A SURVEY OF MULTIVARIATE GARCH MODELS

A Master's Thesis

by
MUSTAFA ANIL TAŞ

Department of
Economics
Bilkent University
Ankara
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by

MUSTAFA ANIL TAŞ

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Dr. Taner Yiğit
Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assoc. Prof. Dr. Fatma Taşkın
Examining Committee Member

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assoc. Prof. Dr. Süheyla Özyıldırım
Examining Committee Member

Approval of the Institute of Economics and Social Sciences

Prof. Dr. Erdal Erel
Director

ABSTRACT

A SURVEY OF MULTIVARIATE GARCH MODELS

Taş, Mustafa Anıl

M.A., Department of Economics

Supervisor: Assist. Prof. Dr. Taner Yiğit

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This paper reviews the recent developments in the multivariate GARCH literature. Most common multivariate GARCH models and their properties are briefly presented.

Keywords: Multivariate GARCH, Volatility

ÖZET

ÇOK DEĞİŞKENLİ GARCH MODELLERİNİN BİR İNCELEMESİ

Taş, Mustafa Anıl

Yüksek Lisans, İktisat Bölümü

Tez Yöneticisi: Yrd. Doç. Dr. Taner Yiğit

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Bu çalışma çok değişkenli GARCH literatüründeki son gelişmeleri incelemiştir. En yaygın çok değişkenli GARCH modelleri ve bunların özellikleri kısaca sunulmuştur.

Anahtar Kelimeler: Çok Değişkenli GARCH, Volatilite

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CHAPTER I

INTRODUCTION

Volatility modelling has been one of the main objects in financial econometrics after the introduction of Autoregressive Conditional Heteroskedasticity (ARCH) in the seminal paper of Engle (1982). It is now widely accepted that understanding the relation between the volatilities and covolatilities of several markets or asset returns are essential. Therefore, univariate models of volatility are inadequate in that sense.

Multivariate GARCH models are often used in applications of asset pricing and asset allocation. Asset pricing depends on the covariances of assets in a portfolio and asset allocation is linked to optimal hedging ratios. Bollerslev et al. (1988), Ng (1991) and Hansson and Hördahl (1998) provide examples of these applications. Multivariate GARCH models are also used to analyze volatility and correlation transmission in studies of contagion, see Tse and Tsui (2002) and Bae et al. (2003).

A multivariate GARCH model should be flexible enough to be able to explain the dynamics of the conditional variances and covariances. However, very high flexibility may hurt the parsimony of the model by increasing the number of parameters. Therefore, the model should be parsimonious enough to allow for easy estimation and easy interpretation of the parameters. Another important issue in a multivariate GARCH model is ensuring the positive definiteness of the conditional covariance matrix. One may derive conditions under which the conditional covariance matrix implied by the model is positive definite. An alternative is to specify the model such that positive definiteness is ensured by construction.

This paper is a brief survey of the recent developments in multivariate GARCH modelling. For similar but more comprehensive surveys, see Bauwens et al. (2006) and Silvennoinen and Teräsvirta (2008). This paper is organized as follows. In section 2, several multivariate GARCH models are reviewed. Section 3 is devoted to hypothesis testing in multivariate GARCH models and section 4 concludes.

CHAPTER II

OVERVIEW OF MODELS

Consider a stochastic vector process $\{\boldsymbol{y}_t\}$ with dimension $N \times 1$. Let $\mathcal{F}_{t-1}$ denote the information set generated by the observed series $\{\boldsymbol{y}_t\}$ until time $t-1$. We have

$$\boldsymbol{y}_t = \boldsymbol{\mu}_t + \boldsymbol{\epsilon}_t, \tag{1}$$

with $\boldsymbol{\mu}_t$ is the conditional mean vector and $\boldsymbol{\epsilon}_t$ is such that

$$\boldsymbol{\epsilon}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t. \tag{2}$$

Thus $\boldsymbol{\epsilon}_t$ is conditionally heteroskedastic given the information set $\mathcal{F}_{t-1}$. The $N \times N$ matrix $\boldsymbol{H}_t$ is the conditional covariance matrix of $\boldsymbol{y}_t$ and $\boldsymbol{\eta}_t$ is an i.i.d. vector error process such that $E[\boldsymbol{\eta}_t\boldsymbol{\eta}_t'] = \boldsymbol{I}$. This defines the standard multivariate GARCH (MGARCH) framework.

