GAMES OF SHARING AIRPORT COSTS

A Master's Thesis

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

GAMES OF SHARING AIRPORT COSTS

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In this study, noncooperative games defined for various cooperative solution concepts in airport problems have been addressed. The existence of a relationship between the design of the games and the equilibrium outcomes has been investigated.

This study explores the conditions where, in a noncooperative game designed with downstream-subtraction consistent or uniform-subtraction consistent solution concept, the cost allocation proposed by this cooperative solution concept appears as the Nash outcome. Then, the uniqueness of this equilibrium has been examined.

Keywords: Airport problems, Downstream-subtraction consistency, Uniform-subtraction consistency, non-cooperative games.

ÖZET

HAVAALANI MALİYETLERİNİN PAYLAŞIMI OYUNLARI

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Bu tez çalışmamızda havaalanı problemlerinde çeşitli işbirlikçi çözüm kavramları için tanımlanan işbirlikçi olmayan oyunlar ele alınmıştır. Oyunun nasıl oluşturulduğu ile ortaya çıkacak denge durumları arasında bir ilişki olup olmadığı araştırılmıştır.

Aşağı akım eksiltmede tutarlı veya birörnek eksiltmede tutarlı olan bir çözüm kavramı ile kurulacak işbirlikçi olmayan bir oyunda Nash çıktısı olarak yine bu işbirlikçi çözüm kavramının önerdiği maliyet dağılımının ortaya çıkacağı koşullar aranmıştır. Ardından bu dengenin tekliği incelenmiştir.

Anahtar kelimeler: Havaalanı problemleri, Aşağı akım eksiltmede tutarlılık, Birörnek eksiltmede tutarlılık, İşbirlikçi olmayan oyunlar

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CHAPTER 1

INTRODUCTION

Airport problems are those problems which have been introduced first by Littlechild and Owen (1973) and illustrated by the airport and the irrigation ditch examples so far. In the airport example, there are several airline companies. These companies are the agents, which need an airstrip for take offs and landings of their planes. Since the companies own different types of planes, they are in need of airstrips of different lengths. However they can also make use of an airstrip which is longer than their needs. Therefore, an airstrip with the length of the need of the company which needs most would fulfill the needs of all of the companies.

Companies, in other words the agents, agree on maintaining a strip jointly and sharing the cost of building the strip. However, the cost of building a strip increases with the length of the strip. They build a strip which has a length equal to the maximum of what each company needs. Now, how much should each of the companies contribute? This is a fair division problem. Should everyone contribute equally or should they contribute proportional to their needs? If not, how should they share the total cost?

Another important example concerning airport problems is about building an irrigation ditch. There is a path with a water source at the beginning of it and along the path there are several fields . In this problem, agents are the ranchers and each of them needs a ditch from the water source to her field. The cost of constructing a ditch goes up with the length of it. Fields are at different distances to the source; hence agents need ditches of different lengths. Agents agree on building a ditch from the source to the furthest field for common use. We are concerned about how the cost of irrigation ditch construction should be distributed among the ranchers.

Two problems mentioned above are identical for a game theorist. In airport problems, when needs of an agent are fulfilled then all agents with smaller needs are also satisfied. This is an important distinction from similar problems, such as bankruptcy problems.

From the structure of the airport problem, we already know that all needs will be met. It is only the amount that each agent will pay, which is left to be determined. An allocation is a plan which indicates how much each agent should pay. Every agent prefers an allocation where she pays less to another allocation where she pays more. Nevertheless, every agent is indifferent to others paying more or less.

An airport problem is represented by the agents and their needs. A solution concept is a rule which gives an allocation for each airport problem. Solution concepts are characterized by its desired properties like efficiency, consistency and monotonicity. In literature, we see numerous solution concepts (see Thomson (2005) for a survey).

For any airport problem, an associated noncooperative game can be defined using a selected solution concept. Now, we will briefly describe the game.

The furthest rancher from the source proposes an allocation to every rancher in sequence starting from the neighbor rancher who owns the field next to hers.

At every stage, the furthest rancher from the source makes a proposal to an agent. The agent in question can either accept or reject. If she accepts, she pays the amount allocated to her in the proposal to be spent on building this amount's worth of ditch. If there is already a piece of ditch built by any other agent in previous stages passing through this rancher's field, then construction starts from the end of that piece towards the source. If such a piece does not exist, her contribution is used for building a piece of ditch starting from her field in the direction of the source.

If she rejects, a two-person airport problem is defined for the proposer and her. Their costs in this problem are calculated as follows: If any of these two ranchers use a piece of the ditch previously built, then the cost of that piece is subtracted from her cost. Also, contributions assigned by the proposal to the agents closer than the responder to the source will be subtracted from the proposer's and the responder's initial costs. Then, the selected solution concept is applied to this two-person problem and the contribution of the responder is determined. With this contribution, a piece of ditch is built as described above. The amount of her contribution replaces the amount allocated to her in the proposal. On the contrary, in the new proposal the contributions of the ranchers except these two do not differ from the previous proposal. Additionally, the proposal is adjusted for the proposer in a way that the total amount of contributions suggested by this new proposal is equal to the total cost of building an irrigation ditch which serves for the needs of each and every agent. In the next stage, the next agent faces this adjusted proposal.

When all agents are done by responding, the furthest rancher pays for the uncovered parts and the ditch is completed. Lastly adjusted proposal shows how much each agent contributed and is referred as the outcome of this game.

For any solution concept and any given airport problem we have a cooperative solution which is chosen by the solution concept and a set of Nash outcomes of the noncooperative game defined using the solution concept. Here, a natural question arises: How are the cooperative solution and the set of Nash outcomes related?

In a recent paper, Arin et al. (2007) define two games for airport problems using slack maximizer and constrained equal contributions solution concepts in turn. They show that under slack maximizer solution concept, the cooperative solution is the unique Nash outcome of the associated noncooperative game. They also show that under constrained equal contributions solution concept, the cooperative solution is in the set of Nash outcomes.

The purpose of this thesis is to investigate if these results hold in a general context. Our results are valid for all solution concepts which satisfy certain conditions, not only for some specific solution concepts. These conditions are downstream-subtraction consistency and some other axioms related to monotonicity, which will be defined later.

Under the conditions mentioned above, the cooperative solution is always in the set of Nash outcomes. However, Nash outcomes are not always unique. Also, if the cooperative solution is always in the set of Nash outcomes, this does not imply that the solution concept satisfies all the conditions mentioned above. These results are presented in chapter 3. Then, three-agent problems are examined as a special case. Questions for further research can be found at the end of the chapter.

In chapter 4, the noncooperative game defined above is modified. In this new game, when an agent makes her contribution a piece of ditch is built starting from the source,or from the point where the previous piece of ditch ends if it exists, towards the furthest rancher. Under specific conditions, the cooperative solution is proven to be in the set of Nash outcomes. Under more restrictive conditions such as uniform-subtraction consistency which will be defined in chapter 4, the cooperative solution is the unique Nash outcome.

Arin et al. (2007) uses the noncooperative games for extending the solution concepts defined for two-person airport problems to n-person case. Nevertheless, those games can also be used as an implementation tool under the settings in which cooperation cannot be attained.

CHAPTER 2

PRELIMINARIES

2.1 The Model

In our model, we have a finite set of agents, represented by $I = \{1, 2, \ldots, n\}$ and a vector of cost parameters, denoted by $c \in \mathbb{R}^I_+$. c_i refers to agent i's cost parameter. For simplicity, we set $c_1 \leq c_2 \leq \cdots \leq c_n$. An airport problem is first mentioned in Littlechild and Owen (1973) and is defined as a pair (I, c) . All airport problems makes up the set $\mathcal{C}$.

An allocation for (I, c) is a vector $x \in \mathbb{R}^I_+$ such that $0 \leqq x \leqq c$ and $\sum_{i \in I} x_i = c_n$, where x_i is the payment made by agent i. This equality is called the efficiency condition . The set of all allocations for (I, c) is denoted by $\mathcal{X}(I, c)$.

An allocation $x \in \mathcal{X}(I, c)$, satisfies no-subsidy requirement if and only if for each $I' \subseteq I$, $\sum_{j \in I'} x_j \leq \max_{j \in I'} c_j$. Equivalently, $x \in \mathcal{X}(I, c)$, satisfies no-subsidy requirement if and only if for each $i \in I$, $\sum_{j \in I: c_j \leq c_i} x_j \leq c_i$.

For any airport problem (I, c),the set of allocations that satisfy no-subsidy requirement, is also called the set of core allocations, denoted by $Core(I, c)$.

2.2 Solution Concepts

A solution concept $S : \mathcal{C} \rightarrow \mathcal{X}(\mathcal{C})$, maps every airport problem (I, c) to an allocation $x \in \mathcal{X}(I, c)$. Note that a solution concept is single-valued.

