UNIVERSALLY SELECTION-CLOSED FAMILIES OF SOCIAL CHOICE FUNCTIONS

A Master's Thesis

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

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In this thesis, we introduce a new notion of consistency for families of social choice functions, called selection-closedness. This concept requires that every member of a family of social choice functions that are to be employed by a society to make its choice from an alternative set it faces, should choose a member of the given family, when it is also employed to choose the social choice function itself in the presence of other rival such functions along with the members of the initial family. We show that a proper subset of neutral social choice functions is universally selection-closed if and only if it is a subset of the set of dictatorial and anti-dictatorial social choice functions. Finally, we introduce a weaker version of selection-closedness and conclude that a "right-extendable scoring correspondence" is strict if and only if the set consisting of its singleton valued refinements is universally weakly selection-closed.

Keywords: Social Choice, Self-selectivity, Selection-Closed, Weakly Selection-Closed, Scoring Correspondence.

ÖZET

EVRENSEL SEÇMEDE-KAPALI SOSYAL SEÇİM FONKSİYONU AİLELERİ

ŞENOCAK, Talat

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Bu tez çalışmamızda, yeni bir tutarlılık ölçütü olarak, seçmede-kapalılık kavramı sunulmaktadır. Bir toplulukça verili bir seçenek kümesinden seçim yapmada kullanılacak olan sosyal seçim kurallarından oluşan bir kümenin seçmede-kapalı olması için, bu kümenin herhangi bir üyesinin, sosyal seçim kuralının kendisinin, bu kümenin içindeki ve dışındaki baz seçme kuralları arasından seçilmesinde kullanılması durumunda, seçilen kuralın başlangıçtaki kümeye ait olması gerekmektedir. Nötr (seçenekler üzerinden permütasyonlar altında değişmez olan) sosyal seçim kuralları kümesinin herhangi bir has alt kümesinin *evrensel seçmede-kapalı* olmasının ancak ve ancak diktatörlük ve karşı diktatörlüklerden oluşan kümenin bir alt kümesi olmasıyla mümkün olacağı gösterilmiştir. Son olarak ise, seçmede-kapalılığın zayıflatılmış bir biçimi sunulmuş ve herhangi bir "sağ-uzatmalı derecelendirme küme değerli sosyal seçim fonksiyonunun" kuvvetli olabilmesinin ancak ve ancak söz konusu kuralın tek değerli bütün inceltmelerini içeren kümenin bu zayıf seçmede-kapalılık koşulunu sağlamasıyla mümkün olacağı gösterilmiştir.

Anahtar Kelimeler: Sosyal Seçim, Kendini-seçerlik, seçmede-kapalılık, zayıf seçmede-kapalılık, derecelendirme küme değerli fonksiyonu.

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CHAPTER 1

INTRODUCTION

Self-selectivity is a kind of consistency pertaining to social choice functions introduced by Koray (2000). We imagine that a society that faces a choice problem on a finite non-empty set A of alternatives is also to choose the social choice function (SCF) that will be used in making the choice from A. A natural question that arises concerns the consistency between these two levels of choice. More specifically, any societal preference profile on A induces a preference profile on any set $\mathscr{A}$ of social choice functions, by ranking SCFs in $\mathscr{A}$ according to the alternatives they choose from A. The question now is which rules from among the available SCFs will choose themselves when they are employed in choosing the choice rule from $\mathscr{A}$.

If an SCF employed to make the social choice from A does not choose itself at the induced preference profile on the set $\mathscr{A}$ of available SCFs, then this phenomenon can be regarded as a lack of consistency on the part of this SCF itself, as it rejects itself according to its own rationale. Roughly speaking, we call an SCF self-selective at a particular preference profile if it selects itself at the induced profile from among any finite number of available SCFs. Moreover, an SCF is said to be universally self-selective if it is self-selective at each preference profile. Koray (2000) shows that a neutral and unanimous SCF is universally self-selective if and only if it is dictatorial.

In this thesis, by using the same framework we introduce a new notion of consistency for sets of social choice functions, called selection-closedness. Let $\mathscr{F}$ be a set of SCFs, which can be considered as a constitution, capturing all possible SCFs that can be used by a society. In our model, we also introduce a different set $\mathscr{A}$ to represent the set of SCFs available to the society at the time of the choice. If an SCF from $\mathscr{F} \cap \mathscr{A}$ is employed to resolve the society's underlying choice problem, our new consistency criterion does not require any more that the chosen SCF selects itself. The yardstick now becomes that it chooses one of its "constitutionally prescribed companions" in $\mathscr{F}$ rather than something in $\mathscr{A} \setminus \mathscr{F}$, when it is used in choosing the choice function. In other words, this new consistency based on the concept of self-selectivity refers to a "group consistency", rather than "individual consistency". Roughly speaking, we call a set $\mathscr{F}$ of SCFs selection-closed at a particular preference profile if each member of $\mathscr{F} \cap \mathscr{A}$ selects a member of $\mathscr{F} \cap \mathscr{A}$ in the presence of all members of $\mathscr{A}$ at the given profile. Moreover, a set $\mathscr{F}$ of SCFs is said to be universally selection-closed if it is selection-closed at each preference profile.

The rest of the thesis is organized as follows: In chapter 2 we introduce the definitions of some basic notions including *self-selectivity* and *selection-closedness*. This new notion of consistency is motivated in this chapter by means of examples. In chapter 3, we introduce one of the main results of the thesis, namely a characterization of *selection-closedness*. In chapter 4, we introduce a weak version of *selection-closedness* and characterize the strictness of right scoring correspondences via this concept. The last chapter summarizes our main results, followed by some concluding remarks.

CHAPTER 2

SELECTION-CLOSEDNESS

2.1 Basic Notions

Let N be a finite non-empty society that will be fixed throughout the paper. For each $m \in \mathbb{N}$, where $\mathbb{N}$ denotes the the set of natural numbers as usual, let $I_m = \{1, \ldots, m\}$ represents the set of alternatives of cardinality m and denote the set of all linear orders on I_m by $\mathscr{L}(I_m)$. A *social choice function* (SCF) F is a function

$$F : \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N \to \mathbb{N}$$

such that for all $m \in \mathbb{N}$ and for all $R \in \mathscr{L}(I_m)^N$, one has $F(R) \in I_m$.