The specification of the matrix process $\boldsymbol{H}_t$ determines the relevant MGARCH model. There are three approaches to the formulation of $\boldsymbol{H}_t$: Parametric, semiparametric and nonparametric formulations. We will mostly deal with the parametric formulations in the following subsections. These models are divided into three categories. In the first one, the conditional covariance matrix $\boldsymbol{H}_t$ is modelled directly. VEC and BEKK models belong to this category. The models in the second category include the factor models. These models assume that the process $\boldsymbol{\epsilon}_t$ is generated by a number of unobserved heteroskedastic factors. In the third category, the conditional variances and correlations are modelled instead of the conditional covariance matrix. This category includes the Constant Conditional Correlation (CCC) model and its extensions. The semiparametric and nonparametric approaches are considered in the last subsection. These approaches can offset the loss of efficiency of the parametric estimators due to misspecification of the conditional covariance matrices.

The number of parameters increases rapidly as the dimension of $\boldsymbol{y}_t$ increases. This creates difficulties in the estimation of the models. Therefore, one of the important objectives in specifying an MGARCH model is to maintain parsimony and flexibility simultaneously. Another goal is to ensure the positive definiteness of the conditional covariance matrices. Doing this through an eigenvalue-eigenvector decomposition is a numerically difficult problem. The numerical optimization of the likelihood function in the case of parametric models is another difficulty in constructing an MGARCH model. The conditional covariance or correlation matrix appearing in the likelihood function depends on the time index t and has to be inverted for all t in every iteration of the numerical optimization. This is a both time consuming and numerically unstable procedure, especially with high dimensions of $\boldsymbol{y}_t$. Therefore, avoiding excessive inversion of matrices is another objective in constructing an MGARCH model.

2.1 Models of the Conditional Covariance Matrix

We start by defining the VEC-GARCH model of Bollerslev et al. (1988), which is the pure multivariate extension of the univariate GARCH model.

In this model, every conditional variance and covariance is a linear function of all lagged conditional variances, covariances and lagged squared errors and cross products of errors. The model is defined as follows

$$\mathrm{vech}(\boldsymbol{H}_t) = \boldsymbol{c} + \sum_{j=1}^{q} \boldsymbol{A}_j \mathrm{vech}(\boldsymbol{\epsilon}_{t-j}\boldsymbol{\epsilon}'_{t-j}) + \sum_{j=1}^{p} \boldsymbol{B}_j \mathrm{vech}(\boldsymbol{H}_{t-j}), \tag{3}$$

where vech$(\cdot)$ is the operator that stacks the lower triangular portion of a $N \times N$ matrix as a $N(N+1)/2 \times 1$ vector. $\boldsymbol{A}_j$ and $\boldsymbol{B}_j$ are $N(N+1)/2 \times N(N+1)/2$ parameter matrices. Although the VEC model is very flexible, estimation of the parameters is quite demanding. The number of parameters equals $(p+q)(N(N+1)/2)^2 + N(N+1)/2$, which is large unless N is small.

The diagonal VEC (DVEC) model, proposed by Bollerslev et al. (1988), assumes that $\boldsymbol{A}_j$ and $\boldsymbol{B}_j$ in (3) are diagonal matrices. Each conditional variance $h_{ii,t}$ depends on its own past squared error $\epsilon^2_{i,t-1}$ and its own lag $h_{ii,t-1}$. Similarly, each conditional covariance $h_{ij,t}$ depends on its own past cross products of errors $\epsilon_{i,t-1}, \epsilon_{j,t-1}$ and its own lag $h_{ij,t-1}$. In this case, the number of parameters drops down to $(p+q+1)N(N+1)/2$ but no interaction is allowed between the different conditional variances and covariances.

As stated previously, estimation of the parameters of the VEC model is computationally demanding. Assuming that the errors $\boldsymbol{\eta}_t$ follow multivariate normal distribution, the log-likelihood of the model (2) has the following form

$$\sum_{t=1}^{T} \ell_t(\boldsymbol{\theta}) = \boldsymbol{c} - \frac{1}{2}\sum_{t=1}^{T} \ln|\boldsymbol{H}_t| - \frac{1}{2}\sum_{t=1}^{T} \boldsymbol{\epsilon}'_t \boldsymbol{H}_t^{-1} \boldsymbol{\epsilon}_t. \tag{4}$$

