} + h_2)$, all of the terms in 3.48 are negative.■

After concluding that, when $S_{21} \leq \check{S_{22}}$, optimal level for primary product is a decreasing function of secondary product order-up-to level, we find upper bound for Y_4 given $S_{21} = 0$ and lower bound given $S_{21} = \infty$ from $\frac{\partial \Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}} = 0$. Closed form expression for lower bound is found as in the following discussion.

Lower bound, $\underline{Y}_4$, is obtained from 3.41 by equating S_{21} to ∞ as:

$$\int_0^{\underline{Y}_4}\int_0^{\infty} f_1(x_{11}, x_{21})dx_{21}dx_{11} = \frac{r_1(p_{11} - c_1 - \alpha(p_{21} + h_2)) + r_2\alpha c_{02}}{r_1(p_{11} + h_1 - \alpha(p_{21} + h_2)) + r_2\alpha c_{02}} \tag{3.49}$$

Therefore, lower bound for Y_4 is equal to the following:

$$\underline{Y}_4 = F_1^{-1}(\frac{r_1(p_{11} - c_1 - \alpha(p_{21} + h_2)) + r_2\alpha c_{02}}{r_1(p_{11} + h_1 - \alpha(p_{21} + h_2)) + r_2\alpha c_{02}}) \tag{3.50}$$

When first period inventory for secondary product is so huge that it is enough to satisfy it is first choice demand, cost of underage for primary product is composed of two parts. First is the cost of not being able to satisfy first choice demand and second is cost because of decreasing initial inventory by using secondary product as a second choice demand. On the other hand, cost of overage is same with first period of base case since there is no second choice demand for primary product.

As we stated previously, upper bound for Y_4 is found from $\frac{\partial \Pi(\overline{Y}_4,0)}{\partial S_{11}}|S_{21} \leq \check{S_{22}} = 0$ and it is shown in the following as:

$$\frac{p_{11} - c_1}{p_{11} + h_1} = Pr(0 \leq \beta x_{21} + x_{11} \leq \overline{Y}_4) \tag{3.51}$$

We interpret this finding as when there is no stock for secondary product the optimal order-up-to level for secondary product is such that the possibility of stocking bigger than the random total first and second choice demand is exactly equal to newsboy ratio for a one period newsboy problem provided that the value found is less than $\check{S_{22}}$. This finding is reasonable since not being able to carry inventory, as a consequence of our rollover strategy, converts the first period optimal levels to myopic levels.

Lemma 3.4.2 $\frac{\partial^2 \Pi}{\partial S_{21}}|S_{21} \leq \check{S_{22}} = 0$ *is a strictly decreasing curve in the* (S_{11}, S_{21}) *plane, given that* $r_1(p_{21} + h_2) > \beta(p_{11} + h_1) + r_2c_{02}$ *and* $\alpha \neq 0$

Proof. We use implicit differentiation. Let dv/dS_{11} be the derivative of $\frac{\partial^2 \Pi}{\partial S_{21}} | S_{21} \leq \check{S_{22}} = 0$ at (S_{11}, S_{21}). Then following holds:

$$\frac{dv}{dS_{11}} = \frac{(r_1(p_{21}+h_2) - r_2 c_{02}) \int_0^{S_{21}} f_1(A, x_{21}) dx_{21} + r_1(p_{11}+h_1) \int_0^{S_{11}} f_1(x_{11}, B) dx_{11}}{\psi} \tag{3.52}$$

where ψ is the following;

$$\begin{aligned} \psi &= (-r_1(p_{21}+h_2) + \beta(p_{11}+h_1) + r_2 c_{02}) \int_0^{S_{11}} f_1(x_{11}, S_{21}) dx_{11} \\ &+ \frac{-r_1(p_{21}+h_2) + r_2 c_{02}}{\alpha} \int_0^{S_{21}} f_1(A, x_{21}) dx_{21} \\ &+ (-r_1 \beta(p_{11}+h_1) \int_0^{S_{11}} f_1(x_{11}, B) dx_{11} \end{aligned} \tag{3.53}$$

Because of the assumptions, ψ is negative while the nominator of $\frac{dv}{dS_{11}}$ is positive.■

After concluding that, when $S_{21} \leq \check{S_{22}}$, optimal level for secondary product is a decreasing function of primary product order-up-to level, we find upper bound for Y_5 given $S_{11} = 0$ and lower bound given $S_{21} = \infty$ from $\frac{\partial^2 \Pi}{\partial S_{21}} | S_{21} \leq \check{S_{22}} = 0 = 0$. Closed form expression for the lower bound is found as in the following discussion.

Lower bound on Y_5, $\underline{Y}_5$, is obtained from 3.42 by equating S_{11} to ∞ as:

$$\underline{Y}_5 = G_1^{-1}\left(\frac{r_1((p_{21} - c_2) - \beta(p_{11}+h_1))}{r_1((p_{21}+h_2) - \beta(p_{11}+h_1)) - r_2 c_{02}}\right) \tag{3.54}$$

When we obtain an upper bound on optimal inventory level for secondary product such that this level is smaller than $\check{S_{22}}$ and there is no second choice demand, cost of underage comes from unsatisfied first choice demand less the portion that is satisfied as second choice demand by secondary product. When we talk about cost of overage, we take into account second period in addition to the first period. Cost of overage is first period cost of average minus benefit of beginning second period with extra inventories.

For upper bound on Y_5, we solve $\frac{\partial^2 \Pi(0,\overline{Y}_5)}{\partial S_{21}} | S_{21} \leq \check{S_{22}} = 0$ and it is shown in the following as:

$$\frac{r_1(p_{21} - c_2)}{r_1(p_{21} + h_2) - r_2 c_{02}} = Pr(0 \leq \beta x_{11} + x_{21} \leq \overline{Y}_5) \tag{3.55}$$

When there is no inventory for the primary product, the total random demand secondary product directly and indirectly face is $\beta x_{11} + x_{21}$. In this case, cost of underage is same with one period newsboy problem but discounted by r_1. Cost of overage, on the other hand, is total of first period cost of overage less the benefit by beginning second period with extra inventories.

We discuss concavity of the total profit function in the region where $S_{21} > \check{S_{22}}$ with the following theorem.

Theorem 3.4.2 *For the region where $S_{21} > \check{S_{22}}$, the expected total profit function, i.e. $\hat{\Pi}(S_{11}, S_{21})$ is jointly strict concave in S_{11} and S_{21} if $r_1(p_{11} + h_1) > \alpha(r_1(p_{21} + h_2) + r_2 h_2)$and $r_1(p_{21} + h_2) > (\beta(p_{11} + h_1) + r_2 c_{02})$. First order conditions of for S_{11} and S_{21} are shown in the following equation as:*

$$
\begin{aligned}
\frac{\partial \Pi}{\partial S_{11}}|S_{21} > \check{S_{22}} &= r_2 \alpha c_{02} \int_{S_{21}-\check{S_{22}}}^{S_{21}} \int_{S_{11}}^{A} f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&+ r_2 \alpha c_{02} \int_{0}^{S_{21}-\check{S_{22}}} \int_{S_{11}+\frac{S_{21}-\check{S_{22}}-x_{21}}{\alpha}}^{A} f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&+ r_2 \int_{0}^{S_{21}-\check{S_{22}}} \int_{S_{11}}^{S_{11}+\frac{S_{21}-\check{S_{22}}-x_{21}}{\alpha}} (\alpha p_{22} - \alpha(p_{22}+h_2) \\
&\int_0^{S_{21}-x_{21}+\alpha(S_{11}-x_{11})} g_2(x_{22}) dx_{22}) f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&+ r_1 \bigg(p_{11} - c_1 - (p_{11}+h_1) \int_0^{S_{11}} \int_0^{B} f_1(x_{11}, x_{21}) dx_{21} dx_{11} \\
&- \alpha(p_{21}+h_2) \int_0^{S_{21}} \int_{S_{11}}^{A} f_1(x_{11}, x_{21}) dx_{11} dx_{21} \bigg) = 0
\end{aligned} \tag{3.56}
$$

$$
\begin{aligned}
\frac{\partial \Pi}{\partial S_{21}}|S_{21} > \check{S_{22}} &= r_2 c_{02} \int_{S_{21}-\check{S_{22}}}^{S_{21}} \int_{0}^{A} f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&+ r_2 c_{02} \int_{0}^{S_{21}-\check{S_{22}}} \int_{S_{11}+\frac{S_{21}-\check{S_{22}}-x_{21}}{\alpha}}^{A} f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&+ r_2 \int_{0}^{S_{21}-\check{S_{22}}} \int_{0}^{S_{11}} \bigg(p_{22} \\
&- (p_{22}+h_2) \int_0^{S_{21}-x_{21}} g_2(x_{22}) dx_{22} \bigg) f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&+ r_2 \int_{0}^{S_{21}-\check{S_{22}}} \int_{S_{11}}^{S_{11}+\frac{S_{21}-\check{S_{22}}-x_{21}}{\alpha}} \bigg(p_{22} \\
&- (p_{22}+h_2) \int_0^{S_{21}-x_{21}+\alpha(S_{11}-x_{11})} g_2(x_{22}) dx_{22} \bigg) f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&+ r_1(p_{21}-c_2) + (r_2 c_{02} - r_1(p_{21}+h_2)) \int_0^{S_{21}} \int_0^{A} f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&- r_1 \beta (p_{11}+h_1) \int_0^{S_{11}} \int_{S_{21}}^{B} f_1(x_{11}, x_{21}) dx_{21} dx_{11} = 0
\end{aligned} \tag{3.57}
$$

Proof.

Second partial derivatives of $\Pi(S_{11}, S_{21})$ that builds up Hessian Matrix are given in the following equations when $S_{21} > \check{S_{22}}$:

$$
\begin{aligned}
\frac{\partial^2\Pi}{\partial {S_{11}}^2} &= \int_{S_{21}-\check{S_{22}}}^{S_{21}} \Big(- r_1(p_{11}+h_1) + r_1\alpha(p_{21}+h_2) - r_2\alpha c_{02}\Big) f_1(S_{11}, x_{21}) dx_{21} \\
&+ \int_0^{S_{21}-\check{S_{22}}} \Big(- r_1(p_{11}+h_1) + r_1\alpha(p_{21}+h_2) + r_2\alpha p_{22} - r_2\alpha(p_{22}+h_2)G_2(S_{21}-x_{21})\Big) \\
&\quad f_1(S_{11}, x_{21}) dx_{21} \\
&+ \int_0^{S_{21}} \alpha(-r_1(p_{21}+h_2) + r_2 c_{02}) f_1(A, x_{21}) dx_{21} \\
&- \int_0^{S_{21}-\check{S_{22}}} \int_{S_{11}}^{S_{11}+\frac{S_{21}-\check{S_{22}}+x_{21}}{\alpha}} r_2(p_{22}+h_2)\alpha^2 g_2(S_{21}-x_{21}+\alpha(S_{11}-x_{11})) \\
&\quad f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&- r_1(p_{11}+h_1)\int_0^{S_{11}} \frac{f_1(x_{11}, B) dx_{11}}{\beta}
\end{aligned}
\tag{3.58}
$$

$$
\begin{aligned}
\frac{\partial^2\Pi}{\partial {S_{21}}^2} &= (-r_1(p_{21}+h_2) + r_2 c_{02}) \int_0^{S_{21}} \frac{f_1(A, x_{21})}{\alpha} dx_{21} \\
&- r_2(p_{22}+h_2) \int_0^{S_{21}-\check{S_{22}}} \int_{S_{11}}^{S_{11}+\frac{S_{21}-\check{S_{22}}+x_{21}}{\alpha}} g_2(S_{21}-x_{21}+\alpha(S_{11}-x_{11})) \\
&\quad f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&- r_2(p_{22}+h_2) \int_0^{S_{21}-\check{S_{22}}} \int_0^{S_{11}} g_2(S_{21}-x_{21}) \\
&\quad f_1(x_{11}, x_{21}) dx_{11} dx_{21} \\
&- (r_1(p_{21}+h_2) - \beta(p_{11}+h_1) - r_2 c_{02}) \int_0^{S_{21}} f_1(x_{11}, S_{21}) dx_{11} \\
&- r_1\beta(p_{11}+h_1) \int_0^{S_{11}} f_1(x_{11}, B) dx_{11}
\end{aligned}
\tag{3.59}
$$

$$\begin{aligned}\frac{\partial^2\Pi}{\partial S_{11}\partial S_{21}} &= \int_0^{S_{21}}(-r_1(p_{21}+h_2)+r_2c_{02})f_1(A,x_{21})dx_{21}\\ &- \int_0^{S_{21}-\check{S_{22}}}\int_{S_{11}}^{S_{11}+\frac{S_{21}-\check{S_{22}}+x_{21}}{\alpha}}(p_{22}+h_2)\alpha g_2(S_{21}-x_{21}+\alpha(S_{11}-x_{11}))\\ &\quad f_1(x_{11},x_{21})dx_{11}dx_{21}\\ &- r_1(p_{11}+h_1)\int_0^{S_{11}}f_1(x_{11},B)dx_{11}\end{aligned} \tag{3.60}$$

$$\begin{aligned}\frac{\partial^2\Pi}{\partial S_{21}\partial S_{11}} &= \int_0^{S_{21}}(-r_1(p_{21}+h_2)+r_2c_{02})f_1(A,x_{21})dx_{21}\\ &- \int_0^{S_{21}-\check{S_{22}}}\int_{S_{11}}^{S_{11}+\frac{S_{21}-\check{S_{22}}+x_{21}}{\alpha}}(p_{22}+h_2)\alpha g_2(S_{21}-x_{21}+\alpha(S_{11}-x_{11}))\\ &\quad f_1(x_{11},x_{21})dx_{11}dx_{21}\\ &- r_1(p_{11}+h_1)\int_0^{S_{11}}f_1(x_{11},B)dx_{11}\end{aligned} \tag{3.61}$$

If $r_1(p_{11}+h_1) > \alpha(r_1(p_{21}+h_2)+r_2h_2)$, then $(p_{11}+h_1) > \alpha(p_{21}+h_2)$ and thus, all of the terms except the second term of 3.58 are negative. We claim that second term is also negative if $r_1(p_{11}+h_1) > \alpha(r_1(p_{21}+h_2)+r_2h_2)$. This claim is proven in the following arguments. Let us denote the term;

$$\Big(-r_1(p_{11}+h_1)+r_1\alpha(p_{21}+h_2)+r_2\alpha p_{22}-r_2\alpha(p_{22}+h_2)G_2(S_{21}-x_{21})dx_{22}\Big) \tag{3.62}$$

with Θ. Then,

$$\Theta < \Big(-r_1(p_{11}+h_1)+r_1\alpha(p_{21}+h_2)+r_2\alpha p_{22}-r_2\alpha(p_{22}+h_2)\Big) \tag{3.63}$$

Our assumption implies that $\frac{r_1(p_{11}+h_1)-r_1\alpha(p_{21}+h_2)+r_2\alpha p_{22}}{r_2\alpha(p_{22}+h_2)}$ is greater than 1. For this reason,

$$\Big(-r_1(p_{11}+h_1)+r_1\alpha(p_{21}+h_2)+r_2\alpha p_{22}-r_2\alpha(p_{22}+h_2)\Big) < 0 \tag{3.64}$$

Thus, we prove that $\Theta < 0$

It is may be shown that determinant of Hessian associated with expected total porfit function is positve because of the assumptions $r_1(p_{11} + h_1) > \alpha(r_1(p_{21} + h_2) + r_2 h_2)$ and $r_1(p_{21} + h_2) > r_1(\beta(p_{11} + h_1)) + r_2 c_{02}$. Thus, we prove concavity in of expected total profit function.■

We denote the solution to the following by system for S_{11} and S_{21} as Y_6 and Y_7, respectively;

$$\begin{aligned} \frac{\partial \Pi}{\partial S_{11}} | S_{21} > \check{S_{22}} = 0 \\ \frac{\partial \Pi}{\partial S_{21}} | S_{21} > \check{S_{22}} = 0 \end{aligned} \tag{3.65}$$

In the appendix it is shown Y_4 andY_5 are optimal levels for primary and secondary product if $Y_5 \leq \check{S_{22}}$. For that case, optimal second period order-up-to-level for the second product is $\check{S_{22}}$. On the other hand, if $Y_4 > \check{S_{22}}$, optimal levels are found as Y_6 and Y_7 respectively for primary and secondary product. In that case, optimal level for the secondary product is found from $max \left\{0, (\check{S_{22}} - Y_7 - x_{11})\right\}$. We show this argument as:

$$({S_{11}}^*, {S_{21}}^*) = \begin{cases} (Y_6, Y_7) & \text{if} \quad Y_5 > \check{S_{22}} \\ (Y_4, Y_5) & \text{if} \quad Y_5 \leq \check{S_{22}} \end{cases}$$

$${S_{22}}^* = \begin{cases} max \left\{I_{21}, \check{S_{22}}\right\} & \text{if} \quad Y_5 > \check{S_{22}} \\ \check{S_{22}} & \text{if} \quad Y_5 \leq \check{S_{22}} \end{cases}$$

3.5 Summary

In this chapter we model four different rollover strategies with stochastic demands, fixed price and period lengths. We decide on stock levels for the products marketed in each of the case.