Firstly, we will define the neutrality of an SCF. Given $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$, for each permutation σ_m on I_m, we define the permuted linear order profile R_{σ_m} on I_m as follows: for each agent $i \in N$, and for each k, $l \in I_m$, $kR^i_{\sigma_m}l$ if and only if $\sigma_m(k)R^i\sigma_m(l)$. An SCF F is said to be *neutral* if and only if for each $m \in \mathbb{N}$ and each permutation σ_m on I_m, one has

$$\sigma_m(F(R_{\sigma_m})) = F(R)$$

for any profile R. We will denote the class of all *neutral* SCFs by $\mathscr{G}$.

We recapped the notion of neutrality to extend the domain of an SCF to linear order profiles on an arbitrary finite set of alternatives A. Given any finite set of alternatives A of cardinality $m \in \mathbb{N}$, consider a bijection $\mu : I_m \to A$. Now, for any linear order profile $L \in \mathscr{L}(A)^N$, we define a new linear order profile L_μ on I_m such that for each agent $i \in N$, and for each k, $l \in I_m$, $kL_\mu^i l$ if and only if $\mu(k)L^i\mu(l)$. Finally, we define $F(L) = \mu(F(L_\mu))$. It is easy to see that, if F is *neutral*, for any two bijections $\mu, v : I_m \to A$, we have $\mu(F(L_\mu)) = v(F(L_v))$. That is, $F(L)$ does not depend upon which bijection μ is employed.

Secondly, we deal with the preferences of agents on any given non-empty finite subset $\mathscr{A}$ of $\mathscr{G}$. Now, from any linear order profile $R \in \mathscr{L}(I_m)^N$, we need to induce a preference profile on $\mathscr{A}$. Given $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$, we define a preference profile $R_\mathscr{A}$ on $\mathscr{A}$ as follows: for each agent $i \in N$ and for each $F_1, F_2 \in \mathscr{G}$, $F_1 R^i_\mathscr{A} F_2$ if and only if $F_1(R)R^iF_2(R)$ (i.e. agents rank SCFs by only considering the outcomes that the SCFs choose at R). Note that $R_\mathscr{A}$ is a complete preorder profile on $\mathscr{A}$, since any two different SCFs may choose the same alternative at R. We will call $R_\mathscr{A}$ the preference profile on $\mathscr{A}$ induced by R.

By definition, the domain of a SCF is the union of linear order profiles on I_m for each $m \in \mathbb{N}$. In order to make a choice from among the set of SCF's in $\mathscr{A}$, we must consider the linear order profiles that are generated by the preference profile $R_\mathscr{A}$ induced by R. Formally, given a complete preorder ρ on a finite, non-empty set A, a linear order λ on A is compatible with by ρ if and only if for all $x, y \in A$, $x\lambda y$ implies $x\rho y$. Now, given $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$ and a non-empty finite subset $\mathscr{A}$ of $\mathscr{G}$, we set $\mathscr{L}(\mathscr{A}, R) = \{L \in \mathscr{L}(\mathscr{A})^N \mid L^i$ is a linear order on $\mathscr{A}$ compatible with $R^i_\mathscr{A}$ for each $i \in N\}$ and call $\mathscr{L}(\mathscr{A}, R)$ the set of all linear order profiles on $\mathscr{A}$ induced by R.

Thirdly, we will define the *self-selectivity* notion for a *neutral* SCF. Given $F \in \mathscr{G}$, $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and a non-empty finite subset $\mathscr{A}$ of $\mathscr{G}$ with

$F \in \mathscr{A}$, we say that F is *self-selective at R relative to $\mathscr{A}$* if and only if there exists some $L \in \mathscr{L}(\mathscr{A}, R)$ such that $F = F(L)$. Moreover, F is *self-selective at R* if and only if F is *self-selective at R relative to* any finite subset $\mathscr{A}$ of $\mathscr{G}$ with $F \in \mathscr{A}$. Lastly, F is *universally self-selective* if and only if F is *self-selective at* each $R \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$. In Koray (2000), it is shown that a *neutral SCF* is *universally self-selective* if and only if it is dictatorial or anti-dictatorial. We will denote the class of all *neutral self-selective SCFs* by $\mathscr{U}$.

Finally, we are ready to define *selection-closedness.* Given $\mathscr{F} \subseteq \mathscr{G}$, $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and a non-empty finite subset $\mathscr{A}$ of $\mathscr{G}$, we say that $\mathscr{F}$ is *selection-closed at R relative to $\mathscr{A}$* if and only if for all $F \in (\mathscr{F} \cap \mathscr{A})$, there exists some $L \in \mathscr{L}(\mathscr{A}, R)$ such that $F(L) \in \mathscr{F}$. Moreover, $\mathscr{F}$ is *selection-closed at R* if and only if $\mathscr{F}$ is *selection-closed at R relative to* any finite subset $\mathscr{A}$ of $\mathscr{G}$. Lastly, $\mathscr{F}$ is *universally selection-closed* if and only if $\mathscr{F}$ is *selection-closed at* each $R \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$.

Remark 1. By definition $\emptyset$ and $\mathscr{G}$ are *universally selection-closed.*

Remark 2. Any subset of $\mathscr{U}$ is *universally selection-closed.*

Remark 3. When $|\mathscr{F}| = 1$, $\mathscr{F}$ is *universally selection-closed* if and only if $F \in \mathscr{F}$ implies F is *universally self-selective.*

2.2 Example

Example 1. (This example is a modified version of an example in Koray (2000)) Let $N = \{\alpha, \beta, \gamma, \delta\}$ be the society endowed with a linear order profile R over the set of alternatives $I_3 = \{1, 2, 3\}$. Let F_1 be the plurality function with tie-breaking in favor of agent α. Given $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$, an outcome $a \in I_m$ is said to be a *Condorcet winners* at R if and only if, for all $b \in I_m \setminus \{a\}$, $|\{i \in N | aR^i b\}| \geq |N|/2 = 2$. In case that the set of *Condorcet winners* at R is non-empty, let F_2 choose the *Condorcet winners* most preferred by α if m is odd, and the *Condorcet winners* most preferred by β if m is even; if there are no *Condorcet winners* at R, let F_2 choose the top ranked alternative by agent α at R^α. Let F_3 be the *Borda* function with tie-breaking in favor of agent γ. Moreover, let F_4 will denote the dictatorial SCF where γ is the dictator, i.e., F_4 assigns the top alternative of R^γ to each $R \in \bigcup_{m\in\mathbb{N}} \mathscr{L}(I_m)^N$. Note that F_1, F_2, F_3 and F_4 are *neutral* SCFs.

