INTEGRABLE SYSTEMS ON REGULAR TIME SCALES

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By
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

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We present two approaches to unify the integrable systems. Both approaches are based on the classical R-matrix formalism. The first approach proceeds from the construction of $(1+1)$-dimensional integrable Δ-differential systems on regular time scales together with bi-Hamiltonian structures and conserved quantities. The second approach is established upon the general framework of integrable discrete systems on R and integrable dispersionless systems. We discuss the deformation quantization scheme for the dispersionless systems. We also apply the theories presented in this dissertation, to several well-known examples.

Keywords: Integrable systems, regular time scale, R-matrix formalism, bi-Hamiltonian structures, conserved quantities, dispersionless systems, deformation quantization scheme.

ÖZET

DÜZGÜN ZAMAN SKALASINDA İNTEGRE EDİLEBİLİR SİSTEMLER

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İntegre edilebilir sistemlerin birleştirilmesi için iki farklı yaklaşım sunuyoruz. Her iki yaklaşım da klasik R- matris formulasyonuna dayanmaktadır. İlk yaklaşım, $(1+1)$ boyutlu integre edilebilir Δ- türevlenebilir sistemlerin, onların ikili Hamilton yapılarının ve korunan niceliklerinin elde edilmesi üstüne kuruludur. İkinci yaklaşım ise $\mathbb{R}$ üzerinde integre edilebilir ayrık sistemlerin ve integre edilebilir dağılımsız sistemlerin genelleştirilmesidir. Dağılımsız sistemler için deformasyon kuvantumlama yöntemi ele alınmaktadır. Ayrıca bu tezde sunulan teoriler çeşitli iyi bilinen örneklere uygulanmaktadır.

Anahtar sözcükler: İntegre edilebilir sistemler, düzgün zaman skalası, R-matris formulasyonu, ikili Hamilton yapıları, korunan nicelikler, dağılımsız sistemler, deformasyon kuvantumlama yöntemi.

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Contents

Chapter 1

Introduction

The theory of integrable systems attracted the attention of many mathematicians and physicists ranging from group theory, topology, algebraic geometry to quantum theory, plasma physics, string theory and applied hydrodynamics. An integrable system of nonlinear partial differential or difference-differential equations arises as a member of an infinite hierarchy. Each member of the hierarchy generates a commuting flow. Additionally, if we transform a solution of the system along a commuting flow, we obtain another solution, which signifies that the equations in the hierarchy are symmetries of the system. Consequently, what we mean by an integrable system is a system of nonlinear partial differential or difference-differential equations which has an infinite-hierarchy of mutually commuting symmetries.

The theory of soliton equations, namely integrable nonlinear evolution equations was initiated in 1895, by Korteweg and de Vries [1] who derived the KdV equation describing the propagation of waves on the surface of a shallow channel. The main core of the theory was created in 1967 in the pioneering article by Gardner, Greene, Kruskal and Miura [2] where the method of inverse scattering transform was introduced. In 1968 Lax [3] and in 1971 Zakharov and Shabat [4] contributed the theory by introducing the Lax pair of KdV and nonlinear Schrödinger equations, respectively. To get rid of the difficulties appearing in the method of Lax, in 1974 Ablowitz, Kaup, Newell and Segur [5] developed an alternative approach called as AKNS scheme, including a wide range of solvable nonlinear evolution equations such as Sine-Gordon and modified KdV equations.

Integrable systems are characterized in $(1+1)$ dimensions, where one of the dimensions stands for the evolution (time) variable and the other one denotes the space variable. The space variable is usually considered on continuous intervals, or both on integer values and on real numbers or on q-numbers. Depending on the space variable, integrable systems are classified as continuous (field) soliton systems, lattice soliton systems and q-discrete soliton systems. The study of continuous soliton systems was initiated from the pioneering article [6] by Gelfand and Dickey. In this article, the authors constructed the soliton systems of KdV type by the use of the so-called R-matrix formalism. This formalism is one of the most powerful and systematic method to construct integrable systems including not only continuous, lattice, q-discrete soliton systems but also dispersionless (or equivalently hydrodynamic) ones. The idea of creating R-matrices is based on decomposition of a given Lie algebra into two Lie subalgebras. Thus, R-matrix formalism allows to produce integrable systems from the Lax equations on appropriate Lie algebras. Apart from the systematic construction of infinite hierarchies of mutually commuting symmetries, the most important advantage of this formalism is the construction of bi-Hamiltonian structures and conserved quantities. The concept of bi-Hamiltonian structures for integrable systems was first introduced by Magri [7], who presented an analysis to find a connection between symmetries and conserved quantities of the evolution equations. Based on the results of Gelfand and Dickey, Adler [8] showed that the considered systems of KdV type are indeed bi-Hamiltonian by using a Lie algebraic setting to describe integrable systems via their Lax representations. This celebrated scheme is now called as Adler-Gelfand-Dickey (AGD) Scheme. The abstract formalism of classical R-matrices on Lie algebras was formulated in [9, 10], which gave rise to many contributions to the theory of continuous soliton systems [11, 12, 13], lattice soliton systems [14, 15, 16, 17], q-discrete soliton systems [18, 19] and dispersionless systems [20, 21].

In order to embed the integrable systems into a more general unifying and extending framework, we establish a new theory, based on two approaches. We illustrate these two approaches in the articles [22, 23, 24, 25]. The first approach is to construct the integrable systems on regular time scales. This approach was initiated in the landmark article [22], where we extended the Gelfand-Dickey approach to obtain integrable nonlinear evolution equations on any regular time scales. The most important advantage of this approach is that it provides not only a unified approach to study on discrete intervals with uniform step size (i.e., lattice $\hbar\mathbb{Z}$), continuous intervals and discrete intervals with non-uniform step size (for instance q-numbers) but more interestingly an extended approach to study on combination of continuous and discrete intervals. Therefore, the concept of time scales can build bridges between the nonlinear evolution equations of type continuous soliton systems, lattice soliton systems and q-discrete systems. The second approach lies in constructing integrable discrete systems on $\mathbb{R}$ [25] which also unifies lattice and q-discrete soliton systems.

In Chapter 2, we give a brief review of time scale calculus. For real valued functions on any time scales, we introduce a derivative and integral notion. We collect the fundamental results concerning differentiability and integrability, crucial throughout this dissertation.

The main goal of Chapter 3, is to present a unified and generalized theory for the systematic construction of $(1+1)$-dimensional integrable Δ-differential systems on regular time scales in the frame of classical R-matrix formalism. For this purpose, we define the δ-differentiation operator and introduce the Lie algebra as an algebra of δ-pseudo-differential operators, equipped with the usual commutator. We observe that, the algebra of δ-pseudo-differential operators turns out to be the algebra of usual pseudo-differential operators in the continuous time scale. Next, we examine the general classes of admissible Lax operators generating consistent Lax hierarchies. We explain the constraints naturally appear between the dynamical fields of finite-field restrictions of Lax operators, which were first observed in [22]. Since generating an infinite hierarchy of symmetries proceeds by applying a recursion operator successively to an initial symmetry, we formulate the construction of recursion operators for Δ-differential systems based on the scheme of [26, 27]. We end up this chapter with illustrations of infinite-field and finite-field integrable hierarchies on regular time scales. The theory and the illustrations presented in this chapter are based on the article [23].

In Chapter 4, we benefit from the R-matrix formalism to present bi-Hamiltonian structures for Δ-differential integrable systems on regular time scales for the first time [24] in the literature. The main result of this chapter, is to establish an appropriate trace form which is well-defined on an arbitrary time scale. More impressively, this trace form unifies and generalizes the trace forms being studied in the literature such as trace forms of algebra of pseudo-differential operators, algebra of shift operators or q-discrete numbers. One of the significant features of integrable systems is having infinitely many mutually commuting symmetries and also infinitely many conserved quantities. For this reason, we construct the Hamiltonians in terms of the trace form and derive the linear Poisson tensors. The construction of the quadratic Poisson tensors is performed by the use of the recursion operators presented in Chapter 3. We state the hereditariness of the recursion operators which assures that both linear and quadratic Poisson tensors are compatible. Finally, we illustrate the theory by bi-Hamiltonian formulation of the two finite-field integrable hierarchies given in Chapter 3, in order to be self-consistent.

Another unifying approach for integrable systems is to formulate different types of discrete dynamics on continuous line. In Chapter 5, a general theory of integrable discrete systems on $\mathbb{R}$ is presented such that it contains lattice soliton systems as well as q-discrete systems as particular cases. The main structure of the theory is hidden in introducing the regular grain structures by one-parameter group of diffeomorphisms in terms of which shift operators are defined. Having introduced one parameter group of diffeomorphisms determined by shift operators, we constitute the algebra of shift operators. Accordingly, the construction of integrable discrete systems on $\mathbb{R}$ follows from the scheme of classical R-matrix formalism and it is parallel to the construction of lattice soliton systems. As illustration, we construct two integrable hierarchies of discrete chains which are counterparts of the original infinite-field Toda and modified Toda chains together with their bi-Hamiltonian structures. We end up this section by presenting the concept of continuous limit. We choose the class of discrete systems in such a way that as the limit of diffeomorphism parameter tends to 0, we obtain the dispersionless systems.

In the last Chapter, a systematic construction of integrable dispersionless systems is presented based on the classical R-matrix approach applied to a commutative Lie algebra equipped with a modified Poisson bracket. We accomplish that the dispersionless systems together with their bi-Hamiltonian structures are continuous (dispersionless) limits of discrete systems derived in previous chapter. One of the most important results, is stating the inverse problem to the dispersionless limit, which is based on the deformation quantization scheme. This scheme enables us to deduce that the quantized algebra is isomorphic to the algebra of shift operators. As a result, we proved that there is a gauge equivalence between integrable discrete systems and their dispersive counterparts of dispersionless systems. We refer to the article [25], for the integrable discrete systems on $\mathbb{R}$, the integrable dispersionless systems and for their correspondence, presented in the last two chapters.

Chapter 2

Time Scale Calculus

The time scales calculus was initiated by Aulbach and Hilger [28], [29] in order to create a theory that can unify and extend differential, difference and q-calculus. What is mentioned as a time scale $\mathbb{T}$, is an arbitrary nonempty closed subset of real numbers. Thus, the real numbers ($\mathbb{R}$), the integers ($\mathbb{Z}$), the natural numbers ($\mathbb{N}$), the non-negative integers ($\mathbb{N}_0$), the h-numbers ($h\mathbb{Z} = \{\hbar k : k \in \mathbb{Z}\}$, where $\hbar > 0$ is a fixed real number), and the q-numbers ($\mathbb{K}_q = q^{\mathbb{Z}} \cup \{0\} \equiv \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, where $q \neq 1$ is a fixed real number), $[0,1] \cup [2,3]$, $[0,1] \cup \mathbb{N}$, and the Cantor set are examples of time scales. However $\mathbb{Q}$, $\mathbb{R} - \mathbb{Q}$ and open intervals are not time scales. Besides unifying discrete intervals with uniform step size (i.e. lattice $\hbar\mathbb{Z}$), continuous intervals and discrete intervals with non-uniform step size (for instance q-numbers $\mathbb{K}_q$), the crucial point of time scales is extending combination of continuous and discrete intervals which are called as *mixed time scales* in the literature.

In [28], [29] Aulbach and Hilger introduced also *dynamic equations* on time scales in order to unify and extend the theory of ordinary differential equations, difference equations, and quantum equations [30] (h-difference and q -difference equations are based on h-calculus and q-calculus, respectively). The existence, uniqueness and properties of the solutions of dynamic equations have become of increasing interest [31, 32]. One of the main contributions to the theory of differential equations is handled by Ahlbrand and Morian [33] who introduced partial differential equations on time scales. Next, Agarwall and O'Regan [34] carried some well-known differential inequalities to time scales to improve the theory. The concept of time scales is utilized not only in dynamic or partial differential equations but it is spread also to other disciplines of mathematics ranging from algebra, topology, geometry to applied mathematics [35, 36, 37, 38].

Throughout this work, we assume that a time scale has the standard topology inherited from real numbers.

2.1 Preliminaries

In this section, we give a brief introduction to the concept of time scales related to our purpose. We refer to the textbooks by Bohner and Peterson [39, 40] for the general theory of time scales.

In order to define the derivative on time scales, which is called as *delta derivative*, we need the following *forward* and *backward jump operators* introduced as follows.

Definition 2.1.1 *For $x \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\sigma(x) = \inf\{y \in \mathbb{T} : y > x\}, \tag{2.1}$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(x) = \sup\{y \in \mathbb{T} : y < x\}. \tag{2.2}$$

Since $\mathbb{T}$ is a closed subset of $\mathbb{R}$, for all $x \in \mathbb{T}$, clearly $\sigma(x), \rho(x) \in \mathbb{T}$.

In this definition, we set in addition $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$, and $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$.

Definition 2.1.2 *The jump operators σ and ρ allow the classification of points $x \in \mathbb{T}$ in the following way: x is called right dense, right scattered, left dense, left scattered, dense and isolated if $\sigma(x) = x$, $\sigma(x) > x$, $\rho(x) = x$, $\rho(x) < x$, $\sigma(x) =$*

$\rho(x) = x$ and $\rho(x) < x < \sigma(x)$, respectively. Moreover, we define the graininess functions μ, $\nu : \mathbb{T} \to [0, \infty)$ as follows

$$\mu(x) = \sigma(x) - x, \quad \nu(x) = x - \rho(x), \quad \text{for all } x \in \mathbb{T}. \tag{2.3}$$

In literature, $\mathbb{T}^\kappa$ denotes Hilger's above truncated set consisting of $\mathbb{T}$ except for a possible left-scattered maximal point while $\mathbb{T}_\kappa$ stands for the below truncated set consisting of points of $\mathbb{T}$ except for a possible right-scattered minimal point.

Definition 2.1.3 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function on a time scale $\mathbb{T}$. For $x \in \mathbb{T}^\kappa$, delta derivative of f, denoted by Δf, is defined as*

$$\Delta f(x) = \lim_{s \to x} \frac{f(\sigma(x)) - f(s)}{\sigma(x) - s}, \qquad s \in \mathbb{T}, \tag{2.4}$$

while for $x \in \mathbb{T}_\kappa$, ∇-derivative of f, denoted by ∇f, is defined as

$$\nabla f(x) = \lim_{s \to x} \frac{f(s) - f(\rho(x))}{s - \rho(x)}, \qquad s \in \mathbb{T}, \tag{2.5}$$

provided that the limits exist. A function $f : \mathbb{T} \to \mathbb{R}$ is called Δ-smooth (∇-smooth) if it is infinitely Δ-differentiable (∇-differentiable).

Similar analogue to calculus is stated in the theorems below.

Theorem 2.1.4 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $x \in \mathbb{T}^\kappa$. Then we have the following:*

(i) If f is Δ-differentiable at x, then f is continuous at x.

(ii) If f is continuous at x and x is right-scattered, then f is Δ-differentiable at x with

$$\Delta f(x) = \frac{f(\sigma(x)) - f(x)}{\mu(x)}. \tag{2.6}$$

(iii) If x is right-dense, then f is Δ-differentiable at x if and only if the limit

$$\lim_{s \to x} \frac{f(x) - f(s)}{x - s} \tag{2.7}$$

exists. In this case, $\Delta f(x)$ is equal to this limit.

(iv) If f is Δ-differentiable at x, then

$$f(\sigma(x)) = f(x) + \mu(x)\Delta f(x). \tag{2.8}$$

Note that, if $x \in \mathbb{T}$ is right-dense, then $\mu(x) = 0$ and the relation (2.8) is trivially satisfied. Otherwise, (2.8) follows from (ii).

The following theorem is ∇ analogue of the previous one.

Theorem 2.1.5 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $x \in \mathbb{T}_\kappa$. Then we have the following:*

(i) If f is ∇-differentiable at x, then f is continuous at x.

(ii) If f is continuous at x and x is left-scattered, then f is ∇-differentiable at x with

$$\nabla f(x) = \frac{f(x) - f(\rho(x))}{\nu(x)}. \tag{2.9}$$

(iii) If x is left-dense, then f is ∇-differentiable at x if and only if the limit

$$\lim_{s \to x} \frac{f(x) - f(s)}{x - s} \tag{2.10}$$

exists. In this case, $\nabla f(x)$ is equal to this limit.

(iv) If f is ∇-differentiable at x, then

$$f(\rho(x)) = f(x) - \nu(x)\nabla f(x). \tag{2.11}$$

In order to be more precise, we clarify the definitions given up to now, for some special time scales.

Example 2.1.6 **(i)** *If $\mathbb{T} = \mathbb{R}$, then $\sigma(x) = \rho(x) = x$ and $\mu(x) = \nu(x) = 0$. Therefore Δ- and ∇-derivatives become ordinary derivative, i.e.*

$$\Delta f(x) = \nabla f(x) = \frac{df(x)}{dx}.$$

(ii) *If* $\mathbb{T} = \hbar\mathbb{Z}$*, then* $\sigma(x) = x + \hbar$*,* $\rho(x) = x - \hbar$ *and* $\mu(x) = \nu(x) = \hbar$*. Thus, it is clear that*

$$\Delta f(x) = \frac{f(x+\hbar) - f(x)}{\hbar} \quad \text{and} \quad \nabla f(x) = \frac{f(x) - f(x-\hbar)}{\hbar}.$$

(iii) *If* $\mathbb{T} = \mathbb{K}_q$*, then* $\sigma(x) = qx$*,* $\rho(x) = q^{-1}x$ *and* $\mu(x) = x(q-1)$*,* $\nu(x) = x(1-q^{-1})$*. Thus*

$$\Delta f(x) = \frac{f(qx) - f(x)}{(q-1)x} \quad \text{and} \quad \nabla f(x) = \frac{f(x) - f(q^{-1}x)}{(1-q^{-1})x},$$

for all $x \neq 0$*, and*

$$\Delta f(0) = \nabla f(0) = \lim_{s \to 0} \frac{f(s) - f(0)}{s}, \quad s \in \mathbb{K}_q,$$

provided that this limit exists.

As an important property of Δ- and ∇-differentiation on $\mathbb{T}$, we state the product rule. If $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable functions at $x \in \mathbb{T}^\kappa$, then their product is also Δ-differentiable and the following Lebniz-like rule hold

$$\begin{aligned} \Delta(fg)(x) &= g(x)\Delta f(x) + f(\sigma(x))\Delta g(x) \\ &= f(x)\Delta g(x) + g(\sigma(x))\Delta f(x). \end{aligned} \tag{2.12}$$

Also, if $f, g : \mathbb{T} \to \mathbb{R}$ are ∇-differentiable functions at $x \in \mathbb{T}_\kappa$, then so is their product fg and the following holds

$$\begin{aligned} \nabla(fg)(x) &= g(x)\nabla f(x) + f(\rho(x))\nabla g(x) \\ &= f(x)\nabla g(x) + g(\rho(x))\nabla f(x). \end{aligned} \tag{2.13}$$

Definition 2.1.7 *A time scale* $\mathbb{T}$ *is regular if both of the following two conditions are satisfied:*

$$(i) \quad \sigma(\rho(x)) = x \quad \text{for all} \quad x \in \mathbb{T} \quad \text{and} \tag{2.14}$$

$$(ii) \quad \rho(\sigma(x)) = x \quad \text{for all} \quad x \in \mathbb{T}, \tag{2.15}$$

The first condition (2.14) implies that the operator $\sigma : \mathbb{T} \to \mathbb{T}$ is surjective while the condition (2.15) implies that σ is injective. Thus σ is a bijection so it is invertible and $\sigma^{-1} = \rho$. Similarly, the operator $\rho : \mathbb{T} \to \mathbb{T}$ is invertible and $\rho^{-1} = \sigma$ if $\mathbb{T}$ is regular.

Set $x_* = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$, and set $x_* = -\infty$ otherwise. Also set $x^* = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$, and set $x^* = \infty$ otherwise.

Proposition 2.1.8 [22] *A time scale $\mathbb{T}$ is regular if and only if the following two conditions hold simultaneously*

(i) the point $x_* = \min \mathbb{T}$ is right dense and the point $x^* = \max \mathbb{T}$ is left-dense;

(ii) each point of $\mathbb{T} \setminus \{x_*, x^*\}$ is either two-sided dense or two-sided scattered.

In particular, $\mathbb{R}, \hbar\mathbb{Z}$ $(\hbar \neq 0)$ and $\mathbb{K}_q$, $[0, 1]$ and $[-1, 0] \cup \{1/k : k \in \mathbb{N}\} \cup \{k/(k+1) : k \in \mathbb{N}\} \cup [1, 2]$ are regular time scale examples.

Throughout this work, we deal with regular time scales since the invertibility of the forward jump operator σ allows us to formulate the Lie algebra, the forthcoming algebra of *δ-pseudo-differential operators*, in a proper way. For this purpose, we need a delta-differentiation operator, which we denote by Δ, assigning each Δ-differentiable function $f : \mathbb{T} \to \mathbb{R}$ to its delta-derivative $\Delta(f)$, defined by

$$[\Delta(f)](x) = \Delta f(x), \quad \text{for} \quad x \in \mathbb{T}^\kappa. \tag{2.16}$$

Furthermore, we define the *shift operator* E by means of the forward jump operator σ as follows

$$(Ef)(x) := f(\sigma(x)), \qquad x \in \mathbb{T}. \tag{2.17}$$

Since σ is invertible, it is possible to formulate the inverse E^{-1} of the shift operator E as

$$(E^{-1} f)(x) = f(\sigma^{-1}(x)) = f(\rho(x)), \tag{2.18}$$

for all $x \in \mathbb{T}$. Note that E^{-1} exists only in the case of regular time scales and in general E and E^{-1} do not commute with Δ and ∇ operators.

The following proposition states the relationship between the Δ- and ∇-derivatives.

Proposition 2.1.9 [32] *Let $\mathbb{T}$ be a regular time scale.*

(i) If $f : \mathbb{T} \to \mathbb{R}$ is a Δ-smooth function on $\mathbb{T}^\kappa$, then f is ∇-smooth and for all $x \in \mathbb{T}_\kappa$ the following relation holds

$$\nabla f(x) = E^{-1} \Delta f(x). \tag{2.19}$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is a ∇-smooth function on $\mathbb{T}_\kappa$, then f is Δ-smooth and for all $x \in \mathbb{T}^\kappa$

$$\Delta f(x) = E \nabla f(x). \tag{2.20}$$

Thus the properties of Δ- and ∇-smoothness for functions on regular time scales are equivalent.

We define the closed interval $[a, b]$ on an arbitrary time scale $\mathbb{T}$, by

$$[a, b] = \{x \in \mathbb{T} : a \leq x \leq b\}, \quad a, b \in \mathbb{T} \tag{2.21}$$

with $a \leq b$. Open and half-open intervals are defined accordingly. In the definitions below, we introduce the integral concept on time scales.

Definition 2.1.10 *(i) A function $F : \mathbb{T} \to \mathbb{R}$ is called a Δ-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided that $\Delta F(x) = f(x)$ holds for all x in $\mathbb{T}^\kappa$. Then we define the Δ-integral from a to b of f by*

$$\int_a^b f(x)\,\Delta\,x = F(b) - F(a) \quad \textit{for all} \;\; a, b \in \mathbb{T}. \tag{2.22}$$

(ii) A function $\bar{F} : \mathbb{T} \to \mathbb{R}$ is called a ∇-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided that $\nabla \bar{F}(x) = f(x)$ holds for all x in $\mathbb{T}_\kappa$. Then we define the ∇-integral from a to b of f by

$$\int_a^b f(x)\,\nabla\,x = \bar{F}(b) - \bar{F}(a) \quad \textit{for all} \;\; a, b \in \mathbb{T}. \tag{2.23}$$

Remark 2.1.11 *Notice that, for every continuous function f we have*

$$\int_x^{\sigma(x)} f(x)\;\Delta x = F(\sigma(x)) - F(x) = \mu(x)\Delta F(x) = \mu(x) f(x). \tag{2.24}$$

Similarly

$$\int_{\rho(x)}^{x} f(x)\ \nabla x = \nu(x) f(x). \tag{2.25}$$

Hence, it is clear that Δ- *and* ∇-*integrals are determined by local properties of a time scale.*

In particular, on a closed interval $[a, b]$ on $\mathbb{T}$, the Δ-integral (2.22) is an ordinary Riemann integral. If all the points between a and b are isolated, then $b = \sigma^n(a)$ for some $n \in \mathbb{Z}_+$ and as a straightforward consequence of (2.24), Δ-integral becomes

$$\int_a^b f(x)\ \Delta x = \sum_{i=1}^{n-1} \mu(\sigma^i(a)) f(\sigma^i(a)).$$

Similar analogue for ∇-integral can be also formulated. For mixed time scales, the integrals can be constructed by appropriate gluing of Riemann integrals and sums.

Proposition 2.1.12 *If the function* $f : \mathbb{T} \to \mathbb{R}$ *is continuous, then for all* $a, b \in \mathbb{T}$ *with* $a < b$ *we have*

$$\int_a^b f(x)\Delta\, x = \int_a^b E^{-1}(f(x))\nabla\, x \quad \textit{and} \quad \int_a^b f(x)\nabla\, x = \int_a^b E(f(x))\Delta\, x. \tag{2.26}$$

Indeed, if $F : \mathbb{T} \to \mathbb{R}$ is a Δ-antiderivative of f, then $\Delta F(x) = f(x)$ for all $x \in \mathbb{T}^\kappa$. By the use of Proposition 2.1.9, we have $E^{-1}f(x) = E^{-1}\Delta F(x) = \nabla F(x)$ for all $x \in \mathbb{T}_\kappa$, which implies that F is a ∇-antiderivative of $E^{-1}f(x)$. Therefore

$$F(b) - F(a) = \int_a^b E^{-1}(f(x))\nabla x = \int_a^b f(x)\Delta x. \tag{2.27}$$

The second part of (2.26) can be derived similarly.

If the functions $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable with continuous derivatives, then by the Leibniz-like rule (2.12) we have the following integration by parts formula,

$$\int_a^b g(x)\Delta f(x)\ \Delta x = f(x)g(x)|_a^b - \int_a^b E(f(x))\Delta g(x)\ \Delta x, \tag{2.28}$$

Furthermore, if the functions $f, g : \mathbb{T} \to \mathbb{R}$ are Δ- and ∇-differentiable with continuous derivatives, from (2.13), (2.19) and (2.20), we have additional integration by parts formulas

$$\int_a^b g(x)\nabla f(x)\nabla x = f(x)\, g(x)|_a^b - \int_a^b E^{-1}(f(x))\nabla g(x)\nabla x, \tag{2.29}$$

$$\int_a^b g(x)\Delta f(x)\Delta x = f(x)\, g(x)|_a^b - \int_a^b f(x)\, \nabla g(x)\nabla x, \tag{2.30}$$

$$\int_a^b g(x)\nabla f(x)\nabla x = f(x)\, g(x)|_a^b - \int_a^b f(x)\, \Delta g(x)\Delta x. \tag{2.31}$$

For Riemann and Lebesgue Δ-integrals on time scales, we refer [41] and [40]. The generalization of the proper integral (2.22) to the improper integral on time scale $\mathbb{T}$ is straightforward.