In the base case, only primary product is marketed. This case is a two period newsboy problem. In the IS strategy, we introduce a secondary product which is in substitution with primary product in terms of price and being available. Price substitution is reflected by dependent random demands. Stock out induced substitution is, on the other hand, analyzed explicitly with rates of customer switching the other product when one is out of stock. In the ISES strategy, we consider a single rollover strategy where only one product is marketed in a period which provides myopic optimal levels. Last strategy, ISEF, is also a dual rollover strategy because there is the first period where both of the products are marketed. Table 3.2 summarizes theoretical findings of this chapter.

In the next chapter, we discuss how to model demand. After modeling demand, we numerically analyze the models we have constructed. We question the changes in optimal levels and try to find the best strategy under different scenarios.

Table 3.2: Summary of Chapter 3

Strategy	S_{11}^*	S_{12}^*	S_{21}^*	S_{22}^*
Base	Y_2 if $Y_1 > \hat{S_{12}}$ Y_1 if $Y_1 \leq \hat{S_{12}}$ Equations 3.11 and 3.10 Follows from Theorem 3.1.1	$max[\hat{S_{12}}, I_{11}]$ if $Y_1 > \hat{S_{12}}$ $\hat{S_{12}}$ if $Y_1 \leq \hat{S_{12}}$ Equation 3.4 Follows from Theorem 3.1.1	NA	NA
IS	Y_3 if $Y_1 > \tilde{S_{12}}$ Y_1 if $Y_1 \leq \tilde{S_{12}}$ Equations 3.31 and 3.32 Follows from Theorem 3.2.1 and Appendix A	$max[\tilde{S_{12}}, I_{11}]$ if $Y_1 > \tilde{S_{12}}$ $\tilde{S_{12}}$ if $Y_1 \leq \tilde{S_{12}}$ Equations 3.31 and 3.18 Follows from Theorem 3.2.1 and Appendix A	NA	$min[\tilde{S}_{22}, S_{22}^*(I_{11})]$ if $Y_1 > \tilde{S_{12}}$ $\tilde{S_{22}}$ if $Y_1 \leq \tilde{S_{12}}$ Equations 3.31 and 3.18 Follows from Theorem 3.2.1 and Appendix A
ISES	$F_1^{-1}(\frac{p_{11}-c_{01}}{p_{11}+h_1})$ Equation 3.34	NA	NA	$G_2^{-1}\left(\frac{p_{22}-c_{02}}{p_{22}+h_2}\right)$ Equation 3.35
IFES	Y_6 if $Y_5 > \tilde{S_{22}}$ Y_4 if $Y_5 \leq \tilde{S_{22}}$ Follows from Theorem 3.4.2	NA	Y_7 if $Y_5 > \tilde{S_{22}}$ Y_5 if $Y_5 \leq \tilde{S_{22}}$ Follows from Theorem 3.4.1	$max[I_{21}, \tilde{S_{22}}]$ if $Y_5 > \tilde{S_{22}}$ $\tilde{S_{22}}$ if $Y_5 \leq \tilde{S_{22}}$ Equation 3.39

Chapter 4

Demand Model

This chapter discusses demand models which show demand as a function of prices. Despite the fact that problem of this study is built under fixed prices, such a relation is significant for our analysis. In numerical analysis we use price data and investigate dependence between price and order-up-to level, price and strategy selected.

We model uncertainty in demand using two alternative models. In the first section, we consider additive randomness whereas in the second section we provide a multiplicative version. We assume that randomness in demand is not dependent with prices as Petruzzi and Dada (1999) does in pricing the newsvendor problem .

In modeling the demand of cases where both of the products sold, we assume dependency between the demands of the products. Lim and Tang (2006) models demand using loyalty factors in existence of both products to model substitution in their analysis for product rollovers. Since we consider stock-out induced substitution and switching rates, we do not use this idea in modeling demand. However, we think that switching rates already include the idea in a modified way. We limit the substitution to the case when one of the products is out of stock. Moreover with dependency between the demands through dependency between the random terms, we include the idea that demand cannibalization may occur due to coexistence of substitute products.

We assume that distributions specified reflect the time length of the each period. We note that demand is independent from one period to another.

Table 4.2 presents notation and Table 4.1 introduces definitions for the concepts used this chapter.

Table 4.1: Definition for Some Terms

Own Price Elasticity	Percentage change in quantity demand with respect to a change of 1% in price of a product
Cross Price Elasticity	Percentage change in quantity demand with respect to a change of 1% in price of a substitute product
Customer Base	Customers avaliable in a market
Effective Demand	Quantity of customers willing to buy the product at the current price levels

Table 4.2: Notation for Demand Functions

ϵ_1	Unit time demand for primary product
ϵ_2	Unit time demand for secondary product
$E[\epsilon_i]$	Mean of ϵ_i
$var[\epsilon_i]$	Variance of ϵ_i
$cov(\epsilon_1, \epsilon_2)$	Covariance between unit time demands
a_i	Customer base for primary ($i = 1$) and secondary ($i = 2$) products
b_i	Own price elasticity for primary ($i = 1$) and secondary ($i = 2$) products
cr_i	Cross price elasticity for primary ($i = 1$) and secondary ($i = 2$) products
x_{ij}	Effective demand for product i in period j

4.1 Additive Demand Model

Petruzzi and Dada (1999) states that randomness in demand is generally attached to different forms of demand: additive randomness with linear demand form and multiplicative randomness with iso-elastic demand form. In this section, we consider a linear demand model with an additive randomness. According to our analysis, linear demand is composed of a deterministic term which is dependent on prices and a probabilistic term independent of prices. Because of this, it may be inferred that market size is stochastic while how demand behaves is not.

Since we define total demand in a period as summation of unit time demand through the period, total period demands are period lengths time unit time demands for each product.

Randomness in demand is defined as ϵ_1 and ϵ_2 for primary and secondary product, respectively and assumed to be normally distributed with mean and variance of $E[\epsilon_i]$ and $var[\epsilon_i]$. We assume dependence between unit time demands which leads to dependence between total demands in coexistence of the products. Covariance between ϵ_1 and ϵ_2 is denoted by $cov(\epsilon_1, \epsilon_2)$.

Effective demand in unit time when products are alone depend on the own price while it also depends on substitute products price when they are together in the market. Thus, formulation of effective demand differs for each rollover strategy. As a result, we show demand for each strategy by Table 4.3. Before that, we show effective demands given prices and randomness. For a product when it is alone in the market unit time effective demand is shown as:

$$d_{it} = a_i - b_i p_{it} + \epsilon_i \tag{4.1}$$

Effective demand for product i when both of the products are in the market is given as:

$$d_{it} = a_i - b_i p_{it} + c r_i p_{jt} + \epsilon_i \tag{4.2}$$

According to our consideration, market size or customer base is composed of both deterministic and probabilistic terms and shown by $a_i+\epsilon_i$. Effective demand is derived according to prices from this customer base. To assure positive demand, $a_i > 0$ should be large enough relative to $var[\epsilon_i]$ as Petruzzi and Dada (1999) puts it. Moreover, b_i should be positive to built a negative relation of own price with demand and cr_i should be positive to assure that demand changes in the same direction with substitute price. After presenting unit time demand form that we utilize, we show effective demands of each period under different rollover strategies with the following table.

Table 4.3: Demand Curves in Additive Model

Strategy	Demand	Formula
Base Case	x_{11}	$l_1(a_1 - b_1p_{11} + \epsilon_1)$
Base Case	x_{12}	$l_2(a_1 - b_1p_{12} + \epsilon_1)$
IS	x_{11}	$l_1(a_1 - b_1p_{11} + \epsilon_1)$
IS	x_{12}	$l_2(a_1 - b_1p_{12} + cr_1p_{22} + \epsilon_1)$
IS	x_{22}	$l_2(a_2 - b_2p_{22} + cr_2p_{12} + \epsilon_2)$
ISES	x_{11}	$l_1(a_1 - b_1p_{11} + \epsilon_1)$
ISES	x_{22}	$l_2(a_2 - b_2p_{22} + \epsilon_2)$
IFES	x_{11}	$l_1(a_1 - b_1p_{11} + cr_1p_{21} + \epsilon_1)$
IFES	x_{21}	$l_1(a_2 - b_2p_{21} + cr_2p_{11} + \epsilon_1)$
IFES	x_{22}	$l_2(a_2 - b_2p_{22} + \epsilon_2)$

Next, we derive mean and variance of each demand under different strategies to define distribution of total demands. For the cases of coexistence, we also need to derive covariance between the demands of products. To ease notation for these statistics, we define following parameters:

$$\begin{aligned}
\lambda_{0t} &= (a_1 - b_1p_{1t}) \\
\lambda_{1t} &= (a_1 - b_1p_{1t} + cr_1p_{2t}) \\
\omega_{0t} &= (a_2 - b_2p_{2t}) \\
\omega_{1t} &= (a_2 - b_2p_{2t} + cr_2p_{1t})
\end{aligned} \tag{4.3}$$

Table 4.4 summarizes the derivations as in the following:

Table 4.4: Demand Parameters in Additive Demand Model

Case	Parameter	Formula
Base Case	μ_{11}	$l_1(\lambda_{01} + E[\epsilon_1])$
Base Case	μ_{12}	$l_2(\lambda_{02} + E[\epsilon_1])$
Base Case	σ_{11}^2	${l_1}^2 var(\epsilon_1)$
Base Case	σ_{12}^2	${l_2}^2 var(\epsilon_1)$
IS	μ_{11}	$l_1(\lambda_{01} + E[\epsilon_1])$
IS	μ_{12}	$l_2(\lambda_{12} + E[\epsilon_1])$
IS	μ_{22}	$l_2(\omega_{12} + E[\epsilon_2])$
IS	σ_{11}^2	${l_1}^2 var(\epsilon_1)$
IS	σ_{12}^2	${l_2}^2 var(\epsilon_1)$
IS	σ_{22}^2	${l_2}^2 var(\epsilon_2)$
IS	$cov(x_{12}, x_{22})$	${l_2}^2 cov(\epsilon_1, \epsilon_2)$
ISES	μ_{11}	$l_1(\lambda_{01} + E[\epsilon_1])$
ISES	μ_{22}	$l_2(\omega_{02} + E[\epsilon_2])$
ISES	σ_{11}^2	${l_1}^2 var(\epsilon_1)$
ISES	σ_{22}^2	${l_2}^2 var(\epsilon_2)$
IFES	μ_{11}	$l_1(\lambda_{11} + E[\epsilon_1])$
IFES	μ_{21}	$l_1(\omega_{11} + E[\epsilon_2])$
IFES	μ_{22}	$l_2(\omega_{02} + E[\epsilon_2])$
IFES	σ_{11}^2	${l_1}^2 var(\epsilon_1)$
IFES	σ_{21}^2	${l_1}^2 var(\epsilon_2)$
IFES	σ_{22}^2	${l_2}^2 var(\epsilon_2)$
IFES	$cov(x_{11}, x_{21})$	${l_1}^2 cov(\epsilon_1, \epsilon_2)$

Discussion for additive demand model ends here. We utilize demand model developed in this section, in the first section of Chapter 5. In the next section, we discuss another commonly used demand model, multiplicative demand model.

4.2 Multiplicative Demand Model

This section provides us multiplicative randomness in demand. We associate multiplicative randomness with constant elasticity demand or power form demand which is commonly used in the literature (Wilkinson, 2005). Similar to the section with additive demand case, we define random variables of ϵ_1 and ϵ_2 and different form that section, we integrate the randomness as a multiplicant. To show our point, we provide following demand curves for a period t when a product $i$is alone in the market and when it is together with its substitute j as in the following, respectively:

$$d_{it} = \epsilon_i(a_i p_{it}^{-b_i}) \tag{4.4}$$

.

$$d_{it} = \epsilon_i(a_i p_{it}^{-b_i}(p_{jt}/p_{it})^{cr_i}) \tag{4.5}$$

For iso-elastic demand curve, we assume that $a_i > 0$ to assure positive market size and $b_i > 1$ to assure that monopoly always produces at a level where demand is elastic. Since cr_i denotes cross price elasticity it should be always positive.

We show demand functions of each strategy in Table 4.5:

Table 4.5: Demand Curves in Multiplicative Model

Strategy	Demand	Formula
Base Case	x_{11}	$l_1\epsilon_1(a_1 p_{11}^{-b_1})$
Base Case	x_{12}	$l_2\epsilon_1(a_1 p_{12}^{-b_1})$
IS	x_{11}	$l_1\epsilon_1(a_1 p_{11}^{-b_1})$
IS	x_{12}	$l_2\epsilon_1(a_1 p_{12}^{-b_1}(p_{22}/p_{12})^{cr_1})$
IS	x_{22}	$l_2\epsilon_2(a_2 p_{22}^{-b_2}(p_{12}/p_{22})^{cr_2})$
ISES	x_{11}	$l_1\epsilon_1(a_1 p_{11}^{-b_1})$
ISES	x_{22}	$l_2\epsilon_2(a_2 p_{22}^{-b_2})$
IFES	x_{11}	$l_1\epsilon_1(a_1 p_{11}^{-b_1}(p_{21}/p_{11})^{cr_1})$
IFES	x_{21}	$l_1\epsilon_2(a_2 p_{21}^{-b_2}(p_{11}/p_{21})^{cr_2})$
IFES	x_{22}	$l_1\epsilon_2(a_2 p_{22}^{-b_2})$

Before showing demand parameters, we define following parameters to ease notation;

$$\begin{aligned}
\Lambda_{0t} &= a_1 p_{1t}^{-b_1} \\
\Lambda_{1t} &= a_1 p_{1t}^{-b_1} (p_{2t}/p_{1t})^{cr_1} \\
\Omega_{0t} &= a_2 p_{2t}^{-b_2} \\
\Omega_{1t} &= a_2 p_{2t}^{-b_2} (p_{1t}/p_{2t})^{cr_2}
\end{aligned} \tag{4.6}$$

Next, we derive expectation, variance and covariance terms of demand under each strategy assuming that randomness of this section is exactly same with randomness of the previous section. We provide demand parameters in the following table as:

4.3 Summary

Our discussion of this chapter includes two demand models commonly used in literature. With these models, we explicitly consider price substitution with cross price elasticity and dependency with correlated demands in addition to stock out based substitution of Chapter 3. Both analyses, additive and multiplicative, provide correlated demands when there is dual existence in the market. Covariance depends on demand functions in multiplicative randomness whereas demand function does not effect covariance in additive case.

In the next chapter, we present a numerical study based on our discussion of Chapter 3 to compare each rollover strategy. In establishment of stochastic demands and constructing relations between parameters such as prices and demand, period lengths and demands, we use the models presented in Chapter 4.