Now, let us consider the linear order profile $R \in \mathscr{L}(I_3)^N$ given through the following table:

R^α	R^β	R^γ	R^δ
2	1	3	1
1	3	2	2
3	2	1	3

Let $\mathscr{F} = \{F_2, F_3\}$ and $\mathscr{A} = \{F_1, F_2, F_3\}$. We have $F_1(R) = 1 = F_3(R)$ and $F_2(R) = 2$. The following table illustrates the complete preorder $R_{\mathscr{A}}$ on $\mathscr{A}$ induced by R where boxes indicates the indifference classes;

R^α	R^β	R^γ	R^δ
F_2	F_1, F_3	F_2	F_1, F_3
F_1, F_3	F_2	F_1, F_3	F_2

Note that, $\mathscr{L}(\mathscr{A}, R)$, the set of all linear order profiles compatible with the above complete preorder $R_{\mathscr{A}}$, has cardinality 2^4. Now, consider the linear order profile $L \in \mathscr{L}(\mathscr{A}, R)$ below;

L^α	L^β	L^γ	L^δ
F_2	F_3	F_2	F_3
F_3	F_1	F_3	F_1
F_1	F_2	F_1	F_2

Note that $F_2(L) = F_2$ and $F_3(L) = F_3$, so we can conclude that both F_2 and

F_3 are *self-selective at R relative to* $\mathscr{A}$. Since $\mathscr{F} = \{F_2, F_3\}$ we can conclude that $\mathscr{F}$ is *selection-closed at R relative to* $\mathscr{A}$.

Now consider the set $\mathscr{A}' = \{F_2, F_3\}$. Since $F_3(R) = 1$ and $F_2(R) = 2$, the set $\mathscr{L}(\mathscr{A}', R)$ has one element, say L_1;

L_1^α	L_1^β	L_1^γ	L_1^δ
F_2	F_3	F_2	F_3
F_3	F_2	F_3	F_2

Note that $F_2(L) = F_3 \neq F_2$ and $F_3(L) = F_2 \neq F_3$, indicating neither F_2 nor F_3 is *self-selective at R relative to* $\mathscr{A}'$. But $F_2(L) = F_3 \in \mathscr{F}$ and $F_3(L) = F_2 \in \mathscr{F}$, so we say that $\mathscr{F}$ is *selection-closed at R relative to* $\mathscr{A}'$.

Moreover, consider the set $\mathscr{A}'' = \{F_2, F_4\}$. Since F_4 is dictatoriality of agent γ, we have $F_4(R) = 3$. So the set $\mathscr{L}(\mathscr{A}'', R)$ has one element, say L_2;

L_2^α	L_2^β	L_2^γ	L_2^δ
F_2	F_4	F_4	F_2
F_4	F_2	F_2	F_4

Note that $F_2(L) = F_4 \neq F_2$, so F_2 is not *self-selective at R relative to* $\mathscr{A}''$. Also since $F_4 \notin \mathscr{F}$, we can say that $\mathscr{F}$ is not *selection-closed at R relative to* $\mathscr{A}''$. Lastly, we conclude that $\mathscr{F}$ is not *universally selection-closed.*

CHAPTER 3

CHARACTERIZATION

3.1 Results

We will define a SCF F_α, that will be used in further discussion. Let $\alpha \in N$ and let D_α, $D_{-\alpha} \in \mathscr{G}$ stand for dictatoriality and anti-dictatoriality of agent α, respectively. Define $F_\alpha \in \mathscr{G}$, for each $m \in \mathbb{N}$ and at each $R \in \mathscr{L}(I_m)^N$, as:

$$F_\alpha(R) = \begin{cases} D_\alpha(R) & \text{if } m \text{ is odd} \\ D_{-\alpha}(R) & \text{if } m \text{ is even} \end{cases}$$

Lemma 1. *If $\mathscr{F}$ is a universally* selection-closed *subset of $\mathscr{G}$ and $F_\alpha \in \mathscr{F}$ for some $\alpha \in N$, then $\mathscr{F} = \mathscr{G}$.*

Proof. Assume not, i.e. $\mathscr{G} \setminus \mathscr{F} \neq \emptyset$. Now, let $C_1 \subseteq \mathscr{G}$ be the set of all SCFs such that for any $F_1 \in \mathscr{G}$, $F_1 \in C_1$ if and only if F_1 is the dictatoriality of agent α for all odd $m \in \mathbb{N}$ and let $C_2 \subseteq \mathscr{G}$ be the set of all SCFs such that for any $F_2 \in \mathscr{G}$, $F_2 \in C_2$ iff there exists an odd $\overline{m} \in \mathbb{N}$ such that F_2 is not coincident with the dictatoriality of agent α for that $\overline{m}$. Now, set $\overline{C_i} = C_i \cap (\mathscr{G} \setminus \mathscr{F})$ for each $i = 1, 2$. Note that $\overline{C_1} \cup \overline{C_2} = \mathscr{G} \setminus \mathscr{F}$. We claim that that $\overline{C_2}$ is empty. Assume not, and take any $F_2 \in \overline{C_2}$. Since $F_2 \notin \mathscr{F}$ we have $F_2 \neq F_\alpha$. Also $F_2 \in \overline{C_2}$ implies there exists an odd $\overline{m} \in \mathbb{N}$ such that F_2 is not the dictatoriality of agent α for that $\overline{m}$. Since $F_2 \neq F_\alpha$, there exists

$R \in \mathscr{L}(I_{\overline{m}})^N$ such that $F_2(R) \neq F_\alpha(R)$. Now, consider the set $\mathscr{A} = \{F_\alpha, F_2\}$. Note that F_α is dictatorial since $\overline{m}$ is odd. Thus for all $L \in \mathscr{L}(\mathscr{A}, R)$, agent α prefers F_α to F_2 and since $|\mathscr{A}| = 2$ we get $F_\alpha(L) = F_2$, contradicting with the *universally selection-closedness* of $\mathscr{F}$. Hence $\overline{C_2} = \emptyset$; i.e. $C_2 \subseteq \mathscr{F}$.