We mainly talk about seven solution concepts. These are: sequential equal contributions (SEC), sequential full contributions (SFC), constrained equal contributions (CEC), constrained equal benefits (CEB), constrained proportional (CP), slack maximizer (also called the nucleolus) and priority rule.

Under sequential equal contributions solution concept, SEC, every agent who uses a given segment contribute equally to the cost of that segment.

Definition. For each airport problem $(I, c) \in \mathcal{C}$ and each $i \in I$,

$$SEC_i(I, c) \equiv \frac{c_1}{n} + \frac{c_2 - c_1}{n-1} + \ldots + \frac{c_i - c_{i-1}}{n-i+1}.$$

Equality is the aim of constrained equal contributions concept, CEC, where no-subsidy requirement is still fulfilled.

Definition. For each problem $(I, c) \in \mathcal{C}$ and each $i \in I$,

$$CEC_i(I, c) \equiv \min \left\{ \frac{c_i - \sum_{j \in I: j<i} x_j}{1}, \ldots, \frac{c_{n-1} - \sum_{j \in I: j<i} x_j}{n-i}, \frac{c_n - \sum_{j \in I: j<i} x_j}{n-i+1} \right\}.$$

Constrained equal benefits solution concept, CEB, focuses on each agent's net benefit instead of her contribution.

Definition. For each game $(I, c) \in \mathcal{C}$ and each $i \in I$,

$$CEB_i(I, c) \equiv \max\{c_i - \beta, 0\}$$

where $\beta \in \mathbb{R}_+$ is chosen such that $\sum_{i \in I} \max\{c_i - \beta, 0\} = c_n$.

Under sequential full contributions solution concept, SFC, agents arrive in the order of increasing cost parameters and each agent pays for the segment which has not been covered before her. Agents with equal cost parameters contribute equally to their common segmental cost.

Definition. For each problem $(I, c) \in \mathcal{C}$ and each $i \in I$, let $I^i(c) \subseteq I$ be defined by $I^i(c) \equiv \{j \in I : c_j = c_i\}$. Then,

$$SFC_i(I, c) \equiv \begin{cases} \dfrac{c_i}{|I^i(c)|} & \text{if } c_i = \min\limits_{j \in I} c_j \\ \dfrac{c_i - \max_{j \in I : c_j < c_i} c_j}{|I^i(c)|} & \text{otherwise} \end{cases}$$

Under constrained proportional solution concept, CP, for a given problem $(I, c) \in \mathcal{C}$, we define

$$\rho^1 \equiv \min_{k \in I} \left\{ \frac{c_k}{\sum_{l \in \{1,\ldots,k\}} c_l} \right\}, \qquad k^1 \equiv \max_{k \in I} \left\{ k : \frac{c_k}{\sum_{l \in \{1,\ldots,k\}} c_l} = \rho^1 \right\}.$$

Each agent $i \in \{1, \ldots, k^1\}$ pays $\rho^1 c_i$. Then we define

$$\rho^2 \equiv \min_{k \in \{k^1+1,\ldots,n\}} \left\{ \frac{c_k - c_{k^1}}{\sum_{l \in \{k^1+1,\ldots,k\}} c_l} \right\},$$

$$k^1 \equiv \max_{k \in \{k^1+1,\ldots,n\}} \left\{ k : \frac{c_k - c_{k^1}}{\sum_{l \in \{k^1+1,\ldots,k\}} c_l} = \rho^1 \right\}.$$

Each agent $i \in \{k^1 + 1, \ldots, k^2\}$ pays $\rho^2 c_i$. We proceed likewise until c_n is collected.

Any permutation of the elements in I is an order on I. In this context, an order $\prec$ shows the order of arrival (or leave). For any two agents $i, j \in I$, $j \prec i$ is interpreted as agent j arrives (or leaves) before agent i.

Under priority rule, D, agents arrive in a given order $\prec$. Each agent fully pays for all segments she needs which has not been covered before.

Definition. Let $\prec$ be a given order on I. For each problem $(I, c) \in \mathcal{C}$, and each $i \in I$,

$$D_i^{\prec}(I, c) \equiv \max\left\{c_i - \max_{j \in I: j \prec i} c_j, 0\right\}.$$

Slack maximizer solution concept, SM, is also called the nucleolus.

Sönmez (1994) shows that slack maximizer of any problem $(I, c) \in \mathcal{C}$ can be computed by the following formula:

Definition. For any $(I, c) \in \mathcal{C}$ and $i \in I$,

$$SM_i(I, c) \equiv \begin{cases} \min\left\{\frac{c_i - \sum_{j \in I: j < i} x_j}{2}, \ldots, \frac{c_{n-1} - \sum_{j \in I: j < i} x_j}{n - i + 1}\right\} & \text{if } i \neq n \\ c_n - \sum_{j \in I \setminus \{n\}} SM_j(I, c) & \text{if } i = n \end{cases}$$

Each of these seven solution concepts always gives us an allocation which satisfies no-subsidy requirement (Thomson (2005)).

A convex combination of solution concepts is also a solution concept, since the set of allocations $\mathcal{X}(I, c)$ is convex for each problem (I, c).

2.3 The Noncooperative Game

We proceed by defining a noncooperative game which was first introduced by Arin et al. (1997).

We start with an airport problem (I, c) and a solution concept S.

Agent n makes a proposal, namely an allocation for the problem, $x^1 \in \mathcal{X}(I, c)$ (In Arin et al. (1997) the proposal needs not to satisfy $x \leqq c$.) Agents respond in an order according to their costs. Agent $n-1$ responds first, agent 1 responds last.

Each agent has two possible actions. She can either accept or reject the proposal she is facing. If she accepts, she pays what is assigned to her by the proposal and leaves. If she rejects, then a new problem for the proposer and the rejector is considered. Their cost parameters are changed as explained below and the solution concept S is applied to this two-agent problem. S determines the amount that the rejector will pay.

At stage t, agent i faces the proposal x^{t-1}. If agent i accepts she pays x_i^{t-1} and leaves. If agent i rejects, then the two-person airport problem is defined for $\{i, n\}$, where costs are as follows:

$$c'_n = x_n^{t-1} + x_i^{t-1} = c_n - \sum_{j:j<i} x_j^{t-1} - \sum_{j:i+1\leq j\leq n-1} x_j^{t-1}$$

$$c'_i = \left[c_i - \sum_{j:j<i} x_j^{t-1} - \max_{l:i+1\leq l\leq n-1} \left(\sum_{j:i+1\leq j\leq l} x_j^{t-1} - (c_l - c_i) \right)_+ \right]_+$$

(We denote $\max\{a, 0\}$ by a_+)

Note that $c'_i \leq c'_n$.

For the next stage, the proposal is adjusted. If agent i accepted, $x^t = x^{t-1}$. If agent i rejected, for any $j \in I$,

$$x_j^t = \begin{cases} x_n^{t-1} + x_i^{t-1} - S_i(\{i, n\}, c') & \text{if } j = n \\ S_i(\{i, n\}, c') & \text{if } j = i \\ x_l^{t-1} & \text{otherwise} \end{cases}$$

When all agents are done with responding at stage T the allocation x^T is realized, the proposer pays the rest. We denote this game by $G(I, c, S)$.

CHAPTER 3

EQUILIBRIA IN THE NONCOOPERATIVE GAME

3.1 Downstream-Subtraction

Given an airport problem (I, c) and a solution concept S, let $x \equiv S(I, c)$. Let $i \in I$ and $I' \equiv I \setminus \{i\}$. Define $d(I, i, c, x)$ as the airport problem with the agent set I', and cost vector $c' \in \mathbb{R}_+^{I'}$, where c' is calculated as follows: For any agent $j \in I'$,

$$c'_j \equiv \begin{cases} \min\{c_j, c_i - x_i\} & \text{if } c_j < c_i \\ c_j - x_i & \text{if } c_j \geq c_i \end{cases}$$

This process is called downstream-subtraction[1]. Note that, for any pair of agents $j, k \in I'$, if $c_j \leq c_k$ then $c'_j \leq c'_k$.

A solution concept S, is called downstream-subtraction consistent if and only if for each airport problem (I, c) and each $i \in I$ with $I' = I \setminus \{i\}$, $x \equiv S(I, c)$ implies $x_{I'} = S(d(I, i, c, x))$

Among the solution concepts we have listed above, four of them satisfies downstream-subtraction consistency. Those are sequential full contributions, constrained equal contributions, slack maximizer and priority rule (Thomson (2005)).

[1] Potters and Sudhölter (1999)

Claim 1. *Let (I, c) be an airport problem, $x \in \mathcal{X}(I, c)$, and $\bar{I} \subset I$. Now agents in $I \backslash \bar{I}$ leave the game in an order $\prec$ and we apply downstream subtraction repeatedly. Let $(\bar{I}, c')$ denote the new problem obtained by this process.*

Let $(\bar{I}, \bar{c})$ denote the reduced problem obtained by applying downstream subtraction repeatedly to the problem (I, c), where agents in $I \backslash \bar{I}$ are leaving in the order $\bar{\prec}$.