Table 4.6: Demand Parameters in Multiplicative Demand Model

Case	Parameter	Formula
Base Case	μ_{11}	$l_1 E[\epsilon_1]\Lambda_{01}$
Base Case	μ_{12}	$l_2 E[\epsilon_1]\Lambda_{02}$
Base Case	σ_{11}^2	$l_1^2\Lambda_{01}^2 var(\epsilon_1)$
Base Case	σ_{12}^2	$l_2^2\Lambda_{02}^2 var(\epsilon_1)$
IS	μ_{11}	$l_1 E[\epsilon_1]\Lambda_{01}$
IS	μ_{12}	$l_2 E[\epsilon_1]\Lambda_{12}$
IS	μ_{22}	$l_2 E[\epsilon_1]\Omega_{12}$
IS	σ_{11}^2	$l_1^2\Lambda_{01}^2 var(\epsilon_1)$
IS	σ_{12}^2	$l_2^2\Lambda_{12}^2 var(\epsilon_1)$
IS	σ_{22}^2	$l_2^2\Omega_{12}^2 var(\epsilon_2)$
IS	$cov(x_{12}, x_{22})$	${l_2}^2\Lambda_{12}\Omega_{12} cov(\epsilon_1, \epsilon_2)$
ISES	μ_{11}	$l_1 E[\epsilon_1]\Lambda_{01}$
ISES	μ_{22}	$l_2 E[\epsilon_2]\Omega_{02})$
ISES	σ_{11}^2	$l_1^2\Lambda_{01}^2 var(\epsilon_1)$
ISES	σ_{22}^2	$l_2^2\Omega_{02}^2 var(\epsilon_2)$
IFES	μ_{11}	$l_1 E[\epsilon_1]\Lambda_{11}$
IFES	μ_{21}	$l_1 E[\epsilon_2]\Omega_{11}$
IFES	μ_{22}	$l_2 E[\epsilon_2]\Omega_{02})$
IFES	σ_{11}^2	$l_1^2\Lambda_{11}^2 var(\epsilon_1)$
IFES	σ_{21}^2	$l_2^2\Omega_{11}^2 var(\epsilon_2)$
IFES	σ_{22}^2	$l_2^2\Omega_{02}^2 var(\epsilon_2)$
IFES	$cov(x_{11}, x_{21})$	${l_1}^2\Lambda_{11}\Omega_{11} cov(\epsilon_1, \epsilon_2)$

Chapter 5

Numerical Analysis

In this chapter, we introduce our hypotheses comparing single rollover strategies with dual rollovers, IFES strategy (early introduction) with ISES (late introduction) and motives of a monopolist to introduce a new product. In the first section, we discuss parameter values and explain our optimization methods. Second section summarizes the hypotheses and numerical test results. Based on these, we compare our findings with literature in Section 5.4. Section 5.5 provides useful comments and implications for management of product rollovers based on our findings.

In this chapter we model demand based on the analysis of Chapter 4. Unit time demands are assumed to be normally distributed because of the following consideration. Normal distribution is a well approximation of variety of real life contexts including additive effect of many independent factors [5]. Demand is one of such cases because underlying utility theory includes several factors such as preferences, tastes, worth, value goodness and any of similar concepts [15]. With this in mind, we assume that demand for each product in each period is normally distributed.

In coding demand models and optimization of order-up-to levels under different rollover strategies, we use MATHEMATICA 7.0 [1] which is one of the

[1]Mathematica for Students Semester Edition

commonly accepted software with its built-in functions in probabilistic models and global optimization.

5.1 Values of Parameters and Optimization

We basically group parameters in two: independent parameters and dependent parameters. Dependent parameters are those for which parameter values are set automatically according to values of independent parameters. Independent parameters are further grouped in two: demand related and general parameters. Demand related parameters are those parameters which build up demand function and most of which are not visible in Chapter 3. Price elasticity, market segment size, cross price elasticity, period lengths, prices mean, variance and covariance of ϵ_i are demand related parameters. Rest of the parameters are called as general parameters which generate instances based on models of Chapter 3 given demand distributions. Cost parameters, switching rates are general parameters. We made this distinction because demand related parameters may change according to the demand model considered. Moreover, to provide values for demand related parameters, we utilize marketing literature. Dependent parameters are mean, variance and covariance of effective demands. Discount factors are also dependent parameters because they are dependent on period lengths. In the following table we provide values for independent parameters. When we test our hypotheses, we use following data set as a reference point and change values according to the hypothesis in consideration.

Before discussing optimization methods, we provide explanation for parameter values. In order to provide reasonable values, investment costs are scaled according to revenues that would be generated at the mean value given rest of the parameters. In IS strategy, we first build a production system that is appropriate for only primary product and face a cost of T and then redesign the system such that it can produce both of the products given a point of differentiation with investment cost of U. In the ISES strategy, we pay an amount of K to convert the system for secondary product in addition to the cost of T that

Table 5.1: Values of Parameters

Parameter	T	U	R	P	K	α	β	p_{11}	p_{12}	p_{21}	p_{22}	c_1	c_2	c_{01}	c_{02}
Value	600	800	300	1300	300	0.4	0.4	10	10	10	10	5	5	2	2
Parameter	h_1	h_2	l_1	l_2	$E[\epsilon_1]$	$E[\epsilon_1]$	$Var(\epsilon_1)$	$Var(\epsilon_1)$	$Cov(\epsilon_1, \epsilon_2)$	a_1	a_2	b_1	b_2	cr_1	cr_2
Value	4	4	10	10	10	10	25	25	-6	20	20	1.5	1.5	1.5	1.5

we face in the first period. We assume that $K < U$ to make sure that design of production in ISES strategy makes sense. In the IFES strategy, we first build a system such that it can handle dual production in the first period with a cost of P. In the second period, with phasing out decision, we redesign the system by paying R to only produce secondary product. We redesign the system to avoid high production related costs of dual production. In our data sets, it is always preferable to redesign the system in IFES strategy. We keep in mind that it may be possible to continue the current dual production system in the second period of IFES strategy under different data sets. Cost of redesigning a system from a dual (individual) production to individual (dual) production is assumed to be smaller than the investment for dual(individual) production made at the beginning, $U < T$ and $R < P$. We assume that secondary product is technologically more advanced or fashioned that its production system is more complex and investment on its production system is larger than the primary product's. When designing a relatively advanced production system to produce both primary and secondary products and investing an amount of P, we assume that we face with investment cost which is smaller than building a system for primary product and redesigning it for dual production and investing an amount of $U + T$. Moreover, we think that converting an individual production system to a dual production system is harder than the vice versa. For this reason, we assume that $R < U$. Producing a product alone is assumed to cause smaller production related costs compared to dual production costs. This is a reasonable assumption because in dual production, production system has common operations and management of these operations would be harder than managing individual operations.

According to Nagarajan and Rajagopalan (2008), the typical stock-out induced substitution rates or switching rates range from 0.3 to 0.6 depending on the market characteristics. We use values of $0.1, 0.2, 0.4, 0.6, 0.8$ in our tests. In general, secondary product customers might be assumed to be more reluctant to switch another product when their first choice is out of stock because secondary product is more advanced compared to primary product and it is less likely that features of primary product will satisfy the needs of secondary product customers. In our tests, such cases are also observed.

We set price levels equal at the reservation point but we let relations of prices to vary in our tests. In general, price of secondary product may be higher than primary product because it is more advanced. It is important to note that we set price levels of primary product equal for first and second period in order not to provide superiority of one strategy over another at the beginning point. In general, it may be assumed that relative price level of primary product falls in the next period because it becomes less desirable in a shorter time period compared to secondary product. This reflects the practice that even, in a monopoly, innovation is necessary for success of firms since customer needs are affected from the current technological level of all other good and services. Moreover, there is always the threat of entry in the long-run if monopoly looses its reputation and borders for entry sooner or later melts down. This case is considered in Hypothesis 5.2.7 through increases in future price levels for secondary product which cause future relative prices of primary to fall.

We assume same length of periods and same unit time demand structure in the products. Unit time demands of product are assume to be negatively correlated with a correlation coefficient of -0.24 which in fact parallel with Nagarajan and Rajagopalan (2008)'s finding that most of the products face with a correlation of demand between 0.5 and -0.4.

We assumed that we face with a price elastic demand by setting parameter b_i to a level greater than 1. According to literature, technological or fashion products tend to have elastic demands. For instance, it is stated that demand is price elastic in Japanese Cell Phone Industry by Iimi (2005), in the long run automobile demand for foreign cars by Alper and Mumcu (2007),in market segments of clothing and footwear of New Zealand by Khaled and Lattimore (2006). We note that b_i is directly price elasticity of demand in multiplicative form. In additive form, demand has elastic and inelastic portion depending on price and quantity of the associated point. In general monopoly reaches profit maximization point at elastic portion of demand. An explanation for monopoly's choosing to produce at inelastic portion of primary product could be possibility of generating satisfactory profits through stock-out induced substitution. Cross price elasticity of demand is also assumed to be elastic inspiring by Khaled and Lattimore (2006)'s empirical findings on cross price elasticity of different market segments of clothing and footwear in New Zealand. Market size for unit demands is assumed to be same for both of the producers in both of the periods.

For integration, we use Global Adaptive strategy with Gauss Kronrod rule. As pointed out in Mathematica Tutorial, global adaptive strategy uses recursive bisectioning of the subregion with the largest error estimate into two halves. According to the tutorial Global adaptive strategies in general provide better results compared to Local Adaptive strategy which partitions all regions the error for which is not small enough. Gaussian Kronrod integration rule generates optimal sampling points with polynomial interpolation, form an average integrand value and update this value by adding new sampling points in between the Gaussian points. We test other integration rules and strategies such as Local adaptive strategy, Monte Carlo rule and Oscillatory strategies and find out that Global Adaptive strategy with Gauss Kronrod rule provides effective results in an efficient time.

For optimization, we use NMaximize which is one of built-in function for constrained optimization of MATHEMATICA 7.0. NMaximize implement global optimization algorithms. It possesses several methods for finding constrained global optima. The methods which are flexible enough to cope with different functions are DifferentialEvolution, NelderMead, RandomSearch, and SimulatedAnnealing. We preferred use Nealder-Mead Method because it works efficiently and effectively when there is relatively small number of local optima. In our problem, most of the time we face functions with few local optima or concave functions. NMaximize needs a rectangular starting point and we provide these points using bounds we have discussed in Chapter 3. However, sometimes providing these bounds becomes as hard as solving the optimization problem since we need to solve complex equations such as in 3.24. In those cases, we provide bounds based on mean of demand or optimal levels for similar instances. We think that this is a reasonable way of bounding because market segments we consider are not very different from each other. There is a section in the Appendix which shows that numerical results with NMaximize are parallel with theoretical work (See Appendix B).

In general, solution algorithm can be summarized as in the following. First, optimal order-up-to levels for second periods from second period expected profit functions (such as Equation 3.13) are found with NMaximize. Accordingly, second period optimal levels are plugged into total expected profit functions (such as Equation 3.28). After these, order-up-to levels for first periods are calculated using NMaximize from expected total profit function. Following, expected total profit function is evaluated in the neighborhood of the first period levels found with NMaximize. Finally, the best point that maximizes expected total profit functions is assigned as optimal first period order-up-to levels for first period. Solution algorithms used in numerical tests specific to each cases are explained in Appendix B.

5.2 Hypotheses

In this section, we compare IS and IFES strategies with single roll strategy ISES, early and late introduction and finally investigate incentives of monopoly driven innovation. Comparing single and dual strategies, we try to find out the conditions under which benefits of coexistences such as stock-out induced substitution overcome the costs associated with demand cannibalization. Erhun et al. (2007) and Billington et al. (1998) evaluate new product introduction strategies under what they call demand/market risks and supply/product risks. We also make comparison of dual versus single rollover strategies under demand or product risks specified in each of the following hypotheses. Each hypothesis is numerically evaluated for data sets provided.

Hypothesis 5.2.1 *Expected relative profit of a dual rollover strategy (single rollover strategy) increases (decreases) as required investment and unit costs for dual production diminish.*

Motivation:

According to Billington et al. (1998), ability to manage technology, responsiveness of supply chain and design of the new product are among product risk factors when managing a rollover. In our study, we assume that we have an amplified supplier and products are marketed and distributed in an effective way. Because of these, we eliminate the risk factor associated with responsiveness of supply chain. Moreover, we assume that both secondary and primary products are developed and ready to be produced. In other words, it is assumed that there is the technology to produce both of the products. In our problem setting, there are investments associated with each primary rollover strategy and production/ordering costs of simultaneous production. Changing these parameters affect our decision regarding dual versus solo rollovers since they change relative cost of dual production.

Numerical Test:

In Table 5.2, parameter values used for this test are shown:

Table 5.2: Parameters for Hypothesis 5.2.1

Parameter	Value
c_1	$6, 5, 4, 3$
c_2	$8, 7, 6, 5, 4, 3$
U	$800, 700, 600, 500, 400, 300$
P	$1100, 1200, 1300, 1400, 1500, 1600$

We pay attention to assumptions regarding the relations of T, U, P, R and K. Thus, there are a total of 504 combinations out of 864 possibilities and a total of 1008 instances are generated. We note that expected profits are calculated using additive demand model. Expected profit levels associated with different unit production/ordering costs are shown in the following figure as:

In general, we see a pattern in the sense that as production/ordering costs increase, expected profit levels decrease for each dual rollover strategy. We also experience that consumer-driven substitution softens this decreasing pattern. When a product becomes relatively cheaper, producing large amounts of that

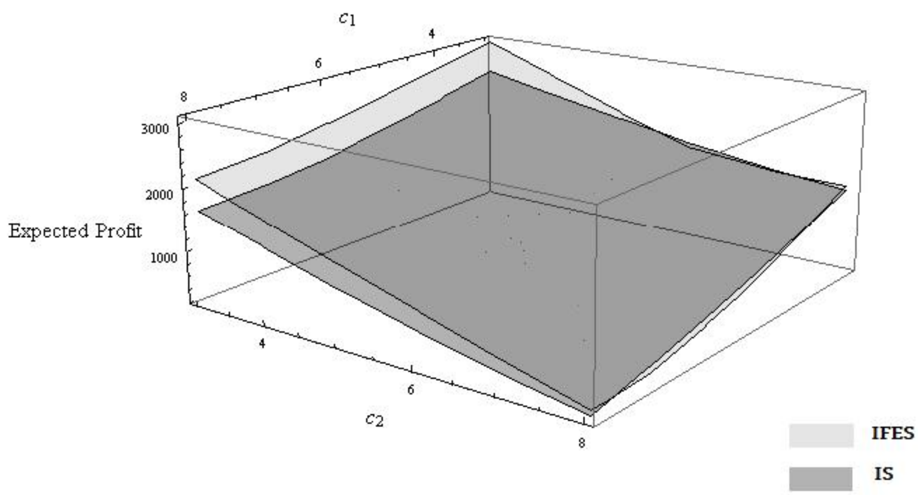

Figure 5.1: Expected Profits v.s. Unit Production/Ordering Costs

product provides improved economy while still enabling us to cover a significant part of loss sales of substitute product through stock-out induced substitution. As unit/production cost of a product increase optimal order-up-to level for the product tend to decrease while causing an increase in substitute product's optimal order-up-to-levels. It is seen that consumer-driven substitution increases sensitivity of a substitute products to changes in unit production/ordering costs of a product. Negative relation between investments costs and expected profit levels are clearer because of investments being fixed costs. Increase/decrease in investment levels do not affect inventory levels in IFES and IS strategies and alters profit and cost levels with the same amount of change in investment levels.

Hypothesis 5.2.2 *Expected relative profit of a dual rollover strategy (single rollover strategy) increases (decreases) as more customers prefer to use second choice products when first choice is out of stock.*

Motivation:

Possibility of stock-out induced substitution decreases demand risk by reducing competition between the two products and decreases supply risk by facilitating stock management. As we note before, stock-out induced substitution is one of advantages of dual rollover strategies. With our numerical experiment, it turns out that the positive effect of switching rates over expected profits of dual rolls is very significant.