Definition 2.1.13 *We define Δ-integral over an whole time scale $\mathbb{T}$ by*

$$\int_{\mathbb{T}} f(x)\ \Delta x := \int_{x_*}^{x^*} f(x)\ \Delta x = \lim_{x \to x^*} F(x) - \lim_{x \to x_*} F(x)$$

provided that the integral converges.

Now, let us constitute the adjoint of Δ-derivative. The integration by parts formula (2.28) on the whole time scale $\mathbb{T}$, leads the following relation

$$\int_{\mathbb{T}} g\Delta(f)\Delta x = -\int_{\mathbb{T}} f\Delta E^{-1}(g)\ \Delta x =: \int_{\mathbb{T}} f\Delta^{\dagger}(g)\ \Delta x, \tag{2.32}$$

if f, g and their Δ-derivatives vanish as $x \to x_*$ or x^*. Thus, we introduce the adjoint of Δ-derivative as

$$\Delta^{\dagger} = -\Delta E^{-1}. \tag{2.33}$$

We figure out that by (2.33), it is clear

$$E^{-1} = 1 + \mu\Delta^{\dagger}. \tag{2.34}$$

We end up this chapter with the examples of Δ- and ∇-integrals for some special time scales.

Example 2.1.14 *(i) If $f : \mathbb{T} \to \mathbb{R}$ then Δ-integral and ∇-integral are nothing but the ordinary integral, i.e.*

$$\int_{\mathbb{R}} f(x)\Delta\, x = \int_{\mathbb{R}} f(x)\, \nabla x = \int_{-\infty}^{\infty} f(x)dx, \tag{2.35}$$

(ii) If $[a, b]$ consists of only isolated points, then

$$\int_a^b f(x)\Delta\, x = \sum_{x\in[a,b)} \mu(x)\, f(x) \quad \textit{and} \quad \int_a^b f(x)\nabla\, x = \sum_{x\in(a,b]} \nu(x)\, f(x). \tag{2.36}$$

In particular, if $\mathbb{T} = \hbar\mathbb{Z}$, then

$$\int_a^b f(x)\Delta\, x = \hbar \sum_{x\in[a,b)} f(x) \quad \textit{and} \int_a^b f(x)\nabla\, x = \hbar \sum_{x\in(a,b]}^{b} f(x), \tag{2.37}$$

while Δ- and ∇-integrals over the whole $\hbar\mathbb{Z}$

$$\int_{\hbar\mathbb{Z}} f(x)\Delta\, x = \hbar \sum_{x\in\hbar\mathbb{Z}} f(x) \quad \textit{and} \int_{\hbar\mathbb{Z}} f(x)\nabla\, x = \hbar \sum_{x\in\hbar\mathbb{Z}} f(x) \tag{2.38}$$

and if $\mathbb{T} = \mathbb{K}_q$, then

$$\begin{aligned} \int_{\mathbb{K}_q} f(x)\Delta\, x &= (q-1) \sum_{x\in\mathbb{K}_q} x f(x), \\ \int_{\mathbb{K}_q} f(x)\nabla\, x &= (1-q^{-1}) \sum_{x\in\mathbb{K}_q} x f(x). \end{aligned} \tag{2.39}$$

Chapter 3

Algebra of δ-pseudo-differential operators

3.1 Leibniz Rule for δ-pseudo-differential operators

In this section, we deal with the algebra of δ-pseudo-differential operators defined on a regular time scale $\mathbb{T}$. We denote the delta differentiation operator by δ instead of Δ, for convenience in the operational relations. The operator δf which is a composition of δ and f, where $f : \mathbb{T} \to \mathbb{R}$, is introduced as follows

$$\delta f := \Delta f + E(f)\delta, \quad \forall f. \tag{3.1}$$

Note that $\delta^{-1} f$ has the form of the formal series

$$\delta^{-1} f = \sum_{k=0}^{\infty} (-1)^k ((E^{-1}\Delta)^k E^{-1}) f \delta^{-k-1}, \tag{3.2}$$

which was previously given in [22], in terms of ∇. Equivalently, (3.2) can be written in terms of the adjoint of the Δ-derivative given in (2.33), as

$$\delta^{-1} f = \sum_{k=0}^{\infty} E^{-1} (\Delta^{\dagger})^k f \delta^{-k-1}. \tag{3.3}$$

Remark 3.1.1 *One can derive the following relations between the operators δ and δ^{-1} which is valid*

$$\delta f \delta^{-1} g = gE(f) + \Delta(f)\delta^{-1} g, \tag{3.4}$$
$$f\delta^{-1} g\delta = fE^{-1}(g) - f\delta^{-1}(\Delta E^{-1}(g)), \tag{3.5}$$

for all f, g.

We introduce the generalized Leibniz rule for the δ-pseudo-differential operators

$$\delta^n f = \sum_{k=0}^{\infty} S_k^n f \delta^{n-k} \qquad n \in \mathbb{Z}, \tag{3.6}$$

where

$$S_k^n = \Delta^k E^{n-k} + \ldots + E^{n-k}\Delta^k \qquad \text{for} \qquad n \geqslant k \geqslant 0,$$

is a sum of all possible strings of length n, containing exactly k times Δ and $n-k$ times E;

$$S_k^n = E^{-1}\left(\Delta^{\dagger^k} E^{n+1} + \ldots + E^{n+1}\Delta^{\dagger^k}\right) \qquad \text{for} \qquad n < 0 \quad \text{and} \quad k \geqslant 0$$

consists of the factor E^{-1} times the sum of all possible strings of length $k-n-1$, containing exactly k times $\Delta^\dagger$ and $-n-1$ times E^{-1}; in all remaining cases $S_k^n = 0$. For the structure constants S_k^n, we have the following recurrence relations

$$S_k^{n+1} = S_k^n E + S_{k-1}^n \Delta \qquad \text{for} \qquad n \geqslant 0 \tag{3.7}$$

and

$$S_k^{n-1} = \sum_{i=0}^{k} S_{k-i}^n E^{-1} \Delta^{\dagger^i} \qquad \text{for} \qquad n < 0. \tag{3.8}$$

Lemma 3.1.2 *For all $n \in \mathbb{Z}$, the relation*

$$\sum_{k \geqslant 0} (-\mu)^k S_k^n = (E - \mu\Delta)^n = 1 \tag{3.9}$$

holds.

Proof. We verify the Lemma 3.1.2 by the use of induction. For this purpose, we consider the positive and negative cases of n separately. By (2.6) and (2.34), we have

$$E - \mu\Delta = E^{-1} - \mu\Delta^{\dagger} = 1.$$

Case $n \geqslant 0$: Now, assume that (3.9) holds for positive n. If we start with expanding $(E-\mu\Delta)^{n+1}$, we have

$$\begin{aligned}(E-\mu\Delta)^{n+1} &= (E-\mu\Delta)^n(E-\mu\Delta)\\ &= (E-\mu\Delta)^n E - \mu(E-\mu\Delta)^n\Delta\\ &= \sum_{k=0}^{n}(-\mu)^k S_k^n E + \sum_{k=0}^{n}(-\mu)^{k+1} S_k^n \Delta\end{aligned}$$

Since $S_{n+1}^n = S_{-1}^n = 0$ and by the use of the recurrence relation (3.7), we have

$$\begin{aligned}(E-\mu\Delta)^{n+1} &= \sum_{k=0}^{n+1}(-\mu)^k S_k^n E + \sum_{k=0}^{n+1}(-\mu)^k S_{k-1}^n \Delta\\ &= \sum_{k=0}^{n+1}(-\mu)^k \left(S_k^n E + S_{k-1}^n\Delta\right) = \sum_{k=0}^{n+1}(-\mu)^k S_k^{n+1} = 1.\end{aligned}$$

Case $n < 0$: First, we show (3.7) for $n = -1$. Thus, using the recursive substitution, we have

$$\begin{aligned}(E-\mu\Delta)^{-1} &= \left(E^{-1}-\mu\Delta^{\dagger}\right)(E-\mu\Delta)^{-1} = E^{-1} - \mu(E-\mu\Delta)^{-1}\Delta^{\dagger}\\ &= E^{-1} - \mu\left(E^{-1} - \mu(E-\mu\Delta)^{-1}\Delta^{\dagger}\right)\Delta^{\dagger}\\ &= E^{-1} - \mu E^{-1}\Delta^{\dagger} + \mu^2(E-\mu\Delta)^{-1}{\Delta^{\dagger}}^2\\ &= E^{-1} - \mu E^{-1}\Delta^{\dagger} + \mu^2 E^{-1}{\Delta^{\dagger}}^2 - \mu^3 E^{-1}{\Delta^{\dagger}}^3 + \dots\\ &= \sum_{k=0}^{\infty}(-\mu)^k E^{-1}{\Delta^{\dagger}}^k = \sum_{k=0}^{\infty}(-\mu)^k S_k^{-1}.\end{aligned}$$

Assume that (3.9) holds for negative n. Then, using the recurrence relation (3.8), we have

$$\begin{aligned}(E-\mu\Delta)^{n-1} &= (E-\mu\Delta)^n(E-\mu\Delta)^{-1}\\ &= \sum_{k=0}^{\infty}(-\mu)^k S_k^n \sum_{i=0}^{\infty}(-\mu)^i S_i^{-1}\\ &= \sum_{k=0}^{\infty}(-\mu)^k S_k^n \sum_{i=0}^{\infty}(-\mu)^i E^{-1}{\Delta^{\dagger}}^i\end{aligned}$$

Playing with indices we obtain the desired result

$$(E-\mu\Delta)^{n-1} = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}(-\mu)^{k+i}S_k^n E^{-1}\Delta^{\dagger^i} = \sum_{k=0}^{\infty}\sum_{i=0}^{k}(-\mu)^k S_{k-i}^n E^{-1}\Delta^{\dagger^i}$$
$$= \sum_{k=0}^{\infty}(-\mu)^k \sum_{i=0}^{k} S_{k-i}^n E^{-1}\Delta^{\dagger^i} = \sum_{k=0}^{\infty}(-\mu)^k S_k^{n-1} = 1.$$

Hence (3.9) holds for $n-1$, which finishes the proof. □

In order to investigate the generalized Leibniz rule for some special cases, it is better to divide the discussion into two cases when $\mu(x)=0$ and when $\mu(x)\neq 0$.

Remark 3.1.3 *(i) When $x \in \mathbb{T}$ is a dense point, i.e. $\mu(x)=0$, then the generalized Leibniz rule* (3.6) *becomes*

$$\delta^n f = \sum_{k=0}^{\infty}\binom{n}{k}\Delta^k f \delta^{n-k} \qquad n \in \mathbb{Z}, \tag{3.10}$$

where $\binom{n}{k}$ is a binomial coefficient $\binom{n}{k} = \frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!}$, and particularly when x is inside of some interval then $\Delta = \partial_x$. Therefore, we recover the generalized Leibniz formula for pseudo-differential operators. One can find the converse formula for (3.10),

$$f\delta^n = \sum_{k=0}^{\infty}\delta^{n-k}\binom{n}{k}\Delta^{\dagger^k} f, \tag{3.11}$$

where the adjoint of Δ is given by (2.33).

(ii) For $x \in \mathbb{T}$ such that $\mu(x)\neq 0$, it is more convenient to deal with the operator

$$\xi := \mu\delta \tag{3.12}$$

instead of δ. By the use of (3.1), *we derive*

$$\xi f = \mu\delta f = (E-1)f + Ef\xi, \qquad \forall f,$$

and the generating rule follows as

$$\xi^n f = \sum_{k=0}^{\infty}\binom{n}{k}(E-1)^k E^{n-k} f \xi^{n-k} \qquad n \in \mathbb{Z}. \tag{3.13}$$

Here, we emphasize that the operator $A = \sum_i a_i\delta^i$ *has a unique* ξ-*representation* $A = \sum_i a_i'\xi^i$, *and there is one-to-one transformation between* a_i *and* a_i'. *Since, it is well-known that*

$$(E^m)^\dagger = E^{-m},$$

the converse formula for (3.13) *yields as*

$$\begin{aligned} f\xi^n &= \sum_{k=0}^{\infty} \xi^{n-k}\binom{n}{k}((E-1)^k E^{n-k})^\dagger f \\ &= \sum_{k=0}^{\infty} \xi^{n-k}\binom{n}{k}\left(E^{-1}-1\right)^k E^{k-n} f \end{aligned} \tag{3.14}$$

We end up this section with the explicit form of the generalized Leibniz rule, essential in our calculations, stated in the following theorem.

Theorem 3.1.4 *The explicit form of the generalized Leibniz rule* (3.6) *on regular time scales is given as follows.*

(i) For $n \geqslant 0$*:*

$$\delta^n f = \sum_{k=0}^{n} \sum_{i_1+i_2+\ldots+i_{k+1}=n-k} (\Delta^{i_{k+1}} E \Delta^{i_k} E \ldots \Delta^{i_2} E \Delta^{i_1}) f \delta^k, \tag{3.15}$$

where $i_\gamma \geqslant 0$ *for all* $\gamma = 1, 2, .., k+1$. *Here the formula includes all possible strings containing* $n-k$ *times* Δ *and* k *times* E.

(ii) For $n < 0$*:*

$$\delta^n f = \sum_{k=-n}^{\infty} \sum_{i_1+i_2+\ldots+i_{k+n+1}=k} (-1)^{k+n} (E^{-i_{k+n+1}} \Delta E^{-i_{k+n}} \Delta \ldots E^{-i_2} \Delta E^{-i_1}) f \delta^{-k}, \tag{3.16}$$

where $i_\gamma > 0$ *for all* $\gamma = 1, 2, .., k+n+1 > 0$. *Here the formula includes strings of length* $2k+2n+1$, *containing* k *times* E^{-1} *with exactly* $k+n+1$ *placement and* $k+n$ *times* Δ.

3.2 Classical R-matrix formalism

In order to construct integrable hierarchies of mutually commuting vector fields on regular time scales, we deal with a systematic method, so-called *the classical R-matrix formalism* [9, 42, 13], presented in the following scheme.

Definition 3.2.1 *[44] A Lie algebra $\mathcal{G}$ is a vector space together with a bilinear operation $[\cdot,\cdot]:\mathcal{G}\times\mathcal{G}\to\mathcal{G}$, which is skew-symmetric*

$$[a,b]=-[b,a],\qquad a,b\in\mathcal{G},\tag{3.17}$$

and satisfies the Jacobi identity

$$[[a,b],c]+[[c,a],b]+[[b,c],a]=0,\qquad a,b,c\in\mathcal{G}.\tag{3.18}$$

Based on the above definition, let $\mathcal{G}$ be an algebra, with an associative multiplication operation, over a commutative field $\mathbb{K}$ of complex or real numbers, based on an additional bilinear product given by a Lie bracket $[\cdot,\cdot]:\mathcal{G}\times\mathcal{G}\to\mathcal{G}$, which is skew-symmetric and satisfies the Jacobi identity.

Definition 3.2.2 *A linear map $R:\mathcal{G}\to\mathcal{G}$ such that the bracket*

$$[a,b]_R:=[Ra,b]+[a,Rb],\tag{3.19}$$

is a second Lie bracket on $\mathcal{G}$, is called the classical R-matrix.

The bracket (3.19) is clearly skew-symmetric. When it comes to discuss the Jacobi identity for (3.19), one finds that

$$\begin{aligned}0=[a,[b,c]_R]_R+\text{cyclic}&=[Ra,[Rb,c]]+[Ra,[b,Rc]]+[a,R[b,c]_R]+\text{cyclic}\\&=[Rb,[Rc,a]]+[Rc,[a,Rb]]+[a,R[b,c]_R]+\text{cyclic}\\&=[a,R[b,c]_R-[Rb,Rc]]+\text{cyclic}\end{aligned}\tag{3.20}$$

Hence, it can be clearly deduced that a sufficient condition for R to be a classical R-matrix is

$$[Ra,Rb]-R[a,b]_R+\alpha[a,b]=0,\tag{3.21}$$

where $\alpha \in \mathbb{K}$. The condition (3.21) is called the *Yang-Baxter equation* YB(α) and there are two relevant cases for YB(α), $\alpha \neq 0$ and $\alpha = 0$. Yang-Baxter equations for $\alpha \neq 0$ are equivalent and can be reparametrized.

Additionally, we assume that the Lie bracket is a derivation of multiplication in $\mathcal{G}$, i.e. the relation

$$[a, bc] = b[a, c] + [a, b]c \qquad a, b, c \in \mathcal{G} \tag{3.22}$$

holds. If the Lie bracket is given by the *commutator*, i.e.

$$[a, b] = ab - ba, \qquad a, b \in \mathcal{G},$$

the condition (3.22) is satisfied automatically, since $\mathcal{G}$ is associative.

Proposition 3.2.3 *Let $\mathcal{G}$ be a Lie algebra fulfilling all the above assumptions and R be the classical R-matrix satisfying the Yang-Baxter equation, YB(α). Let also R commutes with derivatives with respect to these evolution parameters. Then the power functions L^n on $\mathcal{G}$, $L \in \mathcal{G}$ and $n \in \mathbb{Z}_+$, generate the so-called Lax hierarchy*

$$\frac{dL}{dt_n} = [R(L^n), L]\,, \tag{3.23}$$

of pairwise commuting vector fields on $\mathcal{G}$. Here, t_n's are related evolution parameters.

Proof. It is clear that the power functions on $\mathcal{G}$ are well defined. Then

$$\begin{aligned}
(L_{t_m})_{t_n} - (L_{t_n})_{t_m} &= [RL^m, L]_{t_n} - [RL^n, L]_{t_m} \\
&= [(RL^m)_{t_n} - (RL^n)_{t_m}, L] + [RL^m, [RL^n, L]] - [RL^n, [RL^m, L]] \\
&= [(RL^m)_{t_n} - (RL^n)_{t_m} + [RL^m, RL^n], L].
\end{aligned}$$

Hence, the vector fields (3.23) mutually commute if the so-called *zero-curvature condition* (or *Zakharov-Shabat* equation)

$$(RL^m)_{t_n} - (RL^n)_{t_m} + [RL^m, RL^n] = 0,$$

is satisfied. By the Lax hierarchy (3.23) and the Leibniz rule (3.22), we have

$$(L^m)_{t_n} = [R(L^n), L^m].$$

Since R commutes with ∂_{t_n},i.e.

$$(RL)_{t_n} = RL_{t_n},$$

and the Yang-Baxter equation holds for R, we deduce

$$\begin{aligned} R(L^m)_{t_n} - R(L^n)_{t_m} + [RL^m, RL^n] &= R[RL^n, L^m] - R[RL^m, L^n] + [RL^m, RL^n] \\ &= [RL^m, RL^n] - R[L^m, L^n]_R = -\alpha[L^m, L^n] \\ &= 0. \end{aligned}$$

Hence, zero-curvature condition is satisfied which implies that the vector fields pairwise commute. $\square$

In practice, the Lax operators in (3.23) have fractional powers. Notice that, the Yang-Baxter equation is a sufficient condition for mutual commutation of vector fields (3.23), but not necessary. Therefore, choosing the algebra $\mathcal{G}$ properly, the Lax hierarchy produces abstract integrable systems. In practice, the element L of $\mathcal{G}$ must be properly chosen, in such a way that the evolution systems (3.23) are consistent on the subspace of $\mathcal{G}$.

3.3 Classical R-matrix on regular time-scales

The theory and illustrations presented in this section and the forthcoming sections of this chapter are based on the article [23].

We introduce the Lie algebra $\mathcal{G}$ as an associative algebra of formal Laurent series of δ-pseudo-differential operators equipped with a Lie bracket given by the commutator. We define the decomposition of $\mathcal{G}$ in the following form:

$$\mathcal{G} = \mathcal{G}_{\geqslant k} \oplus \mathcal{G}_{<k} = \{\sum_{i\geqslant k} u_i(x)\delta^i\} \oplus \{\sum_{i<k} u_i(x)\delta^i\}, \tag{3.24}$$

where $u_i : \mathbb{T} \to \mathbb{K}$ are Δ-smooth functions additionally depending on the evolution parameters t_n. The subspaces $\mathcal{G}_{\geqslant k}$, $\mathcal{G}_{<k}$ are closed Lie subalgebras of $\mathcal{G}$ only if $k = 0, 1$, i.e., the above decomposition is valid only if $k = 0, 1$. We introduce the classical R-matrix as

$$R := \frac{1}{2}(P_{\geqslant k} - P_{<k}) \qquad k = 0, 1, \tag{3.25}$$

where $P_{\geqslant k}$ and $P_{<k}$ are the projections onto the Lie subalgebras $\mathcal{G}_{\geqslant k}$ and $\mathcal{G}_{<k}$, respectively such that

$$P_{\geqslant k}(A) = \sum_{i \geqslant k} a_i \delta^i, \qquad P_{<k}(A) = \sum_{i<k} a_i \delta^i \qquad \text{for} \qquad A = \sum_i a_i \delta^i \in \mathcal{G}. \tag{3.26}$$

Let $L \in \mathcal{G}$ be the Lax operator of the form

$$L = u_N \delta^N + u_{N-1}\delta^{N-1} + \ldots + u_1 \delta + u_0 + u_{-1}\delta^{-1} + \ldots, \tag{3.27}$$

The Lax hierarchy (3.23), based on the classical R-matrix (3.25), is generated by the fractional powers of the Lax operator L from the algebra of δ-pseudo-differential operators

$$\frac{dL}{dt_n} = \left[\left(L^{\frac{n}{N}}\right)_{\geqslant k}, L\right] = -\left[\left(L^{\frac{n}{N}}\right)_{<k}, L\right] \qquad k = 0, 1 \qquad n \in \mathbb{Z}_+. \tag{3.28}$$

In fact, the Lax hierarchy (3.28) is an infinite hierarchy of mutually commuting vector fields since R satisfies the sufficiency condition *Yang-Baxter equation* (3.21) for $\alpha = \frac{1}{4}$. Moreover, (3.28) represents $(1+1)$-dimensional integrable Δ-differential systems on an arbitrary regular time scale $\mathbb{T}$, involving the time variable t_n and the space variable $x \in \mathbb{T}$ for an infinite number of fields u_i.

The appropriate Lax operators which produce consistent Lax hierarchies (3.28), are given in the following form:

$$k = 0: \qquad L = c_N \delta^N + u_{N-1}\delta^{N-1} + \ldots + u_1 \delta^1 + u_0 + u_{-1}\delta^{-1} + \ldots \tag{3.29}$$

$$k = 1: \qquad L = u_N \delta^N + u_{N-1}\delta^{N-1} + \ldots + u_1 \delta^1 + u_0 + u_{-1}\delta^{-1} + \ldots, \tag{3.30}$$

where c_N is a time-independent field since in the case of $k = 0$, the derivative of the coefficient of the highest order term with respect to time vanishes. Additionally for $k = 0$, one finds that $(u_{N-1})_t = \mu(\ldots)$ and for $k = 1$, $(u_N)_t = \mu(\ldots)$(explicitly presented in the Remarks 3.5.2 and 3.5.3). Thus the fields u_{N-1} (for $k=0$), u_N (for $k=1$) are time-independent for dense points $x \in \mathbb{T}$, as at these points $\mu = 0$.

In order deal with extracted closed finite-field integrable Δ-differential systems on regular time scales, some finite-field restrictions should be imposed on the appropriate infinite-field Lax operators (3.29) and (3.30). The restriction is valid if the commutator on the right-hand side of the Lax equation (3.28) does not produce terms not contained in L_{t_q}. To be more precise, the left- and right-hand of (3.28) have to span the same subspace of $\mathcal{G}$. Simple computation allows to conclude with the most general form of the admissible finite-field Lax operators

$$L = u_N\delta^N + u_{N-1}\delta^{N-1} + \ldots + u_1\delta + u_0 + \delta^{-1}u_{-1} + \sum_s \psi_s\delta^{-1}\varphi_s, \tag{3.31}$$

where for $k=0$, $u_{-1}=0$ and u_N is a non-zero time-independent field, which can be denoted as c_N. Here also the sum is finite and ψ_s, φ_s are arbitrary dynamical fields for all s. When $\mathbb{T} = \mathbb{R}$, i.e in the case of the algebra of pseudo-differential operators the fields ψ_s and φ_s in (3.31) are special dynamical fields and they are so-called source terms, as ψ_s and φ_s are eigenfunctions and adjoint-eigenfunctions, respectively, of the Lax hierarchy (3.28) [12].

Note that, further admissible reductions of the Lax form (3.31) are given by for $k=0$

$$L = c_N\delta^N + u_{N-1}\delta^{N-1} + \ldots + u_1\delta + u_0. \tag{3.32}$$

and for $k=1$

$$L = u_N\delta^N + u_{N-1}\delta^{N-1} + \ldots + u_1\delta + u_0 + \delta^{-1}u_{-1} \tag{3.33}$$

$$L = u_N\delta^N + u_{N-1}\delta^{N-1} + \ldots + u_1\delta + u_0 \tag{3.34}$$

$$L = u_N\delta^N + u_{N-1}\delta^{N-1} + \ldots + u_1\delta. \tag{3.35}$$

respectively, where u_{N-1} (for $k=0$), u_N (for $k=1$) are time-independent at dense points of a time scale.

In general, for an arbitrary regular time scale $\mathbb{T}$, the Lax hierarchies (3.28) represent hierarchies of soliton-like integrable Δ-differential systems. In particular, the Lax hierarchies (3.28) are lattice and q-discrete soliton systems when $\mathbb{T} = \hbar\mathbb{Z}$ or $\mathbb{K}_q$, respectively. When $\mathbb{T} = \mathbb{R}$, i.e. the continuous time scale on the whole $\mathbb{R}$, they are of continuous soliton systems.

