# ASYMPTOTICALLY OPTIMAL ASSIGNMENTS IN ORDINAL EVALUATIONS OF PROPOSALS

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By

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August, 2009

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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# ABSTRACT

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In ordinal evaluations of proposals in peer review systems, a set of proposals is assigned to a fixed set of referees so as to maximize the number of pairwise comparisons of proposals under certain referee capacity and proposal subject constraints. The following two related problems are considered: (1) Assuming that each referee has a capacity to review $k$ out of $n$ proposals, $2 \leq k \leq n$, determine the minimum number of referees needed to ensure that each pair of proposals is reviewed by at least one referee, (2) Find an assignment that meets the lower bound determined in (1). It is easy to see that one referee is both necessary and sufficient when $k = n$, and $n(n\text{-}1)/2$ referees are both necessary and sufficient when $k = 2$. It is shown that 6 referees are both necessary and sufficient when $k = n/2$. Furthermore it is shown that 11 referees are necessary and 12 are sufficient when k = $n/3$, and 18 referees are necessary and 20 referees are sufficient when $k = n/4$. A more general lower bound of $n(n\text{-}1)/k(k\text{-}1)$ referees is also given for any $k$, $2 \leq k \leq n$, and an assignment asymptotically matching this lower bound within a factor of 2 is presented. These results are not only theoretically interesting but they also provide practical methods for efficient assignments of proposals to referees.

*Keywords:* Asymptotically optimal assignment, panel assignment problem, peer review, proposal evaluation.

# ÖZET

## ORDİNAL SIRALAMA YÖNTEMİYLE YAPILAN PROJE ÖNERİSİ DEĞERLENDİRMELERİNDE ASİMTOTİK OLARAK OPTİMUM ATAMA YÖNTEMLERİ

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Hakem değerlendirmelerinin ordinal sıralama yöntemiyle yapılanlarında, belirli bir grup öneri, maksimum sayıda ikili kıyaslama elde etmek için sabit sayıda hakeme, belirli hakem kapasitesi ve öneri konu kısıtları altında atanır. Aşağıda belirtilen iki ilgili problem üzerinde çalışıldı: (1) Her bir hakemin $n$ tane öneri içinden en fazla $k$ tane okuyabilmesi varsayımı altında, öyle ki $2 \leq k \leq n$, tüm öneri çiftlerinin en az bir hakem tarafından okunmasını garanti edebilmek için en az sayıda gereken hakem sayısının hesaplanması, (2) (1)'de hesaplanan en az sayıda hakem sayısına denk gelen atama yapısının bulunması. $k = n$ durumunda 1 hakemin hem gerekli hem de yeterli olduğu ve $k = 2$ durumunda $n(n\text{-}1)/2$ hakemin hem gerekli hem de yeterli olduğu kolaylıkla görülmektedir. $k = n/2$ durumunda 6 tane hakemin hem gerekli hem de yeterli olduğu gösterilmiştir. Ayrıca $k = n/3$ durumunda 11 hakemin gerekli ve 12 hakemin yeterli olduğu ve $k = n/4$ durumunda 18 hakemin gerekli ve 20 hakemin yeterli olduğu gösterilmiştir. Herhangi bir $k$, $2 \leq k \leq n$ değeri için daha genel bir alt sınır hakem sayısı $n(n\text{-}1)/k(k\text{-}1)$ olarak sunulmuştur ve bu alt sınıra en fazla iki katına kadar asimptotik olarak denk gelen bir atama yapısı sunulmuştur. Bu sonuçlar sadece teorik olarak ilgi çekici olmakla kalmayıp, önerilerin hakemlere etkin bir şekilde atanmaları için pratik yöntemler de sağlamaktadır.

*Anahtar sözcükler*: Asimtotik optimum atama yöntemleri, panel oluşturma problemleri, hakem değerlendirmeleri, öneri değerlendirmesi.

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# Chapter 1

# Introduction

A sizeable portion of current academic research is funded through various agencies with specific interests in different areas of research. Each year funding organizations such as the Scientific and Technological Research Council of Turkey (TÜBİTAK), Turkish State Planning Organization (DPT), European Science Foundation (ESF), National Science Foundation (NSF) spend billions of dollars to support research projects. Table 1.1 shows the funding figures in billions of US Dollars for research projects in various countries, including the funding amounts expended by European Union (EU) and Organisation for Economic Co-operation and Development (OECD). Table 1.2 lists the number of projects submitted, number of projects supported and academic R&D expenditures for operative projects in a given year for TÜBİTAK, one of the major funding agencies for academic and industrial research. As we can see from Table 1.2, TÜBİTAK has increased its funding for research more than 20 times from 2000 to 2007. During the same period, the number of proposals has increased by a factor of 5.82 and the number of funded projects has increased only by a factor of 3.81. Since the ratio of the supported projects to submitted projects has been steadily declining, identifying high quality projects to support becomes very important for funding agencies in order to have the highest return for the funds invested.

| | **2000** | **2001** | **2002** | **2003** | **2004** | **2005** | **2006** | **2007** | **2008** |
|---|---|---|---|---|---|---|---|---|---|
| **OECD total** | 607.8 | 643.2 | 658.4 | 682.4 | 714.7 | 769.7 | 829.4 | * | * |
| **United States** | 268.5 | 278 | 277.1 | 290.1 | 301.2 | 323.9 | 348.8 | 368.2 | * |
| **EU27 total** | 183.9 | 196.3 | 205.9 | 210.4 | 217.8 | 229.8 | 247.9 | * | * |
| **Japan** | 98.6 | 103.9 | 108.3 | 112.3 | 117.6 | 128.5 | 138.7 | * | * |
| **China** | * | * | * | * | * | 70.9 | * | 105.1 | * |
| **Germany** | 52.1 | 54.4 | 56.6 | 59.4 | 61.4 | 64.1 | 68.6 | 71.5 | * |
| **France** | * | 35.8 | 38.1 | 36.9 | * | 39.2 | 41.2 | 43.2 | * |
| **United Kingdom** | 28.3 | 29.6 | 31.1 | 31.6 | 32.4 | 34.6 | 36.8 | * | * |
| **Canada** | 16.6 | 19 | 19.1 | 20 | 21.5 | 22.7 | 23.3 | 23.9 | * |
| **Russian Federation** | * | * | * | * | * | 18.1 | * | 23.3 | * |
| **Italy** | 15.2 | 16.8 | 17.3 | 17.3 | 17.5 | 17.9 | 19.7 | * | * |
| **Spain** | 7.8 | 8.3 | 9.8 | 10.9 | 11.7 | 13.3 | 15.6 | * | * |
| **Australia** | 7.9 | * | 9.8 | * | 11.6 | * | 14.9 | * | * |
| **Sweden** | * | 10.3 | * | 10.3 | 10.4 | * | 11.7 | 12.1 | * |
| **Netherlands** | 8.5 | 8.8 | 8.8 | 9 | 9.6 | 9.9 | 10.4 | 11.1 | * |
| **Turkey** | 2.8 | 3 | 3 | 2.8 | 3.5 | 4.6 | 5.1 | * | * |
| **Denmark** | * | 3.7 | 4.1 | 4.2 | 4.3 | 4.4 | 4.6 | 4.9 | * |
| **Norway** | * | 2.6 | 2.7 | 2.9 | 3 | 3.3 | 3.6 | * | * |
| Billion US dollars, current prices and PPPs(Purchasing Power Parity)<br>Source: http://stats.oecd.org (* Data is not available.) | | | | | | | | | |

Table 1.1: Gross domestic expenditure on R&D.

| | **2000** | **2001** | **2002** | **2003** | **2004** | **2005** | **2006** | **2007** |
|---|---|---|---|---|---|---|---|---|
| **Submitted Projects** | 896 | 1069 | 1117 | 831 | 1688 | 4007 | 3933 | 4731 |
| **Supported Projects** | 328 | 407 | 517 | 329 | 472 | 1453 | 1300 | 1250 |
| **Expenditure for Operative Projects (TL in millions)** | 7.225 | 7.842 | 9.342 | 5.836 | 9.518 | 71.22 | 148.014 | 162.032 |
| Source: http://www.tubitak.gov.tr/home.do?ot=1&sid=357 | | | | | | | | |

Table 1.2: Academic Projects and Expenditure for Operative Projects.

## 1.1 Identification of High Quality Proposals

Challenges confronted by funding agencies in identifying high quality proposals are well-documented in the literature, see, for example, [4], [1], [14], [19], [11], [6]. In order to rank proposals, the funding process typically starts with a call for proposals (CFP) to the relevant community. Proposals are then submitted according to guidelines stated in the CFP. These proposals are sent out for a peer review that serves as the core of the entire process [7].

Broadly speaking, the two most challenging problems in peer review are: the assignment of proposals to referees and ranking and selection of proposals. A common practice for the first problem and used by program directors who work in funding agencies is to first partition a set of proposals into a number of groups of proposals according to their subjects areas and then find a sufficient number of experts to assemble a panel for each group of proposals. For the second problem, the ranking and selection of proposals typically rely on cardinal (quantitative) or ordinal (preference)-based comparisons [17], [8], [10]. Recently, Cook et al. [7] demonstrated that cardinal comparisons such as using average scores of proposals can be unreliable especially when referees' scores are not normalized. They suggested that quantifying the intrinsic values of proposals may be difficult, and therefore it is more practical to rely on ordinal rankings. Also, some researchers, especially those working in social sciences and decision-analysis areas, have recently started to ask questions on the reliability of the peer review systems that rely on cardinal ranking. Several studies raised questions about the validity and possible existence of various biases in such peer review processes (e.g., Cicchetti [6], Hodgson [12], Campanario [21], Jayasinghe [22]). They found low degrees of agreement among referees and various kinds of biases. Other studies (e.g., [20]) focused on the criteria that guide the referees' work and reported on a common language—a certain set of criteria (e.g., intellectual merit, broader impact, feasibility, etc) that referees tend to use to evaluate research quality. However, as emphasized by Langfeldt [15], these criteria are often interpreted differently by various referees. Techniques that generate ordinal rankings on the basis of pairwise evaluations may provide some solutions to these difficulties because they are more straightforward and require less effort from the reviewers. As noted in Cook [7], comparing proposals is easier than evaluating their intrinsic merits. Cook also pointed out that cardinal rankings may lead to incorrect conclusions because of the uncalibrated scoring practices of referees. For example consider three reviewers $A$, $B$, $C$ and three proposals $X$, $Y$, $Z$, where the scores of the referees are indicated in Table 1.3. Taking the averages of the reviewers' scores clearly suggests that the proposals must be ranked as $Y > Z > X$, where $>$ means "is better than". On the other hand, if we inspect the scores a little more carefully, the ordinal ranking gives a consistent ranking. That is, $X > Y$ (By referee $A$) and $Y > Z$ (By referee $B$) and $X > Z$ (By referee $C$) imply that

$X > Y > Z$. Obviously the calibration of reviewers' scores is needed and ordinal ranking ensures this by the transitivity of ranking assuming that all reviewers are equally reliable regardless of their scoring habits.

| **Referee/Proposal** | X | Y | Z | Characterization |
|---|---|---|---|---|
| **A** | 6 | 4 | | Average |
| **B** | | 10 | 9 | Generous |
| **C** | 4 | | 3 | Frugal |
| **Average** | 5 | 7 | 6 | |

Table 1.3: Cardinal evaluation of a set of three proposals $\{X,Y,Z\}$ by a set of three referees $\{A,B,C\}$.

Ordinal and cardinal strengths of preferences have also been advocated in [16] as natural extensions of ordinal comparison models. Also Cook [7] introduced a set covering integer programming approach to obtain as many comparisons as possible between the proposals reviewed by a fixed set of referees using ordinal ranking strategy. In [9] a branch-and-bound algorithm was introduced to minimize the number of disagreements among referees based on pairwise comparisons of proposals. More recently, Chen [5] presented a maximum consensus algorithm based on complete rankings of a set of proposals by a set of referees, and Iyer [13], Ahn [3], Sarabando [18] presented dominance-based ordinal ranking and selection algorithms.

The general consensus among these studies is that ordinal ranking is more reliable than cardinal ranking and its variations. Therefore, we will use ordinal ranking in the remainder of the thesis.

## 1.2 Optimal Assignments

This thesis is concerned with the assignment aspect of ordinal evaluations of proposals. In an ordinal ranking, limited coupling of proposals to referees divides proposals into disjoint clusters and makes it impossible to compare proposals between clusters. For example, suppose that 6 proposals are to be assigned to 3 referees under the following incidence (matching) relation with the constraint that no referee can be assigned more than 3 proposals:

$$\text{Referee 1 can review Proposals } 1, 2, 3, 4$$
$$\text{Referee 2 can review Proposals } 2, 3, 4, 5$$
$$\text{Referee 3 can review Proposals } 1, 4, 5, 6$$

It is obvious that it is impossible to cover all $6 \times 5/2 = 15$ possible pairs of proposals under the capacity constraint of 3 proposals per referee. It is still desirable to determine which three proposals should be assigned to each referee so that the number of pairs of proposals covered between the three referees is maximized. In this example, assigning proposals 1,2,3 to referee 1, proposals 3,4,5 to referee 2, and proposals 1,5,6 to referee 3 gives a maximum of 9 pairs of proposals.

As the example illustrates, the underlying assumption of the approach described in [7] is that both proposals and referees are fixed a priori together with an incidence relation to specify which proposals can potentially be assigned to which referees. In contrast, we consider assignment problems in this work with only two parameters of interest: (1) the number of proposals, $n$, and (2) the capacity of each referee, $k$, $2 \leq k \leq n$, i.e., the maximum number of proposals that can be reviewed by each referee. With these two parameters, we consider two related problems: (1) determine the minimum number of referees to ensure that each pair of proposals is reviewed by at least one referee, (2) find an assignment of a set of $n$ proposals to the minimum number of referees determined so that all pairs of proposals are covered. Our interest in these problems is motivated by the fact that referees are generally selected to meet the evaluation needs of a set of proposals rather than randomly assembled together. Thus, unlike in the assignment problems considered in [7] and [9], minimizing the number of referees is the main objective in the assignments of proposals to referees in our work. We consider the assignments of proposals to referees both with and without referee specialties. In the first case, referees may be viewed interchangeable in terms of their expertise. This assumption generally holds for those proposal evaluation processes in which a small set of proposals with identical topics is considered, or for those in which a large set of proposals is pre-screened to identify a small set of proposals for a second stage of a more intense peer review. In the second case, referees with specialties are allowed. This applies to peer review panels in which experts with a multitude of evaluation (research) specialties compare proposals with a multitude of subjects.