Numerical Test:

We use switching rates of $\{0.1, 0.2, 0.4, 0.6, 0.8\}$ for primary product and $\{0.1, 0.2, 0.4, 0.6, 0.8\}$ for secondary product. Demand is modeled with additive demand form in this test. Findings of this test are summarized in the following figure as:

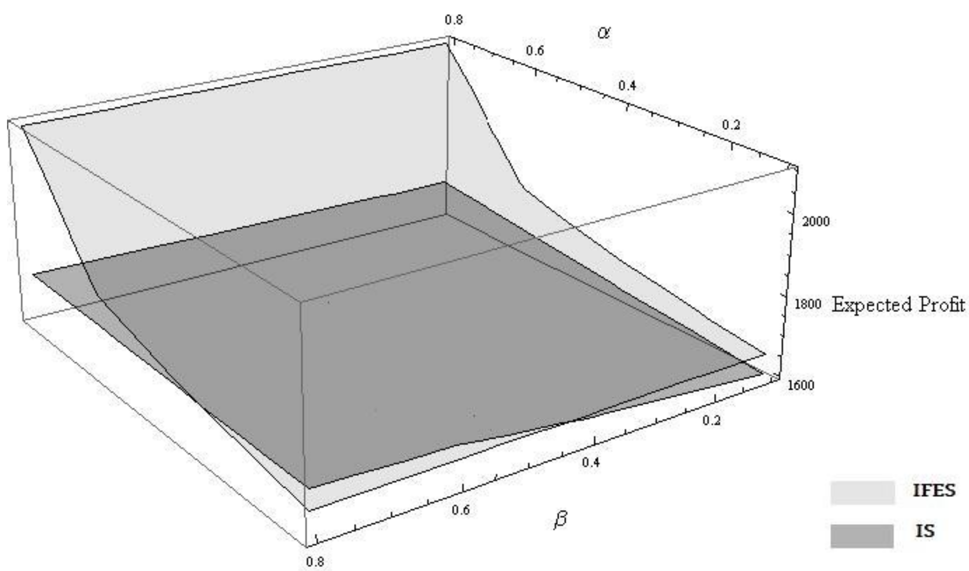

Figure 5.2: Expected Profit vs. Substitution Rates

In general, we find a positive relation between substitution rates and expected profit levels of dual rolls. IS strategy is affected by primary and secondary product substitution rates and if either of them increases, IS strategy experiences higher profit levels. When that is the case, we observe increase in the order-up-to levels for the second choice product and decreasing levels for first choice product in the second period. Li et al. (2010) derives a parallel result in the sense that when there is substitution from old product to new product when old product is out of stock, the need to hold old product diminishes. First period order-up-to-level for primary product seems not to be affected by changes in substitution rates according to our results. Result for IFES strategy is more interesting since expected profit levels are not affected by changes in switching rates of secondary product. We explain this by the tendency of IFES strategy to produce large amounts of secondary product to cover demand of both first and second period. As switching rates associated with primary product increases, expected profit level of IFES strategy increases. When switching rate is almost one, that is most of the primary product customers uses secondary product if they can not find the product, expected profit of IFES jumps up. Thus, results of the numerical study support our hypothesis.

Hypothesis 5.2.3 *Expected relative profit of a dual rollover strategy (single rollover strategy) increases (decreases) as relative prices of secondary product increase.*

Motivation:

Pricing is an important issue in product rollover strategies and there are several papers which deal with pricing issue such as Lim and Tang (2006). In our study, prices are parameters. However, we are aware of importance of pricing in product rollovers (Lim and Tang, 2006). Therefore, we investigate expected profit levels of rollover strategies by changing relative price levels.

Numerical Test:

We change price levels for secondary product. Price levels of $\{10, 12, 14, 16\}$ are used for first period and $\{10, 12, 14, 16\}$ are used for second period. Multiplicative demand model is used for this analysis to assure constant elasticity (detailed information about the values of parameters is given in Hypothesis 5.2.7). Findings when own price elasticity is 1.5 are summarized in the following figures for IFES strategy and ISES strategy, respectively.

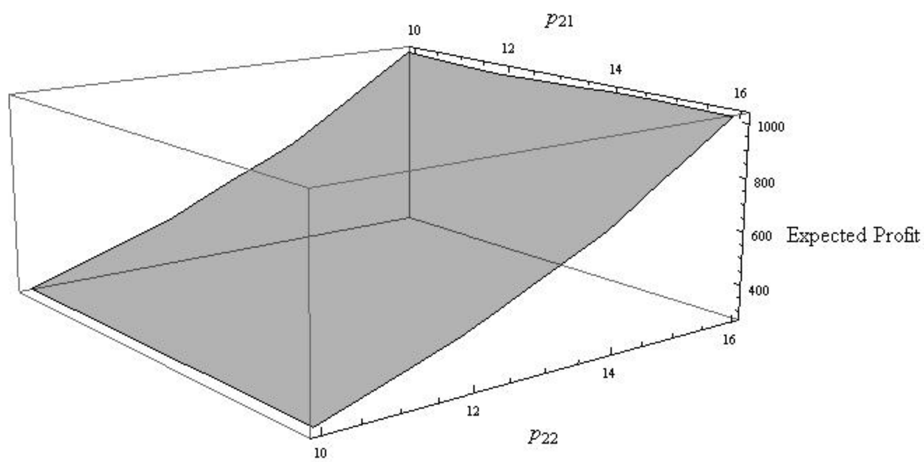

Figure 5.3: Expected Profit vs. Prices (IFES) when $a_2 = 100$, $b_2 = 1.5$, $c_2 = 1.5$

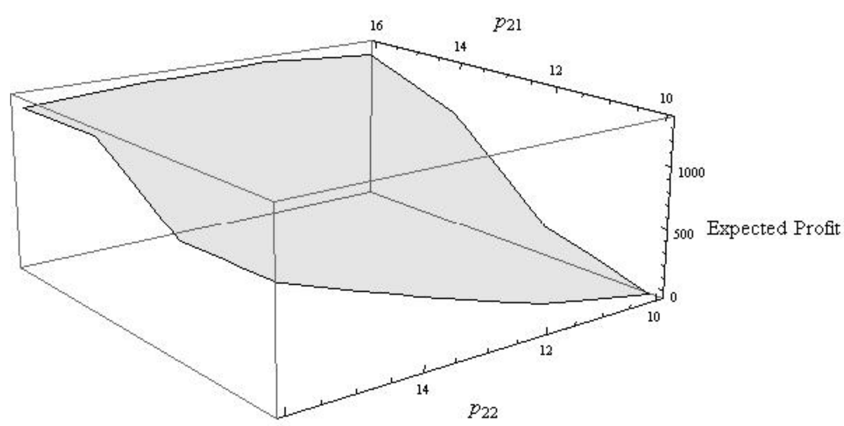

Figure 5.4: Expected Profit vs. Prices (IS) when $a_2 = 100$, $b_2 = 1.5$, $c_2 = 1.5$

In both of Figure 5.2 and Figure 5.2, differences of expected profit levels of dual strategies from the profit levels of ISES strategy are shown.

We observe that as relative prices of secondary product increase in the first period, expected profit levels of IFES strategy increase as well in most of the cases. Moreover, when secondary product becomes expensive in the second period, expected profits rise in general. We explain the few cases where profit levels decrease as prices increase with reduction in demand size. Demand size diminishes because of own and cross price elasticity and reduction in quantity demanded outbalances higher revenues per item sold.

Given fixed prices for first period, it is observed that order-up-to levels for secondary product diminish and levels for primary product increases as second period price increases in most of the instances. As future prices increase, future demand size and uncertainty associated with secondary product decreases. For this reason, order-up-to levels for second period diminish. In fact the levels are so low that it becomes profitable to produce in the first period larger amount and carry it to the next period if there are inventories. However, larger production of first period diminishes as price levels of second period rises up. As ordering for the substitute product diminishes, order-up-to levels for primary product is adjusted by increasing the amount that can not be satisfied with second choice demand of substitute product anymore. Thus, increase in future prices for the secondary product causes resources to be shifted away from secondary product to primary product. At lower price levels, 10, higher future price levels generally indicate higher order-up-to levels for secondary product for first period. In that price level, secondary product is such cheap compared to second period that decision maker prefers to produce large amounts in the first period and if there is any left-over carry it and sell it in the next period. Primary product levels adjust itself according to the inventory levels of secondary product. As it increases with huge amount when future price increases from 10 to 12, stock level of primary product diminishes because of a huge rise in the inventories of secondary product. However, as future prices keep increasing, primary product stock levels increase slightly to be ready for possible loss sales that can not be satisfied by secondary product with increase in its sales.

Given fixed prices for second period, order-up-to levels of secondary product diminish while levels for primary product increase as first period price of secondary product increases in most of the instances. As first period price increases, demand size and uncertainty associated with secondary product decline. These lead to a decrease in the first period ordering levels for the secondary product. Order-up-to levels for primary product is increased to be safe for possible loss sales that could not be satisfied with second choice demand anymore and to produce enough to be able to satisfy unsatisfied customers of substitute product.

Results for IS strategy is illustrated with Figure 5.2. We observe increase in expected profits when second period prices increase. At lower levels of prices , 10, 12 and 14, inventory levels for secondary product increases while levels for primary decreases. Higher marginal revenues outbalances diminishing demand size outbalances at lower price levels. Because primary product adjust itself according to secondary product inventory levels, it decreases stock levels reserved for customers can use secondary as second choice now. At higher price levels, 16, demand size is so small that it is optimal to decrease inventory levels for secondary product. As usual, primary product stock levels adjust by increasing stock levels.

ISES strategy react by decreasing its order-up-to levels for secondary product as second period price rise up. In general, we see that higher price lead lower profits because diminishing expected demand size can not be overcome without consumer-driven substitution or carrying inventory to the lower demand size period.

Our findings support our hypothesis that, as price levels increase for secondary product, relative expected profit levels of dual strategies increase.

Hypothesis 5.2.4 *Expected relative profit of a dual rollover strategy (single rollover strategy) increases (decreases) as negative correlation between the demands of products increases (decreases) or positive correlation between the demands decreases (increases).*

Motivation:

Billington et al. (1998) claims that overlap of market segments decrease demand risk of rollovers. For this reason, dual rollover strategies become more preferable and single rolls become more risky. We think that dependence of demand is related to the overlap of market segments. As overlap between market segments increase, customers becomes more sensitive to alternative product and demands of the products becomes more dependent. Thus, to reflect changes in he overlap between the market segments, we alter correlation of coefficient between demands.

Numerical Test:

Coefficient of correlation is dependent on variances and covariance of unit demands and period lengths. By changing these parameters, we also change mean, variance and discount rates. To focus on overlap of market without any market growth or changes in demand structures, we alter covariance of unit demands. We change the value of covariance for first and second period from their reference point value, -6 to $\{-25, -10, -6, 0, 6, 10, 25\}$. Thus, 49 combinations are tested for each strategy and 154 instances are generated. Additive demand model is used for this analysis. Following figures summarize our findings as:

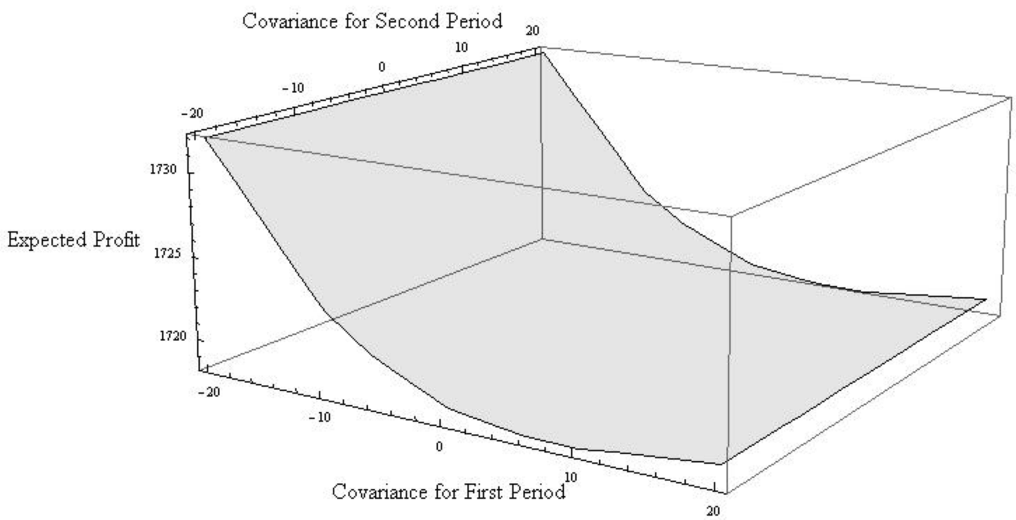

Figure 5.5: Expected Profit vs. Covariance (IFES)

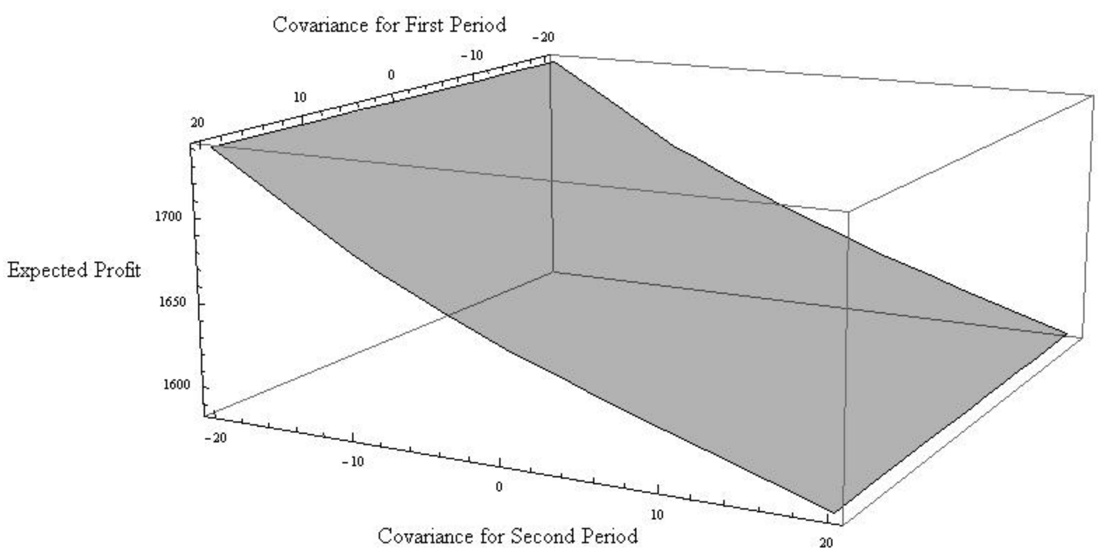

Figure 5.6: Expected Profit vs. Covariance (IS)

It is seen that expected profit level of IS Strategy increases as dependence between two product demands diminishes in the second period when they are negatively correlated. On the contrary, when dependence between demands increase when they are positively correlated, expected profit levels tend to increase. A similar argument can be derived for IFES strategy from Figure 5.2. As negative correlation in the first period, where both products are marketed, expected profit level of IFES increases. However, when positive correlation increases in the first period, expected profits diminish slightly first and then increase. We explain this by tendency to produce large amounts of new product in the first period, use it as a substitute for unsatisfied customers of primary product and carry it to the next period. Expected profit levels for single roll strategy, ISES, remains same since it is not affected by correlation of two products when they are in the market the same time. Thus, relative expected profits of IS strategy moves in the opposite direction with correlation when it is positive while it moves in the same direction when it is negative. Similarly, IFES strategy expected profit levels moves in the same direction with negative correlation but not necessarily move in the opposite direction of positive correlation at low levels of correlation. We conclude that these findings, in general, support our hypothesis. It turns out that, optimal order-up-to levels of dual existence tend to diminish as negative correlation between demands decrease for both strategy. We explain this situation with consumer-driven substitution. When products are not strongly dependent, value of consumer-driven substitution decreases and extra stock carried for substitute product becomes less useful.

Hypothesis 5.2.5 *Expected relative profit of a single rollover strategy (dual rollover strategy) decreases as uncertainty in demand increases (decreases).*

Motivation:

According to Billington et al. (1998) uncertainties in perceived quality of a product, familiarity of market with new technology and diffusion rate of innovation effects market risk associated with a single rollover. According to Raman and Chatterjee (1995) , market acceptance of new products, uncertainty about the new technology and consumer's perception of new technology are among sources of demand uncertainty. Motivated by these ideas; we reflect changes in adoption, quality signal to market and familiarity of new technology with changes in demand variances.