The parameter vector $\boldsymbol{\theta}$ is estimated iteratively. It is apparent from (4) that the conditional covariance matrix $\boldsymbol{H}_t$ has to be inverted for every t at each iteration. This is a computational burden if the number of observations and N are large. A second difficulty is to ensure the positive definiteness of the covariance matrices. The VEC model can be modified such that the conditional covariance matrices are positive definite by construction. This modified model is known as the Baba-Engle-Kraft-Kroner (BEKK) defined in Engle and Kroner (1995). The model is defined as follows

$$\boldsymbol{H}_t = \boldsymbol{C}\boldsymbol{C}' + \sum_{j=1}^{q}\sum_{k=1}^{K} \boldsymbol{A}'_{kj}\boldsymbol{\epsilon}_{t-j}\boldsymbol{\epsilon}'_{t-j}\boldsymbol{A}_{kj} + \sum_{j=1}^{p}\sum_{k=1}^{K} \boldsymbol{B}'_{kj}\boldsymbol{H}_{t-j}\boldsymbol{B}_{kj}, \tag{5}$$

where $\boldsymbol{A}_{kj}$, $\boldsymbol{B}_{kj}$ and $\boldsymbol{C}$ are $N \times N$ parameter matrices, and $\boldsymbol{C}$ is lower triangular. Since $\boldsymbol{C}\boldsymbol{C}' > 0$, positive definiteness of $\boldsymbol{H}_t$ is ensured if $\boldsymbol{H}_0$ is positive definite.

A simplified version of (5) is the diagonal BEKK where $\boldsymbol{A}_j$ and $\boldsymbol{B}_j$ are diagonal matrices. The most simplified version of the BEKK model is the scalar BEKK one with $\boldsymbol{A}_j = a\boldsymbol{I}$ and $\boldsymbol{B}_j = b\boldsymbol{I}$ where a and b are scalars. Despite the advantage of ensuring positive definiteness of $\boldsymbol{H}_t$, the estimation of BEKK model is still a computational difficulty. There are several matrix inversions in the model. The number of parameters in the full BEKK model is $(p+q)KN^2+N(N+1)/2$ and $(p+q)KN+N(N+1)/2$ in the diagonal one, which are quite large. It is usually assumed $p = q = K = 1$ in applications of (5) due to these numerical difficulties.

A recent model proposed by Kawakatsu (2006) is the matrix exponential GARCH (ME-GARCH) model which eliminates parameter restrictions to ensure positive definiteness of $\boldsymbol{H}_t$. It is a generalization of the univariate exponential GARCH (E-GARCH) model proposed by Nelson (1991). The

ME-GARCH model is defined as follows

$$\begin{aligned}\mathrm{vech}(\ln \boldsymbol{H}_t - \boldsymbol{C}) = &\sum_{i=1}^{q} \boldsymbol{A}_i \boldsymbol{\eta}_{t-i} + \sum_{i=1}^{q} \boldsymbol{F}_i(|\boldsymbol{\eta}_{t-i}| - E|\boldsymbol{\eta}_{t-i}|) \\ &+ \sum_{i=1}^{p} \boldsymbol{B}_i \mathrm{vech}(\ln \boldsymbol{H}_{t-i} - \boldsymbol{C}),\end{aligned} \tag{6}$$

where $\boldsymbol{C}$ is a symmetric $N \times N$ matrix and $\boldsymbol{A}_i$, $\boldsymbol{B}_i$ and $\boldsymbol{F}_i$ are parameter matrices of sizes $N(N+1)/2 \times N$, $N(N+1)/2 \times N(N+1)/2$ and $N(N+1)/2 \times N$ respectively. For any symmetric matrix $\boldsymbol{S}$, the matrix exponential is defined as

$$\exp(\boldsymbol{S}) = \sum_{i=0}^{\infty} \frac{\boldsymbol{S}^i}{i!}, \tag{7}$$

which is positive definite. This implies that the covariance matrix $\boldsymbol{H}_t$ is positive definite thus there is no need to impose restrictions on the parameters to ensure positive definiteness. Since the ME-GARCH model also contains a large number of parameters, the need for more parsimonious models is still alive.

2.2 Factor Models

Factor models state that $\boldsymbol{\epsilon}_t$ is generated by a number of underlying conditionally heteroscedastic factors that follow a GARCH type process. The first factor GARCH (F-GARCH) model is introduced by Engle et al. (1990). They assume that $\boldsymbol{H}_t$ is generated by K $(< N)$ underlying factors $f_{k,t}$. The model is defined as follows

$$\boldsymbol{H}_t = \boldsymbol{\Omega} + \sum_{k=1}^{K} \boldsymbol{w}_k \boldsymbol{w}_k' f_{k,t}, \tag{8}$$

where $\boldsymbol{\Omega}$ is an $N \times N$ positive semidefinite matrix, $\boldsymbol{w}_k$ is a set of $N \times 1$ vectors of factor weights which are linearly independent from each other for

