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INDUCTIONS, RESTRICTIONS, EVALUATIONS, AND SUBFUNCTORS OF MACKEY FUNCTORS

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FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

By

Ergün Yaraneri

August, 2008

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. Laurence J. Barker (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

INDUCTIONS, RESTRICTIONS, EVALUATIONS, AND SUBFUNCTORS OF MACKEY FUNCTORS

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P.h.D. in Mathematics
Supervisor: Assoc. Prof. Dr. Laurence J. Barker
August, 2008

In this thesis we try to relate the subfunctor structure of a given Mackey functor M for a finite group G to the submodule structure of the $\mathbb{K}\overline{N}_G(H)$-module $M(H)$ where H is a subgroup of G.

We mainly study the socle and the radical of a Mackey functor M for a finite group G over a field $\mathbb{K}$, (usually, of characteristic $p > 0$). For a subgroup H of G, we construct bijections between some classes of the simple subfunctors of M and some classes of the simple $\mathbb{K}\overline{N}_G(H)$-submodules of $M(H)$. We relate the multiplicity of a simple Mackey functor $S^G_{H,V}$ in the socle of M to the multiplicity of V in the socle of a certain $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$. We also obtain similar results for the maximal subfunctors of M. We specialize our results to some specific kinds of Mackey functors for G that includes the functors obtained by inducing or restricting a simple Mackey functor, Mackey functors for a p-group, the fixed point functor, and the Burnside functor $\mathrm{B}^G_{\mathbb{K}}$.

Let M be the Mackey functor $\uparrow^G_K S^K_{H,W}$ for G obtained by inducing a simple Mackey functor $S^K_{H,W}$ for K. For example, we observe that the socle and the radical of M can be determined from the socle and the radical of the $\mathbb{K}\overline{N}_G(H)$-module $V = \uparrow^{\overline{N}_G(H)}_{\overline{N}_K(H)} W$. We also find similar results for Mackey functors obtained by restricting a simple Mackey functor. Moreover, we derive criterions for a Mackey functor obtained by inducing or restricting a simple Mackey functor to be simple, semisimple or indecomposable.

Our results about induced or restricted Mackey functors include Mackey functor versions of two classical and frequently used results in the representation theory finite groups, namely Clifford's theorem and Green's indecomposibility theorem.

We also apply our general results to Mackey functors satisfying some special conditions such as having a unique maximal or simple subfunctor, being uniserial, and being a functor for a p-group. We give some results about primordial and coprimordial subgroups of G for such kind of functors, and we refine our general results and obtain, for instance, a criterion for a Mackey functor to be a quotient of a projective Mackey functor, and find some information about composition series.

In later chapters the main Mackey functor to which we apply our general results is the Burnside functor $\mathrm{B}_{\mathbb{K}}^{G}$. We first find the maximal subfunctors of $\mathrm{B}_{\mathbb{K}}^{G}$ for any group G, and obtain some results about evaluations of the terms of the radical series of $\mathrm{B}_{\mathbb{K}}^{G}$. We also get some results about simple Mackey functors in radical layers of $\mathrm{B}_{\mathbb{K}}^{G}$ whose minimal subgroups are p-subgroup of G. Assuming that G is a p-group we find the first four top factors of the radical series of $\mathrm{B}_{\mathbb{K}}^{G}$, and assuming further that G is an abelian p-group we find the radical series of $\mathrm{B}_{\mathbb{K}}^{G}$ completely, which means that in this case we find the evaluations of the terms of the radical series, and the simple Mackey functors appearing in radical layers, and the Loewy length of $\mathrm{B}_{\mathbb{K}}^{G}$. We also study the socle series of $\mathrm{B}_{\mathbb{K}}^{G}$. This seems to be harder than the radical series. Nevertheless, we obtain similar results for the socle series of $\mathrm{B}_{\mathbb{K}}^{G}$ assuming mostly that G is an abelian p-group. To illustrate applications of our general results we also study briefly the radical and the socle series of fixed point functors FP_{V}^{G} where V is a one dimensional $\mathbb{K}G$-module.

We finish this thesis by trying to find possible relations between the socles and the radicals of the Mackey functors of the form T and $\mathfrak{F}T$ where T is a Mackey functor and $\mathfrak{F}$ is one of the functors restriction, inflation, evaluation, or adjoints of them, between Mackey functor categories.

Keywords: Mackey functor, Mackey algebra, simple, indecomposable, restriction, induction, evaluation, socle, radical, Clifford's theorem, Green's indecomposibility theorem, maximal subfunctor, Brauer quotient, minimal subfunctor, restriction kernel, primordial, coprimordial, uniserial, Burnside functor, socle series, radical series, composition series, composition factors, Loewy lenght, fixed point functor, functors for p-groups.

ÖZET

MACKEY FUNKTORLARIN GENİŞLETİLMESİ, KISITLANMASI, HESAPLANMASI, VE ALT FUNKTORLARI

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G bir sonlu grup ve H de G nin bir alt grubu olsun. Ayrıca M de bize verilmiş G nin bir Mackey funktoru olsun. Bu tezde, M nin alt funktorlarının yapısıyla M yi H de hesapladığımızda elde ettiğimiz $M(H)$ modülünün, ki $\mathbb{K}\overline{N}_G(H)$ cebirinin bir modülüdür, alt modüllerinin yapısını karşılaştırdık.

Genellikle G nin karakteri 0 dan büyük olan bir cisim üzerinde verilen bir Mackey funktoru M nin sokal ve radikal alt funktorlarına çalıştık. M nin bazı basit alt funktorlarının oluşturduğu sınıflarla $M(H)$ modülünün bazı basit alt modüllerinin oluşturduğu sınıflar arasında bire bir örten gönderimler kurduk. Ayrıca, verilen bir basit Mackey functoru $S_{H,V}^G$ nin M nin sokal alt funktorundaki tekerrür etme sayısını V nin $M(H)$ in bir alt modülünün sokal alt modülünde tekerrür etme sayısıyla ilişkilendirdik. Basit alt funktorlar için yaptığımız çalışmaların benzerlerini basit bölüm funktorları, bir başka deyişle en büyük alt funktorları, için de yaptık. Elde ettiğimiz genel Mackey funktorlar için olan sonuçları bazı özel şartları sağlayan Mackey funktorlarına uyguladık. Mesela, basit Mackey funktorların kısıtlanmasıyla yada genişletilmesiyle elde edilen funktorlara, değişmez eleman funktoruna, ve Burnside funktoruna uyguladık.

K, G nin bir alt grubu olsun. K nin bir basit Mackey funktoru olan $S_{H,W}^K$ den genişletmeyle elde edilen G nin Mackey funktoru $\uparrow_K^G S_{H,W}^K$ yi M ile gösterelim. Örneğin, bu durumda gösterdik ki, M nin sokal ve radikal alt funktorlarını V nin sokal ve radikal alt modüllerini kullanarak bulabiliriz. Benzer bir durumun bir basit Mackey funktorun kısıtlanmasıyla elde edilen Mackey funktorlar için de doğru olduğunu gösterdik. Bunlara ek olarak, bir basit Mackey funktorun genişletilmesiyle yada kısıtlanmasıyla elde edilen Mackey funktorların ne zaman basit, yarı basit, yada parçalanamaz olacağına eşdeğer olan kiriterler bulduk.

Bulduğumuz sonuçlar arasında sonlu grup temsilleri kuramında geçen ve sıkça kullanılan iki önemli klasik teoremin benzerlerinin Mackey funktorlar kuramında da doğru olduğu var. Bu bahsi geçen teoremler Clifford teoremi ve genişletmeyle elde edilen bir modülün ne zaman parçalanamayacağını söyleyen Green teoremidir.

Geliştirdiğimiz genel sonuçları uyguladığımız daha başka Mackey funktorlardan söz etmek gerekirse sadece bir tane en büyük yada en küçük alt funktora sahip olan funktorları, sadece bir tane kompozisyon serisine sahip olan funktorları ve p-grupların funktorlarını sayabiliriz. Örneğin, bu tipteki Mackey funktorların primordiyal alt gruplarıyla alakalı bazı sonuçlar elde ettik.

Tezin sonraki bölümlerinde ise genellikle Burnside funktoru $\mathrm{B}_{\mathbb{K}}^{G}$ ye çalıştık. Öncelikle onun en büyük alt funktorlarını bulduk ve radikal serisinin terimlerinin G nin bazı alt gruplarındaki değerlerini hesapladık. Ek olarak, $\mathrm{B}_{\mathbb{K}}^{G}$ funktorunun radikal katmanlarında bulunan basit Mackey funktorlar hakkında bazı sonuçlara ulaştık. G yi bir p-grup varsaydığımızda ise $\mathrm{B}_{\mathbb{K}}^{G}$ nin ilk dört radikal katmanını hesaplayabildik. G yi bir abelyen p-group varsaydığımızda ise $\mathrm{B}_{\mathbb{K}}^{G}$ funktorunun radikal serisi hakkında tam bir bilgi sahibi olduk. Yani G nin bu durumunda, $\mathrm{B}_{\mathbb{K}}^{G}$ nin radikal katmanlarındaki tüm basit funktorların ne olduklarını, $\mathrm{B}_{\mathbb{K}}^{G}$ nin radikal serisinin terimlerinin G nin alt gruplarındaki değerlerini ve de $\mathrm{B}_{\mathbb{K}}^{G}$ nin Loewy uzunluğunun ne olduğunu bulabildik. Benzer şekilde $\mathrm{B}_{\mathbb{K}}^{G}$ nin sokal serisi hakkında bir calışma yaptık. Fakat bu durumun radikal için yaptığımız çalışmadan daha zor olduğunu gözlemledik, ki bu $\mathrm{B}_{\mathbb{K}}^{G}$ nin kısıtlama çekirdeklerinin hesaplanmasının Brauer bölümlerinin hesaplanmasıyla kıyasladığımızda daha zor olmasından ötürüdür. En azından sokal serisi için G nin abelyen p-group olduğu durumlarda benzer bir çok sonuç çıkardık.

Anahtar sözcükler: Mackey funktoru, Mackey cebiri, basit, parçalanamaz, kısıtlama, genişletme, değer, sokal, radikal, Clifford teoremi, Green parçalanamama teoremi, en büyük alt funktor, Brauer bölümü, en küçük alt funktor, kısıtlama çekirdeği, primordiyal, yardımcı primordiyal, biricik seri funktorları, Burnside funktoru, sokal serisi, radikal serisi, compozisyon serisi, compozisyon faktörleri, Loewy uzunluğu, değişmez eleman funktoru, p grupların funktorları.

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Contents

Chapter 1

Introduction

Let $H \leq G$ be finite groups and $\mathbb{K}$ be a field. Many topics in the representation theory of finite groups deal with the repeated applications of the following three basic functors, namely induction, restriction, and conjugation:

(1) $\uparrow_H^G\colon \mathbb{K}H\text{-mod} \to \mathbb{K}G\text{-mod}, \quad W \mapsto \uparrow_H^G W := \mathbb{K}G \otimes_{\mathbb{K}H} W$

(2) $\downarrow_H^G\colon \mathbb{K}G\text{-mod} \to \mathbb{K}H\text{-mod}, \quad V \mapsto \downarrow_H^G V := \mathbb{K}G \otimes_{\mathbb{K}G} V$

(3) $|_H^g : \mathbb{K}H\text{-mod} \to \mathbb{K}({}^gH)\text{-mod}, \quad U \mapsto |_H^g U := U$ with ${}^gH$-action given by

$$g'u = (g^{-1}g'g)u.$$

Many classical results in the representation theory of finite groups depend only on the properties of the above three functors such as:

(a) (Mackey decomposition formula) If $H \leq G \geq K$ and W is a $\mathbb{K}H$-module then

$$\downarrow_K^G \uparrow_H^G W \cong \bigoplus_{KgH \subseteq G} \uparrow_{K\cap {}^gH}^K \downarrow_{K\cap {}^gH}^{{}^gH} |_H^g W.$$

(b) The pairs $(\uparrow_H^G, \downarrow_H^G)$ and $(\downarrow_H^G, \uparrow_H^G)$ are adjoint pairs.

(c) If $H \leq K \leq G$ then

$$\uparrow_K^G \uparrow_H^K W \cong \uparrow_H^G W.$$

The above properties are formalized in the notion of Mackey functors. At this point one may think of Mackey functors as assigning to each subgroup H of G a $\mathbb{K}$-space (or more generally an R-module where R is a commutative unital ring) $M(H)$ as well as three kinds of R-module homorphisms, a restriction homomorphism $M(H) \to M(K)$ and an induction homomorphism $M(K) \to M(H)$ for $K \leq H$, and a conjugation homorphism transporting the structure of $M(H)$ to $M({}^gH)$. These maps are required to satisfy some natural conditions such as (a)-(c) above. The axioms for a Mackey functor were first formulated by Green [Gr] (1971) and by Dress [Dr2] (1973). A basic example of a Mackey functor for G over $\mathbb{K}$, that motivates also the notion, is the representation ring which is the content of the next example.

Example 1.1 *Representation rings $G_0(\mathbb{K}G)$: the Grothendieck group of the category of finitely generated $\mathbb{K}G$-modules. In characteristic zero this may be identified as the group of characters of $\mathbb{K}G$-modules, and in characteristic p as the group of Brauer characters. More explicitly, for any subgroup H of G if we put*

$$M(H) := G_0(\mathbb{K}H) = \bigoplus_{V \in \mathrm{Irr}(\mathbb{K}H)} \mathbb{Z}[V],$$

then M becomes a Mackey functor for G over $\mathbb{Z}$ with the following maps:

$t_H^K : M(H) \to M(K), \quad [W] \mapsto [\uparrow_H^K W].$

$r_H^K : M(K) \to M(H), \quad [V] \mapsto [\downarrow_H^K V].$

$c_H^g : M(H) \to M({}^gH), \quad [U] \mapsto [{}^gU]$, *where ${}^gU = U$ with ${}^gH$-action given by* $g'u = (g^{-1}g'g)u$.

A Mackey functor is an algebraic structure possessing operations which behave like the induction, restriction and conjugation mappings in the previous example. It can be seen as a category-theoretic approach to various topics where there are notions of induction and restriction. Important examples of Mackey functors are representation rings, induction theory, G-algebras, Burnside rings, algebraic K-theory of group rings, algebraic number theory, group cohomology, equivariant topological K-theory, equivariant L-groups, Witt rings, stable equivariant (co)homology theories, see [We1] and the references in [We1]. It is their widespread occurrence which motivates the study of such operations in abstract. The power of the theory comes from the fact that the category of Mackey functors is fairly well-understood, for instance, the simple Mackey functors have been classified (Thévenaz-Webb [TW]).

We mention now some examples of Mackey functors.

Example 1.2 *Fixed point functors: Let V be an RG-module. For any subgroup H of G, let*

$$M(H) := V^H = \{v \in V : hv = v \ \forall h \in H\}.$$

Then, M is a Mackey functor for G over R with the following maps:

$$t_H^K : M(H) \to M(K), \quad x \mapsto \sum_{gH \subseteq K} gx,$$

$r_H^K : M(K) \to M(H)$ *is the inclusion, and*

$c_H^g : M(H) \to M({}^gH), \quad x \mapsto gx.$

Example 1.3 *Burnside rings: Let H be a subgroup of G. The set of isomorphism classes of finite H-sets form a commutative semiring under the operations disjoint union and cartesian product. The associated Grothendieck ring $B(H)$ is called the Burnside ring of H. Therefore, letting V runs over representatives of the conjugacy classes of subgroups of H, then $[H/V]$ comprise (without repetition) a $\mathbb{Z}$-basis of $B(H)$, where the notation $[H/V]$ denotes the isomorphism class of transitive H-sets whose stabilizers are H-conjugates of V. Thus,*

$$B(H) = \bigoplus_{V \leq_H H} \mathbb{Z}[H/V].$$

Then B becomes a Mackey functor for G over $\mathbb{Z}$ with the maps:

$$t_H^K([H/V]) = [K/V], \quad r_H^K([K/W]) = \sum_{HgW \subseteq K} [H/H \cap {}^gW],$$

$$c_H^g([H/U]) = [{}^gH/{}^gU].$$

Example 1.4 *The commutator functor: Let G be a finite group. For any subgroup H of G we put $M(H) := H/H'$ where H' is the commutator subgroup of H. Then M becomes a Mackey functor for G over $\mathbb{Z}$ with the following maps:*

$t_H^K : H/H' \to K/K'$, $hH' \mapsto hK'$,

$r_H^K : K/K' \to H/H'$, *the map induced by the group theoretical transfer map* $K \to H/H'$.

Example 1.5 *Some other examples:*

$A(G)$: *the Green ring of finitely generated $\mathbb{K}G$-modules.*

$H^n(G;U), H_n(G;U)$: *the cohomology and homology of G in some dimension n with coefficients in the $\mathbb{Z}G$-module U.*

$K_n(\mathbb{Z}G)$: *the algebraic K-theory of $\mathbb{Z}G$, and other related groups such as the Whitehead group.*

$Cl(O(\mathbb{K}G))$: *the class group of the ring of integers of the fixed field $\mathbb{K}^G$ where G is a group of automorphisms of a number field $\mathbb{K}$.*

We call this structure, as in the examples above, a (Mackey) *functor*, because it may be considered as a functor between two categories. Indeed, Dress [Dr2] defined the notion as a bifunctor consisting of a covariant and a contravariant functor from the category of finite G-sets to an abelian category. Another way to see it as a functor between two categories is an instance of a more general observation that any (left) module of a (finite) dimensional R-algebra can be viewed as an R-linear (covariant) functor from a (small) R-linear category to the category of R-modules. The converse is also true that an R-linear (covariant) functor from a (small) R-linear category to the category of R-modules may be viewed as a module of an algebra, called the category algebra, see Webb [We3]. It became apparent after Thévenaz-Webb [TW95] that Mackey functors are algebraic structures in their own right with a theory which fits into the framework of representations of algebras. They may, in fact, be identified with the representations of a certain finite dimensional R-algebra $\mu_R(G)$, called the Mackey algebra, so that a Mackey functor for G is indeed a $\mu_R(G)$-module (and vice versa), and there are simple Mackey functors, projective and injective Mackey functors, resolutions of Mackey functors, and so on. In particular, one may use the Mackey algebra to see a Mackey functor as a functor between two categories. To explain it roughly, let $S(G)$ be the category whose objects are the subgroups of G and for any subgroups H and K the morphisms from H to K are R-linear combinations of symbols $t^K_{gJ}c^g_J r^H_J$ (which are elements of an R-basis of the Mackey algebra $\mu_R(G)$, see 2.1) where $g \in G$ and $J \leq K^g \cap H$, and where t, r, and c satisfy some natural relations as in the examples. Then a Mackey functor M is an R-linear (covariant) functor $M : S(G) \to R$-mod.

In this thesis we mainly study subfunctors and quotient functors of a Mackey functor M and relate them to those of the $\mathbb{K}\overline{N}_G(H)$-module $M(H)$ where H is a subgroup of G. We apply our results to some specific Mackey functors such as Mackey functors obtained by restricting or inducing a simple Mackey functor and the Burnside functor $\mathrm{B}^G_{\mathbb{K}}$. For instance, we obtain in some cases results about the Loewy series and the Loewy layers of $\mathrm{B}^G_{\mathbb{K}}$ and obtain some results about socles and radicals of some specific Mackey functors including the ones obtained by restricting or inducing a simple Mackey functor.

We now want to explain our notations. Let H and K be subgroups of G. By the notation $HgK \subseteq G$ we mean that g ranges over a complete set of representatives of double cosets of (H, K) in G. We write $\overline{N}_G(H)$ for the quotient group $N_G(H)/H$ where $N_G(H)$ is the normalizer of H in G, and write $|G : H|$ for the index of H in G. For a module V of an algebra we denote by $\mathrm{Soc}(V)$ and $\mathrm{Jac}(V)$ the socle and the radical of V, respectively. Most of our other notations are standard and tend to follow [TW, TW95].

Let us finish this chapter by mentioning some (not all) of our main results. Let G be a finite group and let $\mathbb{K}$ be a field.

Theorem A. *Let $H \leq K \leq G$ and let W be a simple $\mathbb{K}\overline{N}_K(H)$-module. Let*

$$M = \uparrow_K^G S_{H,W}^K \quad \text{and} \quad V = \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W.$$

(1) *There is a bijective correspondence (preserving multiplicities in respective socles) between the simple $\mu_{\mathbb{K}}(G)$-submodules of M and the simple $\mathbb{K}\overline{N}_G(H)$-submodules of V.*

(2) *There is a bijective correspondence (preserving multiplicities in respective heads) between the maximal $\mu_{\mathbb{K}}(G)$-submodules of M and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules of V.*

(3) *M is a simple (respectively, semisimple, or indecomposable) $\mu_{\mathbb{K}}(G)$-module if and only if V is a simple (respectively, semisimple, or indecomposable) $\mathbb{K}\overline{N}_G(H)$-module.*

Theorem B. *Let $H \subseteq K$ be subgroups of G and let V be a simple $\mathbb{K}\overline{N}_G(H)$-module. Let*

$$T = \downarrow_K^G S_{H,V}^G \quad \text{and} \quad W = \downarrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} V.$$

(1) *The socle and the radical of T can be determined from the socles and the radicals of the $\mathbb{K}\overline{N}_K({}^gH)$-modules ${}^gV$ where g ranges over all elements of G with ${}^gH \leq K$.*

(2) *The $\mu_{\mathbb{K}}(K)$-module T is semisimple if and only if the $\mathbb{K}\overline{N}_K({}^gH)$-modules ${}^gV$ are all semisimple for any element of G with ${}^gH \leq K$.*

(3) *The $\mu_{\mathbb{K}}(K)$-module T is simple (respectively, indecomposable) if and only if any element of the set $\{{}^gH : {}^gH \leq K, \ g \in G\}$ is a K-conjugate of H and the $\mathbb{K}\overline{N}_K(H)$-module W is simple (respectively, indecomposable).*

Theorem C. *There is a "Clifford's theorem for Mackey functors" and there is a "Green's indecomposibility theorem for Mackey functors."*

Theorem D. *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module. Then:*

(1) *The multiplicity of $S^G_{H,U}$ in $\mathrm{Soc}(M)$ is equal to the multiplicity of U in the socle of the following $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$:*

$$\bigcap_{X/H} \{x \in \underline{M}(H) : \big(\sum_{gH \subseteq X} c^g_H\big)x = 0 \implies t^X_H(x) = 0\}$$

where X/H ranges over all nontrivial p-subgroups of $N_G(H)/H$.

(2) *There is a simple subfunctor of M having H as a minimal subgroup if and only if there is a simple $\mathbb{K}\overline{N}_G(H)$-submodule T of $\underline{M}(H)$ satisfying the following condition for any nontrivial p-subgroup X/H of $N_G(H)/H$:*

$$x \in T, \quad \big(\sum_{gH \subseteq X} c^g_H\big)x = 0 \ \text{ implies } \ t^X_H(x) = 0.$$

(3) *The multiplicity of $S^G_{H,U}$ in $\mathrm{Soc}(M)$ is less than or equal to the multiplicity of U in $\mathrm{Soc}\big(\underline{M}(H)\big)$.*

(4) *The multiplicity of $S^G_{H,U}$ in $\mathrm{Soc}(M)$ is greater than or equal to the multiplicity of U in the socle of the following $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$:*

$$\bigcap_{H < X \leq N_G(H) : |X:H| = p} \mathrm{Ker}\big(t^X_H : \underline{M}(H) \to M(X)\big).$$

(5) *Suppose that $\overline{N}_G(H)$ is a p'-group. Then, the multiplicity of $S^G_{H,U}$ in $\mathrm{Soc}(M)$ is equal to the multiplicity of U in $\underline{M}(H)$.*

Theorem E. *Let M be a $\mu_{\mathbb{K}}(G)$-module, H be a subgroup of G, and V be a simple $\mathbb{K}\overline{N}_G(H)$-module. Put $A = \mu_{\mathbb{K}}(G)$.*

(1) *Suppose that H is maximal subject to the condition $\underline{M}(H) \neq 0$. Then, the multiplicity of $S^G_{H,V}$ in $\mathrm{Soc}(M)$ is equal to the multiplicity of V in $\mathrm{Soc}(\underline{M}(H))$.*

(2) *The multiplicity of V as a composition factor of $\underline{M}(H)$ is equal to the multiplicity of $S^G_{H,V}$ as a composition factor of $A\underline{M}(H)$.*

(3) *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group. Let $H \le K$ be subgroups of G with $|K : H| = p^n$. If $t^K_H(M(H)) \neq 0$ or $r^K_H(M(K)) \neq 0$, then the Loewy length of M is greater than or equal to $n + 1$.*

Theorem F. *Let $\mathbb{K}$ be of characteristic $p > 0$ and $M = \mathrm{B}^G_{\mathbb{K}}$. Let H be a p-subgroup of G, and V be a simple $\mathbb{K}\overline{N}_G(H)$-module, and let n be a natural number with $p^n \le |G|_p$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$. Then:*

(1) *If $S^G_{H,V}$ appears in J_n/J_{n+1} then $|G : H|_p \le p^n$ and $|G : H|_p \neq p^{n-1}$.*

(2) *If $|G : H|_p = p^n$ and $S^G_{H,V}$ appears in J_n/J_{n+1} then $V = \mathbb{K}$.*

(3) *If $|G : H|_p = p^n$ then the multiplicity of $S^G_{H,\mathbb{K}}$ in J_n/J_{n+1} is 1.*

(4) *The multiplicity of $S^G_{1,\mathbb{K}}$ in M is 1, and it appears in J_m/J_{m+1} where $p^m = |G|_p$.*

(5) *The Loewy length of M is greater than or equal to $m + 1$.*

Theorem G. *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| = p^n$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then:*

(1) *For any natural number k with $k \le n$ we have:*

$$J_k/J_{k+1} \cong \bigoplus_{l=0}^{\lfloor k/2 \rfloor} \Big(\bigoplus_{H \le G : |G:H| = p^{k-2l}} \lambda^l_H S^G_{H,\mathbb{K}} \Big)$$

where λ^l_H is the number of elements of the set $\{V \le H : |H : V| = p^l\}$.

(2) *For any natural number k with $k \ge n + 1$ we have:*

$$J_k/J_{k+1} \cong \bigoplus_{l=k-n}^{\lfloor k/2 \rfloor} \Big(\bigoplus_{H \le G : |G:H| = p^{k-2l}} \lambda^l_H S^G_{H,\mathbb{K}} \Big)$$

where λ^l_H is the number of elements of the set $\{V \le H : |H : V| = p^l\}$.