Numerical Test:

Variances of demands for additive demand model are dependent on period lengths and variances of unit demands. Change in period length is discussed in Hypothesis 5.2.5. Thus variances of unit demands are used to alter variances of demands. Data set for variances are $9, 16, 25, 36, 49$. In total, 100 instances are generated and additive form is used to model demand. Findings are summarized in the following figures as:

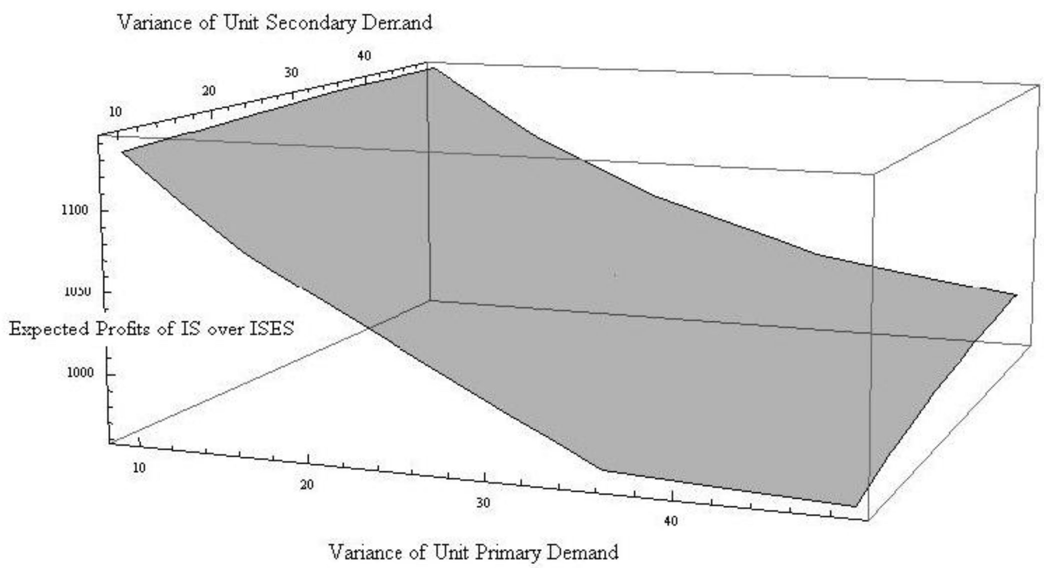

Figure 5.7: Expected Profit vs. Variance (IS)

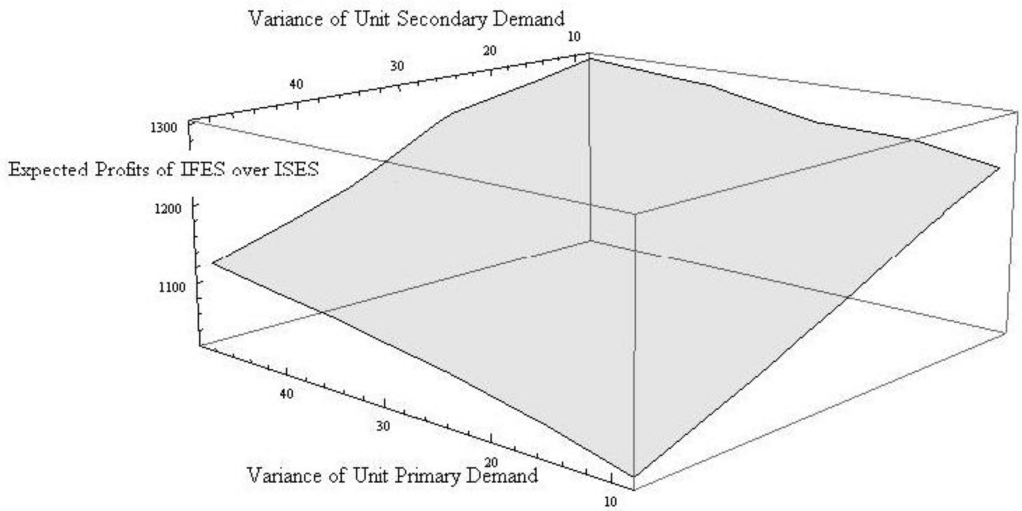

Figure 5.8: Expected Profit vs. Variance (IFES)

We observe that single and dual strategies are affected by uncertainties of demand. To show relative expected profit levels of dual strategies, we plot Figure 5.2 and Figure 5.2. These figures present differences in expected profit levels of each dual rollover strategy and the solo rollover strategy, ISES Strategy. We experience that profit levels of IS strategy are more sensitive to the variance of primary product and this may be due to primary product's being produced in both of the periods. As variance of primary product increases, it becomes optimal to raise inventory level of first period to cope with increasing uncertainty. On the other hand, optimal order-up-to level for primary product decreases in the second period. Since relative uncertainty associated with secondary product demand diminishes as variance of primary increases, it becomes more preferable to produce more of secondary product and use it to satisfy both first and second-choice demand. A similar argument holds when variance associated with secondary product increases. IFES strategy is more sensitive to changes in secondary product demand uncertainty levels since it produces this product for whole time interval. As variance of a product increases, best response of a decision maker in the period of dual existence would be to reduce order-up-to levels for the product and increase levels for substitute product since to use consumer-driven substitution.

We conclude that, our test results support the hypothesis.

Hypothesis 5.2.6 *Expected relative profit of a dual rollover strategy (single rollover strategy) increases (decreases) as periods of dual existence get longer.*

Motivation:

Changes in period lengths affect the values of discount rates, mean, variance and covariance of demands according to our model. Since mean and standard deviation of demands increase at the same rate, coefficient of variation remains same. Similarly, coefficient of correlation remains same. As a result we face with larger demand with similar uncertainty and dependence level. Thus, longer periods imply increase in demand .With larger mean total market segment compared to single segment; decision maker may become more motivated to sell products simultaneously.

Numerical Test:

Data set for peach period length is given as $5, 10, 15, 20$ and a total of 64 instances are developed. Findings are summarized in the following figures for IFES Strategy and IS Strategy, respectively;

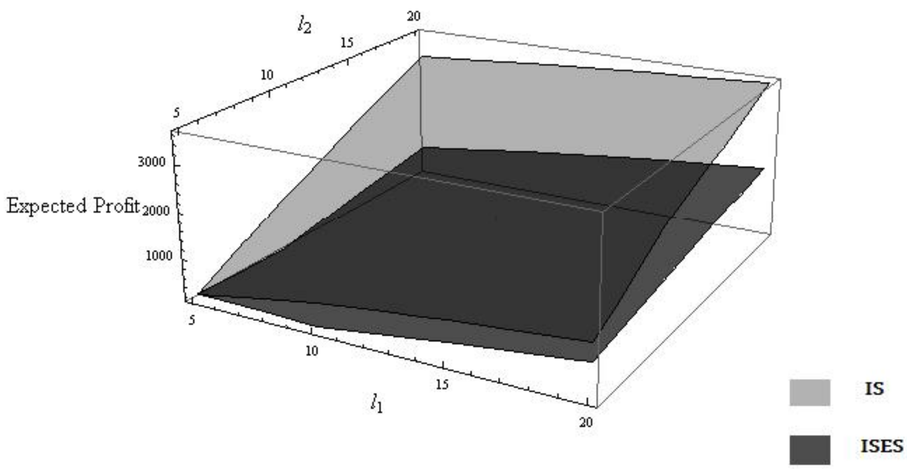

Figure 5.9: Expected Profit vs. Period Lengths (IFES)

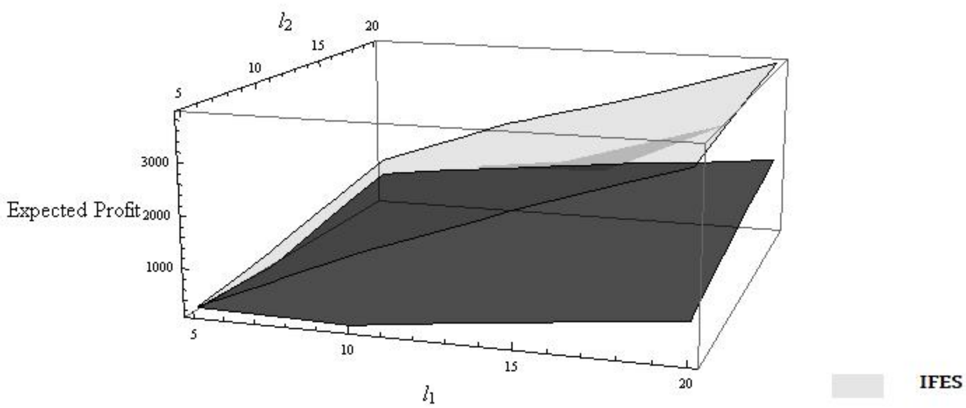

Figure 5.10: Expected Profit vs. Period Lengths (IS)

When the period of which the both products sold becomes longer, relative expected profit levels of dual strategies with respect to ISES strategy increase. Moreover, dual rollovers are also found to be affected by length of the solo existence period in our tests. However, it seems that dual strategies are more sensitive to the lengths of dual existence period. As first period becomes longer, optimal order-up-to levels for secondary product increase and levels for primary product decrease when we look at the test results associated with IFES strategy. Similarly, when second period becomes longer, decision maker improves its relative profit by increasing order-up-to levels for secondary product and adjusting the levels of primary accordingly (decreasing) if we talk about IS strategy. In general, we conclude that results support the hypothesis.

Hypothesis 5.2.7 *Monopoly becomes more willing to introduce a new product as*

- *Relative prices of secondary product increase*
- *New Product demand becomes less price elastic*
- *Market size for new product becomes relatively larger*
- *New product demand becomes more sensitive to substitute prices*
- *Stock-out induced substitution rates increase*
- *Negative correlation between demands decrease*
- *Production technology for new product gets cheaper*

Motivation:

There is np consensus in monopoly and its incentive for innovation (Sastry, 2005). Monopoly firms may be eager to innovation because of market and production factors such as cost reduction effect on production (Reksulak et al., 2008), threats of imitations for existing product from potential competitors (Chen and Schwartz, 2009), short life cycle of existing product, monopoly's financial potential for research and innovation (Blundell et al., 1999). Others claim that competitive environments are more innovative since there are always rival firms. Motivated by these ideas, we observe monopoly profit under Base Case and product rollover strategies and investigate for conditions which may motivate monopoly to introduce new products. In our study there is competition between two products and for this reason their price levels, demand dependence between the two seems to be significant. Moreover, low price levels may be an indicator of short life cycle for primary product. Production technology and easier inventory management by means of stock-out substitution could decrease cost for innovation which in turn motivate monopoly to introduce new products.

Numerical Test:

In this numerical test, we run tests using multiplicative demand model for first, second, third and fourth section of the hypothesis. For remaining section, we use test results from previous hypotheses. We use multiplicative demand model for first four section of the hypothesis since it provides us fixed elasticity. To generate profit levels compatible with additive demand model we use following parameter values:

Table 5.3: Parameters for Hypothesis 5.2.6

Parameter	Value
p_{21}	$10, 12, 14, 16$
p_{22}	$10, 12, 14, 16$
a_2	$100, 150, 180, 200$
b_2	$0.5, 1, 1.5, 2$
cr_2	$0.5, 1, 1.5, 2$

We generate 453 instances and results are summarized in the following figures:

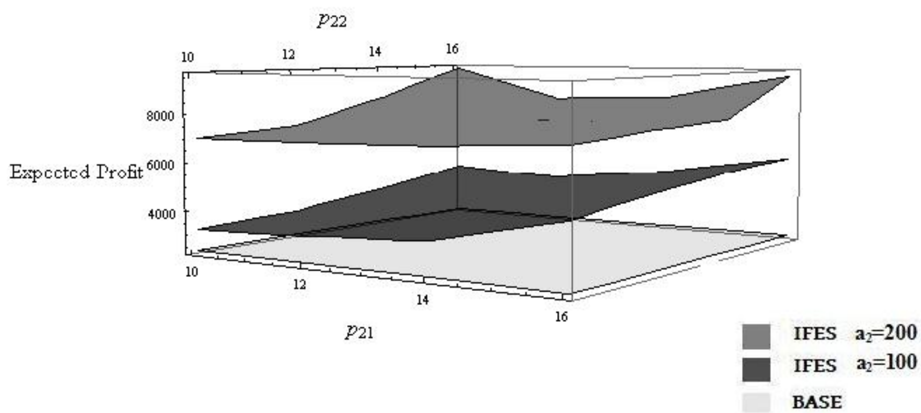

Figure 5.11: Expected Profit vs. Prices given $b_2 = 1.3$ (IFES)

Our test results show that expected profit of IFES strategy increases as second period price levels increase except few cases. In those cases, secondary product is used to be very cheap in the first period compared to its future value and demand is very sensitive to changes in prices. For those exceptions, demand is so elastic in the second period that smaller demand size overcomes high marginal revenue per product. Moreover, before price change, it is economical to produce large amounts in the first period such that remaining amount is carried to the next period. However with price increase, producing smaller amounts in the first period becomes economical with shrinkage in future demand size and uncertainty levels. Changes in the first period price increase expected profit levels in general. However, when first period prices are very cheap compared to second period price (relative price of 0.70), decrease in expected profit is observed. We provide Figure 5.2 as an example to show reaction of expected profit levels to changes in prices with two different demand size at a given own price elasticity.

We claim that if demand becomes more elastic, relative profit levels of IFES strategy is expected to decrease if we compare it with the case where there is no secondary product introduction (Base Case) at given prices and support this claim with our numerical findings. As demand becomes more elastic, customers become more sensitive to the price levels and reflect this by shrinking their demand even more. Thus, expected profit levels at given prices diminishes with sensitive customers and diminishing demand in quantity. Looking from other

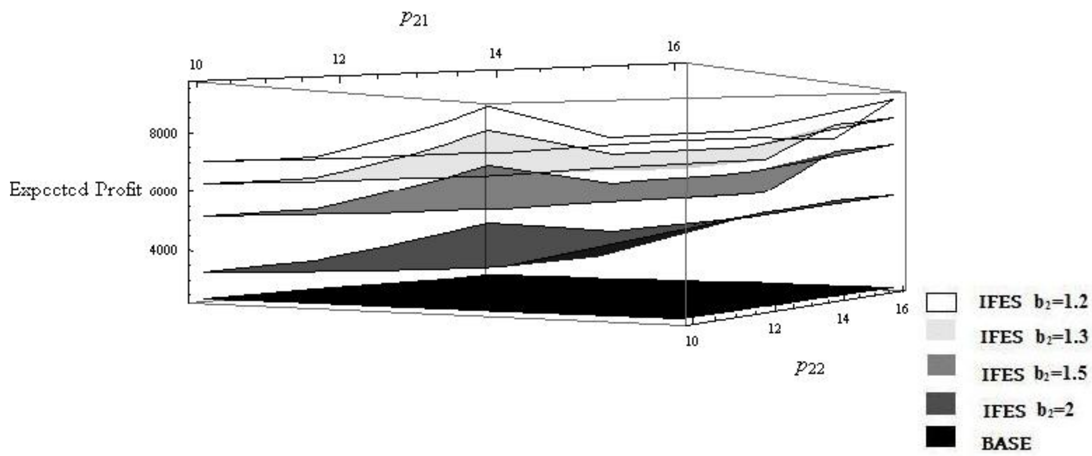

Figure 5.12: Expected Profit vs. Own Price Elasticity given $a_2 = 150$ (IFES)

side, a monopoly would be more willing to introduce a new product if elasticity of demand for new product decreases.

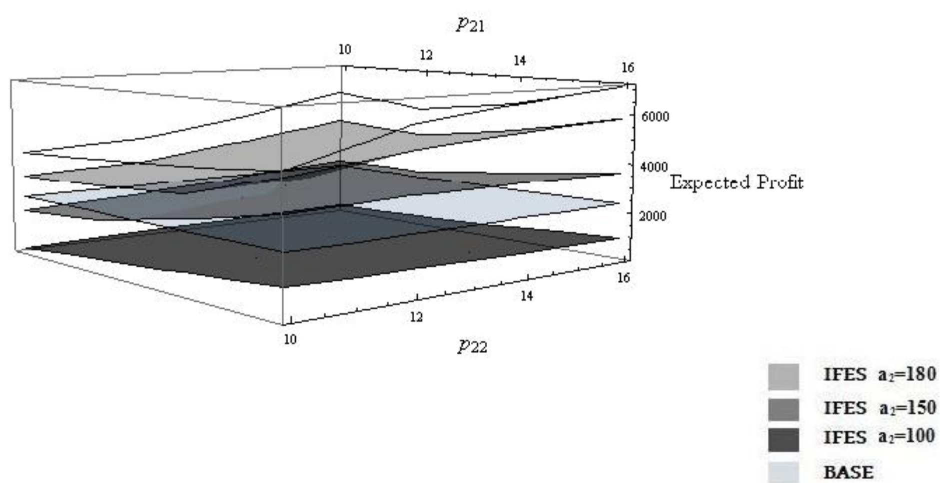

Figure 5.13: Expected Profit vs. Demand Size given $b_2 = 1.2$ (IFES)

We observe that as market size increase, monopoly faces with higher profit levels with IFES strategy at each price levels. Increase in customer base leads increase in the mean of demand. For this reason, opportunity cost of Base Case increases. We provide Figure 5.2 which summarizes results with a given own price elasticity.