Then $c' = \bar{c}$. In other words, order of leave is not important in downstream-subtraction.

Proof. Let (I, c) be an airport problem and x an allocation for this problem. Let $\bar{I} \subseteq$ I and agents in $I \backslash \bar{I}$ leave with an order $\prec$. Pick $i, j \in I \backslash \bar{I}$ such that i and j leaves consecutively in $\prec$. Say, agent i leaves first.

Define $\bar{\prec}$ by keeping everybody in the same order as in $\prec$, but this time replacing the order of agent i and agent j. Without loss of generality assume $i < j$. Fix an arbitrary $k \in I$ such that $k \in \bar{I}$ or k leaves later than agent j with respect to $\prec$. At each stage, one agent leaves and costs are adjusted according to downstream-subtraction. We obtain $c'(t)$ at any stage t by subtraction in order $\prec$ and $\bar{c}(t)$ at stage t by subtraction in order $\bar{\prec}$.

Before it is agent i's turn to leave in $\prec$, the ordering is same for $\prec$ and $\bar{\prec}$, so at that stage there is no difference between $c'(t)$ and $\bar{c}'(t)$. In the following equations, t stands for t steps of reduction before i or j leaves.

$$c'_i(t) = \bar{c}_i(t) = \beta_i$$

$$c'_j(t) = \bar{c}_j(t) = \beta_j$$

$$c'_k(t) = \bar{c}_k(t) = \beta_k$$

We will cover three different cases.

i) Let $k < i < j$.

We then have $c_k \le c_i \le c_j$. Since we reach $c'(t)$ by applying downstream-subtraction t times and downstream subtraction preserves this ordering we know $c'_k(t) \le c'_i(t) \le c'_j(t)$. Similarly, $\bar{c}'_k(t) \le \bar{c}'_i(t) \le \bar{c}'_j(t)$.

Reduction in $\prec$:

When i leaves:

$$
\begin{aligned}
c'_k(t+1) &= \min\{\beta_k, \beta_i - x_i\}, \\
c'_j(t+1) &= \beta_j - x_i.
\end{aligned}
$$

When j leaves:

$$
\begin{aligned}
c'_k(t+2) &= \min\{\min\{\beta_k, \beta_i - x_i\}, \beta_j - x_i - x_j\}, \\
&= \min\{\beta_k, \beta_i - x_i, \beta_j - x_i - x_j,\}.
\end{aligned}
$$

Reduction in $\bar{\prec}$:

When j leaves:

$$
\begin{aligned}
\bar{c}_k(t+1) &= \min\{\beta_k, \beta_j - x_j\}, \\
\bar{c}_i(t+1) &= \min\{\beta_i, \beta_j - x_j\}.
\end{aligned}
$$

When i leaves:

$$
\begin{aligned}
\bar{c}_k(t+1) &= \min\{\min\{\beta_k, \beta_j - x_j\}, \min\{\beta_i, \beta_j - x_j\} - x_i\} \\
&= \min\{\beta_k, \beta_j - x_j, \beta_i - x_i, \beta_j - x_j - x_i\} \\
&= \min\{\beta_k, \beta_i - x_i, \beta_j - x_i - x_j\}.
\end{aligned}
$$

ii) Let $i < k < j$

Reduction in $\prec$:

When i leaves:

$$\begin{aligned} c'_k(t+1) &= \beta_k - x_i, \\ c'_j(t+1) &= \beta_j - x_i. \end{aligned}$$

When j leaves:

$$c'_k(t+2) = \min\left\{\beta_k - x_i, \beta_j - x_i - x_j\right\}.$$

Reduction in $\bar{\prec}$:

When j leaves:

$$\begin{aligned} \bar{c}_k(t+1) &= \min\left\{\beta_k, \beta_j - x_j\right\}, \\ \bar{c}_i(t+1) &= \min\left\{\beta_i, \beta_j - x_j\right\}. \end{aligned}$$

When i leaves:

$$\bar{c}_k(t+2) = \min\left\{\beta_k, \beta_j - x_j\right\} - x_i = \min\left\{\beta_k - x_i, \beta_j - x_j - x_i\right\}.$$

iii) Let $i < j < k$.

Reduction in $\prec$:

When i leaves:

$$\begin{aligned}\bar{c}_k(t+1) &= \beta_k - x_i, \\ \bar{c}_j(t+1) &= \beta_j - x_i.\end{aligned}$$

When j leaves :

$$\bar{c}_k(t+2) = \beta_k - x_j - x_i.$$

Reduction in $\bar{\prec}$:

When j leaves :

$$\bar{c}_k(t+2) = \beta_k - x_i - x_j.$$

We observe that in all cases $c'_k(t+2) = \bar{c}_k(t+2)$.

Since k was chosen arbitrarily, this holds for all remaining agents. Hence we conclude that repeated downstream-subtraction in two different orders give the same reduced problem if one of the orders is obtained from the other order by only switching two agents who leave consecutively.

Since I is finite, $\prec$ is a finite order. Any permutation of a finite order can be obtained by repeated swaps of consecutive elements. This is easy to see. $\prec$ is a permutation of elements in $I \backslash \bar{I}$. Let $\prec'$ be another permutation of the same elements. Take the first element in $\prec'$, say α_1. Start with $\prec$. Replace α_1 with the element that comes before it. Continue until α_1 gets to the first place and call this new ordering $\prec_1$.

Now take the second element, say α_2, in $\prec'$. Start with $\prec_1$. Swap α_2 with the element that comes before it. Continue until α_2 gets to the second place.

If we continue likewise and do this for all elements we obtain $\prec'$ from $\prec$ by repeated swaps of consecutive elements. Therefore, we conclude that no matter in which order the agents leave, we get the same reduced game after repeated downstream-subtraction of a set of agents. □

Claim 2. *Let (I, c) be an airport problem, x be an allocation which satisfies no subsidy requirement for this problem. For any $i \in I \setminus \{n\}$, define c' as:*

$$\begin{aligned} c'_n &= x_n + x_i, \\ c'_i &= \left[c_i - \sum_{j:j<i} x_j - \max_{l:i+1 \leq l \leq n-1} \Big[\sum_{j:i+1 \leq j \leq l} x_j - (c_l - c_i) \Big]_+ \right]_+ . \end{aligned}$$

The game $(\{i, n\}, c')$ is obtained from (I, c) by repeated downstream-subtraction of all agents other than i and n with respect to x.

Proof. Choose an $i \in I \setminus \{n\}$. Let the agents $1, 2, ..., i-1$ leave first, then $i+1, i+2, ..., n-1$ leave in this sequence.

In the parentheses, we state the last agent who left, to show which stage we deal with.

After agents $1, 2, ..., i-1$ leave

$$\begin{aligned} c_i(i-1) &= c_i - \sum_{j:j<i} x_j, \\ c_{i+1}(i-1) &= c_{i+1} - \sum_{j:j<i} x_j, \\ c_{i+2}(i-1) &= c_{i+2} - \sum_{j:j<i} x_j, \\ &\dots \end{aligned}$$

After agent $i+1$ leaves

$$\begin{aligned}
c_i(i+1) &= \min\left\{c_i - \sum_{j:j<i} x_j, c_{i+1} - \sum_{j:j<i} x_j - x_{i+1}\right\}, \\
c_{i+2}(i+1) &= c_{i+2} - \sum_{j:j<i} x_j - x_{i+1}, \\
c_{i+3}(i+1) &= c_{i+3} - \sum_{j:j<i} x_j - x_{i+1}, \\
\ldots
\end{aligned}$$

After an arbitrary agent k leaves (where $k > i$)

$$\begin{aligned}
c_i(k) &= \min_{l:i<l\leq k}\left\{c_i - \sum_{j:j<i} x_j, c_l - \sum_{j:j<i} x_j - \sum_{i+1\leq j\leq l} x_j\right\}, \\
c_n(k) &= c_n - \sum_{j:j<i} x_j - \sum_{j:i+1\leq j\leq k} x_j.
\end{aligned}$$

When all agents except i and n leave

$$c_n(n-1) = c_n - \sum_{j:j<i} x_j - \sum_{j:i+1\leq j\leq k} x_j = x_n + x_i.$$

The last equation is due to $\sum_{j\in I} x_j = c_n$.