$k = 1, \ldots, K$. The factors $f_{k,t}$ are assumed to follow a first order GARCH process:

$$f_{k,t} = w_k + \alpha_k(\boldsymbol{\gamma}_k' \boldsymbol{\epsilon}_{t-1})^2 + \beta_k f_{k,t-1}, \tag{9}$$

where w_k, α_k and β_k are scalars and $\boldsymbol{\gamma}_k$ is an $N \times 1$ vector of weights. There is no restriction on the correlations of factors with each other. If the factors are correlated significantly, several of them yield the same information. If they are uncorrelated, each of them capture a different characteristic of the data. In this case, it is assumed that $\boldsymbol{\epsilon}_t$ is linked to uncorrelated factors, $\boldsymbol{z}_t$ through a linear, invertible transformation matrix $\boldsymbol{W}$:

$$\boldsymbol{\epsilon}_t = \boldsymbol{W} \boldsymbol{z}_t, \tag{10}$$

where $\boldsymbol{W}$ is a nonsingular $N \times N$ matrix. The factors $\boldsymbol{z}_t$ are assumed to follow a GARCH process.

The Generalized Orthogonal GARCH (GO-GARCH) model of van der Weide (2002) is an extension of the Orthogonal GARCH (O-GARCH) model of Alexander and Chibumba (1997). In the GO-GARCH model, the transformation matrix $\boldsymbol{W}$ is invertible but not required to be orthogonal. The uncorrelated factors $\boldsymbol{z}_t$ have unit unconditional variances, that is, $E[\boldsymbol{z}_t \boldsymbol{z}_t'] = \boldsymbol{I}$. The factors are conditionally heteroskedastic and follow a GARCH process. The $N \times N$ diagonal matrix of conditional variances of $\boldsymbol{z}_t$ is defined as follows

$$\boldsymbol{H}_t^z = (\boldsymbol{I} - \boldsymbol{A} - \boldsymbol{B}) + \boldsymbol{A} \odot (\boldsymbol{z}_{t-1} \boldsymbol{z}_{t-1}') + \boldsymbol{B} \boldsymbol{H}_{t-1}^z, \tag{11}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonal $N \times N$ parameter matrices and $\odot$ is the elementwise (Hadamard) product of two matrices.

Vrontos et al. (2003) suggested a slightly different model called the Full Factor GARCH (FF-GARCH) model. In this model, the $N \times N$ transformation matrix $\boldsymbol{W}$ is assumed to be triangular with ones on the main diagonal. The parameters in $\boldsymbol{W}$ are estimated directly using the conditional information. This model is computationally more feasible but also restrictive in the sense that some relationships between the factors and the errors are ignored.

A recent model by Lane and Saikkonen (2007) is the Generalized Orthogonal Factor GARCH (GOF-GARCH) model which assumes that the transformation matrix $\boldsymbol{W}$ is decomposed using the polar decomposition:

$$\boldsymbol{W} = \boldsymbol{C}\boldsymbol{V}, \tag{12}$$

where $\boldsymbol{C}$ is a symmetric positive definite $N \times N$ matrix and $\boldsymbol{V}$ is an orthogonal $N \times N$ matrix. Since $E[\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t'] = \boldsymbol{W}\boldsymbol{W}' = \boldsymbol{C}\boldsymbol{C}'$, the matrix $\boldsymbol{C}$ can be estimated using the spectral decomposition $\boldsymbol{C} = \boldsymbol{U}\boldsymbol{\Lambda}^{1/2}\boldsymbol{U}'$. The columns of $\boldsymbol{U}$ are the eigenvectors of $E[\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t']$ and the diagonal matrix $\boldsymbol{\Lambda}$ contains its eigenvalues. Estimation of $\boldsymbol{V}$ requires the use of conditional information.

2.3 Models of Conditional Variances and Correlations

The models in this section are based on the decomposition of the conditional covariance matrix into conditional standard deviations and correlations. The most basic one of these type of models is the Constant Conditional Correlation GARCH (CCC-GARCH) model of Bollerslev (1990). This model assumes that the conditional correlation matrix is constant, so the conditional covariance matrix is defined as follows

$$\boldsymbol{H}_t = \boldsymbol{D}_t\boldsymbol{P}\boldsymbol{D}_t, \tag{13}$$

where

$$\boldsymbol{D}_t = \mathrm{diag}(h_{1t}^{1/2}, \ldots, h_{Nt}^{1/2}), \tag{14}$$

and $\boldsymbol{P} = [\rho_{ij}]$ is positive definite with $\rho_{ii} = 1$, $i = 1, \ldots, N$. Then the off-diagonal elements of the conditional covariance matrix are defined as follows