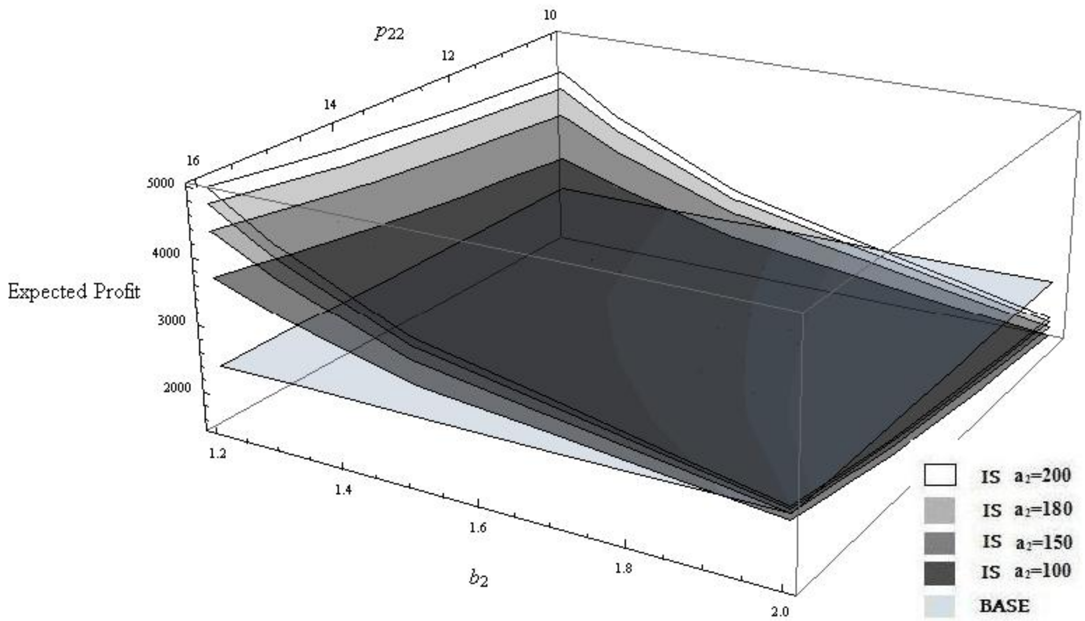

Figure 5.14: Expected Profit vs. Prices, Demand Size, Price Elasticity (IS)

IS strategy reacts to changes in second period prices since it does not have secondary product in the first period. Thus, we are able to provide one figure that summarizes test results for prices, demand size and price elasticity. We observe that as price increases expected profit levels for IS strategy increase even demand is very elastic. We face with lower primary product order-up-to levels and higher levels for secondary product. As market size becomes larger, expected demand increases and monopoly becomes more willing to introduce a new product with IS strategy. Similar to IFES strategy, inelastic demand is an incentive for monopoly to introduce new product according to our test results. Thus, numerical results are parallel with the hypothesis.

Similar to IS strategy, we provide one figure that summarize test results associated with first, second and third sections of this hypothesis. Findings for ISES strategy show that higher price lead lower profits when demand is relatively elastic (for elasticity levels of 1.5 and 2). On the other hand, when demand is less elastic, expected profit levels increase as prices increase. Like IS and IFES

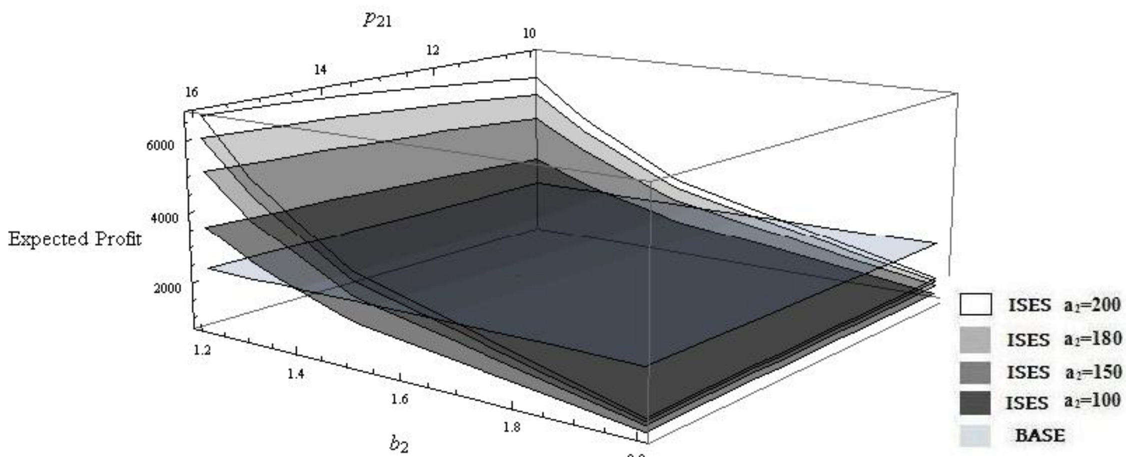

Figure 5.15: Expected Profit vs. Prices, Demand Size, Price Elasticity (ISES)

strategy, customer base is significant in profit levels and as it increase profit levels expected to increase. As shown in the figure, as elasticity increase expected profits decrease and vice versa.

Thus, we conclude that test results are parallel with the first, second and third section of the hypothesis. Next, we investigate effects of cross price elasticity on profits of IS and IFES strategy with the help of following figures:

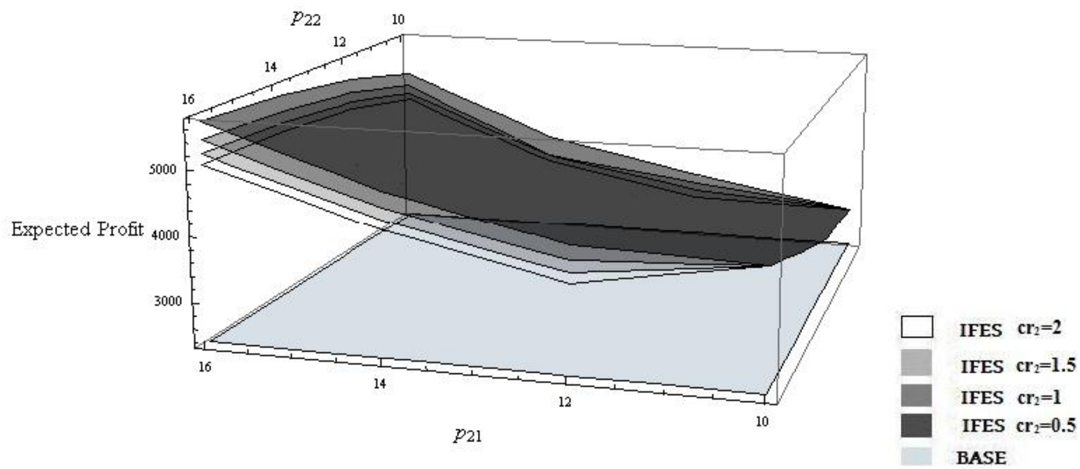

Figure 5.16: Expected Profit vs. Cross Price Elasticity (IFES)

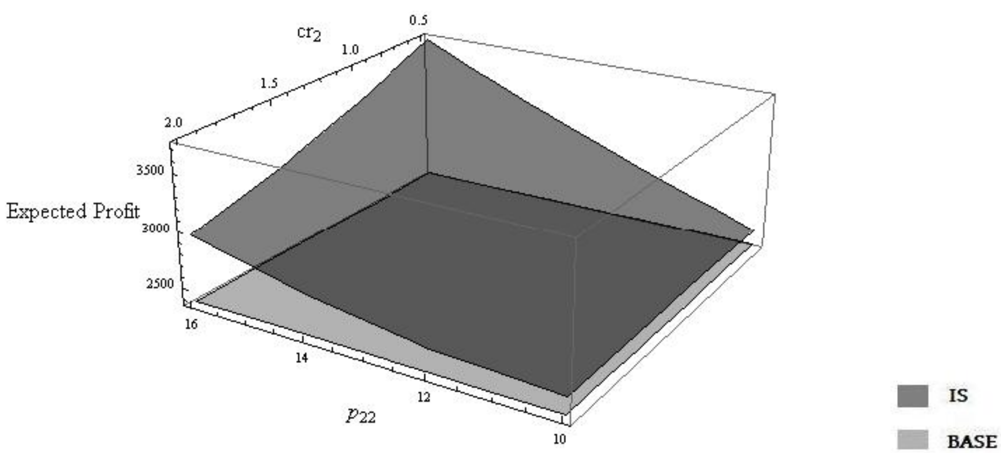

Figure 5.17: Expected Profit vs. Cross Price Elasticity (IS)

Both of the figures show us that as cross price elasticity, sensitivity of customers to the substitute prices, increase, expected profit levels decline at each price levels. This is reasonable finding in the sense that it provocates demand cannibalization of a product by its substitute and very important in our decisions for rollovers. If cross price elasticity decline, monopoly is expected to be motivated to introduce a new product. Moreover, dual product rollovers attractiveness over single roll increase as cross price elasticity decrease based on our findings.

We conclude that as stock-out substitution increase, expected profit levels of dual rollover strategies increase when investigating for Hypothesis 5.2.2 and this result support our hypothesis that monopolist firm would be more willing to introduce new product by using dual rolls as consumer-driven substitution becomes more appealing with increased substitution rates. Similarly, we find a positive relation between expected profits and negative correlation between demands for both of the dual rollover strategies in Hypothesis 5.2.4. Thus, we may conclude that monopolist becomes more willing to innovate with either of dual strategies if negatively correlated demands become more dependent to each other. Last section of Hypothesis 5.2.7 relates monopoly driven innovation to lower cost of simultaneous production and Hypothesis 5.2.1 supports this argument.

Hypothesis 5.2.8 *Early introduction becomes more attractive and late introduction becomes less attractive as;*

- *First period becomes longer and second period becomes shorter*
- *Switching rates for first period increase and rates for second period diminishes*
- *Price of secondary product increases in the first period and decreases in the second period*
- *Negative correlation between the demands decrease in the first period and increase in the second period*

Motivation:

Up to this point, we have considered on single and dual rollovers. In this hypothesis, we compare dual rollovers; IFES and IS and observe how attractiveness of early introduction (IFES) and late introduction (IS) change as relative prices, period lengths, customer loyalty levels and level of dependence between the demands change.

Numerical Test:

We use previous numerical test for this hypothesis. Results are summarized in the following figures:

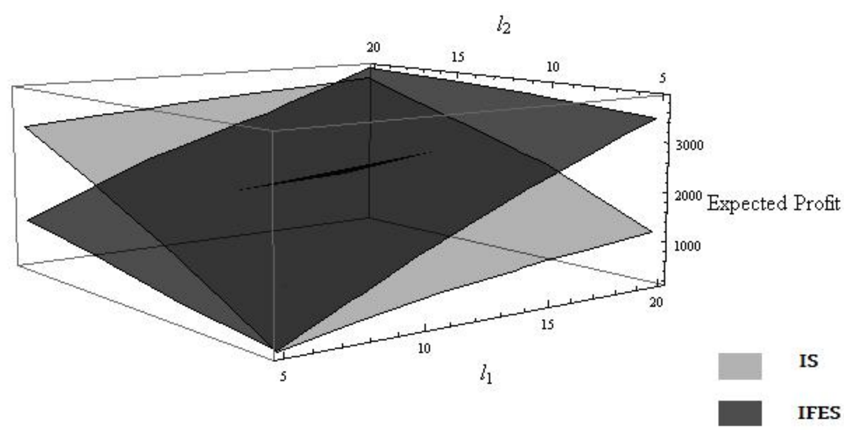

Figure 5.18: Expected Profit vs. Period Lengths

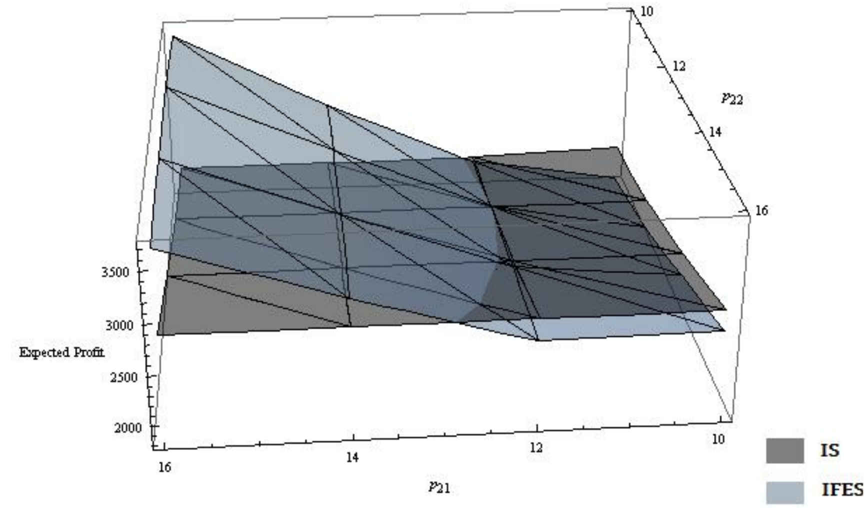

Figure 5.19: Expected Profit vs. Prices

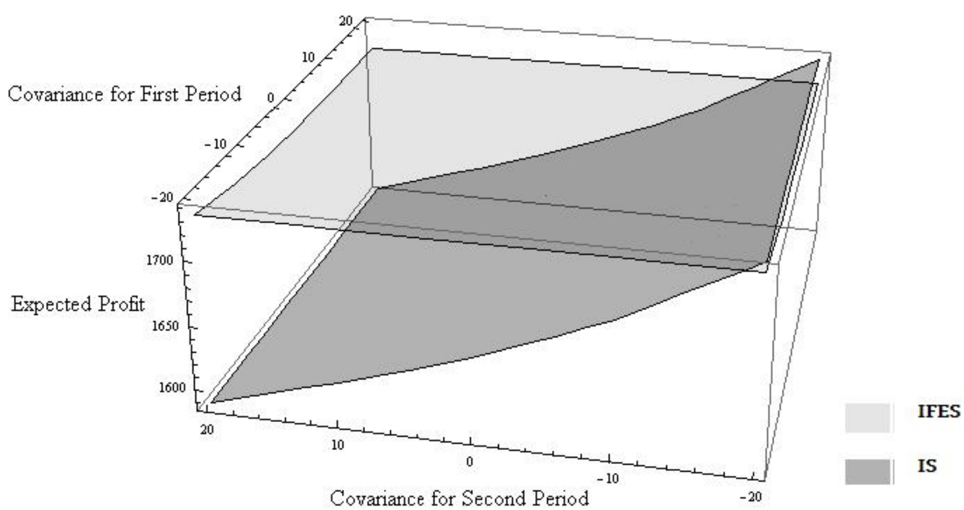

Figure 5.20: Expected Profit vs. Covariance

Figure 5.2 shows the significance of period lengths on our decision for early/late introduction. According to numerical analysis if future period is longer than current period, IS strategy, delaying introduction, becomes attractive. On the other hand, when current period is longer than future period, IFES strategy, immediate introduction, becomes appealing. Figure 5.2 compares profitability of late and early introduction as prices change. If prices of secondary product are going to rise, delaying introduction to the second period is preferable while early introduction becomes more profitable when first period prices are relatively high. Finally, last figure shows dependence between demands and its effect on expected profit levels of dual rollover strategies. To summarize, as products become less dependent in the first period, demand cannibalization effect of IFES strategy decreases while IS strategy becomes more attractive as demands are less likely to steal each other's customers. Rates which are associated with portion of customers who are willing to use second choice demand when their first choice of demand is out are investigated in Hypothesis 5.2.2. When we look at our findings, it is seen that substitution rates are advantages of dual rolls. When customers decide to use more second choice in a period, dual rollover associated with that period becomes favorable.