Let $i < n-1$. After $n-1$ leaves,

$$c_i\left(n-1\right) = \min_{l:i+1\leq l\leq n-1}\left\{c_i - \sum_{j:j<i} x_j c_l - \sum_{j:j<i} x_j - \sum_{j:i+1\leq j\leq l} x_j\right\}.$$

No subsidy requirement is satisfied by x, so $\forall\, l$ s.t. $i < l \leq n-1, \sum_{j:j<i} x_j \leq c_i$ and $\sum_{j:j<l} x_j \leq c_l$. Hence $c_l - \sum_{j:j<i} x_j - \sum_{j:i<j\leq l} x_j \geq 0$. So, $c_i\left(n-1\right)$ is non-negative. Then we have

$$\begin{aligned} c_i(n-1) &= c_i - \sum_{j:j<i} x_j + \min_{l:i+1\le l\le n-1}\left\{0, c_l - c_i - \sum_{j:i+1\le j\le l} x_j\right\} \\ &= c_i - \sum_{j:j<i} x_j - \max_{l:i+1\le l\le n-1}\left(\sum_{j:i+1\le j\le l} x_j - (c_l - c_i)\right)_+ \\ &= \left[c_i - \sum_{j:j<i} x_j - \max_{l:i+1\le l\le n-1}\left(\sum_{j:i+1\le j\le l} x_j - (c_l - c_i)\right)_+\right]_+ \end{aligned}$$

since $c_i(n-1)$ is non-negative.

We have the desired result for the case $i \neq n-1$.

If $i = n-1$ then

$$\left[c_i - \sum_{j<i} x_j - \underbrace{\max_{l:i+1\le l\le n-1}\left(\sum_{j:i+1\le j\le l} x_j - (c_l - c_i)\right)_+}_{0}\right]_+ = c_i - \sum_{j:j<i} x_j = c_{n-1} - \sum_{j:j<n-1} x_j$$

which is equal to what we obtain by repeated subtraction of all agents who come before $n-1$.

Hence for any $i \in I \backslash \{n\}$, $(\{i,n\}, c')$ can be obtained from the original problem (I,c) by repeated downstream-subtraction. We have shown it for a specific order, but using Claim 1 we conclude that this holds for any order.

□

3.2 Axioms

Definition. A solution concept S satisfies weak cost monotonicity if and only if for any pair of airport problems (I,c) and (I,c') s.t. $c' = c + c''$ where $(I,c'') \in \mathcal{C}$, we have $S(c') \geq S(c)$.

Five out of our seven solution concepts satisfy weak cost monotonicity. Only CEB and CP do not satisfy (Thomson (2005)).

Axiom 1. *Let* $(\{i,j\},c)$ *be an airport problem,* $i<j$*,* S *be a solution concept. We assume that* S *satisfies weak cost monotonicity for the two agent case. This means for any* $(\{i,j\},c'')$ *and* $(\{i,j\},c') \in \mathcal{C}$ *s.t.* $c' = c + c''$*,* $S_i(\{i,j\},c') \geq S_i(\{i,j\},c)$ *and* $S_j(\{i,j\},c') \geq S_j(\{i,j\},c)$*.*

Since we have

$$S_i(\{i,j\},c') + S_j(\{i,j\},c') = c'_j \text{ and}$$
$$S_i(\{i,j\},c) + S_j(\{i,j\},c) = c_j$$

we obtain

$$S_i(\{i,j\},c) + c''_j \geq S_i(\{i,j\},c') \text{ and}$$
$$S_j(\{i,j\},c) + c''_j \geq S_j(\{i,j\},c').$$

To check if CP satisfies our axiom, we calculate:

$$CP_1(\{1,2\},c) = \frac{c_1c_2}{c_1+c_2} \leq \frac{(c_1+c''_1)(c_2+c''_2)}{c_1+c''_1+c_2+c''_2} = CP_1(\{1,2\},c'),$$
$$CP_2(\{1,2\},c) = \frac{c^2}{c_1+c_2} \leq \frac{(c_2+c''_2)^2}{c_1+c''_1+c_2+c''_2} = CP_2(\{1,2\},c').$$

Hence CP satisfies Axiom 1.

Note that in weak cost monotonicity and in Axiom 1, we considered a change in cost vector such that for each $i \in I \setminus \{1\}$ the segmental cost, defined as $x_i - x_{i-1}$, ends up at least as much as before. So in Axiom 1, we consider the cost vector changes where $c''_i \leq c''_j$. To check if CEB satisfies our assumption we calculate:

$$CEB\left(\{1,2\},c\right)=\left(\frac{c_1}{2},c_2-\frac{c_1}{2}\right)\text{ and}$$
$$CEB\left(\{1,2\},c'\right)=\left(\frac{c_1+c_1''}{2},c_2+c_2''-\frac{c_1}{2}-\frac{c_1''}{2}\right).$$

Both of them are at least as large as before, since $c_1'', c_2'' \geq 0$ and $c_2'' \geq c_1''$. Therefore all of the seven solution concepts described above satisfy Axiom 1.

Note that, Axiom 1 also implies that if both cost parameters decrease by the same amount, any agent cannot pay more than before.

Definition. A solution concept S satisfies individual cost monotonicity if and only if for any pair of airport problems (I, c) and (I, c'), and for each $i \in I$, where $c_i' \geq c_i$ and for each $j \in I\backslash\{i\}$, $c_j' = c_j$, we have $S_i(c') \geq S_i(c)$.

Axiom 2. *Let* $(\{i,j\}, c)$ *be an airport problem where* $c_i < c_j$*,* S *be a solution concept. We assume that for any airport problem* $(\{i,j\}, c')$ *s.t.* $c_j' = c_j$ *and* $c_i < c_i' \leq c_j$*, we have* $S_i(c') \geq S_i(c)$*.*

Individual cost monotonicity implies Axiom 2, but it is stronger. All seven solution concepts described above except for the sequential full contributions solution concept satisfy individual cost monotonicity (Thomson (2005)), hence they satisfy Axiom 2. SFC does not satisfy Axiom 2 since

$$S_1\left(\{1,2\},(3,4)\right)=3<2=S_1\left(\{1,2\},(4,4)\right).$$

3.3 Results

A noncooperative game $G(I, c, S)$ is defined above, however outcomes, utilities and strategies are not discussed explicitly. Outcome of this game is the payment vector we obtain in the end, x^{n-1}, which is the final version of the proposal adjusted after each rejection, before the next agent makes her move. (Game lasts $n-1$ stages, since at the each stage one agent makes her decision.)

Utilities are equal to the benefits, i.e. for any outcome $x \in \mathcal{X}$ of the game $G(I,c,S)$ and for each $i \in I$, $u_i(x) = c_i - x_i$. Since c_i is given, each agent tries to minimize her payment. Agent n can propose any allocation in the beginning. So the strategy set of agent n is $\mathcal{X}(I,c)$. Agent n has infinitely many possible strategies.

For any other agent $i \in I$, $i \neq n$, possible actions are accept or reject. However, an agent decides to accept or reject any proposal she may face, so her set of strategies is again infinite. A strategy for each agent $i \in I$, is a single-valued function $\delta_i : \mathcal{X}(I,c) \rightarrow \{\text{Accept}, \text{Reject}\}$, which assigns a response to any proposal agent i may face. The strategy set of agent i consists of all such functions.

Proposition 1. *Let (I,c) be an airport problem, $G(I,c,S)$ be a related noncooperative game, where S satisfies the Axiom 1. Let $y \in \mathcal{X}(I,c)$ be offered by agent n in the beginning of the game and $x \in \mathcal{X}(I,c)$ be the outcome, when all agents responded rationally. Then accepting is a best response for all responders $i \in I \backslash \{n\}$, when x is proposed directly in the first stage by agent n.*

Proof. Consider agent $n-1$. When y is proposed by agent n, she chooses between two options: If she accepts, she pays y_{n-1}. If she rejects she pays

$$S_{n-1}\left(\{n-1,n\}, \left(c_{n-1} - \sum_{j:j<n-1} y_j, c_n - \sum_{j:j<n-1} y_j\right)\right).$$

Minimum of these two is equal to x_{n-1}, since it will be adjusted as x_{n-1}.

When x is proposed by agent n, agent $n-1$ again has two options: If she accepts, she pays x_{n-1}. If she rejects she pays

$$S_{n-1}\left(\{n-1,n\}, \left(c_{n-1} - \sum_{j:j<n-1} x_j, c_n - \sum_{j:j<n-1} x_j\right)\right).$$

For all $i \in \{1, 2, \ldots, n-1\}$, we have $y_i \geq x_i$. So, $\sum_{j:j<n-1} y_j \geq \sum_{j:j<n-1} x_j$. Therefore, $\exists \Delta_{n-1} \in \mathbb{R}_+$ s.t. $\sum_{j:j<n-1} y_j = \sum_{j:j<n-1} x_j + \Delta_{n-1}$. Observe that

$$\left(c_{n-1} - \sum_{j:j<n-1} y_j, c_n - \sum_{j:j<n-1} y_j \right) + \Delta_{n-1} = \left(c_{n-1} - \sum_{j:j<n-1} x_j, c_n - \sum_{j:j<n-1} x_j \right).$$

By Axiom 1,

$$S_{n-1}\left(\{n-1, n\}, \left(c_{n-1} - \sum_{j:j<n-1} y_j, c_n - \sum_{j:j<n-1} y_j \right) \right) \leq S_{n-1}\left(\{n-1, n\}, \left(c_{n-1} - \sum_{j:j<n-1} x_j, c_n - \sum_{j:j<n-1} x_j \right) \right).$$

Therefore

$$x_{n-1} \leq S_{n-1}\left(\{n-1, n\}, \left(c_{n-1} - \sum_{j:j<n-1} x_j, c_n - \sum_{j:j<n-1} x_j \right) \right).$$

Hence accepting would be a best response for agent $n-1$, when x is proposed.