Now, we claim that $\overline{C_1}$ is empty as well. Assume not, and take any $F_1 \in \overline{C_1}$. Let F_2, F_3 be SCFs such that F_2 and F_3 are dictatoriality of agent α when m=3 and anti-dictatoriality of agent α when m=5 (construct F_2, F_3 for all other $m \in \mathbb{N}$ such that F_2 and F_3 are *neutral* and different SCFs). Clearly $F_2, F_3 \in C_2 \subseteq \mathscr{F}$. Now let m=5 and $\mathscr{A} = \{F_1, F_2, F_3\}$. Since F_1 is dictatorial and F_2, F_3 are anti-dictatorial, for any R and for any $L \in \mathscr{L}(\mathscr{A}, R)$ F_1 is the top choice of agent α. Since $|\mathscr{A}|$=3, we have $F_2(L) = F_1$, contradicting with the *universally selection-closedness* of $\mathscr{F}$. Hence $\overline{C_1} = \emptyset$; i.e. $C_1 \subseteq \mathscr{F}$.

We have found that $\overline{C_1} = \emptyset$ and $\overline{C_2} = \emptyset$, contradicting $\mathscr{G} \setminus \mathscr{F} \neq \emptyset$. □

Theorem 1. *Let $\mathscr{F} \subseteq \mathscr{G}$ with $\mathscr{F} \nsubseteq \mathscr{U}$ is universally selection-closed, then $\mathscr{F} = \mathscr{G}$.*

Proof. Assume not, i.e. there exists $F_1 \in (\mathscr{G} \setminus \mathscr{U})$ s.t. $F_1 \in \mathscr{F}$. Since F_1 is not *universally self-selective*, there exists $\overline{m} \in \mathbb{N}$, $\overline{\mathscr{A}} = \{F_1, F_2, \ldots, F_{\overline{k}}\}$ and $\overline{R} \in \mathscr{L}(I_{\overline{m}})^N$ such that F_1 is not *self-selective* at $\overline{R}$ relative to $\overline{\mathscr{A}}$; i.e. for all $L \in \mathscr{L}(\overline{\mathscr{A}}, \overline{R})$, we have $F_1(L) \neq F_1$. We know that $\mathscr{F}$ is *universally selection-closed* and $F_1 \in \mathscr{F}$, so there exists some $L \in \mathscr{L}(\overline{\mathscr{A}}, \overline{R})$ such that $F_1(L) \in \mathscr{F}$. Since $F_1(L) \neq F_1$, there exists $i \in \{2, \ldots, \overline{k}\}$ s.t. $F_i \in \mathscr{F}$.

Now, for each $i \in \{2, \ldots, \overline{k}\}$ define $F_i^{'}$ in the following way; when the number of alternatives is equal to $\overline{k}$ or $\overline{m}$, where $\overline{k}$ and $\overline{m}$ are determined above, $F_i^{'}(R)$ is equal to $F_i(R)$, otherwise $F_i^{'}(R)$ is equal to F_α, at each $R \in \mathscr{L}(I_m)^N$; i.e.

$$F_i^{'}(R) = \begin{cases} F_i(R) & \text{if } |I_m| \in \{\overline{k}, \overline{m}\} \\ F_\alpha(R) & \text{otherwise} \end{cases}$$

Now consider the set $\mathscr{B} = \left\{F_1, F_2^{'}, \ldots, F_{\overline{k}}^{'}\right\}$. By the same argument above, there exists some $L \in \mathscr{L}(\mathscr{B}, \overline{R})$ such that $F_1(L) \in \mathscr{F}$. But since, for the values $\overline{k}$ and $\overline{m}$, $F_i^{'}$ is same as F_i and F_1 is not *self-selective* at $\overline{R}$ relative to $\overline{\mathscr{A}}$, we have $F_1(L) \neq F_1$. So there exists some $j \in \left\{2, \ldots, \overline{k}\right\}$ such that $F_j^{'} \in \mathscr{F}$.

Now, we will show that F_α is also an element of $\mathscr{F}$. If $F_j^{'} = F_\alpha$, we are done. Assume $F_j^{'} \neq F_\alpha$. But note that, $F_j^{'}$ and F_α can only differ for the values $\overline{k}$ and $\overline{m}$. For simplicity assume they differ for the value $\overline{m}$. Now consider the case when $\overline{m}$ is odd. Since $F_j^{'} \neq F_\alpha$, there exists some $R \in \mathscr{L}(I_{\overline{m}})^N$ such that $F_j^{'}(R) \neq F_\alpha(R)$. Note that $F_\alpha(R)$ is the top choice of agent α in R since $\overline{m}$ is odd. Now let $t \in \mathbb{N}$ be an odd number such that $t+2 \neq \overline{m}$ and $t+2 \neq \overline{k}$. Also let $\mathscr{B}_1 = \left\{F_j^{'}, F_\alpha, \overline{F_1}, \ldots, \overline{F_t}\right\}$ where $\overline{F_i} \in \mathscr{G}$ for all $i \in \{1, \ldots, t\}$ s.t. $\overline{F_i}(R) \neq F_\alpha(R)$. Now, for any $L \in \mathscr{L}(\mathscr{B}_1, R)$, we have $F_j^{'}(L) = F_\alpha$ since $|\mathscr{B}_1| = t + 2$ is odd and $F_j^{'}$ is the dictatoriality of agent α. Since $F_j^{'} \in \mathscr{F}$ and $\mathscr{F}$ is *universally selection-closed*, we have $F_\alpha \in \mathscr{F}$. Now consider the case when $\overline{m}$ is even. Since $F_j^{'} \neq F_\alpha$, there exists some $R \in \mathscr{L}(I_{\overline{m}})^N$ such that $F_j^{'}(R) \neq F_\alpha(R)$. Note that $F_\alpha(R)$ is the bottom choice of agent α in R since $\overline{m}$ is even. Now let $t \in \mathbb{N}$ be an even number such that $t+2 \neq \overline{m}$ and $t+2 \neq \overline{k}$. Also let $\mathscr{B}_1 = \left\{F_j^{'}, F_\alpha, \overline{F_1}, \ldots, \overline{F_t}\right\}$ where $\overline{F_i} \in \mathscr{G}$ for all $i \in \{1, \ldots, t\}$ s.t. $\overline{F_i}(R) \neq F_\alpha(R)$. Now, for any $L \in \mathscr{L}(\mathscr{B}_1, R)$, we have $F_j^{'}(L) = F_\alpha$ since $|\mathscr{B}_1| = t+2$ is even and $F_j^{'}$ is the anti-dictatoriality of agent α. Since $F_j^{'} \in \mathscr{F}$ and $\mathscr{F}$ is *universally selection-closed*, we have $F_\alpha \in \mathscr{F}$.