$$[\boldsymbol{H}_t]_{ij} = h_{it}^{1/2} h_{jt}^{1/2} \rho_{ij}, \quad i \neq j, \tag{15}$$

where $1 \leq i, j \leq N$. The conditional variances are usually modelled as a GARCH(p, q) model:

$$\boldsymbol{h}_t = \boldsymbol{\omega} + \sum_{j=1}^{q} \boldsymbol{A}_j \boldsymbol{\epsilon}_{t-j}^{(2)} + \sum_{j=1}^{p} \boldsymbol{B}_j \boldsymbol{h}_{t-j}, \tag{16}$$

where $\boldsymbol{\omega}$ is $N \times 1$ vector, $\boldsymbol{A}_j$ and $\boldsymbol{B}_j$ are diagonal $N \times N$ matrices, and $\boldsymbol{\epsilon}_t^{(2)} = \boldsymbol{\epsilon}_t \odot \boldsymbol{\epsilon}_t$. When the conditional correlation matrix $\boldsymbol{P}$ is positive definite and the elements of $\boldsymbol{\omega}$ and the diagonal elements of $\boldsymbol{A}_j$ and $\boldsymbol{B}_j$ are positive, the conditional covariance matrix $\boldsymbol{H}_t$ is positive definite.

Jeantheau (1998) suggested an extension of the CCC-GARCH model, called the Extended CCC-GARCH (ECCC-GARCH) model in which the matrices $\boldsymbol{A}_j$ and $\boldsymbol{B}_j$ in (16) are not required to be diagonal. Then the past squared errors and variances of all series appear in each conditional variance equation. For instance, in the first order ECCC-GARCH model, the i^{th} variance equation is defined as follows

$$h_{it} = \omega_i + a_{11}\epsilon_{1,t-1}^2 + \cdots + a_{1N}\epsilon_{N,t-1}^2 + b_{11}h_{1,t-1} + \cdots + b_{1N}h_{N,t-1}, \quad i = 1, \ldots, N. \tag{17}$$

This extended structure provides a more comprehensive explanation of the autocorrelations between squared observed errors.

After the decomposition in (13), the log-likelihood in (4) takes the following simple form

$$
\begin{aligned}
\sum_{t=1}^{T} \ell_t(\boldsymbol{\theta}) =& \boldsymbol{c} - \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{N} \ln|h_{it}| - \frac{1}{2}\sum_{t=1}^{T} \log|\boldsymbol{P}| \\
& - \frac{1}{2}\sum_{t=1}^{T} \boldsymbol{\epsilon}_t' \boldsymbol{D}_t^{-1} \boldsymbol{P}^{-1} \boldsymbol{D}_t^{-1} \boldsymbol{\epsilon}_t. \qquad (18)
\end{aligned}
$$

It is seen from (18) that the conditional correlation matrix has to be inverted only once per iteration during estimation.

The CCC-GARCH model does not seem realistic because of the assumption of constant conditional correlations. The model can be improved by allowing the conditional correlation matrix in (13) to vary with time:

$$\boldsymbol{H}_t = \boldsymbol{D}_t \boldsymbol{P}_t \boldsymbol{D}_t. \qquad (19)$$

In this case, the positive definiteness of $\boldsymbol{H}_t$ is satisfied if $\boldsymbol{h}_t$ is properly specified and the conditional correlation matrix $\boldsymbol{P}_t$ is positive definite for all t. Furthermore, a computational difficulty arises since the conditional correlation matrix has to be inverted for all t during every iteration.

Tse and Tsui (2002) proposed the Varying Correlation GARCH (VC-GARCH) model in which the conditional correlation matrix follows a GARCH process. In this model, $\boldsymbol{P}_t$ is a function of $\boldsymbol{P}_{t-1}$ and a set of estimated correlations:

$$\boldsymbol{P}_t = (1 - a - b)\boldsymbol{S} + a\boldsymbol{S}_{t-1} + b\boldsymbol{P}_{t-1}, \qquad (20)$$

where $\boldsymbol{S}$ is a contant, positive definite matrix with ones on the diagonal, a and b are nonnegative scalars such that $a+b \leq 1$. The matrix $\boldsymbol{S}_{t-1}$ is a sample correlation matrix of the past M standardized residuals $\hat{\boldsymbol{v}}_{t-1}, \ldots, \hat{\boldsymbol{v}}_{t-M}$ where $\hat{\boldsymbol{v}}_{t-j} = \hat{\boldsymbol{D}}_{t-j}^{-1} \boldsymbol{\epsilon}_{t-j}$, $j = 1, \ldots, M$. The conditional correlation matrix $\boldsymbol{P}_t$ is positive definite provided that $\boldsymbol{P}_0$ and $\boldsymbol{S}_{t-1}$ are positive definite.