(3) *The Loewy length of M is $2n+1$.*

Theorem H. *Let $\mathbb{K}$ be of characteristic $p>0$ and V be a one dimensional $\mathbb{K}G$-module. For any natural number k we put $J_k = \operatorname{Jac}^k(M)$ and $S_k = \operatorname{Soc}^k(M)$ where $M = FP_V^G$. Let n be the natural number satisfying $p^n = |G|_p$. Then:*

(1)

$$J_k/J_{k+1} \cong \bigoplus_{H \leq_G G : |H| = p^{n-k}} S_{H,V}^G.$$

(2)

$$S_{k+1}/S_k \cong \bigoplus_{H \leq_G G : |H| = p^k} S_{H,V}^G.$$

(3) *The Loewy length of M is $n+1$.*

(4) *Let X be a p-subgroup of G. Then, $J_k(X) = 0$ if and only if $|X| \geq p^{n+1-k}$.*

(5) *Let X be a p-subgroup of G. Then, $S_k(X) = 0$ if and only if $|X| \geq p^k$.*

(6) *If G is a p-group then the socle and the radical series of M coincide.*

Throughout this thesis, G is a finite group, $\mathbb{K}$ is an arbitrary field. We consider only finite dimensional Mackey functors.

Next page collects some of our frequently used notations.

Notations

G	:	a finite group
$\|G:H\|$	:	index of the subgroup H in G
$\overline{N}_G(H)$	:	$N_G(H)/H$ where $N_G(H)$ is the normalizer of the subgroup H in G
$HgK \subseteq G$	:	means that g ranges over a complete set of representatives of double cosets of the pair (H,K) of subgroups of G in G
$\mathbb{K}$	:	a field
R	:	a commutative unital ring
$\mu_R(G)$	:	the Mackey algebra of G over the coefficient ring R
$S^G_{H,V}$	:	simple Mackey functor
$P^G_{H,V}$	:	projective cover of $S^G_{H,V}$
$\mathrm{B}^G_{\mathbb{K}}$	:	Burnside functor for G over $\mathbb{K}$
FP^G_V	:	fixed point functor
$\uparrow^G_H$	:	induction of Mackey functors, modules
$\downarrow^G_H$	:	restriction of Mackey functors, modules
$\|^g_H M$	:	conjugation of Mackey functor M, conjugation of module M
${}^gM$	:	conjugation of Mackey functor M, conjugation of module M
$\mathrm{Soc}(M)$	:	socle of Mackey functor M, socle of module M
$\mathrm{Jac}(M)$	:	(Jacobson) radical of Mackey functor M, radical of module M
$\underline{M}(H)$	:	$\bigcap_{J<H} \mathrm{Ker}(r^H_J : M(H) \to M(J))$, called the restriction kernel, where M is a Mackey functor for G and H is a subgroup
$\overline{M}(H)$	:	$M(H)/\sum_{J<H} t^H_J(M(J))$, called the Brauer quotient, where M is a Mackey functor for G and H is a subgroup
n_p	:	p-part of the natural number n
$\lfloor r \rfloor$	:	the largest integer which is less than or equal to the real number r
$\mathrm{Inf}^G_{G/N}$	:	inflation of Mackey functors, modules, from the quotient group G/N to G where N is a normal subgroup of G
${L^+}_{G/N}$	:	left adjoint of the inflation functor from the quotient group G/N to G where N is a normal subgroup of G
${L^-}_{G/N}$	:	right adjoint of the inflation functor from the quotient group G/N to G where N is a normal subgroup of G

Chapter 2

Preliminaries

In this chapter, we briefly summarize some crucial material on Mackey functors. For the proofs, see Thévenaz–Webb [TW, TW95]. Let χ be a family of subgroups of G, closed under taking subgroups and taking G-conjugation. Recall that a Mackey functor for χ over a commutative unital ring R is such that, for each subgroup H of G in χ, there is an R-module $M(H)$; for each pair $H, K \in \chi$ with $H \leq K$, there are R-module homomorphisms $r_H^K : M(K) \to M(H)$ called the restriction map and $t_H^K : M(H) \to M(K)$ called the transfer map or the trace map; for each $g \in G$, there is an R-module homomorphism $c_H^g : M(H) \to M({}^gH)$ called the conjugation map. The following axioms must be satisfied for any $g, h \in G$ and $H, K, L \in \chi$ [Bo, Gr, TW, TW95].

(M$_1$) If $H \leq K \leq L$, then

$$r_H^L = r_H^K r_K^L \quad \text{and} \quad t_H^L = t_K^L t_H^K.$$

Moreover, $r_H^H = t_H^H = id_{M(H)}$.

(M$_2$) $c_K^{gh} = c_{{}^hK}^g c_K^h$.

(M$_3$) If $h \in H$, then $c_H^h : M(H) \to M(H)$ is the identity map.

(M_4) If $H \leq K$, then

$$c_H^g r_H^K = r_{{}^gH}^{{}^gK} c_K^g \quad \text{and} \quad c_K^g t_H^K = t_{{}^gH}^{{}^gK} c_H^g.$$

(M_5) (Mackey Axiom) If $H \leq L \geq K$, then

$$r_H^L t_K^L = \sum_{HgK \subseteq L} t_{H \cap {}^gK}^H r_{H \cap {}^gK}^{{}^gK} c_K^g.$$

When χ is the family of all subgroups of G, we say that M is a Mackey functor for G over R. A homomorphism $f : M \to T$ of Mackey functors for χ is a family of R-module homomorphisms $f_H : M(H) \to T(H)$, where H runs over χ, which commutes with restriction, trace and conjugation. In particular, each $M(H)$ is an $R\overline{N}_G(H)$-module via $\overline{g}.x = c_H^g(x)$ for $\overline{g} \in \overline{N}_G(H)$ and $x \in M(H)$. Also, each f_H is an $R\overline{N}_G(H)$-module homomorphism. By a subfunctor N of a Mackey functor M for χ we mean a family of R-submodules $N(H) \subseteq M(H)$, which is stable under restriction, trace, and conjugation. A Mackey functor M is called simple if it has no proper subfunctor.

Another possible definition of Mackey functors for G over R uses the Mackey algebra $\mu_R(G)$ [Bo, TW95]: $\mu_{\mathbb{Z}}(G)$ is the algebra generated by the elements r_H^K, t_H^K, and c_H^g, where H and K are subgroups of G such that $H \leq K$, and $g \in G$, with the relations (M_1)-(M_7).

(M_6) $\sum_{H \leq G} t_H^H = \sum_{H \leq G} r_H^H = 1_{\mu_{\mathbb{Z}}(G)}$.

(M_7) Any other product of r_H^K, t_H^K and c_H^g is zero.

A Mackey functor M for G, defined in the first sense, gives a left module $\widetilde{M}$ of the associative R-algebra $\mu_R(G) = R \otimes_{\mathbb{Z}} \mu_{\mathbb{Z}}(G)$ defined by $\widetilde{M} = \bigoplus_{H \leq G} M(H)$. Conversely, if $\widetilde{M}$ is a $\mu_R(G)$-module then $\widetilde{M}$ corresponds to a Mackey functor M in the first sense, defined by $M(H) = t_H^H \widetilde{M}$, the maps t_H^K, r_H^K, and c_H^g being defined as the corresponding elements of the $\mu_R(G)$. Moreover, homomorphisms and subfunctors of Mackey functors for G are $\mu_R(G)$-module homorphisms and $\mu_R(G)$-submodules, and conversely.

Theorem 2.1 [TW95] *Letting H and Krun over all subgroups of G, letting g run over representatives of the double cosets $HgK \subseteq G$, and letting J runs over representatives of the conjugacy classes of subgroups of $H^g \cap K$, then $t^H_{gJ} c^g_J r^K_J$ comprise, without repetition, a free R-basis of $\mu_R(G)$.*

For a Mackey functor M for χ over R and a subset E of M, a collection of subsets $E(H) \subseteq M(H)$ for each $H \in \chi$, we denote by $< E >$ the subfunctor of M generated by E.

Proposition 2.2 [TW] *Let M be a Mackey functor for G, and χ be a family of subgroups of G closed under taking subgroups and taking G-conjugation, and let T be a subfunctor of $\downarrow_\chi M$, the restriction of M to χ which is the family $M(H), H \in \chi$, viewed as a Mackey functor for χ. Then*

$$< T > (K) = \sum_{X \in \chi : X \leq K} t^K_X(M(X))$$

for any $K \leq G$. Moreover $\downarrow_\chi < T >= T$.

Let M be a Mackey functor for G and χ be a family of subgroups of G closed under taking subgroups and taking G-conjugation. Then by [TW] we have the following important subfunctors of M, namely $\mathrm{Im}t^M_\chi$ and $\mathrm{Ker}r^M_\chi$ defined by

$$(\mathrm{Im}t^M_\chi)(K) = \sum_{X \in \chi : X \leq K} t^K_X(M(X)),$$

$$(\mathrm{Ker}r^M_\chi)(K) = \bigcap_{X \in \chi : X \leq K} \mathrm{Ker}(r^K_X : M(K) \to M(X)).$$

Let M be a Mackey functor for G over R. A subgroup H of G is called a minimal subgroup of M if $M(H) \neq 0$ and $M(K) = 0$ for every subgroup K of H with $K \neq H$. Given a simple Mackey functor M for G over R, there is a unique, up to G-conjugacy, a minimal subgroup H of M. Moreover, for such an H the $R\overline{N}_G(H)$-module $M(H)$ is simple, where the $R\overline{N}_G(H)$-module structure on $M(H)$ is given by $gH.x = c^g_H(x)$, see [TW].

Proposition 2.3 [TW] *Let S be a simple Mackey functor for G with a minimal subgroup H.*

(1) *S is generated by $S(H)$, that is $S =< S(H) >$.*

(2) *$S(K) \neq 0$ implies that $H \leq_G K$, and so minimal subgroups of S form a unique conjugacy class.*

(3) *$S(H)$ is a simple $R\overline{N}_G(H)$-module.*

Proposition 2.4 [TW] *Let M be a Mackey functor for G over R, and let H be a minimal subgroup of M and $\chi_H = \{X \leq G : X \leq_G H\}$. Then, M is simple if and only if $\mathrm{Imt}^M_{\chi_H} = M$, $\mathrm{Kerr}^M_{\chi_H} = 0$, and $S(H)$ is a simple $R\overline{N}_G(H)$-module.*

Theorem 2.5 [TW] *Given a subgroup $H \leq G$ and a simple $R\overline{N}_G(H)$-module V, then there exists a simple Mackey functor $S^G_{H,V}$ for G, unique up to isomorphism, such that H is a minimal subgroup of $S^G_{H,V}$ and $S^G_{H,V}(H) \cong V$. Moreover, up to isomorphism, every simple Mackey functor for G has the form $S^G_{H,V}$ for some $H \leq G$ and simple $R\overline{N}_G(H)$-module V. Two simple Mackey functors $S^G_{H,V}$ and $S^G_{H',V'}$ are isomorphic if and only if, for some element $g \in G$, we have $H' = {}^gH$ and $V' \cong c^g_H(V)$.*

We now recall the definitions of restriction, induction and conjugation for Mackey functors [Bo, Sa, TW, TW95]. Let M and T be Mackey functors for G and H, respectively, where $H \leq G$.

The restricted Mackey functor $\downarrow^G_H M$ is the $\mu_R(H)$-module $1_{\mu_R(H)}M$ so that

$$(\downarrow^G_H M)(X) = M(X)$$

for $X \leq H$, where $1_{\mu_R(H)}$ denotes the unity of $\mu_R(H)$.

For $g \in G$, the conjugate Mackey functor $|^g_H T = {}^gT$ is the $\mu_R({}^gH)$-module T with the module structure given for any $x \in \mu_R({}^gH)$ and $t \in T$ by

$$x.t = (\gamma_{g^{-1}} x \gamma_g)t,$$

where γ_g is the sum of all c_X^g with X ranging over subgroups of G. Therefore, $(|_H^g T)({}^gX) = T(X)$ for all $X \leq H$ and the maps $\tilde{t}, \tilde{r}, \tilde{c}$ of $|_H^g T$ satisfy

$$\tilde{t}_B^A = t_{B^g}^{A^g}, \quad \tilde{r}_B^A = r_{B^g}^{A^g}, \quad \text{and} \quad \tilde{c}_A^x = c_{A^g}^{x^g}$$

where t, r, c are the maps of T. Obviously, one has $|_L^g S_{H,V}^L \cong S_{{}^gH, c_H^g(V)}^{{}^gL}$.

The induced Mackey functor $\uparrow_H^G T$ is the $\mu_R(G)$-module

$$\mu_R(G)1_{\mu_R(H)} \otimes_{\mu_R(H)} T,$$

where $1_{\mu_R(H)}$ denotes the unity of $\mu_R(H)$. It may be useful to express the $\mu_R(G)$-module $\uparrow_H^G T$ as a Mackey functor in the first sense which is the context of the next result. By the axioms (M_1)-(M_7) defining the Mackey algebra, it can be seen easily that for any $K \leq G$ we have:

$$t_K^K \mu_R(G)1_{\mu_R(H)} = \bigoplus_{KgH \subseteq G} c_{K^g}^g t_{H\cap K^g}^{K^g} \mu_R(H).$$

Therefore

$$(\uparrow_H^G T)(K) = t_K^K\big(\mu_R(G)1_{\mu_R(H)} \otimes_{\mu_R(H)} T\big) = \bigoplus_{KgH \subseteq G} c_{K^g}^g t_{H\cap K^g}^{K^g} \otimes_{\mu_R(H)} t_{H\cap K^g}^{H\cap K^g} T.$$

The following result is clear now.

Proposition 2.6 [Sa, TW] *Let H be a subgroup of G and T be a Mackey functor for H. Then for any subgroup K of G*

$$(\uparrow_H^G T)(K) \cong \bigoplus_{KgH \subseteq G} T(H \cap K^g)$$

as R-modules. In particular, if $T(X) \neq 0$ for some subgroup X of H then

$$(\uparrow_H^G T)(X) \neq 0.$$

The induced Mackey functor $\uparrow_H^G T$ can also be defined by giving its values on subgroups K of G as the R-modules in the right hand side of the isomorphism in 2.6, and by giving its maps t, r, c in terms of the maps of T. See [Sa, TW].

Indeed, let $H \leq G$ and let M be a Mackey functor for H. Then for any $K \leq G$ the induced Mackey functor $\uparrow_H^G M$ for G is given by

$$(\uparrow_H^G M)(K) = \bigoplus_{KgH \subseteq G} M(H \cap K^g)$$

where, if we write m_g for the component in $M(H \cap K^g)$ of $m \in (\uparrow_H^G M)(K)$, the maps $\tilde{t}, \tilde{r}, \tilde{c}$ of $\uparrow_H^G M$ are given as follows:

$$\tilde{r}_L^K(m)_g = r_{H\cap L^g}^{H\cap K^g}(m_g),$$

$$\tilde{t}_L^K(n)_g = \sum_{Lu(K\cap {}^gH)\subseteq K} t_{H\cap L^{ug}}^{H\cap K^{ug}}(n_{ug}),$$

$$\tilde{c}_K^y(m)_g = m_{y^{-1}g}$$

for $L \leq K$, $n \in (\uparrow_H^G M)(L)$ and $y \in G$.

We next record the Mackey decomposition formula for Mackey functors, which can be found (for example) in [TW95].

Theorem 2.7 *Given $H \leq L \geq K$ and a Mackey functor M for K over R, we have*

$$\downarrow_H^L \uparrow_K^L M \cong \bigoplus_{HgK \subseteq L} \uparrow_{H\cap {}^gK}^H \downarrow_{H\cap {}^gK}^{{}^gK} |_K^g M.$$

Theorem 2.8 [Sa] *Let H be a subgroup of G. Then $\uparrow_H^G$ is both left and right adjoint of $\downarrow_H^G$.*

We finally recall some facts from [TW] about inflated Mackey functors. Let N be a normal subgroup of G. Given a Mackey functor $\widetilde{M}$ for G/N, we define a Mackey functor $M = \mathrm{Inf}_{G/N}^G \widetilde{M}$ for G, called the inflation of $\widetilde{M}$, as

$$M(K) = \widetilde{M}(K/N) \text{ if } K \geq N, \qquad \text{and} \qquad M(K) = 0 \text{ otherwise.}$$

The maps t_H^K, r_H^K, c_H^g of M are zero unless $N \leq H \leq K$ in which case they are the maps

$$\tilde{t}_{H/N}^{K/N}, \quad \tilde{r}_{H/N}^{K/N}, \quad \tilde{c}_{H/N}^{gN}$$

of $\widetilde{M}$. Evidently, one has $\mathrm{Inf}_{G/N}^{G} S_{H/N,V}^{G/N} \cong S_{H,V}^{G}$.

Given a Mackey functor M for G we define Mackey functors

$$L^{+}{}_{G/N}M \qquad \text{and} \qquad L^{-}{}_{G/N}M$$

for G/N as follows:

$$(L^{+}{}_{G/N}M)(K/N) = M(K) \Big/ \sum_{J \leq K : J \not\geq N} t_J^K(M(J))$$

$$(L^{-}{}_{G/N}M)(K/N) = \bigcap_{J \leq K : J \not\geq N} \mathrm{Ker} r_J^K.$$

The maps on these two new functors come from those on M. They are well defined because the maps on M preserve the sum of images of traces and the intersection of kernels of restrictions, see [TW].

Theorem 2.9 [TW] *For any normal subgroup N of G, $L^{+}{}_{G/N}$ is a left adjoint of $\mathrm{Inf}_{G/N}^{G}$ and $L^{-}{}_{G/N}$ is a right adjoint of $\mathrm{Inf}_{G/N}^{G}$.*

Theorem 2.10 [TW] *For any simple $\mu_{\mathbb{K}}(G)$-module $S_{H,V}^{G}$, we have*

$$S_{H,V}^{G} \cong \uparrow_{N_G(H)}^{G} \mathrm{Inf}_{N_G(H)/H}^{N_G(H)} S_{1,V}^{\overline{N}_G(H)} \cong \uparrow_{N_G(H)}^{G} S_{H,V}^{N_G(H)}.$$

Chapter 3

Our approach

In this chapter we explain our main methods that we will apply to Mackey functors in this work.

There are several equivalent definitions of Mackey functors two of them we explained in Chapter 2. We mainly view Mackey functors as modules of Mackey algebras.

Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. We will usually compare the properties of the $\mu_{\mathbb{K}}(G)$-module M with the properties of the $\mathbb{K}\overline{N}_G(H)$-module $M(H)$. As t_H^H is an idempotent of $\mu_{\mathbb{K}}(G)$ and as $M(H) = t_H^H M$, the evaluation $M(H)$ of M at H has a natural $t_H^H \mu_{\mathbb{K}}(G) t_H^H$-module structure. However, the structure of the algebra $t_H^H \mu_{\mathbb{K}}(G) t_H^H$ is usually not easier than the structure of the Mackey algebra $\mu_{\mathbb{K}}(G)$. Moreover, one may see that the algebra $t_H^H \mu_{\mathbb{K}}(G) t_H^H$ decomposes as

$$t_H^H \mu_{\mathbb{K}}(G) t_H^H = A_H \oplus I_H$$

where A_H is a subalgebra of $t_H^H \mu_{\mathbb{K}}(G) t_H^H$ isomorphic to $\mathbb{K}\overline{N}_G(H)$ and I_H is a two sided ideal of $t_H^H \mu_{\mathbb{K}}(G) t_H^H$. Therefore, it may be fruitful to compare the properties of the $\mu_{\mathbb{K}}(G)$-module with the properties of the $\mathbb{K}\overline{N}_G(H)$-module $M(H)$. Most of our results comes from this approach.

Let us recall some general facts related to above paragraph. Let A be a finite dimensional algebra and e be a nonzero idempotent of A. We collect in the following result some general facts about module categories of the algebra A and its corner algebra eAe. We have the following functors some of whose properties are recalled in the next result:

$$R_e : \mathrm{Mod}(A) \to \mathrm{Mod}(eAe) \qquad \text{and} \qquad C_e, I_e : \mathrm{Mod}(eAe) \to \mathrm{Mod}(A)$$

given on the objects by

$$R_e(V) = eV, \quad C_e(W) = \mathrm{Hom}_{eAe}(eA, W) \;\; \text{and} \;\; I_e(W) = Ae \otimes_{eAe} W.$$

The definitions on morphisms of these functors are obvious (and well-known).

Theorem 3.1 *Let A be a finite dimensional algebra over a field and e be an idempotent of A. Then:*

(1) *I_e and C_e are full and faithful linear functors such that both of the functors R_eI_e and R_eC_e are naturally isomorphic to the identity functor.*

(2) *(I_e, R_e) and (R_e, C_e) are adjoint pairs.*

(3) *Both of I_e and C_e send indecomposable modules to indecomposable modules.*

(4) *Any simple eAe-module is of the form eS for some simple A-module S, and conversely for any simple A-module S the eAe-module eS is either zero or simple.*

(5) *Given simple A-modules S and S' that are not annihilated by e, one has $S \cong S'$ as A-modules if and only if $eS \cong eS'$ as eAe-modules.*

(6) *Given a simple eAe-module T, the A-module $I_e(T)$ has a unique maximal A-submodule J_T and one has $R_e(I_e(T)/J_T) \cong T$ and J_T is the sum of all A-submodules of $I_e(T)$ annihilated by e.*

The above fact is well-known, and can be found in [Gr2, pp. 83-87].

We usually apply the above theorem to the Mackey algebra $A = \mu_{\mathbb{K}}(G)$ by choosing an idempotent

$$e = \sum_{X \in \chi} t_X^X$$

of A where χ is a set of subgroups of G. This method is used for instance in [Yar1] and [Yar4]. For instance, if χ is the set of all subgroups of a normal subgroup N of G, then $e\mu_{\mathbb{K}}(G)e$ is a crossed product of G/N over $\mu_{\mathbb{K}}(N)$ so that one may use the theory of group graded algebras to study Mackey functors, see [Yar1]. For another example, if χ is the set of all subgroups of G containing a normal subgroup N of G, then $e\mu_{\mathbb{K}}(G)e$ decomposes as $e\mu_{\mathbb{K}}(G)e = B \oplus I$ where B is a subalgebra of $e\mu_{\mathbb{K}}(G)e$ isomorphic to $\mu_{\mathbb{K}}(G/N)$and I is a two sided ideal of $e\mu_{\mathbb{K}}(G)e$ so that one may derive some results about inflations of Mackey functors by using the above theorem, see [Yar4].

To illustrate the usefulness of studying functors by viewing them as a module of the category algebra and by using the idempotents of the category algebra, we want to mention what comes next. Let R be a commutative unital ring, $\mathfrak{A}$ is an (small) R-linear category, and $\mathfrak{F}$ be the category of R-linear (covariant) functors from $\mathfrak{A}$ to the category of left R-modules. The following result (see, for instance, [Yar3, Proposition 3.5]) is proved in some slightly special contexts assuming $\mathfrak{A}$ to be some specific category satisfying some conditions by using the methods and constructions of the each context:

Fact: *Let $M \in \mathfrak{F}$ be a functor and X be an object of $\mathfrak{A}$ such that $M(X)$ is nonzero. Then, M is simple if and only if* $\mathrm{Im}^M_{X,M(X)} = M$, $\mathrm{Ker}^M_{X,0} = 0$, *and $M(X)$ is a simple* $\mathrm{End}_{\mathfrak{A}}(X)$*-module. Here,*

$$\mathrm{Im}^M_{X,W}(Y) = \sum_{f \in \mathrm{Hom}_{\mathfrak{A}}(X,Y)} M(f)(W) \quad \textit{and} \quad \mathrm{Ker}^M_{X,0}(Y) = \bigcap_{f \in \mathrm{Hom}_{\mathfrak{A}}(Y,X)} \mathrm{Ker}(M(f)).$$

One may view M, by identifying it with

$$\bigoplus_{X \in \mathfrak{A}} M(X),$$

as a left module of the algebra

$$A_{\mathfrak{A}} = \bigoplus_{Y,Z} \mathrm{Hom}_{\mathfrak{A}}(Y,Z)$$

where Y, Z ranges over the objects of the category $\mathfrak{A}$, and where the multiplication in the algebra is induced from the composition of morphisms in $\mathfrak{A}$. See [We3] for more details. Note that the identity morphisms 1_X of $\mathrm{End}_{\mathfrak{A}}(X)$ is an idempotent of $A_{\mathfrak{A}}$ such that

$$M(X) = 1_X M \quad \text{and} \quad 1_X A_{\mathfrak{A}} 1_X = \mathrm{End}_{\mathfrak{A}}(X).$$

Letting $A = A_{\mathfrak{A}}$, $e = 1_X$, and $V = M$, Fact becomes:

Fact′: *Let V be an A-module and e be an idempotent of A such that eV is nonzero. Then V is simple if and only if $AeV = V$, $eAv = 0 \implies v = 0$, and eV is a simple eAe-module.*
Fact′ is almost trivial by using Theorem 3.1. Therefore, Fact can readily be obtained from 3.1.

In this work we usually obtain some results connecting modules of an algebra A and its corners eAe, and we translate these results to Mackey functors and try to refine them by using the extra structures in the context of Mackey functors.

We end this chapter with explaining the well known converse situation in which one may view module of an algebra as a functor. Indeed, any left module W of an R-algebra B can be viewed as a functor between two categories. Indeed, one may choose a collection of mutually orthogonal idempotents $f_1, f_2, ..., f_n$ of B whose sum is the identity of B, and may view W as a functor from the category $\mathfrak{B}$ to the category of R-modules. Here, the objects of $\mathfrak{B}$ are the idempotents f_i, and $\mathrm{Hom}_{\mathfrak{B}}(f_i, f_j) = f_j B f_i$, and the composition is the multiplication in the algebra B.

Chapter 4

Inducing and restricting simple functors

Almost all the materials in this chapter comes from [Yar5, Section 3].

Our main aim in this chapter is to study the subfunctors, especially the socle and the radical, of a Mackey functor obtained by restricting or inducing a simple functor. Let S be a simple $\mu_{\mathbb{K}}(H)$-module and T be a simple $\mu_{\mathbb{K}}(G)$-module where H is a subgroup of G. For example, we determine the socles and radicals of the functors $\uparrow_H^G S$ and $\downarrow_H^G T$ (in terms of the socles and the radicals of some modules of group algebras), and obtain some criterions for $\uparrow_H^G S$ and $\downarrow_H^G T$ to be simple, semisimple, or indecomposable.

We begin by a preliminary result, see for instance [Yar4, Lemma 7.2 and Lemma 6.12].