5.3 An example for Selecting Best Strategy

In this section, we compare primary strategies and choose best strategy to maximize the expected profit level under different data sets. We change demand base, potential customer levels associated with each product in each period and show the optimal rollover strategy associated with each instance. We denote customer base of product i in period j as a_{ij} when both of the products are in the market and product 1 (product 2) and primary product (secondary product) are used interchangeably. When primary product is alone in the market in period j, demand base is shown with a_{pj}. Similarly, when secondary product is alone in the market in the second period, demand base is shown with a_{s2}. Table 5.4 summarizes results associated with each instance we test for as in the following:

Table 5.4: Comparing Primary Strategies when Demand Bases Change

Parameters							Expected Profits			
a_{p1}	a_{p2}	a_{s2}	a_{11}	a_{12}	a_{21}	a_{22}	BASE	ISES	IS	IFES
100	100	100	50	50	50	50	**2355**	1826	1425	811
100	90	100	50	40	40	60	**2247**	1826	1382	703
100	80	120	50	30	50	60	**2165**	2101	710	1087
100	80	120	50	30	50	80	**2165**	2101	1238	1087
80	50	120	30	20	60	80	1551	**1797**	933	997
80	50	120	30	20	40	100	1551	**1797**	1099	790
80	50	140	30	30	40	120	1551	**2072**	1414	1065
100	50	150	5	30	80	150	2276	**2514**	1990	1816
80	60	110	50	60	50	110	1551	1659	**1704**	949
80	70	120	50	70	50	110	1619	1797	**1808**	1087
80	80	130	50	80	50	125	1724	1935	**2028**	1224
70	80	120	50	80	50	115	1466	1645	**1746**	1087
50	50	80	50	30	75	80	778	790	567	**818**
50	50	80	50	20	75	70	778	790	358	**818**
50	50	90	50	20	80	70	778	928	358	**994**
50	50	90	50	20	85	70	778	928	358	**1051**

Table 5.4 shows expected total profit function of each strategy for instances derived with the changes in demand bases. Rest of the parameter values are same with the values in Table 5.1. For each instance, optimal product rollover strategy is highlighted.

Solo rollover strategies, Base and ISES Strategies, are affected from changes of demand base associated with the products when they are single in the market, i.e. a_{p1}, a_{p2}, a_{s2}. On the other hand, dual rollover strategies are affected from changes in demand bases for the products when they exist in the market with substitute product in addition to changes in solo demand bases associated with each strategy. In particular, IS Strategy is affected in changes from solo demand base, a_{p1} of the first period and demand bases, a_{12} and a_{22} of the second period. Similarly, IFES Strategy is affected from changes in a_{s2} , a_{11} and a_{21}.

We begin our tests with instance where solo demand bases are equal to total demand bases in dual rollover strategies and we observe superiority of Base Strategy over other primary strategies. Another solo rollover strategy is second best strategy in this instance. It seems that dual rollover strategies are not preferable when introduction of a new product to the market when primary product exists does not bring improvement in the number of potential customers in the market than the case where they are marketed uniquely.

Introduction of a new product generally brings improvement in the demand bases because new product features or technology may attract population that used not to be part of market of the monopoly. Moreover, keeping primary product in the market may save loyal customers of primary product, which are reluctant to new features or technology, and as a result of this bring improved total demand base in comparison to solo secondary product demand base. Thus, we derive instances where dual market bases do not necessarily add up to solo demand bases but takes values in between solo demand bases of new and old product. Moreover, we also derived instances where solo demand bases add up to the values over solo bases. In these cases, we are able to find other primary strategies as optimal strategies.

We observe that when total demand bases of dual existence of second period are in between solo demand bases of the same period for primary product , a_{p2} and for secondary product, a_{s2}, Base Strategy still remain as optimal primary strategy. Instances from 1 to 4 show such cases.

When solo demand base of primary product in the first period, a_{p2}, becomes smaller with respect to solo demand base of secondary product, a_{s2}, ISES strategy becomes superior over Base strategy and this is what we observe in the instances from 5 to 8.

When we let dual demand bases of second period to add up more than the solo demand bases in addition to shrinkage in solo demand bases for primary and secondary product in the second period, we are able to observe IS Strategy as optimal primary strategy. Instances from 9 to 12 show such cases. We may observe these cases in new product introductions, when an unfamiliar and distinct feature is added with secondary product and there is room for new customers being invited to the market with innovation and primary product customers are composed of customer profile skeptical to new technology.

Finally, if dual demand bases of first period to add up more than the solo demand bases, there are shrinkages in solo demand bases for primary and secondary product in the first period and primary product suffers from losses in demand bases from first period to the second period, we are able to observe IFES Strategy as optimal primary strategy. Instances from 13 to 16 shows such cases. We may observe these cases in new product introductions, when primary product has such a short life cycle that it looses its popularity in the future periods dramatically.

5.4 Comparing Findings with Literature

In this section, we compare hypotheses and findings with literature of product rollover, consumer driven substitution and innovation in monopoly markets.

First hypothesis is related to investment and production/ordering costs associated with delayed product differentiation. We find a negative correlation between investment costs or unit production/ordering costs and expected profits of dual rollover strategies. According to, Billington et al. (1998), distinction in possibility of delayed differentiation decrease product/supply risk of single product rollovers. We interpret this argument as in the following: increase in costs and investments associated with delayed product differentiation decreases attractiveness of dual rollovers. Similarly, Erhun et al. (2007) considers manufacturing as internal execution risks which adds to supply risks of rollovers and argue that risks affect how we choose between rollover strategies.

To our knowledge, there is not a direct conclusion related to switching rates, portion of customers shifting to the other product when their first choice is out of stock, with dual/solo rollover strategies. However, Rajaram and Tang (2001) concludes that gains from consumer-driven substitution increase as switching rates increase in their comparison of cases with and without stock-out induced substitution. Moreover, Li et al. (2010) shows that profit levels are expected to increase if there is substitution when introduction time of new product is known. We think that these conclusion support Hypothesis 5.2.2 since it could be interpreted as in the following: dual rollovers with substitution might enjoy increasing profit as substitution increases if there is stock-out induced substitution. In other words, consumer driven substitution is an advantage to cope with price substitution and it serves as a tool to reduce management of product risk and market risk of dual rollovers.

Billington et al. (1998) claims that intersection of the customer bases of new product with the old product contribute to marketing risk associated with dual product rollovers. We think that correlation of demand is related to overlap of market bases since the following idea makes sense: as customers base coincide dependence between demands increase. As it can be in numerical tests, our findings are parallel to Billington et al. (1998) in the sense that when correlation of demands are positive, attractiveness of dual rolls increase and single rolls decrease as dependence between demands decreases. However, for negative correlation the opposite seems to be valid that is correlation is negatively related with expected profit levels of dual rollover strategies.

As we have pointed out before, customers perception about new technology and new product are among factors that affect demand uncertainty (Raman and Chatterjee, 1995). Billington et al. (1998) discusses uncertainties in perceived quality, market's being unfamiliar with new technology and slow adoption rates to innovation increases demand risk with single rollovers. Moreover, Erhun et al. (2007) considers new product characteristics compared to the previous version among factors determining market risk with rollovers. We think that demand uncertainty indicates uncertainties of customers about the product and its attributes. In that case, supplying the new product with the old one becomes less risky. Our findings are parallel to these arguments. Moreover, we also observe effects of the gap between the variances on rollover strategy.

We discuss period lengths effect on dual versus single rollover strategy selection. In our problem setting period lengths affect demand size. According to Lim and Tang (2006) a dual strategy is optimal if large profits are promising. In our setting, with larger demand for both of the products and stock-out substitution between them creates such an environment.

Monopoly driven innovation is a popular subject but there is no consensus on it. According to literature, short life cycle of existing product and low costs of new product introduction and production (Blundell et al., 1999) are among factors that facilitate innovation in monopoly. Short product life cycle of a product is related to its demand structure and price levels. Investment costs for delayed product differentiation technology and costs related to the production of products simultaneously are related to discussion of Blundell et al. (1999). Previous hypothesis support the motivation of a monopoly to introduce new product with high substitution rates, long dual existence periods and lower dependence between demands.

Arslan et al. (2009), Koca et al. (2010) and Lim and Tang (2006) are among papers which consider pricing and timing decision in product rollovers. Lim and Tang (2006) discusses optimal pricing and timing strategy when one of single and dual rollover strategy is optimal. They claim that in pricing decision profit margins and demand densities are critical. Their findings show that in a dual rollover, product with lower demand density should be priced higher if there is a significant difference of demand densities between the old and new product. In our analysis, price levels are parameters and we try to find effect of reservation price of customers on the strategy we choose. Thus, price levels of our study indicate market perception about the new product. Thus, high reservation price levels for new product may indicate product capability (Erhun et al., 2007) and customer base characteristics (Billington et al. 1998) and lowers market risk associated with dual rollovers.

5.5 Managerial Insights and Summary

Consumer-driven substitution, price substitution, correlation of demands, price levels, period lengths and costs and investment associated with delayed product differentiation technology, affect our decision regarding early/late introduction and dual/solo rollovers. Moreover, we correlate relative prices, demand elasticity, period lengths, switching rates, dependence of demands and costs/investments for dual production to the monopoly driven innovation.

We compare IS and IFES strategy with the Base Case in strategy selection for dual/solo rollovers. When selecting appropriate strategy for timing of new product introduction in dual rollovers, we compare IS and IFES strategies. Finally, for monopoly incentive to introduce new product we compare IS, ISES and IFES strategies with Base Case in which there is no secondary product introduction.

Based on our test results, it is seen that there are several forces determining expected profit levels and superiority of a strategy over another. To show nature of strategies and summarize forces which dominate in each strategy, we provide following figure.

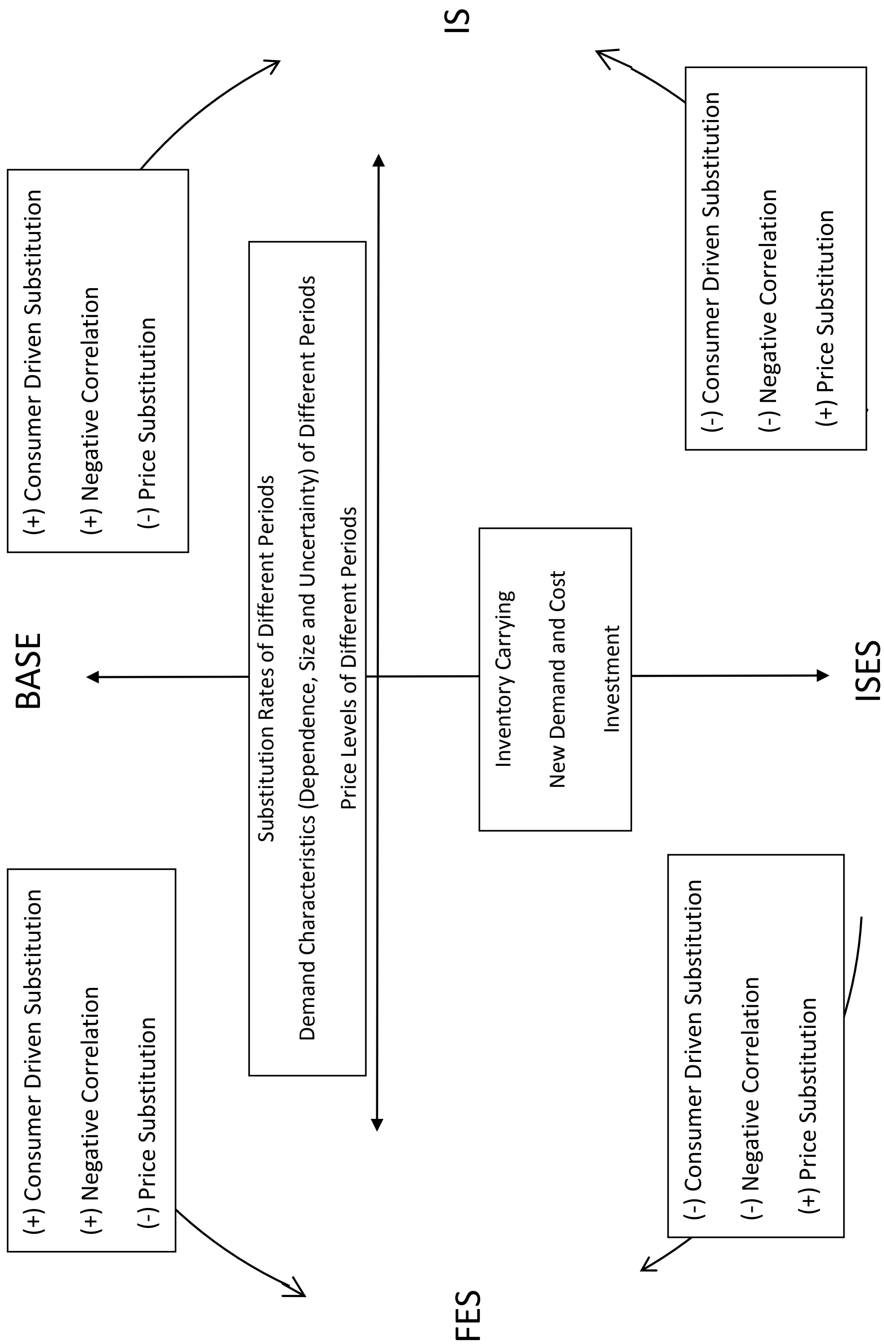

Figure 5.21: Comparing Rollover Strategies

We explain Figure 5.5 in this paragraph as in the following. Consumer-driven substitution combined with negative correlation provides dual rollover strategies; IFES and IS to be more preferable over solo rollover strategies, ISES and Base Strategy. Price substitution through elastic demands, on the other hand, decreases attractiveness of dual rollover strategies. When we compare two dual rollover strategies, IS and IFES, we implicitly compare early and late introduction. It is found in this study that substitution rates, demand characteristics and price levels of different periods are critical in choosing between early and late introduction. Another direct relation shown in the Figure 5.5 is between Base Strategy and ISES Strategy. Innovating with a solo rollover strategy becomes more appealing as new product demand is associated with lower uncertainty levels, higher market base, more inelastic form with respect to price changes. Moreover, as cost/investments to afford ISES strategy diminish, innovating with solo rollover strategy becomes more attractive. Monopoly may be reluctant to innovate with ISES Strategy if advantages of carrying inventory to the next period with BASE Strategy overweigh the advantages of ISES Strategy. We summarize direct relations shown with the figure and we note that using this Figure and by following arrows from one strategy to another, it is possible to derive tradeoff between other strategies.

Chapter 6

Conclusion and Further Research

We consider a monopoly introducing a primary product and need to decide on its strategy for a new, secondary, product introduction. We discuss various issues including product rollover strategy, consumer driven substitution, price substitution and inventory control.

Our study brings different literature together. We incorporate several components such as correlated demands, price substitution and consumer-driven substitution. To our knowledge, we are the first to integrate consumer-driven substitution into comparison of solo/dual rollovers and investigation of monopoly incentives to innovate.

We investigate the conditions that affect the choice of single vs. dual rollover strategy and early vs. late introduction of a secondary product in a dual rollover strategy. Moreover, we observe the behaviors of optimal order-up-to levels when parameters and strategies change. It turns out that our findings are parallel with literature of product rollovers (see Chapter 5.4). We also discuss the factors that motivate a monopoly firm to innovate and introduce new products.

We acknowledge the limitations posed by the assumptions of our model. We do not take into account competition by considering a monopoly market. Competition is a significant factor that affects decision maker's choice of product rollovers as pointed out by Billington et al. (1998) and Erhun et al. (2007). For this reason, a future study we are planning to conduct is an extension of this model with a competitive environment such as oligopoly or monopolistic competition.

Pricing is a significant factor in deciding for rollover strategies as pointed out in the previous section. A future extension of this study could be transforming our model by including price as a decision variable. Period lengths are also shown to be critical in managing rollovers in numerical analysis and converting parameter period lengths into decision variables in our model could be framework for a future study. Another future variant of our study would be using correlated demand between periods.

Another limitation of our model arises from incomplete supply chain consideration. A more practical model could be developed by considering retailers and suppliers. In that case, issues such as centralized vs. decentralized decision making and contract management could be integrated to the model and provide a bigger picture for decision makers.