Moreover, in some special cases, continuous soliton systems can be obtained from the continuous limit of integrable systems on time scales. Indeed, if the deformation parameter is properly introduced, it is possible to deal with a continuous limit of a time scale. For instance, the continuous limit of $\hbar\mathbb{Z}$ is the whole real line $\mathbb{R}$, i.e.

$$\mathbb{T} = \hbar\mathbb{Z} \longrightarrow \mathbb{T} = \mathbb{R}, \qquad \text{as} \quad \hbar \to 0, \tag{3.36}$$

and the continuous limit of $\mathbb{K}_q$ is the closed half line $\mathbb{R}_+ \cup \{0\}$, i.e

$$\mathbb{T} = \mathbb{K}_q \longrightarrow \mathbb{T} = \mathbb{R}_+ \cup \{0\}, \qquad \text{as} \quad q \to 1. \tag{3.37}$$

In the case of continuous time scale, the algebra of δ-pseudo-differential operators (3.24) turns out to be the algebra of pseudo-differential operators

$$\mathcal{G} = \mathcal{G}_{\geqslant k} \oplus \mathcal{G}_{<k} = \{\sum_{i\geqslant k} u_i(x)\partial^i\} \oplus \{\sum_{i<k} u_i(x)\partial^i\}, \tag{3.38}$$

where ∂ acts as $\partial u = \partial_x u + u\partial = u_x + u\partial$. In this case, the decomposition is valid for $k = 0, 1$ and 2. However, the algebra $\mathcal{G}$ (3.24) of δ-pseudo-differential operators does not decompose into closed Lie subalgebras for $k = 2$ on an arbitrary time scale. To be more precise, the decomposition of the Lie algebra is valid when $\mathbb{T} = \mathbb{R}$, in the case of $k = 2$, while this case disappears for the rest of the time scales. Therefore, in the general theory of integrable systems on time scales, we loose one case contrary to the ordinary soliton systems constructed by the frame of pseudo-differential operators.

For appropriate Lax operators, finite field restrictions and more information about the algebra of pseudo-differential operators, we refer to [11, 12, 13, 42].

3.4 Recursion operators

One of the characteristic features of integrable systems is the existence of a recursion operator. A recursion operator [43] of a given system, is an operator such that when it acts on one symmetry of the system, it produces another symmetry, i.e.

$$\Phi(L_{t_n}) = L_{t_{n+N}}, \qquad n \in \mathbb{Z}_+.$$

Hence it allows to reconstruct the whole hierarchy (3.28) when applied to the first $(N-1)$ symmetries. Gürses *et al.* [26] presented a very efficient general method to construct recursion operators for Lax hierarchies and the authors illustrated the method on finite-field reductions of the KP hierarchy. In [27] the method was applied to the reductions of modified KP hierarchy as well as to the lattice systems. Our further considerations are based on the scheme from [26] and [27].

Lemma 3.4.1 *The recursion operator of the related Lax operator* (3.31) *is constructed by solving the recursion relation*

$$L_{t_{n+N}} = L_{t_n} L + [R, L], \tag{3.39}$$

where R is the remainder operator of the form

$$R = a_N \delta^N + a_{N-1}\delta^{N-1} + \cdots + a_0 + \sum_s a_{-1,s}\delta^{-1}\varphi_s, \tag{3.40}$$

which has the same degree as the Lax operator L (3.31)*. Here $a_N = 0$ for the case $k = 0$.*

Proof. We prove the Lemma, by the continuous analogue presented in [26]. Consider the case $k = 0$. In this case, $u_{-1} = 0$ and u_N is time-independent in the Lax operator (3.31). Since $((L^{\frac{n}{N}})_{\geqslant 0} L)_{\geqslant 0}$ has only positive powers, we have

$$\begin{aligned}
(L^{\frac{n+N}{N}})_{\geqslant 0} &= ((L^{\frac{n}{N}})_{\geqslant 0} L)_{\geqslant 0} + ((L^{\frac{n}{N}})_{<0} L)_{\geqslant 0} \\
&= (L^{\frac{n}{N}})_{\geqslant 0} L - \sum_s [(L^{\frac{n}{N}})_{\geqslant 0}\psi_s]_0 \delta^{-1}\varphi_s + ((L^{\frac{n}{N}})_{<0} L)_{\geqslant 0} \\
&= (L^{\frac{n}{N}})_{\geqslant 0} L + R,
\end{aligned}$$

where R is of order $N-1$ and we substituted $R = ((L^{\frac{n}{N}})_{<0}L)_{\geqslant 0}$, which is exactly of the form (3.40) with $a_N = 0$. Similarly for $k = 1$, we have

$$
\begin{aligned}
(L^{\frac{n+N}{N}})_{\geqslant 1} &= ((L^{\frac{n}{N}})_{\geqslant 1}L)_{\geqslant 1} + ((L^{\frac{n}{N}})_{<1}L)_{\geqslant 1} \\
&= (L^{\frac{n}{N}})_{\geqslant 1}L - [(L^{\frac{n}{N}})_{\geqslant 1}L]_0 - \sum_s [(L^{\frac{n}{N}})_{\geqslant 0}\psi_s]_0 \delta^{-1}\varphi_s + ((L^{\frac{n}{N}})_{<1}L)_{\geqslant 1} \\
&= (L^{\frac{n}{N}})_{\geqslant 1}L + R,
\end{aligned}
$$

where R has the form (3.40). Thus, in both cases (3.39) follows from (3.28). Hence we can extract the recursion operator from (3.39). $\square$

Note that in general, recursion operators on time scales are non-local., i.e., they contain non-local terms with Δ^{-1} being formal inverse of Δ operator. However, such recursion operators acting on an appropriate domain produce only local hierarchies.

3.5 Infinite-field integrable systems on time scales

In this section, we illustrate the theory of integrable Δ-differential systems on regular time scales by two-infinite field integrable hierarchies which are Δ-differential counterparts of Kadomtsev-Petviashvili (KP) and modified Kadomtsev-Petviashvili (mKP).

3.5.1 Δ-differential KP, $k = 0$:

Consider the following infinite field Lax operator

$$
L = \delta + u_0 + \sum_{i \geqslant 1} u_i \delta^{-i}, \tag{3.41}
$$

which generates the Lax hierarchy (3.28) as the Δ-differential counterpart of the KP hierarchy. For $(L)_{\geqslant 0} = \delta + u_0$, the first flow is given by

$$\begin{aligned}
\frac{du_0}{dt_1} &= \mu\Delta u_1 \\
\frac{du_i}{dt_1} &= \sum_{k=0}^{i-1}(-1)^{k+1}u_{i-k}\sum_{j_1+j_2+\ldots+j_{k+1}=i}(E^{-j_{k+1}}\Delta E^{-j_k}\Delta\ldots E^{-j_2}\Delta E^{-j_1})u_0 \qquad (3.42)\\
&\quad + \mu\Delta u_{i+1} + \Delta u_i + u_i u_0 \qquad \forall i > 0,
\end{aligned}$$

where $j_\gamma > 0$ for all $\gamma \geqslant 1$.

Similarly, by the use of $(L^2)_{\geqslant 0} = \delta^2 + \xi\delta + \eta$, where

$$\xi := Eu_0 + u_0 \qquad \eta := \Delta u_0 + u_0^2 + u_1 + Eu_1, \tag{3.43}$$

the second flow yields as

$$\begin{aligned}
\frac{du_0}{dt_2} &= \mu\Delta(E+1)u_2 + \mu\Delta(\Delta u_1 + u_1 u_0 + u_1 E^{-1}u_0) \\
\frac{du_i}{dt_2} &= \sum_{k=-1}^{i-1}(-1)^{k+2}u_{i-k}\sum_{j_1+j_2+\ldots+j_{k+2}=i+1}(E^{-j_{k+2}}\Delta E^{-j_{k+1}}\Delta\ldots E^{-j_2}\Delta E^{-j_1})\xi \\
&+ \sum_{k=0}^{i-1}(-1)^{k+1}u_{i-k}\sum_{j_1+j_2+\ldots+j_{k+1}=i}(E^{-j_{k+1}}\Delta E^{-j_k}\Delta\ldots E^{-j_2}\Delta E^{-j_1})\eta \qquad (3.44)\\
&+ \Delta^2 u_i + (E\Delta + \Delta E)u_{i+1} + \mu\Delta(E+1)u_{i+2} + \xi(\Delta u_i + Eu_{i+1}) + \eta u_i,
\end{aligned}$$

where $j_\gamma > 0$ for all $\gamma \geqslant 1$.

Example 3.5.1 *The simplest case in $(2+1)$ dimensions: We rewrite the first two members of the first flow by setting $u_0 = w$ and $t_1 = y$ and the first member of the second flow by setting $t_2 = t$. Since E and Δ do not commute, note that in the calculations up to the last step, we use $E - 1$ instead of $\mu\Delta$, in order to avoid confusion.*

$$w_y = (E-1)u_1, \tag{3.45}$$

$$u_{1,y} = (E-1)u_2 + \Delta u_1 + u_1(1 - E^{-1})(w), \tag{3.46}$$

$$w_t = (E^2-1)u_2 + (E-1)(\Delta u_1 + u_1 w + u_1 E^{-1}(w)) \tag{3.47}$$

Applying $(E+1)$ *to* (3.46) *from left we have*

$$(E^2-1)u_2 = (E+1)u_{1,y} - (E+1)\Delta u_1 - (E-1)u_1(1-E^{-1})w. \tag{3.48}$$

Applying $(E-1)$ *to* (3.47) *from left and substituting* (3.45) *and* (3.48) *into the new derived equation we finally obtain the* $(2+1)$*-dimensional one-field system of the form*

$$\mu\Delta w_t = (E+1)w_{yy} - 2\Delta w_y + 2\mu\Delta(ww_y). \tag{3.49}$$

which does not have a continuous counterpart. For the case of $\mathbb{T}=h\mathbb{Z}$*, one can show that* (3.49) *is equivalent to the* $(2+1)$*-dimensional Toda lattice system.*

The Δ-differential analogue of one-field continuous KP equation is too complicated to write it down explicitly.

Remark 3.5.2 *Here we want to illustrate the behavior of* u_0 *in all symmetries of the difference KP hierarchy. Let* $(L^n)_{<0} = \sum_{i\geqslant 1} v_i^{(n)}\delta^{-i}$*, then by the right-hand of the Lax equation* (3.28)*, we obtain the first members of all flows*

$$\frac{du_0}{dt_n} = \mu\Delta v_1^{(n)}. \tag{3.50}$$

Thus u_0 *is a time-independent field for dense points* $x\in\mathbb{T}$ *since* $\mu=0$*. Hence, in the case of* $\mathbb{T}=\mathbb{R}$*,* u_0 *appears to be a constant.*

In $\mathbb{T}=\mathbb{R}$ case, or in the continuous limit of some special time scales, with the choice $u_0=0$, the Lax operator (3.41) turns out to be a Laurent series of pseudo-differential operators

$$L = \partial + \sum_{i\geqslant 1} u_i\partial^{-i}. \tag{3.51}$$

Moreover, the first flow (3.42) turns out to be exactly the first flow of the KP system

$$\frac{du_i}{dt_1} = u_{i,x}, \qquad \forall i\geqslant 1 \tag{3.52}$$

while the second flow (3.44) becomes exactly the second flow of the KP system

$$\frac{du_i}{dt_2} = (u_i)_{2x} + 2(u_{i+1})_x + 2\sum_{k=1}^{i-1}(-1)^{k+1}\binom{i-1}{k}u_{i-k}(u_1)_{kx} \qquad \forall i\geqslant 1. \tag{3.53}$$

3.5.2 Δ-differential mKP, $k = 1$:

Consider the Lax operator of the form

$$L = u_{-1}\delta + \sum_{i\geqslant 0} u_i \delta^{-i} \tag{3.54}$$

which generates the Δ-differential counterpart of the mKP hierarchy. Then, $(L)_{\geqslant 1} = u_{-1}\delta$ implies the first flow

$$\begin{aligned}
\frac{du_{-1}}{dt_1} &= \mu u_{-1}\Delta u_0 \\
\frac{du_i}{dt_1} &= \sum_{k=-1}^{i-1}(-1)^{k+2}u_{i-k}\sum_{j_1+j_2+\cdots+j_{k+2}=i+1}(E^{-j_{k+2}}\Delta E^{-j_{k+1}}\Delta\ldots E^{-j_2}\Delta E^{-j_1})u_{-1} \\
&+ u_{-1}Eu_{i+1} + u_{-1}\Delta u_i \qquad \forall i \geqslant 0,
\end{aligned} \tag{3.55}$$

where $j_\gamma > 0$, $\gamma = 1, 2, \ldots, k+2$.

Next, for $(L^2)_{\geqslant 1} = \xi\delta^2 + \eta\delta$, where

$$\xi := u_{-1}Eu_{-1}, \qquad \eta := u_{-1}\Delta u_{-1} + u_{-1}Eu_0 + u_0u_{-1}, \tag{3.56}$$

we have the second flow as follows

$$\begin{aligned}
\frac{du_{-1}}{dt_2} &= \xi(E\Delta u_0 + E^2(u_1)) + \mu u_{-1}\Delta u_0^2 - u_1E^{-1}\xi - u_{-1}^2\Delta u_0 \\
\frac{du_i}{dt_2} &= \sum_{k=-2}^{i-1}(-1)^{k+3}u_{i-k}\sum_{j_1+j_2+\ldots+j_{k+3}=i+2}(E^{-j_{k+3}}\Delta E^{-j_{k+2}}\Delta\ldots\Delta E^{-j_1})\xi \\
&+ \sum_{k=-1}^{i-1}(-1)^{k+2}u_{i-k}\sum_{j_1+j_2+\ldots+j_{k+2}=i+1}(E^{-j_{k+2}}\Delta E^{-j_{k+1}}\Delta\ldots\Delta E^{-j_1})\eta \\
&+ \xi_2(\Delta^2u_i + (E\Delta + \Delta E)u_{i+1} + E^2u_{i+2}) + \eta(\Delta u_i + Eu_{i+1}),
\end{aligned} \tag{3.57}$$

where $i \geqslant 0$ and $j_\gamma > 0$ for all $\gamma \geqslant 1$.

Remark 3.5.3 *Similarly we illustrate the behavior of u_{-1} in all symmetries of the Δ-differential mKP hierarchy by considering $(L^n)_{<1} = \sum_{i\geqslant 0} v_i^{(n)}\delta^{-i}$. Then we obtain the first members of all flows*

$$\frac{du_{-1}}{dt_n} = \mu u_{-1}\Delta v_0^{(n)}, \tag{3.58}$$

Thus, u_{-1} is time-independent for dense $x \in \mathbb{T}$. Hence when $\mathbb{T} = \mathbb{R}$, u_{-1} appears to be a constant.

In $\mathbb{T} = \mathbb{R}$ case, or in the continuous limit of some special time scales, with the choice of $u_{-1} = 1$, the Lax operator (3.54) turns out to be the pseudo-differential operator

$$L = \partial + \sum_{i \geqslant 0} u_i \partial^{-i}, \tag{3.59}$$

Furthermore, the system of equations (3.55) is exactly the first flow of the mKP system

$$\frac{du_i}{dt_1} = u_{i,x}, \qquad \forall i \geqslant 0, \tag{3.60}$$

while the second flow (3.57) turns out to be the second flow of the mKP system

$$\begin{aligned} \frac{du_i}{dt_2} &= (u_i)_{2x} + 2(u_{i+1})_x + 2u_0(u_i)_x + 2u_0u_{i+1} \\ &+ 2\sum_{k=0}^{i} (-1)^{k+1} \binom{i}{k} u_{i+1-k}(u_0)_{kx} \qquad \forall i \geqslant 0. \end{aligned} \tag{3.61}$$

3.6 Constraints

There appear natural constraints between the dynamical fields of the admissible finite-field Lax restrictions (3.31) fulfilling the Lax hierarchy (3.28). We determine these constraints in the following theorem, which is a consequence of the property of the algebra of δ-pseudo-differential operators. The property is illustrated in the following proposition.

Proposition 3.6.1 *Let $L_1, L_2 \in \mathcal{G}$ be*

$$L_1 = \sum_{i=0}^{r} a_i \delta^{r-i}, \quad L_2 = \sum_{i=0}^{s} b_i \delta^{s-i} + \psi \delta^{-1} \varphi,$$

with

$$[L_1, L_2] = \sum_{i=0}^{r+s} C_i \delta^{r+s-i} + \hat{C}_{r+s+1} \delta^{-1} \varphi + \psi \delta^{-1} C_{r+s+1}, \tag{3.62}$$

then

$$\sum_{i=0}^{r+s}(-1)^i\mu^iC_i + (-1)^{r+s+1}\mu^{r+s+1}(\varphi\hat{C}_{r+s+1} + \psi C_{r+s+1}) = 0. \tag{3.63}$$

We verify the Proposition 3.6.1, by the use of the Lemma's stated and proved below.

Lemma 3.6.2 *Let* $\delta^r\psi\delta^{-1}\varphi = \sum_{i=0}^{r-1} C_i\delta^{r-i-1} \quad + C_r\delta^{-1}\varphi, \quad r \geqslant 0,$ *then*

$$\sum_{i=0}^{r-1}(-1)^i\mu^iC_i \quad + (-1)^r\mu^r\varphi \ \ C_r = \psi\varphi. \tag{3.64}$$

Proof. In order to prove the Lemma we make use of induction. Consider

$$\delta^{r+1}\psi\delta^{-1}\varphi = \sum_{i=0}^{r} D_i\delta^{r-i} \quad + D_{r+1}\delta^{-1}\varphi, \tag{3.65}$$

where

$$\begin{aligned}
D_0 &= E(C_0),\\
D_i &= \Delta C_{i-1} + E(C_i) = \frac{E-1}{\mu}C_{i-1} + E(C_i), \quad i = 1, 2, ..., r-1,\\
D_r &= \Delta C_{r-1} + \varphi E(C_r) = \frac{E-1}{\mu}C_{r-1} + \varphi E(C_r),\\
D_{r+1} &= \Delta C_r = \frac{E-1}{\mu}C_r.
\end{aligned}$$

Next, we consider

$$\begin{aligned}
\sum_{i=0}^{r}(-1)^i\mu^iD_i + (-1)^{r+1}\mu^{r+1}\varphi \ \ D_{r+1} &= D_0 + \sum_{i=1}^{r-1}(-1)^i\mu^iD_i\\
&+ (-1)^r\mu^rD_r + (-1)^{r+1}\mu^{r+1}\varphi D_{r+1}\\
&= E(C_0) + \sum_{i=1}^{r-1}(-1)^i\mu^i(\frac{E-1}{\mu}C_{i-1} + E(C_i))\\
&+ (-1)^r\mu^r(\frac{E-1}{\mu}C_{r-1} + \varphi E(C_r))\\
&+ (-1)^{r+1}\mu^{r+1}\varphi(\frac{E-1}{\mu}C_r)
\end{aligned}$$

$$\begin{aligned}
\sum_{i=0}^{r}(-1)^i\mu^iD_i+(-1)^{r+1}\mu^{r+1}\varphi\ D_{r+1} &= \sum_{i=0}^{r-1}(-1)^i\mu^iE(C_i)\\
&+ \sum_{i=1}^{r}(-1)^i\mu^{i-1}(E-1)C_{i-1}\\
&+ (-1)^r\mu^r\varphi\ C_r
\end{aligned}$$

Thus

$$\sum_{i=0}^{r}(-1)^i\mu^iD_i+(-1)^{r+1}\mu^{r+1}\varphi\ D_{r+1} = \sum_{i=0}^{r-1}(-1)^i\mu^iC_i \quad +(-1)^r\mu^r\varphi\ C_r=\psi\varphi.$$

□

Lemma 3.6.3 *Assume*

$$[\delta^r,\psi\delta^{-1}\varphi]=\sum_{i=0}^{r-1}C_i\delta^{r-i-1}+\hat{C}_r\delta^{-1}\varphi+\psi\delta^{-1}C_r,\quad r\geqslant 0. \tag{3.66}$$

Then

$$\sum_{i=0}^{r-1}(-1)^i\mu^iC_i+(-1)^r\mu^r(\varphi\hat{C}_r+\psi C_r)=0. \tag{3.67}$$

Proof. Similarly, we use induction. The assumption hypothesis of Lemma 3.6.3 implies

$$[\delta^{r+1},\psi\delta^{-1}\varphi]=\sum_{i=0}^{r}F_i\delta^{r-i} \quad +\hat{F}_{r+1}\delta^{-1}\varphi+\psi\delta^{-1}F_{r+1}, \tag{3.68}$$

By (3.65) and the relation (3.5), we have

$$\begin{aligned}
[\delta^{r+1},\psi\delta^{-1}\varphi] &= \delta^{r+1}\psi\delta^{-1}\varphi-\delta^r\psi\delta^{-1}\varphi\ \delta+[\delta^r,\psi\delta^{-1}\varphi]\delta\\
&= \sum_{i=0}^{r}D_i\delta^{r-i}+D_{r+1}\delta^{-1}\varphi-(\sum_{i=0}^{r-1}K_i\delta^{r-i}+K_rE^{-1}(\varphi)-K_r\delta^{-1}(\Delta E^{-1}\varphi))\\
&+ \sum_{i=0}^{r-1}C_i\delta^{r-i}+\hat{C}_rE^{-1}(\varphi)-\hat{C}_r\delta^{-1}(\Delta E^{-1}\varphi)+\psi E^{-1}C_r\\
&- \psi\delta^{-1}(\Delta E^{-1}C_r).
\end{aligned}$$

Now consider

$$
\begin{aligned}
& \sum_{i=0}^{r}(-1)^i\mu^i F_i + (-1)^{r+1}\mu^{r+1}(\varphi\hat{F}_{r+1} + \psi F_{r+1}) = \sum_{i=0}^{r}(-1)^i\mu^i D_i + (-1)^{r+1}\mu^{r+1}\varphi D_{r+1} \\
- & \sum_{i=0}^{r-1}(-1)^i\mu^i K_i - (-1)^r\mu^r(K_rE^{-1}(\varphi)) - (-1)^{r+1}\mu^{r+1}(-K_r\Delta E^{-1}(\varphi)) \\
+ & \sum_{i=0}^{r-1}(-1)^i\mu^i C_i + (-1)^r\mu^r(\hat{C}_rE^{-1}(\varphi) + \psi E^{-1}(C_r)) + (-1)^{r+1}\mu^{r+1}(-\hat{C}_r\Delta E^{-1}(\varphi) \\
- & \psi\Delta E^{-1}(C_r)).
\end{aligned}
$$

Then the result of Lemma 3.6.2 and (3.67) implies that

$$
\begin{aligned}
\sum_{i=0}^{r}(-1)^i\mu^i F_i + (-1)^{r+1}\mu^{r+1}(\varphi\hat{F}_{r+1} + \psi F_{r+1}) &= \psi\varphi - \psi\varphi \\
&+ (-1)^r\mu^r(\varphi K_r - K_rE^{-1}(\varphi) \\
&- \mu K_r\Delta E^{-1}(\varphi)) \\
&+ (-1)^r\mu^r(-\varphi\hat{C}_r - \psi C_r) \\
&+ (-1)^r\mu^r(\hat{C}_rE^{-1}(\varphi) + \psi E^{-1}(C_r) \\
&+ \mu(\hat{C}_r\Delta E^{-1}(\varphi) + \psi\Delta E^{-1}(C_r))) \\
&= 0.
\end{aligned}
$$

$\square$

Lemma 3.6.4 *:Assume*

$$[A\delta^r, B\delta^s] = \sum_{i=0}^{r} C_i\delta^{r+s-i}, \tag{3.69}$$

for all $r \geqslant s \geqslant 0$, then

$$\sum_{i=0}^{r}(-1)^i\mu^i C_i = 0. \tag{3.70}$$

Proof. If $\delta^r F = \sum_{i=0}^{r} C_i\delta^{r-i}$, for all $r > 0$, then the following holds

$$\sum_{i=0}^{r}(-1)^i\mu^i C_i = F, \tag{3.71}$$

which can be proved easily similar to the proof of the Lemma 3.6.2. The proof of the lemma proceeds by making use of the following expansion

$$[A\delta^{r+1}, B\delta^s] = A\delta^r(\Delta B)\delta^s + A\delta^r(\mu\Delta B)\delta^{s+1} + [A\delta^r, B\delta^s]\delta,$$

and induction. □

Hence, the virtue of Lemma 3.6.2, Lemma 3.6.3 and Lemma 3.6.4 straightforwardly imply the proof of the Proposition 3.6.1.

In order to explain the source of the Proposition 3.6.1, it is much simpler to consider the Lemma 3.6.4. Let A be a purely δ-differential operator such that

$$A = \sum_{i\geqslant 0} a_i\delta^i, \tag{3.72}$$

where the sum is finite. In order to expand A with respect to the shift operator $\mathcal{E}$: $\mathcal{E}u = E(u)\mathcal{E}$, we need an explicit relation between the shift operator $\mathcal{E}$ and δ-pseudo-differential operator δ, which is presented below.

Proposition 3.6.5 *[22] The operator formula*

$$\mathcal{E} = I + \mu\,\delta, \tag{3.73}$$

holds, where I denotes the identity operator.

The equality (3.70) from Lemma 3.6.4 is trivially satisfied for dense $x \in \mathbb{T}$, since in this case $\mu = 0$. Therefore, it is enough to consider remaining points in a time scale so assume that $\mu \neq 0$. Thus, the operator formula (3.73) implies the relation for $\mu \neq 0$,

$$\delta = \mu^{-1}\mathcal{E} - \mu^{-1}. \tag{3.74}$$

The relation (3.72) can be rewritten, by the use of (3.74), as

$$A(\mathcal{E}) = \sum_i a_i'\mathcal{E}^i. \tag{3.75}$$

Thus, the constant term of the polynomial A in $\mathcal{E}$ can be obtained by substituting $\mathcal{E} = 0$, which implies the replacement δ with $-\mu^{-1}$ by (3.74). Replacing δ

with $-\mu^{-1}$ in the assumption hypothesis (3.69) of Lemma 3.6.4, the commutator vanishes and this allows us to find

$$\sum_{i=0}^{r}(-\mu)^{-r-s+i}C_i = 0, \tag{3.76}$$

which is equivalent to (3.70).

The above procedure can be also extended to the operators A which are not purely δ-differential and contain finitely many negative ordered terms . For this purpose consider the Proposition 3.6.1. Replacing δ with $-\mu^{-1}$ in (3.62) the commutator vanishes, and we obtain (3.63).

Such behavior of the algebra of δ-pseudo-differential operators leads us to determine the general form of the constraints between the dynamical fields of the Lax operators (3.31), stated in the following theorem.