## 1.3 Contributions and Outline of the Thesis

This thesis makes the following contributions:

(1) It derives a generic lower bound of $n(n-1)/k(k-1)$ on the number of referees to cover all pairs of $n$ proposals for any referee capacity, $k$.

(2) Lower bounds on the number of referees needed to cover all pairs of proposals for referee capasities of $n/2$, $n/3$ and $n/4$.

(3) It presents an optimal assignment of proposals to referees where the referee capacity is $n/2$ and asymptotically optimal assignment for referee capacities of $n/3$ and $n/4$.

(4) It extends the assignments in (3) to arbitrary capacity case with two asymptotically optimal designs. More specifically, the referee complexity of one of those designs tends to $n(2n-k)/k^2$. The second design makes use of balanced incomplete block designs and uses $\frac{n}{k}\left(\frac{n}{k}+1\right)$ referees with a capacity of $k$ to cover all pairs of $n$ proposals.

The rest of the thesis is organized as follows. In Chapter 2, we present the mathematical facts that will be used in the thesis. In Chapter 3, we derive our lower bounds. In Chapter 4, assignments that match these lower bounds are presented. These results are extended to distinguishable referees in Chapter 5 and we conclude the thesis in Chapter 6.

# Chapter 2

# Combinatorial Designs

In this chapter, we present a set of mathematical facts that are related to block designs and Latin squares, and pertinent to establishing the main results of the thesis in Chapters 3 and 4.

## 2.1 Block Designs

A block design is a pair ($P$,$B$) where $P = \{p_1, p_2, ..., p_v\}$ is a set of $v$ elements called points and $B = \{B_1, B_2, ..., B_b\}$ is a collection of $b$ subsets of $P$, called blocks. We will denote the blocks in block designs using a parenthesis notation. For example, if $P = \{1, 2, 3, 4, 5, 6\}$, then {(1,3,5), (2,4,5,6)} is a block design with the two blocks (1,3,5) and (2,4,5,6).

A block design is called balanced if all blocks are of equal size and the pairs of points occur in all of the blocks an equal number of times. It is called incomplete if the number of elements in every block is less than $v$. Let $\lambda, k$, and $r$ be positive integers, where $v > k \geq 2$. A block design is called a $(v, b, r, k, \lambda)$-balanced and incomplete block design (BIBD) if (1) all blocks have $k$ points, (2) each point appears in exactly $r$ blocks and (3) each pair of points appears in exactly $\lambda$ blocks. It is easy to see that $vr = bk$ since each of the $v$ points in $P$ appears $r$ times in all the blocks and the union of the blocks as a multiset contains exactly $bk$ elements. It can further be shown that $\lambda(v-1) = r(k-1)$. Solving for $b$ and

$r$ in terms of $\lambda$ and $k$, we have

$$\begin{aligned} b &= \frac{v(v-1)\lambda}{k(k-1)} \\ r &= \frac{(v-1)\lambda}{(k-1)} \end{aligned} \tag{2.1}$$

and thus a $(v, b, r, k, \lambda)$-BIBD design is often referred to as a $(v, k, \lambda)$-BIBD design as will be done in this thesis.

**Proposition 1** *Let P = {1,2,3,4,5,6,7,8,9}. The following is a (9,3,1)-BIBD design: {(1, 2, 3), (4, 5, 6), (7, 8, 9), (1, 4, 7), (1, 5, 8), (1, 6, 9), (2, 4, 9), (2, 5, 7), (2, 6, 8), (3, 4, 8), (3, 5, 9), (3, 6, 7)}.*

**Proof** The proof immediately follows from a direct inspection of the blocks. ∎

**Remark 1** *By Equation 2.1, b = (9*8)/(3*2) = 12 and r = 8/2 = 4 and hence we have 12 blocks of 4 points each as the design gives.* ∎

**Proposition 2** *Let P = {1,2,3,4,5,6,7,8,9,10,11,12,13,14,16}. The following is a (16,4,1)-design with 20 blocks each having 4 points :*

$$\begin{aligned} \{&(1,2,3,4),(5,6,7,8),(9,10,11,12),(13,14,15,16), \\ &(1,5,9,13),(1,6,11,16),(1,7,12,14),(1,8,10,15), \\ &(2,6,10,14),(2,5,12,15),(2,8,11,13),(2,7,9,16), \\ &(3,7,11,15),(3,5,10,16),(1,6,12,13),(3,8,9,14), \\ &(4,8,12,16),(4,5,11,14),(4,7,10,13),(4,6,9,15)\} \end{aligned}$$

**Proof** Again, the proof immediately follows from a direct inspection of the blocks. ∎

## 2.2 Orthogonal Latin Squares

The following facts about orthogonal Latin squares will also be useful in the sequel [23].

A Latin square of order $n$ is an $n \times n$ matrix in which each row and each column is a permutation of a set of $n$ symbols. Two Latin squares $L_1$ and $L_2$ are said to be orthogonal if the matrix obtained by superposing $L_1$ and $L_2$ entry by entry contains each of the possible $n^2$ ordered pairs exactly once. For example, the Latin squares in Figure 2.1 are orthogonal. A set of Latin squares $\{L_1, L_2, ..., L_t\}$ are said to be mutually orthogonal if each pair of Latin squares is orthogonal, i.e., $L_i$ and $L_j$ are orthogonal whenever $i \neq j$. We call such a set of Latin squares mutually orthogonal Latin squares or MOLS. It is not too difficult to show that the maximum number of Latin squares of order $n$ that can be mutually orthogonal to one another cannot exceed $n$-1. Such a set of mutually orthogonal Latin squares is said to be a complete MOLS.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} 4 & 5 & 6 \\ 6 & 4 & 5 \\ 5 & 6 & 4 \end{pmatrix} = \begin{pmatrix} (1,4) & (2,5) & (3,6) \\ (2,6) & (3,4) & (1,5) \\ (3,5) & (1,6) & (2,4) \end{pmatrix}$$

Figure 2.1: A 3×3 Latin square example with symbols 1, 2, 3 and 4, 5, 6 with their superposed matrix.

There are many of methods for constructing a complete set of $n$-1 MOLS of order $n$. The method described in the following theorem is taken from [23].

**Theorem 1** *If $n$ is a prime power, a complete set of $n-1$ MOLS can be found by taking the nonzero elements $a \in F_n$ (the finite field of order $n$) and setting the entry in the $x^{th}$ row and $y^{th}$ column in the $a^{th}$ Latin square to $f_a(x,y) = ax + y$ (mod $n$).*

**Proof** The proof basically follows from the fact that every nonzero element $a$ in a finite field is invertible. ∎

**Example 1** *For $n = 3$, we compute the Latin squares as follows*[1]*:*

$$\begin{array}{lll} f_1(1,1)=2 & f_1(1,2)=3 & f_1(1,3)=1 \\ f_1(2,1)=3 & f_1(2,2)=1 & f_1(2,3)=2 \\ f_1(3,1)=1 & f_1(3,2)=2 & f_1(3,3)=3 \end{array} \Rightarrow \quad L_1 = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\begin{array}{lll} f_2(1,1)=3 & f_2(1,2)=1 & f_2(1,3)=2 \\ f_2(2,1)=2 & f_2(2,2)=3 & f_2(2,3)=1 \\ f_2(3,1)=1 & f_2(3,2)=2 & f_2(3,3)=3 \end{array} \Rightarrow \quad L_2 = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

The following results will be useful in the sequel.

**Corollary 1** *The intersection of any two columns in any given row or the intersection of any two rows in any given column in the $n \times n$ matrix which is obtained by superposing a complete set of MOLS is empty.*

**Proof** The entries in any column or any row of an $n \times n$ Latin square is a permutation of the $n$ symbols it uses. Therefore the intersections of the juxtapositions of the symbols across the rows or columns of the Latin squares in the MOLS must be empty. ∎

**Corollary 2** *For any prime power $n$, there exists a $(n^2, n, 1)$-BIBD.*

**Proof** By Theorem 1, we can construct a complete set of $n-1$ MOLS using $n-1$ Latin squares of order $n$, $L_i, 1 \leq i \leq n-1$. Let $U_i = \{(i-1)n+1, (i-1)n+2, ..., (i-1)n+n\}$ denote the set of symbols used in $L_i, 1 \leq i \leq n-1$, and let $M = L_1 \times L_2 \times ... \times L_{n-1}$ denote the $n \times n$ matrix obtained by the superposition of $n-1$ Latin squares. Let $U_n = \{(n-1)n+1, (n-1)n+2, ..., (n-1)n+n\}$. Clearly $U_n \cap U_i = \emptyset$ for $i = 1, 2, ..., n-1$ and suppose that the matrix $M$ is modified by concatenating $(n-1)n+i$ to all its columns in row $i$, $1 \leq i \leq n$. Denote this new matrix by $M_a$. By Corollary 1, the intersection of any two rows in any given column must be empty. Furthermore, the intersection of any two columns

[1] *Note that $0 \equiv 3$ (mod 3)*

in any given row cannot have more than one symbol in common. It follows that the possible pairs of symbols which can be formed from the entries in each row and each column of matrix $M_a$ must all be different.

Let the entries of matrix $M_a$ be the blocks of a block design. By the construction of $M$, each symbol in the $U_1 \cup U_2 \cup \ldots \cup U_{n-1}$ must appear exactly $n$ times among these blocks. Since each symbol in $U_n$ is inserted into the columns of a distinct row, the symbols in $U_n$ must also appear exactly $n$ times among the blocks.

Now to complete this to a $(n^2, n, 1)$-BIBD, it suffices to add $U_i$, $1 \leq i \leq n$ as blocks to this block design and note that (a) that this block design consists of $n^2 + n$ blocks each of which comprising $n$ symbols, (b) each symbol appears exactly in $n + 1$ blocks, (c) each pair of symbols appears in exactly one block. $\blacksquare$

**Remark 2** *We note that the equation* $n^2\binom{n}{2} + n\binom{n}{2} = \binom{n^2}{2}$ *represents the fact that the set of all pairs of* $n^2$ *symbols is obtained by the union of set of all pairs all symbols in all the blocks of* $M_a$ *plus the set of all pairs of symbols generated by* $U_i$, $1 \leq i \leq n$.

*We further note that*

$$\begin{aligned} b &= \frac{v(v-1)\lambda}{k(k-1)} = \frac{n^2(n^2-1)\lambda}{n(n-1)} = n(n+1) \\ r &= \frac{(v-1)\lambda}{(k-1)} = \frac{(n^2-1)\lambda}{(n-1)} = n+1 \end{aligned} \qquad \blacksquare \tag{2.2}$$

**Example 2** *Let* $n = 5$. *The following four Latin squares form a complete set of MOLS of order 5.*

$$L_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 2 & 5 & 3 \\ 5 & 4 & 1 & 3 & 2 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 6 & 7 & 8 & 9 & 10 \\ 8 & 10 & 9 & 7 & 6 \\ 9 & 6 & 7 & 10 & 8 \\ 10 & 9 & 6 & 8 & 7 \\ 7 & 8 & 10 & 6 & 9 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 11 & 12 & 13 & 14 & 15 \\ 14 & 11 & 12 & 15 & 13 \\ 15 & 14 & 11 & 13 & 12 \\ 12 & 13 & 15 & 11 & 14 \\ 13 & 15 & 14 & 12 & 11 \end{pmatrix} \qquad L_4 = \begin{pmatrix} 16 & 17 & 18 & 19 & 20 \\ 20 & 19 & 16 & 18 & 17 \\ 17 & 18 & 20 & 16 & 19 \\ 18 & 20 & 19 & 17 & 16 \\ 19 & 16 & 17 & 20 & 18 \end{pmatrix}$$

*The matrices $M$ and $M_a$ are constructed as follows:*

$$M = \begin{pmatrix} (1,6,11,16) & (2,7,12,17) & (3,8,13,18) & (4,9,14,19) & (5,10,15,20) \\ (2,8,14,20) & (3,10,11,19) & (5,9,12,16) & (1,7,15,18) & (4,6,13,17) \\ (3,9,15,17) & (5,6,14,18) & (4,7,11,20) & (2,10,13,16) & (1,8,12,19) \\ (4,10,12,18) & (1,9,13,20) & (2,6,15,19) & (5,8,11,17) & (3,7,14,16) \\ (5,7,13,19) & (4,8,15,16) & (1,10,14,17) & (3,6,12,20) & (2,9,11,18) \end{pmatrix}$$

$$M_a = \begin{pmatrix} (1,6,11,16,\underline{21}) & (2,7,12,17,\underline{21}) & (3,8,13,18,\underline{21}) & (4,9,14,19,\underline{21}) & (5,10,15,20,\underline{21}) \\ (2,8,14,20,\underline{22}) & (3,10,11,19,\underline{22}) & (5,9,12,16,\underline{22}) & (1,7,15,18,\underline{22}) & (4,6,13,17,\underline{22}) \\ (3,9,15,17,\underline{23}) & (5,6,14,18,\underline{23}) & (4,7,11,20,\underline{23}) & (2,10,13,16,\underline{23}) & (1,8,12,19,\underline{23}) \\ (4,10,12,18,\underline{24}) & (1,9,13,20,\underline{24}) & (2,6,15,19,\underline{24}) & (5,8,11,17,\underline{24}) & (3,7,14,16,\underline{24}) \\ (5,7,13,19,\underline{25}) & (4,8,15,16,\underline{25}) & (1,10,14,17,\underline{25}) & (3,6,12,20,\underline{25}) & (2,9,11,18,\underline{25}) \end{pmatrix}$$