Proposition 4.1 *Let $H \leq K \leq G$ and let W be a simple $R\overline{N}_K(H)$-module. Then:*

(1) *We have the direct sum decomposition*

$$t_H^H \mu_R(G) t_H^H = A_H \oplus I_H$$

where A_H is a unital subalgebra of $t_H^H \mu_R(G) t_H^H$ isomorphic to $R\overline{N}_G(H)$ (via the map $c_H^g \mapsto gH$) and I_H is a two sided ideal of $t_H^H \mu_R(G) t_H^H$ with the R-basis consisting of the elements of the form

$$t^H_{{}^gJ} c^g_J r^H_J$$

where $J \neq H$.

(2)

$$(\uparrow_K^G S_{H,W}^K)(H) \cong \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$$

as $R\overline{N}_G(H)$-modules.

Proof: (1) The basis theorem 2.1 implies that

$$t_H^H \mu_{\mathbb{K}}(G) t_H^H = \Big(\bigoplus_{gH \subseteq N_G(H)} \mathbb{K} c_H^g \Big) \oplus J_H$$

as $\mathbb{K}$-spaces, where J_H is the $\mathbb{K}$-subspace with basis elements of the desired form. We see easily that

$$\bigoplus_{gH \subseteq N_G(H)} \mathbb{K} c_H^g \quad \text{and} \quad \mathbb{K}\overline{N}_G(H)$$

are isomorphic algebras with isomorphism given by $c_H^g \leftrightarrow gH$. Finally, using the axioms in the definition of Mackey algebras we observe that J_H is a two sided ideal of $t_H^H \mu_{\mathbb{K}}(G) t_H^H$.

(2) Because of $T = S_{H,W}^K$, for a $g \in G$ we see that $T(K \cap H^g) \neq 0$ if and only if $K \cap H^g$ is equal to H^g and H^g is a K-conjugate of H, which is equivalent to $g \in N_G(H)K$. Moreover,

$$T(K \cap H^g) = c_H^{g^{-1}}(W)$$

if $g \in N_G(H)K$ where c is the conjugation map for T. Then using the explicit formula for the induced Mackey functors given in [Sa, TW] we obtain

$$(\uparrow_K^G T)(H) = \bigoplus_{gK \subseteq N_G(H)K} c_H^{g^{-1}}(W).$$

If $\tilde{c}$ denotes the conjugation map for $\uparrow_K^G T$ then $k \in N_G(H)$ acts on an element

$$x = \bigoplus_{gK \subseteq N_G(H)K} x_g \in (\uparrow_K^G T)(H) \quad \text{as}$$

$$k.x = \tilde{c}_H^k(x) = \bigoplus_{gK \subseteq N_G(H)K} \left(\tilde{c}_H^k(x)\right)_g,$$

where

$$\left(\tilde{c}_H^k(x)\right)_g = x_{k^{-1}g},$$

see [Sa, TW]. Therefore $\overline{N}_G(H)$ permutes the summands $c_H^{g^{-1}}(W)$ of $(\uparrow_K^G T)(H)$ transitively, and the stabilizer of the summand $c_H^1(W) = W$ is

$$N_G(H) \cap K = N_K(H).$$

This proves the result. □

Let T be a Mackey functor for a subgroup K of G. Relating $\mathrm{Soc}(\uparrow_K^G T)$ to $\mathrm{Soc}(T)$ may require finding a relation between the minimal subgroups of the functors $\uparrow_K^G T$ and T. It is not true in general that any minimal subgroup of T is also a minimal subgroup of $\uparrow_K^G T$. For instance, if the subgroup K have subgroups A and B satisfying $A <_G B$ but $A \not<_K B$ then we may take $T = S_{A,\mathbb{K}}^K \oplus S_{B,\mathbb{K}}^K$ so that, by the explicit description of an induced functor given in 2.6, the minimal subgroup B of T is not a minimal subgroup of $\uparrow_K^G T$. However if T is simple then it is clear by 2.6 that the minimal subgroups of $\uparrow_K^G T$ are precisely the G-conjugates of the minimal subgroups of T. Thus part (6) of [Yar4, Lemma 6.1] is true only when T is simple, and must be corrected as the first part of the following result. However the results of [Yar4] depending on it remain true because they made use of it when T is simple.

Lemma 4.2 *Let K be a subgroup of G.*

(1) *If T is a $\mu_{\mathbb{K}}(K)$-module, then the minimal subgroups of $\uparrow_K^G T$ are precisely the smallest elements (with respect to $\subseteq$) of the set of all G-conjugates of the minimal subgroups of T.*

(2) *If M be a $\mu_{\mathbb{K}}(G)$-module, then the minimal subgroups of $\downarrow_K^G M$ are precisely the minimal subgroups of M that are contained in K.*

Proof: (1) We will argue as in the proof of part (6) of [Yar4, Lemma 6.1]. Let X be a minimal subgroup of $\uparrow_K^G T$. Then by 2.6 there is a $g \in G$ such that $T(K \cap X^g) \neq 0$ so that we can find a minimal subgroup Y of T satisfying $Y \leq K \cap X^g$. As $T(Y) \neq 0$ we see by 2.6 that $(\uparrow_K^G T)(Y) \neq 0$. Since X and hence X^g is a minimal subgroup of $\uparrow_K^G T$, we must have that $X^g = Y$ is a minimal subgroup of T. Moreover, if there is a minimal subgroup Z of T such that $Z^h \leq X$ for some $h \in G$ then 2.6 implies that $(\uparrow_K^G T)(Z^h) \neq 0$, because $T(Z) \neq 0$. As X is a minimal subgroup of $\uparrow_K^G T$, we must have that $Z^h = X$. Hence any minimal subgroup X of $\uparrow_K^G T$ is a smallest element of the set of all G-conjugates of the minimal subgroup of T.

Conversely, let Y be a minimal subgroup of T such that for some $g \in G$ the group Y^g is a smallest element of the set of all G-conjugates of the minimal subgroups of T. Then $T(Y) \neq 0$ and 2.6 implies that $(\uparrow_K^G T)(Y^g) \neq 0$. Thus we can find a minimal subgroup X of $\uparrow_K^G T$ such that $X \leq Y$. By the what we have shown in above there is a $k \in G$ such that X^k is a minimal subgroup of T. But then X^g is a G-conjugate of the minimal subgroup X^k of T such that $X^g \leq Y^g$. The condition on Y^g shows that $Y^g = X^g$. Thus Y^g is a minimal subgroup of $\uparrow_K^G T$.

(2) This is obvious. □

Lemma 4.3 *Let K be a subgroup of G. Then*

(1) *For any simple $\mu_{\mathbb{K}}(K)$-module $S_{H,W}^K$, the minimal subgroups of any nonzero $\mu_{\mathbb{K}}(G)$-submodule of $\uparrow_K^G S_{H,W}^K$ are precisely the G-conjugates of H.*

(2) *For any simple $\mu_{\mathbb{K}}(G)$-module $S_{L,V}^G$ with $L \leq_G K$, any minimal subgroup of any nonzero $\mu_{\mathbb{K}}(K)$-submodule of $\downarrow_K^G S_{L,V}^G$ is a G-conjugate of L.*

Proof: (1) Let M be a nonzero $\mu_{\mathbb{K}}(G)$-submodule of $\uparrow_K^G S_{H,W}^K$, and let X be a minimal subgroup of M. As $(\uparrow_K^G S_{H,W}^K)(X) \neq 0$, we can find a minimal subgroup of $\uparrow_K^G S_{H,W}^K$ contained in X. Part (1) of 4.2 implies that $H \leq_G X$. From the adjointness of the pair $(\downarrow_K^G, \uparrow_K^G)$ we see the existence of a $\mu_{\mathbb{K}}(K)$-epimorphism

$$\downarrow_K^G M \to S_{H,W}^K.$$

This implies that $M(H) \neq 0$. Since X is a minimal subgroup of the Mackey functor M for G, we conclude that $X =_G H$.

(2) Let T be a nonzero $\mu_{\mathbb{K}}(K)$-submodule of $\downarrow^G_K S^G_{L,V}$, and let Y be a minimal subgroup of T. Then $(\downarrow^G_K S^G_{L,V})(Y) \neq 0$ implying that $L \leq_G Y$.

Let T' denote the functor $\downarrow^K_Y T$. Then T' is a nonzero $\mu_{\mathbb{K}}(Y)$-submodule of $\downarrow^G_Y S^G_{L,V}$. From the adjointness of the pair $(\uparrow^G_Y, \downarrow^G_Y)$ we see the existence of a $\mu_{\mathbb{K}}(G)$-epimorphism

$$\uparrow^G_Y T' \to S^G_{L,V}.$$

This implies that $(\uparrow^G_Y T')(L) \neq 0$ from which we see by 2.6 that

$$0 \neq T'(Y \cap L^g) = T(Y \cap L^g)$$

for some $g \in G$. Since Y is a minimal subgroup of T we conclude that $Y \leq Y \cap L^g$.

□

The above lemma is a combination of [Yar4, Lemma 6.13] and [Yar1, Remark 3.1].

For an algebra A and an idempotent e of A, there are some well known relations between the module categories of the algebras A and eAe. In particular, the map $S \mapsto eS$ define a bijective correspondence between the isomorphism classes of simple A-modules not annihilated by e and the isomorphism classes of simple eAe-modules. Most of these can be found in [Gr2, pp. 83-87] from which the following lemma follows easily. For any subset X of the A-module V we denote by AX the A-submodule of V generated by X.

Lemma 4.4 *Let A be a finite dimensional $\mathbb{K}$-algebra and let e be a nonzero idempotent of A. If V is a nonzero A-module having no nonzero A-submodule annihilated by e, then:*

(1) *The maps*

$$S \to eS \qquad \text{and} \qquad AT \leftarrow T$$

define a bijective correspondence between the simple A-submodules of V and the simple eAe-submodules of eV.

(2) $\mathrm{Soc}_{eAe}(eV) = e\mathrm{Soc}_A(V)$ *and* $\mathrm{Soc}_A(V) = A\mathrm{Soc}_{eAe}(eV)$.

Proof: By the help of the results in [Gr2, pp. 83-87], it remains to prove that $AT = AeT$ is a simple A-submodule of V for any simple eAe-submodule T of eV. In general AT may not be simple, but our hypothesis on V forces it to be simple because any nonzero A-submodule U of AT is not annihilated by e so that $eU = T$ implying $U = AT$. □

Let S and V be modules of an algebra A where S is simple and V is finite dimensional. By the multiplicity of S in V we mean the number of composition factors of V isomorphic to S.

Theorem 4.5 *Let $H \leq K \leq G$ and let W be a simple $\mathbb{K}\overline{N}_K(H)$-module. Let*

$$M = \uparrow_K^G S_{H,W}^K \quad \text{and} \quad V = \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W.$$

Then, there is a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules of M and the simple $\mathbb{K}\overline{N}_G(H)$-submodules of V. More precisely, any simple $\mu_{\mathbb{K}}(G)$-submodule of M is isomorphic to a simple functor of the form $S_{H,U}^G$ where U is a simple $\mathbb{K}\overline{N}_G(H)$-submodule of V, and conversely any simple $\mathbb{K}\overline{N}_G(H)$-submodule of V is isomorphic to a simple module of the form $S(H)$ where S is a simple $\mu_{\mathbb{K}}(G)$-submodules of M. Furthermore, for any simple $\mathbb{K}\overline{N}_G(H)$-module U, the multiplicity of $S_{H,U}^G$ in $\mathrm{Soc}(M)$ *is equal to the multiplicity of U in* $\mathrm{Soc}(V)$.

Proof: Let $A = \mu_{\mathbb{K}}(G)$, $B = \mathbb{K}\overline{N}_G(H)$ and $e = t_H^H$. By 4.1 the B-modules $eM = M(H)$ and V are isomorphic. We also see by using 4.3 that the ideal I_H of $eAe = A_H \oplus I_H$ given in 4.1 annihilates eM where the algebra A_H is isomorphic to B via $c_H^g \leftrightarrow gH$. Therefore, the (simple) eAe-submodules of eM and the (simple) B-submodules of eM coincide. 4.3 implies that any nonzero A-submodule of M has H as a minimal subgroup. In particular, M has no nonzero A-submodule annihilated by e so that 4.4 may be applied to deduce that there is a bijection between the simple A-submodules of M and the simple B-submodules of $eM \cong V$. Moreover, the B-modules $e\mathrm{Soc}(M)$ and $\mathrm{Soc}(V)$ are isomorphic.

Any simple subfunctor S of M has H as a minimal subgroup (by 4.3), and by part (1) of 4.4 the B-module $eS = S(H)$ is a simple B-submodule of $eM \cong V$. So, any simple A-submodule of M is isomorphic to a simple functor of the form $S^G_{H,U}$ where U is a simple B-submodule of V. Conversely, if U is a simple B-submodule of $V \cong eM$ then again by part (1) of 4.4 there is a simple A-submodule S of M such that $S(H) \cong U$.

Let U be a simple B-module. $e\mathrm{Soc}(M)$ and $\mathrm{Soc}(V)$ are isomorphic B-modules and any simple A-submodule of M is of the form $S^G_{H,U'}$. By 2.5 we see that the isomorphisms of the simple functors of the forms $S^G_{H,U'}$ and $S^G_{H,U''}$ is equivalent to the isomorphisms of the simple B-modules U' and U''. Therefore, the statement about the multiplicities must be true because $S^G_{H,U'}(H) \cong U'$ and because the left multiplication by the idempotent e respects the direct sums. □

Lemma 4.6 *Let K be a subgroup of G. Then*

(1) *Let $\mathcal{X}$ be a set of subgroups of K and let T be a $\mu_{\mathbb{K}}(K)$-module. If T is generated as a $\mu_{\mathbb{K}}(K)$-module by its values on $\mathcal{X}$, then $\uparrow^G_K T$ is generated as a $\mu_{\mathbb{K}}(G)$-module by its values on $\mathcal{X}$. In particular, for any simple $\mu_{\mathbb{K}}(K)$-module $S^K_{H,W}$ and any proper $\mu_{\mathbb{K}}(G)$-submodule M of $\uparrow^G_K S^K_{H,W}$, the minimal subgroups of*

$$(\uparrow^G_K S^K_{H,W})/M$$

are precisely the G-conjugates of H.

(2) *Let $\mathcal{Y}$ be a set of subgroups of G and let M be a $\mu_{\mathbb{K}}(G)$-module. If M is generated as a $\mu_{\mathbb{K}}(G)$-module by its values on $\mathcal{Y}$, then $\downarrow^G_K M$ is generated as a $\mu_{\mathbb{K}}(K)$-module by its values on the elements of the set*

$$\{X \leq K : X \leq_G Y,\ Y \in \mathcal{Y}\}.$$

In particular, for any simple $\mu_{\mathbb{K}}(G)$-module $S^G_{L,V}$ with $L \leq_G K$ and any proper $\mu_{\mathbb{K}}(K)$-submodule T of $\downarrow^G_K S^G_{L,V}$, there is a minimal subgroup of

$$(\downarrow^G_K S^G_{L,V})/T$$

which is a G-conjugate of L.

Proof: (1) Let S be a $\mu_{\mathbb{K}}(G)$-submodule of $\uparrow_K^G T$ such that $S(X) = (\uparrow_K^G T)(X)$ for all X in $\mathcal{X}$. To show that $\uparrow_K^G T$ is generated by its values on $\mathcal{X}$ it suffices to show that $S = \uparrow_K^G T$.

If S is not equal to $\uparrow_K^G T$ then by the adjointness of the pair $(\uparrow_K^G, \downarrow_K^G)$ there is a nonzero $\mu_{\mathbb{K}}(K)$-module homomorphism

$$\pi : T \to \downarrow_K^G \big((\uparrow_K^G T)/S\big)$$

whose L-component

$$\pi_L : T(L) \to \downarrow_K^G \big((\uparrow_K^G T)/S\big)(L)$$

is nonzero for some subgroup L of K. So there is a $t \in T(L)$ such that $\pi_L(t) \neq 0$. As T is generated by its values on $\mathcal{X}$,

$$T(L) = \sum_{X \in \mathcal{X}} t_L^L \mu_{\mathbb{K}}(K) t_X^X T$$

so that t can be written as a sum of elements of the form

$$t_{{}^kJ}^L c_J^k r_J^X t_X$$

where $k \in K$, $J \leq K$, and $t_X \in T(X)$. Since π commutes with the maps t, r, c of T, it follows that $\pi_L(t)$ can be written as a sum of elements of the form

$$t_{{}^kJ}^L c_J^k r_J^X \pi_X(t_X).$$

But then $\pi_X(t_X)$ and hence $\pi_L(t)$ is 0 because $S(X) = (\uparrow_K^G T)(X)$. Consequently, $S = \uparrow_K^G T$.

For the second statement, let M be a proper $\mu_{\mathbb{K}}(G)$-submodule of $\uparrow_K^G S_{H,W}^K$. As $S_{H,V}^K$ is generated by its value on H, it follows by what we have showed above that the quotient

$$(\uparrow_K^G S_{H,W}^K)/M$$

is nonzero at H. Moreover, if Y is a minimal subgroup of the quotient then $\uparrow_K^G S_{H,W}^K$ is nonzero at Y so that $H \leq_G Y$ by part (1) of 4.3. Hence, the minimal subgroups of the quotient are precisely the G-conjugates of H.

(2) The first statement is obvious. For the second statement, let T be a proper $\mu_{\mathbb{K}}(K)$-submodule of $\downarrow_K^G S_{L,V}^G$. If the quotient

$$(\downarrow_K^G S_{L,V}^G)/T$$

is nonzero at a subgroup X of K then $\downarrow_K^G S_{L,V}^G$ is nonzero at X so that $L \leq_G X$. On the other hand, $\downarrow_K^G S_{L,V}^G$ is generated by its values on G-conjugates of L that are in K and so, by the first statement, the quotient cannot be 0 at every G-conjugate of L that is in K. Consequently, a minimal subgroup of the quotient must ba a G-conjugate of L. □

Theorem 4.7 *Let $H \leq K \leq G$ and let W be a simple $\mathbb{K}\overline{N}_K(H)$-module. Then,*

$$\uparrow_K^G S_{H,W}^K$$

is a simple (respectively, semisimple) $\mu_{\mathbb{K}}(G)$-module if and only if

$$\uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$$

is a simple (respectively, semisimple) $\mathbb{K}\overline{N}_G(H)$-module.

Proof: Let $M = \uparrow_K^G S_{H,W}^K$, $V = \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$, $A = \mu_{\mathbb{K}}(G)$, and $B = \mathbb{K}\overline{N}_G(H)$. It follows by 4.1 that $M(H) \cong V$ as B-modules. We note also that the ideal I_H in 4.1 annihilates $M(H)$ which is a consequence of 4.3.

Suppose that M is a simple (respectively, semisimple) A-module. Then 4.3, 4.4 and 4.1 imply that $M(H)$ is a simple (respectively, semisimple) A_H-module. Since A_H and B are isomorphic algebras via $c_H^g \mapsto gH$, we can conclude that V is a simple (respectively, semisimple) B-module.

Suppose that V is a simple (respectively, semisimple) B-module. Then 4.1 implies that $M(H)$ is a simple (respectively, semisimple) eAe-module where $e = t_H^H$. From 4.4 we see that $\mathrm{Soc}_A(M) = AM(H)$ is a simple (respectively, semisimple) A-module. As $S_{H,W}^K$ is generated as a $\mu_{\mathbb{K}}(K)$-module by its value on H, it follows by 4.6 that M is generated as an A-module by $M(H)$. This shows that $M = AM(H) = \mathrm{Soc}_A(M)$. □

The previous result generalizes [Yar1, Proposition 3.5 and Corollary 3.7].

Let e be an idempotent of an algebra A, and let V be an A-module, and T be an eAe-submodule of eV. We denote by the notation $(V :_e T)$ the subset

$$\{v \in V : eAv \subseteq T\}$$

of V. It is clear that $(V :_e T)$ is an A-submodule of V such that $e(V :_e T) = T$.

Lemma 4.8 *Let A be a finite dimensional $\mathbb{K}$-algebra and let e be a nonzero idempotent of A. If V is a nonzero A-module having no nonzero quotient module annihilated by e (equivalently, $AeV = V$) then:*

(1) *The maps*

$$J \to eJ \qquad \text{and} \qquad (V :_e I) \leftarrow I$$

define a bijective correspondence between the maximal A-submodules of V and the maximal eAe-submodules of eV.

(2) $\mathrm{Jac}_{eAe}(eV) = e\mathrm{Jac}_A(V)$ *and* $\mathrm{Jac}_A(V) = (V :_e \mathrm{Jac}_{eAe}(eV))$.

Proof: (1) For any maximal eAe-submodule I of eV, we must show that $(V :_e I)$ is a maximal A-submodule of V and that $e(V :_e I) = I$:

For any eAe-submodule I' of eV it is obvious that $AI' \subseteq (V :_e I')$ and that $e(V :_e I') \subseteq I'$. From these two the equality $e(V :_e I') = I'$ follows for any eAe-submodule (not necessarily maximal) I' of eV.

It follows from $e(V :_e I) = I$ that $(V :_e I)$ is a proper A-submodule of V. Let T be a proper A-submodule of V containing $(V :_e I)$. Then $I \subseteq eT$. Moreover, V/T, being nonzero, is not annihilated by e so that $eT \neq eV$. Now $I = eT$ by the maximality of I. This implies that $T \subseteq (V :_e I)$. Consequently, $(V :_e I)$ is a maximal A-submodule of V.

For any maximal A-submodule J of V, we must show that eJ is a maximal eAe-submodule of eV and that $(V :_e eJ) = J$:

As V/J is a simple A-module not annihilated by e, the eAe-module $eV/eJ \cong e(V/J)$ is simple so that eJ is a maximal eAe-submodule of eV.

The containment $J \subseteq (V :_e eJ)$ is clear. If $(V :_e eJ)$ is equal to V then $eJ = e(V :_e eJ) = eV$ which is not the case. Hence $(V :_e eJ) = J$ by the maximality of J.

(2) This is obvious from the first part. □

Theorem 4.9 *Let $H \leq K \leq G$ and let W be a simple $\mathbb{K}\overline{N}_K(H)$-module. Let*

$$M = \uparrow_K^G S_{H,W}^K \quad \text{and} \quad V = \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W.$$

Then, there is a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules of M and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules of V. In particular, any simple quotient of M is isomorphic to a simple functor of the form $S_{H,U}^G$ where U is a simple quotient of V, and conversely any simple quotient of V is isomorphic to a simple module of the form $S(H)$ where S is a simple quotient of M. Furthermore, for any simple $\mathbb{K}\overline{N}_G(H)$-module U, the multiplicity of $S_{H,U}^G$ in $M/\mathrm{Jac}(M)$ is equal to the multiplicity of U in $V/\mathrm{Jac}(V)$.

Proof: Let $A = \mu_{\mathbb{K}}(G)$, $B = \mathbb{K}\overline{N}_G(H)$, and $e = t_H^H$. Firstly, we note that the ideal I_H of eAe given in 4.1 annihilates the eAe-module eM (by 4.3) so that the (maximal) eAe and (maximal) eAe/I_H-submodules of eM coincide. As B and eAe/I_H are isomorphic algebras (by 4.1), we see that there is a bijective correspondence between the maximal B and eAe-submodules of eM. From 4.6 any nonzero quotient of M has H as a minimal subgroup. In particular, there is no nonzero quotient of M annihilated by e so that 4.8 gives a bijective correspondence between the maximal A-submodules of M and the maximal B-submodules of V. Moreover, the B-modules $e\mathrm{Jac}(M) = \mathrm{Jac}(eM)$ and $\mathrm{Jac}(V)$ are isomorphic so that, from the B-module isomorphism $eM \cong V$ we obtain that

$$eM/e\mathrm{Jac}(M) \cong V/\mathrm{Jac}(V)$$

as B-modules.

Any simple quotient M/J of M has H as a minimal subgroup (by 4.6), and by part (1) of 4.8 the B-module eM/eJ is a simple quotient of $eM \cong V$. So, any simple quotient of M is isomorphic to a simple functor of the form $S^G_{H,U}$ where U is a simple quotient of V. Conversely, for any simple quotient of V/I of $V \cong eM$ then again by by part (1) of 4.8 there is a simple quotient $S = M/J$ such that $S(H) \cong V/I$.

Let U be a simple B-module. The there B-modules $e\big(M/\mathrm{Jac}(M)\big)$, $eM/e\mathrm{Jac}(M)$ and $V/\mathrm{Jac}(V)$ are isomorphic, and any simple quotient of the A-module of M is of the form $S^G_{H,U'}$. By 2.5 we see that the isomorphisms of the simple functors of the forms $S^G_{H,U'}$ and $S^G_{H,U''}$ is equivalent to the isomorphisms of the simple B-modules U' and U''. Therefore, the statement about the multiplicities must be true because $S^G_{H,U'}(H) \cong U'$ and because the left multiplication by the idempotent e respects the direct sums. □

Lemma 4.10 *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that V and W be nonzero A-modules. Let*

$$\phi : \mathrm{Hom}_A(V, W) \to \mathrm{Hom}_{eAe}(eV, eW), \quad f \mapsto f|_{eV},$$

be the $\mathbb{K}$-space ($\mathbb{K}$-algebra if $W = V$) homomorphism sending f to $f|_{eV}$ where $f|_{eV}$ denotes the restriction of f to eV. Then:

(1) *ϕ is a monomorphism if and only if W has no nonzero A-submodule annihilated by e and isomorphic to a quotient of V.*

(2) *If V has no nonzero quotient module annihilated by e (equivalently, $AeV = V$) and if W has no nonzero A-submodule annihilated by e (equivalently, $(W :_e 0) = 0$), then ϕ is an isomorphism.*

Proof: (1) Firstly, it is obvious that ϕ is not injective if and only if $ef(V) = 0$ for some nonzero f in $\mathrm{Hom}_A(V, W)$. For any A-submodule W_0 of W isomorphic to a quotient V/V_0 of V, it is clear that there is an f in $\mathrm{Hom}_A(V, W)$ with the kernel equal to V_0 and the image equal to W_0. And conversely, any A-module homomorphism gives such submodules. Thus the result follows.