Theorem 3.6.6 *The constraint between the dynamical fields of Lax operators* (3.31), *generating the Lax hierarchy* (3.28), *has the following form*

$$\sum_{i=-k}^{N+k-1}(-\mu)^{N+k-1-i}u_i + (-\mu)^{N+k}\sum_s \psi_s\varphi_s = a, \quad k = 0, 1, \tag{3.77}$$

where a is a time-independent function. (for $k = 1$, a is nonzero when $\mu = 0$)

Proof. Clearly, the right-hand side of (3.28) can be represented in the form of L_{t_n}. If we replace δ with $-\mu^{-1}$ in both sides of (3.28), we deduce that

$$L_{t_n}|_{\delta=-\mu^{-1}} = [(L^n)_{\geqslant k}, L]|_{\delta=-\mu^{-1}} = 0, \quad k = 0, 1. \tag{3.78}$$

since the commutator vanishes. Analysing furthermore, we obtain

$$(-\mu)^{N+k-1}L_{t_n}\big|_{\delta=-\mu^{-1}} = 0, \quad k = 0, 1. \tag{3.79}$$

For $k = 1$, applying (3.79) on the Lax operator (3.31), the constraint

$$\sum_{i=-1}^{N}(-\mu)^{N-i}u_i + (-\mu)^{N+1}\sum_s \psi_s\varphi_s = a \tag{3.80}$$

follows. Similarly for $k = 0$, we have the following constraint

$$\sum_{i=0}^{N-1}(-\mu)^{N-1-i}u_i + (-\mu)^N \sum_s \psi_s\varphi_s = a \tag{3.81}$$

since u_N is time-independent and $u_{-1} = 0$ in this case. The constraints (3.80) and (3.81) imply the general form of the constraint between the dynamical fields of (3.31) as (3.77). □

As a consequence, the constraint (3.77) with a fixed value of a, is valid for the whole Lax hierarchy (3.28) which allows to generalize the above theorem for further finite-field restrictions.

3.7 Finite-field integrable systems on time scales

3.7.1 Δ-differential AKNS, $k = 0$:

Let the Lax operator (3.31) for $N = 1$ and $u_1 = c_1 = 1$ be of the form

$$L = \delta + u + \psi\delta^{-1}\varphi. \tag{3.82}$$

The constraint (3.77) between fields, with $a = 0$, becomes

$$u = \mu\psi\varphi. \tag{3.83}$$

For $(L)_{\geqslant 0} = \delta + u$, one finds the first flow

$$\begin{aligned}
\frac{du}{dt_1} &= \mu\Delta(\psi E^{-1}\varphi),\\
\frac{d\psi}{dt_1} &= \Delta\psi + u\psi,\\
\frac{d\varphi}{dt_1} &= \Delta E^{-1}\varphi - u\varphi.
\end{aligned} \tag{3.84}$$

Eliminating field u by (3.83), we have

$$\begin{aligned}\frac{d\psi}{dt_1} &= \Delta\psi + \mu\psi^2\varphi,\\ \frac{d\varphi}{dt_1} &= \Delta E^{-1}\varphi - \mu\varphi^2\psi.\end{aligned} \tag{3.85}$$

Next we calculate $(L^2)_{\geqslant 0} = \delta^2 + \xi\delta + \eta$ where

$$\xi := (E+1)u, \quad \eta := \Delta u + u^2 + \varphi E(\psi) + \psi E^{-1}(\varphi). \tag{3.86}$$

Thus, the second flow takes the form

$$\begin{aligned}\frac{du}{dt_2} &= \mu\Delta\left[\Delta(\psi E^{-1}(\varphi)) + \psi E^{-1}(u\varphi) + u\psi E^{-1}\varphi\right] - \mu\Delta(E+1)\psi E^{-1}\Delta E^{-1}(\varphi),\\ \frac{d\psi}{dt_2} &= \Delta^2\psi + \psi\eta + \xi\Delta\psi,\\ \frac{d\varphi}{dt_2} &= -(\Delta E^{-1})^2\varphi - \varphi\eta + \Delta E^{-1}(\varphi\xi).\end{aligned} \tag{3.87}$$

By the use of the constraint (3.83), the second flow can be written as

$$\begin{aligned}\frac{d\psi}{dt_2} &= \Delta^2\psi + \psi\bar{\eta} + \bar{\xi}\Delta\psi,\\ \frac{d\varphi}{dt_2} &= -(\Delta E^{-1})^2\varphi - \varphi\bar{\eta} + \Delta E^{-1}(\varphi\bar{\xi}),\end{aligned} \tag{3.88}$$

where

$$\bar{\xi} := (E+1)\mu\psi\varphi, \quad \bar{\eta} := \Delta\mu\psi\varphi + (\mu\psi\varphi)^2 + \varphi E(\psi) + \psi E^{-1}(\varphi). \tag{3.89}$$

In order to obtain higher elements of the hierarchy of Δ-differential AKNS, it is much simpler to derive the recursion operator. For this purpose, one finds that the appropriate reminder (3.40) has the form

$$R = \Delta^{-1}\left(\mu^{-1}u_{t_n}\right) - \psi_{t_n}\delta^{-1}\varphi. \tag{3.90}$$

Then, (3.39) implies the following recursion formula

$$\begin{pmatrix} u\\ \psi\\ \varphi\end{pmatrix}_{t_{n+1}} = \begin{pmatrix} u-\mu^{-1} & \phi E & \psi E^{-1}\\ \psi + \psi\Delta^{-1}\mu^{-1} & \Delta + u + \psi\Delta^{-1}\varphi & \psi\Delta^{-1}\psi\\ -\varphi\Delta^{-1}\mu^{-1} & -\varphi E\Delta^{-1}\varphi & u - \Delta E^{-1} - \varphi E\Delta^{-1}\psi\end{pmatrix}\begin{pmatrix} u\\ \psi\\ \varphi\end{pmatrix}_{t_n} \tag{3.91}$$

which is valid for isolated points $x \in \mathbb{T}$, i.e. when $\mu \neq 0$. For dense points, its reduction by the constraint (3.83) follows as,

$$\begin{pmatrix} \psi \\ \varphi \end{pmatrix}_{t_{n+1}} = \begin{pmatrix} \Delta + 2\mu\psi\varphi + 2\psi\Delta^{-1}\varphi & \mu\psi^2 + 2\psi\Delta^{-1}\psi \\ -\mu\varphi^2 - 2\varphi\Delta^{-1}\varphi & -\Delta E^{-1} - 2\varphi\Delta^{-1}\psi \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix}_{t_n} . \tag{3.92}$$

In the case of $\mathbb{T} = \mathbb{R}$, or in the continuous limit of some special time scales, the recursion formula (3.92) turns out to be:

$$\begin{pmatrix} \psi \\ \varphi \end{pmatrix}_{t_{n+1}} = \begin{pmatrix} \partial_x + 2\psi\partial_x^{-1}\varphi & 2\psi\partial_x^{-1}\psi \\ -2\varphi\partial_x^{-1}\varphi & -\partial_x - 2\varphi\partial_x^{-1}\psi \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix}_{t_n} . \tag{3.93}$$

Using the recursion operator (3.92), the third flow is calculated in the form

$$\begin{aligned} \frac{d\psi}{dt_3} =& \Delta^3\psi + \Delta(\psi\bar{\eta} + \bar{\xi}\Delta\psi) + 2\psi\Delta^{-1}(\varphi\Delta^2\psi - \psi(\Delta E^{-1})^2\varphi) \\ &+ 2\psi\Delta^{-1}(\varphi\bar{\xi}\Delta\psi + \psi\Delta E^{-1}(\varphi\bar{\xi})) + 2\mu\psi\varphi(\Delta^2\psi + \bar{\xi}\Delta\psi) \\ &+ \mu\psi^2(\varphi\bar{\eta} + \Delta E^{-1}\varphi\bar{\xi} - (\Delta E^{-1})^2\varphi) \\ \frac{d\varphi}{dt_2} =& (\Delta E^{-1})^3\varphi + 2\varphi\Delta^{-1}(\psi(\Delta E^{-1})^2\varphi - \varphi\Delta^2\psi) + \Delta E^{-1}\varphi\bar{\eta} \\ &- 2\varphi\Delta^{-1}(\psi\Delta E^{-1}\varphi\bar{\xi} + \varphi\bar{\xi}\Delta\psi) - (\Delta E^{-1})^2\varphi\bar{\xi} \\ &- \mu\varphi^2(\Delta^2\psi + \psi\bar{\eta} + \bar{\xi}\Delta\psi). \end{aligned} \tag{3.94}$$

where $\bar{\xi}, \bar{\eta}$ are given in (3.89). In $\mathbb{T} = \mathbb{R}$ case, or in the continuous limit of some special time scales, with the apparent choice $u = 0$ (the constraint (3.83) implies that $u = 0$ since $\mu = 0$), the Lax operator (3.82) takes the form $L = \partial + \psi\partial^{-1}\varphi$. Then, the continuous limits of (3.84) and (3.87) respectively, imply that the first flow is the translational symmetry

$$\begin{aligned} \frac{d\psi}{dt_1} &= \psi_x \\ \frac{d\varphi}{dt_1} &= \varphi_x \end{aligned}$$

and the first non-trivial equation from the hierarchy is the AKNS equation

$$\begin{aligned} \frac{d\psi}{dt_2} &= \psi_{xx} + 2\psi^2\varphi, \\ \frac{d\varphi}{dt_2} &= -\varphi_{xx} - 2\varphi^2\psi. \end{aligned} \tag{3.95}$$

Furthermore, the continuous limit of the third flow (3.94) of Δ-differential AKNS becomes

$$\begin{aligned} \frac{d\psi}{dt_3} &= \psi_{3x} + 6\psi\varphi\psi_x, \\ \frac{d\varphi}{dt_3} &= \varphi_{3x} + 6\psi\varphi\varphi_x, \end{aligned} \tag{3.96}$$

which can be also derived by directly applying the recursion operator (3.93) to the continuous second flow (3.95). Note that, the choice of $\varphi = 1$ in (3.96) implies the usual KdV-equation while setting $\psi = \varphi$ in (3.96) yields the usual modified KdV-equation.

In $\mathbb{T} = \mathbb{R}$ case, the first nontrivial flow is the second one (3.88), i.e. the AKNS system. When $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{K}_q$ we get the lattice and q-discrete counterparts of the AKNS hierarchy where the first nontrivial flow is (3.85).

3.7.2 Δ-differential KdV, $k = 0$:

A further admissible reduction of the Lax operator (3.31) for $k = 0$ is given by (3.32). Consider the following finite-field Lax operator, with $N = 2$ and $c_2 = 1$

$$L = \delta^2 + v\delta + u, \tag{3.97}$$

which generates the Δ-differential counterpart of KdV hierarchy. The constraint (3.77) between the dynamical fields, with $a = \lambda$, where λ is an arbitrary time independent function, becomes

$$v = \mu u + \lambda. \tag{3.98}$$

Straightforward calculation for

$$L^{1/2} = \delta + \alpha_0 + \alpha_1\,\delta^{-1} + \alpha_2\,\delta^{-2} + \cdots \tag{3.99}$$

allows to obtain the coefficients α_i, $i \geqslant 0$ in terms of the dynamical fields u and v, as

$$E(\alpha_0) + \alpha_0 = v, \tag{3.100}$$

$$E(\alpha_1) + \alpha_1 + \Delta\alpha_0 + (\alpha_0)^2 = u, \tag{3.101}$$

$$E(\alpha_2) + \alpha_2 + \alpha_1\,E^{-1}(\alpha_0) + \Delta\alpha_1 = 0. \tag{3.102}$$

We obtain the members of the KdV hierarchy by the choice of $n = \{2k+1 : k \in \mathbb{N}_0\}$.

(1). Let $n = 1$. Then Lax hierarchy (3.28)

$$L_t = [(L^{1/2})_{\geq 0}, L] \tag{3.103}$$

implies the first flow as

$$\begin{aligned} \frac{du}{dt} &= \Delta u - v\Delta\alpha_0 - \Delta^2\alpha_0, && (3.104) \\ \frac{dv}{dt} &= \Delta v + (E-1)u - v(E-1)\alpha_0 - E\Delta\alpha_0 - \Delta E\alpha_0, \\ &= \mu(\Delta u - v\Delta\alpha_0 - \Delta^2\alpha_0). && (3.105) \end{aligned}$$

By the constraint (3.98) the first flow can be rewritten as

$$\frac{du}{dt} = \Delta u - (\mu\, u + \lambda)\, \Delta\alpha_0 - \Delta^2\alpha_0, \tag{3.106}$$

where α_0 is,

$$E(\alpha_0) + \alpha_0 = \mu\, u + \lambda. \tag{3.107}$$

We investigate the reduced first flow (3.106) for particular cases of $\mathbb{T}$ with the ansatz $\lambda = 0$.

(i) In $\mathbb{T} = \mathbb{R}$ case, or in the continuous limit of some special time scales, with the choice $v = 0$ (in this case, $\mu = 0$ and by the assumption $\lambda = 0$, the constraint (3.98) implies that $v = 0$), the Lax operator (3.97) takes the form $L = \partial^2 + u$. Then, the relations (3.100), (3.101), (3.102) imply the first three coefficients of the operator $L^{1/2}$

$$\alpha_0 = 0, \quad \alpha_1 = \frac{1}{2}u, \quad \alpha_2 = -\frac{1}{4}u_x, \tag{3.108}$$

and the continuous limit of (3.106) becomes

$$u_t = u_x, \tag{3.109}$$

which is a linear equation explicitly solvable:

$$u(x,t) = \varphi(x+t), \tag{3.110}$$

where φ is an arbitrary differentiable function.

(ii) In $\mathbb{T} = \mathbb{Z}$ case, we have $\mu = 1$ and (3.107) is satisfied by

$$\alpha_0(n) = -\sum_{k=-\infty}^{n-1} (-1)^{n+k}\, u(k), \quad n \in \mathbb{Z} \tag{3.111}$$

and therefore the equation (3.106) becomes

$$\frac{du(n)}{dt} = -u^2(n) + 2u(n) + 2(-1)^n\,[2 + u(n)] \sum_{k=-\infty}^{n-1} (-1)^k\, u(k), \tag{3.112}$$

for $n \in \mathbb{Z}$.

(iii) In $\mathbb{T} = \mathbb{K}_q$ case, we have $\mu(x) = (q-1)x$ and (3.107) is satisfied by $\alpha_0(0) = 0$ and

$$\alpha_0(x) = -(q-1) \sum_{y \in (0, q^{-1}x]} (-1)^{log_q(xy)}\, y u(y) \tag{3.113}$$

for $x \in \mathbb{K}_q$ and $x \neq 0$. Substituting (3.113) into (3.106) we obtain an evolution equation for u.

(iv) Let $\mathbb{T} = (-\infty, 0) \cup \mathbb{K}_q = (-\infty, 0] \cup q^{\mathbb{Z}}$. Here, by the choice of this special time scale we have two different types of graininess functions. If $x \in (-\infty, 0]$, we have $\mu(x) = 0$ which implies clearly $\alpha_0 = 0$. On the other hand, if $x \in q^{\mathbb{Z}}$, the graininess function is $\mu(x) = (q-1)x$ and thus $\alpha_0(x)$ is exactly equivalent to the form given as (3.113). As a result, (3.106) produces an evolution equation that coincides on $(-\infty, 0]$ and $q^{\mathbb{Z}}$ with the evolution equations described in the examples (i) and (iii), respectively. Note that the solution u has to satisfy the smoothness conditions

$$u(0^-) = u(0^+), \quad u'(0^-) = \Delta u(0^+), \tag{3.114}$$

at $x = 0$.

(2). Let $n = 3$, we obtain

$$(L^{3/2})_{\geqslant 0} = \delta^3 + p\,\delta^2 + q\,\delta + r \tag{3.115}$$

where

$$p = \alpha_0 + E(v), \tag{3.116}$$

$$q = \Delta v + E(u) + \alpha_0 \, v + \alpha_1, \tag{3.117}$$

$$r = \Delta u + \alpha_0 \, u + \alpha_1 \, E^{-1}(v) + \alpha_2, \tag{3.118}$$

and the Lax equation (3.28) implies the second flow as

$$\frac{du}{dt} = \Delta^3 u + p \, \Delta^2 u + q \, \Delta u - \Delta^2 r - v \, \Delta r, \tag{3.119}$$

$$\begin{aligned} \frac{dv}{dt} = \; & \Delta^3 v + E\Delta^2 u + \Delta E \Delta u + \Delta^2 E u + p \, [\Delta^2 v + E\Delta u \\ & + \Delta E u] + q \, (\Delta v + E(u) - u) + rv - \Delta^2 q - E\Delta r \\ & - \Delta E r - v \, \Delta q - vE(r). \end{aligned} \tag{3.120}$$

By the use of the constraint (3.98) with the choice λ = constant, the reduced second flow yields as

$$\frac{du}{dt} = \Delta^3 u + p \, \Delta^2 u + q \, \Delta u - \Delta^2 r - v \, \Delta r, \tag{3.121}$$

Similar to the discussions given in part (i), when $\mathbb{T} = \mathbb{R}$, or in the continuous limit of some special time scales, the relations (3.116), (3.117), (3.118) imply the first three coefficients of the operator $L^{3/2}$

$$p = 0, \quad q = \frac{3}{2} u, \quad r = \frac{3}{4} u_x \tag{3.122}$$

and the continuous limit of (3.121) becomes the KdV equation

$$u_t = \frac{1}{4} u_{3x} + \frac{3}{2} u u_x. \tag{3.123}$$

Remark 3.7.1 *The recursion operator of KdV hierarchy can be calculated by taking the square of the recursion operator* (3.92) *of AKNS hierarchy. Note that, such a behavior leads us to deduce that the Lax hierarchies and bi-Hamiltonian structures of Δ-differential KdV are hidden inside of Δ-differential AKNS.*

3.7.3 Δ-differential Kaup-Broer, $k = 1$:

The admissible finite field restrictions (3.31) with $N = 1$ and without the finite sum on the right hand side of (3.31) leads to consider the simplest Lax operator of the form

$$L = u\delta + v + \delta^{-1}w. \tag{3.124}$$

The constraint (3.77), with $a = 1$, implies

$$u = 1 + \mu v - \mu^2 w. \tag{3.125}$$

Calculating $(L)_{\geqslant 1} = u\delta$, the Lax equation (3.28) implies the first flow as

$$\begin{aligned}
\frac{du}{dt_1} &= \mu u \Delta v, \\
\frac{dv}{dt_1} &= u\Delta v + \mu \Delta E^{-1}(uw), \\
\frac{dw}{dt_1} &= \Delta E^{-1}(uw).
\end{aligned} \tag{3.126}$$

By the use of the constraint (3.125), one can rewrite the first flow in the form

$$\begin{aligned}
\frac{dv}{dt_1} &= (1 + \mu v - \mu^2 w)\Delta v + \mu \Delta E^{-1}(w(1 + \mu v - \mu^2 w)), \\
\frac{dw}{dt_1} &= \Delta E^{-1}\left(w + \mu v w - \mu^2 w^2\right).
\end{aligned} \tag{3.127}$$

Next, we calculate $(L^2)_{\geqslant 1} = \xi\delta^2 + \eta\delta$, where

$$\xi := uEu, \quad \eta := u\Delta u + uEv + vu, \tag{3.128}$$

that yields the second flow

$$\begin{aligned}
\frac{du}{dt_2} &= \mu u \Delta(E^{-1} + 1)uw + \mu u \Delta v^2 + \mu u \Delta(u\Delta v), \\
\frac{dv}{dt_2} &= \xi(\Delta^2 v + \Delta w) + \mu \Delta E^{-1}(w\eta) + E^{-1}\Delta E^{-1}(w\xi) + \eta \Delta v, \\
\frac{dw}{dt_2} &= -\Delta E^{-1}\Delta E^{-1}(w\xi) + \Delta E^{-1}(w\eta).
\end{aligned} \tag{3.129}$$

Since it is cumbersome to write the second flow in terms of the constraint, we skip this complicated coupled equation.

In the case of $\mathbb{T} = \mathbb{R}$, or in the continuous limit of some special time scales, with the apparent choice $u = 1$ (the constraint (3.125) implies that $u = 1$ since $\mu = 0$), the Lax operator (3.124) takes the form $L = \partial + v + \partial^{-1}w$. Then the similar continuous analogue allows us to obtain the first flow

$$\begin{aligned} \frac{dv}{dt_1} &= v_x, \\ \frac{dw}{dt_1} &= w_x, \end{aligned} \tag{3.130}$$

and the first non-trivial equation from the hierarchy is the Kaup-Broer equation

$$\begin{aligned} \frac{dv}{dt_2} &= v_{2x} + 2w_x + 2vv_x, \\ \frac{dw}{dt_2} &= -w_{2x} + 2(vw)_x. \end{aligned} \tag{3.131}$$

The appropriate remainder (3.40) of Δ-differential KB is given by

$$R = u\Delta^{-1}(\mu u)^{-1}u_{t_n}\delta - v_{t_n} - \Delta^{-1}w_{t_n}. \tag{3.132}$$

Hence, from (3.39) we have the following recursion formula

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_{n+1}} = \begin{pmatrix} R_{uu} & uE & \mu u \\ R_{vu} & v + u\Delta & (1+E^{-1})u \\ R_{wu} & w & -\Delta E^{-1}u + v - \mu w \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_n}, \tag{3.133}$$

which is valid when $\mu \neq 0$ and

$$\begin{aligned} R_{uu} &= E(v) - \mu^{-1}u + \mu u\Delta(v)\Delta^{-1}(\mu u)^{-1} \\ R_{vu} &= \Delta(v) + w + u\Delta(v)\Delta^{-1}(\mu u)^{-1} + (1 - E^{-1})uw\Delta^{-1}(\mu u)^{-1} \\ R_{wu} &= \Delta E^{-1}uw\Delta^{-1}(\mu u)^{-1}. \end{aligned} \tag{3.134}$$

Its reduction by the constraint (3.125) is

$$\begin{pmatrix} v \\ w \end{pmatrix}_{t_{n+1}} = \begin{pmatrix} v + u\Delta + R_{vu}\mu & (1+E^{-1})u - R_{vu}\mu^2 \\ w + R_{wu}\mu & -\Delta E^{-1}u + v - \mu w - R_{wu}\mu^2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_{t_n}, \tag{3.135}$$

with u given by (3.125). In the case of $\mathbb{T} = \mathbb{R}$, or in the continuous limit of some special time scales, the recursion formula (3.135) turns out to be

$$\begin{pmatrix} v \\ w \end{pmatrix}_{t_{n+1}} = \begin{pmatrix} \partial_x + v + v_x\partial_x^{-1} & 2 \\ w + \partial_x w\partial_x^{-1} & -\partial_x + v \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_{t_n}. \tag{3.136}$$

3.7.4 Δ-differential Burgers equation, $k = 1$:

A further admissible reduction of the Lax operator (3.31) for $k = 1$ is given by (3.34). Thus we can consider the reduction of the Lax operator (3.124) as

$$L = u\delta + v. \tag{3.137}$$

By eliminating the field w from (3.129), we derive the second flow,

$$\begin{aligned} \frac{du}{dt_2} &= \mu u\Delta(u\Delta v + v^2), \\ \frac{dv}{dt_2} &= uE(u)\Delta^2 v + (u\Delta u + uE(v) + uv)\Delta v = u\Delta(u\Delta v + v^2). \end{aligned} \tag{3.138}$$

which is equivalent to Burgers equation on time scales

$$\frac{dv}{dt_2} = (1 + \mu v)\Delta((1 + \mu v)\Delta v + v^2), \tag{3.139}$$

together with the constraint

$$u = 1 + \mu v. \tag{3.140}$$

(i) In the case of $\mathbb{T} = \mathbb{R}$, or in the continuous limit of some special time scales, (3.139) becomes the standard Burgers equation on $\mathbb{R}$

$$\frac{dv}{dt_2} = v_{2x} + 2vv_x. \tag{3.141}$$

(ii) When $\mathbb{T} = \hbar\mathbb{Z}$ then $\mu(x) = \hbar$. Then considering the constraint $u = \mu v = \hbar v$ (i.e $a = 0$ in the constraint (3.77)), we find

$$\frac{dv(x)}{dt} = v(x)\, v(x + h)\, [v(x + 2h) - v(x)], \tag{3.142}$$

where $x \in \hbar\mathbb{Z}$. The evolution equation (3.142) represents the difference version of the Burgers equation.

(iii) When $\mathbb{T} = \mathbb{K}_q$, we have $\mu(x) = (q - 1)x$. If we consider the constraint $u = \mu v = (q - 1)xv$, we get from (3.139)

$$\frac{dv(x)}{dt} = v(x)v(qx)[v(q^2x) - v(x)]. \tag{3.143}$$

The evolution equation (3.143) represents the q-difference version of the Burgers equation.

Remark 3.7.2 *Note that, we obtained the Burgers hierarchy directly in [22] rather than eliminating the field w from KB hierarchy. By the frame of the reduced Lax operator* (3.137)*, the Lax equation* (3.28) *turns out to be*

$$\frac{dL}{dt_n} = \left[(L^n)_{\geqslant 1}, L\right] = -\left[(L^n)_{<1}, L\right] = \left[-(L^n)_0, L\right], \qquad n \geqslant 1. \tag{3.144}$$

Since $(L^n)_0$ is a scalar function, letting $(L^n)_0 = \rho_n$ implies the general form of all flows as

$$\frac{du}{dt_n} = \mu\, u\, \Delta\rho_n, \tag{3.145}$$

$$\frac{dv}{dt_n} = u\, \Delta\rho_n, \tag{3.146}$$

where the first three ρ_n are given by

$$\rho_1 = v, \tag{3.147}$$

$$\rho_2 = u\, \Delta v + v^2, \tag{3.148}$$

$$\rho_3 = (v + u\Delta)(u\Delta v + v^2). \tag{3.149}$$

The above hierarchy reduces to a single evolution equation

$$\frac{dv}{dt_n} = (1 + \mu\, v)\Delta\rho_n, \quad n \geqslant 1, \tag{3.150}$$

with the constraint (3.140)*.*

Chapter 4

Bi-Hamiltonian Theory

4.1 Classical bi-Hamiltonian structures

In this section we collect the fundamental notions and definitions in the theory of bi-Hamiltonian structures for the algebra of pseudo-differential operators, i.e. in $\mathbb{R}$ [42, 44, 13].