Now consider an arbitrary agent $k \in I,\ k \neq n$. When y is proposed by agent n in the first stage, agent k faces the proposal

$$(y_1, y_2, \ldots, y_k, x_{k+1}, \ldots, x_{n-1}, c_n - \sum_{j:k+1 \leq j \leq n-1} x_j - \sum_{j:1 \leq j \leq k} y_j).$$

If she accepts, she pays y_k. If she rejects, she pays $S_k(\{k, n\}, c')$, where

$$c'_n = c_n - \sum_{j:j<k} y_j - \sum_{j:k+1 \leq j \leq n-1} x_j,$$

$$c'_k = \left[c_k - \sum_{j:j<k} y_j - \max_{l:k+1 \leq l \leq n-1} \left(\sum_{j:k+1 \leq j \leq l} x_j - (c_l - c_k) \right)_+ \right]_+.$$

When x is proposed, assume that all agents from $n-1$ to $k+1$ accepted. So x is not changed. Agent k faces x. She has two options: If she accepts she pays

x_k. If she rejects she pays $S_k(\{k,n\},\bar{c})$, where

$$\bar{c_n} = c_n - \sum_{j:j<k} x_j - \sum_{j:k+1\leq j\leq n-1} x_j,$$

$$\bar{c_k} = \left[c_k - \sum_{j:j<k} x_j - \max_{l:k+1\leq l\leq n-1} \left(\sum_{j:k+1\leq j\leq l} x_j - (c_l - c_k) \right)_+ \right]_+ .$$

The same logic applies here.

$$\sum_{j:j<k} x_j \leq \sum_{j:j<k} y_j, \text{ so } \exists \Delta_k \in \mathbb{R}_+ \text{ s.t. } \sum_{j:j<k} x_j + \Delta_k = \sum_{j:j<k} y_j.$$

Observe that $\bar{c_n} = c'_n + \Delta_k$ and $\bar{c_k} \geq c'_k$, $\bar{c_k} - c'_k \leq \Delta_k$. Hence, $S_k(\{k,n\},c') \leq S_k(\{k,n\},\bar{c})$ by Axiom 1.

This implies $x_k \leq S_k(\{k,n\},\bar{c})$ therefore accepting would be a best response for agent k, when x is proposed.

As a result, x would be accepted if it is proposed by agent n in stage 1. □

Corollary. *Let (I,c) be an airport problem, $G(I,c,S)$ be a related noncooperative game, where S satisfies Axiom 1. If x is a SPNE outcome of $G(I,c,S)$, then there exists a SPNE where x is proposed by n and accepted by all agents.*

Proof. If x is a SPNE outcome, that means agent n is able to collect at most $\sum_{i\in I\setminus\{n\}} x_i$ from the rest. He collects the same amount by proposing x. So, this corollary directly follows from Proposition 1. □

Claim 3. *If a solution concept S satisfies downstream-subtraction consistency, then for any problem (I,c), S always give us an allocation which satisfies no-subsidy requirement.*

Proof. Let (I,c) be an arbitrary airport problem, S be a solution concept which satisfies DS consistency. Denote $S(I,c) = x$.

A solution concept always give an allocation, so by definition $0 \leq x \leq c$. Therefore $x_1 \leq c_1$.

Let agents leave the game in an increasing order. At each stage we apply downstream-subtraction w.r.t. x. At an arbitrary stage k, agent k's cost parameter is $c_k - \sum_{i:i<k} x_i$. Since S is DS consistent agent k pays x_k in the reduced game too. Therefore $x_k \leq c_k - \sum_{i:i<k} x_i$. This implies $\forall k \in I \backslash \{n\}, \sum_{i:i\leq k} x_i \leq c_k$. Also $\sum_{j:j\leq n} x_j = c_n$. We conclude that $\forall\ i \in I, \sum_{j\leq i} x_i \leq c_i$, i.e. x satisfies no subsidy requirement. □

Theorem 1. *Let (I, c) be an airport problem, $G(I, c, S)$ be a related noncooperative game, where S satisfies Axiom 1,2 and downstream-subtraction consistency. Now $S(I, c)$ is a SPNE outcome of the game $G(I, c, S)$.*

Proof. Consider a strategy profile in which agent n proposes $s = S(I, c)$ at stage 1, each responder chooses the option which asks her to pay less, and she chooses 'Accept' if she is indifferent. Using Claim 2, the small problem defined for agents n and $n-1$ is a reduced problem of (I, c) obtained by repeated downstream-subtraction with respect to s. Since S is DS consistent, agent $n-1$ will be indifferent, so she will choose accept, proposal will not be changed. Proceeding in this manner, we see that each responder will be indifferent, so accepting is a best response.

We assigned the option 'Accept' when an agent is indifferent. Note that this choice was arbitrary. If some or even all of the agents reject, the proposal again will not be changed.

Let agent n collect the maximum amount she can collect from the rest when responders act rationally, by proposing an allocation z, where the outcome of the game is y. By Proposition 1, we know that there is a SPNE where y is proposed by n and accepted by all the other agents.

Assume $s_n \neq y_n$. Define $k = \{\max_{j\in I} j : j \neq n, y_j \neq s_j\}$. Agent k pays $\min\{y_k, s_k(\{k, n\}, c')\} = y_k$ where

$$c'_n = c_n - \sum_{j:j<k} y_j - \sum_{j:k+1\leq j\leq n-1} s_j,$$

$$c'_k = \left[c_k - \sum_{j:j<k} y_j - \max_{l:k+1\leq l\leq n-1}\left(\sum_{j:k+1\leq j\leq l} s_j - (c_l - c_k)\right)_+\right]_+.$$

when y is proposed.

Agent k pays $\min\{s_k, s_k(\{k,n\},\bar{c})\} = s_k = S_k(\{k,n\},\bar{c})$ where

$$c'_n = c_n - \sum_{j:j<k} s_j - \sum_{j:k+1\leq j\leq n-1} s_j,$$

$$c'_k = \left[c_k - \sum_{j:j<k} s_j - \max_{l:k+1\leq l\leq n-1}\left(\sum_{j:k+1\leq j\leq l} s_j - (c_l - c_k)\right)_+\right]_+.$$

when s is proposed.

i) Let $y_k > s_k$.

If $\sum_{j:j<k} s_j \leq \sum_{j:j<k} y_j$, then $\exists \Delta \in \mathbb{R}_+$ s.t $\sum_{j:j<k} y_j \leq \sum_{j:j<k} s_j + \Delta$. Now $\bar{c}_n = c'_n + \Delta$, $\bar{c}_k \geq c'_k$ and $\bar{c}_k - c'_k \leq \Delta$. By Axiom 1, $S_k(\{n,k\},c') \leq S_k(\{n,k\},\bar{c}) = s_k$. So agent k rejects when y is proposed, contradiction.

If $\sum_{j:j<k} s_j > \sum_{j:j<k} y_j$ then $\exists \Delta \in \mathbb{R}_{++}$ s.t $\sum_{j:j<k} s_j > \sum_{j:j<k} y_j + \Delta$. Now $c'_n = \bar{c}_n + \Delta$, $c'_k \geq \bar{c}_k$ and $c'_k - \bar{c}_k \leq \Delta$. By Axiom 1, $S_k(\{n,k\},c') \leq S_k(\{n,k\},\bar{c}) + \Delta$. Therefore $y_k \leq s_k + \Delta$ and

$$\begin{aligned}\sum_{j\in I\setminus\{n\}} s_j &= \sum_{j:j<k} s_j + s_k + \sum_{j:k+1\leq j\leq n-1} s_j = \sum_{j:j<k} y_j + \Delta + s_k + \sum_{j:k+1\leq j\leq n-1} y_j \\ &\geq \sum_{j:j<k} y_j + y_k + \sum_{j:k+1\leq j\leq n-1} y_j = \sum_{j\in I\setminus\{n\}} y_j.\end{aligned}$$

So agent n cannot collect more by proposing y instead of s, contradiction.