Hence we observe that for any *universally selection-closed* subset $\mathscr{F}$ of $\mathscr{G}$ s.t. $\mathscr{F} \nsubseteq \mathscr{U}$, we have $F_\alpha \in \mathscr{F}$, by previous lemma, we conclude that $\mathscr{F} = \mathscr{G}$. $\square$

CHAPTER 4

WEAKLY SELECTION-CLOSEDNESS

4.1 Preliminaries

In this chapter, we will define a weak version of the *selection-closedness* and characterize a special class of scoring correspondences via this concept.

Given $\mathscr{F} \subseteq \mathscr{G}$, $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and a non-empty finite subset $\mathscr{A}$ of $\mathscr{G} \setminus \mathscr{F}$, we say that $\mathscr{F}$ is *weakly selection-closed at R relative to $\mathscr{A}$* if and only if for each $F \in \mathscr{F}$, there exists some $\mathscr{A}' \subseteq \mathscr{F}$ with $F \in \mathscr{A}'$ such that each $F' \in \mathscr{A}'$ is *self-selective at R relative to $\mathscr{A} \cup \mathscr{A}'$*. Moreover, $\mathscr{F}$ is *weakly selection-closed at R* if and only if $\mathscr{F}$ is *weakly selection-closed at R relative to* any finite subset $\mathscr{A}$ of $\mathscr{G} \setminus \mathscr{F}$. Lastly, $\mathscr{F}$ is *universally weakly selection-closed* if and only if $\mathscr{F}$ is *weakly selection-closed at* each $R \in \bigcup_{m\in\mathbb{N}} \mathscr{L}(I_m)^N$.

4.2 Examples

Example 2. Let $CW(R)$ denotes the set of all *Condorcet winners* at each $R \in \mathscr{L}(I_m)^N$ and let $C : \bigcup_{m\in\mathbb{N}} \mathscr{L}(I_m)^N \to 2^{\mathbb{N}}$ be a social choice correspondence

(SCC) defined for each $m \in \mathbb{N}$ and each $R \in \mathscr{L}(I_m)^N$ by,

$$C(R) = \begin{cases} CW(R) & \text{if } CW(R) \neq \emptyset \\ I_m & \text{if } CW(R) = \emptyset, \end{cases}$$

Let $\mathscr{C}$ stands for the set of all singleton-valued refinements of C.

Let $N = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ be the society endowed with a linear order profile R over the set of alternatives $I_3 = \{1, 2, 3\}$. In case that the set of *Condorcet winner* at R is non-empty, let F_1 choose the *Condorcet winner* least preferred by α_1; if there are no *Condorcet winners* at R, let F_1 choose the top ranked alternative by agent α_1 at R^{α_1}. Moreover, let F_2 be the dictatoriality of α_4. Note that F_1 is a singleton-valued refinement of C, i.e. $F_1 \in \mathscr{C}$.

Let $R \in \mathscr{L}(I_3)^N$ be the linear order profile represented by the table below:

R^{α_1}	R^{α_2}	R^{α_3}	R^{α_4}	R^{α_5}	R^{α_6}
1	1	2	2	1	3
3	3	1	1	3	2
2	2	3	3	2	1

Let $\mathscr{A} = \{F_2\}$. Note that we have $F_1(R) = 1$, $F_2(R) = 2$. Moreover, for any $F_j \in \mathscr{C}$, $F_j(R) = 1$ since $CW(R) = 1$. So, for any $\mathscr{A}' \subseteq \mathscr{C}$, with $F_1 \in \mathscr{A}'$, the following table illustrates the complete preorder $R_{\mathscr{A} \cup \mathscr{A}'}$ on $\mathscr{A} \cup \mathscr{A}'$ induced by R where boxes indicates the indifference classes;

R^{α_1}	R^{α_2}	R^{α_3}	R^{α_4}	R^{α_5}	R^{α_6}
$\boxed{F_j \in \mathscr{A}'}$	$\boxed{F_j \in \mathscr{A}'}$	F_2	F_2	$\boxed{F_j \in \mathscr{A}'}$	F_2
F_2	F_2	$\boxed{F_j \in \mathscr{A}'}$	$\boxed{F_j \in \mathscr{A}'}$	F_2	$\boxed{F_j \in \mathscr{A}'}$

Note that, for any linear order profile $L \in \mathscr{L}(\mathscr{A} \cup \mathscr{A}', R)$, we have $F_2 \in CW(L)$. Since F_1 choose the *Condorcet winner* least preferred by α_1, we have $F_1(L) = F_2$ for any $L \in \mathscr{L}(\mathscr{A} \cup \mathscr{A}', R)$. So, we can conclude that F_1 is not *self-selective at R relative to* $\mathscr{A} \cup \mathscr{A}'$ for any $\mathscr{A}' \subseteq \mathscr{C}$. Since $F_1 \in \mathscr{C}$, we can conclude that $\mathscr{C}$ is not *weakly selection-closed at R relative to* $\mathscr{A}$. Thus, we conclude that $\mathscr{C}$ is not *universally weakly selection-closed.*

Example 3. Let $\mathscr{P}$ stands for the set of all singleton-valued refinements of the *Pareto Correspondence* P. Let $N = \{\alpha, \beta\}$ be the society endowed with a linear order profile R over the set of alternatives $I_2 = \{1, 2\}$. Let F_1 be the dictatoriality of α if m is 2, and F_1 choose the *Pareto optimal* outcome which is least preferred by α otherwise. Note that F_1 is a singleton-valued refinement of P, i.e. $F_1 \in \mathscr{P}$.