Let $\mathcal{U}$ be a linear space of N tuples

$$u := (u_1(x), u_2(x), ..., u_N(x))^T, \tag{4.1}$$

of smooth functions $u_i : \Omega \to \mathbb{K}$, where $\mathbb{K}$ is a field of complex or real numbers and the space $\Omega \subseteq \mathbb{R}$ is chosen such that u_i and all derivatives are *rapidly decaying* functions, i.e. u_i and all derivatives tend to 0 as $|x| \to \infty$. Then, $\mathcal{U}$ arises as an infinite dimensional phase space with local coordinates $\{u, u_x, u_{2x}, ...\}$. A smooth vector field on $\mathcal{U}$ is given by a system of differential equations

$$\mathbf{u}_t = K[\mathbf{u}], \tag{4.2}$$

where $u_t := \frac{\partial u}{\partial t}$ and

$$K[\mathbf{u}] := (K_1[\mathbf{u}], K_2[\mathbf{u}], ..., K_N[\mathbf{u}])^{\mathrm{T}}.$$

The scalar fields on $\mathcal{U}$ are functionals $F : \mathcal{U} \to \mathbb{K}$ of the form

$$F(u) = \int_\Omega f[u]dx. \tag{4.3}$$

Let $\mathcal{F} = \{F : \mathcal{U} \to \mathbb{K}\}$ be a space of functions on $\mathcal{U}$, defined through functionals (4.3). Let $\mathcal{V}$ be a linear space over $\mathbb{K}$ of smooth vector fields on $\mathcal{U}$. and $\mathcal{V}^*$ be the dual space to $\mathcal{V}$ with respect to the duality map

$$\langle \cdot, \cdot \rangle : \mathcal{V}^* \times \mathcal{V} \to \mathbb{K}.$$

Then, the dual space $\mathcal{V}^*$ is a space of all linear maps $\eta : \mathcal{V} \to \mathbb{K}$ and the action of $\eta = (\eta_1, \eta_2, ..., \eta_N)^T \in \mathcal{V}^*$ on $K \in \mathcal{V}$ can be defined through a duality map given by

$$\langle \eta, K \rangle = \int_\Omega \sum_{i=1}^{N} \eta_i K_i dx = \int_\Omega \eta^T.K.dx, \tag{4.4}$$

Definition 4.1.1 *The directional derivative of an arbitrary tensor field F at $u \in \mathcal{U}$ in the direction of the vector field $K \in \mathcal{V}$ is defined by*

$$F'(u)[K] = \frac{d}{d\epsilon} F(u + \epsilon K) \mid_{\epsilon=0} \tag{4.5}$$

Remark 4.1.2 *By the above definition, the directional derivative of the functional F (4.3) is written as*

$$F'[K] = \langle dF, K \rangle = \int_\Omega dF^T.K.dx, \tag{4.6}$$

and one can derive the related differential (or gradient) $dF \in \mathcal{V}^*$ of F, in the following scheme: If we differentiate F with respect to t in the direction of K (4.2), we find out that

$$F'(u)[u_t] = \frac{dF(u)}{dt} = \int_\Omega \frac{\partial f}{\partial (u_i)_{jx}} ((u_i)_{jx})_t.dx = \int_\Omega \sum_{i=1}^{N} \frac{\delta F}{\delta u_i} (u_i)_t.dx. \tag{4.7}$$

Here by the use of integration by parts, variational derivative is as follows

$$\frac{\delta F}{\delta u_i} = \sum_{j \geqslant 0} (-\partial)^j \frac{\partial f}{\partial (u_i)_{jx}} \tag{4.8}$$

and the differential of F yields as

$$dF(u) = \left(\frac{\delta F}{\delta u_1}, \frac{\delta F}{\delta u_2}, ..., \frac{\delta F}{\delta u_N} \right)^T. \tag{4.9}$$

Note that, the above scheme is valid only for the algebra of pseudo-differential operators. Now, we can pass through the remarkable concept of bi-Hamiltonian structures.

Definition 4.1.3 *A bilinear product* $\{\cdot,\cdot\} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ *which defines the Lie algebra structure on* $\mathcal{F}$ *is called a Poisson bracket. A linear operator* $\pi : \mathcal{V}^* \to \mathcal{V}$ *is called Poisson operator if the bracket*

$$\{H, F\}_\pi = \langle dF, \pi dH \rangle = \int_\Omega dF^T.\pi dH.dx \qquad F, H \in \mathcal{F} \tag{4.10}$$

is a Poisson bracket.

Definition 4.1.4 *A vector field* $K \in \mathcal{V}$ *is called a bi-Hamiltonian with respect to Poisson operators* π_0 *and* π_1*, if there exists functionals* $H_0, H_1 \in \mathcal{F}$ *such that*

$$K = \pi_0 dH_1 = \pi_1 dH_0. \tag{4.11}$$

Definition 4.1.5 *The pair of Poisson tensors* π_0 *and* π_1 *is called compatible if* $\pi_0 + \lambda\pi_1$ *is also a Poisson tensor for any constant* λ*.*

Definition 4.1.6 *[44] A linear operator* $\pi : \mathcal{V}^* \to \mathcal{V}$ *is degenerate if there is a nonzero operator* $\bar{\pi} : \mathcal{V} \to \mathcal{V}^*$ *such that* $\bar{\pi}.\pi = 0$*.*

The following theorem summarizes the main properties of bi-Hamiltonian systems.

Theorem 4.1.7 *[44] Assume*

$$u_{t_1} = K_1[u] = \pi_0 dH_1 = \pi_1 dH_0$$

be a bi-Hamiltonian system of evolution equations. Let the operator π_0 *be nondegenerate and* $\Phi : \mathcal{V} \to \mathcal{V}$ *be of the form*

$$\Phi = \pi_1.\pi_0^{-1} \tag{4.12}$$

(which is so-called recursion operator). Let us also define recursively

$$u_{t_0} = K_0[u] := \pi_0 dH_0 \Rightarrow K_i = \Phi.K_{i-1} \tag{4.13}$$

for each $i = 1, 2, ...$, *i.e. for each* i, K_{i-1} *lies in the image of* π_0. *Then for all* $i \geqslant 0$, *there exists a sequence of functionals* H_i *satisfying*

(i) *For each* $i \geqslant 1$, *the evolution equation*

$$u_{t_i} = K_i[u] = \pi_0 dH_i = \pi_1 dH_{i-1}, \tag{4.14}$$

is a bi-Hamiltonian system.

(ii) *The evolutionary vector fields* K_i *mutually commute*

$$[K_i, K_j] = 0, \quad \forall i, j \geqslant 0. \tag{4.15}$$

(iii) *The Hamiltonian functionals* H_i *are all in involution with respect to each Poisson bracket, i.e.*

$$\{H_i, H_j\}_{\pi_0} = \{H_i, H_j\}_{\pi_1} = 0, \qquad i, j \geqslant 0. \tag{4.16}$$

Hence, the Hamiltonian functionals H_i *is an infinite collection of conserved quantities for each of the bi-Hamiltonian systems* (4.14).

Remark 4.1.8 *Since we have defined integrable systems as systems which has infinite hierarchy of mutually commuting symmetries (all symmetries in the hierarchy are Hamiltonian), the theorem 4.1.7 ensures that bi-Hamiltonian system of evolution equations are completely integrable.*

4.2 Δ-differential systems

We present now the theory of bi-Hamiltonian structures on an arbitrary regular time scale, based on the article [24].

Let $\mathcal{U}$ be the linear space of N-tuples

$$u := (u_1, \ldots, u_N)^{\mathrm{T}}$$

of Δ-smooth functions $u_k : \mathbb{T} \to \mathbb{R}$, on a regular time scale $\mathbb{T}$ and assuming values on the field $\mathbb{R}$. Additionally assume that, u_k's depend on an appropriate set of evolution parameters, i.e. u_k's are dynamical fields. Consider the set of Δ-differential smooth functions

$$\mathcal{C} = \{\Lambda u_k(x) : k = 1, \ldots, N; \Lambda \in S\},$$

where

$$S = \left\{\Delta^{i_1}\Delta^{\dagger j_1} \cdot \ldots \cdot \Delta^{i_n}\Delta^{\dagger j_n} : n \in \mathbb{N}_0, i_1, j_1 \ldots, i_n, j_n \in \mathbb{N}\right\}.$$

and $\Delta^\dagger$ is given in (2.33). Note that, S is the set of all possible strings of Δ and $\Delta^\dagger$ operators which do not commute.

Definition 4.2.1 *A system of evolution equations of the form*

$$u_t = K[u], \tag{4.17}$$

is called a Δ-differential system, where $u_t := \frac{\partial u}{\partial t}$ and $K := (K_1, K_2, ..., K_N)^{\mathrm{T}}$ with K_i being finite order polynomials of elements from $\mathcal{C}$, with coefficients that might be time independent Δ-smooth functions.

The system (4.17) represents a $(1+1)$ dimensional dynamical system since $t \in \mathbb{R}$ can be treated as an evolution (time) parameter and x as a spatial (space) one on an arbitrary regular time scale. Furthermore, the linear space $\mathcal{U}$ defines an infinite-dimensional phase space which assures that the system of evolution equations (4.17) creates a vector space on this phase space of Δ-differential smooth functions of elements from $\mathcal{C}$.

We have an additional assumption on the fields such that all fields $u_k : \mathbb{T} \to \mathbb{R}$ together with their Δ derivatives are *rapidly decaying* functions, i.e. all fields and their Δ-derivatives tend to zero sufficiently rapidly as x goes to x_* or x^*, where $x_* = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$ and $x_* = -\infty$ otherwise, $x^* = \max \mathbb{T}$

if there exists a finite $\max \mathbb{T}$ and $x^* = \infty$ otherwise. Thus, we define a space $\mathcal{F} = \{F : \mathcal{U} \to \mathbb{R}\}$ of functions on $\mathcal{U}$ through linear functionals

$$F(u) = \int_{\mathbb{T}} f[u] \ \Delta x, \tag{4.18}$$

where $f[u]$ are polynomial functions of $\mathcal{C}$. Let $\mathcal{V}$ be a linear space of all vector fields on $\mathcal{U}$. Then, the dual space $\mathcal{V}^*$ is a space of all linear maps $\eta : \mathcal{V} \to \mathbb{R}$ and the action of $\eta \in \mathcal{V}^*$ on $K \in \mathcal{V}$ can be defined through a duality map by means of the functionals (4.18). Moreover, since (4.17) are evolution equations, the concept of variational derivative which is defined by

$$\frac{\delta F}{\delta u_k} := \sum_{\Lambda \in S} \Lambda^\dagger \frac{\partial f[u]}{\partial(\Lambda u_k)} \qquad k = 1, \ldots, N. \tag{4.19}$$

is well-posed. Therefore, the notions of directional derivative and the differential of a functional (4.18) is also well-posed. Hence, we follow the procedure presented in the previous section, for Δ-differential systems on regular time scales. Note that, since $\frac{\delta}{\delta u}\Delta = 0$, the definition of variational derivative (4.19) is consistent with the definition of functionals (4.18).

4.3 The Trace Functional

In this section, we will introduce a trace form which is well-defined on an arbitrary time scale and at the same time which recovers in $\mathbb{T} = \mathbb{R}$ case the trace form of pseudo-differential operators, in $\mathbb{T} = \mathbb{Z}$ case the trace form of shift operators and in $\mathbb{T} = \mathbb{K}_q$ case the one of q-numbers after constraints are taken into consideration.

Definition 4.3.1 *The trace form* $\mathrm{Tr} : \mathcal{G} \to \mathbb{K}$ *is introduced by*

$$\mathrm{Tr} A := -\int_{\mathbb{T}} \frac{1}{\mu} \left.(A_{<0})\right|_{\delta = -\frac{1}{\mu}} \ \Delta x \equiv \int_{\mathbb{T}} \sum_{i<0} (-\mu)^{-i-1} a_i \ \Delta x, \tag{4.20}$$

where $A_{<0} = \sum_{i<0} a_i \delta^i$ *for the* δ*-pseudo-differential operator* $A = \sum_i a_i \delta^i$*.*

In order to show that the substitution $\delta = -\frac{1}{\mu}$ given in the trace form (4.20) is well-posed, we state the following proposition.

Proposition 4.3.2 *Let A and B be δ-differential operators such that the following relation holds*

$$(AB)_{<0} = AB.$$

Then the multiplication operation in the algebra $\mathcal{G}$ of δ-pseudo-differential operators commutes with the substitution $\delta = -\frac{1}{\mu}$, i.e,

$$\int_{\mathbb{T}} \frac{1}{\mu} \left.(AB)\right|_{\delta=-\frac{1}{\mu}} \Delta x = \int_{\mathbb{T}} \frac{1}{\mu} \left.(A)\right|_{\delta=-\frac{1}{\mu}} \left.(B)\right|_{\delta=-\frac{1}{\mu}} \Delta x. \tag{4.21}$$

Proof. It is sufficient to prove (4.21) for the monomials $A = a\delta^m$ and $B = b\delta^n$ such that $m + n < 0$. Substituting the monomials into the left-hand-side of the expression (4.21) and using the Leibniz rule (3.6) we obtain

$$\begin{aligned}\mathrm{Tr}(AB) &= -\int_{\mathbb{T}} \frac{1}{\mu} a\delta^m b\delta^n\big|_{\delta=-\frac{1}{\mu}} \Delta x = -\int_{\mathbb{T}} \frac{1}{\mu} a \sum_{k\geqslant 0} S_k^m b\delta^{m+n-k}\Bigg|_{\delta=-\frac{1}{\mu}} \Delta x \\ &= \int_{\mathbb{T}} a \sum_{k\geqslant 0} (-\mu)^{k-m-n-1} S_k^m b \,\Delta x = \int_{\mathbb{T}} ab(-\mu)^{-m-n-1} \Delta x,\end{aligned}$$

where the last equality follows from the relation (3.9). Consequently, (4.21) follows. □

Remark 4.3.3 *Here, we want to investigate the trace form* (4.20) *for two particular cases by reconsidering the Remark 3.1.3. The trace functional* (4.20)

$$\mathrm{Tr}A := \int_{\mathbb{T}} \sum_{i<0} (-\mu)^{-i-1} a_i \,\Delta x = \int_{\mathbb{T}} [a_{-1} + (-\mu)a_{-2} + (-\mu)^2 a_{-3} + \ldots]\Delta x, \tag{4.22}$$

turns out to be the following form when $\mu(x) = 0$;

$$\mathrm{Tr}A = \int_{\mathbb{T}} a_{-1} \,\Delta x. \tag{4.23}$$

Thus, when $\mathbb{T} = \mathbb{R}$, we recover the trace formula for the algebra of pseudo-differential operators [8].

For the case $\mu(x) \neq 0$, by the definition of ξ-operator (3.12)*, the substitution $\delta = -\frac{1}{\mu}$ implies $\xi = -1$ and the trace form* (4.20) *within the algebra of ξ-operators* *is given by*

$$\mathrm{Tr}A := -\int_{\mathbb{T}} \frac{1}{\mu}\, A_{<0}\big|_{\xi=-1}\ \Delta x \equiv -\int_{\mathbb{T}} \frac{1}{\mu} \sum_{i<0} (-1)^i a_i'\ \Delta x, \tag{4.24}$$

with ξ-representation $A = \sum_i a_i'\xi^i$.

The simplest way to define an appropriate inner product is to identify it by a trace form. Thus, we introduce the inner product on $\mathcal{G}$ by the bilinear map $(\cdot,\cdot)_{\mathcal{G}} : \mathcal{G}\times\mathcal{G} \to \mathbb{K}$ and in terms of the trace form (4.20) as follows

$$(A,B)_{\mathcal{G}} := \mathrm{Tr}(AB). \tag{4.25}$$

Theorem 4.3.4 *The inner product* (4.25) *is*

(i) *nondegenerate, i.e. $A = 0$ is the only element of $\mathcal{G}$ fulfilling*

$$(A,B)_{\mathcal{G}} = 0, \quad \forall B \in \mathcal{G}.$$

(ii) *symmetric, i.e.*

$$(A,B)_{\mathcal{G}} = (B,A)_{\mathcal{G}}, \quad \forall A,B \in \mathcal{G}.$$

(iii) ad-*invariant, i.e.*

$$(A,[B,C])_{\mathcal{G}} + ([B,A],C)_{\mathcal{G}} = 0, \quad \forall A,B,C \in \mathcal{G}.$$

Proof. The nondegeneracy of (4.25) follows immediately from the definition of the trace.

In order to show that (4.20) is symmetric, it is enough to make use of the monomials $A = a\delta^m$ and $B = b\delta^n$ once again. Then, depending on $m+n$, we have three cases. If $m,n \geqslant 0$, obviously by the definition of the trace we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA) = 0$. If $m,n < 0$, the Proposition 4.3.2 implies the symmetricity. Therefore, it remains to prove the case when $m.n < 0$. Without loss of generality, let $m > 0$ and $n < 0$. Now, we consider the cases $\mu(x) = 0$ and $\mu(x) \neq 0$, separately.

(i) For $\mu(x) = 0$, applying the generalized Leibniz rule (3.10) to the term $\delta^m b$ below, we have

$$\mathrm{Tr}(AB) = \mathrm{Tr}\left(a\delta^m b\delta^n\right) = \mathrm{Tr}\left(\sum_{k=0}^{m} \tbinom{m}{k} a\Delta^k b\delta^{m+n-k}\right).$$

Since, in this case the trace functional is of form (4.23), $k = m+n+1$ and trace form becomes

$$\mathrm{Tr}(AB) = \int_{\mathbb{T}} \tbinom{m}{m+n+1} a\Delta^{m+n+1} b \;\Delta x. \tag{4.26}$$

Applying the converse formula (3.11) to the below term $a\delta^m$ and using (4.23), we obtain

$$\begin{aligned}\mathrm{Tr}(BA) &= \mathrm{Tr}\left(b\delta^n a\delta^m\right) = \mathrm{Tr}\left(\sum_{k=0}^{m} \tbinom{m}{k} b\delta^{m+n-k} {\Delta^{\dagger}}^{k} a\right) \\ &= \int_{\mathbb{T}} \tbinom{m}{m+n+1} b {\Delta^{\dagger}}^{m+n+1} a \;\Delta x\end{aligned}$$

Using the integration by parts formula (2.32), finally we have

$$\mathrm{Tr}(BA) = \int_{\mathbb{T}} \tbinom{m}{m+n+1} a\Delta^{m+n+1} b \;\Delta x = \mathrm{Tr}(AB) \tag{4.27}$$

which immediately follows the symmetricity.

(ii) For $\mu(x) \neq 0$, we pass to the ξ-pseudo-differential operators. Let $A = a\xi^m$ and $B = b\xi^n$ with $m > 0$ and $n < 0$. Applying the generalized Leibniz rule (3.13) to the below term $\xi^m b$ and using the trace form (4.24), we have

$$\begin{aligned}\mathrm{Tr}(AB) &= \mathrm{Tr}\left(a\xi^m b\xi^n\right) = \mathrm{Tr}\left(\sum_{k=0}^{m} \tbinom{m}{k} a(E-1)^k E^{m-k} b\xi^{m+n-k}\right) \\ &= -\int_{\mathbb{T}} \frac{1}{\mu} \sum_{k=m+n+1}^{m} \tbinom{m}{k} (-1)^{m+n-k} a(E-1)^k E^{m-k} b \;\Delta x.\end{aligned}$$

Applying the converse formula (3.14) to the below term $a\xi^m$ and using (4.24), we have

$$\begin{aligned}\mathrm{Tr}(BA) &= \mathrm{Tr}\left(b\xi^n a\xi^m\right) = \mathrm{Tr}\left(\sum_{k=0}^{m} \tbinom{m}{k} b\xi^{m+n-k} \left(E^{-1}-1\right)^k E^{k-m} a\right) \\ &= -\int_{\mathbb{T}} \frac{1}{\mu} \sum_{k=0}^{m} \tbinom{m}{k} (-1)^{m+n-k} b \left(E^{-1}-1\right)^k E^{k-m} a \;\Delta x\end{aligned}$$

At this point notice the following remark.

Remark 4.3.5 *By* (2.6)*, we have* $E = 1+\mu\Delta$ *and by the use of* (2.33) *we derive the following relation*

$$\begin{aligned}(E\mu)^{\dagger} \quad &= \mu E^{\dagger} = \mu(1+\mu\Delta)^{\dagger} = \mu(1+\Delta^{\dagger}\mu) = \mu - \mu\Delta E^{-1}\mu \\ &= \mu - (E-1)E^{-1}\mu = E^{-1}\mu. \end{aligned} \tag{4.28}$$

Let $f(E)$ be a polynomial function of E. Then by (4.28), it follows that

$$\left(\frac{1}{\mu}f\left(E\right)\right)^{\dagger} = \frac{1}{\mu}f\left(E^{-1}\right)$$

and finally trace form yields as

$$\mathrm{Tr}(BA) = -\int_{\mathbb{T}} \frac{1}{\mu} \sum_{k=m+n+1}^{m} \binom{m}{k}(-1)^{m+n-k} a(E-1)^k E^{m-k} b \; \Delta x = \mathrm{Tr}(AB).$$

The symmetricity of the trace functional on the algebra of ξ-pseudo-differential operators implies the symmetricity of the trace functional on the algebra of δ-pseudo-differential operators for $\mu(x) \neq 0$.

Hence, the inner product (4.25) is symmetric. Finally, since the inner product (4.25) is from now on symmetric and the multiplication operation defined on the algebra $\mathcal{G}$ of δ-pseudo-differential operators is associative, then the inner product (4.25) is ad-invariant. $\square$

The following proposition provides us to interrelate the trace form (4.20) with the ones that will be defined in this section.

Proposition 4.3.6 *The expansion of* $(1+\mu\delta)^{-1}$ *into non-negative order terms of* δ*-pseudo-differential operators is given by*

$$(1+\mu\delta)^{-1} := \sum_{k=0}^{\infty}(-\delta)^k(\mu^k + \Delta\mu^{k+1}) \equiv \sum_{k=0}^{\infty}(-\delta)^k \frac{E\mu^{k+1}}{\mu}, \tag{4.29}$$

while its expansion into negative order terms is

$$(1+\mu\delta)^{-1} := -\sum_{k=1}^{\infty}(-\delta)^{-k}\frac{1}{\mu E\mu^{k-1}}. \tag{4.30}$$

Note that, the first expansion (4.29) is valid for all points of $\mathbb{T}$ including the dense points, however the second expansion (4.30) is valid only for $\mu \neq 0$. It is sufficient to prove the first expansion (4.29).

Proof. We verify the Proposition 4.3.6 by multiplying both sides of the expression (4.29) with $(1+\mu\delta)$ from right-hand side. Then, using (3.1) we have

$$\begin{aligned}
(1+\mu\delta)^{-1}(1+\mu\delta) &= \sum_{k=0}^{\infty}(-\delta)^k \frac{E\mu^{k+1}}{\mu} + \sum_{k=0}^{\infty}(-\delta)^k E\mu^{k+1}\delta \\
&= \sum_{k=0}^{\infty}(-\delta)^k \frac{E\mu^{k+1}}{\mu} + \sum_{k=0}^{\infty}(-\delta)^k (\delta\mu^{k+1} - \Delta\mu^{k+1}) \\
&= \sum_{k=0}^{\infty}(-\delta)^k \frac{E\mu^{k+1}}{\mu} - \sum_{k=0}^{\infty}(-\delta)^{k+1}\mu^{k+1} - \sum_{k=0}^{\infty}(-\delta)^k \Delta\mu^{k+1} \\
&= \frac{E\mu}{\mu} - \Delta\mu + \sum_{k=1}^{\infty}(-\delta)^k \left(\frac{E\mu^{k+1}}{\mu} - \mu^k - \Delta\mu^{k+1}\right) = 1.
\end{aligned}$$

Similarly one can verify the second expansion (4.30). □

Proposition 4.3.7 *The trace form* (4.20) *is equivalent to the following trace form*

$$\operatorname{Tr} A = \int_{\mathbb{T}} \frac{E^{-1}\mu}{\mu} \operatorname{res}\left(A(1+\mu\delta)^{-1}\right)\Delta x, \tag{4.31}$$

where

$$\operatorname{res} A := a_{-1} \qquad \textit{for} \qquad A = \sum_i a_i \delta^i.$$

Proof. First we calculate the residue term $\operatorname{res}\left(A(1+\mu\delta)^{-1}\right)$, by assuming the expansion (4.29) of $(1+\mu\delta)^{-1}$ into nonnegative terms.

$$\begin{aligned}
\operatorname{res}\left(A(1+\mu\delta)^{-1}\right) &= \operatorname{res}\left(\sum_{k=0}^{\infty}\sum_i (-1)^k a_i \delta^{i+k} \frac{(E\mu)^{k+1}}{\mu}\right) \\
&= \operatorname{res}\left(\sum_{i<0}(-1)^{-i-1} a_i \delta^{-1} \frac{(E\mu)^{-i}}{\mu} + \dots\right)
\end{aligned}$$

Using the rule (3.2), residue follows as

$$\begin{aligned}\operatorname{res}\left(A(1+\mu\delta)^{-1}\right) &= \operatorname{res}\left(\sum_{i<0}(-1)^{-i-1}a_i\frac{\mu^{-i}}{E^{-1}\mu}\delta^{-1}+\dots\right)\\ &= -\sum_{i<0}\frac{(-\mu)^{-i}}{E^{-1}\mu}a_i.\end{aligned}$$

Substituting the residue into the trace form (4.31), we obtain

$$\begin{aligned}\mathrm{Tr}A &= \int_{\mathbb{T}}\frac{E^{-1}\mu}{\mu}\operatorname{res}\left(A(1+\mu\delta)^{-1}\right)\Delta x = \int_{\mathbb{T}}\sum_{i<0}(-\mu)^{-i-1}a_i\ \Delta x\\ &= -\int_{\mathbb{T}}\frac{1}{\mu}\left(A_{<0}\right)\big|_{\delta=-\frac{1}{\mu}}\ \Delta x\end{aligned}$$

which ensures that the trace forms (4.20) and (4.31) are equivalent. □

Remark 4.3.8 *The trace form* (4.31) *is the most general form for the trace functional. We proved in the previous Proposition that the trace form* (4.31) *is equivalent to the form* (4.20) *if the expansion* (4.29) *of* $(1+\mu\delta)^{-1}$ *into nonnegative order terms are considered. If on the contrary, we make use of the expansion* (4.30) *of* $(1+\mu\delta)^{-1}$ *into negative order terms, the trace formula* (4.31) *yields the following trace form*

$$\mathrm{Tr}'A := \int_{\mathbb{T}}\frac{1}{\mu}\ A_{\geqslant 0}\big|_{\delta=-\frac{1}{\mu}}\ \Delta x \equiv -\int_{\mathbb{T}}\sum_{i\geqslant 0}(-\mu)^{-i-1}a_i\ \Delta x. \tag{4.32}$$

Observe that, this alternative trace form (4.32) *is valid on regular-discrete time scales, i.e. when* $\mu\neq 0$.