(2) By the first part, it is enough to show that ϕ is surjective:

Let g be in $\mathrm{Hom}_{eAe}(eV, eW)$. We want to construct an element f in $\mathrm{Hom}_A(V, W)$ whose restriction to eV is equal to g. As $V = AeV$, any element of V can be written as a sum of elements of the form aev where each a in A and each v in V. Letting

$$v = a_1ev_1 + ... + a_nev_n,$$

it is natural to define

$$f(v) = a_1g(ev_1) + ... + a_ng(ev_n).$$

By its construction, we only need to show that f is well-defined because there may be some elements of V which can be expressed as a sum of elements of the form aev in different ways. Suppose that

$$b_1eu_1 + ... + b_meu_m = 0$$

for some natural number m and some elements $u_i \in V$ and $b_i \in A$. Then for any a in A we have

$$0 = g(0) = g\big(ea(b_1eu_1 + ... + b_meu_m)\big) = ea\big(b_1g(eu_1) + ... + b_mg(eu_m)\big).$$

Thus $eAw = 0$ where $w = b_1g(eu_1) + ... + b_mg(eu_m)$, implying that Aw is an A-submodule of W annihilated by e. By the condition on W we must have that $w = 0$, as desired. □

Lemma 4.11 *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Let V be a nonzero A-module satisfying $AeV = V$ and $(V :_e 0) = 0$. Suppose*

$$V = V_1 \oplus ... \oplus V_n$$

is a decomposition of V into nonzero A-modules. Then,

$$eV = eV_1 \oplus ... \oplus eV_n$$

is a decomposition of eV into nonzero eAe-modules such that the A-modules V_i and V_j are isomorphic if and only if the eAe-modules eV_i and eV_j are isomorphic. Moreover, V_i is an indecomposable A-module if and only if eV_i is an indecomposable eAe-module.

Proof: This is obvious because the endomorphism algebras of V and eV are isomorphic by part (2) of 4.10. □

Using 4.11, one may lift most of the results about induction of simple modules of group algebras to the results about induction of simple Mackey functors. As an example, in part (3) of the next result we want to lift a part of the result [Ha, Theorem 7] which says that if N is a normal subgroup of G and W is a simple $\mathbb{K}N$-module, then, for any indecomposable direct summand P of $\uparrow_N^G W$, there is a simple $\mathbb{K}G$-module V satisfying

$$\mathrm{Soc}(P) \cong P/\mathrm{Jac}(P) \cong V$$

(where W is necessarily a direct summand of $\downarrow_N^G V$). The first two parts of the following result are slight generalizations of 4.5 and 4.9.

Corollary 4.12 *Let $H \leq K$ be subgroups of G and let W be a simple $\mathbb{K}\overline{N}_K(H)$-module. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$. Then, for any nonzero $\mu_{\mathbb{K}}(G)$-module M,*

(1) *If M is isomorphic to a $\mu_{\mathbb{K}}(G)$-submodule of $\uparrow_K^G S_{H,W}^K$, then the maps*

$$S \to S(H) \qquad \textit{and} \qquad AT \leftarrow T$$

define a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules of M and the simple $\mathbb{K}\overline{N}_G(H)$-submodules of $M(H)$.

(2) *If M is isomorphic to a quotient functor of $\uparrow_K^G S_{H,W}^K$, then the maps*

$$J \to J(H) \qquad \textit{and} \qquad (M :_e I) \leftarrow I$$

define a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules of M and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules of $M(H)$.

(3) *Suppose that $N_K(H)$ is normal in $N_G(H)$. If M is an indecomposable $\mu_{\mathbb{K}}(G)$-module which is a direct summand of $\uparrow_K^G S_{H,W}^K$, then*

$$\mathrm{Soc}(M) \qquad \textit{and} \qquad M/\mathrm{Jac}(M)$$

are isomorphic simple functors having H as minimal subgroups.

Proof: Firstly, in all cases the ideal I_H of eAe given in 4.1 annihilates the eAe-module eM so that the eAe-submodules of M and the eAe/I_H-submodules of M are the same, where from 4.1 we also have that $eAe/I_H \cong \mathbb{K}\overline{N}_G(H)$.

(1) Any A-submodule of M is isomorphic to an A-submodule of $\uparrow_K^G S_{H,W}^K$. So 4.3 implies that M has no nonzero A-submodule annihilated by e. The result follows by 4.4.

(2) Any quotient functor of M is isomorphic to a quotient functor of $\uparrow_K^G S_{H,W}^K$. So 4.6 implies that M has no nonzero quotient module annihilated by e. The result follows by 4.8.

(3) In this case any subfunctor and any quotient functor of M are isomorphic to a subfunctor and a quotient functor of $\uparrow_K^G S_{H,W}^K$, respectively. This means that

$$AeM = M \qquad \text{and} \qquad (M :_e 0) = 0$$

implying applicability of 4.11. Now, 4.11 implies that $M(H)$ is an indecomposable $\mathbb{K}\overline{N}_G(H)$-module which is a direct summand of

$$(\uparrow_K^G S_{H,W}^K)(H),$$

isomorphic by 4.1 to

$$\uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W.$$

Then the result [Ha, Theorem 7], mentioned above, implies that

$$\mathrm{Soc}(M(H)) \cong M(H)/\mathrm{Jac}(M(H)) \cong V$$

where V is a simple $\mathbb{K}\overline{N}_G(H)$-module. The bijective correspondences given in the first two parts now imply that

$$\mathrm{Soc}(M) \cong S_{H,V}^G \cong M/\mathrm{Jac}(M).$$

□

Theorem 4.13 *Let $K \le G \ge L$ and $H \le K \cap L$. Then, for any simple $\mathbb{K}\overline{N}_K(H)$-module W and any simple $\mathbb{K}\overline{N}_L(H)$-module U,*

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(\uparrow_K^G S_{H,W}^K, \uparrow_L^G S_{H,U}^L\big) \cong \mathrm{Hom}_{\mathbb{K}\overline{N}_G(H)}\big(\uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W, \uparrow_{\overline{N}_L(H)}^{\overline{N}_G(H)} U\big)$$

as $\mathbb{K}$-spaces ($\mathbb{K}$-algebras if $L = K$ and $U = W$).

Proof: Let $M_1 = \uparrow_K^G S_{H,W}^K$, $M_2 = \uparrow_L^G S_{H,U}^L$, $A = \mu_{\mathbb{K}}(G)$, and $e = t_H^H$. It is a consequence of 4.3 and 4.6 that both of the modules M_1 and M_2 have no nonzero quotient modules annihilated by e and no nonzero submodules annihilated by e. Thus part (2) of 4.10 implies that

$$\mathrm{Hom}_A(M_1, M_2) \qquad \text{and} \qquad \mathrm{Hom}_{eAe}(eM_1, eM_2)$$

are isomorphic. Moreover, as the ideal I_H of eAe in 4.1 annihilates both of the eAe-modules eM_1 and eM_2, it follows that

$$\mathrm{Hom}_{eAe}(eM_1, eM_2) \qquad \text{and} \qquad \mathrm{Hom}_{eAe/I_H}(eM_1, eM_2)$$

are isomorphic. The result follows from 4.1. □

For $L = K = G$, the previous theorem reduces to [Bo, Lemma 11.6.6, page 302] proved (more conceptually) by using the G-set definition of Mackey functors.

The results 4.5 and 4.9 follows also (more quickly) from the previous theorem.

Corollary 4.14 *Let $H \subseteq K$ be subgroups of G and W simple $\mathbb{K}\overline{N}_K(H)$-module. Then, the $\mu_{\mathbb{K}}(G)$-module*

$$\uparrow_K^G S_{H,W}^K$$

is indecomposable if and only if the $\mathbb{K}\overline{N}_G(H)$-module

$$\uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$$

is indecomposable.

Proof: This follows from 4.13 stating that endomorphism algebras of $\uparrow_K^G S_{H,W}^K$ and $\uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$ are isomorphic, and hence they both local or not local. □

See [Yar4, Theorem 6.15] which is related to the above result.

Corollary 4.15 *Let M be a $\mu_{\mathbb{K}}(G)$-module, H be a subgroup of G, and U be a simple $\mathbb{K}\overline{N}_G(H)$-module. Then, the multiplicity of $S_{H,U}^G$ in the socle (respectively, in the head) of M is equal to the multiplicity of $S_{H,U}^{N_G(H)}$ in the socle (respectively, in the head) of $\downarrow_{N_G(H)}^G M$.*

Proof: As a consequence of 4.13 the endomorphism algebra of the the $\mu_{\mathbb{K}}(G)$-module $S^G_{H,V}$ is isomorphic to the endomorphism algebra of the $\mu_{\mathbb{K}}\big(N_G(H)\big)$-module $S^{N_G(H)}_{H,V}$. Using the isomorphism

$$S^G_{H,V} \cong \uparrow^G_{N_G(H)} S^{N_G(H)}_{H,V}$$

given in 2.10, we see that the result follows by the adjointness of the pair

$$(\uparrow^G_{N_G(H)}, \downarrow^G_{N_G(H)})$$

(respectively, of the pair

$$(\downarrow^G_{N_G(H)}, \uparrow^G_{N_G(H)})).$$

□

It may be thought that 4.13 is a very restrictive result dealing with simple functors whose minimal subgroups are equal (or conjugate). Indeed, the next result indicates that it is not so.

Proposition 4.16 *Let $A \le K \le G \ge L \ge B$. Then, for any simple $\mathbb{K}\overline{N}_K(A)$-module W and any simple $\mathbb{K}\overline{N}_L(B)$-module U, if*

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(\uparrow^G_K S^K_{A,W}, \uparrow^G_L S^L_{B,U}\big) \neq 0,$$

then $B = A^g$ for some $g \in G$ (so that $\uparrow^G_L S^L_{B,U}$ and $\uparrow^G_{{}^gL} S^{{}^gL}_{A,{}^gU}$ are isomorphic).

Proof: Let $M_1 = \uparrow^G_K S^K_{A,W}$ and $M_2 = \uparrow^G_L S^L_{B,U}$. Suppose that

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(M_1, M_2) \neq 0.$$

Then, using the adjointness of the pairs $(\uparrow^G_K, \downarrow^G_K)$ and $(\downarrow^G_L, \uparrow^G_L)$, we see that there are (nonzero) maps

$$S^K_{A,W} \to \downarrow^G_K M_2 \quad \text{and} \quad \downarrow^G_L M_1 \to S^L_{B,U},$$

which are necessarily a $\mu_{\mathbb{K}}(K)$-module monomorphism and a $\mu_{\mathbb{K}}(L)$-module epimorphism, respectively. From these morphisms of functors we obtain that $M_2(A) \neq 0$ and $M_1(B) \neq 0$. So it follows by 2.6 that $B \le_L L \cap A^x$ and that

$A \leq_K K \cap B^y$ for some x and y in G. Hence, $B = A^g$ for some $g \in G$. Furthermore, the g conjugate $|_G^g M_2$ of the functor M_2 for G is isomorphic to M_2, and hence M_2 is isomorphic to

$$|_G^g M_2 \cong \uparrow_{^gL}^G S_{^gB,^gU}^{^gL}.$$

□

One may want to obtain results similar to 4.5, 4.7, 4.9 and 4.13 for restrictions of simple functors. The results similar to 4.5 and 4.9 can be readily given by using 4.13 and using the adjointness property of induction and restriction.

Theorem 4.17 *Let $K \leq L \leq G$ and let V be a simple $\mathbb{K}\overline{N}_G(K)$-module. Let*

$$M = \downarrow_L^G S_{K,V}^G.$$

Then, any simple $\mu_{\mathbb{K}}(L)$-submodule of M is isomorphic to a simple functor of the form

$$S_{^gK,W}^L$$

where g is an element of G with $^gK \leq L$ and W is a simple $\mathbb{K}\overline{N}_L(^gK)$-submodule of ^gV. Conversely, for any element g of G with $^gK \leq L$, any simple $\mathbb{K}\overline{N}_L(^gK)$-submodule of ^gV is isomorphic to a simple module of the form $S(^gK)$ where S is a simple $\mu_{\mathbb{K}}(L)$-submodule of M. Moreover, for any element g of G with $^gK \leq L$ and any simple $\mathbb{K}\overline{N}_L(^gK)$-module of W, the multiplicity of $S_{^gK,W}^L$ in

$$\mathrm{Soc}(M)$$

is equal to the multiplicity of U in

$$\mathrm{Soc}\big(\downarrow_{\overline{N}_L(^gK)}^{\overline{N}_G(^gK)} {}^gV\big).$$

Proof: It follows by part (2) of 4.3 that any simple $\mu_{\mathbb{K}}(L)$-submodule of M has a minimal subgroup which is a G-conjugate of K so that it must be of the form $S_{^gK,W}^L$ where g is an element of G with $^gK \leq L$ and W is simple $\mathbb{K}\overline{N}_L(^gK)$-submodule of $^gV \cong M(^gK)$.

What remain will follow easily from the following isomorphism of $\mathbb{K}$-spaces. Let $g \in G$ with $^gK \leq L$ and let W be a simple $\mathbb{K}\overline{N}_L(^gK)$-module. Put $x = g^{-1}$

to simplify the notation. Using the adjointness of the pair $(\uparrow_L^G, \downarrow_L^G)$ and 4.13 we have the following isomorphisms of $\mathbb{K}$-spaces:

$$\begin{aligned}
\mathrm{Hom}_{\mu_{\mathbb{K}}(L)}\big(S^L_{{}^gK,W}, M\big) &\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(\uparrow_L^G S^L_{{}^gK,W}, S^G_{K,V}\big)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(\uparrow_L^G |^g_L S^{L^g}_{K,{}^xW}, S^G_{K,V}\big)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(|^g_G \uparrow^G_{L^g} S^{L^g}_{K,{}^xW}, S^G_{K,V}\big)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(\uparrow^G_{L^g} S^{L^g}_{K,{}^xW}, S^G_{K,V}\big)\\
&\cong \mathrm{Hom}_{\mathbb{K}\overline{N}_G(K)}\big(\uparrow^{\overline{N}_G(K)}_{\overline{N}_{L^g}(K)} {}^xW, V\big)\\
&\cong \mathrm{Hom}_{\mathbb{K}\overline{N}_{L^g}(K)}\big({}^xW, \downarrow^{\overline{N}_G(K)}_{\overline{N}_{L^g}(K)} V\big)\\
&\cong \mathrm{Hom}_{\mathbb{K}\overline{N}_L({}^gK)}\big(W, \downarrow^{\overline{N}_G({}^gK)}_{\overline{N}_L({}^gK)} {}^gV\big)
\end{aligned}$$

We also used the following obvious properties of conjugation which transports the structure. Firstly, the Mackey functors $S^L_{{}^gK,W}$ and $|^g_L S^{L^g}_{K,{}^xW}$, where $x = g^{-1}$, are isomorphic. Secondly, given subgroups $A \leq B \leq G$, an element $g \in G$, and $\mathbb{K}A$-modules U_1 and U_2, the functors $|^g_B \uparrow^B_A$ and $\uparrow^{{}^gB}_{{}^gA} |^g_A$ are naturally isomorphic, the $\mathbb{K}$-spaces $\mathrm{Hom}_{\mathbb{K}A}(U_1, U_2)$ and $\mathrm{Hom}_{\mathbb{K}({}^gA)}({}^gU_1, {}^gU_2)$ are isomorphic, and moreover $|^g_G$ and the identity functor are naturally isomorphic. □

The previous theorem remains true if we replace simple $\mu_{\mathbb{K}}(L)$ and $\mathbb{K}\overline{N}_L({}^gK)$-submodules with simple quotients, and replace socles with heads.

Theorem 4.18 *Let $K \leq L \leq G$ and let V be a simple $\mathbb{K}\overline{N}_G(K)$-module. Let*

$$M = \downarrow^G_L S^G_{K,V}.$$

Then, any simple quotient of M is isomorphic to a simple functor of the form

$$S^L_{{}^gK,W}$$

where g is an element of G with ${}^gK \leq L$ and W is a simple quotient of ${}^gV$. Conversely, for any element g of G with ${}^gK \leq L$, any simple quotient of ${}^gV$ is isomorphic to a simple module of the form $S({}^gK)$ where S is a simple quotient of M. Moreover, for any element g of G with ${}^gK \leq L$ and any simple $\mathbb{K}\overline{N}_L({}^gK)$-module of W, the multiplicity of $S^L_{{}^gK,W}$ in

$$M/\mathrm{Jac}(M)$$

is equal to the multiplicity of U in

$$\big(\downarrow^{\overline{N}_G({}^gK)}_{\overline{N}_L({}^gK)} {}^gV\big)/\mathrm{Jac}\big(\downarrow^{\overline{N}_G({}^gK)}_{\overline{N}_L({}^gK)} {}^gV\big).$$

Proof: It follows by part (2) of 4.6 that any simple quotient of M has a minimal subgroup which is a G-conjugate of K so that it must be of the form $S^L_{{}^gK,W}$ where g is an element of G with ${}^gK \le L$ and W is simple quotient of ${}^gV \cong M({}^gK)$.

What remain will follow easily from the following isomorphism of $\mathbb{K}$-spaces. Let $g \in G$ with ${}^gK \le L$ and let W be a simple $\mathbb{K}\overline{N}_L({}^gK)$-module. Put $x = g^{-1}$ to simplify the notation. Using the adjointness of the pair $(\downarrow^G_L, \uparrow^G_L)$ and 4.13 we have the following isomorphisms of $\mathbb{K}$-spaces:

$$\begin{aligned}
\mathrm{Hom}_{\mu_{\mathbb{K}}(L)}\big(M, S^L_{{}^gK,W}\big) &\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(S^G_{K,V}, \uparrow^G_L S^L_{{}^gK,W}\big)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(S^G_{K,V}, \uparrow^G_L |^g_L S^{L^g}_{K,{}^xW}\big)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(S^G_{K,V}, |^g_G \uparrow^G_{L^g} S^{L^g}_{K,{}^xW}\big)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(S^G_{K,V}, \uparrow^G_{L^g} S^{L^g}_{K,{}^xW}\big)\\
&\cong \mathrm{Hom}_{\mathbb{K}\overline{N}_G(K)}\big(V, \uparrow^{\overline{N}_G(K)}_{\overline{N}_{L^g}(K)} {}^xW\big)\\
&\cong \mathrm{Hom}_{\mathbb{K}\overline{N}_{L^g}(K)}\big(\downarrow^{\overline{N}_G(K)}_{\overline{N}_{L^g}(K)} V, {}^xW\big)\\
&\cong \mathrm{Hom}_{\mathbb{K}\overline{N}_L({}^gK)}\big(\downarrow^{\overline{N}_G({}^gK)}_{\overline{N}_L({}^gK)} {}^gV, W\big)
\end{aligned}$$

We also used the following obvious properties of conjugation which transports the structure. Firstly, the Mackey functors $S^L_{{}^gK,W}$ and $|^g_L S^{L^g}_{K,{}^xW}$, where $x = g^{-1}$, are isomorphic. Secondly, given subgroups $A \le B \le G$, an element $g \in G$, and $\mathbb{K}A$-modules U_1 and U_2, the functors $|^g_B \uparrow^B_A$ and $\uparrow^{{}^gB}_{{}^gA} |^g_A$ are naturally isomorphic, the $\mathbb{K}$-spaces $\mathrm{Hom}_{\mathbb{K}A}(U_1, U_2)$ and $\mathrm{Hom}_{\mathbb{K}({}^gA)}({}^gU_1, {}^gU_2)$ are isomorphic, and moreover $|^g_G$ and the identity functor are naturally isomorphic. □

Theorem 4.19 *Let $K \le L \le G$ and let V be a simple $\mathbb{K}\overline{N}_G(K)$-module. Let $M = \downarrow^G_L S^G_{K,V}$.*

(1) *M is a semisimple $\mu_{\mathbb{K}}(L)$-module if and only if ${}^gV$ is a semisimple $\overline{N}_L({}^gK)$-module for every element g of G with ${}^gK \le L$.*

(2) *M is a simple $\mu_{\mathbb{K}}(L)$-module if and only if any element of the set*

$$\{{}^gK : \ {}^gK \le L, \ g \in G\}$$

is an L-conjugate of K and the $\mathbb{K}\overline{N}_L(K)$-module V is simple.

Proof: As a consequence of 4.17, for any $g \in G$ with ${}^gK \le L$ we have

$$\big(\mathrm{Soc}(M)\big)({}^gK) \cong \mathrm{Soc}\big(\downarrow^{\overline{N}_G({}^gK)}_{\overline{N}_L({}^gK)} {}^gV\big).$$

(1) Suppose that M is semisimple. Then $M = \mathrm{Soc}(M)$ so that the socle of the $\overline{N}_L({}^gK)$-module ${}^gV$ is isomorphic to $M({}^gK)$. As $M({}^gK) \cong {}^gV$, the $\overline{N}_L({}^gK)$-module ${}^gV$ must be semisimple. Suppose that ${}^gV$ is semisimple for every g in G with ${}^gK \le L$. Since $M({}^gK) \cong {}^gV$, we must have that

$$\big(\mathrm{Soc}(M)\big)({}^gK) = M({}^gK).$$

It follows by part (2) of 4.6 that M is generated by its values on ${}^gK$ where g ranges over elements of G satisfying ${}^gK \le L$. This shows that $M = \mathrm{Soc}(M)$.

(2) This is clear from 4.17 and from the isomorphism given at the beginning of the proof. □

The following immediate consequence of 4.17 and 4.19 generalizes part (ii) of [Yar2, Corollary 3.5].

Corollary 4.20 *Let $K \le_G L$ be subgroups of G and let V be a simple $\mathbb{K}\overline{N}_G(K)$-module such that $\dim_{\mathbb{K}} V = 1$. Then, the $\mu_{\mathbb{K}}(L)$-module $\downarrow^G_L S^G_{K,V}$ is semisimple and satisfies*

$$\downarrow^G_L S^G_{K,V} \cong \bigoplus_{LgN_G(K) \subseteq G : {}^gK \le L} S^L_{{}^gK, {}^gV}.$$

We now want to obtain an analogous of 4.13 for restrictions of simple functors. It seems that such a result is not an immediate consequences of 4.13, the Mackey decomposition formula, the formula in 2.10, and the adjointness properties of restriction and induction. Instead of using 4.13 we may try to adopt the proof of 4.13. Therefore, given a simple functor M for G and a subgroup L of G, we must find an appropriate idempotent e of $\mu_{\mathbb{K}}(L)$ such that $\downarrow_L^G M$ has no nonzero quotient module annihilated by e and no nonzero submodule annihilated by e. We must also relate the algebra $e\mu_{\mathbb{K}}(L)e$ to some group algebras.

Lemma 4.21 *Let $\mathcal{X}$ be a set of subgroups of G and let*

$$e_{\mathcal{X}} = \sum_{X\in\mathcal{X}} t_X^X.$$

Then we have the direct sum decomposition

$$e_{\mathcal{X}}\mu_{\mathbb{K}}(G)e_{\mathcal{X}} = A_{\mathcal{X}} \oplus I_{\mathcal{X}}$$

where $A_{\mathcal{X}}$ is a unital subalgebra of $e_{\mathcal{X}}\mu_{\mathbb{K}}(G)e_{\mathcal{X}}$ and $I_{\mathcal{X}}$ is a $(A_{\mathcal{X}}, A_{\mathcal{X}})$-bisubmodule of $e_{\mathcal{X}}\mu_{\mathbb{K}}(G)e_{\mathcal{X}}$. The elements of the form

$$t_{{}^gJ}^X c_J^g r_J^Y$$

with X and Y are different elements of $\mathcal{X}$ and the elements of the form

$$t_{{}^gJ}^X c_J^g r_J^X$$

with $X \in \mathcal{X}$ and $J \neq X$ form a $\mathbb{K}$-basis of $I_{\mathcal{X}}$. Moreover we have the following $\mathbb{K}$-algebra isomorphism

$$A_{\mathcal{X}} = \bigoplus_{X\in\mathcal{X}} A_X, \quad A_X = \Big(\bigoplus_{gX\subseteq N_G(X)} \mathbb{K}c_X^g\Big) \cong \mathbb{K}\overline{N}_G(X), \quad c_X^g \leftrightarrow gX,$$

where A_X are two sided ideals of $A_{\mathcal{X}}$ so that the identities $t_X^X = c_X^1$ of the algebras A_X, $X \in \mathcal{X}$, are mutually orthogonal central idempotents of $A_{\mathcal{X}}$ whose sum is equal to the identity $e_{\mathcal{X}}$ of $A_{\mathcal{X}}$. Furthermore, $I_{\mathcal{X}}$ is a two sided ideal of $e_{\mathcal{X}}\mu_{\mathbb{K}}(G)e_{\mathcal{X}}$ if and only if there no elements X and Y of $\mathcal{X}$ with $X < Y$.

Proof: Follows easily by the axioms defining the Mackey algebra and by the basis theorem 2.1. See also 4.1. □

Using the previous result and 4.10, we sometimes can reduce hom spaces of Mackey functors to hom spaces of $A_{\mathcal{X}}$-modules. Moreover, as the algebra direct summands of $A_{\mathcal{X}}$ given in 4.21 are actually two sided ideals of $A_{\mathcal{X}}$, using the next result, hom spaces can be reduced further to direct sums of hom spaces of group algebras.

Remark 4.22 *Let $1 = e_1 + ... + e_n$ be a decomposition of the unity of a finite dimensional $\mathbb{K}$-algebra A into orthogonal central idempotents. Then, for any A-modules V and W,*

$$\mathrm{Hom}_A(V, W) \to \bigoplus_{i=1}^{n} \mathrm{Hom}_{Ae_i}(e_i V, e_i W), \quad f \mapsto \oplus_{i=1}^{n} f|_{e_i V},$$

is a $\mathbb{K}$-space ($\mathbb{K}$-algebra if $V = W$) isomorphism.

Proof: Well-known and easy. □

Theorem 4.23 *Let $K \leq L \leq A \leq G \geq B \geq L$. Let $Y_1, Y_2, ..., Y_n$ be a complete list of representatives of L-orbits (i.e., L-conjugacy classes) of the L-set*

$$\{{}^aK : {}^aK \leq L,\ a \in A\} \bigcap \{{}^bK : {}^bK \leq L,\ b \in B\}$$

on which L acts by conjugation. Suppose that

$$Y_i = {}^{a_i}K = {}^{b_i}K; \quad a_i \in A,\ b_i \in B,\ i = 1, 2, ..., n.$$

Then, for any simple $\mathbb{K}\overline{N}_A(K)$-module W and any simple $\mathbb{K}\overline{N}_B(K)$-module U,

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(L)}\big(\downarrow_L^A S_{K,W}^A, \downarrow_L^B S_{K,U}^B \big) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathbb{K}\overline{N}_L(Y_i)}({}^{a_i}W, {}^{b_i}U)$$

as $\mathbb{K}$-spaces ($\mathbb{K}$-algebras if $B = A$ and $W = U$ and if we choose $b_i = a_i$).