Let $R \in \mathscr{L}(I_2)^N$ be the linear order profile represented by the table below:

R^α	R^β
1	2
2	1

Let $\mathscr{A} = \{F_2, F_3\}$ where $F_2(R) = F_3(R) = 2$ and $F_2, F_3 \in \mathscr{G} \setminus \mathscr{P}$. Note that $F_1(R) = 1$. Moreover, for any $F_j \in \mathscr{P}$, $F_j(R) \in \{1, 2\}$ since $P(R) = \{1, 2\}$. But, for any $\mathscr{A}' \subseteq \mathscr{P}$, with $F_1 \in \mathscr{A}'$ and for any linear order profile $L \in \mathscr{L}(\mathscr{A} \cup \mathscr{A}', R)$, we have either $F_2 \in P(L)$ or $F_3 \in P(L)$ or $F_j \in P(L)$ for some $F_j \in \mathscr{A}'$ with $F_j \neq F_1$. For all cases, F_1 is not the *Pareto optimal* outcome which is least preferred by α. So, we have $F_1(L) \neq F_1$ for any $L \in \mathscr{L}(\mathscr{A} \cup \mathscr{A}', R)$. So, we can conclude that F_1 is not *self-selective at R relative to* $\mathscr{A} \cup \mathscr{A}'$ for any $\mathscr{A}' \subseteq \mathscr{P}$. Since $F_1 \in \mathscr{P}$, we can conclude that $\mathscr{P}$ is not *weakly selection-closed at R relative to* $\mathscr{A}$. Thus, we conclude that $\mathscr{P}$ is not *universally weakly selection-closed.*

Example 4. Let $F \in \mathscr{G}$ be a social choice function. Then $\mathscr{F} = \mathscr{G} \setminus F$ is *universally weakly selection-closed.*

4.3 Results

To state our results we need further definitions. For any $m \in \mathbb{N}$, a *score vector* is an m-tuple $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ with $s_i \geq s_{i+1}$ for all $i \in \{1, \ldots, m-1\}$ and $s_1 > s_m$.

Given some alternative $a \in I_m$ and $R \in \mathscr{L}(I_m)^N$, let $\sigma(a, R_\alpha)$ denote the ranking of a in agent α's ordering, i.e. $\sigma(a, R_\alpha) = |\{b \in A | bR_\alpha a\}|$.

We say that F is a *scoring correspondence* if and only if for all $m \in \mathbb{N}$ there exists a score vector $s \subseteq \mathbb{R}^m$ such that for any $R \in \mathscr{L}(I_m)^N$ we have,

$$F(R) = \left\{ a \in I_m | \sum_{\alpha \in N} s_{\sigma(a,R_\alpha)} \geq \sum_{\alpha \in N} s_{\sigma(b,R_\alpha)} \quad \text{for any} \quad b \in I_m \right\}$$

Denote the set of all scoring correspondences by $\mathscr{S}$.

A *scoring correspondence* $F \in \mathscr{S}$ is said to be *strict* if and only if for all $m \in \mathbb{N}$ and for the associated score vector $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ we have $s_i > s_{i+1}$ for all $i \in \{1, \ldots, m-1\}$.

A *scoring correspondence* $F \in \mathscr{S}$ is said to be *right-extendable* if and only if for any $m, m+1 \in \mathbb{N}$ and for the associated score vectors $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ and $s' = (s'_1, \ldots, s'_m, s'_{m+1}) \in \mathbb{R}^{m+1}$ we have $s'_i = s_i$ for all $i \in \{1, \ldots, m\}$.

A *scoring correspondence* $F \in \mathscr{S}$ is said to be *left-extendable* if and only if for any $m, m+1 \in \mathbb{N}$ and for the associated score vectors $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ and $s' = (s'_1, \ldots, s'_m, s'_{m+1}) \in \mathbb{R}^{m+1}$ we have $s'_{i+1} = s_i$ for all $i \in \{1, \ldots, m\}$.

Note that most common scoring rules, e.g., Borda, plurality, inverse plurality, and any vote for k alternatives rule are either *right-extendable* or *left-extendable* (Borda is both *right-extendable* and *left-extendable*).

Lemma 2. *Let $C \in \mathscr{S}$ be a scoring correspondence and let $\mathscr{C}$ stands for the set of all singleton-valued refinements of this correspondence. If $\mathscr{C}$ is universally weakly selection-closed, then for any $m \in \mathbb{N}$ and for any $R \in \mathscr{L}(I_m)^N$ we have $C(R) \subseteq P(R)$.*

Proof. Assume not, i.e. there exists an $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$ such that $C(R) \nsubseteq P(R)$. Let $a \in C(R)$ and $a \notin P(R)$. Since $a \notin P(R)$, there exists $b \in I_m$ which Pareto dominates a at R. Let $F \in \mathscr{C}$ be such that $F(R) = a$ and for other $m' \in \mathbb{N}$ let F choose the most preferred alternative by agent 1 with respect to C. Clearly F is a singleton valued refinement of C. Now, let $\mathscr{A} = \{G_1, G_2, \ldots, G_m\} \subseteq \mathscr{G} \setminus \mathscr{C}$ be such that $G_i(R) = b$ for all $i \in \{1, 2, \ldots, m\}$. Since $\mathscr{C}$ is *universally weakly selection-closed*, there exist $\mathscr{A}' \subseteq \mathscr{C}$, with $F \in \mathscr{A}'$ such that F is *self selective* at R *relative* to $\mathscr{A} \cup \mathscr{A}'$, i.e. there exists $L \in \mathscr{L}(\mathscr{A} \cup \mathscr{A}', R)$ such that $F(L) = F$. But for any $L \in \mathscr{L}(\mathscr{A} \cup \mathscr{A}', R)$, F is Pareto dominated by any $G_j \in \mathscr{A}$. Moreover, since C is a scoring correspondence and $F(L) = F \in C(L)$, we must have $G_j \in C(L)$, i.e. there exists an alternative $G_j \in C(L)$ which Pareto dominates F at L. But, since the number of alternatives is strictly greater than m, F must choose the most preferred alternative by agent 1 with respect to C, i.e. $F(L) \neq F$, contradiction. □