In order to show the correspondence between the trace form (4.32) and the trace form of the algebra of shift operators explicitly, we make use of the relation (3.74), which is valid $\mu\neq 0$. Now, if we assume that δ^{-1} expands into negative order terms of shift operator $\mathcal{E}$ and we expand the operator A by means of shift operators $\mathcal{E}$, as $A=\sum_i a_i'\mathcal{E}^i$, then from the alternative trace form (4.32) we regain the standard trace form of the algebra of shift operators

$$\mathrm{Tr}'A := \int_{\mathbb{T}}\frac{1}{\mu}a_0'\ \Delta x.$$

The traces (4.20) and (4.32) are not equivalent in general, although they are produced from the most general trace form (4.31). This lies in using different expansions of $(1+\mu\delta)^{-1}$. Nevertheless, they are closely related to each other on regular-discrete time scales. To be more precise, consider the constraints (3.77) for the Lax operators of the form (3.31). For an arbitrary constrained operator $A|_{\delta=-\frac{1}{\mu}} = const$, it is clear that,

$$A|_{\delta=-\frac{1}{\mu}} = (A_{\geqslant 0} + A_{<0})|_{\delta=-\frac{1}{\mu}} = A_{\geqslant 0}|_{\delta=-\frac{1}{\mu}} + A_{<0}|_{\delta=-\frac{1}{\mu}} \tag{4.33}$$

which implies that

$$A_{\geqslant 0}|_{\delta=-\frac{1}{\mu}} = -\, A_{<0}|_{\delta=-\frac{1}{\mu}} + const.$$

Thus, on regular-discrete time scales if we apply the traces (4.20) and (4.32) to the constrained operator $A|_{\delta=-\frac{1}{\mu}} = const$, then both traces yield the same results up to a constant. Hence, the traces (4.20) and (4.32) are equivalent up to a constant if the constraints are taken into consideration.

Note that, by similar observations for $\mathbb{T} = \mathbb{K}_q$, one recovers from (4.32) the trace form of q-discrete numbers (we refer the appendix of [45]).

As a summary, we state the following Remark involving the relationships between the trace forms introduced in this section.

Remark 4.3.9 *The trace form* (4.20) *is valid on arbitrary regular time scales and in particular for $\mathbb{T} = \mathbb{R}$, it produces the standard trace form of pseudo-differential operators. Furthermore, if the appropriate constraints are taken into consideration,* (4.20) *also recovers the trace forms for $\mathbb{T} = \mathbb{Z}$ of lattice shift operators and for $\mathbb{T} = \mathbb{K}_q$ of q-discrete numbers.*

Hence, we establish an appropriate trace form which is well-defined on an arbitrary regular time scale. More impressively, in this work, we fulfill the gap of a trace form which unifies and generalizes the trace forms being studied in the literature.

4.4 Bi-Hamiltonian structures on regular time scales

In order to define the Hamiltonian structures for the Lax hierarchy (3.28), we need to derive the Poisson tensors. For this purpose, by the use of the relation

$$(A, RB)_{\mathcal{G}} = (R^{\dagger}A, B)_{\mathcal{G}},$$

the adjoint of R-matrices (3.25), $R^{\dagger}$, is found as

$$R^{\dagger} = P^{\dagger}_{\geqslant k} - \frac{1}{2} \qquad k = 0, 1 \ , \tag{4.34}$$

where trace form (4.31) implies

$$P^{\dagger}_{\geqslant k}A = \left(A(1+\mu\delta)^{-1}\right)_{<-k}(1+\mu\delta). \tag{4.35}$$

Here the projections are of the form

$$B_{<-k} = \sum_{i<-k} \delta^i b_i \qquad \text{for} \qquad B = \sum_i \delta^i b_i.$$

which are hardly different than the projections performed in (3.26).

The existence of the well-defined inner product (4.25) allows us to identify the Lie algebra $\mathcal{G}$ of δ-pseudo-differential operators with its dual $\mathcal{G}^*$.

Remark 4.4.1 *The general theory of bi-Hamiltonian structures are presented in section 4.1 due to the linear space $\mathcal{U}$ of smooth functions which corresponds, in our case, to the space $\mathcal{U}$ of Δ-smooth functions on regular time scales. In order to utilize a very essential tool, classical R-matrix formalism, which allows to produce infinite hierarchy of mutually commuting symmetries together with bi-Hamiltonian structures at once, we have to pass from the linear space $\mathcal{U}$ of Δ-smooth functions on regular time scales to a Lie algebraic setting. For this purpose, let $\iota : \mathcal{U} \to \mathcal{G}^*$, be the embedding of the linear space $\mathcal{U}$ into the algebra $\mathcal{G} \cong \mathcal{G}^*$ of δ-pseudo-differential operators*

$$\iota : \mathcal{U} \to \mathcal{G}^* \cong \mathcal{G} \qquad u \to \iota(u) = \eta$$
$$d\iota : \mathcal{V} \to \mathcal{G}^* \cong \mathcal{G} \qquad u_t \to d\iota(u_t) = \eta_t,$$

where $d\iota$ is the differential of the embedding. Then every functional $F : \mathcal{U} \to \mathbb{R}$ can be extended to a Δ-smooth function on $\mathcal{G}^* \cong \mathcal{G}$. Therefore, let $\mathcal{F}(\mathcal{G} \cong \mathcal{G}^*)$ be the space of smooth function on $\mathcal{G}^* \cong \mathcal{G}$ of the form $F \circ \iota^{-1} : \mathcal{G} \cong \mathcal{G}^* \to \mathbb{R}$, consisting of functionals (4.18). Then, the differentials $dF(\eta)$ of $F(\eta) \in \mathcal{F}(\mathcal{G} \cong \mathcal{G}^*)$, at the point $\eta \in \mathcal{G}^* \cong \mathcal{G}$ belong to $\mathcal{G}$.

Hence, we formulate the bi-Hamiltonian system of integrable Δ-differential equations (3.28) as follows

$$L_{t_n} = \pi_0 dH_n = \pi_1 dH_{n-N}, \tag{4.36}$$

where $H \in \mathcal{F}(\mathcal{G} \cong \mathcal{G}^*)$ are constructed in terms of (4.20)

$$H_n(L) = \frac{N}{n+N} \mathrm{Tr}\left(L^{\frac{n}{N}+1}\right) \tag{4.37}$$

and the differentials dH belong to $\mathcal{G} \cong \mathcal{G}^*$.

Note that, the functionals (4.37) are the related Hamiltonians (conserved quantites) (integrals of motion) since the derivative of H_n with respect to time parameter t vanishes. They are such that $dH_n = L^{\frac{n}{N}}$.

The linear Poisson tensor [13, 46] has the general form;

$$\pi_0 : dH \to [RdH, L] + R^\dagger [dH, L].$$

Then, the R-matrix and its adjoint allows us to derive the linear Poisson tensor as follows:

$$\begin{aligned} \pi_0 dH &= [RdH, L] + R^\dagger [dH, L] \\ &= [L, dH_{<k}] + \left([dH, L](1+\mu\delta)^{-1}\right)_{<-k} (1+\mu\delta) \qquad k = 0, 1. \end{aligned} \tag{4.38}$$

Since there appears additional conditions on R and $R^\dagger$ (4.34) with the chaotic projection (4.35), we do not construct the quadratic Poisson tensor by proceeding the R-matrix scheme [13, 46, 14, 15]. Thus, rather than the standard procedure, we utilize the recursion operators Φ, derived for the Lax hierarchies (3.28) such that

$$\Phi L_{t_n} = L_{t_{n+N}}. \tag{4.39}$$

Since, the linear Poisson tensor π_0 is formulated as in (4.38), the quadratic Poisson tensor π_1 can be reconstructed by the frame of π_0 and the recursion operator Φ, i.e.

$$\pi_1 = \Phi\pi_0. \tag{4.40}$$

The recursion operator Φ is hereditary [47], [48] at least on the vector space spanned by the symmetries from the related Lax hierarchy (3.28). In some particular degenerated cases, the recursion operator Φ may not be hereditary and therefore equivalently π_1 may not be compatible with π_0 or in a worse case, π_1 may not even a Poisson tensor. In general, showing the fact that an operator is hereditary, i.e it is an operator with vanishing Nijenhuis torsion [49] is so troublesome that we omit this calculation. The following remark guarantees the hereditariness property of Φ, which is closely related with the compatibility of Poisson tensors.

Remark 4.4.2 *We consider the quadratic Poisson tensor π_1 for dense points $(\mu = 0)$ and for regular-discrete points $\mu \neq 0$, separately. When $\mu = 0$, the construction of π_1 within the algebra of δ-pseudo-differential operators, using the generalized Leibniz rule* (3.10)*, proceeds parallel to the construction by the frame of the algebra of pseudo-differential operators [11, 12]. On the other hand, when $\mu \neq 0$, the construction of π_1 on regular-discrete time scales, using* (3.13)*, is completely parallel to the construction by means of the algebra of shift operators [15]. In this case, note that dependence on μ, different than a scalar, should be taken into consideration. Therefore, the construction of π_1 in both cases assures that it is a Poisson tensor and furthermore it is compatible with π_0. Hence, the recursion operator $\Phi = \pi_1\pi_0^{-1}$, fulfilling* (4.39) *is hereditary.*

When it comes to derive the differentials dH with respect to Lax operators (3.31), we present them in an implicit form given in the following scheme. Let

$$dH = \sum_{i=1}^{n} \delta^{i-N-k}\gamma_i, \tag{4.41}$$

where N is the order of the Lax operator (3.31), n is the number of the rest of the dynamical fields of (3.31) after taking the constraint (3.77) into consideration and clearly k is either 0 or 1. Our aim is to express γ_i's in terms of dynamical fields of (3.31) and their variational derivatives. For this purpose, we assume that

$$(dH, L_t)_{\mathcal{G}} = \int_{\mathbb{T}} \left(\sum_{i=k}^{N+k-2} \frac{\delta H}{\delta u_i}(u_i)_t + \sum_s \left(\frac{\delta H}{\delta \psi_s}(\psi_s)_t + \frac{\delta H}{\delta \phi_s}(\psi_s)_t \right) \right) \Delta x. \qquad (4.42)$$

where the dynamical fields u, ψ, φ belong to the Lax operator (3.31). Therefore, substituting the form (4.41) into the ansatz (4.42), the terms γ_i ' s can be written in terms of the related dynamical fields.

We end up this section with some formulae used in the calculations of the linear Poisson tensor.

$$\begin{aligned} P^{\dagger}_{\geqslant 0}(a\delta^{-1}b) &= a\delta^{-1}b + \mu ab \\ P^{\dagger}_{\geqslant 1}(a\delta^{-1}b) &= a\delta^{-1}b - \delta^{-1}ab \\ P^{\dagger}_{\geqslant 1}(\delta^{-1}a\delta^{-1}b) &= \delta^{-1}a\delta^{-1}b + \delta^{-1}\mu ab. \end{aligned}$$

4.5 Examples: Δ-differential AKNS and Kaup-Broer

In this section, we fill the gap of the bi-Hamiltonian structures of the finite-field examples Δ-differential AKNS and Δ-differential Kaup-Broer, presented in Chapter 3. These Δ-differential illustrations are chosen in such a way that they are the simplest Δ-differential examples and at the same time they are counterparts of famous field and lattice soliton systems.

Example 4.5.1 *Δ-differential AKNS, $k = 0$: For the Lax operator* (3.82) *with the constraint* (3.83), *we have $N = 1$, $n = 2$. Thus, in this case,* (4.41) *implies that the differential for Δ-differential AKNS is of the form*

$$dH = \gamma_1 + \delta\gamma_2, \qquad (4.43)$$

where

$$\begin{aligned}
\gamma_1 &= \frac{1}{\varphi}\frac{\delta H}{\delta\psi} + \frac{\Delta^\dagger(\varphi)}{\psi\varphi E^{-1}(\varphi)}\Delta^{-1}(A),\\
\gamma_2 &= -\frac{1}{\psi E^{-1}(\varphi)}\Delta^{-1}(A),
\end{aligned}$$

and

$$A = \psi\frac{\delta H}{\delta\psi} - \varphi\frac{\delta H}{\delta\varphi}$$

Here Δ^{-1} *is a formal inverse of* Δ *and adjoint* $\Delta^\dagger$ *is applied to only* φ*. Then, the general form* (4.38) *implies the linear Poisson tensor*

$$\pi_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.44}$$

The quadratic Poisson tensor based on the recursion operator (3.92) *is*

$$\pi_1 = \Phi\pi_0 = \begin{pmatrix} -\mu\psi^2 - 2\psi\Delta^{-1}\psi & \Delta + 2\mu\psi\varphi + 2\psi\Delta^{-1}\varphi \\ -\Delta^\dagger + 2\varphi\Delta^{-1}\psi & -\mu\varphi^2 - 2\varphi\Delta^{-1}\varphi \end{pmatrix}. \tag{4.45}$$

The first three Hamiltonians are

$$\begin{aligned}
H_0 &= \int_{\mathbb{T}} \psi\varphi\ \Delta x\\
H_1 &= \int_{\mathbb{T}} \left(\frac{1}{2}\mu\psi^2\varphi^2 + \varphi\Delta\psi\right)\Delta x\\
H_2 &= \int_{\mathbb{T}} \left(\frac{1}{3}\mu^2\psi^3\varphi^3 + \psi^2\varphi^2 + \varphi\Delta^2\psi + \mu\psi\varphi^2\Delta\psi + \mu\psi^2\varphi\Delta^\dagger\varphi\right)\Delta x
\end{aligned}$$

In order to check the bi-Hamiltonian property (4.36) *for this example, let us rewrite the first two flows* (3.84), (3.87) *in terms of* Δ *and* $\Delta^\dagger$ *only. Thus, we have*

$$\begin{aligned}
\psi_{t_1} &= \mu\psi^2\varphi + \Delta\psi,\\
\varphi_{t_1} &= -\mu\varphi^2\psi - \Delta^\dagger\varphi.
\end{aligned} \tag{4.46}$$

and

$$\begin{aligned}
\psi_{t_2} &= \mu^2\psi^3\varphi^2 + 2\psi^2\varphi + \Delta^2\psi + \Delta\left(\mu\psi^2\varphi\right) + 2\mu\psi\varphi\Delta\psi + \mu\psi^2\Delta^\dagger\varphi\\
\varphi_{t_2} &= -\mu^2\psi^2\varphi^3 - 2\psi\varphi^2 - {\Delta^\dagger}^2\varphi - \Delta^\dagger\left(\mu\psi\varphi^2\right) - \mu\varphi^2\Delta\psi - 2\mu\psi\varphi\Delta^\dagger\varphi.
\end{aligned} \tag{4.47}$$

In particular, when $\mathbb{T} = \mathbb{R}$ the above bi-Hamiltonian hierarchy becomes exactly the bi-Hamiltonian field soliton AKNS hierarchy [12]. For $\mathbb{T} = \mathbb{Z}$, the system (4.46)*, together with its bi-Hamiltonian structure, is equivalent to the system considered in [15].*

Example 4.5.2 *Δ-differential Kaup-Broer, $k = 1$: For the Lax operator* (3.124) *with the constraint* (3.125)*, it is clear that $N = 1$ and $n = 2$. Then, from the implicit form* (4.41)*, the differentials yields as*

$$dH = \delta^{-1}\gamma_1 + \gamma_2,$$

where

$$\gamma_1 = \frac{\delta H}{\delta v}, \qquad \gamma_2 = \frac{\delta H}{\delta w} + \mu\frac{\delta H}{\delta v}. \tag{4.48}$$

Thus, the general form (4.38) *implies the linear Poisson tensor*

$$\pi_0 = \begin{pmatrix} u\Delta\mu - \mu\Delta^\dagger u & u\Delta \\ -\Delta^\dagger u & 0 \end{pmatrix}.$$

The recursion operator has the form

$$\Phi = \begin{pmatrix} w + u\Delta + R & \mu v - \mu w + (2 + \mu\Delta^\dagger)u - R\mu \\ w - \Delta^\dagger uw\Delta^{-1}u^{-1} & \Delta^\dagger u + v - \mu w\Delta^\dagger uw\Delta^{-1}\mu u^{-1} \end{pmatrix},$$

where

$$R = u\Delta v\Delta^{-1}u^{-1} - \mu\Delta^\dagger uw\Delta^{-1}u^{-1}.$$

Hence

$$\pi_1 = \Phi\pi_0 = \begin{pmatrix} \pi_{vv} & u\Delta v + u\Delta u\Delta + \mu uw\Delta - \mu\Delta^\dagger uw \\ -v\Delta^\dagger u - \Delta^\dagger u\Delta^\dagger u + uw\Delta\mu - \Delta^\dagger\mu uw & uw\Delta - \Delta^\dagger uw \end{pmatrix},$$

where

$$\pi_{vv} = u\Delta\mu v - \mu v\Delta^\dagger u + u\Delta u - u\Delta^\dagger u + u\Delta u\Delta\mu - \mu\Delta^\dagger u\Delta^\dagger u + \mu uw\Delta\mu - \mu\Delta^\dagger\mu uw.$$

The first three Hamiltonians are calculated as

$$\begin{aligned} H_0 &= \int_{\mathbb{T}} w\ \Delta x \\ H_1 &= \int_{\mathbb{T}} \left(vw - \frac{1}{2}\mu w^2\right)\Delta x \\ H_2 &= \int_{\mathbb{T}} \left(w^2 + v^2 w + (w + \mu vw - \mu^2 w^2)\Delta v - \frac{2}{3}\mu^2 w^3\right)\Delta x \end{aligned}$$

Similarly, let us rewrite the first two flows (3.127), (3.129) *in terms of* Δ *and* $\Delta^\dagger$, *as follows*

$$\begin{aligned} v_{t_1} &= (1+\mu v-\mu^2 w)\Delta v-\mu\Delta^\dagger(w+\mu vw-\mu^2w^2),\\ w_{t_1} &= -\Delta^\dagger\left(w+\mu vw-\mu^2w^2\right), \end{aligned} \tag{4.49}$$

and

$$\begin{aligned} v_{t_2} &= u\Delta\left(v^2+2uw+u\Delta v+\mu\Delta^\dagger(uw)\right)-\mu\Delta^\dagger\left(2uvw+\mu uw\Delta v+u\Delta^\dagger(uw)\right),\\ w_{t_2} &= -\Delta^\dagger\left(2uvw+\mu uw\Delta v+u\Delta^\dagger(uw)\right). \end{aligned} \tag{4.50}$$

When $\mathbb{T}=\mathbb{R}$ *the above construction recovers the field Kaup-Broer hierarchy with its bi-Hamiltonian structure [11]. In* $\mathbb{T}=\mathbb{Z}$ *case, the above bi-Hamiltonian hierarchy is equivalent to the relativistic Toda hierarchy considered in [15] and,* (4.49) *is equivalent to the relativistic Toda system.*

Chapter 5

Integrable discrete systems on $\mathbb{R}$

5.1 One-parameter regular grain structures on $\mathbb{R}$

The main goal of this chapter is the formulation of a general unifying framework of integrable discrete systems on $\mathbb{R}$, that contains lattice soliton systems as well as q-discrete systems as particular cases, in such a way that the domain of dynamical fields u is always $\mathbb{R}$. The theory presented in this chapter is based on the article [25]. We first introduce the concept of a regular grain structure on $\mathbb{R}$ which is described by discrete one-parameter groups of diffeomorphisms $\sigma_{m\hbar}(x)$. We construct the shift operators by means of forward jump operator, i.e. $\sigma(x) = \sigma_{\hbar}(x)$.

Reconsider the forward (2.1) and backward jump operators (2.2) such that they are the maps of the form $\sigma : \mathbb{R} \to \mathbb{R}$ and $\rho : \mathbb{R} \to \mathbb{R}$, (i.e. here $\mathbb{T} = \mathbb{R}$). We introduce the range of all possible points to which we map x by forward and backward operators (including x), as

$$\mathbb{G}_x := \{\rho^n(x) : n \in \mathbb{Z}_+\} \cup \{x\} \cup \{\sigma^n(x) : n \in \mathbb{Z}_+\}. \tag{5.1}$$

Thus, for each point x of $\mathbb{R}$, we associate a set $\mathbb{G}_x$.

Definition 5.1.1 *The union of all* $\mathbb{G}_x$*, given by*

$$\mathbb{G} := \bigcup_{x\in\mathbb{R}} \mathbb{G}_x.$$

is called the grain structure on $\mathbb{R}$*.*

Similar to the definition 2.1.7 of regularity, we define the regular grain structures.

Definition 5.1.2 *If there exist inverse maps* σ^{-1} *and* ρ^{-1}*, such that* $\sigma(x) = \rho^{-1}(x)$ *and* $\rho(x) = \sigma^{-1}(x)$ *for all* $x \in \mathbb{R}$*, then* $\mathbb{G}$ *is called as the regular grain structure.*

If we assume that $\sigma^0 \equiv \mathrm{id}_{\mathbb{R}}$, then the set (5.1) turns out to be $\mathbb{G}_x = \{\sigma^n(x) : n \in \mathbb{Z}\}$. Note that, bijective σ defines a discrete one-parameter group of bijections on $\mathbb{R}$:

$$\mathbb{Z} \ni m \mapsto \{\sigma_m : \mathbb{R} \to \mathbb{R}\},$$

such that $\sigma_m := \sigma^m$, and on the other direction one-parameter group of bijections on $\mathbb{R}$ defines the regular grain structure on $\mathbb{R}$ with the forward jump operator defined by $\sigma := \sigma_1$. Therefore, it is clear that the regular grain structure introduces equivalence classes between points of $\mathbb{R}$, such that

$$x \sim y; \quad \text{if} \quad \mathbb{G}_x = \mathbb{G}_y \qquad x, y \in \mathbb{R},$$

in other words, there exists $k \in \mathbb{Z}$ such that $y = \sigma^k(x)$.

As we are interested in infinite-dimensional systems of smooth dynamical fields, it is better to introduce a regular grain structure $\mathbb{G}$ on $\mathbb{R}$ by one-parameter group of diffeomorphisms instead of bijections. Let $\mathbb{Z} \ni m \mapsto \sigma_{m\hbar}$ be a discrete one-parameter group of diffeomorphisms on $\mathbb{R}$: $\sigma_{m\hbar} : \mathbb{R} \to \mathbb{R}$, i.e

$$\sigma_0(x) = x \quad \text{and} \quad \sigma_{m\hbar}(\sigma_{n\hbar}(x)) = \sigma_{(m+n)\hbar}(x) \qquad m, n \in \mathbb{Z},$$

where $\hbar > 0$ is some deformation parameter. It follows that $(\sigma_{n\hbar})^{-1}(x) = \sigma_{-n\hbar}(x)$. Consider the continuous one-parameter group of diffeomorphisms

$$t \mapsto \sigma_t$$

with the deformation parameter $t \in \mathbb{R}$. By Taylor expansion of σ_t around $t = 0$,

$$\sigma_t(x) = \sigma_0(x) + t.\left.\frac{d\sigma_t(x)}{dt}\right|_{t=0} + O(t^2) = x + t.\left.\frac{d\sigma_t(x)}{dt}\right|_{t=0} + O(t^2), \qquad (5.2)$$

it is clear that one-parameter group of diffeomorphism is generated by a vector field which is called as the *infinitesimal generator*, denoted by $\mathcal{X}(x)\partial_x$;

$$\sigma_t(x) = x + t.\mathcal{X}(x) + O(t^2). \qquad (5.3)$$

We assume that the component $\mathcal{X}(x)$ is smooth on $\mathbb{R}$ except at most finite number of points. Thus, there is a one-to-one correspondence between one-parameter group of diffeomorphisms and their infinitesimal generators,

$$\mathcal{X}(x) = \left.\frac{d\sigma_t(x)}{dt}\right|_{t=0} \qquad \Leftrightarrow \qquad \frac{d\sigma_t(x)}{dt} = \mathcal{X}\left(\sigma_t(x)\right). \qquad (5.4)$$

Arbitrary $\mathcal{X}\partial_x$ generates a continuous one-parameter group of diffeomorphisms only when it is a *complete* vector field, for which maximal integrals are defined on the whole $\mathbb{R}$, i.e. $\mathbb{R}$ is a domain of the mapping $t \mapsto \sigma_t$. In such a case, the above discrete one-parameter group is well-defined since it is enough to consider subgroup $\mathbb{Z}$ of $\mathbb{R}$. Incomplete $\mathcal{X}\partial_x$ might still well define a discrete group of diffeomorphisms, if $\hbar$ is properly chosen.

Lemma 5.1.3 *Let $\sigma_t(x)$ be a one-parameter group of diffeomorphisms generated by $\mathcal{X}(x)\partial_x$. Then, the following relation is valid*

$$\mathcal{X}(x)\frac{d\sigma_t(x)}{dx} = \mathcal{X}\left(\sigma_t(x)\right). \qquad (5.5)$$

Proof. From (5.4) one observes that

$$\mathcal{X}\left(\sigma_{s+t}(x)\right) = \mathcal{X}\left(\sigma_s(\sigma_t(x))\right) = \frac{d\sigma_{s+t}(x)}{ds}. \qquad (5.6)$$

Acting σ_s on the left-hand-side of (5.5), we have the following relation

$$\mathcal{X}\left(\sigma_s(x)\right)\frac{d\sigma_{s+t}(x)}{d\sigma_s(x)} = \frac{d\sigma_s(x)}{ds}.\frac{d\sigma_{s+t}(x)}{d\sigma_s(x)} = \frac{d\sigma_{s+t}(x)}{ds} = \mathcal{X}\left(\sigma_{s+t}(x)\right),$$

which implies the desired relation (5.5). □

The computation of one-parameter group generated by a vector field is often referred to as an exponentiation of the infinitesimal generator. Therefore, an element of one-parameter group of diffeomorphisms, $\sigma_{\hbar}(x)$ is computed as

$$\sigma_{\hbar}(x) = e^{\hbar\mathcal{X}(x)\partial_x}x, \tag{5.7}$$

equivalently we have

$$e^{\hbar\mathcal{X}(x)\partial_x}u(x) = u\left(e^{\hbar\mathcal{X}(x)\partial_x}x\right). \tag{5.8}$$

Similar to (2.17), we use the following notation for the shift operator E

$$E^m u(x) := (E^m u)(x) = u\left(\sigma_{m\hbar}(x)\right), \qquad m \in \mathbb{Z}, \tag{5.9}$$

where $u(x)$ is a dynamical field. Note that, the shift operator E is compatible with the grain structure defined by $\sigma_{\hbar}(x)$. The formulae (5.7) and (5.8) are valid on the whole real line if $\mathcal{X}(x)\partial_x$ is complete or where a discrete one-parameter group of diffeomorphisms is well defined. Thus, the shift operator E can be identified with $e^{\hbar\mathcal{X}(x)\partial_x}$, i.e.

$$E^m \equiv e^{m\hbar\mathcal{X}(x)\partial_x}. \tag{5.10}$$

Example 5.1.4 *Consider vector fields of the form*

$$\mathcal{X}(x)\partial_x = x^{1-n}\partial_x, \qquad n \in \mathbb{Z}.$$

(i) For $n = 0$, let $y = \ln x$, then $\partial_y = x\partial_x$, which implies that

$$\sigma_t(x) = e^{tx\partial_x} = e^{t\partial_y}x = e^{t\partial_y}e^y = e^{y+t} = e^t.x$$

and

$$\sigma_{m\hbar}(x) = e^{m\hbar}x = q^m x \qquad q \equiv e^{\hbar},$$

which is defined for all $t \in \mathbb{R}$ and so $\mathcal{X}\partial_x = x\partial_x$ is a complete vector field. Therefore, when $n = 0$, we deal with systems of 'q-discrete' type.