Proof: Let $X_1, X_2, ..., X_m$ be a complete list of representatives of L-orbits (i.e., L-conjugacy classes) of the L-set

$$\{{}^aK : {}^aK \leq L,\ a \in A\} \bigcup \{{}^bK : {}^bK \leq L,\ b \in B\}$$

on which L acts by conjugation. We may assume that

$$\{Y_1, Y_2, ..., Y_n\} \subseteq \{X_1, X_2, ..., X_m\}.$$

We also let $M_1 = \downarrow_L^A S_{K,W}^A$, $M_2 = \downarrow_L^B S_{K,U}^B$, $E = \mu_{\mathbb{K}}(L)$, and $\mathcal{X} = \{X_1, X_2, ..., X_m\}$.

Letting $e_{\mathcal{X}}$ be the idempotent of E defined as in 4.21, it follows by part (2) of 4.6 that the E-module M_1 has no nonzero quotient module annihilated by $e_{\mathcal{X}}$, because any quotient of M_1 must be nonzero at some element of $\mathcal{X}$. And similarly, it follows by part (2) of 4.3 that M_2 has no nonzero E-submodule annihilated by $e_{\mathcal{X}}$. Then 4.10 implies that

$$\mathrm{Hom}_E(M_1, M_1) \cong \mathrm{Hom}_{e_{\mathcal{X}} E e_{\mathcal{X}}}(e_{\mathcal{X}} M_1, e_{\mathcal{X}} M_2).$$

If $M_1(J) \neq 0$ for some subgroup J of L then $S_{K,W}^A(J) \neq 0$ implying that $K \leq_A J$. This shows that the ideal $I_{\mathcal{X}}$ of $e_{\mathcal{X}} E e_{\mathcal{X}}$ given in 4.21 annihilates the $e_{\mathcal{X}} E e_{\mathcal{X}}$-module $e_{\mathcal{X}} M_1$, because if a basis element

$$t_{{}^g J}^X c_J^g r_J^Y$$

of $I_{\mathcal{X}}$ does not annihilate $e_{\mathcal{X}} M_1$ then, as $g \in L$, we must have that $X =_L Y$, which is not the case by the choice of the set $\mathcal{X}$. In a similar way, we see also that $I_{\mathcal{X}}$ annihilates $e_{\mathcal{X}} M_2$. Therefore,

$$\mathrm{Hom}_{e_{\mathcal{X}} E e_{\mathcal{X}}}(e_{\mathcal{X}} M_1, e_{\mathcal{X}} M_2) \cong \mathrm{Hom}_{A_{\mathcal{X}}}(e_{\mathcal{X}} M_1, e_{\mathcal{X}} M_2)$$

where $A_{\mathcal{X}}$ is the subalgebra of $e_{\mathcal{X}} E e_{\mathcal{X}}$ given in 4.21.

The unities $t_X^X = c_X^1$ of the algebras A_X are central idempotents of $A_{\mathcal{X}}$ which are mutually orthogonal. That is,

$$\sum_{X \in \mathcal{X}} t_X^X = e_{\mathcal{X}}$$

is a decomposition of the unity $e_{\mathcal{X}}$ of the algebra $A_{\mathcal{X}}$ into central orthogonal idempotents of $A_{\mathcal{X}}$. Now it follows by 4.22 that

$$\mathrm{Hom}_{A_{\mathcal{X}}}(e_{\mathcal{X}} M_1, e_{\mathcal{X}} M_2) \cong \bigoplus_{X \in \mathcal{X}} \mathrm{Hom}_{A_{\mathcal{X}} t_X^X}\big(M_1(X), M_2(X)\big).$$

Let $X \in \mathcal{X}$. Then $X = {}^g K$ for some $g \in A \cup B$ with ${}^g K \leq L$. If

$$\mathrm{Hom}_{A_{\mathcal{X}} t_X^X}\big(M_1(X), M_2(X)\big) \neq 0$$

then both of $M_1(X)$ and $M_2(X)$ must be nonzero. Thus $S^A_{K,W}({}^gK) \neq 0$ and $S^B_{K,U}({}^gK) \neq 0$. This gives that ${}^gK =_A K$ and ${}^gK =_B K$. Consequently, X must be an L-conjugate of Y_i for some $i \in \{1, 2, ..., n\}$. Thus,

$$\bigoplus_{X \in \mathcal{X}} \mathrm{Hom}_{A_{\mathcal{X}} t^X_X}\big(M_1(X), M_2(X)\big) \cong \bigoplus_{i=1}^{n} \mathrm{Hom}_{A_{\mathcal{X}} t^{Y_i}_{Y_i}}\big(M_1(Y_i), M_2(Y_i)\big).$$

The algebras $A_{\mathcal{X}} t^{Y_i}_{Y_i} = A_{Y_i}$ and $\mathbb{K}\overline{N}_L(Y_i)$ are isomorphic via $c^{y_i}_{Y_i} \mapsto y_i Y_i$ by 4.21. Moreover,

$$M_1(Y_i) = M_1({}^{a_i}K) \cong {}^{a_i}W \quad \text{and} \quad M_2(Y_i) = M_2({}^{b_i}K) \cong {}^{b_i}U.$$

As a result, for each $i \in \{1, 2, ..., n\}$, we have that

$$\mathrm{Hom}_{A_{\mathcal{X}} t^{Y_i}_{Y_i}}\big(M_1(Y_i), M_2(Y_i)\big) \cong \mathrm{Hom}_{\mathbb{K}\overline{N}_L(Y_i)}\big({}^{a_i}W, {}^{b_i}U\big).$$

□

Using 4.23 we can find a criterion about the indecomposibilty of a functor obtained by restricting a simple Mackey functor.

Corollary 4.24 *Let $H \subseteq K$ be subgroups of G and let V be a simple $\mathbb{K}\overline{N}_G(H)$-module. Then, the $\mu_{\mathbb{K}}(K)$-module*

$$\downarrow^G_K S^G_{H,V}$$

is indecomposable if and only if any element of the set

$$\{{}^gH : {}^gH \leq K, \ g \in G\}$$

is a K-conjugate of H and the $\mathbb{K}\overline{N}_K(H)$-module

$$\downarrow^{\overline{N}_G(H)}_{\overline{N}_K(H)} V$$

is indecomposable.

Proof: The $\mu_{\mathbb{K}}(K)$-module $\downarrow^G_K S^G_{H,V}$ is indecomposable if and only if its endomorphism algebra is local. By using the isomorphism of the endomorphism algebras given 4.23 one concludes the result. □

Proposition 4.25 *Let $L \leq A \leq G \geq B \geq L$ and $X \leq A$ and $Y \leq B$. Then, for any simple $\mathbb{K}\overline{N}_A(X)$-module W and any simple $\mathbb{K}\overline{N}_Y(K)$-module U, if*

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(L)}\big(\downarrow_L^A S_{X,W}^A, \downarrow_L^B S_{Y,U}^B \big) \neq 0$$

then $X^a = Y^b \leq L$ for some $a \in A$ and $b \in B$.

Proof: Similar to the proof of 4.16. □

As a consequence of 4.25, hom spaces of restrictions of any simple functors can be related to hom spaces of modules of some group algebras.

Given any simple $\mathbb{K}\overline{N}_K(H)$-module W, we have seen that the socle and head of the $\mu_{\mathbb{K}}(G)$-module

$$M = \uparrow_K^G S_{H,W}^K$$

can be determined by the socle and head of the $\mathbb{K}\overline{N}_G(H)$-module

$$V = \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W.$$

As M may have composition factors with minimal subgroups not G-conjugates of H, we do not expect a connection between (say) the socle series of M and V (except when the socle length of M is 2).

Example 4.26 *Let $\mathbb{K}$ be a field of characteristic 2 and G be a 2-group. Let K be a subgroup of G with $|G : K| = 2$ and let $W = \mathbb{K}$ be the trivial module $\mathbb{K}(G/K)$-module. Put*

$$M = \uparrow_K^G S_{H,W}^K \qquad \text{and} \qquad V = \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} \mathbb{K} \cong \mathbb{K}(G/K).$$

Then:

(1) *The factors of the socle series of (the uniserial $\mathbb{K}(G/K)$-module) V are*

$$\mathbb{K} \qquad \textit{and} \qquad \mathbb{K}.$$

(2) *The factors of the socle series of (the uniserial $\mu_{\mathbb{K}}(G)$-module) M are*

$$S_{K,\mathbb{K}}^G, \qquad S_{G,\mathbb{K}}^G, \qquad \textit{and} \qquad S_{K,\mathbb{K}}^G.$$

Proof: (1) As $|G : K| = 2$ it is clear that V is a uniserial $\mathbb{K}(G/K)$-module factors of whose socle series are $\mathbb{K}$ and $\mathbb{K}$.

(2) Using 4.5 and 4.9 we see by part (1) that M has a unique simple subfunctor S and has a unique maximal subfunctor J where

$$S \cong S^G_{K,\mathbb{K}} \cong M/J.$$

If $M(X) \neq 0$ for a subgroup X of G, then it is clear that $|G : X| \leq 2$. Moreover, it follows by 2.6 that

$$\dim_{\mathbb{K}} M(G) = 1 \quad \text{and} \quad \dim_{\mathbb{K}} M(Y) = 2 \;\; \text{for } Y \leq G \text{ with } |G : Y| = 2.$$

Now $S \leq J$, and using the above dimensions we see that

$$\dim_{\mathbb{K}} J/S = 1 \quad \text{and} \quad J(G) \neq S(G).$$

Hence J/S must be isomorphic to $S^G_{G,\mathbb{K}}$. $\square$

Chapter 5

Clifford's theorem for functors

Almost all the materials in this chapter comes from [Yar1, Section 3]

In this chapter we prove that restriction of a simple functor to a normal subgroup is semisimple and simple summands of it are conjugate, obtaining a Mackey functor version of Clifford's theorem occurring in representation theory of finite groups.

Lemma 5.1 *Let N be a normal subgroup of G and L be a subgroup of G with $N \leq L$. Let $M = S^G_{H,V}$ be a simple $\mu_{\mathbb{K}}(G)$-module such that $H \leq N$. For any $\mathbb{K}\overline{N}_L(H)$-submodule U of $M(H) = V$ and any $g \in G$, we denote by*

$$T^L_{{}^gH, c^g_H(U)}$$

the $\mu_{\mathbb{K}}(L)$-submodule of $\downarrow^G_L M$ generated by $c^g_H(U)$. Then:

(1) *For any $K \leq L$,*

$$T^L_{{}^gH, c^g_H(U)}(K) = \sum_{x \in L : {}^x({}^gH) \leq K} t^K_{{}^{xg}H} c^x_{{}^gH} c^g_H(U).$$

(2)

$$T^L_{{}^gH, c^g_H(U)}({}^gH) = c^g_H(U).$$

(3) *For any* $x \in L$

$$T^L_{{}^gH,c^g_H(U)} = T^L_{{}^{xg}H,c^{xg}_H(U)}.$$

(4) *Any minimal subgroup of* $T^L_{{}^gH,c^g_H(U)}$ *is an* L*-conjugate of* ${}^gH$.

(5) $T^L_{{}^gH,c^g_H(U)}$ *is simple if and only if* U *is simple* $\mathbb{K}\overline{N}_L(H)$*-module.*

Proof: (1) As $T^L_{{}^gH,c^g_H(U)}$ is the $\mu_{\mathbb{K}}(L)$-submodule of $\downarrow^G_L M$ generated by $c^g_H(U)$,

$$T^L_{{}^gH,c^g_H(U)}(K) = t^K_K\mu_{\mathbb{K}}(L)t^{{}^gH}_{{}^gH}c^g_H(U).$$

By the basis theorem 2.1 any element of $t^K_K\mu_{\mathbb{K}}(L)t^{{}^gH}_{{}^gH}$ is a $\mathbb{K}$-linear combination of elements of the form

$$t^K_{{}^xJ}c^x_Jt^{{}^gH}_J$$

where $x \in L$. It follows by 4.3 that if $J \neq {}^gH$ then

$$t^K_{{}^xJ}c^x_Jt^{{}^gH}_J c^g_H(U) = 0.$$

Therefore, $T^L_{{}^gH,c^g_H(U)}(K)$ is the sum of $\mathbb{K}$-spaces of the form

$$t^K_{{}^x({}^gH)}c^x_{{}^gH}t^{{}^gH}_{{}^gH}c^g_H(U),$$

with $x \in L$. So the result follows.

(2) This follows from part (1).

(3) For any $x \in L$, it is obvious that the subsets

$$c^g_H(U) \qquad \text{and} \qquad c^x_{{}^gH}c^g_H(U) = c^{xg}_H(U)$$

of $\downarrow^G_L M$ generate the same L-subfunctor of $\downarrow^G_L M$.

(4) It follows by 4.3 that any minimal subgroup of $T^L_{{}^gH,c^g_H(U)}$ is a G-conjugate of H. Moreover, if ${}^aH$ is a minimal subgroup of $T^L_{{}^gH,c^g_H(U)}$, then

$$T^L_{{}^gH,c^g_H(U)}({}^aH) \neq 0$$

so that part (1) implies that ${}^gH =_L {}^aH$, proving the result.

(5) If $T^L_{{}^gH,c^g_H(U)}$ is simple, then 2.3 implies that U is simple $\mathbb{K}\overline{N}_N(H)$-module. Suppose now U is simple. If S is a nonzero L-subfunctor of $T^L_{{}^gH,c^g_H(U)}$ then S is a nonzero L-subfunctor of $\downarrow^G_L M$, and hence, by 4.3, $S({}^yH) \neq 0$ for some $y \in G$. Then, $S({}^yH)$ is a nonzero submodule of $T^N_{{}^gH,c^g_H(U)}({}^yH)$, implying that the index set

$$\{x \in L : {}^x({}^gH) \leq {}^yH\}$$

of the sum expressing $T^L_{{}^gH,c^g_H(U)}({}^yH)$ is nonempty, and so $xg = yu$ for some $x \in L$ and $u \in N_G(H)$. Then, by part (3), we have

$$T^L_{{}^gH,c^g_H(U)} = T^L_{{}^{xg}H,c^{xg}_H(U)} = T^L_{{}^{yu}H,c^{yu}_H(U)} = T^L_{{}^yH,c^y_H(U)}.$$

Thus, S is a nonzero subfunctor of $T^L_{{}^yH,c^y_H(U)}$, and so $S({}^yH)$ is a nonzero submodule of $c^y_H(U)$. Then simplicity of U implies that $S({}^yH) = c^y_H(U)$. Now,

$$T^L_{{}^yH,c^y_H(U)} =< c^y_H(U) >=< S({}^yH) >$$

implies that

$$T^L_{{}^gH,c^g_H(U)} = T^L_{{}^yH,c^y_H(U)} = S.$$

Hence, $T^L_{{}^gH,c^g_H(U)}$ is simple. □

We state below the main result of this chapter, which is Clifford's theorem for Mackey functors. We state it over a filed, but it is true over any commutative base ring. Of course, restriction of a simple Mackey functor may be 0. Indeed, $\downarrow^G_K S^G_{H,V} \neq 0$ implies that $H \leq_G K$. And note that if $H \leq N \trianglelefteq G$ then

$$\overline{N}_N(H) \trianglelefteq \overline{N}_G(H).$$

Theorem 5.2 (Clifford's theorem for Mackey functors)
Let N be a normal subgroup of G, and let $S^G_{H,V}$ be a simple $\mu_{\mathbb{K}}(G)$-module such that $H \leq N$. Then:

(1) *There is a simple $\mu_{\mathbb{K}}(N)$-submodule of $\downarrow^G_N S^G_{H,V}$ isomorphic to $S^N_{H,W}$ where W is a simple $\mathbb{K}\overline{N}_N(H)$-submodule of V.*

(2) *Let*

$$L = \{g \in G : S^N_{{}^gH, c^g_H(W)} \cong S^N_{H,W}\}$$

be the inertia group of $S^N_{H,W}$*. Then, there is a positive integer* d*, called the ramification index of* $S^G_{H,V}$ *relative to* N*, such that*

$$\downarrow^G_N S^G_{H,V} \cong d \bigoplus_{gL \subseteq G} |^g_N \, S^N_{H,W} \cong d \bigoplus_{gL \subseteq G} S^N_{{}^gH, c^g_H(W)}.$$

Moreover, if

$$\overline{T} = \{\overline{g} \in \overline{N}_G(H) : c^g_H(W) \cong W\}$$

is the inertia group of the $\overline{N}_N(H)$*-module* W *in* $\overline{N}_G(H)$*, then*

$$L = NT \qquad \text{and} \qquad \downarrow^{\overline{N}_G(H)}_{\overline{N}_N(H)} V \cong d \bigoplus_{gT \subseteq N_G(H)} c^g_H(W).$$

Furthermore

$$S^N_{{}^gH, c^g_H(W)}, \qquad for \qquad gL \subseteq G,$$

form, without repetition, a complete set of nonisomorphic G*-conjugates of* $S^N_{H,W}$*. And*

$$c^g_H(W), \qquad for \qquad gT \subseteq N_G(H),$$

form, without repetition, a complete set of nonisomorphic $\overline{N}_G(H)$*-conjugates of* W*.*

(3) $N_L(H) = T$ *and there is a simple* $\mu_{\mathbb{K}}(L)$*-submdule* S *for* L *such that*

$$S \cong S^L_{H,U}$$

where U *is the sum of all* $\mathbb{K}\overline{N}_N(H)$*-submodules of* $\downarrow^{\overline{N}_G(H)}_{\overline{N}_N(H)} V$ *isomorphic to* W*. Moreover,* S *is a simple* L*-subfunctor of* $\downarrow^G_L S^G_{H,V}$ *such that*

$$\downarrow^L_N S \cong dS^N_{H,W} \qquad \text{and} \qquad \uparrow^G_L S \cong S^G_{H,V}.$$

Furthermore U *is a simple* $\mathbb{K}\overline{N}_L(H)$*-submodule of* $\downarrow^{\overline{N}_G(H)}_{\overline{N}_L(H)} V$ *satisfying*

$$\downarrow^{\overline{N}_L(H)}_{\overline{N}_N(H)} U \cong dW \qquad \text{and} \qquad \uparrow^{\overline{N}_G(H)}_{\overline{N}_L(H)} U \cong V.$$

Proof: We begin by explaining a notation we used in the proof. Let $K \leq G$ and $g \in G$. If U is a $\mathbb{K}\overline{N}_G(K)$-module then we use the notation $c_K^g(U)$ to denote the its conjugate ${}^gU$ which is a $\mathbb{K}\overline{N}_G({}^gK)$-module

As V is a simple $\mathbb{K}\overline{N}_G(H)$-module and $\overline{N}_N(H) \trianglelefteq \overline{N}_G(H)$, by Clifford's theorem for group algebras, see for instance [Na], there is a positive integer d, and a simple $\mathbb{K}\overline{N}_N(H)$-submodule W of V such that

$$\downarrow_{\overline{N}_N(H)}^{\overline{N}_G(H)} V \cong d \bigoplus_{\overline{g}\overline{T} \subseteq \overline{N}_G(H)} c_H^g(W) = d \bigoplus_{gT \subseteq N_G(H)} c_H^g(W),$$

where $\overline{T} = \{\overline{g} \in \overline{N}_G(H) : c_H^g(W) \cong W\}$ is the inertia group of the $\overline{N}_N(H)$-module W in $\overline{N}_G(H)$. Moreover $c_H^g(W)$, $gT \subseteq N_G(H)$, form, without repetition, a complete set of nonisomorphic $\overline{N}_G(H)$-conjugates of W. Also, if U is the sum of all $\mathbb{K}\overline{N}_N(H)$-submodules of $\downarrow_{\overline{N}_N(H)}^{\overline{N}_G(H)} V$ isomorphic to W then U is a simple $\mathbb{K}\overline{T}$-module such that

$$\downarrow_{\overline{N}_N(H)}^{\overline{T}} U \cong dW \qquad \text{and} \qquad \uparrow_{\overline{T}}^{\overline{N}_G(H)} U \cong V.$$

For any $x \in G$, it is clear that

$$\downarrow_{\overline{N}_N({}^xH)}^{\overline{N}_G({}^xH)} c_H^x(V) = c_H^x(\downarrow_{\overline{N}_N(H)}^{\overline{N}_G(H)} V) \cong d \bigoplus_{gT \subseteq N_G(H)} c_H^{xg}(W).$$

Firstly, the $\mu_{\mathbb{K}}(N)$-module $\downarrow_N^G S_{H,V}^G$ is semisimple by the virtue of 4.19. Moreover, it follows by 4.17 that any simple $\mu_{\mathbb{K}}(N)$-submodule of $\downarrow_N^G S_{H,V}^G$ is of isomorphic to a functor of the form $S_{{}^aH,U'}^N$ where a is an element of G and U' is a simple $\mathbb{K}\overline{N}_N({}^aH)$-submodule of

$$\downarrow_{\overline{N}_N({}^aH)}^{\overline{N}_G({}^aH)} c_H^a(V) \cong d \bigoplus_{gT \subseteq N_G(H)} c_H^{ag}(W),$$

and conversely, for any $a \in G$ and any $g \in N_G(H)$ the simple $\mu_{\mathbb{K}}(N)$-module $S_{{}^{ag}H,c_H^{ag}(W)}^N$ is isomorphic to a submodule of $\downarrow_N^G S_{H,V}^G$.

Let $a \in G$ and let U' be a simple $\mathbb{K}\overline{N}_N({}^aH)$-module. Then 4.17 implies that the multiplicity of the simple $\mu_{\mathbb{K}}(N)$-module $S_{{}^aH,c_H^a(W)}^N$ in the semisimple

$\mu_{\mathbb{K}}(N)$-module $\downarrow_N^G S_{H,V}^G$ is equal to the multiplicity of $c_H^a(U')$ in

$$\downarrow_{\overline{N}_N({}^aH)}^{\overline{N}_G({}^aH)} c_H^a(V) \cong d \bigoplus_{gT \subseteq N_G(H)} c_H^{ag}(W).$$

Therefore, simple $\mu_{\mathbb{K}}(N)$-submodules of $\downarrow_N^G S_{H,V}^G$ are precisely of the form $S^N_{{}^bH, c_H^b(W)}$ where b ranges in G, moreover each simple summand appears with multiplicity equal to d.

Using 2.5, we see that $S^N_{{}^gH, c_H^g(W)} \cong S^N_{H,W}$ if and only if,

$$\text{for some } \; n \in N, \;\; {}^{ng}H = H \quad \text{ and } \quad c_H^{ng}(W) \cong W,$$

equivalently $g \in NT$. In particular, $L = NT$. Hence, $S^N_{{}^gH, c_H^g(W)}$, $gL \subseteq G$, form, without repetition, a complete set of nonisomorphic G-conjugates of $S^N_{H,W}$.

Now U is a simple $\mathbb{K}\overline{T}$-submodule of $M(H) = V$. If we apply the modular law to the tower $T \leq N_G(H) \leq G \geq N$ we see that

$$N_L(H) = N_G(H) \cap L = N_G(H) \cap TN = T(N_G(H) \cap N) = TN_N(H) = T.$$

As a result, U is a simple $\mathbb{K}\overline{N}_L(H)$-submodule of V. We put

$$S = T^L_{H,U}$$

where $T^L_{H,U}$ is defined as in 5.1. Using 5.1 we see that S is a simple L-subfunctor of $\downarrow_L^G M$ and

$$S \cong S^L_{H,U}.$$

As $\uparrow_{\overline{N}_L(H)}^{\overline{N}_G(H)} U \cong V$ is simple, 4.5 and 4.7 imply that $\uparrow_L^G S^L_{H,U} \cong S^G_{H,V}$.

Finally, it follows by 4.19 that

$$\downarrow_N^L S \cong \downarrow_N^L S^L_{H,U}$$

is a semisimple $\mu_{\mathbb{K}}(N)$-module. Moreover, it is a consequence of 4.17 that any simple $\mu_{\mathbb{K}}(N)$-submodule of $\downarrow_N^L S^L_{H,U}$ is of the form

$$S^N_{{}^lH, W'}$$

for some $l \in L$ and some simple $\mathbb{K}\overline{N}_N({}^lH)$-submodule W' of

$$\downarrow^{\overline{N}_L({}^lH)}_{\overline{N}_N({}^lH)} {}^lU \cong {}^l\big(\downarrow^{\overline{N}_L(H)}_{\overline{N}_N(H)} U\big) \cong {}^l\big(\downarrow^{\overline{T}}_{\overline{N}_N(H)} U\big) \cong d({}^lW).$$

Therefore, any simple $\mu_{\mathbb{K}}(N)$-submodule of $\downarrow^L_N S^L_{H,U}$ is of the form

$$S^N_{{}^lH,{}^lW} \cong S^N_{H,W}.$$

Since

$$\downarrow^{\overline{N}_L(H)}_{\overline{N}_N(H)} U \cong dW,$$

it is clear that

$$\downarrow^L_N S \cong \downarrow^L_N S^L_{H,U} \cong dS^N_{H,W}.$$

□

The socle and the head of the functor M in the example 4.26 are isomorphic. This can also be seen as an immediate consequence of part (3) of 4.12. Indeed, it can also be derived from the next result.

Corollary 5.3 *Let N be a normal subgroup of G and S be a $\mu_{\mathbb{K}}(N)$-module. If* $\mathrm{Soc}(S) \cong S/\mathrm{Jac}(S)$, *then*

$$\mathrm{Soc}(\uparrow^G_N S) \cong (\uparrow^G_N S)/\mathrm{Jac}(\uparrow^G_N S).$$

Proof: Take any simple $\mu_{\mathbb{K}}(G)$-module T. The $\mu_{\mathbb{K}}(N)$-module $\downarrow^G_N T$ is semisimple (if nonzero) by Clifford's theorem for Mackey functors. Using the adjointness of the pairs $(\downarrow^G_N, \uparrow^G_N)$ and $(\uparrow^G_N, \downarrow^G_N)$ we see that:

$$\begin{aligned}
\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(T, \mathrm{Soc}(\uparrow^G_N S)\big) &\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(T, \uparrow^G_N S\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(N)}\big(\downarrow^G_N T, S\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(N)}\big(\downarrow^G_N T, \mathrm{Soc}(S)\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(N)}\big(\mathrm{Soc}(S), \downarrow^G_N T\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(N)}\big(S/\mathrm{Jac}(S), \downarrow^G_N T\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(N)}\big(S, \downarrow^G_N T\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(\uparrow^G_N S, T\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big((\uparrow^G_N S)/\mathrm{Jac}(\uparrow^G_N S), T\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(T, (\uparrow^G_N S)/\mathrm{Jac}(\uparrow^G_N S)\big),
\end{aligned}$$

from which the result follows. □

Chapter 6

Green's theorem for functors

All the materials in this chapter comes from [Yar1, Sections 5 and 6].