ii) Let $y_k < s_k$

But $\sum_{j\in I\setminus\{n\}} s_j < \sum_{j\in I\setminus\{n\}} y_j$. So there must be $m \in I$ s.t $y_m > s_m$. Now take the largest such m. When y is proposed, agent m pays $\min\{y_m, s_m(\{m,n\}, c')\} = y_k$, where

$$c'_n = c_n - \sum_{j:j<m} y_j - \sum_{j:m+1\le j\le k} y_j - \sum_{j:k+1\le j\le n-1} s_j,$$

$$c'_m = \left[c_m - \sum_{j:j<m} y_j - \max_{l:m+1\le l\le n-1} \left(\sum_{j:m+1\le j\le l} y_j - (c_l - c_m) \right)_+ \right]_+ .$$

When s is proposed,agent m pays $\min\{s_m, S_k(\{m,n\}, \bar{c})\} = s_k = S_k(\{n,k\}, \bar{c})$, where

$$\bar{c}_n = c_n - \sum_{j:j<m} s_j - \sum_{j:m+1\le j\le k} s_j - \sum_{j:k+1\le j\le n-1} s_j,$$

$$\bar{c}_m = \left[c_m - \sum_{j:j<m} s_j - \max_{l:m+1\le l\le n-1} \left(\sum_{j:m+1\le j\le l} s_j - (c_l - c_m) \right)_+ \right]_+ .$$

Denote

$$\epsilon = \sum_{j\in I\setminus\{n\}} y_j - \sum_{j\in I\setminus\{n\}} s_j, \qquad \Delta = y_m - s_m, \qquad e = \sum_{j:m+1\le j\le k} s_j - \sum_{j:m+1\le j\le k} s_j$$

where $\epsilon, \Delta, e \in \mathbb{R}_{++}$. Then,

$$\sum_{j\in I\setminus\{m,n\}} y_j = \sum_{j\in I\setminus\{m,n\}} s_j + (\epsilon - \Delta),$$

$$\bar{c}_n = c'_n + (\epsilon - \Delta) \text{ and}$$

$$\sum_{j:j<m} y_j + y_m + \sum_{j:m+1\le j\le k} y_j + \sum_{j:k+1\le j\le n-1} y_j + y_n =$$
$$\sum_{j:j<m} s_j + s_m + \sum_{j:m+1\le j\le k} s_j + \sum_{j:k+1\le j\le n-1} s_j + s_n.$$

So,

$$\sum_{j:j<m} y_j = \sum_{j:j<m} s_j + e + (\epsilon - \Delta)$$

whereas

$$0 \le \max_{l:m+1\le l\le n-1} \left(\sum_{j:m+1\le j\le l} s_j - (c_l - c_m) \right)_+ -$$
$$\max_{l:m+1\le l\le n-1} \left(\sum_{j:m+1\le j\le l} y_j - (c_l - c_m) \right)_+ \le e.$$

Case 1) If $\epsilon > \Delta$, then $\bar{c}_m \ge c'_m$, $\bar{c}_n = c'_n + (\epsilon - \Delta)$. Define

$$c'' = (c'_m, c'_n + (\epsilon - \Delta)) = (c'_m, \bar{c}_n)$$

Then, $S_m(\{m,n\}, c'') \ge S_m(\{m,n\}, c')$ by Axiom 1. By Axiom 2, $S_m(\{m,n\}, \bar{c}) \ge S_m(\{m,n\}, c'')$. We know that $s_m = S_m(\{m,n\}, \bar{c}) \ge S_m(\{m,n\}, c') \ge y_m$. But $y_m > s_m$, contradiction.

Case 2) If $\epsilon \le \Delta$

$$c'_m - (\Delta - \epsilon) = \max \left\{ 0, c_m - \sum_{j:j<m} y_j - \max_{l:m+1\le l\le n-1} \left(\sum_{j:m+1\le j\le l} y_j - (c_l - c_m) \right)_+ \right\}$$

$$= \max\left\{ -(\Delta-\epsilon), c_m - \sum_{j:j<m} s_j - e + (\Delta-\epsilon) \right.$$
$$\left. - \max_{l:m+1\le l\le n-1}\left(\sum_{j:m+1\le j\le l} y_j - (c_l - c_m)\right)_+ - (\Delta-\epsilon)\right\}$$
$$= \max\left\{ -(\Delta-\epsilon), c_m - \sum_{j:j<m} s_j - e - \max_{l:m+1\le l\le n-1}\left(\sum_{j:m+1\le j\le l} y_j - (c_l - c_m)\right)_+\right\}$$
$$\le \max\left\{ 0, c_m - \sum_{j:j<m} s_j - \max_{l:m+1\le l\le n-1}\left(\sum_{j:m+1\le j\le l} y_j - (c_l - c_m)\right)_+\right\}.$$

$c'_m - (\Delta-\epsilon) \le \bar{c}_m$. So, $c'_m - \bar{c}_m \le \Delta - \epsilon$. We also have $c'_n = \bar{c}_n + (\Delta-\epsilon)$.

Define $c'' = \bar{c} + (\Delta-\epsilon)$. By Axiom 1, $S_m(\{m,n\}, c'') \le S_m(\{m,n\}, \bar{c}) + (\Delta-\epsilon)$ and by Axiom 2, $S_m(\{m,n\}, c') \le S_m(\{m,n\}, c'')$.

Then, $S_m(\{m,n\}, c') \le S_m(\{m,n\}, \bar{c}) + (\Delta-\epsilon)$. Therefore, $S_m(\{m,n\}, c') \le S_m + \Delta - \epsilon$ and $S_m(\{m,n\}, c') \le y_m - \epsilon < y_m$, contradiction.

So agent n can collect the maximum amount she can collect in this game by proposing s. Therefore $S(I,c)$ is a subgame perfect Nash equilibrium, SPNE, outcome of the game. □

Corollary. *Axioms 1, 2 and downstream subtraction consistency are preserved in all convex combinations of solution concepts which satisfy them. CEC, slack maximizer, priority rule and their convex combinations give us a SPNE outcome of the noncooperative game for any given airport problem.*

Proposition 2. *For any airport problem (I,c), sequential full contributions solution concept gives $SFC(I,c)$, which is a SPNE outcome of the game $G(I,c,SFC)$.*

Proof. Let (I,c) be an airport problem. Assume $\exists i \in I\backslash\{n\}$ such that $c_i = c_n$. Now denote $k = \min_{i\in I\backslash\{n\}}\{i \ : c_i = c_n\}$. Let y be a proposal which is accepted by all responders. Then for each agent $i \in \{k,\ldots,n-1\}$, we have

$$y_i \le \frac{1}{2}\left(c_n - \sum_{j:k\le j\le n+1} y_j + y_i - \sum_{j:j<k} y_j\right).$$

This inequality implies

$$\sum_{j:k\leq j\leq n-1} y_j \leq \frac{1}{2}\left((n-k-2)\left(c_n - \sum_{j:k\leq j\leq n-1} y_j - \sum_{j:j<k} y_j\right) + \sum_{j:k\leq j\leq n-1} y_j\right),$$

$$\sum_{j:k\leq j\leq n-1} y_j \leq \frac{n-k-2}{n-k-1}\left(c_n - \sum_{j:j<k} y_j\right),$$

$$\sum_{j\in I\setminus\{n\}} y_j \leq \frac{n-k-2}{n-k-1}\left(c_n - \sum_{j:j<k} y_j\right) + \sum_{j:j<k} y_j \text{ and}$$

$$\sum_{j\in I\setminus\{n\}} y_j \leq \frac{n-k-2}{n-k-1}c_n + \frac{1}{n-k-1}\sum_{j:j<k} y_j.$$

Right side increases with $\sum_{j:j<k} y_j$. So, in order to collect the maximum amount, agent n should collect the maximum $\sum_{j:j<k} y_j$.

For any agent $i \in \{1,\ldots,k-1\}$, $c_i - \sum_{j:j<i} y_j \geq y_i$ Hence the maximum amount to be collected is $c_k - 1$. By proposing $SFC(I,c)$ agent n collects

$$\frac{n-k-2}{n-k-1}c_n + \frac{1}{n-k-1}c_{k-1}.$$

We have already shown that agent n cannot collect more than this amount. If $\nexists i \in I\setminus\{n\}$ s.t. $c_i = c_n$, then $c_i - \sum_{j:j<i} y_j \geq y_i, \forall i \in I\setminus\{n\}$.

So agent n cannot collect more than c_{n-1}. In both cases agent n collects the maximum amount by proposing $SFC(I,c)$. Therefore $SFC(I,c)$ is a SPNE outcome of the game $G(I,c,SFC)$.

□

Corollary. *The set of axioms in Theorem 1 is sufficient, but not all of them are necessary. Proposition 2 is an example, since SFC solution concept does not satisfy Axiom 2.*

Definition. For each airport problem (I, c) and each $i \in I$ where i is not the last agent with $I' \equiv I \backslash \{i\}$, if $x = S(I, c)$ implies $x'_I = S(d(I, i, c, x))$, we call the solution concept S, DS semiconsistent.