*We obtain the blocks of the (25,5,1)-BIBD by combining the entries of $M_a$ with the set $U_1 = \{1,2,3,4,5\}, U_2 = \{6,7,8,9,10\}, U_3 = \{11,12,13,14,15\}, U_4 = \{16,17,18,19,20\}, U_5 = \{21,22,23,24,25\}$ as*

$$\begin{array}{ccccc} \{(1,2,3,4,5) & (6,7,8,9,10) & (11,12,13,14,15) & (16,17,18,19,20) & (21,22,23,24,25) \\ (1,6,11,16,21) & (2,7,12,17,21) & (3,8,13,18,21) & (4,9,14,19,21) & (5,10,15,20,21) \\ (2,8,14,20,22) & (3,10,11,19,22) & (5,9,12,16,22) & (1,7,15,18,22) & (4,6,13,17,22) \\ (3,9,15,17,23) & (5,6,14,18,23) & (4,7,11,20,23) & (2,10,13,16,23) & (1,8,12,19,23) \\ (4,10,12,18,24) & (1,9,13,20,24) & (2,6,15,19,24) & (5,8,11,17,24) & (3,7,14,16,24) \\ (5,7,13,19,25) & (4,8,15,16,25) & (1,10,14,17,25) & (3,6,12,20,25) & (2,9,11,18,25)\} \end{array}$$

*As mentioned in Remark 2, this BIBD construction satisfies the equation:*

$$25\binom{5}{2} + 5\binom{5}{2} = 300 = \binom{25}{2}$$

# Chapter 3

# Lower Bounds

Let $P = \{p_1,\ p_2, ...,\ p_n\}$ be a set of proposals, $n \geq 2$, and let $R = \{r_1,\ r_2, ...,\ r_m\}$ be a set of referees. The referees in $R$ are said to cover all $n(n\text{-}1)/2$ pairs of $n$ proposals if each pair of proposals is reviewed by at least one referee in $R$. Suppose that each referee is willing to review $k$ proposals, where $k$, $2 \leq k \leq n$. Then, for all $n(n\text{-}1)/2$ pairs of proposals to be covered by the $m$ referees, the following inequality must clearly hold:

$$m\binom{k}{2} \geq \binom{n}{2}, \quad k \geq 2$$

Simplifying this inequality gives the following lower bound on the number of referees:

$$m = \left\lceil \frac{n(n-1)}{k(k-1)} \right\rceil, \quad k \geq 2 \tag{3.1}$$

In particular, when $k = 2$, that is, when each referee reviews 2 proposals, a minimum of $n(n\text{-}1)/2$ referees is required, and when $k = n$, one referee is required. Other constraints can be derived from this inequality.

| Referee Capacity | Minimum Number of Referees($m$) | | | | | | |
|---|---|---|---|---|---|---|---|
| | Equation 3.1 | $n = 2$ | $n = 4$ | $n = 8$ | $n = 16$ | $n = 32$ | $n \to \infty$ |
| $k = n$, $n \geq 2$ | $m \geq 1$ | $k$ =2, $m \geq 1$ | $k$ = 4, $m \geq 1$ | $k$ = 8, $m \geq 1$ | $k$= 16, $m \geq 1$ | $k$ =32, $m \geq 1$ | $m \to 1$ |
| $k = n/2$, $n \geq 4$ | $m \geq \left\lceil \frac{4(n-1)}{(n-2)} \right\rceil$ | N/A | $k$ = 2, $m \geq 6$ | $k$ = 4, $m \geq 5$ | $k$ = 8, $m \geq 5$ | $k$ =16, $m \geq 5$ | $m \to 5$ |
| $k = n/3$, $n \geq 6$ | $m \geq \left\lceil \frac{9(n-1)}{(n-3)} \right\rceil$ | N/A | N/A | $k$ = 3, $m \geq 15$ | $k$ = 6, $m \geq 11$ | $k$ =12, $m \geq 10$ | $m \to 10$ |
| $k = n/4$, $n \geq 8$ | $m \geq \left\lceil \frac{16(n-1)}{(n-4)} \right\rceil$ | N/A | N/A | $k$ = 2, $m \geq 28$ | $k$ = 4, $m \geq 20$ | $k$ = 8, $m \geq 18$ | $m \to 17$ |

Table 3.1: Minimum numbers of referees with specified capacities for $n$ proposals.

Table 3.1 lists the capacities of referees versus minimum numbers of referees for various values of $n$. It is obvious that when $k = n$, and $n \geq 2$, one referee will also suffice, and hence $m = 1$ is always achievable. For even $n$ and $k = n/2$, the table shows that $m$ tends to 5 as $n \to \infty$. However, for $n = 4$, Equation 3.1 implies that $m = 6$. We strengthen the lower bound to 6 for other values of $n$ as follows.

**Theorem 2** *For all even $n = 2k \geq 4$, if each referee is assigned $k$ proposals, at least 6 referees are needed to cover all pairs of $n$ proposals.*

**Proof** For $n = 4$, $k = 2$, each referee is assigned two proposals, and can therefore cover only one pair. Since there are 6 pairs of proposals in all, 6 referees are clearly necessary. For any even $n \geq 6$, without loss of generality, suppose that the first 2 referees are assigned $k$ proposals as shown below with $u$ proposals shared between them, where $u$ is an integer between 0 and $k$, and the shaded areas represent the sets of proposals assigned to the two referees:

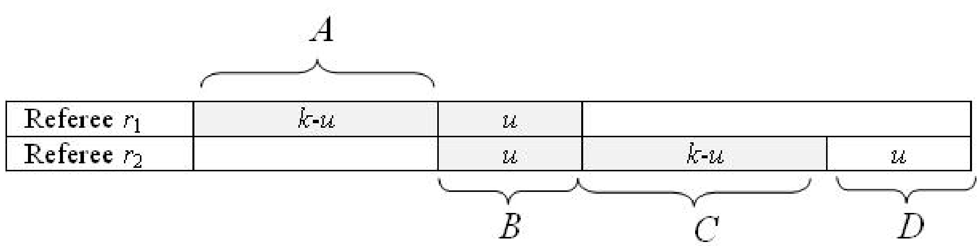


Figure 3.1: The assignment of proposals to the first two referees.

Then we have the following sets of pairs of proposals that remain to be covered:

$$
\begin{aligned}
A \times C &= \{(a, c) : a \in A, c \in C\} \\
A \times D &= \{(a, d) : a \in A, d \in D\} \\
B \times D &= \{(b, d) : b \in B, d \in D\} \\
C \times D &= \{(c, d) : c \in C, d \in D\} \\
D \times D &= \{(d_1, d_2) : d_1, d_2 \in D, d_1 < d_2\}
\end{aligned} \tag{3.2}
$$

If $u = 0$ then $B$ and $D$ vanish, and $|A| = |C| = k$ so that the number of additional pairs of proposals that remain to be covered is given by $k^2$. Furthermore, in order to cover these $k^2$ pairs of proposals, each additional referee must be assigned at least one proposal from each of $A$ and $C$. Therefore, the number of additional referees cannot be less than

$$\left\lceil \frac{k^2}{w(k-w)} \right\rceil$$

where $w$ denotes the number of proposals in $A$ and $k - w$ denotes the number of proposals in $C$. Since the denominator is maximized when $w = k/2$, the number of additional referees cannot be less than 4 implying that 6 referees are necessary in this case.

On the other hand, if $u = k$ then $A$ and $C$ vanish, and $|B| = |D| = k$ so that the number of additional pairs of proposals to be covered is given by $k^2$+$k$($k$-1)/2. But since each new referee can cover at most $k$($k$-1)/2 proposals, we need at least

$$\left\lceil \frac{k^2 + k(k-1)/2}{k(k-1)/2} \right\rceil = \left\lceil \frac{3k^2 - k}{k^2 - k} \right\rceil \geq 4, for\ k > 1$$

more referees[1]. Therefore, at least 6 referees are needed to cover all pairs of n proposals in this case as well.

To complete the proof, suppose that $1 \leq u < k$. In this case, we must cover the pairs of proposals in all the sets stated above. In particular, we must cover the pairs of proposals in the sets $A \times C$, $A \times D$, $B \times D$, and $C \times D$. This leads to the assignment pattern for the subsequent referees as follows:

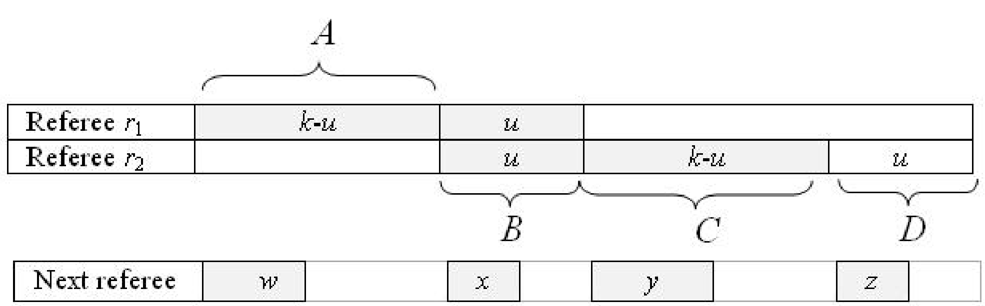


Figure 3.2: The assignment of proposals to the third referee.

[1] $\frac{3k^2-k}{k^2-k} > 3$ for all $k > 1$ . Therefore, $\left\lceil \frac{3k^2-k}{k^2-k} \right\rceil \geq 4, for\ k > 1$.

Therefore, the number of additional referees cannot be less than

$$\left\lceil \frac{(k-u)(k-u+u)+(u+k-u)u}{wy+wz+xz+yz} \right\rceil = \left\lceil \frac{k^2}{wy+wz+xz+yz} \right\rceil$$

where $w$, $x$, $y$, $z$ are the numbers of proposals assigned to a new referee from the subsets, $A$, $B$, $C$, and $D$, respectively. It can be shown that, under the constraint $w+x+y+z=k$, the denominator of this expression has a maximum at $w=y=z=k/3$, and $x=0$ and is given by $k^2/3$ (See Lemma 1). However, unless $u=0$, the value of $x$ cannot be zero for all additional referees as this will leave out one or more pairs of proposals one of which belongs to $B$. Therefore, the maximum number of pairs generated by at least one of the additional referees must be less than $k^2/3$, and hence the number of additional referees cannot be less than 4. Adding these to the first two referees shows that 6 referees are necessary in this case as well and this completes the proof. ∎

**Corollary 3** *For all odd $n$ = 2k+1 $\geq$ 5, suppose that each of the half of the referees is assigned k+1 proposals, and each of the other half of the referees is assigned $k$ proposals. Then at least 6 referees are needed to cover all pairs of $n$ proposals.*

**Proof** Let $n$ = 2$k$+1, where $k \geq 2$. Consider any 2$k$ of the $n$ proposals, and let $p$ be the proposal that is left out. By Theorem 2, at least 6 referees must be used, with each assigned to $k$ proposals, to cover all 2$k$(2$k$-1)/2 = $k$(2$k$-1) pairs of these 2$k$ proposals. This leaves

$$(2k+1)2k/2 - k(2k-1) = k\{(2k+1)-(2k-1)\} = 2k$$

pairs of proposals still to be covered. Suppose that one of the referees is removed and proposal $p$ is assigned to 3 of the remaining 5 referees each, in addition to their $k$ proposals which they had been originally assigned. Now, with one of the referees removed, at least one pair of proposals among the first 2$k$ proposals, previously covered by the 6 referees must clearly be left uncovered. Otherwise, 5 referees would have been sufficient to cover the original 2$k$ proposals. Therefore at least 2$k$+1 pairs of proposals must be covered by the 3 referees whose assignments have been increased by one proposal. However, with one new proposal, i.e., proposal $p$, these three referees can collectively increase the number of pairs of proposals by at most $2k$ since the three referees were assigned their $k$ proposals from the original set of $2k$ proposals prior to the assignment of proposal $p$. But, this is less than the $2k+1$ pairs of proposals still to be covered and the statement follows. $\blacksquare$

These results can be extended to assignments where each referee can review $k = n/3$ proposals. For $n = 6$ ($k = 2$) and $n = 9$ ($k = 3$), it is easily verified that 15 and 12 referees are required. For all $n = 3k \geq 12$, where $k$ is a positive even integer, we can improve the lower bound of 10 referees in Table 3.1 to 11 as follows:

**Theorem 3** *For all $n = 3k \geq 12$, and even $k$, if each referee is assigned $k$ proposals then at least 11 referees are needed to cover all pairs of $n$ proposals*[2].

**Proof** The proof proceeds as in the proof of Theorem 2 with the following modified diagram. The only change in the set up is that the cardinality of $D$ is now $k + u$ and $k = n/3$.

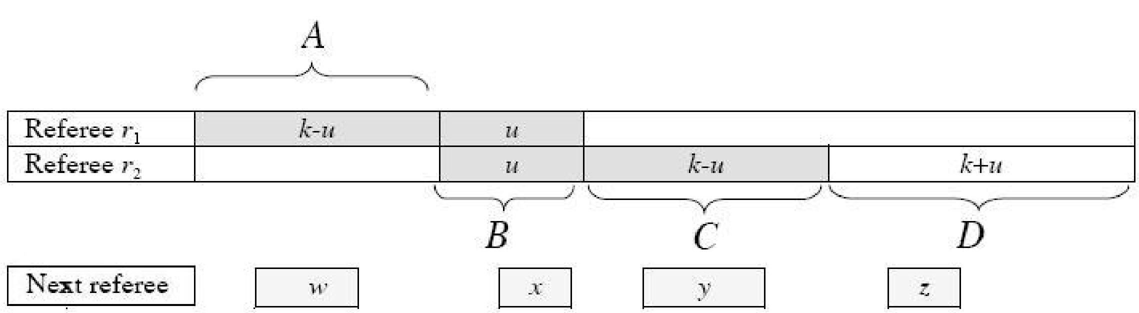


Figure 3.3: The assignments of proposals to the referees for capacity, $k = n/3$.