We will illustrate in this chapter that Green's indecomposibility theorem in the context of group algebras (or more generally for group graded algebras) has an analogue in the context of Mackey algebras.

An R-algebra A is called strongly G-graded algebra if $A = \bigoplus_{x \in G} A_x$, direct sum of R-submodules of A, and $A_x A_y = A_{xy}$ for all $x, y \in G$; here $A_x A_y$ is the R-submodule of A consisting of all finite sums $\sum_i a_i b_i$ with $a_i \in A_x$ and $b_i \in A_y$. The trivial component A_1 is a unital subring of A. Let $U(A)$ be the set of all units of the algebra A. If $u \in U(A)$ lies in A_x for some $x \in G$ then u is called graded unit and x is called the degree of u, written $\deg(u) = x$. Letting $GrU(A)$ be the set of all graded units of A we see that $GrU(A)$ is a subgroup of $U(A)$ and $\deg : GrU(A) \to G$, $u \mapsto \deg(u)$, is a group homomorphism with kernel $U(A_1)$. If $U(A) \cap A_x$ is nonempty for all $x \in G$ then A is called a crossed product of G over A_1. Let A be a crossed product of G over A_1, choosing $u_x \in U(A) \cap A_x$ for any $x \in G$, we see that $A_x = A_1 u_x = u_x A_1$.

From now on in this chapter, for $K \leq G$ we let 1_K denote the unity of $\mu_R(K)$ which is a nonunital subring of $\mu_R(G)$, if $K \neq G$, and a unital subring of $1_K \mu_R(G) 1_K$. Moreover, for an element g of G and for a normal subgroup N of G

we let

$$\gamma_g = \sum_{L \leq G} c_L^g, \qquad \text{and} \qquad \beta_g = \sum_{L \leq N} c_L^g \in 1_N \mu_R(G) 1_N.$$

Lemma 6.1 *Let N be a normal subgroup of G. Then,*

(1) $\gamma_g \gamma_{g^{-1}} = 1_G$ *and* $\beta_g \beta_{g^{-1}} = 1_N$ *for any* $g \in G$. *In particular,* γ_g *is a unit of* $\mu_{\mathbb{K}}(G)$ *and* β_g *is a unit of* $1_N \mu_{\mathbb{K}}(G) 1_N$.

(2) $\beta_x \mu_R(N) = \beta_y \mu_R(N)$ *if and only if* $xN = yN$.

(3) $\beta_x \mu_R(N) = \mu_R(N) \beta_x$

(4)

$$1_N \mu_R(G) 1_N = \bigoplus_{gN \in G/N} \beta_{\bar{g}} \mu_R(N).$$

Proof: (1) This is obvious.

(2) Noting that $\beta_x 1_N = \beta_x = 1_N \beta_x$ for any $x \in G$, we see that

$$\beta_x \mu_R(N) = \beta_y \mu_R(N)$$

if and only if $\beta_{y^{-1}x} \mu_R(N) = \mu_R(N)$, and so $\beta_{y^{-1}x} = \beta_{y^{-1}x} 1_N \in \mu_R(N)$, implying that $y^{-1}x \in N$.

Conversely, $y^{-1}x \in N$ implies that $\beta_{y^{-1}x}$ is a unit of $\mu_R(N)$. Thus $\beta_{y^{-1}x} \mu_R(N) = \mu_R(N)$.

(3) By 2.1, an R-basis element of $\mu_R(N)$ is of the form

$$t_{nJ}^H c_J^n r_J^K$$

where $H \leq N \geq K$, $n \in N$, and $J \leq H^n \cap K$. For any $x \in G$ we have

$$\beta_x t_{nJ}^H c_J^n r_J^K = c_H^x t_{nJ}^H c_J^n r_J^K = t_{xnx^{-1}(^xJ)}^{^xH} c_{(^xJ)}^{xnx^{-1}} r_{(^xJ)}^{^xK} c_K^x = t_{xnx^{-1}(^xJ)}^{^xH} c_{(^xJ)}^{xnx^{-1}} r_{(^xJ)}^{^xK} \beta_x.$$

Using the normality of N we see that

$$t_{nJ}^H c_J^n r_J^K$$

is an element of $\mu_R(N)$ if and only if

$$t^{^xH}_{xnx^{-1}(^xJ)} c^{xnx^{-1}}_{(^xJ)} r^{^xK}_{(^xJ)}$$

is an element of $\mu_R(N)$. Therefore, $\beta_x \mu_R(N) = \mu_R(N)\beta_x$.

(4) It follows by 2.1 that the elements

$$t^H_{gJ} c^g_J r^K_J,$$

where $H \leq N \geq K$, $HgK \subseteq G$, and J is a subgroup of $H^g \cap K$ up to conjugacy, form, without repetition, a free R-basis of $1_N \mu_R(G) 1_N$. Now $g \in G$ is in a unique coset xN, and if $g = xn$ with $n \in N$ then

$$t^H_{gJ} c^g_J r^K_J = c^x_{H^x} t^{H^x}_{nJ} c^n_J r^K_J = \beta_x t^{H^x}_{nJ} c^n_J r^K_J \in \beta_x \mu_R(N).$$

Hence,

$$1_N \mu_R(G) 1_N = \sum_{gN \in G/N} \beta_{\bar{g}} \mu_R(N).$$

Furthermore, since β_x is a unit of $1_N \mu_R(G) 1_N$ we see that the elements

$$\beta_x t^H_{nJ} c^n_J r^K_J,$$

where $H \leq N \geq K$, $HnK \subseteq N$, and J is a subgroup of $H^n \cap K$ up to conjugacy, form, without repetition, a free R-basis of $\beta_x \mu_R(N)$. If

$$\beta_x t^H_{nJ} c^n_J r^K_J = \beta_y t^{H'}_{mI} c^m_I r^{K'}_I$$

then

$$\beta_{y^{-1}x} t^H_{nJ} c^n_J r^K_J = t^{^{y^{-1}x}H}_{y^{-1}xnJ} c^{y^{-1}xn}_J r^K_J = t^{H'}_{mI} c^m_I r^{K'}_I.$$

Then, by 2.1, we get that $K' = K$, ${}^{y^{-1}x}H = H'$ and $H'mK' = H'y^{-1}xnK'$, implying that $N = y^{-1}xN$. So part (1) implies that $\beta_x \mu_R(N) = \beta_y \mu_R(N)$. Hence, any basis element of $1_N \mu_R(G) 1_N$ lies in a unique summand $\beta_{\bar{x}} \mu_R(N)$. Therefore the sum

$$\sum_{gN \in G/N} \beta_{\bar{g}} \mu_R(N)$$

must be direct. $\square$

Lemma 6.1 implies

Theorem 6.2 *If $N \trianglelefteq G$ then*

$$1_N \mu_R(G) 1_N = \bigoplus_{gN \in G/N} \beta_{\bar{g}} \mu_R(N)$$

is a crossed product of G/N over $\mu_R(N)$.

If $A = \bigoplus_{g \in G} A_g$ is a strongly G-graded algebra and W is an A_1-module, the conjugate of W is defined to be the A_1-module $A_g \otimes_{A_1} W$ with obvious A_1-action. Let $A_1 = \mu_R(N)$ and $A = 1_N \mu_R(G) 1_N$. Then, by 6.2, A is a strongly G/N-graded algebra, and note that the notion of conjugation of A_1-modules described above coincides with the conjugation of $\mu_R(N)$-modules defined in Chapter 2, because if S is a $\mu_{\mathbb{K}}(N)$-module we defined its conjugate $|_N^g S$ in Chapter 2 as $|_N^g S = S$ with $\mu_{\mathbb{K}}(N)$ action given as $x.s = (\gamma_{g^{-1}} x \gamma_g)s$ for $x \in \mu_{\mathbb{K}}(N), s \in S$. On the other hand, we defined its conjugate here as ${}^g S = \beta_{\bar{g}} \mu_{\mathbb{K}}(N) \otimes_{\mu_{\mathbb{K}}(N)} S$. Now it is clear that there is a $\mu_{\mathbb{K}}(N)$- module isomorphism $|_N^g S \rightarrow {}^g S$ given by $s \mapsto \beta_{\bar{g}} \otimes s$.

Proposition 6.3 *Let N be a normal subgroup of G. Given a Mackey functor S for N over R,*

$$\uparrow_N^G S$$

is an indecomposable $\mu_R(G)$-module if and only if

$$1_N \uparrow_N^G S$$

is an indecomposable $1_N \mu_R(G) 1_N$-module.

Proof: Let $A = \mu_{\mathbb{K}}(G)$, $M = \uparrow_N^G S$, and let $e = 1_N$.

We first observe that $AeM = M$ and $(M :_e 0) = 0$: Indeed these are immediate consequences of the adjointness of the pairs $(\uparrow_N^G, \downarrow_N^G)$ and $(\downarrow_N^G, \uparrow_N^G)$.

Now 4.10 implies that the endomorphism algebras $\mathrm{End}_A(M)$ and $\mathrm{End}_{eAe}(eM)$ are isomorphic, from which we may conclude the result because indecomposibilitiy of a (finite) dimensional module is equivalent to the locality of the corresponding endomorphism algebra. $\square$

The following result contains a Mackey functor version of Green's indecomposability theorem.

Theorem 6.4 (Green's indecomposibility criterion for Mackey functors)
Let R be a commutative complete noetherian local ring whose residue field $R/\mathrm{Jac}(R)$ is algebraically closed and is of characteristic $p > 0$, and N be a normal subgroup of G. Let S be a finitely generated indecomposable Mackey functor for N over R, and let L be the inertia group of S. Then, $\uparrow_N^G S$ is an indecomposable Mackey functor for G over R if and only if L/N is a p-group.

Proof: Let $A = 1_N \mu_R(G) 1_N$ and $A_1 = \mu_R(N)$. Then we know that A is a crossed product of G/N over A_1, and in the context of crossed products S is an indecomposable A_1-module whose inertia group is L/N. Then, Green's theorem in the context of group graded algebras implies that $A \otimes_{A_1} S$ is indecomposable if and only if $(L/N)/(N/N)$ is a p-group. Moreover,

$$A \otimes_{A_1} S = 1_N \uparrow_N^G S.$$

Now, it follows by 6.3 that $1_N \uparrow_N^G S$ is indecomposable if and only if $\uparrow_N^G S$ is indecomposable. Hence the result is proved. $\square$

Chapter 7

Maximal subfunctors

Almost all the materials in this chapter comes from [Yar5, Sections 4 and 5].

Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. The purpose of this chapter is to find some relations between the maximal $\mu_{\mathbb{K}}(G)$-submodules of M and the $\mathbb{K}\overline{N}_G(H)$-submodules of the coordinate module $M(H)$ of M. For instance, we construct a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules J of M satisfying $M/J \cong S^G_{H,V}$ for some simple $\mathbb{K}\overline{N}_G(H)$-module V and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules $\overline{I}$ of the Brauer quotient $\overline{M}(H)$ satisfying some certain conditios. Using this bijective correspondence, we show that the multiplicity of a simple $\mu_{\mathbb{K}}(G)$-module $S^G_{H,V}$ in the head of M is equal to the multiplicity of the simple $\mathbb{K}\overline{N}_G(H)$-module V in the head of a certain quotient module of $\overline{M}(H)$.

We begin with recording some properties of the submodules $(V :_e T)$ defined at the beginning of 4.8. Recall that

$$(V :_e T) = \{v \in V : eAv \subseteq T\}$$

where A is an algebra, e is an idempotent of A, V is an A-module, and T is an eAe-submodule of eV.

Lemma 7.1 *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that $W \subseteq V$ are A-modules with $eV \neq 0$ and suppose that*

I, I_1, and I_2 are eAe-submodules of eV. Then:

(1) *$(V :_e I)$ is the largest A-submodule of V subject to the condition $e(V :_e I) = I$. In particular, $W \subseteq (V :_e eW)$.*

(2) *If $I_1 \subseteq I_2$ then $(V :_e I_1) \subseteq (V :_e I_2)$.*

(3) *$(V :_e I_1) \cap (V :_e I_2) = (V :_e I_1 \cap I_2)$.*

(4) *$(V :_e I) \cap W = (W :_e I \cap eW)$.*

(5) *$\big(V/W :_e (I + W)/W\big) = (V :_e I + eW)/W$.*

(6) *$(V \times V' :_e I \times I') = (V :_e I) \times (V' :_e I')$ for any A-module V' and any eAe-submodule I' of eV'.*

(7) *V is a simple A-module if and only if $AeV = V$, $(V :_e 0) = 0$, and eV is a simple eAe-module.*

(8) *Let $(V :_e 0) = 0$. Then, V is a semisimple A-module if and only if $AeV = V$ and eV is a semisimple eAe-module.*

Proof: (1) Let S be an A-submodule of V such that $eS = I$. Take any $s \in S$. Then, $As \subseteq S$ implies that $eAs \subseteq eS = I$; so $s \in (V :_e I)$. Hence, $S \subseteq (V :_e I)$.

(2) This is obvious.

(3) Let $v \in V$ be such that $v \in (V :_e I_i)$ for $i = 1, 2$. Then $eAv \subseteq I_1 \cap I_2$, implying that $v \in (V :_e I_1 \cap I_2)$.

The converse inclusion follows by part (2).

(4) This is clear.

(5) Let $v \in V$.

Suppose that $v + W \in \big(V/W :_e (I + W)/W\big)$. Then, $eA(v+W) \subseteq (I+W)/W$ and so $eAv \subseteq I + W$, implying that

$$eAv = e^2Av \subseteq e(I + W) = eI + eW = I + eW.$$

Thus, $v \in (V :_e I + eW)/W$.

Suppose that $v \in (V :_e I + eW)$. Then,

$$eAv \subseteq I + eW = e(I + W) \subseteq (I + W),$$

and hence $eA(v + W) \subseteq (I + W)/W$. This shows that

$$v + W \in \big(V/W :_e (I + W)/W\big).$$

(6) This is straightforward because A acts on $V \times V'$ diagonally, that is, $a(v, v') = (av, av')$ for $a \in A, v \in V, v' \in V'$.

(7) Suppose that V is a simple A-module. As eV is nonzero, the A-submodules AeV and $(V :_e 0)$ of V are nonzero and proper, respectively. Thus $AeV = V$ and $(V :_e 0) = 0$. The simplicity of the eAe-module eV is well-known (from [Gr2, pp. 83-87] or 4.4). Conversely, suppose V is an A-module satisfying $AeV = V$, $(V :_e 0) = 0$, and eV simple. Let U be a nonzero A-submodule of V. Then it follows from $(V :_e 0) = 0$ that eU is a nonzero eAe-submodule of eV so that $eU = eV$ by the simplicity of eV. Now $AeV = V$ implies that $V = U$. Hence V is a simple A-module.

(8) As $(V :_e 0) = 0$, the A-module V has no nonzero A-submodule annihilated by e so that 4.4 may be applied to see the result. $\square$

Let $\mathcal{X}$ be a set of subgroups of G and M be a $\mu_{\mathbb{K}}(G)$-module. If we put

$$A = \mu_{\mathbb{K}}(G) \qquad \text{and} \qquad e = e_{\mathcal{X}}$$

where the idempotent $e_{\mathcal{X}}$ is defined as in 4.21, then the module

$$(M :_e 0)$$

becomes an already familiar subfunctor of M. Indeed, assuming that $\mathcal{X}$ is closed under taking subgroups and taking G-conjugates, we have

$$\begin{aligned}(M :_e 0) &= \{m = \bigoplus_{H \leq G} m_H \in M : eAm = 0\} \\ &= \bigoplus_{H \leq G} \{m_H \in M(H) : t_X^X \mu_{\mathbb{K}}(G) t_H^H m_H = 0 \ \forall X \in \mathcal{X}\}.\end{aligned}$$

The basis theorem 2.1 and the conditions on $\mathcal{X}$ imply that

$$t_X^X \mu_{\mathbb{K}}(G) t_H^H m_H = 0$$

for all $X \in \mathcal{X}$ if and only if $r_X^H m_H = 0$ for all $X \in \mathcal{X}$ satisfying $X \leq H$. Consequently,

$$(M :_e 0)(H) = \bigcap_{X \in \mathcal{X}: X \leq H} \mathrm{Ker}\big(r_X^H : M(H) \to M(X)\big).$$

Thus $(M :_e 0)$ is the subfunctor $\mathrm{Ker} r_{\mathcal{X}}^M$ of M defined in [TW, Section 3]. This observation shows that part (7) of 7.1 implies the characterization of simple functors in [TW, (3.1) Theorem].

Moreover, for any set $\mathcal{X}$ of subgroups of G and any $\mu_{\mathbb{K}}(G)$-module M, a $\mu_{\mathbb{K}}(G)$-submodule $R_{\mathcal{X}} M$ of M defined in [We2] to be the largest $\mu_{\mathbb{K}}(G)$-submodule of M subject to the condition $r_J^K(R_{\mathcal{X}} M(K)) = 0$ for all $J \in X$ with $J \leq K$. It can be seen easily that $R_{\mathcal{X}} M = (M :_{e_{\mathcal{X}}} 0)$.

For an algebra A and its idempotent e we want to relate the maximal A-submodules of an A-module V to the maximal eAe-submodules of eV. Although we gave such a relation in 4.8, some modules we want to consider may not satisfy the conditions of 4.8. For this reason we next state the following result.

Lemma 7.2 *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that V is a nonzero A-module, J is an A-submodule of V, and I is an eAe-submodule of eV. Then:*

(1) *I is a maximal eAe-submodule of eV if and only if $(V :_e I)$ is a largest element of the set of all A-submodules of V not containing AeV.*

(2) *$(V :_e I)$ is a maximal A-submodule of V if and only if I is a maximal eAe-submodule of eV and $AeV + (V :_e I) = V$.*

(3) *J is a largest element of the set of all A-submodules of V not containing AeV if and only if eJ is a maximal eAe-submodule of eV and $J = (V :_e eJ)$.*

(4) *$J = (V :_e eJ)$ if and only if V/J has no nonzero A-submodule annihilated by e, equivalently $(V/J :_e 0) = 0$.*

(5) *Suppose that J does not contain AeV. Then, J is a maximal A-submodule of V if and only if eJ is a maximal eAe-submodule of eV, $AeV + J = V$, and $(V :_e eJ) = J$.*

Proof: (1) Let I be a maximal eAe-submodule of eV. As I is not equal to eV, the A-module $(V :_e I)$ can not contain AeV. Let W be an A-submodule of V containing $(V :_e I)$ but not containing AeV. Then eW is a proper eAe-submodule of eV containing I. This implies that $eW = I$ because I is a maximal eAe-submodule of eV. Hence $W = (V :_e I)$.

Let $(V :_e I)$ be a largest among all the A-submodules of V not containing AeV. Then I must be a proper eAe-submodule of eV. Let T be a maximal eAe-submodule of eV that contains I. By using 7.1 we see that $(V :_e T)$ contains $(V :_e I)$ but does not contain AeV. Because of the condition on $(V :_e I)$, this implies that $(V :_e T) = (V :_e I)$. Thus $T = I$.

(2) We may assume that I is not equal to eV, because $(V :_e I) = V$ if and only if $I = eV$. Thus, $V/(V :_e I)$ is not annihilated by e so that part (7) of 7.1 is applicable.

$(V :_e I)$ is a maximal A-submodule of V if and only if $V/(V :_e I)$ is a simple A-module. This is equivalent to the conditions:

$$Ae\big(V/(V :_e I)\big) = V/(V :_e I),$$

$$\text{the } eAe - \text{module } e\big(V/(V :_e I)\big) \text{ is simple,}$$

$$\text{and } \big(V/(V :_e I) :_e 0\big) = 0.$$

The result follows by 7.1.

(3) Let J be such a largest element. As J does not contain AeV, the eAe-module eJ is not equal eV. Let I' be a maximal eAe-submodule of eV containing eJ. It follows by part (1) that the A-module $(V :_e I')$ is also a largest element of the

set of all A-submodules of V not containing AeV. This shows that $(V :_e I') = J$ because $(V :_e I')$ contains J. Hence $I' = eJ$ is a maximal eAe-submodule of eV and $J = (V :_e eJ)$. The converse direction follows from the first part of this lemma.

(4) Follows from part (5) of 7.1 which implies that $(V :_e eJ)/J = (V/J :_e 0)$.

(5) Follows from part (7) of 7.1 because the maximality of J is equivalent to simplicity of V/J, which is not annihilated by e. □

From 7.2 the following is immediate.

Proposition 7.3 *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that V is a nonzero A-module. Then:*

(1) *The maps*

$$J \to eJ \qquad \text{and} \qquad (V :_e I) \leftarrow I$$

define a bijective correspondence between the largest elements of the set of all A-submodules of V not containing AeV and the maximal eAe-submodules of eV.

(2) *The maps*

$$J \to eJ \qquad \text{and} \qquad (V :_e I) \leftarrow I$$

define a bijective correspondence between the maximal A-submodules of V that are containing $A(1-e)V$ (so, necessarily not containing AeV) and the maximal eAe-submodules of eV that are containing $eA(1-e)V$.

(3) *The maps*

$$J \to eJ \qquad \text{and} \qquad (V :_e I) \leftarrow I$$

define a bijective correspondence between the maximal A-submodules of V that are not containing AeV and the maximal eAe-submodules of eV that satisfy

$$AeV + (V :_e I) = V.$$

We next need to recall the notion of the Brauer quotient of a Mackey functor, see [Th, TW95, We2]. Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. We put

$$b_H(M) = \sum_{S<H} t_S^H(M(S)).$$

It is clear that $b_H(M)$ is a $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$. The quotient module

$$M/b_H(M)$$

is called the Brauer quotient (or the residue module) of $M(H)$ and denoted by

$$\overline{M}(H).$$

Given any $\mu_{\mathbb{K}}(G)$-module M and any subgroup H of G we will observe in the proof of the next result that if I is a (maximal) $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$ containing $b_H(M)$ then it is also a $t_H^H\mu_{\mathbb{K}}(G)t_H^H$-submodule of $M(H)$ so that the notation $(M :_e I)$ in the next result makes sense (see also part (1) of 7.5).

Theorem 7.4 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $e = t_H^H$. Then, the maps*

$$J \to J(H) \qquad \text{and} \qquad (M :_e I) \leftarrow I$$

define a bijective correspondence between the largest elements of the set of all subfunctors J of M whose quotient functor M/J has H as a minimal subgroup and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules $I/b_H(M)$ of $\overline{M}(H)$. In particular, $\overline{M}(H) = 0$ if and only if M has no quotient functor having H as a minimal subgroup.

Proof: Let $A = \mu_{\mathbb{K}}(G)$, $B = \mathbb{K}\overline{N}_G(H)$, $\mathcal{X} = \{X \le G : X < H\}$, and let the idempotent $f = e_{\mathcal{X}}$ of A be defined as in 4.21. By 4.21 or part (1) of 4.1 we have the direct sum decomposition $eAe = A_H \oplus I_H$ where the algebra A_H can be identified with B via the isomorphism given by $c_H^g \leftrightarrow gH$.

We also define five sets $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ as follows:

$\mathcal{A}$ is the set of all subfunctors of M whose quotient has H as a minimal subgroup,

$\mathcal{B}$ is the set of all A-submodules of M containing AfM but not containing AeM,

$\mathcal{C}$ is the set of all eAe-submodules of eM containing $eAfM$,

$\mathcal{D}$ is the set of all B-submodules of eM containing $eAfM$, and

$\mathcal{E}$ is the set of all $\mathbb{K}\overline{N}_G(H)$-submodules of $M(H)$ containing $b_H(M)$.

We first show that the sets $\mathcal{A}$ and $\mathcal{B}$ are equal: Let J be a subfuncor of M. Then, H is a minimal subgroup of M/J if and only if $(M/J)(X) = 0$ for all $X < H$ and $(M/J)(H) \neq 0$. This is equivalent to the conditions $f(M/J) = 0$ and $e(M/J) \neq 0$. Note that $f(M/J) = 0$ if and only if $AfM \subseteq J$, and that $e(M/J) \neq 0$ if and only if J does not contain AeM. Thus the sets $\mathcal{A}$ and $\mathcal{B}$ are equal.

Let J be an A-submodule of M and I be an eAe-submodule of eM. If J contains AfM then eJ contains $eAfM$, and conversely if I contains $eAfM$ then, by its definition, $(M :_e I)$ contains AfM. Therefore, it follows by part (1) of 7.3 that the maps $J \to eJ$ and $(M :_e I) \leftarrow I$ define a bijective correspondence between the maximal elements of the sets $\mathcal{B}$ and $\mathcal{C}$.

We finish the proof by showing the equality of the sets $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$: By the basis theorem 2.1 it is clear that any element of eAf can be written as a linear combination of the elements of the form

$$t^H_{{}^gA} c^g_A r^X_A$$

where $X < H$ so that ${}^gA < H$. Moreover, it is obvious that t^H_S is in eAf for any $S < H$. Consequently, $eAfM = b_H(M)$. The elements of the form

$$t^H_{{}^xB} c^x_B r^H_B$$

with $B \neq H$ form a $\mathbb{K}$-basis of the two sided ideal I_H of eAe, see 4.1. This shows that $I_HeM = I_HM$ is in $b_H(M)$. Therefore,

$$I_HM \subseteq eAfM = b_H(M) \subseteq eM.$$

By the correspondence theorem, there is a bijection between the eAe-submodules of eM containing $eAfM$ and eAe-submodules of M/I_HM containing $eAfM/I_HM$. As the ideal I_H annihilates the eAe-module M/I_HM and as $eAe = B \oplus I_H$, the eAe-submodules of M/I_HM and the B-submodules of M/I_HM are the same. By another usage of the correspondence theorem, we see that the eAe-submodules of eM containing $eAfM$ and the B-submodules of eM containing $eAfM$ are the same. As $eAfM = b_H(M)$, the sets $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are equal.

□

Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. A consequence of 7.4 is that the number of maximal $\mu_{\mathbb{K}}(G)$-submodules J of M such that

$$M/J \cong S^G_{H,V}$$

for some simple $\mathbb{K}\overline{N}_G(H)$-module V is less than or equal to the number of maximal $\mathbb{K}\overline{N}_G(H)$-submodules of $\overline{M}(H)$. Indeed, by part (2) of 7.2 (or part (3) of 7.3) we see that the maps in 7.4 define a bijection between the maximal $\mu_{\mathbb{K}}(G)$-submodules J of M that satisfies the given condition in 7.4 and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules $I/b_H(M)$ of $\overline{M}(H)$ that satisfies

$$AeM + (M :_e I) = M,$$

where $A = \mu_{\mathbb{K}}(G)$ and $e = t^H_H$.