A similar model by Engle (2002) is the Dynamic Conditional Correlation GARCH (DCC-GARCH) model. The conditional correlation matrix of the DCC-GARCH model is defined as follows

$$\boldsymbol{P}_t = (\boldsymbol{I} \odot \boldsymbol{Q}_t)^{-1/2} \boldsymbol{Q}_t (\boldsymbol{I} \odot \boldsymbol{Q}_t)^{-1/2}, \tag{21}$$

where the matrix process $\boldsymbol{Q}_t$ is defined as

$$\boldsymbol{Q}_t = (1 - a - b)\boldsymbol{S} + a\boldsymbol{\upsilon}_{t-1}\boldsymbol{\upsilon}'_{t-1} + b\boldsymbol{Q}_{t-1}. \tag{22}$$

Here a is a positive and b a nonnegative scalar such that $a + b < 1$, $\boldsymbol{S}$ is the unconditional correlation matrix of the standardized errors $\boldsymbol{\upsilon}_t$ and $\boldsymbol{Q}_0$ is positive definite.

Both the VC-GARCH and DCC-GARCH models assume that the conditional correlation matrix is a function of past errors $\boldsymbol{\epsilon}_{t-j}$. There is another class of models that constructs the conditional correlation matrix using an exogeneous variable. This variable may be either an observed variable or a latent variable. The first one of these models is the Smooth Transition Conditional Correlation GARCH (STCC-GARCH) by Silvennoinen and Teräsvirta (2005). They state that the conditional correlation matrix varies between two extreme states according to a transition variable:

$$\boldsymbol{P}_t = (1 - G(s_t))\boldsymbol{P}_{(1)} + G(s_t)\boldsymbol{P}_{(2)}, \tag{23}$$

where $\boldsymbol{P}_{(1)}$ and $\boldsymbol{P}_{(2)}$ are positive definite extreme correlation matrices and $G(\cdot) : \mathbb{R} \rightarrow (0, 1)$ is a monotonic function of an observable transition variable s_t. The function $G(\cdot)$ is defined as follows

$$G(s_t) = \left(1 + e^{-\gamma(s_t - c)}\right)^{-1}, \quad \gamma > 0, \tag{24}$$

where γ and c are the speed and location parameters respectively. The STCC-GARCH model has $N(N-1)+2$ parameters excluding the univariate GARCH equations. It is important to note that $\boldsymbol{P}_t$ is positive definite since $\boldsymbol{P}_{(1)}$ and $\boldsymbol{P}_{(2)}$ are positive definite. The transition variable s_t is chosen properly according to the application. A special case occurs when s_t is calendar time. This model is known as the Time Varying Conditional Correlation GARCH (TVCC-GARCH) introduced by Berben and Jansen (2005).

The Double Smooth Transition Conditional Correlation GARCH (DSTCC-GARCH) model by Silvennoinen and Teräsvirta (2007) extends the STCC-GARCH model by allowing a variation between two STCC-GARCH models:

$$\begin{aligned}\boldsymbol{P}_t =&(1-G_2(s_{2t}))\{(1-G_1(s_{1t}))\boldsymbol{P}_{(11)}+G_1(s_{1t})\boldsymbol{P}_{(21)}\}\\&+G_2(s_{2t})\{(1-G_1(s_{1t}))\boldsymbol{P}_{(12)}+G(s_{1t})\boldsymbol{P}_{(22)}\}.\end{aligned} \tag{25}$$

If one of the transition variables is calendar time, the model is known as the Time Varying Smooth Transition Conditional Correlation GARCH (TVSTCC-GARCH) model. This model allows the extreme states to vary with time, thus enhances flexibility. However, the number of parameters, excluding the univariate GARCH equations, increases to $2N(N-1)+4$ which is inconvenient in very large dimensions.