As before, if $u = 0$ then $B$ vanishes, $A$, $C$, and $D$ contain $k$ proposals each without any overlap with one another. Hence, the number of pairs of proposals

[2] *The statement can be extended to odd $n$ using a similar argument as in Corollary 3.*

that remains to be covered is given by $3k^2 + k(k\text{-}1)/2$. Just considering the first term, the number of additional referees cannot be less than

$$\left\lceil \frac{3k^2}{wy + wz + yz} \right\rceil,$$

where $0 < w$, $y$, $z < k$ are the numbers of proposals in sets $A$, $C$, and $D$, and $w + y + z = k$. It can be shown that $wy+wz+yz$ is maximized when $w = y = z = k/3$. Therefore, the minimum value of the expression above is given by

$$\left\lceil \frac{3k^2}{\frac{k^2}{9} + \frac{k^2}{9} + \frac{k^2}{9}} \right\rceil = 9$$

proving that the total number of referees cannot be less than 11 in this case.

On the other hand, if $u = k$ then $A$ and $C$ vanish, and $|B| = k$, $|D| = 2k$ so that the number of proposals that remains to be covered is given by $2k^2$+$2k(2k$-$1)/2 = 4k^2$-$k$. But since each new referee can cover at most $k(k\text{-}1)/2$ proposals, we need at least

$$\left\lceil \frac{4k^2 - k}{k(k-1)/2} \right\rceil = \left\lceil \frac{8k^2 - 2k}{k^2 - k} \right\rceil \geq 9$$

more referees[3] . Adding these to the first two referees, at least 11 referees are needed to cover all pairs of $n$ proposals, in this case as well.

Finally, suppose that $1 \leq u < k$. As in Theorem 2, we must cover the pairs of proposals in all the sets described in Equation 3.2. In particular, the number of pairs in the first four sets must be covered, where $A$, $B$, $C$, and $D$ are defined as in Figure 3.3. The number of these pairs of proposals is given by

$$(k - u)(k - u + k + u) + (u + k - u)(k + u) = 3k^2 - ku$$

With the distribution of $k$ proposals of each additional referee into the sets $A$, $B$, $C$, and $D$ as shown in Figure 3.3, the number of pairs of proposals covered by each additional referee is given by $wy + wz + xz + yz$. Furthermore, as shown in Lemma 1, $wy + wz + xz + yz$ is maximized when $x = 0$, and $w = y$ for any

[3] $\frac{8k^2-2k}{k^2-k} > 8$ for all $k > 1$. Therefore, $\left\lceil \frac{8k^2-k}{k^2-k} \right\rceil \geq 9, for\ k > 1$.

given $z$. Therefore, the maximum number of pairs of proposals covered by each such referee is given by

$$wy + wz + xz + yz = w^2 + wz + wz = w^2 + 2wz$$

where $w + y + z = k$, or $z = k - w - y = k - 2w$. Replacing $z$ by $k$-$2w$ in the above equation, the maximum number of pairs of proposals that can be generated by any additional referee becomes

$$w^2 + 2w(k - 2w) = 2kw - 3w^2.$$

Let $a$ denote the minimum number of additional referees to cover the missing $3k^2$-$ku$ pairs of proposals, and let $w_i$ denote the number of proposals assigned to the $i^{\text{th}}$ referee, $1 \leq i \leq a$ under this maximality constraint. Then the maximum number of pairs of proposals covered by $a$ referees is given by

$$\sum_{i=1}^{a} 2kw_i - 3w_i^2$$

Therefore, to cover the missing $3k^2$-$ku$ pairs, the following inequality must hold:

$$\sum_{i=1}^{a} 2kw_i - 3w_i^2 \geq 3k^2 - ku$$

Dividing both sides of the inequality by $k^2$ and rewriting the argument of the sum on the left, we get

$$\sum_{i=1}^{a} 2\frac{w_i}{k} - 3\frac{w_i^2}{k^2} = \sum_{i=1}^{a} \frac{w_i}{k}\left(2 - 3\frac{w_i}{k}\right) \geq 3 - \frac{u}{k}$$

It is easy to verify that the argument of the sum is maximized if

$$\frac{w_i}{k} = 2 - 3\frac{w_i}{k} \text{ or } w_i = \frac{k}{2}$$

Therefore,

$$\sum_{i=1}^{a} \frac{1}{4} \geq 3 - \frac{u}{k} \text{ or } a \geq 12 - 4\frac{u}{k}$$

Given that

$$\left\lceil 12 - 4\frac{u}{k} \right\rceil \geq 9 \text{ if } u < k$$

the number of additional referees cannot be less than 9, leading to a lower bound of 11 referees in this case as well. ∎

The next theorem extends these results to referees with a capacity of $n/4$ for $n$ proposals:

**Theorem 4** *For all $n = 4k \geq 16$, if each referee is assigned $k$ proposals then at least 18 referees must be used to cover all pairs of $n$ proposals.*

**Proof** Let $n = 4k$, where $k \geq 4$ is an integer. The proof proceeds as in the proof of earlier theorems with the following modified diagram. The only change in the set up of the proof is that the cardinality of $D$ is now $2k + u$ and $k = n/4$.

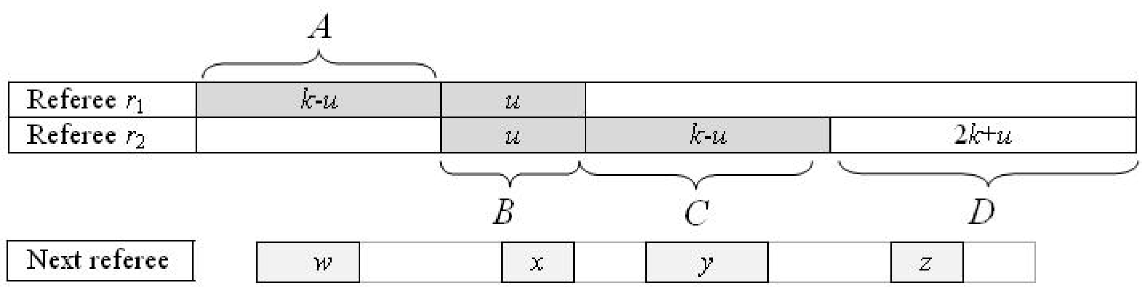


Figure 3.4: The assignment of proposals to the referees for capacity, $k = n/4$.

If $u = 0$ then $B$ vanishes, $A$ and $C$ contain $k$ proposals each and $D$ contains $2k$ proposals without any overlap with one another. Hence, the number of proposals that remains to be covered is given by $5k^2$+ $k(2k$-1). To cover the first $5k^2$ of these proposals, let $w, y, z$ be the number of proposals assigned to each additional referee from sets $A$, $C$, and $D$. Therefore, the number of additional referees cannot be less than

$$\left\lceil \frac{5k^2}{wy + wz + yz} \right\rceil,$$

where $w + y + z = k$. As before, the maximum value of $wy+wz+yz$ is maximized when $w = y = z = k/3$, and is given by $k^2/3$. Therefore, the minimum value of the expression above cannot be smaller than

$$\left\lceil \frac{5k^2}{\frac{k^2}{9} + \frac{k^2}{9} + \frac{k^2}{9}} \right\rceil = 15$$

However, this assumes that the pairs of proposals generated by cross multiplying the sets of $k/3$ proposals from $A$, $C$, and $D$ can all be different. But, this is not possible since if we just consider the sets $A$ and $C$, and partition each into subsets of $k/3$ proposals then the maximum number of non-overlapping pairs of such subsets cannot exceed 9. Therefore, at least one pair of proposals must be covered more than once if we were to use more than 9 referees. This implies that the number of distinct pairs of proposals covered by cross multiplying subsets of $k/3$ proposals from each of $A$, $B$, and $C$ must be strictly less than $k^2/3$. Therefore, at least 16 new referees are needed and adding this to the first two referees gives at least 18 referees.

On the other hand, if $u = k$ then $A$ and $C$ vanish, and $|B| = k$, $|D| = 3k$ so that the number of additional pairs of proposals to be covered is given by $3k^2$+$3k(3k$-$1)/2 = (15k^2$-$3k)/2$. Dividing this by the maximum number of pairs of proposals that can be covered by a referee gives at least

$$\left\lceil \frac{(15k^2-3k)/2}{k(k-1)/2} \right\rceil = \left\lceil \frac{15k^2-3k}{k^2-k} \right\rceil \geq 16$$

new referees[4] or a total of 18 referees with the first two referees added.

Finally, suppose that $1 \leq u < k$. Given the distribution of the proposals to the sets $A, B, C$, and $D$ as shown in Figure 3.4, the number of pairs of proposals that remain to be covered is given by

$$\binom{4k}{2} - \left\{ 2\binom{k}{2} - \binom{u}{2} \right\} = 7k^2 - k + u^2/2 - u/2$$

Now arbitrarily divide the set $D$ into three subgroups of $D_1$, $D_2$ and $D_3$ where the sizes of these subgroups are $k$, $k$ and $u$ respectively and the sum of their sizes is $k + k + u = 2k + u$, the size of the set $D$. Suppose that the pairs of proposals within each of the sets $D_1$, $D_2$ and $D_3$ are already covered without using any new referees. Then, the number of pairs of proposals that remain to be covered is given by

$$7k^2 - k + u^2/2 - u/2 - \left\{ \binom{k}{2} + \binom{k}{2} + \binom{u}{2} \right\} = 6k^2$$

[4] $\frac{15k^2-3k}{k^2-k} > 15$ for all $k > 1$. Therefore, $\left\lceil \frac{15k^2-3k}{k^2-k} \right\rceil \geq 16, for\ k > 1$.

Now, any additional referee can generate at most $wy+wz+ws+wt+xz+xs+xt+yz+ys+yt+zs+zt+st$ new pairs of proposals, as shown in Figure 3.5.

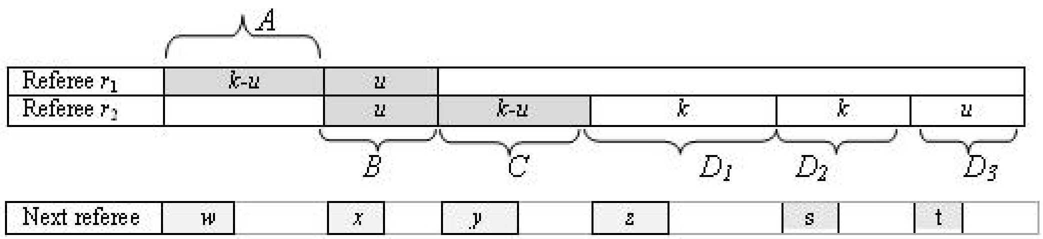


Figure 3.5: The assignment of proposals to the third referee.

Therefore the number of additional referees cannot be less than

$$\left\lceil \frac{6k^2}{wy+wz+ws+wt+xz+xs+xt+yz+ys+yt+zs+zt+st} \right\rceil$$

where $w$, $x$, $y$, $z$, $s$, and $t$ are the numbers of proposals assigned to a new referee from the subsets, $A$, $B$, $C$, $D_1$, $D_2$ and $D_3$, respectively. It can be shown that, under the constraint $w+x+y+z+s+t=k$, the denominator of this expression has a maximum at $w=y=z=s=t=k/5$ and $x=0$, and is given by $2k^2/5$ (See Lemma 2). Hence the number of additional referees cannot be less than

$$\frac{6k^2}{2k^2/5} = 15.$$

However, this assumes that the pairs of proposals generated by cross multiplying the sets of $k/5$ proposals from $A$, $C$, $D_1$, $D_2$ and $D_3$ can all be different. But this is not possible since the number of non-overlapping pairs of subsets of size $k/5$ between $A$ and $D_3$ is strictly less than 15. To see this, just note that the number of non-overlapping pairs of subsets of size $k/5$ in $A$ is given by $\frac{k-u}{k/5}$ and similarly those in $D_3$ is $\frac{u}{k/5}$. Therefore, the maximum number of non-overlapping pairs of subsets of size $k/5$ is given by $\frac{(k-u)u}{k^2/25} = \frac{25(k-u)u}{k^2}$ and it is easy to see that this is always less than 15 for any $u$, $1 \le u < k$. It follows that the number of distinct pairs of proposals covered by cross multiplying subsets of $k/5$ proposals from each of $A$, $C$, $D_1$, $D_2$ and $D_3$ must be strictly less than $2k^2/5$ for at least one of the additional referees. Hence the number of additional referees cannot be less than 16. Since we need 16 referees in order to cover $6k^2$ pairs of proposals, then we also need at least 16 referees in order to cover $7k^2 - k + u^2/2 - u/2$ pairs of proposals. Adding these to the first two referees shows that 18 referees are necessary in this case as well and this completes the proof. ∎

# Chapter 4

# Optimal Assignments

In this section, we provide explicit assignments of proposals to referees to cover all pairs of proposals using 6 referees for $n = 2k$, 12 referees for $n = 3k$, and 20 referees for $n = 4k$. We further prove that the lower bound of $\lceil n(n-1)/k(k-1) \rceil$ referees is asymptotically optimal within a factor of 2 by giving an actual assignment for capacity $k$ for all other $k$, $2 \leq k \leq n$.

## 4.1 Referees With Half Capacity

We first present an optimal assignment of $n$ proposals to referees with a capacity of $n/2$.

**Theorem 5**

(*a*) *For any even integer $n = 2k \geq 4$, if 4 referees are assigned $k$ proposals each, one referee is assigned $2\lceil k/2 \rceil$ proposals and one referee is assigned $2\lfloor k/2 \rfloor$ proposals, then 6 referees are sufficient to cover all pairs of $n$ proposals.*

(*b*) *For any odd integer $n = 2k+1 \geq 5$, if one half of referees are assigned $\lceil n/2 \rceil$ proposals and the other half of referees are assigned $\lfloor n/2 \rfloor$ proposals then 6 referees are sufficient to cover all pairs of $n$ proposals.*

**Proof**

a) For even $n$, we give one possible assignment that uses 6 referees below.

| Proposals | $p_1, p_2, .., p_{\lceil k/2 \rceil}$ | $p_{\lceil k/2 \rceil+1}, .., p_k$ | $p_{k+1}, .., p_{k+\lceil k/2 \rceil}$ | $p_{k+\lceil k/2 \rceil+1}, .., p_{2k}$ |
|---|---|---|---|---|
| Referee $r_1$ | $k$ proposals | | | |
| Referee $r_2$ | | | $k$ proposals | |
| Referee $r_3$ | $\lceil k/2 \rceil$ proposals | | $\lceil k/2 \rceil$ proposals | |
| Referee $r_4$ | $\lceil k/2 \rceil$ proposals | | | $\lfloor k/2 \rfloor$ proposals |
| Referee $r_5$ | | $\lfloor k/2 \rfloor$ proposals | $\lceil k/2 \rceil$ proposals | |
| Referee $r_6$ | | $\lfloor k/2 \rfloor$ proposals | | $\lfloor k/2 \rfloor$ proposals |

Table 4.1: Assignment of $n = 2k$ proposals to 6 referees, each with a capacity of $k$.