Lemma 7.5 *Let M be a $\mu_{\mathbb{K}}(G)$-module, H be a subgroup of G, and let $e = t^H_H$. Then:*

(1) *$I_HM \subseteq b_H(M)$ so that $b_H(M)$ is a $e\mu_{\mathbb{K}}(G)e$-submodule of $M(H)$ where I_H is the ideal of $e\mu_{\mathbb{K}}(G)e$ given in 4.1. In particular, any $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$ containing $b_H(M)$ is an $e\mu_{\mathbb{K}}(G)e$-submodule of $M(H)$.*

(2) *If $H \not\leq_G X$ then $\big(M :_e b_H(M)\big)(X) = M(X)$, and if $H \leq_G X$ then*

$$\big(M :_e b_H(M)\big)(X) =$$

$$\{x \in M(X) : c^g_{H^g} r^X_{H^g}(x) \in b_H(M)\ \forall g \in G \text{ with } H^g \leq X\}.$$

(3) *Let $\mathcal{X} = \{X \leq G : H \not\leq_G X\}$ and $I/b_H(M)$ be a $\mathbb{K}\overline{N}_G(H)$-submodule of $\overline{M}(H)$. Then, for any subset $\mathcal{Y}$ of $\mathcal{X}$ containing a G-conjugate of H we have*

$$AeM + (M :_e I) = Ae_{\mathcal{Y}}M + (M :_e I),$$

where $A = \mu_{\mathbb{K}}(G)$ and $e_{\mathcal{Y}}$ is the idempotent of A defined as in 4.21. In particular, the evaluations of the functors

$$AeM + (M :_e I) \qquad and \qquad M$$

at subgroups of G in $\mathcal{X}$ are all equal.

Proof: (1) It is obtained in the proof of 7.4.

(2) As

$$\big(M :_e b_H(M)\big)(X) = \{x \in M(X) : t_H^H \mu_{\mathbb{K}}(G) t_X^X x \subseteq b_H(M)\},$$

by the basis theorem 2.1 we see that $\big(M :_e b_H(M)\big)(X)$ is the set of all elements $x \in M(X)$ satisfying

$$t_{{}^gJ}^H c_J^g r_J^X(x) \in b_H(M)$$

for all $g \in G$ and all $J \leq H^g \cap X$. Note that if ${}^gJ < H$ then this condition is satisfied trivially for all $x \in M(X)$. Thus, the result follows.

(3) Since $\big(M :_e b_H(M)\big) \subseteq (M :_e I)$, it follows by part (2) that

$$(M :_e I)(Y) = M(Y)$$

for all $Y \in \mathcal{Y}$ with $Y \neq_G H$. If $Y =_G H$ then it is clear that

$$(AeM)(Y) = M(Y)$$

(because $e = t_H^H$). Therefore,

$$AeM \subseteq Ae_{\mathcal{Y}}M \subseteq AeM + (M :_e I),$$

from which the result follows. □

Corollary 7.6 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $e = t_H^H$. Then:*

(1) *The maps*

$$J \to J(H) \qquad \text{and} \qquad (M :_e I) \leftarrow I$$

define a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules J of M such that

$$M/J \cong S_{H,V}^G$$

for some simple $\mathbb{K}\overline{N}_G(H)$-module V and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules $I/b_H(M)$ of $\overline{M}(H)$ that satisfies

$$M(X) =$$

$$\sum_{g\in G} t_{X\cap {}^gH}^X\big(M(X\cap {}^gH)\big) + \{x \in M(X) : c_{H^g}^g r_{H^g}^X(x) \in I,\ \forall g \in G,\ H^g \leq X\}$$

for all $X \leq G$ with $H < X$.

(2) *Let M be a semisimple $\mu_{\mathbb{K}}(G)$-module. Then, the maps*

$$J \to J(H) \qquad \text{and} \qquad (M :_e I) \leftarrow I$$

define a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules J of M such that

$$M/J \cong S_{H,V}^G$$

for some simple $\mathbb{K}\overline{N}_G(H)$-module V and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules $I/b_H(M)$ of $\overline{M}(H)$.

Proof: Let $A = \mu_{\mathbb{K}}(G)$, $\mathcal{Y} = \{Y \leq G : Y \leq_G H\}$, and let the idempotent $e' = e_{\mathcal{Y}}$ be defined as in 4.21.

(1) Let $I/b_H(M)$ be a $\mathbb{K}\overline{N}_G(H)$-submodule of $\overline{M}(H)$. It follows by part (3) of 7.5 that the A-modules

$$AeM + (M :_e I) \text{ and } Ae'M + (M :_e I)$$

are equal. Since $\mathcal{Y}$ is closed under taking subgroups and taking G-conjugates, we see easily by using the basis theorem 2.1 that

$$(Ae'M)(X) = \sum_{Y \in \mathcal{Y} : Y \leq X} t_Y^X(M(Y))$$

for any $X \leq G$. Part (3) of 7.5 implies that the evaluations of

$$AeM + (M :_e I) \ \text{ and } \ M$$

at subgroups X of G for which $H \not<_G X$ are all equal. Thus, to justify that

$$AeM + (M :_e I) = M$$

it is enough to see that

$$(Ae'M)(X) + (M :_e I)(X) = M(X)$$

for all X with $H <_G X \leq G$. As the conjugation maps c_X^g of M are $\mathbb{K}$-space isomorphism, it is enough to see the equality of the above evaluations at subgroups X satisfying $H < X \leq G$.

Let $H < X \leq G$. If $Y \in \mathcal{Y}$ with $Y \leq X$ then there is a $g \in G$ such that $Y \leq X \cap {}^gH \in \mathcal{Y}$. By the transitivity property (M_1) of the trace maps on M (see the definition of a Mackey functor given in Chapter 2) we have

$$t_Y^X(M(Y)) \subseteq t_{X \cap {}^gH}^X\big(M(X \cap {}^gH)\big).$$

Therefore,

$$(Ae'M)(X) = \sum_{g \in G} t_{X \cap {}^gH}^X\big(M(X \cap {}^gH)\big).$$

Moreover, since $b_H(M) \subseteq I$, we see as in the proof of part (2) of 7.5 that

$$(M :_e I)(X) = \{x \in M(X) : c_{H^g}^g r_{H^g}^X(x) \in I, \ \forall g \in G, \ H^g \leq X\}.$$

The result now follows by the explanation given at the beginning of 7.5.

(2) By the explanation given at the beginning of 7.5, it suffices to prove that

$$AeM + (M :_e I) = M$$

for any maximal $\mathbb{K}\overline{N}_G(H)$-submodule $I/b_H(M)$ of $\overline{M}(H)$. Indeed, this is true for any (not necessarily maximal) $\mathbb{K}\overline{N}_G(H)$-submodule $I/b_H(M)$. To see this, we first note by part (5) of 7.1 that

$$\big(M/(M :_e I) :_e 0\big) = 0.$$

As M is semisimple, part (8) of 7.1 implies the result. □

The condition on I given in part (1) of 7.6 becomes slightly simpler if we assume that H is normal in G. Using 4.15 we see that the existence of a maximal subfunctor J of M such that

$$M/J \cong S^G_{H,V}$$

for some simple $\mathbb{K}\overline{N}_G(H)$-module V is equivalent to the existence of a maximal subfunctor J' of $\downarrow^G_{N_G(H)} M$ such that

$$(\downarrow^G_{N_G(H)} M)/J' \cong S^{N_G(H)}_{H,V}.$$

Corollary 7.7 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put*

$$M' = \downarrow^G_{N_G(H)} M \qquad \textit{and} \qquad e = t^H_H.$$

Then:

(1) *The maps*

$$J \to \big(M' :_e J(H)\big) \qquad \textit{and} \qquad \big(M :_e J'(H)\big) \leftarrow J'$$

define a bijective correspondence between the largest elements of the set of all subfunctors J of M whose quotient functor M/J has H as a minimal subgroup and the largest elements of the set of all subfunctors J' of M' whose quotient functor M'/J' has H as a minimal subgroup.

(2) *The map*

$$J \to \big(M' :_e J(H)\big)$$

define an injection from the set of all maximal $\mu_{\mathbb{K}}(G)$-submodules J of M such that H is a minimal subgroup of the simple functor M/J to the set of all maximal $\mu_{\mathbb{K}}\big(N_G(H)\big)$-submodules J' of M' such that H is a minimal subgroup of the simple functor M'/J'.

(3) *For any maximal $\mu_{\mathbb{K}}(G)$-submodule J of M such that H is a minimal subgroup of the simple functor M/J, there is a maximal $\mu_{\mathbb{K}}\big(N_G(H)\big)$-submodule J' of M' such that H is a minimal subgroup of the simple functor M'/J' and $J = \big(M :_e J'(H)\big)$.*

Proof: (1) This follows from 7.4 because, for any $H \leq K \leq G$, it follows by the definition of the Brauer quotient that

$$b_H(M) = b_H(\downarrow^G_K M) \text{ and } \overline{M}(H) = \overline{(\downarrow^G_K M)}(H).$$

(2) and (3) Let $A = \mu_{\mathbb{K}}(G)$, $B = \mu_{\mathbb{K}}\big(N_G(H)\big)$, and $L = N_G(H)$. Let J be a maximal A-submodule of M such that H is a minimal subgroup of the simple A-module M/J. Then M/J must be isomorphic to a simple functor of the form $S^G_{H,V}$. Using 4.15 we see that the multiplicity of the simple B-module $S^L_{H,V}$ in the head of

$$\downarrow^G_L (M/J) \cong M'/(\downarrow^G_L J)$$

is nonzero (indeed one) where the isomorphism of the B-modules follows from the exactness of the functor $\downarrow^G_L$. Therefore there is a maximal B-submodule J' of M' containing $\downarrow^G_L J$ such that M'/J' is isomorphic to $S^L_{H,V}$. In particular, $J(H) = J'(H)$. Moreover, part (4) of 7.2 implies that

$$J' = \big(M' :_e J'(H)\big) \quad \text{and} \quad J = \big(M :_e J(H)\big).$$

Using the equality $J(H) = J'(H)$ we obtain that

$$J' = \big(M' :_e J(H)\big) \quad \text{and} \quad J = \big(M :_e J'(H)\big).$$

As $\big(M' :_e J(H)\big)$ is equal to the maximal B-submodule J' of M', part (2) follows. As the maximal B-submodule J' of M' satisfies

$$J = \big(M :_e J'(H)\big),$$

part (3) follows. □

Lemma 7.8 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a normal subgroup of G. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t^H_H$. The following conditions are equivalent for any maximal $\mathbb{K}(G/H)$-submodule $\overline{I} = I/b_H(M)$ of $\overline{M}(H)$:*

(i) *$(M :_e I)$ is a maximal $\mu_{\mathbb{K}}(G)$-submodule of M.*

(ii) *$AeM + (M :_e I) = M$.*

(iii) *$M(X) = t_H^X(M(H)) + \{x \in M(X) : r_H^X(x) \in I\}$ for all $X \leq G$ with $H < X$.*

(iv) *For all $X \leq G$ with $H < X$,*

$$r_H^X(M(X)) \subseteq \big(\sum_{gH\subseteq X} c_H^g\big)M(H) + I$$

(v) *For all $X \leq G$ with $H < X$,*

$$\big(r_H^X(M(X)) + b_H(M)\big)/b_H(M) \subseteq \big(\sum_{gH\subseteq X} c_H^g\big)\overline{M}(H) + \overline{I}$$

(vi) *There is a simple $\mathbb{K}(G/H)$-module U and a nonzero $\alpha \in \mathrm{Hom}_{\mathbb{K}(G/H)}(\overline{M}(H), U)$ with kernel equal to $\overline{I}$ and such that*

$$\alpha \circ \pi_H \circ r_H^X(M(X)) \subseteq \big(\sum_{gH\subseteq X} c_H^g\big)U$$

for all $X \leq G$ with $H < X$, where $\pi_H : M(H) \to M(H)/b_H(M)$ is the natural epimorphism.

Proof: (i), (ii), and (iii) are equal: Follows from 7.2 and 7.6.

(iv) equals to (v): Clear.

(iii) implies (iv): Take any $x \in M(X)$. Then $x = t_H^X(a)+b$ for some $a \in M(H)$ and $b \in M(X)$ with $r_H^X(b) \in I$. By the Mackey axiom

$$r_H^X(x) = r_H^X t_H^X(a) + r_H^X(b) = \big(\sum_{gH\subseteq X} c_H^g\big)a + r_H^X(b) \in \big(\sum_{gH\subseteq X} c_H^g\big)M(H) + I.$$

(iv) implies (iii): Take any $x \in M(X)$. Then, there is a $u \in M(H)$ and $v \in I$ such that

$$r_H^X(x) = \big(\sum_{gH\subseteq X} c_H^g\big)u + v.$$

By the Mackey axiom

$$r_H^X(x) = r_H^X t_H^X(u) + v, \quad \text{implying that} \quad r_X^H\big(x - t_H^X(u)\big) = v \in I.$$

Consequently,

$$x = t_H^X(u) + \big(x - t_H^X(u)\big) \in t_H^X(M(H)) + \{x \in M(X) : r_H^X(x) \in I\}.$$

(v) implies (vi): Put $U = \overline{M}(H)/\overline{I}$ and let $\alpha : \overline{M}(H) \to U$ be the natural surjection. Then, U is a simple $\mathbb{K}(G/H)$-module and α is a (nonzero) $\mathbb{K}(G/H)$-module epimorphism with kernel equal to $\overline{I}$. Moreover, using (v) we have:

$$\begin{aligned} \alpha \circ \pi_H \circ r_H^X(M(X)) &= \alpha\Big(\big(r_H^X(M(X)) + b_H(M)\big)/b_H(M)\Big) \\ &\subseteq \big(\sum_{gH \subseteq X} c_H^g\big)U + \alpha(\overline{I}) \\ &= \big(\sum_{gH \subseteq X} c_H^g\big)U. \end{aligned}$$

(vi) implies (iv): Take $x \in M(X)$. As a result of (vi) there is a $y \in M(H)$ such that

$$\alpha\big(r_H^X(x) + b_H(M)\big) = \alpha \circ \pi_H \circ r_H^X(x) = \big(\sum_{gH \subseteq X} c_H^g\big)\alpha\big(y + b_H(M)\big).$$

This shows that

$$r_H^X(x) - \big(\sum_{gH \subseteq X} c_H^g\big)y + b_H(M) \in \mathrm{Ker}\alpha = \overline{I},$$

implying

$$r_H^X(x) \in \big(\sum_{gH \subseteq X} c_H^g\big)M(H) + I.$$

□

Corollary 7.9 [TW95, (15.4) Proposition] *Let M be a $\mu_{\mathbb{K}}(G)$-module, H be a subgroup of G, and U be a simple $\mathbb{K}\overline{N}_G(H)$-module. Then,* $\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(M, S_{H,U}^G) \neq 0$ *if and only if there is a nonzero $\alpha \in \mathrm{Hom}_{\mathbb{K}\overline{N}_G(H)}(\overline{M}(H), U)$ such that*

$$\alpha \circ \pi_H \circ r_H^X(M(X)) \subseteq \big(\sum_{gH \subseteq X} c_H^g\big)U$$

for all $X \leq G$ *with* $H < X \leq N_G(H)$*, where* $\pi_H : M(H) \to M(H)/b_H(M)$ *is the natural epimorphism.*

Proof: By 4.15 we may assume that H is normal in G, because

$$\overline{M}(H) = \overline{(\downarrow_K^G M)}(H)$$

for any $H \leq K \leq G$. Put $e = t_H^H$.

Suppose that $\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(M, S_{H,U}^G) \neq 0$. There is a maximal subfunctor J of M such that $M/J \cong S_{H,U}^G$. Moreover,

$$U \cong M(H)/I \cong \overline{M}(H)/\overline{I}$$

as $\mathbb{K}(G/H)$-modules, where $I = J(H)$. It follows by 7.6 that $J = (M :_e I)$ and that $\overline{I}$ is a maximal $\mathbb{K}(G/H)$-submodule of $\overline{M}(H)$ satisfying the equivalent conditions (in particular (vi)) of 7.8. Thus there is a simple $\mathbb{K}(G/H)$-module U' and a (nonzero) $\mathbb{K}(G/H)$-module epimorphism $\alpha' : \overline{M}(H) \to U'$ with kernel equal to $\overline{I}$ so that $U \cong U'$, and such that

$$\alpha' \circ \pi_H \circ r_H^X(M(X)) \subseteq \Big(\sum_{gH \subseteq X} c_H^g\Big)U'.$$

Let $f : U' \to U$ be a $\mathbb{K}(G/H)$-module isomorphism. Put $\alpha = f \circ \alpha'$ which is a nonzero element of $\mathrm{Hom}_{\mathbb{K}(G/H)}(\overline{M}(H), U)$. Now,

$$\begin{aligned} \alpha \circ \pi_H \circ r_H^X(M(X)) &= f \circ \alpha' \circ \pi_H \circ r_H^X(M(X)) \\ &\subseteq \Big(\sum_{gH \subseteq X} c_H^g\Big) f(U') \\ &= \Big(\sum_{gH \subseteq X} c_H^g\Big) U. \end{aligned}$$

Conversely, assume that there is a nonzero $\alpha \in \mathrm{Hom}_{\mathbb{K}(G/H)}(\overline{M}(H), U)$ satisfying the required conditions. Letting $\overline{I} = \mathrm{Ker}\alpha$, we see that $\overline{I}$ is a maximal $\mathbb{K}(G/H)$-submodule of $\overline{M}(H)$ satisfying the condition (vi) of 7.8 and such that $\overline{M}(H)/I \cong U$. Thus $J = (M :_e I)$ is a maximal $\mu_{\mathbb{K}}(G)$-submodule of M, and H is a minimal subgroup of M/J, and $J(H) = I$ so that $M/J \cong S_{H,U}^G$. □

Given a $\mu_{\mathbb{K}}(G)$-module M and a subgroup of H of G, we want to find a quotient module of the $\mathbb{K}\overline{N}_G(H)$-module $\overline{M}(H)$ such that the multiplicity of a simple $\mathbb{K}\overline{N}_G(H)$-module V in the head of it is equal to the multiplicity of the simple $\mu_{\mathbb{K}}(G)$-module $S_{H,V}^G$ in the head of M. For this end we first need some trivial remarks.

Remark 7.10 *Let V be a finite dimensional $\mathbb{K}$-vector space. For any $\mathbb{K}$-subspaces A, B, and W of V :*

(1) *Let $\mathcal{A} = \{f \in \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}) : f(W) = 0\}$. Then*

$$W = \bigcap_{f \in \mathcal{A}} \mathrm{Ker}(f : V \to \mathbb{K})$$

where $\mathrm{Ker}(f : V \to \mathbb{K})$ denotes the kernel of f.

(2) *Let $\mathcal{B} = \{f \in \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}) : f(B) = 0 \implies f(A) = 0\}$. Then, $A \subseteq B + W$ if and only if*

$$\bigcap_{f \in \mathcal{B}} \mathrm{Ker}(f : V \to \mathbb{K}) \subseteq W.$$

Proof: (1) Let

$$v \in \bigcap_{f \in \mathcal{A}} \mathrm{Ker} f.$$

Write $V = W \oplus W'$ as $\mathbb{K}$-spaces for some subspace W' of V. Then, there are elements $w \in W$ and $w' \in W'$ such that $v = w + w'$. If $w' \neq 0$ then we can find an f in $\mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ such that $f(W) = 0$ and $f(w') = 1$, implying that

$$0 = f(v) = f(w) + f(w') = 1.$$

Thus, $w' = 0$ so that $v \in W$. This proves that

$$\bigcap_{f \in \mathcal{A}} \mathrm{Ker} f \subseteq W.$$

The reverse inclusion is clear.

(2) Suppose that $A \subseteq B + W$. Then,

$$f(A) \subseteq f(B) + f(W) = f(B)$$

for any $f \in \mathcal{A}$. This shows that $\mathcal{A} \subseteq \mathcal{B}$. Thus,

$$\bigcap_{f \in \mathcal{B}} \mathrm{Ker} f \subseteq \bigcap_{f \in \mathcal{A}} \mathrm{Ker} f = W$$

where the last equality follows from the first part.

Conversely, suppose that

$$\bigcap_{f \in \mathcal{B}} \mathrm{Ker} f \subseteq W.$$

Take any $a \in A$. Write $V = (B + W) \oplus C$ as $\mathbb{K}$-spaces for some subspace C of V. Then, there are elements $u \in (B + W)$ and $r \in C$ such that $a = u + r$. Assume for a moment that there is an $f_0 \in \mathcal{B}$ such that $f_0(r) \neq 0$. As the codimension of f_0 is 1, we must have that

$$f_0(B) \subseteq f_0(B + W) = 0.$$

So $f_0(A) = 0$ because $f_0 \in \mathcal{B}$. But now,

$$0 = f_0(a) = f_0(u) + f_0(r) = f_0(r) \neq 0.$$

Therefore, $f(r) = 0$ for all $f \in \mathcal{B}$ implying that $r \in W$ so that

$$a = u + r \in (B + W).$$

□

Let M be a $\mu_{\mathbb{K}}(G)$-module. For a restriction map r^Y_X on M, it may not be true that $r^Y_X(b_Y(M)) \subseteq b_X(M)$. So, in general, r^Y_X does not induce a well defined map from $\overline{M}(Y)$ to $\overline{M}(X)$. However, we will use the notations $r^X_H t^X_H(\overline{M}(H))$ and $r^X_H(\overline{M}(X))$ in some of our later results to indicate the subspaces

$$\big(r^X_H t^X_H(M(H)) + b_H(M)\big)/b_H(M) \quad \text{and} \quad \big(r^X_H(M(X)) + b_H(M)\big)/b_H(M)$$

of $\overline{M}(H)$, respectively.

Lemma 7.11 *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module and H is a normal subgroup of G. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t^H_H$. The following conditions are equivalent for any maximal $\mathbb{K}(G/H)$-submodule $\overline{I} = I/b_H(M)$ of $\overline{M}(H)$:*

(i) *$(M :_e I)$ is a maximal $\mu_{\mathbb{K}}(G)$-submodule of M.*

(ii) *$M(X) = t_H^X(M(H)) + \{x \in M(X) : r_H^X(x) \in I\}$ for any nontrivial p-subgroup X/H of G/H.*

(iii)
$$\bigcap_f \mathrm{Ker}(f : \overline{M}(H) \to \mathbb{K}) \subseteq \overline{I}$$
for any nontrivial p-subgroup X/H of G/H, where f ranges over all elements of the set

$$\{f \in \mathrm{Hom}_{\mathbb{K}}(\overline{M}(H), \mathbb{K}) : f\big(r_H^X t_H^X(\overline{M}(H)\big) = 0 \implies f\big(r_H^X(\overline{M}(X))\big) = 0\}.$$

Proof: (i) equals to (ii): By the virtue of 7.8, it suffices to show that part (ii) of the present result implies the part (ii) of 7.8. Let Y/H be any nontrivial subgroup of G/H. Take any $y \in M(Y)$. We need to show that

$$y \in t_H^Y(M(H)) + \{y \in M(Y) : r_H^Y(y) \in I\}.$$

Let X/H be a Sylow p-subgroup of Y/H and let $n = 1/|Y : X|$. As X/H is a (nontrivial) p-subgroup of G/H,

$$M(X) = t_H^X(M(H)) + \{x \in M(X) : r_H^X(x) \in I\}$$

so that
$$r_X^Y(y) = t_H^X(a) + b$$
for some $a \in M(H)$ and $b \in M(X)$ with $r_H^X(b) \in I$. (Note also that, for $X = H$, such a decomposition of $r_X^Y(y)$ holds trivially, in which $b = 0$). Now we can write

$$y = t_H^Y(na) + \big(y - t_H^Y(na)\big).$$

Thus, we may finish the proof by indicating that

$$r_H^Y\big(y - t_H^Y(na)\big) \in I.$$

Indeed, by using the axioms in the definition of a Mackey functor,

$$\begin{aligned} r_H^Y\big(y - t_H^Y(na)\big) &= r_H^Y(y) - r_H^Y t_H^Y(na) \\ &= r_H^Y(y) - \sum_{gX\subseteq Y} c_H^g r_H^X t_H^X(na) \\ &= n\Big(|Y:X| r_H^Y(y) - \sum_{gX\subseteq Y} c_H^g r_H^X t_H^X(a)\Big) \\ &= n\Big(\sum_{gX\subseteq Y} c_H^g r_H^Y(y) - \sum_{gX\subseteq Y} c_H^g r_H^X t_H^X(a)\Big) \\ &= n\sum_{gX\subseteq Y} c_H^g r_H^X\big(r_X^Y(y) - t_H^X(a)\big) \\ &= n\sum_{gX\subseteq Y} c_H^g r_H^X(b) \in I, \end{aligned}$$

as desired.

(ii) equals to (iii): Part (2) of 7.10 implies that (iii) equals to the condition

$$r_H^X(\overline{M}(X) \subseteq r_H^X t_H^X(\overline{M}(H) + \overline{I}$$

where X is any subgroup satisfying the required condition. From the Mackey axiom,

$$r_H^X t_H^X = \sum_{gH\subseteq X} c_H^g,$$

implying by (the proof) of 7.8 that the above containment relation is equivalent to (ii). □

Proposition 7.12 *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module and H is a subgroup of G.*

(1) *The map*

$$J \to J(H)$$

define an injection from the set of all maximal $\mu_{\mathbb{K}}(G)$-submodules J of M such that H is a minimal subgroup of the simple functor M/J to the set of all maximal $\mathbb{K}\overline{N}_G(H)$-submodules $I/b_H(M)$ of $\overline{M}(H)$ satisfying the following condition for any nontrivial p-subgroup X/H of $N_G(H)/H$:

$$M(X) = t_H^X(M(H)) + \{x \in M(X) : r_H^X(x) \in I\}.$$

(2) *For any maximal $\mu_{\mathbb{K}}(G)$-submodule J of M such that H is a minimal subgroup of the simple functor M/J, there is a maximal $\mathbb{K}\overline{N}_G(H)$-submodule $I/b_H(M)$ of $\overline{M}(H)$ satisfying the condition given in the first part such that $J = (M :_e I)$ where $e = t_H^H$.*

Proof: Follows by 7.7, 7.6, 7.8, and 7.11. □

Remark 7.13 *Let A be a finite dimensional algebra, V be a finite dimensional A-module, and e be a nonzero idempotent of A.*

(1) *Let $V_1, V_2, ..., V_n$ be A-submodules of V. For each i, we put*

$$\widetilde{V_i} = \bigcap_{j=1:j\neq i}^{n} V_j.$$

For the map

$$\psi : V \to \prod_{i=1}^{n} V/V_i, \quad v \mapsto \prod_{i=1}^{n} v + V_i$$

we have:

(i) *If $V_i + \widetilde{V_i} = V$ for each i then ψ is surjective.*

(ii) *Suppose further that each V_i is a maximal A-submodule of V. If ψ is surjective then $V_i + \widetilde{V_i} = V$ for each i.*

(2) *Let $I_1, I_2, ..., I_n$ be maximal eAe-submodules of eV. Suppose that each $(V :_e I_i)$ is a maximal A-submodule of V. If the product of natural epimorphisms*

$$eV \to \prod_{i=1}^{n} eV/I_i$$

is surjective then the product of natural epimorphisms

$$V \to \prod_{i=1}^{n} V/(V :_e I_i)$$

is surjective.