Theorem 2. *A right-extendable scoring correspondence is strict if and only if the set of all singleton valued refinements of the correspondence is universally weakly selection-closed.*

Proof. Let C be a *right-extendable scoring correspondence*. Assume C is *strict*, then we will show that the set of all singleton-valued refinements of the correspondence, $\mathscr{C}$, is *universally weakly selection-closed*.

Take any $m \in \mathbb{N}$ and any $R \in \mathscr{L}(I_m)^N$. First, we claim that $C(R) \subseteq P(R)$. Suppose not, then there exists $a \in C(R)$ and $b \in I_m$ such that b Pareto dominates a. But, since C is *strict*, we must have $\sum_{\alpha \in N} s_{\sigma(b,R_\alpha)} > \sum_{\alpha \in N} s_{\sigma(a,R_\alpha)}$, contradicting with $a \in C(R)$.

Now, take any $F \in \mathscr{C}$ and any $\mathscr{A} \subseteq \mathscr{G} \setminus \mathscr{C}$. Since $F(R) \in C(R) \subseteq P(R)$ (by previous claim), there exists a linear order profile $L \in \mathscr{L}(\mathscr{A} \cup \{F\}, R)$ such that $F \in P(L)$. We will construct the set $\mathscr{A}' \subseteq \mathscr{C}$ by replicating the role of F in the linear order profile L. Consider the set $\mathscr{A}' = \{F, F_1\}$ where $F_1 \in \mathscr{C}$

and $F(R) = F_1(R)$. Now construct $L' \in \mathscr{L}(\mathscr{A} \cup \mathscr{A}', R)$ such that F_1 is just below F for all agents and the other alternatives remain in the same position as in L. Since the correspondence C is *right-extendable*, *strict* and $F \in P(L)$, we have $\sum_{\alpha \in N} s_{\sigma(F,L_\alpha)} = \sum_{\alpha \in N} s_{\sigma(F,L'_\alpha)}$ but $\sum_{\alpha \in N} s_{\sigma(G,L_\alpha)} < \sum_{\alpha \in N} s_{\sigma(G,L'_\alpha)}$ for all $G \in \mathscr{A}$. By continuing in this way, we can find $\mathscr{A}' = \{F, F_1, \dots, F_k\}$ such that $F_i \in \mathscr{C}$ and $F(R) = F_i(R)$ for all $i \in \{1, \dots, k\}$. Now, let $\hat{L} \in \mathscr{L}(\mathscr{A} \cup \mathscr{A}', R)$ such that $\sum_{\alpha \in N} s_{\sigma(F,L_\alpha)} = \sum_{\alpha \in N} s_{\sigma(F,\hat{L}_\alpha)} > \sum_{\alpha \in N} s_{\sigma(G,\hat{L}_\alpha)}$ for all $G \in \mathscr{A}$. Since C is *strict* and F *Pareto* dominates all F_i's for all $i \in \{1, \dots, k\}$ at $\hat{L}$ we have $C(\hat{L}) = F$. Since F is a singleton-valued refinement of C, we have $F(\hat{L}) = F$. For any $F' \in \mathscr{A}'$, we can construct $\hat{L}'$ by changing the position of F with F' in $\hat{L}$ such that $F'(\hat{L}') = F'$. Moreover, since F and $\mathscr{A}$ are arbitrary, we can conclude that $\mathscr{C}$ is *universally weakly selection-closed.*

Conversely let C be a *right-extendable scoring correspondence* and assume the set of all singleton-valued refinements of the correspondence, $\mathscr{C}$, is *universally weakly selection-closed.*

Take any $m \in \mathbb{N}$ and let $s = (s_1, \dots, s_m) \in \mathbb{R}^m$ be the score vector for that m. We claim that if $s_i = s_{i+1}$ for some $i \in \{1, \dots, m-1\}$, then for any $m' \geq m$ we must have $s_j = s_{j+1}$ for all $j \in \{i, \dots, m'-1\}$. Suppose not, i.e. $s_i = s_{i+1}$ for some $i \in \{1, \dots, m-1\}$ and there exists $k \geq i+1$ such that $s_k > s_{k+1}$ (wlog assume k is the smallest integer with this property). Now, let $|N|$ be the number of agents such that $|N|s_i > s_1 + (|N|-1)s_{k+1}$ and consider the following profile.

R_1	R_2	$\dots$	$R_{\lvert N\rvert-1}$	$R_{\lvert N\rvert}$
a_{11}	a_{12}	$\dots$	$a_{1(\lvert N\rvert-1)}$	$a_{1(\lvert N\rvert)}$
a_{21}	a_{22}	$\dots$	$a_{2(\lvert N\rvert-1)}$	$a_{2(\lvert N\rvert)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_{(i-1)1}$	$a_{(i-1)2}$	$\dots$	$a_{(i-1)(\lvert N\rvert-1)}$	$a_{(i-1)(\lvert N\rvert)}$
b_i	b_i	$\dots$	b_i	b_i
b_{i+1}	b_{i+1}	$\dots$	b_{i+1}	b_{i+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
b_{k-1}	b_{k-1}	$\dots$	b_{k-1}	b_{k-1}
b_k	b_k	$\dots$	b_k	b_k
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Note that for any a_{jl}, where $j \in \{1, \dots, i-1\}$, $l \in \{1, \dots, |N|\}$, we have $\sum_{\alpha \in N} s_{\sigma(a_{jl}, R_\alpha)} \leq s_1 + (|N|-1)s_{k+1}$. Moreover, for any b_j, where $j \in \{i, \dots, k\}$, we have $\sum_{\alpha \in N} s_{\sigma(b_j, R_\alpha)} = |N|s_i$. By construction we have $|N|s_i > s_1 + (|N|-1)s_{k+1}$, so we can conclude that $\{b_i, b_{i+1}\} \subseteq C(R)$. But, b_{i+1} is *Pareto* dominated by b_i, contradicting the previous lemma. So, if $s_i = s_{i+1}$ for some $i \in \{1, \dots, m-1\}$, then for any $m' \geq m$ we must have $s_j = s_{j+1}$ for all $j \in \{i, \dots, m'-1\}$.