That this assignment covers all $n(n\text{-}1)/2$ pairs of proposals can be seen as follows. The first referee covers the $k(k\text{-}1)/2$ pairs of the first $k$ proposals and the second referee covers the $k(k\text{-}1)/2$ pairs of the second $k$ proposals, and therefore they are disjoint. The third referee covers $\lceil k/2 \rceil \times \lceil k/2 \rceil$ pairs of proposals and clearly, these pairs are all different from those covered by the first two referees. Likewise, the fourth, fifth, and sixth referees, cover $\lceil k/2 \rceil \times \lfloor k/2 \rfloor$, $\lfloor k/2 \rfloor \times \lceil k/2 \rceil$, $\lfloor k/2 \rfloor \times \lfloor k/2 \rfloor$ pairs of proposals which are all distinct from one another and those covered by the first three referees. Hence, the number of pairs covered by the 6 referees is given by

$$\begin{aligned}
& 2k(k-1)/2 + \left\lceil \frac{k}{2} \right\rceil \times \left\lceil \frac{k}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil \times \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor \times \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor \times \left\lfloor \frac{k}{2} \right\rfloor \\
= \; & k(k-1) + \left\lceil \frac{k}{2} \right\rceil \times \left\{ \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor \right\} + \left\lfloor \frac{k}{2} \right\rfloor \times \left\{ \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor \right\} \\
= \; & k(k-1) + \left\{ \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor \right\} \times k \\
= \; & k(k-1) + k^2 = \binom{2k}{2} = \binom{n}{2}
\end{aligned}$$

as desired.

b) For odd $n = 2k{+}1$, we give the following assignment that also uses 6 referees.

| Proposals | $p_1, p_2, .., p_{\lceil (k+1)/2 \rceil}$ | $p_{\lceil (k+1)/2 \rceil + 1}, .., p_{k+1}$ | $p_{k+2}, .., p_{k+1+\lceil k/2 \rceil}$ | $p_{k+2+\lceil k/2 \rceil}, .., p_{2k+1}$ |
|---|---|---|---|---|
| Referee $r_1$ | *k+1* proposals | | | |
| Referee $r_2$ | | | $k$ proposals | |
| Referee $r_3$ | $\lceil (k+1)/2 \rceil$ proposals | | $\lceil k/2 \rceil$ proposals | |
| Referee $r_4$ | $\lceil (k+1)/2 \rceil$ proposals | | | $\lfloor k/2 \rfloor$ proposals |
| Referee $r_5$ | | $\lfloor (k+1)/2 \rfloor$ proposals | $\lceil k/2 \rceil$ proposals. | |
| Referee $r_6$ | | $\lfloor (k+1)/2 \rfloor$ proposals | | $\lfloor k/2 \rfloor$ proposals |

Table 4.2: Assignment of $n$ proposals to 6 referees, each with a capacity of $n/2$, $n = 2k{+}1$.

As before, adding all the pairs of proposals contributed by the 6 referees, we obtain

$$\begin{aligned}
\Rightarrow \quad & \frac{(k+1)k}{2} + \frac{k(k-1)}{2} + \left\lceil \frac{k+1}{2} \right\rceil \times \left\lceil \frac{k}{2} \right\rceil + \left\lceil \frac{k+1}{2} \right\rceil \times \left\lfloor \frac{k}{2} \right\rfloor \\
& + \left\lfloor \frac{k+1}{2} \right\rfloor \times \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{k+1}{2} \right\rfloor \times \left\lfloor \frac{k}{2} \right\rfloor \\
= \quad & k^2 + \left\lceil \frac{k+1}{2} \right\rceil \times \left\{ \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor \right\} + \left\lfloor \frac{k+1}{2} \right\rfloor \times \left\{ \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor \right\} \\
= \quad & k^2 + \left\{ \left\lceil \frac{k+1}{2} \right\rceil + \left\lfloor \frac{k+1}{2} \right\rfloor \right\} \times k \\
= \quad & k^2 + (k+1)k = \binom{2k+1}{2} = \binom{n}{2}
\end{aligned}$$

and the statement follows. ∎

**Example 3** *Optimal assignments of proposals to referees for n = 5,6,7,8.*

*a)* $n = 5,\ k = 2$

| | | | | | |
|---|---|---|---|---|---|
| $r_1$ | $p_1$ | $p_2$ | $p_3$ | | |
| $r_2$ | | | | $p_4$ | $p_5$ |
| $r_3$ | $p_1$ | $p_2$ | | $p_4$ | |
| $r_4$ | $p_1$ | $p_2$ | | | $p_5$ |
| $r_5$ | | | $p_3$ | $p_4$ | |
| $r_6$ | | | $p_3$ | | $p_5$ |

*b)* $n = 6,\ k = 3$

| | | | | | | |
|---|---|---|---|---|---|---|
| $r_1$ | $p_1$ | $p_2$ | $p_3$ | | | |
| $r_2$ | | | | $p_4$ | $p_5$ | $p_6$ |
| $r_3$ | $p_1$ | $p_2$ | | $p_4$ | $p_5$ | |
| $r_4$ | $p_1$ | $p_2$ | | | | $p_6$ |
| $r_5$ | | | $p_3$ | $p_4$ | $p_5$ | |
| $r_6$ | | | $p_3$ | | | $p_6$ |

*c)* $n=7, k=3$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $r_1$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | | | |
| $r_2$ | | | | | $p_5$ | $p_6$ | $p_7$ |
| $r_3$ | $p_1$ | $p_2$ | | | $p_5$ | $p_6$ | |
| $r_4$ | $p_1$ | $p_2$ | | | | | $p_7$ |
| $r_5$ | | | $p_3$ | $p_4$ | $p_5$ | $p_6$ | |
| $r_6$ | | | $p_3$ | $p_4$ | | | $p_7$ |

*d)* $n=8, k=4$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $r_1$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | | | | |
| $r_2$ | | | | | $p_5$ | $p_6$ | $p_7$ | $p_8$ |
| $r_3$ | $p_1$ | $p_2$ | | | $p_1$ | $p_6$ | | |
| $r_4$ | $p_1$ | $p_2$ | | | | | $p_7$ | $p_8$ |
| $r_5$ | | | $p_3$ | $p_4$ | $p_5$ | $p_6$ | | |
| $r_6$ | | | $p_3$ | $p_4$ | | | $p_7$ | $p_8$ |

**Remark 3** *When n and $k = n/2$ are both even, each referee is assigned exactly k proposals in Theorem 5 and this conforms to the hypothesis of Theorem 2. However, when k is odd, this happens only in an average sense. That is, the average number of proposals assigned to the 6 referees is still k with one of the referees receiving k+1 proposals and another referee k-1 proposals as in (a) in the example above. We conjecture that it is impossible to cover all pairs of proposals if all 6 referees are assigned exactly k proposals. For odd n = 2k + 1, the assignments of proposals to the 6 referees conforms to the hypothesis of Theorem 2 for both even and odd k as can be seen in (c) and (d) in the example above. In particular, when k is even, referees $r_3$ and $r_4$ are assigned k+1 proposals each and $r_5$ and $r_6$ are assigned k proposals each. When k is odd, referees $r_3$ and $r_5$ are assigned k+1 proposals each and $r_4$ and $r_6$ are assigned k proposals each.*

*We also note that the assignments of the proposals to the 6 referees in Theorem 5 are not unique. For even n, there exist $\binom{2k}{k}\binom{k}{k/2}\binom{k}{k/2}$ such assignments, where $\binom{2k}{k}$ represents the number of choices for the first two referees, and the last two terms represent the number of choices for the last four referees. Similarly, for odd n, there exist $\binom{2k+1}{k+1}\binom{k+1}{k/2}\binom{k}{k/2}$ such assignments.* ∎

## 4.2 Referees With One-Third Capacity

These construction can be extended to assignments where each referee can review $k = n/3$ proposals.

**Theorem 6** *Suppose that n is divisible by 9, and let $k = n/3$. Then 12 referees are sufficient to cover all pairs of n proposals.*

**Proof** Let the set of $n$ proposals be partitioned into 9 subsets of $k/3$ proposals each and denote them by $G_i$, $1 \leq i \leq 9$. Let $P = \{i : 1 \leq i \leq 9\}$ where $i$ denotes the index of $G_i$. By Proposition 1, the indices in $P$ form a (9,3,1)-BIBD with the blocks given in that proposition. Let these blocks be denoted by $B_j$ and let $B_j$ be assigned to referee j, $1 \leq j \leq 12$. Since the blocks form a (9,3,1)-BIBD, each pair of indices $i$ and $j$ in $P$ appear together in blocks. Therefore all possible pairs of proposals ($x$,$y$) where x is in $G_i$ and y is in $G_j$, $1 \leq i \neq j \leq 9$ is covered by one of the referees. This covers

$$\binom{9}{2}(\frac{k}{3})^2$$

pairs of proposals and

$$\binom{n}{2} - \binom{9}{2}(\frac{k}{3})^2 = 9\binom{k/3}{2}$$

remains to be covered. And this corresponds to the pairs of proposals generated within $G_i$, $1 \leq i \leq 9$, given that each $G_i$ appears four times among the blocks of the design. These pairs of proposals are clearly generated more than once by the twelve referees and this completes the proof. ∎

**Example 4** *The assignment below covers all 153 pairs of 18 proposals with 12 referees with each referee assigned 6 proposals. Whether it is possible to use 11 referees to cover all 153 pairs remains an open problem.*

## 4.3 Referees With One Fourth Capacity

The previous two theorems can be extended to $n$ proposals and referees with capacity $n/4$.

**Theorem 7** *Suppose that n is divisible by 16, and let $k = n/4$. Then 20 referees are sufficient to cover all pairs of n proposals.*

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r_1$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | | | | | | | | | | | | |
| $r_2$ | | | | | | | $p_7$ | $p_8$ | $p_9$ | $p_{10}$ | $p_{11}$ | $p_{12}$ | | | | | | |
| $r_3$ | | | | | | | | | | | | | $p_{13}$ | $p_{14}$ | $p_{15}$ | $p_{16}$ | $p_{17}$ | $p_{18}$ |
| $r_4$ | $p_1$ | $p_2$ | | | | | $p_7$ | $p_8$ | | | | | $p_{13}$ | $p_{14}$ | | | | |
| $r_5$ | | | $p_3$ | $p_4$ | | | | | $p_9$ | $p_{10}$ | | | $p_{13}$ | $p_{14}$ | | | | |
| $r_6$ | | | | | $p_5$ | $p_6$ | | | | | $p_{11}$ | $p_{12}$ | $p_{13}$ | $p_{14}$ | | | | |
| $r_7$ | $p_1$ | $p_2$ | | | | | | | $p_9$ | $p_{10}$ | | | | | $p_{15}$ | $p_{16}$ | | |
| $r_8$ | | | $p_3$ | $p_4$ | | | | | | | $p_{11}$ | $p_{12}$ | | | $p_{15}$ | $p_{16}$ | | |
| $r_9$ | | | | | $p_5$ | $p_6$ | $p_7$ | $p_8$ | | | | | | | $p_{15}$ | $p_{16}$ | | |
| $r_{10}$ | $p_1$ | $p_2$ | | | | | | | | | $p_{11}$ | $p_{12}$ | | | | | $p_{17}$ | $p_{18}$ |
| $r_{11}$ | | | $p_3$ | $p_4$ | | | $p_7$ | $p_8$ | | | | | | | | | $p_{17}$ | $p_{18}$ |
| $r_{12}$ | | | | | $p_5$ | $p_6$ | | | $p_9$ | $p_{10}$ | | | | | | | $p_{17}$ | $p_{18}$ |

Table 4.3: Assignment of 18 proposals to 12 referees, each with a capacity of 6.