(ii) For $n \neq 0$, let $y = \frac{1}{n}x^n$. Then, $\partial_y = x^{1-n}\partial_x$ and $\sigma_t(x)$ has the general implicit form

$$\sigma_t(x) = e^{tx^{1-n}\partial_x}x = e^{t\partial_y}x = e^{t\partial_y}(ny)^{\frac{1}{n}} = (n(y+t))^{\frac{1}{n}} = (x^n + nt)^{\frac{1}{n}},$$

which implies that

$$(\sigma_t(x))^n = x^n + nt.$$

As a subcase when $n = 1$, we have

$$\sigma_t(x) = x + t \qquad \Rightarrow \qquad \sigma_{m\hbar}(x) = x + m\hbar,$$

and $\mathcal{X}\partial_x = \partial_x$ is obviously complete. In this case we deal with systems of 'lattice' type. However, when $n = -1$, the related vector field $\mathcal{X}\partial_x = x^2\partial_x$ is incomplete as $t \neq \frac{1}{x}$:

$$\sigma_t(x) = \frac{x}{1 - tx} \qquad \Rightarrow \qquad \sigma_{m\hbar}(x) = \frac{x}{1 - m\hbar\, x}.$$

On the other hand, if $x \neq \frac{1}{m\hbar}$, the related discrete one-parameter group of diffeomorphisms is well-defined. When n is odd, it is always possible to define discrete one-parameter group of diffeomorphisms generated by $\mathcal{X}\partial_x = x^{1-n}\partial_x$.

Remark 5.1.5 *All integrable discrete systems defined by different vector fields $\mathcal{X}(x)\partial_x$ are not equivalent. However, it is possible to find a local transformation relating respective vector fields. If we reconsider $\mathcal{X}(x) = x^{1-n}$ for odd $n \neq 0$ and another component $\mathcal{X}'(x') = 1$ (the lattice case), we deduce that $x' = \frac{1}{n}x^n$ is a bijection on $\mathbb{R} \setminus \{0\}$. Hence, all discrete systems generated by $\mathcal{X}\partial_x = x^{1-n}\partial_x$, with odd n, can be reduced to the original lattice Toda type systems, excluding the point $x = 0$. However, for $n = 0$, $\mathcal{X}(x) = x$ (the q-discrete case) and let $\mathcal{X}'(x') = 1$. Then, we have $x = e^{x'}$ which is not a bijection. However, if the domain of dynamical fields of q-discrete systems is restricted to $x \in \mathbb{R}_+$, then the above map is a bijection and q-discrete systems on $\mathbb{R}_+$ are equivalent to the lattice systems on $\mathbb{R}$.*

5.2 Difference-differential-Systems

In order to introduce a phase space related to discrete systems, consider an infinite-tuple of dynamical fields

$$\mathbf{u} := (u_0(x), u_1(x), u_2(x), ...)^{\mathrm{T}}$$

where each $u_i : \mathbb{R} \to \mathbb{K}$ assumes values on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{U}$ be a linear topological space, with local independent coordinates $\mathbf{u}\left(\sigma_{m\hbar}(x)\right)$ for all $m \in \mathbb{Z}$, which defines infinite-dimensional phase space. Let $\mathcal{C}$ be the algebra over $\mathbb{K}$ of functions on $\mathcal{U}$ of the form

$$f\left[\mathbf{u}\right] := \sum_{m \geqslant 0} \sum_{i_1,\ldots,i_m \geqslant 0} \sum_{s_1,\ldots,s_m \in \mathbb{Z}} a^{i_1 i_2 \ldots i_m}_{s_1 s_2 \ldots s_m} (E^{s_1} u_{i_1})(E^{s_2} u_{i_2}) \cdots (E^{s_m} u_{i_m}) \qquad (5.11)$$

where $f\left[\mathbf{u}\right]$ are polynomials in $\mathbf{u}\left(\sigma_{m\hbar}(x)\right)$ of finite order, with coefficients $a^{i_1 i_2 \ldots i_m}_{s_1 s_2 \ldots s_m} \in \mathbb{K}$ and the shift operator E is defined in (5.9). The algebra $\mathcal{C}$ can be extended into operator algebra $\mathcal{C}\left[E, E^{-1}\right]$ ($\mathcal{C}[x, y, \ldots]$ stands for the linear space of polynomials in $x, y, \ldots$ with coefficients from $\mathcal{C}$). A space $\mathcal{F} = \{F : \mathcal{U} \to \mathbb{K}\}$ of functions on $\mathcal{U}$ is defined through linear functionals

$$\int (\cdot)\, d_\hbar x : \mathcal{C} \to \mathbb{K} \qquad f[\mathbf{u}] \mapsto F(\mathbf{u}) := \int f[\mathbf{u}]\, d_\hbar x, \qquad (5.12)$$

such that the following property is fulfilled

$$\int E f[\mathbf{u}]\, d_\hbar x = \int f[\mathbf{u}]\, d_\hbar x. \qquad (5.13)$$

Here $\int d_\hbar x$ is a formal integration symbol.

Definition 5.2.1 *The explicit form of appropriate functionals can be introduced in two ways.*

(i) A discrete representation is defined as

$$F(\mathbf{u}) = \int f[\mathbf{u}]\, d_\hbar x := \hbar \sum_{n \in \mathbb{Z}} f\left[\mathbf{u}\left(\sigma_{n\hbar}(x)\right)\right]. \qquad (5.14)$$

(ii) A continuous representation is given as

$$F(\mathbf{u}) = \int f[\mathbf{u}]\, d_\hbar x := \int_{-\infty}^{\infty} f[\mathbf{u}(x)] \frac{dx}{\mathcal{X}(x)}, \qquad (5.15)$$

where we assume that $u_i(x)$ vanishes as $|x| \to \infty$ (if $\mathcal{X}(x) \to 0$ for $|x| \to \infty$, then $u_i(x)$ must vanish faster then $\mathcal{X}(x)$ does). The above integral is in general improper, so additionally we assume that $u_i(x)$ behave properly as x tends to critical points x_c of $\mathcal{X}(x)$ ($\mathcal{X}(x_c) = 0$). Then, evaluating the integral we take its principal value.

We have explicitly defined the functionals in two ways reflecting two different approaches developed for the lattice soliton systems. The first one is with the domain of dynamical fields $\mathbb{Z}$ [16, 15], the second one with $\mathbb{R}$ [50, 51]. So, the functionals (5.14) and (5.15) are appropriate generalizations of these two approaches.

Proposition 5.2.2 *Both functionals* (5.14) *and* (5.15) *are well defined and satisfy the property* (5.13).

Proof. Clearly both functionals are linear. The discrete case trivially satisfy the property (5.13) by changing freely the boundaries of the sum over the whole $\mathbb{Z}$. For the continuous case, we have

$$\begin{aligned}\int Ef[\mathbf{u}]\, d_\hbar x &= \int_{-\infty}^{\infty} f[\mathbf{u}\,(\sigma_\hbar(x))]\frac{dx}{\mathcal{X}(x)} = \int_{-\infty}^{\infty} f[\mathbf{u}(x)]\frac{d\sigma_{-\hbar}(x)}{\mathcal{X}\,(\sigma_{-\hbar}(x))}\\ &= \int_{-\infty}^{\infty} f[\mathbf{u}(x)]\frac{d\sigma_{-\hbar}(x)}{dx}\frac{dx}{\mathcal{X}\,(\sigma_{-\hbar}(x))}\\ &= \int_{-\infty}^{\infty} f[\mathbf{u}(x)]\frac{\mathcal{X}\,(\sigma_{-\hbar}(x))}{\mathcal{X}(x)}\frac{dx}{\mathcal{X}\,(\sigma_{-\hbar}(x))}\\ &= \int_{-\infty}^{\infty} f[\mathbf{u}(x)]\frac{dx}{\mathcal{X}(x)} = \int f[\mathbf{u}]\, d_\hbar x,\end{aligned}$$

where the third equality is obtained by the change of variables $x \mapsto \sigma_\hbar(x)$, while the fifth one follows from Lemma 5.1.3, as

$$\mathcal{X}(x)\frac{d\sigma_{-\hbar}(x)}{dx} = \mathcal{X}\,(\sigma_{-\hbar}(x))\,. \tag{5.16}$$

$\square$

Definition 5.2.3 *A system of equations of the form*

$$\mathbf{u}_t = K(\mathbf{u}), \tag{5.17}$$

is called a difference-differential system if the difference calculus is performed with respect to the grain structure defined by $\sigma_\hbar$ while the first order differential calculus is with respect to the evolution parameter t, where $u_t := \frac{\partial u}{\partial t}$ and

$$K(\mathbf{u}) := \frac{1}{\hbar}(K_1[\mathbf{u}], K_2[\mathbf{u}], ...)^{\mathrm{T}}$$

with $K_i[\mathbf{u}] \in \mathcal{C}$.

Remark 5.2.4 *The particular choice of the algebra $\mathcal{C}$, and consequently the algebra $\mathcal{C}[E, E^{-1}]$, determines the class of discrete systems. The class of the discrete systems is chosen in such a way that in the continuous limit $\hbar \to 0$, we obtain differential systems of first order (dispersionless systems)(systems of hydrodynamic type). This assumption explains the appearance of the factor $\hbar$ in K.*

The following definition introduces the form of the duality maps.

Definition 5.2.5 *Let $\mathcal{V}$ be a linear space over $\mathbb{K}$, of all such vector fields on $\mathcal{U}$. The dual space $\mathcal{V}^*$ is a space of all linear maps $\eta : \mathcal{V} \to \mathbb{K}$. The action of $\eta \in \mathcal{V}^*$ on $K \in \mathcal{V}$ is defined through a duality map (bilinear functional) $\langle \cdot, \cdot \rangle : \mathcal{V}^* \times \mathcal{V} \to \mathbb{K}$ given by functional* (5.12) *as*

$$\langle \eta, K \rangle = \int \sum_{i=0}^{\infty} \eta_i K_i \, d_\hbar x = \int \eta^{\mathrm{T}} \cdot K \, d_\hbar x, \tag{5.18}$$

where the components of $\eta := (\eta_1, \eta_2, ...)^{\mathrm{T}}$ belong to $\mathcal{C}$.

The duality map (5.18) implies the adjoint of E^m as $(E^m)^\dagger = E^{-m}$.

Proposition 5.2.6 *The differential*

$$dF(\mathbf{u}) = \left(\frac{\delta F}{\delta u_0}, \frac{\delta F}{\delta u_1}, ... \right)^{\mathrm{T}} \in \mathcal{V}^*$$

of a functional $F(\mathbf{u}) = \int f[\mathbf{u}] \, d_\hbar x$, such that its pairing with $K \in \mathcal{V}$ assumes the usual Euclidean form

$$F'[K] = \langle dF, K \rangle = \int \sum_{i=0}^{\infty} \frac{\delta F}{\delta u_i} (u_i)_t \, d_\hbar x, \tag{5.19}$$

where $F'[K]$ is the directional derivative, is defined by variational derivatives of the form

$$\frac{\delta F}{\delta u_i} := \sum_{m \in \mathbb{Z}} E^{-m} \frac{\partial f[\mathbf{u}]}{\partial u_i \left(\sigma_{m\hbar}(x) \right)}.$$

Proof. Let $\mathbf{u}_t = K(\mathbf{u})$, then

$$F'(\mathbf{u})[\mathbf{u}_t] \equiv \frac{dF(\mathbf{u})}{dt} = \int \sum_{i=0}^{\infty} \sum_{m\in\mathbb{Z}} \frac{\partial f[\mathbf{u}]}{\partial u_i\left(\sigma_{m\hbar}(x)\right)} \frac{du_i\left(\sigma_{m\hbar}(x)\right)}{dt}\, d_\hbar x = \int \sum_{i=0}^{\infty} \frac{\delta F}{\delta u_i}(u_i)_t\, d_\hbar x,$$

where the last equality follows from (5.13). □

Having introduced the form of the differentials, it is time to present the related Poisson tensors in order to deal with bi-Hamiltonian structures. Consider bi-vector fields on $\mathcal{U}$ defined through linear operators $\pi : \mathcal{V}^* \to \mathcal{V}$, which are matrices with coefficients from $\mathcal{C}[E, E^{-1}]$ multiplied by $\frac{1}{\hbar}$ in a local representation.

Remark 5.2.7 *Alternative approach for the construction of discrete systems on* $\mathbb{R}$ *with the grain structure* $\mathbb{G}$*, is based on the* Δ*-derivative* (2.6)*, instead of the shift operator. In this case, the restriction* (5.13) *on the functional is replaced by*

$$\int' \Delta f[u]\, d_\hbar x = 0. \tag{5.20}$$

We denote $\int'$ *in* (5.20) *to differ the functional satisfying property* (5.20) *from the functional satisfying property* (5.13)*. Nevertheless, both functionals are interrelated by the relation*

$$\int' (\cdot)\, d_\hbar x = \int (\cdot) \mu_\hbar(x)\, d_\hbar x,$$

which is a consequence of the restrictions imposed on them.

5.3 R-matrix approach to integrable discrete systems on $\mathbb{R}$

The construction of integrable discrete systems following from the scheme of classical R-matrix formalism presented in chapter 3, is parallel to the procedure used in the case of lattice soliton systems [16, 13, 50].

Let $\mathbb{G}$ be the grain structure defined by some diffeomorphism $\sigma_\hbar$. We denote the shift operator by $\mathcal{E}$, rather than E for convenience in operational relations. Thus, by the use of (5.9), we have the following notation

$$\mathcal{E}^m u(x) := (E^m u)(x)\mathcal{E}^m \equiv u(\sigma_{m\hbar}(x))\mathcal{E}^m \qquad \sigma_{m\hbar} := \sigma_\hbar^m, \qquad m \in \mathbb{Z}, \tag{5.21}$$

We introduce the algebra of shift operators of finite highest order as follows

$$\mathcal{G} = \mathcal{G}_{\geqslant k-1} \oplus \mathcal{G}_{<k-1} = \left\{ \sum_{i \geqslant k-1}^{N} u_i(x)\mathcal{E}^i \right\} \oplus \left\{ \sum_{i<k-1} u_i(x)\mathcal{E}^i \right\} \tag{5.22}$$

equipped with the Lie bracket

$$[A, B] = \frac{1}{\hbar}(AB - BA), \qquad A, B \in \mathcal{G}. \tag{5.23}$$

Here $u_i(x)$ are smooth dynamical fields additionally depending on the evolution parameters t_n.

Proposition 5.3.1 *The multiplication operation on $\mathcal{G}$ defined by* (5.21) *is non-commutative and associative.*

Proof. Non-commutativity is obvious. Associativity follows from straightforward calculation and from the fact that $\sigma_{m\hbar}$ is a one-parameter group of diffeomorphisms. □

The subspaces $\mathcal{G}_{\geqslant k-1}$ and $\mathcal{G}_{<k-1}$ of $\mathcal{G}$ are closed Lie subalgebras of $\mathcal{G}$ only if $k = 1$ and $k = 2$. As a result, we define the classical R-matrices as

$$R = \frac{1}{2}(P_{\geqslant k-1} - P_{<k-1}) = P_{\geqslant k-1} - \frac{1}{2}, \tag{5.24}$$

where $P_{\geqslant k-1}$ and $P_{<k-1}$ are the projections onto the Lie subalgebras $\mathcal{G}_{\geqslant k-1}$ and $\mathcal{G}_{<k-1}$, respectively such that

$$P_{\geqslant k-1}(A) = \sum_{i \geqslant k-1} a_i\mathcal{E}^i, \qquad P_{<k}(A) = \sum_{i<k-1} a_i\mathcal{E}^i \qquad \text{for} \qquad A = \sum_i a_i\mathcal{E}^i \in \mathcal{G}. \tag{5.25}$$

The Lax hierarchy (3.23), based on the classical R-matrix (5.24) is generated by the integer powers (in general rational powers) of the Lax operator L from the algebra of shift operators

$$L_{t_n} = \left[(L^n)_{\geqslant k-1}, L\right] = \pi_0 dH_n = \pi_1 dH_{n-1} \qquad n \in \mathbb{Z}_+ \qquad k = 1, 2. \tag{5.26}$$

And indeed it is an infinite hierarchy of mutually commuting symmetries. The appropriate Lax operators producing consistent Lax hierarchies are given in the form:

$$k = 1: \qquad L = \mathcal{E} + \sum_{i \geqslant 0} u_i \mathcal{E}^{-i} \tag{5.27}$$

$$k = 2: \qquad L = \sum_{i \geqslant 0} u_i \mathcal{E}^{1-i}. \tag{5.28}$$

Note that, we do not consider finite-field reductions of (5.27), (5.28) since the procedure immediately follows from [16, 50]. We calculate the first chains

$$(u_i)_{t_1} = \frac{1}{\hbar}\left[(E-1)u_{i+1} + u_i(1-E^{-i})u_0\right] \tag{5.29}$$

$$\begin{aligned}(u_i)_{t_2} = \frac{1}{\hbar}\Big[&(E^2-1)u_{i+2} + Eu_{i+1}(E+1)u_0 - u_{i+1}(E^{-i}+E^{-i-1})u_0 \\ &+u_i(1-E^{-i})u_0^2 + u_i(E+1)(1-E^{-i})u_1\Big]\\ &\vdots\end{aligned}$$

for $k = 1$, and

$$(u_i)_{t_1} = \frac{1}{\hbar}\left[u_0 E u_{i+1} - u_{i+1}E^{-i}u_0\right] \tag{5.30}$$

$$\begin{aligned}(u_i)_{t_2} &= \frac{1}{\hbar}[u_0Eu_0E^2u_{i+2} - u_{i+2}E^{-i-1}u_0E^{-i}u_0 \\ &+ u_0(E+1)u_1Eu_{i+1} - u_{i+1}E^{-i}u_0(E^{1-i}+E^{-i})u_1 \qquad (5.31)\\ &\vdots\end{aligned}$$

for $k = 2$. Throughout this chapter, the shift operators E^m in the evolution equations and conserved quantities act only on the nearest field on the right and in Poisson operators act on everything on the right of the symbol E^m.

Example 5.3.2 *The lattice case: $\mathcal{X} = 1$. Let $\hbar = 1$. The first flows* (5.29) *and* (5.30) *yields the Toda and modified Toda chains, respectively,*

$$k = 1: \qquad u_i(x)_{t_1} = u_{i+1}(x+1) - u_{i+1}(x) + u_i(x)\left(u_0(x) - u_0(x-i)\right),$$

$$k = 2: \qquad u_i(x)_{t_1} = u_0(x)u_{i+1}(x+1) - u_0(x-i)u_{i+1}(x).$$

Example 5.3.3 *The q-discrete case:* $\mathcal{X} = x$ $(q \equiv e^{\hbar})$*. In this case, we obtain q-deformed analogues of the same flows* (5.29) *and* (5.30)

$$\begin{aligned} k = 1: \qquad & u_i(x)_{t_1} = u_{i+1}(qx) - u_{i+1}(x) + u_i(x)\left(u_0(x) - u_0(q^{-i}x)\right) \\ k = 2: \qquad & u_i(x)_{t_1} = u_0(x)u_{i+1}(qx) - u_0(q^{-i}x)u_{i+1}(x), \end{aligned}$$

where the constant factor $\hbar$ is absorbed into the evolution parameter t_1 through simple rescaling.

To construct bi-Hamiltonian structures for the Lax hierarchy (5.26), we have to define an appropriate inner product on $\mathcal{G}$. Since the simplest way to define an inner product is identifying it by a trace functional, we state the following definition.

Definition 5.3.4 *Let* $\mathrm{Tr} : \mathcal{G} \to \mathbb{K}$ *be a trace form, being a linear map, such that*

$$\mathrm{Tr}(A) := \int \mathrm{res}(A\mathcal{E}^{-1})\, d_{\hbar}x,$$

where $\mathrm{res}(A\mathcal{E}^{-1}) := a_0$ *for* $A = \sum_i a_i \mathcal{E}^i$*. Then, the bilinear map* $(\cdot,\cdot) : \mathcal{G} \times \mathcal{G} \to \mathbb{K}$ *defined as*

$$(A, B) := \mathrm{Tr}\,(AB)\,, \tag{5.32}$$

is an inner product on $\mathcal{G}$.

Proposition 5.3.5 *The inner product* (5.32) *is nondegenerate, symmetric and* ad*-invariant.*

Proof. The nondegeneracy of (5.32) is obvious by the definition. The symmetricity follows from (5.13). Finally, the ad-invariance is a consequence of the symmetricity of the inner product and the associativity of the multiplication operation defined in $\mathcal{G}$. $\square$

We present the explicit form of the differentials dH of functionals $H(L) \in \mathcal{F}(\mathcal{G})$ with respect to the Lax operators (5.27), (5.28) by assuming that the inner product on $\mathcal{G}$ is compatible with (5.19), i.e.

$$(dH, L_t) = \int \sum_{i=0}^{\infty} \frac{\delta H}{\delta u_i} (u_i)_t \, d_\hbar x.$$

Thus, the differentials have the following forms

$$\begin{aligned} k = 1: \qquad & dH = \sum_{i \geqslant 0} \mathcal{E}^i \frac{\delta H}{\delta u_i}, \\ k = 2: \qquad & dH = \sum_{i \geqslant 0} \mathcal{E}^{i-1} \frac{\delta H}{\delta u_i}, \end{aligned}$$

We present the linear Poisson tensor for both cases $k = 1, 2$

$$\pi_0 : dH \mapsto [L, (dH)_{<k-1}] + ([dH, L])_{<2-k}$$

and the quadratic Poisson tensors as follows

$$\begin{aligned} k = 1: \qquad & \pi_1 : dH \mapsto \frac{1}{2} \left(\left[L, (LdH + dHL)_{<0} \right] + L \left([dH, L] \right)_{<1} + \left([dH, L] \right)_{<1} L \right) \\ & \qquad + \hbar \left[(E+1)(E-1)^{-1} \mathrm{res} \left([dH, L] \, \mathcal{E}^{-1} \right), L \right] \\ k = 2: \qquad & \pi_1 : dH \mapsto \frac{1}{2} \left(\left[L, (LdH + dHL)_{<1} \right] + L \left([dH, L] \right)_{<0} + \left([dH, L] \right)_{<0} L \right). \end{aligned}$$

Here the operation $(E-1)^{-1}$ is the formal inverse of $(E-1)$ and note that

$$(E+1)(E-1)^{-1} = \sum_{i=1}^{\infty} (E^{-i} - E^i).$$

The appropriate Hamiltonians are

$$H_n(L) = \frac{1}{n+1} \mathrm{Tr} \left(L^{n+1} \right),$$

such that

$$dH_n(L) = L^n$$

and the explicit bi-Hamiltonian structure of the Lax hierarchies (5.26) is given by

$$(u_i)_{t_n} = \sum_{j \geqslant 0} \pi_0^{ij} \frac{\delta H_n}{\delta u_j} = \sum_{j \geqslant 0} \pi_1^{ij} \frac{\delta H_{n-1}}{\delta u_j} \qquad i \geqslant 0.$$

The Poisson tensors for $k = 1$ are

$$
\begin{aligned}
\pi_0^{ij} &= \frac{1}{\hbar}\left[E^j u_{i+j} - u_{i+j}E^{-i}\right] \\
\pi_1^{ij} &= \frac{1}{\hbar}\Big[\sum_{k=0}^{i}\left(u_k E^{j-k}u_{i+j-k} - u_{i+j-k}E^{k-i}u_k + u_i\left(E^{j-k} - E^{-k}\right)u_j\right) \\
&\qquad + u_i\left(1 - E^{j-i}\right)u_j + E^{j+1}u_{i+j+1} - u_{i+j+1}E^{-i-1}\Big]
\end{aligned}
$$

together with the first three Hamiltonians

$$
\begin{aligned}
H_0 &= \int u_0\, d_\hbar x \\
H_1 &= \int \left(u_1 + \frac{1}{2}u_0^2\right) d_\hbar x \\
H_2 &= \int \left(u_2 + u_0(E+1)u_1 + \frac{1}{3}u_0^3\right) d_\hbar x \\
&\ \ \vdots\ .
\end{aligned}
$$

For $k = 2$, the linear Poisson tensor is the form

$$
\begin{aligned}
\pi_0^{10} &= \frac{1}{\hbar}(1 - E^{-1})u_0, \qquad \pi_0^{01} = \frac{1}{\hbar}u_0(E-1), \\
\pi_0^{ij} &= \frac{1}{\hbar}\left[E^{j-1}u_{i+j-1} - u_{i+j-1}E^{1-i}\right], \quad i,j \geqslant 2,
\end{aligned}
$$

with all remaining π_0^{ij} equal to zero. The quadratic Poisson tensor is

$$
\pi_1^{ij} = \frac{1}{\hbar}\Big[\sum_{k=0}^{i-1}\left(u_k E^{j-k}u_{i+j-k} - u_{i+j-k}E^{k-i}u_k\right) + \frac{1}{2}u_i(E^{1-i} - 1)(E^{j-1} + 1)u_j\Big]
$$

and the first Hamiltonians are

$$
\begin{aligned}
H_0 &= \int u_1\, d_\hbar x \\
H_1 &= \int \left(\frac{1}{2}u_1^2 + u_0 E u_2\right) d_\hbar x \\
H_2 &= \int \left(\frac{1}{3}u_1^3 + u_0 E u_0 E^2 u_3 + u_0 u_1 E u_2 + u_0 E u_1 E u_2\right) d_\hbar x \\
&\ \ \vdots\ .
\end{aligned}
$$

5.4 The continuous limit

The aim of this section is to consider the limit of discrete systems (5.17) as $\hbar \to 0$. We explain the continuous limit first emphasized in Remark 5.2.4 and determine the class of discrete systems by the choice of the algebra $\mathcal{C}$. Now, assume that the dynamical fields from $\mathcal{C}$ depend on $\hbar$ in such a way that the expansion, with respect to $\hbar$ near zero, is of the form

$$u_i(x) = u_i^{(0)}(x) + u_i^{(1)}(x)\hbar + O\left(\hbar^2\right),$$

i.e. u_i tends to $u_i^{(0)}$ as $\hbar \to 0$. In further considerations we use u_i instead of $u_i^{(0)}$. In the continuous limit, the algebra $\mathcal{C}$ of functions (5.11) turns out to be the algebra of polynomial functions in $u_i(x)$, denoted by $\mathcal{C}_0$,

$$\mathcal{C}_0 \ni f(\mathbf{u}) := \sum_{m\geqslant 0} \sum_{i_1,\dots,i_m\geqslant 0} a^{i_1 i_2 \dots i_m}\, u_{i_1}(x) u_{i_2}(x) \cdots u_{i_m}(x).$$

In general, the limit of discrete systems (5.17) does not have to exist. For the limit procedure, we should first expand the coefficients of $K(\mathbf{u})$ into a Taylor series with respect to $\hbar$ near 0, i.e.

$$E^m u = e^{m\hbar\mathcal{X}\partial_x} u = u + m\hbar\mathcal{X}u_x + \frac{m^2}{2}\hbar^2\left(\mathcal{X}\mathcal{X}_x u_x + \mathcal{X}^2 u_{2x}\right) + O\left(\hbar^3\right).$$

Thus, the continuous limit of (5.17) exists only if zero order terms in $\hbar$ will mutually cancel in the above expansion . In this case, as $\hbar \to 0$, the discrete systems (5.17) tend to the systems of hydrodynamic type given in the following form

$$\mathbf{u}_t = \mathcal{X}\,\mathbf{A}(\mathbf{u})\mathbf{u}_x, \tag{5.33}$$

where $\mathbf{A}(\mathbf{u})$ is the matrix with coefficients from $\mathcal{C}_0$, and the continuous limit is indeed the dispersionless limit.