A recent model by Pelletier (2006) is the Regime Switching Dynamic Correlation GARCH (RSDC-GARCH) model which assumes constant conditional correlations within a regime. The conditional correlation matrix is defined as follows

$$\boldsymbol{P}_t = \sum_{r=1}^{R} \mathbb{I}_{\{\Delta_t=r\}}\boldsymbol{P}_{(r)}, \tag{26}$$

where Δ_t is a Markov chain that can take R possible values, $\mathbb{I}$ is the indicator function and $\boldsymbol{P}_{(r)}$, $r = 1, \ldots, R$ are positive definite regime specific correlation matrices. The correlation component of the model has $RN(N-1)/2 - R(R-1)$ parameters. The model can be restricted to have less parameters such that R possible states are linear combinations of a state of zero correlations and that of high correlations. The restricted conditional correlations can be defined explicitly as follows

$$\boldsymbol{P}_t = (1 - \lambda(\Delta_t))\boldsymbol{I} + \lambda(\Delta_t)\boldsymbol{P}, \tag{27}$$

where $\boldsymbol{I}$ is the identity matrix meaning zero correlations, $\boldsymbol{P}$ is the correlation matrix with highly correlated states and $\lambda(\cdot) : \{1, \ldots, R\} \rightarrow [0, 1]$ is a monotonic function of Δ_t. The conditional correlation matrix is positive definite at each point in time by construction both in the restricted and unrestricted model.

2.4 Nonparametric and Semiparametric Models

Parametric MGARCH models are usually preferred in applications because of their advantage both in estimation and interpretation of parameters. The quasi-maximum likelihood (QML) estimator is consistent when the errors are assumed multivariate normal. However, this is a very restrictive assumption. Serious efficiency losses occur if the data is not normally distributed. Semiparametric models are invariant to distributional misspecification while preserving consistency and interpretability. Nonparametric models does not perform well in estimation due to the curse of dimensionality.

Hafner and Rombouts (2007) specify a parametric MGARCH model for the conditional covariance structure but estimate the error distribution nonparametrically. Then the log-likelihood becomes:

$$\sum_{t=1}^{T} \ell_t(\boldsymbol{\theta}) = \boldsymbol{c} - \frac{1}{2}\sum_{t=1}^{T} \ln|\boldsymbol{H}_t| + \sum_{t=1}^{T} \ln g(\boldsymbol{H}_t^{-1/2}\boldsymbol{\epsilon}_t), \tag{28}$$

where $g(\cdot)$ is the density function of the standardized residuals $\boldsymbol{\eta}_t$ such that $E[\boldsymbol{\eta}_t] = \mathbf{0}$ and $E[\boldsymbol{\eta}_t\boldsymbol{\eta}_t'] = \boldsymbol{I}$. In this semiparametric model, nonparametric error distribution offsets some of the misspecification of conditional covariance structure.

In a similar model by Long and Ullah (2005), a parametric model is estimated and the estimated standardized residuals $\hat{\boldsymbol{\eta}}_t$ are extracted. Then the conditional covariance matrix is estimated using the Nadaraya-Watson estimator:

$$\boldsymbol{H}_t = \hat{\boldsymbol{H}}_t^{1/2} \frac{\sum_{\tau=1}^{T} \hat{\boldsymbol{\eta}}_t\hat{\boldsymbol{\eta}}_t' K_h(s_\tau - s_t)}{\sum_{\tau=1}^{T} K_h(s_\tau - s_t)} \hat{\boldsymbol{H}}_t^{1/2}, \tag{29}$$

where $\hat{\boldsymbol{H}}_t$ is the conditional covariance matrix estimated parametrically form an MGARCH model, s_t is the conditioning variable, $K(\cdot)$ is a kernel function and h is the bandwidth parameter. The semiparametric estimator $\boldsymbol{H}_t$ is also positive definite since $\hat{\boldsymbol{H}}_t$ is positive definite.

Hafner et al. (2006) suggest the Semi-Parametric Conditional Correlation GARCH (SPCC-GARCH) model in which the conditional variances are modelled parametrically by a univariate GARCH model. Then the conditional correlations $\boldsymbol{P}_t$ are estimated using a transformed Nadaraya-Watson estimator:

$$\boldsymbol{P}_t = (\boldsymbol{I} \odot \boldsymbol{Q}_t)^{-1/2}\boldsymbol{Q}_t(\boldsymbol{I} \odot \boldsymbol{Q}_t)^{-1/2}, \tag{30}$$

where

$$\boldsymbol{Q}_t = \frac{\sum_{\tau=1}^{T} \hat{\boldsymbol{v}}_t\hat{\boldsymbol{v}}_t' K_h(s_\tau - s_t)}{\sum_{\tau=1}^{T} K_h(s_\tau - s_t)}. \tag{31}$$

In (31), $\hat{\boldsymbol{v}}_t = \hat{\boldsymbol{D}}_t^{-1}\boldsymbol{\epsilon}_t$ is the vector of the standardized residuals, s_t is a conditioning variable, $K(\cdot)$ is a kernel function and h is the bandwidth parameter.