**Proof** As in Theorem 6, let the set of $n$ proposals be patitioned into 16 subsets of $k/4$ proposals each and denote them by $G_i$, $1 \leq i \leq 16$. Let $P = \{i : 1 \leq i \leq 16\}$ where $i$ denotes the index of $G_i$. By Proposition 2, the indices in $P$ form a (16,4,1)-BIBD with the blocks given in that proposition. Let these blocks be denoted by $B_j$ and let $B_j$ be assigned to referee j, $1 \leq j \leq 20$. Since the blocks form a (16,4,1)-BIBD, each pair of indices $i$ and $j$ in $P$ appear together in blocks. Therefore all possible pairs of proposals ($x$,$y$) where x is in $G_i$ and y is in $G_j$, $1 \leq i \neq j \leq 16$ is covered by one of the referees. This covers

$$\binom{16}{2}(\frac{k}{4})^2$$

pairs of proposals and

$$\binom{n}{2} - \binom{16}{2}(\frac{k}{4})^2 = 16\binom{k/4}{2}$$

remains to be covered. And this corresponds to the pairs of proposals generated within $G_i$, $1 \leq i \leq 16$, given that each $G_i$ appears five times among the blocks of the design. This pairs of proposals are clearly generated more than once by the twenty referees and this completes the proof. ∎

**Example 5** *The assignment below covers all 496 pairs of 32 proposals with 20 referees with each referee assigned 8 proposals.*

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r_1$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | $p_6$ | $p_7$ | $p_8$ | | | | | | | | | | | | | | | | | | | | | | | | |
| $r_2$ | | | | | | | | | $p_9$ | $p_{10}$ | $p_{11}$ | $p_{12}$ | $p_{13}$ | $p_{14}$ | $p_{15}$ | $p_{16}$ | | | | | | | | | | | | | | | | |
| $r_3$ | | | | | | | | | | | | | | | | | $p_{17}$ | $p_{18}$ | $p_{19}$ | $p_{20}$ | $p_{21}$ | $p_{22}$ | $p_{23}$ | $p_{24}$ | | | | | | | | |
| $r_4$ | | | | | | | | | | | | | | | | | | | | | | | | | $p_{25}$ | $p_{26}$ | $p_{27}$ | $p_{28}$ | $p_{29}$ | $p_{30}$ | $p_{31}$ | $p_{32}$ |
| $r_5$ | $p_1$ | $p_2$ | | | | | | | $p_9$ | $p_{10}$ | | | | | | | $p_{17}$ | $p_{18}$ | | | | | | | $p_{25}$ | $p_{26}$ | | | | | | |
| $r_6$ | | | $p_3$ | $p_4$ | | | | | | | | | $p_{13}$ | $p_{14}$ | | | | | | | | | $p_{23}$ | $p_{24}$ | | | $p_{27}$ | $p_{28}$ | | | | |
| $r_7$ | | | | | $p_5$ | $p_6$ | | | | | | | | | $p_{15}$ | $p_{16}$ | | | $p_{19}$ | $p_{20}$ | | | | | | | | | $p_{29}$ | $p_{30}$ | | |
| $r_8$ | | | | | | | $p_7$ | $p_8$ | | | $p_{11}$ | $p_{12}$ | | | | | | | | | $p_{21}$ | $p_{22}$ | | | | | | | | | $p_{31}$ | $p_{32}$ |
| $r_9$ | | | $p_3$ | $p_4$ | | | | | | | $p_{11}$ | $p_{12}$ | | | | | | | $p_{19}$ | $p_{20}$ | | | | | $p_{25}$ | $p_{26}$ | | | | | | |
| $r_{10}$ | $p_1$ | $p_2$ | | | | | | | | | | | | | $p_{15}$ | $p_{16}$ | | | | | $p_{21}$ | $p_{22}$ | | | | | $p_{27}$ | $p_{28}$ | | | | |
| $r_{11}$ | | | | | | | $p_7$ | $p_8$ | | | | | $p_{13}$ | $p_{14}$ | | | $p_{17}$ | $p_{18}$ | | | | | | | | | | | $p_{29}$ | $p_{30}$ | | |
| $r_{12}$ | | | | | $p_5$ | $p_6$ | | | $p_9$ | $p_{10}$ | | | | | | | | | | | | | $p_{23}$ | $p_{24}$ | | | | | | | $p_{31}$ | $p_{32}$ |
| $r_{13}$ | | | | | $p_5$ | $p_6$ | | | | | | | $p_{13}$ | $p_{14}$ | | | | | | | $p_{21}$ | $p_{22}$ | | | $p_{25}$ | $p_{26}$ | | | | | | |
| $r_{14}$ | | | | | | | $p_7$ | $p_8$ | $p_9$ | $p_{10}$ | | | | | | | | | $p_{19}$ | $p_{20}$ | | | | | | | $p_{27}$ | $p_{28}$ | | | | |
| $r_{15}$ | $p_1$ | $p_2$ | | | | | | | | | $p_{11}$ | $p_{12}$ | | | | | | | | | | | $p_{23}$ | $p_{24}$ | | | | | $p_{29}$ | $p_{30}$ | | |
| $r_{16}$ | | | $p_3$ | $p_4$ | | | | | | | | | | | $p_{15}$ | $p_{16}$ | $p_{17}$ | $p_{18}$ | | | | | | | | | | | | | $p_{31}$ | $p_{32}$ |
| $r_{17}$ | | | | | | | $p_7$ | $p_8$ | | | | | | | $p_{15}$ | $p_{16}$ | | | | | | | $p_{23}$ | $p_{24}$ | $p_{25}$ | $p_{26}$ | | | | | | |
| $r_{18}$ | | | | | $p_5$ | $p_6$ | | | | | $p_{11}$ | $p_{12}$ | | | | | $p_{17}$ | $p_{18}$ | | | | | | | | | $p_{27}$ | $p_{28}$ | | | | |
| $r_{19}$ | | | $p_3$ | $p_4$ | | | | | $p_9$ | $p_{10}$ | | | | | | | | | | | $p_{21}$ | $p_{22}$ | | | | | | | $p_{29}$ | $p_{30}$ | | |
| $r_{20}$ | $p_1$ | $p_2$ | | | | | | | | | | | $p_{13}$ | $p_{14}$ | | | | | $p_{19}$ | $p_{20}$ | | | | | | | | | | | $p_{31}$ | $p_{32}$ |

Table 4.4: Assignment of 32 proposals to 20 referees, each with a capacity of 8.

## 4.4 Arbitrary Capacity Case

The assignments described in Theorems 5, 6, and 7 will work for effectively for small values of $n$. In particular, 6-referee assignments in Theorem 5 can handle up to 20 proposals where each referee may be assigned up to 10 proposals. However, for larger $n$, it will be impractical for referees to review $n/2$, $n/3$, or $n/4$ proposals and the number of proposals assigned to each referee may have to be decreased as needed. To deal with larger numbers of proposals, we present another assignment using an asymptotically minimum number of referees. The following theorem describes this assignment for any even $k$ that divides $n$. The theorem is easily extended to odd $k$ as described in the remark that follows the theorem.

**Theorem 8** *Let n and k be positive integers, where k is even and divides n. It is sufficient to have $n(2n\text{-}k)/k^2$ referees, each with capacity k to cover all $n(n\text{-}1)/2$ pairs of n proposals.*

**Proof** Divide the set of $n$ proposals into $n/k$ groups, and use a different referee to review the $k$ proposals in each group. This covers $(n/k)\binom{k}{2}$ pairs with $n/k$ referees. Now, use four more referees to cover the pairs of proposals between every two groups of $k$ proposals as shown in Table 4.4 for one such pair of groups.

This gives

$$4\binom{n/k}{2}\frac{k^2}{4} = \binom{n/k}{2}k^2$$

more distinct pairs, making the total number of pairs equal to

$$\frac{n}{k}\binom{k}{2} + \binom{n/k}{2}k^2 = \frac{n(k-1)}{2} + \frac{n(n-k)}{2} = \frac{n(n-1)}{2} = \binom{n}{2}$$

as desired. Since there are $\binom{n/k}{2}$ such pairs of groups, the number of referees we need to cover the pairs of proposals generated by these pairs of groups is given by $4\binom{n/k}{2}$. Therefore, the total number of referees to cover all $n(n\text{-}1)/2$ pairs of proposals is given by

$$\frac{n}{k} + 4\binom{n/k}{2} = \frac{n}{k} + 2\frac{n}{k}\left(\frac{n}{k} - 1\right) = \frac{n(2n-k)}{k^2}$$

and the statement follows. ∎

**Corollary 4** *The number of referees used in the assignment described in Theorem 8 is within a factor of 2 of the lower bound given in Equation 3.1 and therefore is asymptotically optimal.*

**Proof** Dividing the number of referees obtained in Theorem 8 by the lower bound on the number of referees given in Equation 3.1, we get

$$\frac{n(2n-k)}{k^2} \times \frac{k(k-1)}{n(n-1)} = \frac{(2n-k)}{k} \times \frac{(k-1)}{(n-1)} < \frac{(2n-k)}{(n-1)} \leq 2, \text{for} k \geq 2$$

and the statement follows. ∎

**Example 6** *(Even k): Let $n = 6$ and $k = 2$. By Theorem 8, $n(2n\text{-}k)/k^2 = 15$ referees are sufficient as illustrated in Table 4.6 below. In this case, the number of referees used does exactly match the minimum number of referees given in Equation 3.1.*

| | | | | | | |
|---|---|---|---|---|---|---|
| Referee $r_1$ | $p_1$ | $p_2$ | | | | |
| Referee $r_2$ | | | $p_3$ | $p_4$ | | |
| Referee $r_3$ | | | | | $p_5$ | $p_6$ |
| Referee $r_4$ | $p_1$ | | $p_3$ | | | |
| Referee $r_5$ | $p_1$ | | | $p_4$ | | |
| Referee $r_6$ | | $p_2$ | $p_3$ | | | |
| Referee $r_7$ | | $p_2$ | | $p_4$ | | |
| Referee $r_8$ | $p_1$ | | | | $p_5$ | |
| Referee $r_9$ | $p_1$ | | | | | $p_6$ |
| Referee $r_{10}$ | | $p_2$ | | | $p_5$ | |
| Referee $r_{11}$ | | $p_2$ | | | | $p_6$ |
| Referee $r_{12}$ | | | $p_3$ | | $p_5$ | |
| Referee $r_{13}$ | | | $p_3$ | | | $p_6$ |
| Referee $r_{14}$ | | | | $p_4$ | $p_5$ | |
| Referee $r_{15}$ | | | | $p_4$ | | $p_6$ |

Table 4.6: Assignment of n = 6 proposals to $n(2n\text{-}k)/k^2$ referees, each with capacity $k = 2$.

**Remark 4** *For odd k, partition the n proposals into n/k groups of k proposals each as in Theorem 8 and assign each group to a different referee. Assign k+1 proposals to each of the rest of referees and divide each group of k proposals into two overlapping groups of (k+1)/2 proposals as in the example below. The rest of the proof applies as it is.* ∎

**Example 7** *(Odd k): Let n = 6 and k = 3. By Equation 3.1, 5 referees are necessary and by Theorem 8, n(2n-k)/k² = 6 referees are sufficient, as shown in Table 4.7. As seen in the table, the proposals assigned to referees $r_3$, $r_4$, $r_5$, and $r_6$ overlap. This results in some of the pairs of proposals to be covered more than once but it does not increase the number of referees in the assignment. However, it also makes the assignment asymmetric with respect to the number of referees assigned to the proposals (proposals$p_2$ and $p_5$ are reviewed by 5 referees whereas the rest of proposals are reviewed by 3 referees each). This can be avoided by removing the last referee and reassigning the proposals to remaining referees as shown in Table 4.8.* ∎

| | | | | | | |
|---|---|---|---|---|---|---|
| Referee $r_1$ | $p_1$ | $p_2$ | $p_3$ | | | |
| Referee $r_2$ | | | | $p_4$ | $p_5$ | $p_6$ |
| Referee $r_3$ | $p_1$ | $p_2$ | | $p_4$ | $p_5$ | |
| Referee $r_4$ | $p_1$ | $p_2$ | | | $p_5$ | $p_6$ |
| Referee $r_5$ | | $p_2$ | $p_3$ | $p_4$ | $p_5$ | |
| Referee $r_6$ | | $p_2$ | $p_3$ | | $p_5$ | $p_6$ |

Table 4.7: Assignment of 6 proposals to 6 referees with a capacity of 3.

| | | | | | | |
|---|---|---|---|---|---|---|
| Referee $r_1$ | $p_1$ | $p_2$ | $p_3$ | | | |
| Referee $r_2$ | | | | $p_4$ | $p_5$ | $p_6$ |
| Referee $r_3$ | $p_1$ | | $p_3$ | $p_4$ | $p_5$ | |
| Referee $r_4$ | $p_1$ | $p_2$ | | | $p_5$ | $p_6$ |
| Referee $r_5$ | | $p_2$ | $p_3$ | $p_4$ | | $p_6$ |

Table 4.8: Assignment of 6 proposals to 5 referees with a capacity of 3.

Theorem 8 provides asymptotically optimal assignment for referees with arbitrary capacity to cover all pairs of $n$ proposals. This assignment can be improved by using the BIBD-design described in Corollary 2

**Theorem 9** *Let $n$ and $k$ be positive integers, where $n/k$ is a prime power, and $n$ divides $k^2$. Then $\frac{n}{k}\left(\frac{n}{k}+1\right)$ referees are sufficient to cover all pairs of $n$ proposals.*

**Proof** As in Theorem 6, let the set of $n$ proposals be patitioned into $n^2/k^2$ subsets of $k^2/n$ proposals each and denote them by $G_i$, $1 \leq i \leq n^2/k^2$. Let $P = \{i : 1 \leq i \leq n^2/k^2\}$ where $i$ denotes the index of $G_i$. By Corrollary 2, the indices in $P$ form a ($n^2/k^2$,$n/k$,1)-BIBD with the blocks given in that corollary. Let these blocks be denoted by $B_j$ and let $B_j$ be assigned to referee j, $1 \leq j \leq n^2/k^2 + n/k$. Since the blocks form a ($n^2/k^2$,$n/k$,1)-BIBD, each pair of indices $i$ and $j$ in $P$ appear together in blocks. Therefore all possible pairs of proposals ($x$,$y$) where x is in $G_i$ and y is in $G_j$, $1 \leq i \neq j \leq n^2/k^2$ is covered by one of the referees. This covers

$$\binom{n^2/k^2}{2}\left(\frac{k^2}{n}\right)^2$$

pairs of proposals and

$$\binom{n}{2} - \binom{n^2/k^2}{2}(\frac{k^2}{n})^2 = n^2/k^2\binom{k^2/n}{2}$$

remains to be covered. And this corresponds to the pairs of proposals generated within $G_i$, $1 \leq i \leq n^2/k^2$, given that each $G_i$ appears $(n/k)+1$ times among the blocks of the design. This pairs of proposals are clearly generated more than once by the $\frac{n}{k}\left(\frac{n}{k}+1\right)$ referees and this completes the proof. ∎

**Example 8** *Let $n = 25$ and $k = 5$. Since $n/k = 25/5 = 5$ is a prime power, by Theorem 9, $\frac{n}{k}\left(\frac{n}{k}+1\right) = \frac{25}{5}\left(\frac{25}{5}+1\right) = 30$ referees are sufficient. The first five of the 30 referees are assigned pairwise disjoint sets of $k = 5$ proposals. The remaining 25 referees are also assigned five proposals each but the proposals are spread across the $n/k = 5$ groups of 5 proposals which have been assigned to the first five referees as shown below. Inspecting the assignments for referees $r_6$ through $r_{30}$ shows that no two referees are allocated the same two or more subgroups of $k^2/n = 1$ proposals. This guarantees that all pairs of proposals that are not covered by the first 5 referees are covered by the last 25 referees as shown in Table 4.9.*