Clearly DS consistency implies DS semiconsistency.

So far, we only used DS consistency. But in all our results, agent n was not leaving the problem. If DS consistency in previous results is replaced by DS semiconsistency, they would still hold.

Definition. For each airport problem $(I, c) \in C \subset \mathcal{C}$ and each $i \in I$ where i is not the last agent with $I' \equiv I \backslash \{i\}$, if $x = S(I, c)$ implies $d(I, i, c, x) \in C$ and $x'_I = S(d(I, i, c, x))$, we call the solution concept S, DS semiconsistent for C.

3.4 Uniqueness of the Equilibria

Now let us check the uniqueness of the SPNE outcomes discussed above.

Example 1. Consider the airport problem $(I, c) = (\{1, 2, 3, 4\}, (2, 3, 6, 7))$

$CEC(I, c) = (\frac{3}{2}, \frac{3}{2}, 2, 2)$.

For two agents problems, under CEC first agent pays the minimum of her own cost and half of the total cost and second agent pays the rest.

If agent 4 offers $x = (1, 2, 2, 2)$ in the game above, in the first stage,

$$\begin{aligned} c'_4 &= 7 - 3 = 4, \\ c'_3 &= 6 - 3 = 3. \end{aligned}$$

Agent 3 pays $\frac{4}{2} = 2$. In the second stage,

$$\begin{aligned} c'_4 &= 7 - 3 = 4, \\ c'_2 &= 3 - 1 = 2. \end{aligned}$$

Agent 2 pays 2. In the third stage,

$$\begin{aligned} c'_4 &= 7 - 4 = 3, \\ c'_1 &= 2 - 1 = 1. \end{aligned}$$

Agent 1 pays 1. Hence this offer will be accepted. Since agent 4 collects as much as she does by offering $CEC(I, c)$, this allocation is also a best response for her. Therefore $(1, 2, 2, 2)$ is a SPNE outcome of the game although it is not the $CEC(I, c)$. So $CEC(I, c)$ is not the unique SPNE outcome of the game.

Example 2. Consider the airport problem $(I, c) = (\{1, 2, 3\}), (5, 10, 15)\}$ and the order $1 \prec 2 \prec 3$. $D^{\prec}(I, c) = (5, 5, 5)$. In the noncooperative game where conflicts are resolved according to this priority rule, let agent 3 propose $(3, 7, 5)$. In the first stage,

$$\begin{aligned} c'_3 &= 15 - 3 = 12, \\ c'_2 &= 10 - 3 = 7. \end{aligned}$$

So agent 2 pays 7. In the second stage,

$$\begin{aligned} c'_3 &= 15 - 7 = 8, \\ c'_1 &= 5 - 2 = 3. \end{aligned}$$

So agent 1 pays 3. $(3, 7, 5)$ is accepted by all agents and agent 3 collects as much as she does by offering $D^{\prec}(I, c)$. Therefore, $(3, 7, 5)$ is a SPNE outcome.

Hence $D^{\prec}(I,c)$ is not the unique SPNE outcome of the game.

Example 3. Consider the airport problem $(I,c) = (\{1,2,3\},(3,5,10))$. $SFC(I,c)$ $=(3,2,5)$.

If agent 3 offers $(1,4,5)$; in the first stage,

$$\begin{aligned} c'_3 &= 10-3=7, \\ c'_2 &= 5-1=4. \end{aligned}$$

So agent 2 pays 4. In the second stage,

$$\begin{aligned} c'_3 &= 10-2=8, \\ c'_2 &= 3-2=1. \end{aligned}$$

So agent 1 pays 1. $(1,4,5)$ is accepted and agent 3 collect 5 from the rest. She could collect 5 if she proposes SFC(I,c). Therefore, $(3,7,5)$ is a SPNE outcome. Hence SFC(I,c) is not the unique SPNE outcome of the game.

For two of the solution concepts we have discussed before, we already have a characterization of the SPNE's.

Theorem 2 (Arin et.al. 1997)**.** *Let (I,c) be an airport problem and $G(I,c,SM)$ its associated noncooperative game where every two-agent problem is solved by applying the slack maximizer solution. Then the unique SPNE outcome of $G(I,c,SM)$ is $SM(I,c)$, i.e. the nucleolus of (I,c).*

Given an airport problem (I,c), define $B(I,c)$ such that $B(I,c) = \{x \in Core(I,c) : x_i \leq x_n, \forall i \in I\}$.

Theorem 3 (Arin et. al. , 1997)**.** *Let (I,c) be an airport problem and $G(I,c,CEC)$ its associated noncooperative game where every two-agent problem is solved by applying the constrained equal contributions solution concept. Then, z is a SPNE outcome if and only if $z \in B(I,c)$ and $z_n = x_n$ where $x = CEC(I,c)$.*

3.5 Three-Agent Case

Proposition 3. *Let S be a solution concept which satisfies Axiom 1. If $S(I,c)$ is the unique SPNE outcome of the associated noncooperative game $G(I,c,S)$ for any airport problem (I,c) where $|I| \leq 3$, then S is DS semiconsistent for all airport problems (I,c) where $|I| \leq 3$.*

Proof. Now assume that S is not DS semiconsistent for all airport problems (I,c) where $|I| \leq 3$. Then $\exists$ an airport problem (I,c) and $i \in I$ such that i is not the last agent in I and $x = S(I,c)$ does not imply $S_{I'}(d(I,i,c,x)) = x_{I'}$ i.e. $\exists j \in I \backslash \{i\}$ s.t. $x_j \neq S_j(d(I,i,c,x))$.

i) Let $|I| = 2$. This is not possible due to the efficiency condition.

ii) Let $|I| = 3$. Namely $I = \{i,j,n\}$.

Case 1) $i < j < n$

$c'_n = c_n - x_i$

$c'_j = c_j - x_i$

$S_j(\{j,n\},(c'_n,c'_j)) = s_j \neq x_j$. Since x is a SPNE outcome, $s_j > x_j$.

$c'_n = c_n - x_j$,

$c'_i = c_i - (x_j - c_j + c_i)_+$.

If agent n proposes $(x_i, s_j, c_n - x_i - s_j)$, agent j agrees to pay s_j. In the second stage,

$\bar{c}_n = c_n - s_j$,

$\bar{c}_i = c_i - (s_j - c_j + c_i)_+$.

Denote $s_j - x_j = \Delta$ where $\Delta \in \mathbb{R}_+$. By Axiom 1, $S_i(\{i,n\},c') \leq S_i(\{i,n\},\bar{c}) + \Delta$ so $S(\{i,n\},\bar{c}) - S(\{i,n\},c') \leq \Delta$

If this inequality holds strictly, then x is not a SPNE. If they are equal, then x is not the unique SPNE, contradiction.

Case 2) $j < i < n$

This case is almost the same. $(s_j, x_i, c_n - x_i - s_j) \neq x$ and $(s_j, x_i, c_n - x_i - s_j)$

is a SPNE outcome, contradiction.

Therefore S must be DS semiconsistent for all airport problems (I, c) where $|I| \leq 3$.

□

Axiom 3. *Let $(\{i, j\}, c)$ be an airport problem, $i < j$, S be a solution concept. We assume that for any $(\{i, j\}, c'')$ and $(\{i, j\}, c') \in \mathcal{C}$ s.t. $c' = c + c''$, $c''_j > 0$, we have $S_i(\{i, j\}, c') < S_i(\{i, j\}, c) + c''_j$.*

Proposition 4. *Let S be a solution concept which satisfies Axiom 3. If $S(I, c)$ is a SPNE outcome of the associated noncooperative game $G(I, C, S)$ for any airport problem (I, c) where $|I| \leq 3$, then S is DS semiconsistent for all airport problems (I, c) where $|I| \leq 3$.*

Proof. This proof is very similar to the proof of Proposition 3. □

3.6 First Agent Proposer Game

Given an airport problem (I, c) and a solution concept S, we can change the associated noncooperative game described above and define a new game $H(I, C, S)$ where agent 1 makes the proposal.

By applying downstream-subtraction repeatedly, we can suggest that it is convenient to define two agent problems in the game as $(\{1, i\}, c')$ where

$c'_1 = \min_{l:l \neq 1 \neq i} \left\{ c_1, c_l - \sum_{j:2 \leq j \leq i-1} x_j - \sum_{j:i+1 \leq j \leq l} x_j \right\}$ and

$c'_i = \min_{l:i<l} \left\{ c_i - \sum_{j:2 \leq j \leq i-1} x_j, c_l - \sum_{j:2 \leq j \leq i-1} x_j - \sum_{j:i+1 \leq j \leq l-1} x_j \right\}$.

It can also be checked if our results still hold, if not which new assumptions should be made. The question is left here for further research.