Long and Ullah (2005) also suggest a full nonparametric model which is not an MGARCH model but a parameter free multivariate model. The conditional covariance matrix estimator is defined as follows

$$\boldsymbol{H}_t = \frac{\sum_{\tau=1}^{T} \boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t' K_h(s_\tau - s_t)}{\sum_{\tau=1}^{T} K_h(s_\tau - s_t)}, \tag{32}$$

where s_t is a conditioning variable, $K(\cdot)$ is a kernel function and h is the bandwidth parameter. The positive definiteness of $\boldsymbol{H}_t$ is ensured in this model.

CHAPTER III

HYPOTHESIS TESTING

General misspecification tests are used to check the adequacy of an estimated model. Ling and Li (1997) proposed a misspecification test which is applicable for many GARCH models. The test statistic is defined as follows

$$Q(k) = T\boldsymbol{\gamma}_k'\hat{\boldsymbol{\Omega}}_k^{-1}\boldsymbol{\gamma}_k, \tag{33}$$

where $\boldsymbol{\gamma}_k = (\gamma_1, \ldots, \gamma_k)'$ with

$$\gamma_j = \frac{\sum_{t=j+1}^{T}(\boldsymbol{\epsilon}_t'\hat{\boldsymbol{H}}_t^{-1}\boldsymbol{\epsilon}_t - N)(\boldsymbol{\epsilon}_{t-j}'\hat{\boldsymbol{H}}_{t-j}^{-1}\boldsymbol{\epsilon}_{t-j} - N)}{\sum_{t=1}^{T}(\boldsymbol{\epsilon}_t'\hat{\boldsymbol{H}}_t^{-1}\boldsymbol{\epsilon}_t - N)^2}, \quad j = 1, \ldots, k, \tag{34}$$

$\hat{\boldsymbol{H}}_t$ is an estimator of $\boldsymbol{H}_t$ and $\hat{\boldsymbol{\Omega}}_k$ is the estimated covariance matrix of $\boldsymbol{\gamma}_k$. The null hypothesis is $H_0 = \boldsymbol{\eta}_t \sim \text{i.i.d.}(\boldsymbol{0}, \boldsymbol{I})$ meaning that the GARCH model is corectly specified. Under the null hypothesis, the test statistic in (33) has an asymptotic χ^2 distribution with k degrees of freedom. Since $E[\boldsymbol{\epsilon}_t'\boldsymbol{H}_t^{-1}\boldsymbol{\epsilon}_t] = N$ under the null, then (34) boils down to the j^{th} order sample autocorrelation between $\boldsymbol{\epsilon}_t'\boldsymbol{H}_t^{-1}\boldsymbol{\epsilon}_t = \boldsymbol{\eta}_t'\boldsymbol{\eta}_t$ and $\boldsymbol{\epsilon}_{t-j}'\boldsymbol{H}_{t-j}^{-1}\boldsymbol{\epsilon}_{t-j} = \boldsymbol{\eta}_{t-j}'\boldsymbol{\eta}_{t-j}$. This test is a generalization of the univariate portmanteau test of Li and Mak (1994).

The CCC-GARCH model assumes that the conditional correlation matrix is constant. Therefore, it is crucial to test whether this is statistically true. The Lagrange multiplier (LM) test by Tse (2000) adopts the null hypothesis of constant correlations against the following alternative

$$\boldsymbol{P}_t = \boldsymbol{P} + \boldsymbol{\Delta} \odot \boldsymbol{\epsilon}_{t-1}\boldsymbol{\epsilon}_{t-1}', \tag{35}$$

where $\mathbf{\Delta}$ is a symmetric matrix with zeros on the main diagonal. The null hypothesis can be expressed as $\mathbf{\Delta} = \mathbf{0}$. Under the alternative, the correlations depend on the previous observations.

CHAPTER IV

CONCLUSION

This paper analyzes a number of multivariate GARCH models. The VEC model can be considered as the base model. However, this model contains too many parameters which leads to inapplicability especially in large dimensions. The BEKK model is developed as an alternative to the VEC model. Despite its flexibility, the BEKK model is still not parsimonious enough. Diagonal VEC and BEKK models are much more parsimonious but very restrictive for the cross dynamics. Another set of alternatives is the factor GARCH models which allow the conditional variances and covariances to depend on their lagged values.

Direct modelling of conditional covariances through conditional variances and correlations leads to a number of new models which are more popular now. The conditional correlation models are more feasible both in estimation and interpretation of parameters. The DCC-GARCH model is more realistic than the CCC-GARCH model since the conditional correlations are time varying. Recent research has focused on modelling the conditional correlation matrix with utmost flexibility and parsimony.

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