**Corollary 5** *The number of referees used in the assignment described in Theorem 9 is asymptotically minimum with respect to the number of referees given in the lower bound of Equation 3.1 and therefore is asymptotically optimal.*

**Proof** Dividing the number of referees obtained in Theorem 9 by the lower bound on the number of referees given in Equation 3.1, we get

$$\frac{n}{k}\left(\frac{n}{k}+1\right) \times \frac{k(k-1)}{n(n-1)} = \frac{(n+k)}{k} \times \frac{(k-1)}{(n-1)} \approx \frac{(n+k)}{n} = 1 + \frac{k}{n}$$

and the statement follows. ∎

**Corollary 6** *The number of referees used in the assignment described in Theorem 9 is optimum with respect to the number of referees if $n = k^2$.*

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r_1$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | | | | | | | | | | | | | | | | | | | | |
| $r_2$ | | | | | | $p_6$ | $p_7$ | $p_8$ | $p_9$ | $p_{10}$ | | | | | | | | | | | | | | | |
| $r_3$ | | | | | | | | | | | $p_{11}$ | $p_{12}$ | $p_{13}$ | $p_{14}$ | $p_{15}$ | | | | | | | | | | |
| $r_4$ | | | | | | | | | | | | | | | | $p_{16}$ | $p_{17}$ | $p_{18}$ | $p_{19}$ | $p_{20}$ | | | | | |
| $r_5$ | | | | | | | | | | | | | | | | | | | | | $p_{21}$ | $p_{22}$ | $p_{23}$ | $p_{24}$ | $p_{25}$ |
| $r_6$ | $p_1$ | | | | | $p_6$ | | | | | $p_{11}$ | | | | | $p_{16}$ | | | | | $p_{21}$ | | | | |
| $r_7$ | $p_1$ | | | | | | | | $p_9$ | | | | $p_{13}$ | | | | | | | $p_{20}$ | | $p_{22}$ | | | |
| $r_8$ | $p_1$ | | | | | | $p_7$ | | | | | | | | $p_{15}$ | | | | $p_{19}$ | | | | $p_{23}$ | | |
| $r_9$ | $p_1$ | | | | | | | | | $p_{10}$ | | $p_{12}$ | | | | | | $p_{18}$ | | | | | | $p_{24}$ | |
| $r_{10}$ | $p_1$ | | | | | | | $p_8$ | | | | | | $p_{14}$ | | | $p_{17}$ | | | | | | | | $p_{25}$ |
| $r_{11}$ | | $p_2$ | | | | | $p_7$ | | | | | $p_{12}$ | | | | | $p_{17}$ | | | | | $p_{22}$ | | | |
| $r_{12}$ | | $p_2$ | | | | | | | | $p_{10}$ | | | | $p_{14}$ | | $p_{16}$ | | | | | | | $p_{23}$ | | |
| $r_{13}$ | | $p_2$ | | | | | | $p_8$ | | | $p_{11}$ | | | | | | | | | $p_{20}$ | | | | $p_{24}$ | |
| $r_{14}$ | | $p_2$ | | | | $p_6$ | | | | | | | $p_{13}$ | | | | | | $p_{19}$ | | | | | | $p_{25}$ |
| $r_{15}$ | | $p_2$ | | | | | | | $p_9$ | | | | | | $p_{15}$ | | | $p_{18}$ | | | $p_{21}$ | | | | |
| $r_{16}$ | | | $p_3$ | | | | | $p_8$ | | | | | $p_{13}$ | | | | | $p_{18}$ | | | | | $p_{23}$ | | |
| $r_{17}$ | | | $p_3$ | | | $p_6$ | | | | | | | | | $p_{15}$ | | $p_{17}$ | | | | | | | $p_{24}$ | |
| $r_{18}$ | | | $p_3$ | | | | | | $p_9$ | | | $p_{12}$ | | | | $p_{16}$ | | | | | | | | | $p_{25}$ |
| $r_{19}$ | | | $p_3$ | | | | $p_7$ | | | | | | | $p_{14}$ | | | | | | $p_{20}$ | $p_{21}$ | | | | |
| $r_{20}$ | | | $p_3$ | | | | | | | $p_{10}$ | $p_{11}$ | | | | | | | | $p_{19}$ | | | $p_{22}$ | | | |
| $r_{21}$ | | | | $p_4$ | | | | | $p_9$ | | | | | $p_{14}$ | | | | | $p_{19}$ | | | | | $p_{24}$ | |
| $r_{22}$ | | | | $p_4$ | | | $p_7$ | | | | $p_{11}$ | | | | | | | $p_{18}$ | | | | | | | $p_{25}$ |
| $r_{23}$ | | | | $p_4$ | | | | | | $p_{10}$ | | | $p_{13}$ | | | | $p_{17}$ | | | | $p_{21}$ | | | | |
| $r_{24}$ | | | | $p_4$ | | | | $p_8$ | | | | | | | $p_{15}$ | $p_{16}$ | | | | | | $p_{22}$ | | | |
| $r_{25}$ | | | | $p_4$ | | $p_6$ | | | | | | $p_{12}$ | | | | | | | | $p_{20}$ | | | $p_{23}$ | | |
| $r_{26}$ | | | | | $p_5$ | | | | | $p_{10}$ | | | | | $p_{15}$ | | | | | $p_{20}$ | | | | | $p_{25}$ |
| $r_{27}$ | | | | | $p_5$ | | | $p_8$ | | | | $p_{12}$ | | | | | | | $p_{19}$ | | $p_{21}$ | | | | |
| $r_{28}$ | | | | | $p_5$ | $p_6$ | | | | | | | | $p_{14}$ | | | | $p_{18}$ | | | | $p_{22}$ | | | |
| $r_{29}$ | | | | | $p_5$ | | | | $p_9$ | | $p_{11}$ | | | | | | $p_{17}$ | | | | | | $p_{23}$ | | |
| $r_{30}$ | | | | | $p_5$ | | $p_7$ | | | | | | $p_{13}$ | | | $p_{16}$ | | | | | | | | $p_{24}$ | |

Table 4.9: Assignment of 25 proposals to 30 referees, each with a capacity of 5.

**Proof** Proof is similar to the one that is done for Corollary 5. Dividing the number of referees obtained in Theorem 9 by the lower bound on the number of referees given in Equation 3.1, we get

$$\frac{n}{k}\left(\frac{n}{k}+1\right) \times \frac{k(k-1)}{n(n-1)} = \frac{(n+k)}{k} \times \frac{(k-1)}{(n-1)}, \text{ substitute } n \text{ by } k^2$$

$$\frac{k^2+k}{k} \times \frac{k-1}{k^2-1} = 1$$

and the statement follows. ∎

**Remark 5** *The assignment given in Theorem 9 is more efficient than the one given in Theorem 8 when $k < n/2$. This can be seen from*

$$\frac{n}{k}\left(\frac{n}{k}+1\right) < \frac{n(2n-k)}{k^2}$$

$$n + k < 2n - k$$

$$2k < n$$

$$k < n/2.$$

∎

# Chapter 5

# Assignments with Distinguishable Referees

In the assignment problems considered thus far we have not taken into account the specialties of referees in handling proposals. It is often desirable to assign proposals to referees who are experts or specialists on the subjects of proposals they review. The assignment methods in Section 4 can still be applied if the specialties of referees satisfy certain constraints. In what follows, we describe some of these extensions.

**Corollary 7** *Suppose that a set of $n$ proposals can be partitioned into two specialty areas of $n/2$ proposals, $S_1$ and $S_2$. Further suppose that, among some 6 referees, (a) one is able to review the proposals in $S_1$ and another is able to review the proposals in $S_2$, and (b) the other four are each able to review $n/4$ proposals in each of $S_1$ and $S_2$. Then all pairs of $n$ proposals can be covered by the 6 referees with the side condition that each proposal is reviewed by three referees in its subject area.*

**Proof** It follows directly from Theorem 5 as shown in Table 5.1. ∎

This corollary can be generalized to $n/k$ specialty areas of $k$ proposals and $\frac{n}{k} + 4\binom{n/k}{2}$ referees for any integer $k$, $2 \leq k \leq n$ that divides $n$.

| Proposals | $p_1, p_2, .., p_{n/4}$ | $p_{(n/4)+1}, .., p_{n/2}$ | $p_{(n/2)+1}, .., p_{3n/4}$ | $p_{(3n/4)+1}, .., p_n$ |
|---|---|---|---|---|
| Referee 1 | Specialty $S_1$ | | | |
| Referee 2 | | | Specialty $S_2$ | |
| Referee 3 | Specialty $S_1$ | | Specialty $S_2$ | |
| Referee 4 | Specialty $S_1$ | | | Specialty $S_2$ |
| Referee 5 | | Specialty $S_1$ | Specialty $S_2$ | |
| Referee 6 | | Specialty $S_1$ | | Specialty $S_2$ |

Table 5.1: Assignment of proposals to 6 referees with 2 specialties, each with a capacity of $n/2$.

**Corollary 8** *Suppose that a set of $n$ proposals can be partitioned into $n/k$ subject areas of $k$ proposals, $S_1, S_2, ..., S_{n/k}$. Suppose also that (a) there exist $n/k$ referees with $n/k$ different specialties matching the $n/k$ subject areas of these $n/k$ sets of $k$ proposals and each is able to review $k$ proposals, (b) the remaining $4\binom{n/k}{2}$ referees can be partitioned into $\binom{n/k}{2}$ groups of 4 referees so that the referees in each group have two specialties matching the specialties of a distinct pair of sets of $k$ proposals. Then $n/k + 4\binom{n/k}{2}$ referees are sufficient to cover all pairs of $n$ proposals.*

**Proof** The proof immediately follows from Theorem 8. ∎

The example below illustrates the corollary.

**Example 9** *$n = 12$, $k = 4$. A possible assignment is shown in Table 5.2 with the following specialties:*

Referee 1: Specialty in $p_4$, $p_8$, $p_2$, $p_{11}$

Referee 2: Specialty in $p_7$, $p_3$, $p_5$, $p_{10}$

Referee 3: Specialty in $p_6$, $p_1$, $p_{12}$, $p_9$

Referee 4: Specialty in $p_4$, $p_8$, $p_2$, $p_{11}$ and in $p_7$, $p_3$, $p_5$, $p_{10}$

⋮

Referee 14: Specialty in $p_7$, $p_3$, $p_5$, $p_{10}$ and in $p_6$, $p_1$, $p_{12}$, $p_9$

Referee 15: Specialty in $p_7$, $p_3$, $p_5$, $p_{10}$ and in $p_6$, $p_1$, $p_{12}$, $p_9$

With $n = 12$, $k = 4$, Equation 3.1 gives a lower bound of 11 on the number of referees. This assignment uses four more referees than the minimum number of referees needed.

| | | | | | | |
|---|---|---|---|---|---|---|
| Referee 1 | $p_4$, $p_8$, $p_2$, $p_{11}$ | | | | | |
| Referee 2 | | | $p_7$, $p_3$, $p_5$, $p_{10}$ | | | |
| Referee 3 | | | | | $p_6$, $p_1$, $p_{12}$, $p_9$ | |
| Referee 4 | $p_4$, $p_8$ | | $p_7$, $p_3$ | | | |
| Referee 5 | $p_4$, $p_8$ | | | $p_5$, $p_{10}$ | | |
| Referee 6 | | $p_2$, $p_{11}$ | $p_7$, $p_3$ | | | |
| Referee 7 | | $p_2$, $p_{11}$ | | $p_5$, $p_{10}$ | | |
| Referee 8 | $p_4$, $p_8$ | | | | $p_6$, $p_1$ | |
| Referee 9 | $p_4$, $p_8$ | | | | | $p_{12}$, $p_9$ |
| Referee 10 | | $p_2$, $p_{11}$ | | | $p_6$, $p_1$ | |
| Referee 11 | | $p_2$, $p_{11}$ | | | | $p_{12}$, $p_9$ |
| Referee 12 | | | $p_7$, $p_3$ | | $p_6$, $p_1$ | |
| Referee 13 | | | | $p_5$, $p_{10}$ | $p_6$, $p_1$ | |
| Referee 14 | | | $p_7$, $p_3$ | | | $p_{12}$, $p_9$ |
| Referee 15 | | | | $p_5$, $p_{10}$ | | $p_{12}$, $p_9$ |

Table 5.2: Assignment of 12 proposals to 15 referees satisfying specialty constrains.

# Chapter 6

# Conclusion and Future Work

We have explored the referee complexity of covering all pairs of $n$ proposals. A lower bound on the referee complexity of covering all pairs of $n$ proposals has been derived for any $n \geq 2$, and this lower bound has been strengthened for referee capacities, $n/2$, $n/3$, and $n/4$. Explicit assignments which are asymptotically optimal with respect to the derived lower bounds have been given for proposals with and without specialty classifications. Table 6.1 lists the number of referees facilitated by these assignments and their simple extensions for typical panel sizes used in peer-review systems. The numbers in parentheses denote the minimum number of referees required by the lower bound in Equation 3.1, except for the cases when $k = n/2$, $n/3$, $n/4$, $n = 20$ and $k = 15$, and $n = 30$ and $k = 20$. The lower bounds for the former cases are derived from Theorems 2, 3, and 4. In the latter two cases, assigning two referees 15 (or 20) proposals each leaves 10 (or 20) proposals unpaired as illustrated below for $n = 15$.

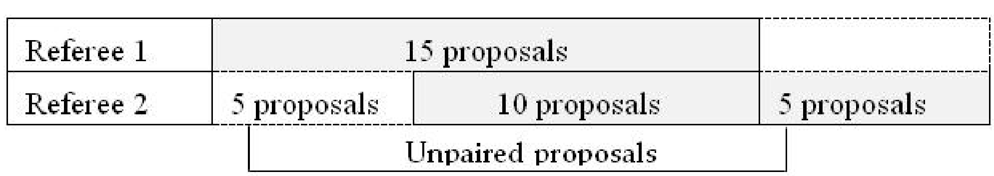


Figure 6.1: Illustration of assigning two referees 15 proposals each leaves 10 proposals unpaired.