Proposition 5.4.1 *Assume that the fields $u_i(x)$ vanish as $|x| \to \infty$, in the continuous limit. Then the functionals from Definition 5.2.1 turn out to be*

$$\int (\cdot)\, d_0x : \mathcal{C}_0 \to \mathbb{K} \qquad f[\mathbf{u}] \mapsto F(\mathbf{u}) = \int f(\mathbf{u})\, d_0x = \int_{-\infty}^{\infty} f(\mathbf{u}(x)) \frac{dx}{\mathcal{X}(x)}. \tag{5.34}$$

Proof. For the continuous case (5.15) the proof is straightforward. In the case of discrete functionals (5.14), by the concept of Riemann integral construction, we have

$$\begin{aligned}\int f[\mathbf{u}]\, d_0x &\equiv \lim_{\hbar\to 0}\int f[\mathbf{u}]\, d_\hbar x = \lim_{\hbar\to 0}\sum_{n\in\mathbb{Z}}\hbar\, f\left[\mathbf{u}\left(\sigma_{n\hbar}(x)\right)\right]\\ &= \lim_{\hbar\to 0}\sum_{n\in\mathbb{Z}} f\left[\mathbf{u}\left(\sigma_{n\hbar}(x)\right)\right]\frac{\hbar}{\mu_\hbar(x)}.\mu_\hbar(x)\\ &= \lim_{\hbar\to 0}\sum_{n\in\mathbb{Z}} f\left[\mathbf{u}\left(\sigma_{n\hbar}(x)\right)\right]\left(\frac{\mu_\hbar(x)}{\hbar}\right)^{-1}\mu_\hbar(x) = \int_{-\infty}^{\infty} f(\mathbf{u}(x))\frac{dx}{\mathcal{X}(x)}.\end{aligned}$$

□

Thus, π are matrices with coefficients of the operator form $a\mathcal{X}\partial_x b$, where $a, b\in\mathcal{C}_0$. With respect to the duality map defined by the 'dispersionless' functional (5.34), the adjoint of the operator ∂_x is given as

$$(\partial_x)^\dagger = \frac{\mathcal{X}_x}{\mathcal{X}} - \partial_x. \tag{5.35}$$

Consequently, the variational derivatives of functionals $F = \int f d_0x = \int_{-\infty}^{\infty} f\frac{dx}{\mathcal{X}}$ are given by the derivatives of densities f with respect to the fields u_i, i.e.

$$\frac{\delta F}{\delta u_i} = \frac{\partial f}{\partial u_i}.$$

Example 5.4.2 *The dispersionless limit of the system* (5.29) *together with its Hamiltonian structure with respect to the first Poisson tensor is given by*

$$(u_i)_{t_1} = \mathcal{X}\left[(u_{i+1})_x + iu_i(u_0)_x\right] = \pi_0^{ij}\frac{\delta H_1}{\delta u_j}, \tag{5.36}$$

where

$$\pi_0^{ij} = j\mathcal{X}\partial_x u_{i+j} + iu_{i+j}\mathcal{X}\partial_x \qquad \textit{and} \qquad H_1 = \int\left(u_1 + \frac{1}{2}u_0^2\right) d_0x.$$

The Hamiltonian representation of the systems (5.33) with the functional (5.34) follows directly from the continuous limit and leads to the nonstandard form with the adjoint operator of the differential operator given by (5.35). A more natural representation is the one with the components $\mathcal{X}(x)$ included in the densities of functionals given in the standard form

$$F(\mathbf{u}) = \int_{-\infty}^{\infty} \mathcal{X}(x)^{-1} f(\mathbf{u}(x))\, dx \equiv \int_{-\infty}^{\infty} \varphi(\mathbf{u}(x))\, dx,$$

for which the variational derivatives preserve the form $\frac{\delta F}{\delta u_i} = \frac{\partial \varphi}{\partial u_i}$. As a consequence, π from the previous representation must be multiplied on the right-hand side by $\mathcal{X}$. Now, the adjoint of the operator ∂_x takes the standard form $(\partial_x)^\dagger = -\partial_x$. Therefore, we use only the natural Hamiltonian representation of dispersionless systems (5.33).

Example 5.4.3 *The natural Hamiltonian structure of* (5.36) *is given by*

$$\pi_0^{ij} = j\mathcal{X}\partial_x\mathcal{X}u_{i+j} + iu_{i+j}\mathcal{X}\partial_x\mathcal{X} \qquad \text{and} \qquad H_1 = \int_{-\infty}^{\infty} \mathcal{X}^{-1}\left(u_1 + \frac{1}{2}u_0^2\right) dx.$$

Chapter 6

Integrable dispersionless systems on $\mathbb{R}$

6.1 R-matrix approach to integrable dispersionless systems on $\mathbb{R}$

The theory of classical R-matrices on commutative algebras, with the multi-Hamiltonian formalism, was given in [20]. In this section we consider the R-matrix formalism of the dispersionless systems (5.33), which were developed in [21, 52, 25].

Before passing through the details, let us introduce what we mean by a *dispersionless system.* First order partial differential equations of the form

$$(u_i)_t = \sum_j A_i^j(u)(u_j)_x, \qquad i, j = 1, 2, ..., n \tag{6.1}$$

where A_i^j is a square matrix with coefficients in u, are called *hydrodynamic type* or *dispersionless systems* in $(1+1)$ dimension. Now, let $\mathcal{G}$ be the algebra of Laurent series in the auxiliary variable p

$$\mathcal{G} = \left\{ \sum_{i=-\infty}^{\infty} u_i(x)p^i \right\}, \tag{6.2}$$

where u_i's are smooth functions, equipped with a Poisson bracket

$$\{f,g\}_k := p^k \left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} \right), \qquad f,g \in \mathcal{G}, \quad k \in \mathbb{Z}. \tag{6.3}$$

Recall that, for $k=0$ the Lie bracket is called the standard Poisson bracket. The Lax equation on the algebra $\mathcal{G}$ is given by,

$$L_t = \{A, L\}_k \tag{6.4}$$

with appropriate functions $L, A \in \mathcal{G}$. Such non-standard Lax representations are called *dispersionless Lax equations.* Throughout this section, we deal with the algebra $\mathcal{A}$ of polynomials in p of finite highest order, equipped with the Lie bracket induced by the Poisson bracket of the form

$$\{f,g\} := p\mathcal{X}(x) \left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} \right), \qquad f,g \in \mathcal{A}, \tag{6.5}$$

with such a decomposition:

$$\mathcal{A} = \mathcal{A}_{\geqslant k-1} \oplus \mathcal{A}_{<k-1} = \left\{ \sum_{i \geqslant k-1}^{N} u_i(x)p^i \right\} \oplus \left\{ \sum_{i<k-1} u_i(x)p^i \right\}. \tag{6.6}$$

The subspaces $\mathcal{A}_{\geqslant k-1}$ and $\mathcal{A}_{<k-1}$ are closed Lie subalgebras of $\mathcal{A}$ only if $k=1$ and $k=2$. We define the classical R-matrices as $R = P_{\geqslant k-1} - \frac{1}{2}$. Thus, the Lax hierarchy (3.23), based on this classical R-matrix, is generated by the integer powers of the Lax operator L from the algebra of polynomials

$$L_{t_n} = \left\{ (L^n)_{\geqslant k-1}, L \right\} = \pi_0 dH_n = \pi_1 dH_{n-1}, \qquad n \in \mathbb{Z}_+, \qquad k = 1,2. \tag{6.7}$$

The appropriate Lax operators producing consistent Lax hierarchies (6.7) are given below

$$k=1: \qquad L = p + \sum_{i \geqslant 0} u_i p^{-i} \tag{6.8}$$

$$k=2: \qquad L = \sum_{i \geqslant 0} u_i p^{1-i}. \tag{6.9}$$

The first dispersionless chains for $k=1$, take the following form

$$\begin{aligned} (u_i)_{t_1} &= \mathcal{X}\left[(u_{i+1})_x + iu_i(u_0)_x \right] \qquad (6.10)\\ (u_i)_{t_2} &= 2\mathcal{X}\left[(u_{i+2})_x + u_0(u_{i+1})_x + (i+1)u_{i+1}(u_0)_x + iu_iu_0(u_0)_x + iu_i(u_1)_x \right] \\ &\ \ \vdots\ , \end{aligned}$$

while for $k = 2$, we derive

$$
\begin{aligned}
(u_i)_{t_1} &= \mathcal{X}\left[u_0(u_{i+1})_x + iu_{i+1}(u_0)_x\right] \qquad (6.11)\\
(u_i)_{t_2} &= 2\mathcal{X}\left[u_0^2(u_{i+2})_x + (i+1)u_0u_{i+2}(u_0)_x + u_0u_1(u_{i+1})_x + iu_{i+1}(u_0u_1)_x\right]\\
&\vdots\ .
\end{aligned}
$$

Example 6.1.1 *For $\mathcal{X} = 1$ the chains* (6.10) *and* (6.11) *are dispersionless Toda and modified Toda chains, respectively, while for $\mathcal{X} = x$ the chains* (6.10) *and* (6.11) *are dispersionless limits of the q-analogues of Toda and modified Toda.*

In order to discuss bi-Hamiltonian structures for the algebra of polynomials, we should define the appropriate trace form on this algebra.

Definition 6.1.2 *Let* $\mathrm{Tr} : \mathcal{A} \to \mathbb{K}$ *be a trace form, being a linear map, such that*

$$
\mathrm{Tr}(A) := \int_{-\infty}^{\infty} \mathcal{X}^{-1}\mathrm{res}(Ap^{-1})\,dx,
$$

where $\mathrm{res}(A) := a_{-1}$ *for* $A = \sum_i a_i p^i$*. Then, the bilinear map* $(\cdot,\cdot) : \mathcal{A}\times\mathcal{A} \to \mathbb{K}$ *defined as*

$$
(A, B) := \mathrm{Tr}\,(AB)\,, \qquad (6.12)
$$

is an inner product on $\mathcal{A}$*.*

Proposition 6.1.3 *The inner product* (6.12) *is nondegenerate, symmetric and* ad*-invariant with respect to the Poisson bracket* (6.5)*, i.e.*

$$
(\{A, B\}\,, C) = (A, \{B, C\}) \qquad A, B, C \in \mathcal{A}.
$$

Proof. The nondegeneracy and symmetricity is obvious. The ad-invariance is a consequence of the following equality: $\mathrm{Tr}\,\{A, B\} = 0$, which is valid for arbitrary $A, B \in \mathcal{A}$. $\square$

We give the explicit form of the differentials $dH(L)$ of functionals $H(L) \in \mathcal{F}(\mathcal{A})$ related to the Lax operators (6.8-6.9) as follows:

$$k = 1: \qquad dH = \mathcal{X} \sum_{i \geqslant 0} \frac{\delta H}{\delta u_i} p^i$$
$$k = 2: \qquad dH = \mathcal{X} \sum_{i \geqslant 0} \frac{\delta H}{\delta u_i} p^{i-1}.$$

The bi-Hamiltonian structure of the Lax hierarchies (6.7) is defined through the compatible (for fixed k) Poisson tensors:

$$k = 1, 2: \qquad \pi_0 : dH \mapsto \{L, (dH)_{<k-1}\} + (\{dH, L\})_{<2-k}$$

and

$$k = 1: \qquad \pi_1 : dH \mapsto \left\{L, (dHL)_{<0}\right\} + L\left(\{dH, L\}\right)_{<1} + \left\{\partial_x^{-1}\mathrm{res}(\mathcal{X}^{-1}p^{-1}\{dH, L\}), L\right\}$$
$$k = 2: \qquad \pi_1 : dH \mapsto \left\{L, (dHL)_{<1}\right\} + L\left(\{dH, L\}\right)_{<0}.$$

Then, for Hamiltonians

$$H_n(L) = \frac{1}{n+1}\mathrm{Tr}\left(L^{n+1}\right) \qquad dH_n(L) = L^n,$$

the explicit bi-Hamiltonian structure of (6.7) is given by

$$(u_i)_{t_n} = \sum_{j \geqslant 0} \pi_0^{ij} \frac{\delta H_n}{\delta u_j} = \sum_{j \geqslant 0} \pi_1^{ij} \frac{\delta H_{n-1}}{\delta u_j} \qquad i \geqslant 0.$$

The Poisson tensors for $k = 1$ are given by

$$\pi_0^{ij} = \mathcal{X}\left[j\partial_x u_{i+j} + i u_{i+j}\partial_x\right]\mathcal{X}$$
$$\pi_1^{ij} = \mathcal{X}\Big[\sum_{k=0}^{i}\left((j-k)u_k\partial_x u_{i+j-k} + (i-k)u_{i+j-k}\partial_x u_k\right) + i(j+1)u_i\partial_x u_j$$
$$+ (j+1)\partial_x u_{i+j+1} + (i+1)u_{i+j+1}\partial_x\Big]\mathcal{X}$$

where the related Hamiltonians are

$$H_0 = \int_{-\infty}^{\infty} \mathcal{X}^{-1} u_0 \, dx$$
$$H_1 = \int_{-\infty}^{\infty} \mathcal{X}^{-1}\left(u_1 + \frac{1}{2}u_0^2\right) dx$$
$$H_2 = \int_{-\infty}^{\infty} \mathcal{X}^{-1}\left(u_2 + 2u_0u_1 + \frac{1}{3}u_0^3\right) dx$$
$$\vdots\,.$$

For $k = 2$, the linear Poisson tensor yields as

$$
\begin{aligned}
\pi_0^{10} &= \mathcal{X}\partial_x\mathcal{X}u_0 \qquad \pi_0^{01} = u_0\mathcal{X}\partial_x\mathcal{X} \\
\pi_0^{ij} &= \mathcal{X}\left[(j-1)\partial_x u_{i+j-1} + (i-1)u_{i+j-1}\partial_x\right]\mathcal{X} \quad i,j \geqslant 2,
\end{aligned}
$$

where all remaining π_0^{ij} are equal to zero, while the quadratic Poisson tensor is as follows

$$
\pi_1^{ij} = \mathcal{X}\Big[\sum_{k=0}^{i-1}((j-k)u_k\partial_x u_{i+j-k} + (i-k)u_{i+j-k}\partial_x u_k) + (1-i)u_i\partial_x u_j\Big]\mathcal{X}.
$$

Finally the related Hamiltonians are

$$
\begin{aligned}
H_0 &= \int_{-\infty}^{\infty} \mathcal{X}^{-1}u_1\,dx \\
H_1 &= \int_{-\infty}^{\infty} \mathcal{X}^{-1}\left(\frac{1}{2}u_1^2 + u_0u_2\right)dx \\
H_2 &= \int_{-\infty}^{\infty} \mathcal{X}^{-1}\left(\frac{1}{3}u_1^3 + u_0^2u_3 + 2u_0u_1u_2\right)dx \\
&\vdots\,.
\end{aligned}
$$

Remark 6.1.4 *Here we emphasize that the R-matrix formalism of the dispersionless systems together with their bi-Hamiltonian structures are the continuous limit of the formalisms of the integrable discrete systems presented in Section 5.3.*

6.2 Deformation quantization procedure

The aim of this section is to formulate the inverse procedure of the continuous limit (dispersionless limit) presented in Section 5.2. This inverse procedure is based on the quantization deformation formalism which is a unified approach to the lattice and field soliton systems. We follow the procedure presented in [50].

The most important point of the deformation quantization theory is that a classical system can be obtained from a quantum system by the quasi-classical limit as $\hbar \to 0$, where $\hbar$ is the related deformation parameter. In other words, the quantization of classical systems is performed by appropriate deformations depending on $\hbar$. The idea of deformation quantization is based on the deformation of the usual multiplication to a new non-commutative, associative product, which is called as $\star$-*product*. Here the crucial point is that as $\hbar \to 0$, the $\star$-product reduces to the standard multiplication and the deformed Poisson bracket reduces to the Poisson bracket. To have a well-defined $\star$-product, let us state the following definition.

Definition 6.2.1 *A deformed multiplication $\star$ is a formal quantization of the algebra and is called the $\star$-product if all the following relations are satisfied*

(i) $\lim_{\hbar\to 0} f \star g = fg$,

(ii) $c \star f = f \star c = cf$, $c \in \mathbb{R}$ *or* $\mathbb{C}$,

(iii) $\lim_{\hbar\to 0} \{f, g\}_\star = \{f, g\}$,

where the deformed Poisson bracket is given by

$$\{f, g\}_\star = \frac{1}{\hbar}(f \star g - g \star f). \tag{6.13}$$

Since an arbitrary Poisson bracket can be written by a wedge product of appropriate commuting vector fields, the Poisson bracket (6.5) can be written in the form

$$\{f, g\} := f\,(p\partial_p \wedge \mathcal{X}(x)\partial_x)\,g \qquad f, g \in \mathcal{A},$$

where the derivations $p\partial_p$ and $\mathcal{X}(x)\partial_x$ obviously commute. Hence, the Poisson bracket on $\mathcal{A}$ can be quantized in infinitely many ways via the $\star$-products satisfying the conditions presented in the Definition 6.2.1. We consider the following non-commutative $\star^\alpha$-product

$$f \star^\alpha g = f \exp\left[\frac{\hbar}{2}\left((\alpha+1)p\partial_p \otimes \mathcal{X}(x)\partial_x + (\alpha-1)\mathcal{X}(x)\partial_x \otimes p\partial_p\right)\right] g. \tag{6.14}$$

whose associativity follows from purely algebraic consequence of the construction (for the proof we refer [50]). The algebra $\mathcal{A}$ (6.6) with the multiplication defined by (6.14), with a fixed α, is an associative, non-commutative algebra with the deformed Poisson bracket,

$$\{f, g\}_{\star^\alpha} = \frac{1}{\hbar}\left(f \star^\alpha g - g \star^\alpha f\right). \tag{6.15}$$

Then, as $\hbar \to 0$, we have

$$\lim_{\hbar\to 0} f \star^\alpha g = fg,$$
$$\lim_{\hbar\to 0} \{f, g\}_{\star^\alpha} = \{f, g\},$$

which implies that the $\star^\alpha$-product (6.14) is well-defined. From now on, we will denote the algebra $\mathcal{A}$ equipped with the deformed bracket (6.15), based on $\star^\alpha$-product (6.14) as $\mathcal{A}_\alpha$. Note that, the $\star^\alpha$-product (6.14) for $\alpha = 0$ and $\alpha = 1$ is the generalization of the Moyal and Kuperschmidt-Manin products, respectively. In order to make the $\star^\alpha$-products (6.14) consistent with the introduced grain structures, we assume that $\mathcal{X}\partial_x$ is complete or it generates well defined discrete one-parameter group of diffeomorphisms. By simple observation, one can derive the following properties.

Proposition 6.2.2 *For the $\star^\alpha$-product* (6.14) *the following properties hold*

$$(i)(p\partial_p)^k p^m = m^k p^m,$$
$$(ii)p^m \star^\alpha u(x) = \sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!}(\alpha+1)^k m^k (\mathcal{X}\partial_x)^k u(x)\, p^m = e^{m(\alpha+1)\frac{\hbar}{2}\mathcal{X}\partial_x} u(x)\, p^m = E^{m\frac{\alpha+1}{2}} u(x)\, p^m,$$
$$(iii)u(x) \star^\alpha p^m = \sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!}(\alpha-1)^k m^k (\mathcal{X}\partial_x)^k u(x)\, p^m = e^{m(\alpha-1)\frac{\hbar}{2}\mathcal{X}\partial_x} u(x)\, p^m = E^{m\frac{\alpha-1}{2}} u(x)\, p^m,$$

Expanding (6.14) one finds that

$$f \star^\alpha g = \sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!} \sum_{j=0}^{k} (\alpha+1)^{k-j}(\alpha-1)^j \left[(p\partial_p)^{k-j}(\mathcal{X}\partial_x)^j f\right] \left[(\mathcal{X}\partial_x)^{k-j}(p\partial_p)^j g\right] \tag{6.16}$$

Here we do not require a convergence of the sum in (6.16).

Remark 6.2.3 *The decomposition of* (6.6) *into Lie subalgebras with respect to the Lie bracket* (6.15) *is preserved after deformation quantization. Hence, the Lax hierarchies, generated by powers of the Lax operators of the form* (6.8) *and* (6.9)*, with respect to $\star^{\alpha}$-products, i.e. $L^n = L \star^{\alpha} ... \star^{\alpha} L$ are as follows*

$$L_{t_n} = \left\{(L^n)_{\geqslant k-1}, L\right\}_{\star^{\alpha}}, \qquad n \in \mathbb{Z}_+, \qquad k = 1, 2. \tag{6.17}$$

We calculate the first chains from the Lax hierarchies (6.17)

$$\begin{aligned} k = 1: \qquad (u_i)_{t_1} &= \frac{1}{\hbar}\left[(E-1)E^{\frac{\alpha-1}{2}}u_{i+1} + u_i(1-E^{-i})E^{i\frac{1-\alpha}{2}}u_0\right] \\ k = 2: \qquad (u_i)_{t_1} &= \frac{1}{\hbar}\left[E^{i\frac{1-\alpha}{2}}u_0E^{\frac{\alpha+1}{2}}u_{i+1} - E^{\frac{\alpha-1}{2}}u_{i+1}E^{-i(\frac{\alpha+1}{2})}u_0\right]. \end{aligned}$$

which coincide with the discrete systems (5.29) and (5.30) for $\alpha = 1$.

Let $D^{\alpha'-\alpha} : \mathcal{A}_{\alpha} \to \mathcal{A}_{\alpha'}$ be an isomorphism of the form

$$D^{\alpha'-\alpha} = \exp\left[(\alpha-\alpha')\frac{\hbar}{2}\,\mathcal{X}(x)\partial_x\,p\partial_p\right], \tag{6.18}$$

The isomorphism (6.18) allows to produce a new $\star^{\alpha'}$-product

$$f \star^{\alpha'} g = D^{\alpha'-\alpha}\left[D^{\alpha-\alpha'}f \star^{\alpha} D^{\alpha-\alpha'}g\right], \tag{6.19}$$

which is well-defined and associative ensured by the associativity of the $\star^{\alpha}$-product. These two products $\star^{\alpha}$ and $\star^{\alpha'}$ are gauge equivalent under the isomorphism (6.18) which implies the gauge equivalence of all algebras $\mathcal{A}_{\alpha}$.

Note that, under the above isomorphism the Lax hierarchy structure is preserved. Let $L_{\alpha} = \sum_i u_i p^i \in \mathcal{A}_{\alpha}$ and $L_{\alpha'} = \sum_i u'_i p^i \in \mathcal{A}_{\alpha'}$. Then, the transformation between fields is as follows

$$L_{\alpha'} = D^{\alpha'-\alpha}L_{\alpha} \qquad \Rightarrow \qquad u'_i = E^{i\frac{\alpha-\alpha'}{2}}u_i.$$

On the other hand, (6.14) implies the following commutation rules:

$$\begin{aligned} &u \star v = uv \\ &p^m \star p^n = p^{m+n} \\ &p^m \star u = \left(e^{m\hbar\mathcal{X}\partial_x}u\right) \star p^m = E^m u \star p^m \\ &u \star p^m = p^m \star \left(e^{-m\hbar\mathcal{X}\partial_x}u\right) = p^m \star E^{-m}u, \end{aligned}$$

which are independent of the choice of $\star^{\alpha}$-product. Therefore we skip the related index.

Now we consider the following algebra

$$\mathfrak{a} = \left\{ \sum_i u_i \star p^i \right\},$$

which is clearly associative by the commutation rules presented above. Here instead of standard multiplication we make use of the $\star$-product and we quantize the algebra $\mathcal{A}$ of polynomials in p to the algebra $\mathfrak{a}$ separately [53].

Remark 6.2.4 *Note that*

$$u \star^1 p^m = up^m, \qquad p^m \star^1 u = E^m up^m, \tag{6.20}$$

which implies that the algebra $\mathfrak{a}$ is trivially equivalent to the algebra $\mathcal{A}_1$.

Moreover, $\mathfrak{a}$ is isomorphic to the algebra of shift operators $\mathcal{G}$ (5.22) defined on the grain structure by some discrete one-parameter group of diffeomorphisms on $\mathbb{R}$. As a conclusion, the algebra (6.6) with Poisson bracket (6.5) is the limit of the algebra of shift operators (5.22) with the Lie bracket (5.23) as $\hbar \to 0$, which assures the existence of the inverse problem to the dispersionless limit.

Chapter 7

Conclusion

In order to embed the integrable systems into a more general unifying framework, we established two approaches depending on construction of integrable systems either on regular time scales or discrete ones on $\mathbb{R}$. We made use of R-matrix formalism not only to construct systematically integrable systems but also to present the related bi-Hamiltonian structures and conserved quantities. The main result of this dissertation is to present a unified and generalized theory for $(1+1)$-dimensional integrable Δ-differential systems which builds bridges between continuous, lattice and q-discrete soliton systems.

We would like to emphasize that for both unifying approaches, by the use of the continuous limit, discrete systems give rise to either continuous systems (dispersive counterpart of continuous systems) or dispersionless systems (systems of hydrodynamic type). In the first approach algebra of δ-pseudo-differential operators on regular discrete time scales are utilized while in the second approach algebra of shift operators being introduced on one parameter group of diffeomorphisms, which unifies lattice and q-discrete soliton systems, are considered. Therefore, these two results are hidden in the particular choice of the Lie algebras for discrete systems.

There are many open problems such as generalization and unification of soliton solutions of the nonlinear equations or construction of integrable systems on commutative algebras to find soliton solutions to construct Bäcklund transformations on arbitrary time scales.

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