Now, assume C is not *strict*, then for some $m \in \mathbb{N}$, there exists $k{+}1, k{+}2 \in \{1, \dots, m-1\}$ such that $s_{k+1} = s_{k+2}$. By previous claim, we know that for any $m' \geq m$ we have $s_j = s_{j+1}$ for all $j \in \{k+1, \dots, m'-1\}$. Now, let $|N| = 5$ and consider the following profile.

R_1	R_2	R_3	R_4	R_5
a	a	b_1	c_{14}	c_{15}
c_{21}	c_{22}	b_2	c_{24}	c_{25}
c_{31}	c_{32}	b_3	c_{34}	c_{35}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_{k1}	c_{k2}	b_k	c_{k4}	c_{k5}
b_1	b_1	a	b_1	b_1
b_2	b_2	$c_{(k+2)3}$	b_2	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
b_k	b_k	$c_{(2k)3}$	b_k	b_k
$\vdots$	$\vdots$	$\vdots$	a	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Note that for any c_{jl}, where $j \in \{1, \ldots, 2k\}$, $l \in \{1, \ldots, 5\}$, we have $\sum_{\alpha \in N} s_{\sigma(c_{jl}, R_\alpha)} \leq s_1 + (|N| - 1)s_{k+1}$ (note that c_{jl} is not defined for all j and l, but for simplicity in writing we will disregard this). Moreover, for any b_j, where $j \in \{1, \ldots, k\}$, we have $\sum_{\alpha \in N} s_{\sigma(b_j, R_\alpha)} \leq s_1 + (|N| - 1)s_{k+1}$. Lastly, we have $\sum_{\alpha \in N} s_{\sigma(a, R_\alpha)} = 2s_1 + (|N| - 2)s_{k+1}$. So $C(R) = a$, since $s_1 > s_{k+1}$.

Now, let $F \in \mathscr{C}$ and let $\mathscr{A} = \{G_1, G_2, \ldots G_k\}$ such that $G_j(R) = b_j$ and $G_j \in \mathscr{G} \setminus \mathscr{C}$. Moreover, for any $F_j \in \mathscr{C}$, $F_j(R) = a$ since $C(R) = a$. So, for any $\mathscr{A}' \subseteq \mathscr{C}$, with $F \in \mathscr{A}'$, the following table illustrates the complete preorder profile $R_{\mathscr{A} \cup \mathscr{A}'}$ on $\mathscr{A} \cup \mathscr{A}'$ induced by R where boxes indicates the indifference classes;

L_1	L_2	L_3	L_4	L_5
$\boxed{F_j \in \mathscr{A}'}$	$\boxed{F_j \in \mathscr{A}'}$	G_1	G_1	G_1
G_1	G_1	G_2	G_2	G_2
G_2	G_2	G_3	G_3	G_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
G_k	G_k	$\boxed{F_j \in \mathscr{A}'}$	$\boxed{F_j \in \mathscr{A}'}$	$\boxed{F_j \in \mathscr{A}'}$

Note that, for any linear order profile $L \in \mathscr{L}(\mathscr{A} \cup \mathscr{A}', R)$ and for any $F_j \in \mathscr{A}'$, we have $\sum_{\alpha \in N} s_{\sigma(F_j, L_\alpha)} \leq 2s_1 + 3s_{k+1}$. Moreover, we have $\sum_{\alpha \in N} s_{\sigma(G_1, L_\alpha)} = 3s_1 + 2s_{k+1}$. Since $s_1 > s_{k+1}$, we have $C(L) = G_1$. Since F is a singleton valued refinement of C, we must have $F(L) = G_1$. So, we can conclude that F is not *self-selective at R relative to $\mathscr{A} \cup \mathscr{A}'$* for any $\mathscr{A}' \subseteq \mathscr{C}$. Since $F \in \mathscr{C}$, we can conclude that $\mathscr{C}$ is not *weakly selection-closed at R relative to $\mathscr{A}$*, contradiction. □

Remark 4. If the set of all singleton valued refinements of a *left-extendable scoring correspondence* is *universally weakly selection-closed* then the correspondence is *strict* (this result is immediate from lemma 2). But there exist some *left-extendable scoring correspondence*s which are strict but the set of all singleton valued refinements of the correspondence is not *universally weakly selection-closed.*

CHAPTER 5

CONCLUSION

In this thesis, we introduced the notion of selection-closedness as a new criterion of consistency, and provided a characterization for the class of selection-closed families of SCFs. Moreover, we also introduced a weaker version of selection-closedness and characterized when right-extendable scoring correspondences are strict via this concept. A nonempty-valued social choice correspondence can be conceived as a constitutional construct which recommends its singleton-valued refinements as SCFs that can be employed in resolving particular social choice problems. We conjecture that most well-behaved constitutional social choice correspondences can be classified via different weak versions of selection-closedness. In particular, it might be interesting to look into the class of Condorcet consistent rules to find the kinds of weakened selection-closedness that fit the spirit of such rules.

Our work in this thesis also paves the way for the analysis of selection-closedness in restricted domains. According to our definition of selection-closedness, a member of a selection-closed family of SCFs must select a member of its own family even when rivaled by extremely nonstandard functions (such as F_α) that no modern society would think of employing. Such unnatural functions play an essential role in the proof of main theorem of the thesis yielding an impossibility result. Therefore, the use of restricted domains for the SCFs seems worth being considered to escape from this pessimistic result. Finally, if “self-selectiviy” is deemed to be a desirable property by a society, different relaxations of universal selection-closedness may lead to further interesting and valuable results.

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