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Appendix A

Model

A.1 Second Region of Total Profit Function (IS)

Second derivative of total profit function with respect to S_{11} is given as:

$$\begin{aligned}
\frac{\partial \hat{\Pi}^2}{\partial {S_{11}}^2} &= -f_1(S_{11})\left\{r_1(h_1+p_{11})-r_2c_1\right\} \\
&+ r_2 f_1(S_{11}-\tilde{S_{12}})\frac{\partial L(\tilde{S_{12}},\tilde{S_{22}})}{\partial S_{12}} \\
&+ r_2\int_0^{S_{11}-\tilde{S_{12}}}\Bigg(-(p_{12}+h_1)\int_0^{S_{11}-x_{11}}(\frac{1}{\beta}+\frac{dS^*_{22}(S_{11}-x_{11})}{dS_{11}})f_2(x_{12},B^*(S_{11}-x_{11}))dx_{12}) \\
&+ (\alpha(p_{22}+h_2)-(p_{12}+h_1))\int_0^{S^*_{22}(S_{11}-x_{11})}f_2(S_{11}-x_{11},x_{22})dx_{22} \\
&- \alpha(p_{22}+h_2)\int_0^{S^*_{22}(S_{11}-x_{11})}(1+\frac{\frac{dS^*_{22}(S_{11}-x_{11})}{dS_{11}})}{\alpha}f_2(A^*(S_{11}-x_{11}),x_{22})dx_{22}\Bigg)
\end{aligned} \tag{A.1}$$

First line is negative because of our assumptions regarding parameters and second one is zero because $\frac{\partial L(\tilde{S_{12}},\tilde{S_{22}})}{\partial S_{12}}=0$ is zero. We can rewrite remaining lines as in the following:

$$
\begin{aligned}
& \Big((p_{12}+h_1) \int_0^{S_{11}-x_{11}} f_2(x_{12}, B^*(S_{11}-x_{11}))dx_{12}))dx_{12} \\
+ \quad & (p_{22}+h_2) \int_0^{S_{22}^*(S_{11}-x_{11})} f_2(A^*(S_{11}-x_{11}), x_{22})dx_{22} \frac{dS_{22}^*(S_{11}-x_{11})}{dS_{11}} \Big) \\
+ \quad & (\alpha(p_{22}+h_2)-(p_{12}+h_1)) \int_0^{S_{22}^*(S_{11}-x_{11})} f_2(S_{11}-x_{11}, x_{22})dx_{22} \\
- \quad & \frac{(p_{12}+h_1)}{\beta} \int_0^{S_{11}-x_{11}} f_2(x_{12}, B^*(S_{11}-x_{11}))dx_{12} \\
- \quad & \alpha(p_{22}+h_2) \int_0^{S_{22}^*(S_{11}-x_{11})} f_2(A^*(S_{11}-x_{11}), x_{22})dx_{22}
\end{aligned} \tag{A.2}
$$

Using Equation 3.22 and Equation 3.23, we conclude that if $\frac{dv}{dS_{12}}$ is smaller than $\frac{du}{dS_{12}}$ at the points of $(S_{11}-x_{11}, S_{22}^*(S_{11}-x_{11}))$, then the total profit function is strictly concave in S_{11}. Thus, we conclude that concavity of total profit funciton is dependent on demand distribution, price and holding cost parameters associated with the second period as well as susbtitution rates.

When $(\frac{\alpha-\beta}{\beta}+\alpha)(p_{22}+h_2) > (p_{12}+h_1)$ and $\alpha > \beta$, the expression $\alpha(p_{22}+h_2)\int_0^{S_{22}^*(S_{11}-x_{11})}(1+\frac{\frac{dS_{22}^*(S_{11}-x_{11})}{dS_{11}})}{\alpha} f_2(A^*(S_{11}-x_{11}), x_{22})dx_{22}$, from Equation A.1, is positive and this guarantees concavity of total profit function.

A.2 Total Profit Function IFES

We make the following argument;

$$
({S_{11}}^*, {S_{21}}^*) = \begin{cases} (Y_6, Y_7) & \text{if} \quad Y_5 > \check{S_{22}} \\ (Y_4, Y_5) & \text{if} \quad Y_5 \leq \check{S_{22}} \end{cases}
$$

In the proof of this argument we utilize first derivatives of total profit funciton in the regions of

First, we make following definitions;

$$W_1 = r_2 \int_0^{S_{21}-\check{S_{22}}} \int_{S_{11}}^{S_{11}+\frac{S_{21}-\check{S_{22}}-x_{21}}{\alpha}} (\alpha(p_{22}-c_{02}) - \alpha(p_{22}+h_2)$$
$$\int_0^{S_{21}-x_{21}+\alpha(S_{11}-x_{11})} g_2(x_{22})dx_{22}) f_1(x_{11},x_{21})dx_{11}dx_{21} \tag{A.3}$$

$$W_2 = r_2 \int_0^{S_{21}-\check{S_{22}}} \int_0^{S_{11}+\frac{S_{21}-\check{S_{22}}-x_{21}}{\alpha}} (-c_{02}+p_{22}-(p_{22}+h_2)$$
$$\int_0^{S_{21}-x_{21}+\alpha min\{0,(S_{11}-x_{11})\}} g_2(x_{22})dx_{22}\Big) f_1(x_{11},x_{21})dx_{11}dx_{21} \tag{A.4}$$

Then $T_i(S_{11}, S_{21})$ can be rewritten for $i = \{1, 2\}$ as:

$$\frac{\partial \Pi}{\partial S_{11}}|S_{21} > \check{S_{22}} = \frac{\partial \Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}} + W_1 \tag{A.5}$$

$$\frac{\partial \Pi}{\partial S_{21}}|S_{21} > \check{S_{22}} = \frac{\partial \Pi}{\partial S_{21}}|S_{21} \leq \check{S_{22}} + W_2 \tag{A.6}$$

Secondly, inspired by [41], we demonstrate $\frac{\partial \Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}} = 0$ and $\frac{\partial \Pi}{\partial S_{21}}|S_{21} \leq \check{S_{22}} = 0$ based on Theorem 3.4.1, Lemmata 3.4.1 and 3.4.2 as:

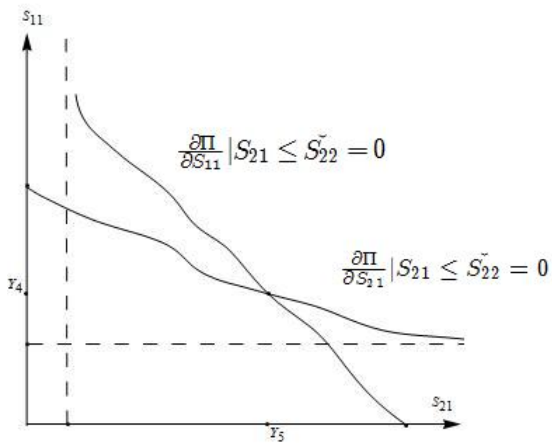

Figure A.1: First order conditions, i.e $\frac{\partial\Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}} = 0$ and $\frac{\partial\Pi}{\partial S_{21}}|S_{21} \leq \check{S_{22}} = 0$

Moreover, we note that, as these curves shifts downwards to the origin, they get negative values. On the other hand, if they shift outwards they get positive values.

If $Y_5 \leq \check{S_{22}}$, we claim that the system $\frac{\partial\Pi}{\partial S_{11}}|S_{21} > \check{S_{22}} = 0, \frac{\partial\Pi}{\partial S_{21}}|S_{21} > \check{S_{22}} = 0$ has no solution for $S_{21} > \check{S_{22}}$ and $\forall S_{11}$. The proof to this claim is stated as in the following arguments.

For $S_{21} > \check{S_{22}} \geq Y_5$, $\frac{\partial\Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}} = 0$ and $\frac{\partial\Pi}{\partial S_{21}}|S_{21} \leq \check{S_{22}} = 0$ do not coincide and one of the following situations occurs; both of the functions are negative, one is negative and the other is positive or both of the functions are negative (see Figure A.2. For the first two cases, we guarantee that $\frac{\partial\Pi}{\partial S_{11}}|S_{21} > \check{S_{22}} =, \frac{\partial\Pi}{\partial S_{21}}|S_{21} > \check{S_{22}} = 0$ does not have a solution because both W_1 and W_2 are negative. The case where both of the functions are positive occurs in the right upward direction. In this case, $\frac{\partial\Pi}{\partial S_{21}}|S_{21} \leq \check{S_{22}}$ is bigger than $\frac{\partial\Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}}$ because $\frac{\partial\Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}} = 0$ is under the curve $\frac{\partial\Pi}{\partial S_{21}}|S_{21} \leq \check{S_{22}} = 0$. Since $W_2 < W_1$, $\frac{\partial\Pi}{\partial S_{11}}|S_{21} > \check{S_{22}} = 0, \frac{\partial\Pi}{\partial S_{21}}|S_{21} > \check{S_{22}} = 0$ does not have a solution.

If $Y_5 > \check{S_{22}}$, we claim that there is a solution to the system of $\frac{\partial\Pi}{\partial S_{11}}|S_{21} > \check{S_{22}} = 0, \frac{\partial\Pi}{\partial S_{21}}|S_{21} > \check{S_{22}} = 0$ in the region where $S_{21} > \check{S_{22}}$. To the left of the line $S_{21} = Y_5$, first derivative functions the region $S_{21} \leq \check{S_{22}}$ can take positive values at the same time by shifting both of the curves upwards. For the case where both of the functions having positive values, $\frac{\partial \Pi}{\partial S_{11}}|S_{21} \leq \check{S_{22}}$ is bigger than $\frac{\partial \Pi}{\partial S_{21}}|S_{21} \leq \check{S_{22}}$. Thus, there is a solution where the system has a solution and we denote this point by (Y_6, Y_7).$\blacksquare$

Appendix B

Computational Algorithm and Results

Chapter 3 provides conditions under which expected profit function is concave. To ensure genarility of our algorithm, we use Global Optimization Built-in Function NMaximize of Mathematica 7.0 in our numerical studies. Although computation time varies from instance to instance, average time is about 10 minutes per instance. Computational results for the instances of Chapter 5 and associated optimal primary strategies with each instance can be found in the website, *http://www.ie.bilkent.edu.tr/ ~ esmakoca*.

In each of the following sections, we explain the algorithm and compare our findings with theoritical findings for Base Case, IS Strategy and IFES Strategy. Since ISES Strategy gives closed form expressions, we need not use Global Optimization.

B.1 BASE Strategy

We prove that Base Strategy is concave with our general assumptions. In finding optimal values for Base Strategy, we use following algorithm;

1. Compute optimal order-up-to level for primary product in the second period, $\hat{S_{12}}$
2. Compute optimal point in the first region of total profti funciton, Y_1
3. If $Y_1 < \hat{S_{12}}$, optimal primary product order-up-to-level in the first period is Y_1
4. Otherwise, find Y_2 from total profit function in the second region, Equation 3.6 with NMaximize and assign Y_2 as optimal primary product in the first period

To show that NMaximize provides us effective results, we present first derivatives of total profit function (second region) with respect to S_{11} for three instances with Table B.2. Parameter values of the intsances are shown in Table B.1 and rest of the parameter values are identical with Table 5.1.

Table B.1: Instances (IBase)

Parameter	Instance 1	Instance 2	Instance 3
p_{11}	10	10	12
p_{12}	10	9	10
c_{01}	2	2	2
h_1	4	4	4

First derivative , $\frac{\partial \Pi}{\partial S_{11}} | S_{11} \leq \hat{S_{12}}$, associated with each instance are shown as:

Table B.2: First Derivatives (BASE)

Values	Instance 1	Instance 2	Instance 3
Y_2	170.296	170.304	146.906
$\frac{\partial \Pi}{\partial S_{11}} \| S_{11} \leq \hat{S_{12}}$	-0.0156869	-0.0149	-0.00799855

Thus, we see that NMaximize is an efficient optimization tool for our problem setting.

B.2 IS Strategy

First, we present the algorithm we utilize when findings optimal values in numerical studies as:

1. Compute optimal order-up-to levels for primary and secondary products in the second period, $\hat{S_{12}}$ and $\hat{S_{22}}$
2. Compute optimal point in the first region of total profit function, Y_1 and check if Y_1 is feasible.
3. Find the feasible point, Y_3 and Y_4 that maximizes total profit function in the second region, Equation 3.28, with NMaximize.
4. Compute and compare total profit function at the points (Y_3, Y_4 and (Y_1, $\hat{S_{22}}$) (if this point is feasible)
5. Choose the values which gives improved results as optimal values.
6. Evalute total profit function in the neighboorhood of (Y_3, Y_4 and (Y_1, $\hat{S_{22}}$) and update optimal values if better points are found

We provide three instances which show that NMaximize gives us theoretical results under associated conditions. In the first section, we show that when $(\frac{\alpha-\beta}{\beta}+\alpha)(p_{22}+h_2) > (p_{12}+h_1)$ together with the assumptions of $(p_{12}+h_1) > \alpha(p_{22}+h_2)$ and $(p_{22}+h_2) > \beta(p_{12}+h_1)$, NMaximize gives us results that could be verified with theoretical findings of 3.2.

Rest of the parameter values are same with the values of Table 5.1 and demand is modeled with additive demand form. We ensure that assumptions of concavity hold with these parameter values. To check optimality of the values given with NMaximize we use first order functions,i.e. 3.47 for second period profit function and 3.32 for total profit function in the region where $S_{11} > \tilde{S_{12}}$. Results are shown in the following table as:

Thus, we conclude that results of NMaximize are good approximations of the theory when functions are concave.

Table B.3: Instances (IS)

Parameter	Instance 1	Instance 2	Instance 3
p_{12}	10	10	16
p_{22}	10	10	14
α	0.8	0.7	0.1
β	0.1	0.2	0.8
Optimal Values	Instance 1	Instance 2	Instance 3
Y_3	444.358	451.991	476
$\tilde{S_{12}}$	168.926	202.568	219.898
$\tilde{S_{22}}$	394.574	356.404	334.368

Table B.4: First Derivatives (IS)

Parameter	Instance 1	Instance 2	Instance 3
$\frac{\partial L}{\partial S_{12}}$	$-1.86141.10^{-8}$	$-3.01721.10^{-9}$	$-1.23377.10^{-8}$
$\frac{\partial L}{\partial S_{22}}$	$-8.9561.10^{-8}$	$3.19867.10^{-8}$	$1.36272.10^{-8}$
$\frac{\partial \Pi}{\partial S_{11}}$	-0.00165471	-0.00775654	0.0014702

B.3 IFES Strategy

Algorithm used in computing for IFES is as follows:

1. Compute optimal order-up-to level for secondary products in the second period, $\check{S_{22}}$
2. Compute the feasible point that maximizes total profit function in the first region of total profit function, $(Y_4,Y_{5)})$with NMaximize
3. Find the feasible point, (Y_5,Y_6) that maximizes total profit function in the second region with NMaximize
4. Compute and compare total profit function at the points $(Y_4$, Y_5 and (Y_5,Y_6)
5. Choose the values which gives improved results as optimal values
6. Evalute total profit function in the neighboorhood of $(Y_4$, Y_5 and (Y_5,Y_6) and update optimal values if better points are found

We provide critical parameter values and optimal values calculated with NMaximize when associated expected total profit function is concave. Expected total profit function is concave in S_{11} if $(p_{11}+h_1) > \alpha(p_{21}+h_2)$ and in S_{21} if $r_1(p_{21}+h_2) > r_1\beta(p_{11}+h_1)+r_2c_{02}$ in the region where $S_{21} > \check{S_{22}}$. Similarly, expected total profit function is concave in S_{21} if $r_1(p_{21}+h_2) > r_1(\beta(p_{11}+h_1))+r_2c_{02}$ and in S_{11} if $r_1(p_{11}+h_1) > \alpha(r_1(p_{21}+h_2)+r_2h_2)$ in the remaining region.

Table B.5: Instances (IFES)

Parameter	Instance 1	Instance 2	Instance 3
p_{11}	10	10	10
p_{21}	10	12	14
α	0.8	0.2	0.4
β	0.1	0.4	0.2
Optimal Values	Instance 1	Instance 2	Instance 3
Y_4	257.379	328.029	290.155
Y_5	344.69	230.443	268.452
Y_6	222.465	325.982	282.11
Y_7	430.64	239.113	282.982

First order derivatives of expected total profit function is calculated with values generated by NMaximize in each of the region. Results are shown in the following table as:

Table B.6: First Derivatives (IFES)

Parameter	Instance 1	Instance 2	Instance 3
$\frac{\partial \Pi}{\partial S_{11}} \vert S_{21} \leq \check{S_{22}}$	-0.00282782	-0.00181196	-0.000498651
$\frac{\partial \Pi}{\partial S_{21}} \vert S_{21} \leq \check{S_{22}}$	-0.0099077	-0.00137076	-0.00164165
$\frac{\partial \Pi}{\partial S_{11}} \vert S_{21} > S_{22}$	-0.0000257695	-0.000162702	-0.00387625
$\frac{\partial \Pi}{\partial S_{21}} \vert S_{21} > S_{22}$	-0.00748324	0.00229398	0.000211895

Thus, we conclude that results of NMaximize gives consistent results with theory when functions are concave.