Therefore, at least three referees are needed, and adding a third referee is sufficient to cover the missing pairs of proposals. The upper bounds are derived from Theorems 5, 6, 7, and 8. The shaded entries indicate the optimal assignments. The lower and upper bounds on the lower left are both unreasonably large and this is due to the fact that $k$ is very small compared to $n$. The upper bounds in this case are computed using Theorem 8 and both lower and upper bounds tend to $O(n^2)$ as $k$ tends to $O(1)$. On the other hand, as $k$ tends to $O(n)$, the lower and upper bounds both tend to $O(1)$. In particular, when $k = n/2$, the lower and upper bounds become 5 and 6, when $k = n/3$, they become 10 and 15, and when k $= n/4$, they become 17 and 28. Figure 6.2 depicts the lower and upper bounds based on the formulas $n(n\text{-}1)/k(k\text{-}1)$ and $n(2n\text{-}k)/k^2$ for $n = 50$ and $2 \leq k \leq 50$. It can be shown that the ratio of the upper bound to the lower bound reaches a maximum when $k = \sqrt{2n}$ for any $n$. It remains open if the lower and upper bounds can be made any closer, especially, for values of $k$ in the neighborhood of $\sqrt{2n}$.

| Number of proposals($n$) | Referee capacity( $k$) 5 | 10 | 15 | 20 |
|---|---|---|---|---|
| 20 | 20(19) | 6(6) | 3(3) | 1(1) |
| 30 | 66(44) | 12(11) | 6(6) | 3(3) |
| 40 | 120(78) | 20(18) | 10(8) | 6(6) |
| 50 | 190(123) | 45(28) | 17(12) | 10(7) |

Table 6.1: Minimum and maximum numbers of referees to cover all pairs of proposals in typical proposal panels.

Even though our results have been presented for assignments of proposals to referees, they can directly be applied to other assignment problems with similar constraints.

The results of this thesis may be extended in a number of directions. One such direction is to study the referee complexity when the referee capacity exceeds $n/2$. Another direction would be to obtain more direct lower and upper bound for assignments with referee specialty constraints. More precisely, the following variant of Cook et al [7], problem remains to be solved: Consider a set of $m$ referees with their specialties specified on a set of $n$ proposals by an incident matrix $M_{m \times n}$. The original problem addressed in Cook [7] seeks to find an assignment of referees to proposals so as to cover all pairs of proposals under a

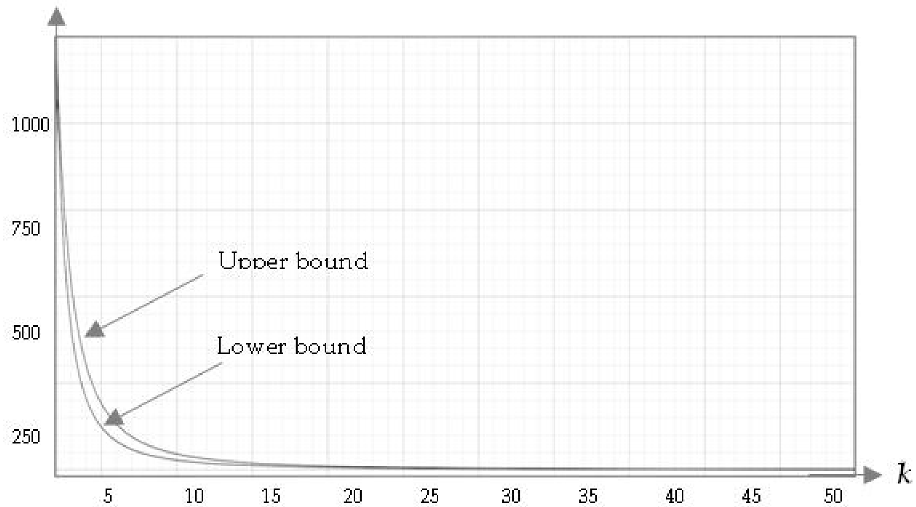


Figure 6.2: Lower and upper bounds for $n = 50$ and $2 \leq k \leq 50$.

capacity constraint of $k$ proposals. An equally interesting problem would be to determine the minimum number of referees to obtain such an assignment with the specified capacity constraint of $k$. In this setting, we wish to minimize the number of referees without violating the capacity constraints of referees. A more general version of the problem would be to allow the referees to have different capacity constraints.

# Bibliography


[1] A. Y. Oruç, N. Yetiş, and O. Cebeci, Effective evaluation and funding of research projects, *Peer review: Its present and future state*, Prague, pp. 74–84, 2006.

[2] A. Y. Oruç, A. Atmaca, Asymptotically Optimal Assignments in Ordinal Evaluations of Proposals, submitted to *IEEE Transactions on Systems, Man, and Cybernetics Part A: Systems and Humans.*

[3] B. S. Ahn and K. S. Park, Comparing methods for multiattribute decision making with ordinal weights, *Computers and Operations Research*, Vol. 35, No. 5, pp. 1660–1670, 2008.

[4] L. Bornmann, M. Ruediger, D. Hans-Dieter, Row-column (RC) association model applied to grant peer review, *Scientometrics*, Vol. 73, No. 2, pp. 139–147, 2007.

[5] Y.L. Chen and Li-C. Cheng, Mining maximum consensus sequences from group ranking data, *European Journal of Operational Research*, Vol. 198, No. 1, pp. 241–251, 2009.

[6] D.V. Cicchetti, Peer review for manuscript and grants submissions: A cross-disciplinary investigation, *Behavioral and Brain Sciences*, Vol. 14, No. 1, pp. 119–186, 1991.

[7] W.D. Cook, B. Golany, M. Kress, M. Penn, and T. Raviv, Optimal allocation of proposals to reviewers to facilitate effective ranking, *Management Science*, Vol. 51, No. 4, pp. 655–661, 2005.

[8] W.D. Cook, Distance-based and ad hoc consensus models in ordinal preference ranking, *European Journal of Operational Research*, Vol. 172, No. 2, pp. 369–385, 2006.

[9] W. D. Cook, B. Golany, M. Kress, M. Penn, and T. Raviv, Creating a consensus ranking of proposals from reviewers' partial ordinal rankings, *Computers and Operations Research*, Vol. 34, No. 4, pp. 954–965, 2007.

[10] D. S. Hochbaum and A. Levin, Methodologies and algorithms for group rankings decision, *Management Science*, Vol. 52, No. 9, pp. 1394–1408, 2006.

[11] C. Hodgson, How reliable is peer review? An examination of operating grant proposals simultaneously submitted to two similar peer review systems, *Journal of Clinical Epidemiology*, Vol. 50, No. 11, pp. 1189–1195, 1997.

[12] C. Hodgson, Evaluation of cardiovascular grant-in-aid applications by peer review-Influence of internal and external reviewers and committees, *Canadian J. Cardiology*, Vol. 11, No. 10, pp. 864–868, 1995.

[13] N. S. Iyer, A family of dominance rules for multiattribute decision making under uncertainty, *IEEE Transactions on Systems, Man, and Cybernetics Part A: Systems and Humans*, Vol. 33, No. 4, pp. 441–450, 2003.

[14] L. Langfeldt, Expert panels evaluating research: decision making and sources of bias, *Research Education*, Vol. 13, No. 1, pp. 51–62, 2004.

[15] L. Langfeldt, The decision-making constraints and processes of grant peer review, and their effects on the review outcomes, *Social Stud. Sci.*, Vol. 31, No. 6, pp. 820–841, 2001.

[16] B. Malakooti, Ranking and screening multiple criteria alternatives with partial information and use of ordinal and cardinal strength of preferences, *IEEE Transactions on Systems, Man, and Cybernetics Part A: Systems and Humans*, Vol. 30, No. 3, pp. 355–568, 2000.

[17] K.S. Park, S.H. Kim, and W.C. Yoon, Establishing strict dominance between alternatives with special type of incomplete information, *European Journal of Operational Research*, Vol. 96, No. 2, pp. 398–406, 1997.

[18] P. Sarabando and L.C. Dias, Multiattribute choice with ordinal Information: A comparison of different decision rules, *IEEE Transactions on Systems, Man, and Cybernetics Part A: Systems and Humans*, Vol. 39, No. 3, pp. 545–554, 2009.

[19] S. Wessely, Peer review of grant applications: what do we know?, *The Lancet*, Vol. 352, No.9124, pp. 301–305, 1998.

[20] L. Dirk, A measure of originality: The elements of science source, *Social Stud. Sci.*, Vol. 29, No. 5, pp. 765–776, 1999.

[21] J. M. Campanario, Peer review for journals as it stands today, *Science Communication*, Vol. 19, No. 3, pp. 181–211, 1998.

[22] U. W. Jayasinghe, H. W. Marsh, N. Bond, Peer review in the funding of research in higher education: The Australian experience, *Ed. Eval. Policy Anal.*, Vol. 23, No. 4, pp. 343–364, 2001.

[23] Charles F. Laywine and Gary L. Mullen, Discrete Mathematics Using Latin Squares, *A Wiley Interscience Publication*, 1998.

# Appendix

**Lemma 1** *Let f(w,x,y,z) = wy + wz + xz + yz.*

*a) Under the constraint* $w + x + y + z = k$*, the maximum value of f(w,x,y,z) occurs at* $x = 0$*,* $w = y = z = k/3$ *and is equal to* $k^2/3$*.*

*b) For any fixed z, and under the constraint* $w + x + y + z = k$*, the maximum value of f(w,x,y,z) occurs when* $w = y$*, and* $x = 0$*.*

**Proof**

a) Rearranging the terms in $f(w, x, y, z)$, we have $f(w, x, y, z) = wy + (w + x + y)z$ and since the second term $w + x + y$ can be increased arbitrarily by increasing $w$ and/or $y$ while also increasing the first term, setting $x = 0$ maximizes the value of $f(w, x, y, z)$. Now to find the maximum value of the function $f(w, 0, y, z) = wy + wz + yz$ under the constraint $w + y + z = k$, it is sufficient to note that $f(w, 0, y, z)$ is a symmetric function of $w$, $y$, and $z$, and therefore has its maximum when[1] $w = y = z = k/3$, and $f(k/3, 0, k/3, k/3) = k^2/3$. Given that any value of $x$ other than 0 makes the product $wy$ less than $k^2/9$, at any global maximum of $f(w, x, y, z)$, $x$ must be 0. Similarly, since $f(w, 0, y, z)$ is symmetric, any values of $w$, $y$, and $z$ other than $k/3$ should make $f(w, 0, y, z)$ strictly less than $k^2/9$. Therefore, $f(w, x, y, z)$ has a unique maximum at $x = 0$, $w = y = z = k/3$.

[1] If $k$ is not evenly divisible by 3, the maximum occurs at either $w = (k-1)/3 + 1$, $y = (k-1)/3$, $z = (k-1)/3$, or $w = (k-2)/3 + 1$, $y = (k-2)/3 + 1$, $z = (k-1)/3$ up to a permutation of $w$, $y$, and $z$. Direct substitution of $w$, $y$, and $z$ into $f(w, 0, y, z)$ in each case shows that the maximum is $(k^2 - 1)/3$, and therefore, cannot exceed $k^2/3$.

b) Using the same argument as in (**a**), for any $w$, $y$, and $z$, the maximum value of $f(w, x, y, z)$ must occur when $x = 0$. Then, for any fixed $z$, the constraint equation reduces to $w + y = k - z$. We can now determine the maximum value of $f(w, 0, y, z)$ by setting up the Lagrangian,

$$L(w, y) = f(w, y) - \lambda(k - z - w - y)$$

and examining its derivatives with respect to $w$, $y$, and $\lambda$. This reveals that $f(w, 0, y, z)$ assumes its maximum when $w = y = (k - z)/2$.

**Lemma 2** *Let f(w,x,y,z,s,t) = wy + wz + ws + wt + xz + xs + xt + yz + ys + yt + zs + zt + st. Under the constraint w + x + y + z + s + t = k, the maximum value of f(w,x,y,z,s,t) occurs at x = 0, w = y = z = s = t = k/5 and is equal to* $2k^2/5$.

**Proof** Rearranging the terms in $f(w, x, y, z, s, t)$, we have $f(w, x, y, z, s, t) = wy + (w + x + y)(z + s + t) + (s + t)z + st$ and by using the same approach given in Lemma 1, setting $x = 0$ maximizes the value of $f(w, x, y, z, s, t)$. Now to find the maximum value of the function $f(w, 0, y, z, s, t) = wy + (w + 0 + y)(z + s + t) + (s + t)z + st = wy + wz + ws + wt + yz + ys + yt + zs + zt + st$ under the constraint $w + y + z + s + t = k$, it is sufficient to note that $f(w, 0, y, z, s, t)$ is a symmetric function of $w$, $y$, $z$, $s$ and $t$, and therefore has its maximum when[2] $w = y = z = s = t = k/5$, and $f(k/5, 0, k/5, k/5, k/5, k/5) = 2k^2/5$. ∎

[2]If $k$ is not evenly divisible by 5, the maximum occurs at one of the following: $w = (k - 1)/5 + 1$, $y = z = s = t = (k - 1)/5$; $w = y = (k - 2)/5 + 1$, $z = s = t = (k - 2)/5$; $w = y = z = (k-3)/5+1$, $s = t = (k-3)/5$; $w = y = z = s = (k-4)/5+1$, $t = (k-4)/5$ up to a permutation of $w$, $y$, $z$, $s$, and $t$. Direct substitution of $w$, $y$, $z$, $s$, and $t$ into $f(w, 0, y, z, s, t)$ in each case shows that the maximum values are $(2k^2 - 2)/5$, $(2k^2 - 3)/5$, $(2k^2 - 3)/5$, $(2k^2 - 2)/5$ respectively and therefore, cannot exceed $2k^2/5$.