CHAPTER 4

ANOTHER NONCOOPERATIVE GAME

4.1 New Game

An airport problem (I, c) and a solution concept S is given. We define a noncooperative game $\Gamma(I, c, s)$ which is very similar to the ones above. Agent n makes a proposal x^1 starting with agent $n-1$ agents respond in an order according to their costs.

If agent i rejects at stage t, then a two-person problem is defined as $(\{i, n\}, c')$ where

$$\begin{aligned} c'_n &= c_n - \sum_{j \neq i,n} x_j^{t-1}, \\ c'_i &= \left(c_i = \sum_{j \neq i,n} x_j^{t-1} \right)_+ . \end{aligned}$$

Note that $c'_i \leq c'_n$.

Then S is applied to this small problem, agent i pays and leaves, the proposal is adjusted accordingly.

4.2 Uniform-Subtraction

Given an airport problem (I, c), an allocation $x \in \mathcal{X}(I, c)$ and $i \in I$, the uniform - subtraction[2] reduced problem $r(I, i, c, x)$ is an airport problem with the agent set $I' \equiv I \setminus \{i\}$ and cost vector $c' \in \mathbb{R}_+^{I'}$ where c' is defined as follows: For any $j \in I'$

$$c'_j \equiv \begin{cases} \max\{c_j - x_i, 0\} & \text{if } c_i < c_j \\ c_j - x_i & \text{if } c_i \geq c_j \end{cases}$$

Let (I, c) be an airport problem. For each $i \in I$, if agent i is not the unique agent such that $\max_{j \in I} c_j = c_i$, if $x \equiv S(I, c)$ implies $x_{I'} = S(r(I, i, c, x))$, then we call the solution concept S uniform-subtraction consistent.

4.3 Results

Claim 4. *Let (I, c) be an airport problem and let $\bar{I} \subset I$. Now agents in $I \backslash \bar{I}$ leave the game in an order $\prec$ and we apply uniform-subtraction repeatedly. We obtain a new problem $(\bar{I}, c')$.*

If agents in $I \backslash \bar{I}$, instead leave in an order $\bar{\prec}$ and we apply uniform-subtraction repeatedly. We obtain a new problem $(\bar{I}, \bar{C})$.

Then $c' = \bar{c}$. In other words, order of leave is not important in uniform-subtraction.

Proof. Straightforward □

Claim 5. *Let (I, c) be an airport problem, x be an allocation for this problem. For any $i \in I \backslash \{n\}$, define c' as*

$$\begin{aligned} c_{n'} &= c_n - \sum_{j \neq i,n} x_j, \\ c_{i'} &= (c_i - \sum_{j \neq i,n} x_j)_+. \end{aligned}$$

[2]This concept is first defined by Potters and Sudhölter (1999).

Now the problem $(\{i,n\},c')$ is obtained from (I,c) by repeated Uniform-Subtraction of all agents other than i and n w.r.t. x.

Proof. Straightforward. □

Proposition 5. *Let (I,c) be an airport problem, $\Gamma(I,c,s)$ be an associated non-cooperative game, where S satisfies Axiom 1. Let $y \in \mathcal{X}(I,c)$ be offered by agent n in the beginning of the game and $x \in \mathcal{X}(I,c)$ be the outcome, when all agents respond rationally. Then accepting is a best response for all responders $i \in I\backslash\{n\}$ when x is proposed directly in the first stage by agent n.*

Proof. Very similar to the proof of Proposition 1. □

Corollary. *Let (I,c) be an airport problem, $\Gamma(I,c,s)$ be an associated noncooperative game, where S satisfies Axiom 1. If x is a SPNE outcome of $\Gamma(I,c,S)$, then there exists a SPNE where x is proposed by n and accepted by all agents.*

Theorem 4. *Let (I,c) be an airport problem, $\Gamma(I,c,s)$ be an associated noncooperative game, where S satisfies Axiom 1. Now $S(I,c)$ is a SPNE outcome of the game $\Gamma(I,c,S)$.*

Proof. Denote $s \equiv S(I,c)$.

Let z be a proposal, where the outcome of the game is y, when responders act rationally. Then by Proposition 5, y will be accepted if it is proposed in the beginning.

Let $y \neq s$ and $y_n < s_n$.

So $\sum_{j:j\neq n} s_j < \sum_{j:j\neq n} y_j$. Therefore, there must be an agent $k \in I\backslash\{n\}$ such that $y_k > s_k$.

When y is proposed agent k pays $min\{y_k, s_k(\{k,n\},c')\} = y_k$ where

$$c'_n = c_n - \sum_{j\neq k,n} y_j,$$

$$c'_k = \left(c_k - \sum_{j\neq k,n} y_j\right)_+.$$

When s is proposed agent k pays $min\{s_k, s_k(\{k,n\},\bar{c})\} = s_k = s_k(\{k,n\},\bar{c})$ where

$$\bar{c_n} = c_n - \sum_{j\neq k,n} s_j,$$

$$c'_k = \left(c_k - \sum_{j\neq k,n} s_j\right)_+ .$$

If $\sum_{j:j\neq n,k} s_j \leq \sum_{j:j\neq n,k} y_j$, then by Axiom 1 $y_k \leq s_k(\{k,n\},c') \leq s_k(\{k,n\},\bar{c}) = s_k$, contradiction.

If $\sum_{j:j\neq n,k} s_j > \sum_{j:j\neq n,k} y_j$, then denote $\Delta \equiv \sum_{j:j\neq n,k} s_j - \sum_{j:j\neq n,k} y_j$. By Axiom 1, $s_k(\{k,n\},c') \leq S_k(\{k,n\},\bar{c}) + \Delta$. Therefore $y_k \leq s_k + \Delta$.

$$\sum_{j:j\neq n} y_j = \sum_{j:j\neq n,k} s_j - \Delta + y_k \leq \sum_{j:j\neq n,k} s_j - \Delta + s_k + \Delta = \sum_{j:j\neq n} s_j.$$

But $\sum_{j:j\neq n} y_j > \sum_{j:j\neq n} s_j$, contradiction.

Therefore $S(I,c)$ is a SPNE outcome of the game $\Gamma(I,c,S)$.

□

Theorem 5. *Let (I,c) be an airport problem, $\Gamma(I,c,S)$ be an associated noncooperative game, where S satisfies uniform-subtraction consistency, Axiom 1 and Axiom 3. Now $S(I,c)$ is the unique SPNE outcome of the game $\Gamma(I,c,S)$.*

Proof. Similar to the proof of Theorem 4.

Among the seven solution concepts described above, only CEB satisfies uniform-subtraction consistency (Thomson (2005)).

Now let us check if it satisfies Axiom 3.

For the two agent case CEB gives us $\left(\frac{c_1}{2}, c_2 - \frac{c_1}{2}\right)$.

$CEB(\{1,2\},c') = \left(\frac{c_1+c''_1}{2}, c_2 + c''_2 - \frac{c_1}{2} - \frac{c''_1}{2}\right)$ where $c' = c + c''$, (I,c), (I,c'), $(I,c'') \in \mathcal{C}$.

$c''_1 \geq 0$ $c''_2 > 0$ and $c''_2 \geq c''_1$

$\frac{c_1+c''_1}{2} < \frac{c_1}{2} + c''_2 = \frac{c_1+2c''_2}{2}$. Therefore CEB satisfies Axiom 3.

□

Corollary. *Let* (I, c) *be an airport problem and* $\Gamma(I, c, CEB)$ *its associated non-cooperative game where every two-agent problem is solved by applying the constrained equal benefits solution. Then the unique SPNE outcome of* $\Gamma(I, c, CEB)$ *is* $CEB(I, c)$.

It is an interesting observation that CEB and Slack Maximizer solutions gives us the same allocation in two-agent problems.

Due to the definition of Uniform-Subtraction consistency, we can only consider last agent proposer games if we want to utilize this concept.

CHAPTER 5

CONCLUSION

In this thesis, we combined the cooperative and the noncooperative approach to airport problems. We defined noncooperative games in which conflicts are resolved using a cooperative solution concept. We investigated how the cooperative solution and the set of Nash equilibria are related. Instead of characterizing the Nash equilibria of the game associated to a specific solution concept selected from the literature, we obtained general results.

We showed that if a solution concept satisfies downstream-subtraction consistency, weaker versions of individual cost monotonicity and weak cost monotonicity, then cooperative solutions are in the set of the Nash equilibrium outcomes of the associated game.

Additionally, we modified the game. For this modified game, we proved that if a solution concept satisfies a weaker version of weak cost monotonicity, then cooperative solutions are in the set of the Nash equilibrium outcomes of the associated game. We also showed that if a solution concept satisfies uniform-subtraction consistency, a weak version of weak cost monotonicity and a related version of cost monotonicity, then the cooperative solution is the unique Nash equilibrium outcome of the associated game.

We concluded that noncooperative games can be used as an implementation tool for cooperative solutions in airport problems.

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