Proof: (1)(i) Take any elements $v_1, v_2, ..., v_n$ of V. As $V_i + \widetilde{V}_i = V$ for each i, we may find $u_i \in \widetilde{V}_i$ satisfying $u_i + V_i = v_i + V_i$. Let

$$v_0 = u_1 + u_2 + ... + u_n.$$

By the definition of $\widetilde{V}_i$ we see that $u_i \in \widetilde{V}_i \subseteq V_k$ for any k with $k \neq i$. Hence,

$$v_0 + V_i = u_i + V_i = v_i + V_i.$$

Consequently, ψ is surjective.

(1)(ii) For each i, let

$$\widetilde{\psi}_i : V \to \prod_{j=1:j\neq i}^{n} V/V_j$$

be the product of natural epimorphisms. Note that $\widetilde{V}_i$ is equal to the kernel the map $\widetilde{\psi}_i$. If there is an s such that $V_s + \widetilde{V}_s \neq V$ then by the maximality of V_s we see that $\widetilde{V}_s \subseteq V_s$. Then, the kernel of ψ is equal to $\widetilde{V}_s \cap V_s = \widetilde{V}_s$. Therefore, there must be an A-module monomorphism from the image

$$\prod_{i=1}^{n} V/V_i$$

of ψ to

$$\prod_{i=1:i\neq s}^{n} V/V_i.$$

This is impossible, because V is finite dimensional and $V/V_s \neq 0$.

(2) For each i we use the notation V_i to denote $(V :_e I_i)$. If the product of natural epimorphisms

$$V \to \prod_{i=1}^{n} V/(V :_e I_i)$$

is not surjective, then by part (1)(i) there is a j such that $V_j + \widetilde{V}_j \neq V$ where $\widetilde{V}_j$ is defined as in the first part. By the maximality of V_j we obtain that $\widetilde{V}_j \subseteq V_j$. Multiplying this containment by the idempotent e, we get that

$$I_j = eV_j \supseteq e\widetilde{V}_j = e(V :_e \widetilde{I}_j) = \widetilde{I}_j$$

where the equalities follows from 7.1 and $\widetilde{I}_j$ is defined as in the first part. On the other hand, using part (1)(ii) we see that the containment $I_j \subseteq \widetilde{I}_j$ contradicts the surjectivity of the product of the natural epimorphisms

$$eV \to \prod_{i=1}^{n} eV/I_i.$$

$\square$

Theorem 7.14 *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module. Then, the multiplicity of $S^G_{H,U}$ in*

$$M/\mathrm{Jac}(M)$$

is equal to the multiplicity of U in the head of the following quotient module of the $\mathbb{K}\overline{N}_G(H)$-module $\overline{M}(H)$:

$$\overline{M}(H)\Big/\sum_{X/H}\Big(\bigcap_{f\in\mathcal{A}_X}\mathrm{Ker}(f:\overline{M}(H)\to\mathbb{K})\Big)$$

where X/H ranges over all nontrivial p-subgroups of $N_G(H)/H$, and for each X,

$$\mathcal{A}_X = \{f \in \mathrm{Hom}_{\mathbb{K}}(\overline{M}(H),\mathbb{K}) : f\big(r^X_H t^X_H(\overline{M}(H)\big) = 0 \implies f\big(r^X_H(\overline{M}(X))\big) = 0\}.$$

Proof: Let

$$V_X = \bigcap_{f\in\mathcal{A}_X}\mathrm{Ker}f \qquad \text{and} \qquad \mathcal{M}_H = \sum_{X/H} V_X$$

where X/H ranges over all nontrivial p-subgroups of $N_G(H)/H$.

We first note that $\mathcal{M}_H$ is a $\mathbb{K}\overline{N}_G(H)$-submodule $\overline{M}(H)$: As conjugation maps of a Mackey functor are $\mathbb{K}$-space isomorphism, it is clear for any $K \le G$ and $a \in G$ that c^a_K induces a $\mathbb{K}$-space isomorphism $\overline{M}(K) \to \overline{M}({}^gK)$. Moreover, for any $g \in N_G(H)$, it can be seen by the definition of a Mackey functor that $f \in \mathcal{A}_X$ if and only if $f \circ c^g_H \in \mathcal{A}_{X^g}$. So, $c^g_H(V_X) \subseteq V_{{}^gX}$, proving that $\mathcal{M}_H$ is a $\mathbb{K}\overline{N}_G(H)$-module.

As $\overline{M}(H) = \overline{(\downarrow^G_K M)}(H)$ for any $H \le K \le G$, it follows by 4.15 that we may (and will do) assume that H is normal in G. Let n be the multiplicity of $S^G_{H,U}$ in the head of M, and let m be the multiplicity of U in the head of $\overline{M}(H)/\mathcal{M}_H$.

Let $A = \mu_{\mathbb{K}}(G)$, $B = \mathbb{K}\overline{N}_G(H)$, and $e = t_H^H$. There are n maximal A-submodules J_1, J_2,...,J_n of M such that all of the quotients M/J_i are isomorphic to $S_{H,U}^G$ and that the product of natural epimorphisms

$$M \to \prod_{i=1}^{n} M/J_i$$

is surjective. By 7.6, 7.8 and 7.11 we know that each $J_i(H)/b_H(M)$ is a maximal B-submodule of $\overline{M}(H)$ containing $\mathcal{M}_H$. As the multiplication by the idempotent e is an exact functor (from A-mod to eAe-mod), we see that it induces a surjective eAe-module homomorphism

$$\overline{M}(H) \to \prod_{i=1}^{n} M(H)/J_i(H)$$

with kernel containing $\mathcal{M}_H$. The last surjection induces a surjective B-module homomorphism

$$\overline{M}(H)/\mathcal{M}_H \to nU,$$

because B is a unital subalgebra of eAe and each B-module $M(H)/J_i(H)$ is isomorphic to U. This shows that $n \leq m$.

Conversely, there are m maximal B-submodules $T_1, T_2, ..., T_m$ of $\overline{M}(H)$ containing $\mathcal{M}_H$ such that each B-module $\overline{M}(H)/T_i$ is isomorphic to U and that the product of all natural epimorphisms

$$\overline{M}(H) \to \prod_{i=1}^{m} \overline{M}(H)/T_i$$

is surjective. By the correspondence theorem, there are maximal B-submodules $\overline{I_i} = I_i/b_H(M)$ of $\overline{M}(H)$ such that $T_i = \overline{I_i}/\mathcal{M}_H$. Using the canonical isomorphisms

$$\overline{M}(H)/T_i \cong M(H)/I_i,$$

we see that the product of the natural epimorphisms

$$M(H) \to \prod_{i=1}^{m} M(H)/I_i$$

is surjective. By part (1) of 7.5, each I_i is a maximal eAe-submodule of $eM = M(H)$. Moreover, using 7.11 we see that each $(M :_e I_i)$ is a maximal A-submodule of M. So, we may apply part (2) of 7.13 to deduce that the product of natural epimorphism

$$M \to \prod_{i=1}^{m} M/(M :_e I_i)$$

is surjective. This shows that $m \leq n$, because

$$M/(M :_e I_i) \cong S^G_{H,U}$$

for each i. □

Corollary 7.15 *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module.*

(1) *There is a maximal subfunctor of M whose quotient has H as a minimal subgroup if and only if there is a maximal $\mathbb{K}\overline{N}_G(H)$-submodule $I/b_H(M)$ of $\overline{M}(H)$ satisfying the following condition for any nontrivial p-subgroup X/H of $N_G(H)/H$:*

$$M(X) = t^X_H(M(H)) + \{x \in M(X) : r^X_H(x) \in I\}.$$

(2) *The multiplicity of $S^G_{H,U}$ in $M/\mathrm{Jac}(M)$ is less than or equal to the multiplicity of U in $\overline{M}(H)\big/\mathrm{Jac}\big(\overline{M}(H)\big)$.*

(3) *The multiplicity of $S^G_{H,U}$ in $M/\mathrm{Jac}(M)$ is greater than or equal to the multiplicity of U in the head of the following $\mathbb{K}\overline{N}_G(H)$-module:*

$$\overline{M}(H)\Big/\sum_{H<X\leq N_G(H):|X:H|=p} r^X_H\big(\overline{M}(X)\big).$$

(4) *Suppose that $\overline{N}_G(H)$ is a p'-group. Then, the multiplicity of $S^G_{H,U}$ in $M/\mathrm{Jac}(M)$ is equal to the multiplicity of U in $\overline{M}(H)$.*

Proof: (1) It follows by (the proof of) 7.14 and 7.11.

(2) and (4) They are immediate from 7.14.

(3) We use the notations $\mathcal{A}_X$, V_X, and $\mathcal{M}_H$ defined in 7.14 and its proof. For any nontrivial p-subgroup X/H of $N_G(H)/H$, part (2) of 7.10 implies that $V_X \subseteq r_H^X(\overline{M}(X))$. Therefore,

$$\mathcal{M}_H \subseteq \sum_{X/H} r_H^X(\overline{M}(X))$$

where X/H ranges over all nontrivial p-subgroup X/H of $N_G(H)/H$. From the transitivity of restriction maps on M, we see that

$$r_H^X(M(X)) = r_H^Y r_Y^X(M(X)) \subseteq r_H^Y(M(Y)),$$

implying that

$$r_H^X(\overline{M}(X)) \subseteq r_H^Y(\overline{M}(Y))$$

for any subgroup Y/H of X/H of order p. Therefore,

$$\mathcal{M}_H \subseteq \mathcal{N}_H, \quad \text{where } \mathcal{N}_H = \sum_{H<X\leq N_G(H):|X:H|=p} r_H^X\big(\overline{M}(X)\big).$$

This proves that $\overline{M}(H)/\mathcal{N}_H$ is isomorphic to a quotient module of $\overline{M}(H)/\mathcal{M}_H$. The result now follows from 7.14. □

Proposition 7.16 *Let $\mathbb{K}$ be of characteristic $p>0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module. If all the elements of $\overline{N}_G(H)$ of order p acts on U trivially, then the multiplicity of $S_{H,U}^G$ in*

$$M/\mathrm{Jac}(M)$$

is equal to the multiplicity of U in the head of the following $\mathbb{K}\overline{N}_G(H)$-module:

$$\overline{M}(H)\Big/\sum_{H<X\leq N_G(H):|X:H|=p} r_H^X\big(\overline{M}(X)\big).$$

Proof: We use the notations $\mathcal{A}_X$, V_X, $\mathcal{M}_H$ and $\mathcal{N}_H$ defined in 7.14 and 7.15 and their proofs. By 7.14 the multiplicity of $S_{H,U}^G$ in the head of M is equal to the multiplicity of U in the head of $\overline{M}(H)/\mathcal{M}_H$. Let $\varphi : \overline{M}(H) \to U$ be any (nonzero) $\mathbb{K}\overline{N}_G(H)$-module homomorphism whose kernel contains $\mathcal{M}_H$. As

$$\mathcal{M}_H = \sum_{X/H} V_X$$

where X/H ranges over all nontrivial p-subgroups of $N_G(H)/H$, it follows that $V_X \subseteq \mathrm{Ker}\varphi$, in particular, for any subgroup X/H of $N_G(H)/H$ of order p. Using part (2) of 7.10 we see that the containment $V_X \subseteq \mathrm{Ker}\varphi$ is equivalent to the condition

$$r^X_H(\overline{M}(X)) \subseteq r^X_H t^X_H(\overline{M}(H)) + \mathrm{Ker}\varphi.$$

We will show that $r^X_H(\overline{M}(X)) \subseteq \mathrm{Ker}\varphi$: It follows by the assumption on U that

$$\varphi\big(r^X_H t^X_H(\overline{M}(H))\big) = \varphi\big(\sum_{gH\subseteq X} c^g_H(\overline{M}(H))\big) = \sum_{gH\subseteq X}(gH)U = |X:H|U = 0.$$

Therefore,

$$r^X_H(\overline{M}(X)) \subseteq r^X_H t^X_H(\overline{M}(H)) + \mathrm{Ker}\varphi = \mathrm{Ker}\varphi.$$

Since this is true for any subgroup X/H of $N_G(H)/H$ of order p, we obtain that $\mathcal{N}_H \subseteq \mathrm{Ker}\varphi$. Consequently, the $\mathbb{K}$-spaces

$$\mathrm{Hom}_B(\overline{M}(H)/\mathcal{M}_H, U) \qquad \text{and} \qquad \mathrm{Hom}_B(\overline{M}(H)/\mathcal{N}_H, U)$$

must be isomorphic where $B = \mathbb{K}\overline{N}_G(H)$. This finishes the proof. □

If $\overline{N}_G(H)$ is a nilpotent group (or more generally, a group with normal Sylow p-subgroup), then (Clifford's theorem implies that) the hypothesis of 8.15 is satisfied for any simple $\mathbb{K}\overline{N}_G(H)$-module U. For another example, the hypothesis of 8.15 is satisfied for any group G and for any simple $\mathbb{K}\overline{N}_G(H)$-module U with $\dim_{\mathbb{K}} U = 1$.

We finish this chapter by giving some conditions on a $\mu_{\mathbb{K}}(G)$-module M equivalent to the condition $\overline{M}(H) \neq 0$ where H is a subgroup of G.

Remark 7.17 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Then the following conditions are equivalent:*

(i) $\overline{M}(H) \neq 0$.

(ii) *M has a quotient functor having H as a minimal subgroup.*

(iii) *$\uparrow^G_H\downarrow^G_H M$ has a simple quotient having H as a minimal subgroup.*

(iv) $\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(M, \uparrow^G_H S^H_{H,\mathbb{K}}) \neq 0$.

Proof: (i) equals to (ii): Follows by 7.4.

(ii) implies (iii): Let J be a subfunctor of M such that M/J has H as a minimal subgroup. It is clear that

$$\downarrow_H^G (M/J) \cong nS_{H,\mathbb{K}}^H$$

where $n = \dim_{\mathbb{K}}(M/J)(H)$. Therefore, it follows by the exactness of the functors $\uparrow$ and $\downarrow$ that $\uparrow_H^G S_{H,\mathbb{K}}^H$ is an epimorphic image of $\uparrow_H^G\downarrow_H^G M$. By 4.6 we know that the minimal subgroups of any nonzero quotient functor of $\uparrow_H^G S_{H,\mathbb{K}}^H$ is a G-conjugate of H. As a result, $\uparrow_H^G S_{H,\mathbb{K}}^H$ and hence $\uparrow_H^G\downarrow_H^G M$ has a simple quotient having H as a minimal subgroup.

(iii) implies (iv): Suppose that $\uparrow_H^G\downarrow_H^G M$ has a simple quotient having H as a minimal subgroup. Then there is a simple $\mathbb{K}\overline{N}_G(H)$-module V such that

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(\uparrow_H^G\downarrow_H^G M, S_{H,V}^G) \neq 0.$$

It is clear that $\downarrow_H^G S_{H,V}^G \cong nS_{H,\mathbb{K}}^H$ where $n = \dim_{\mathbb{K}} V$. Using the adjointness of the pairs $(\uparrow_H^G, \downarrow_H^G)$ and $(\downarrow_H^G, \uparrow_H^G)$ we see that

$$\begin{aligned}
0 &\neq \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(\uparrow_H^G\downarrow_H^G M, S_{H,V}^G)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}(\downarrow_H^G M, \downarrow_H^G S_{H,V}^G)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}(\downarrow_H^G M, nS_{H,\mathbb{K}}^H)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(M, \uparrow_H^G nS_{H,\mathbb{K}}^H)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(M, n\uparrow_H^G S_{H,\mathbb{K}}^H)\\
&\cong n\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(M, \uparrow_H^G S_{H,\mathbb{K}}^H)
\end{aligned}$$

(iv) implies (i): Firstly, using the adjointness of the pairs

$$(\downarrow_H^G, \uparrow_H^G) \quad \text{and} \quad (L^+{}_{H/H}, \mathrm{Inf}_{H/H}^H)$$

and using the obvious isomorphisms

$$S_{H,\mathbb{K}}^H \cong \mathrm{Inf}_{H/H}^H S_{H/H,\mathbb{K}}^{H/H}, \quad \mu_{\mathbb{K}}(H/H) \cong \mathbb{K}, \quad S_{H/H,\mathbb{K}}^{H/H} \cong \mathbb{K},$$

$$L^+{}_{H/H} \downarrow_H^G M = (L^+{}_{H/H} \downarrow_H^G M)(H/H) = \overline{M}(H),$$

we obtain that

$$\begin{aligned}
\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(M, \uparrow_H^G S_{H,\mathbb{K}}^H) &\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}(\downarrow_H^G M, S_{H,\mathbb{K}}^H) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}(\downarrow_H^G M, \mathrm{Inf}_{H/H}^H S_{H/H,\mathbb{K}}^{H/H}) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H/H)}(L^+{}_{H/H} \downarrow_H^G M, S_{H/H,\mathbb{K}}^{H/H}) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(\overline{M}(H), \mathbb{K}).
\end{aligned}$$

□

Chapter 8

Minimal subfunctors

Almost all the materials in this chapter comes from [Yar5, Section 5].

Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. This chapter deals with the simple $\mu_{\mathbb{K}}(G)$-submodules of M and the $\mathbb{K}\overline{N}_G(H)$-submodules of the coordinate module $M(H)$ of M. We want to obtain results similar to the ones obtained in the previous chapter. For example, we show that if $\mathbb{K}$ is of characteristic $p > 0$ and U is a simple $\mathbb{K}\overline{N}_G(H)$-module, then the multiplicity of the simple $\mu_{\mathbb{K}}(G)$-module $S^G_{H,U}$ in the socle of a $\mu_{\mathbb{K}}(G)$-module M is equal to the multiplicity of the simple $\mathbb{K}\overline{N}_G(H)$-module U in the socle of the following $\mathbb{K}\overline{N}_G(H)$-submodule of the restriction kernel $\underline{M}(H)$:

$$\bigcap_{X/H} \{x \in \underline{M}(H) : \big(\sum_{gH \subseteq X} c^g_H\big)x = 0 \implies t^X_H(x) = 0\}$$

where X/H ranges over all nontrivial p-subgroups of $N_G(H)/H$.

We begin with a general easy lemma.

Lemma 8.1 *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that V is a nonzero A-module, S is an A-submodule of V, and T is an eAe-submodule of eV. Then:*

(1) *T is a simple eAe-submodule of eV if and only if AT is a smallest element* *of the set of all A-submodules of V not contained in $(V :_e 0)$.*

(2) *AT is a simple A-submodule of V if and only if T is a simple eAe-submodule of eV and $(AT :_e 0) = 0$.*

(3) *S is a smallest element of the set of all A-submodules of V not contained in $(V :_e 0)$ if and only if eS is a simple eAe-submodule of eV and $S = AeS$.*

(4) *$S = AeS$ if and only if S has no nonzero quotient module annihilated by e.*

Proof: (1) Let T be a simple eAe-submodule of eV. We want to show that AT is a smallest element of the set of all A-submodules of V not contained in $(V :_e 0)$:

As T is nonzero, AT is not contained in $(V :_e 0)$. Let W be an A-submodule of V contained in V but not contained in $(V :_e 0)$. Then eW is a nonzero eAe-submodule of T. This implies that $eW = T$ because T is simple. Hence, $AT \subseteq W$ implying that $W = AT$.

Let AT be a smallest element of the set of all A-submodules of V not contained in $(V :_e 0)$. We want to show that T is a simple eAe-submodule of eV :

As $eAT = T$ and as $AT \nsubseteq (V :_e 0)$, it is clear that T is nonzero. Let T' be a simple eAe-submodule of eV that is contained in T. Then, we get by what we have shown above that AT' is not contained in $(V :_e 0)$. As AT' is contained in AT, we conclude by using the condition on AT that $AT' = AT$. Thus $T' = T$.

(2) We may assume that $T \neq 0$, because $T = 0$ if and only if $AT = 0$.

As $T \neq 0$, the idempotent e does not annihilate AT so that we may apply part (7) of 7.1. Therefore, AT is a simple A-module if and only if $Ae(AT) = AT$, $(AT :_e 0) = 0$, and $e(AT) = T$ is a simple eAe-module.

(3) Let S be a smallest element of the set of all A-submodules of V not contained in $(V :_e 0)$. As S is not contained in $(V :_e 0)$, the eAe-module eS is nonzero. Let T' be a simple eAe-submodule of eV contained in eS. It follows by part (1) that the A-module AT' is also a smallest element of the set of all A-submodules of V not contained in $(V :_e 0)$. This shows that $AT' = S$ because AT'

is in S. Hence, $eS = T'$ is a simple eAe-submodule of eV and $S = AT' = AeS$.

The converse direction follows by part (1).

(4) Let S' be an A-submodule of S. If $AeS \subseteq S'$ then multiplying the containment $AeS \subseteq S' \subseteq S$ by the idempotent e we obtain that $eS' = Se$ or $e(S/S') = 0$. Conversely, if $e(S/S') = 0$ then $eS = eS'$ so that $AeS = AeS' \subseteq S'$. Hence we seen that S/S' is annihilated by e if and only if $AeS \subseteq S'$. The result is clear now. □

From 8.1 the following is immediate.

Proposition 8.2 *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Suppose that V is a nonzero A-module. Then:*

(1) *The maps*

$$S \to eS \qquad \textit{and} \qquad AT \leftarrow T$$

define a bijective correspondence between the smallest elements of the set of all A-submodules of V not contained in $(V :_e 0)$ and the simple eAe-submodules of eV.

(2) *The maps*

$$S \to eS \qquad \textit{and} \qquad AT \leftarrow T$$

define a bijective correspondence between the simple A-submodules of V that are contained in $(V :_{1-e} 0)$ (so, necessarily not contained in $(V :_e 0)$) and the simple eAe-submodules of eV that are contained in $e(V :_{1-e} 0)$.

(3) *The maps*

$$S \to eS \qquad \textit{and} \qquad AT \leftarrow T$$

define a bijective correspondence between the simple A-submodules of V that are not contained in $(V :_e 0)$ and the simple eAe-submodules of eV that satisfy $(AT :_e 0) = 0$.

We now need to recall the notion of the restriction kernel of a Mackey functor, see [Th, We2]. Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. By the restriction kernel of M at H we mean the following module

$$\underline{M}(H) = \bigcap_{J<H} \mathrm{Ker}\big(r^H_J : M(H) \to M(J)\big).$$

It is clear that $\underline{M}(H)$ is a $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$. Moreover, there is a $\mathbb{K}\overline{N}_G(H)$-module isomorphism

$$\underline{M}(H) \cong \big(\overline{(M^*)}(H)\big)^*$$

obtained by taking $\mathbb{K}$-duals, see [We2]. Thus, every result concerning Brauer quotients has a dual result concerning restriction kernels. In this section we obtain these dual results and refine them. However we will not make use of this duality property here.

Lemma 8.3 *Let M be a $\mu_{\mathbb{K}}(G)$-module, H be a subgroup of G, and T be a $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t^H_H$. Then:*

(1) *The ideal I_H of eAe defined in 4.1 annihilates $\underline{M}(H)$ so that $\underline{M}(H)$ is also an eAe-submodule of $M(H)$ whose $\mathbb{K}\overline{N}_G(H)$-submodules and eAe-submodules are the same.*

(2) *Let $\mathcal{X}$ be a set of subgroups of G. If*

$$\{X \le G : X < H\} \subseteq \mathcal{X} \subseteq \{X \le G : H \not\le_G X\},$$

then $(M :_{e_{\mathcal{X}}} 0)(H) = \underline{M}(H)$, where $e_{\mathcal{X}}$ is the idempotent of A defined as in 4.21.

(3) *If $H \not\le_G X$ then $(AT)(X) = 0$, and if $H \le_G X$ then*

$$(AT)(X) = \sum_{g\in G:\, {}^gH \le X} t^X_{{}^gH} c^g_H(T).$$

(4) *If $H \not<_G X$ then $(AT :_e 0)(X) = 0$, and if $H <_G X$ then*

$$(AT :_e 0)(X) =$$

$$\Big(\sum_{g\in G:\, {}^gH \le X} t^X_{{}^gH} c^g_H(T)\Big) \bigcap \Big(\bigcap_{g\in G:\, {}^gH \le X} \mathrm{Ker}\big(r^X_{{}^gH} : M(X) \to M({}^gH)\big)\Big).$$

Proof: (1) Follows from 4.1 because any element of I_H is a linear combination of elements of the form $t^H_{{}^gJ}c^g_J r^H_J$ with $J \neq H$.

(2) By its definition

$$(M :_{e_{\mathcal{X}}} 0)(H) = \{x \in M(H) : t^X_X \mu_{\mathbb{K}}(G) t^H_H x = 0, \forall X \in \mathcal{X}\}.$$

The basis theorem 2.1 implies the result. Because, $J < H$ for any basis element

$$t^X_{{}^gJ}c^g_J r^H_J$$

of $t^X_X \mu_{\mathbb{K}}(G) t^H_H$, and because if $J < H$ then $J \in \mathcal{X}$ so that r^H_J is in $t^X_X \mu_{\mathbb{K}}(G) t^H_H$ for some $X \in \mathcal{X}$.

(3) It is clear that

$$(AT)(X) = t^X_X \mu_{\mathbb{K}}(G) t^H_H T.$$

As $T \subseteq \underline{M}(H)$, if $J < H$ then r^H_J annihilates T. The result follows by the basis theorem 2.1.

(4) Part (3) implies that if $H \not\leq_G X$ then

$$(AT :_e 0)(X) \subseteq (AT)(X) = 0.$$

Moreover,

$$(AT :_e 0)(H) = e(AT :_e 0) = 0.$$

So we now assume that $H <_G X$. For any $g \in G$ and any $J \leq H^g \cap X$, we see that if $x \in (AT)(X)$ then

$$t^H_{{}^gJ}c^g_J r^X_J x \in t^H_{{}^gJ}\big((AT)({}^gJ)\big) = 0$$

in the case ${}^gJ \neq H$. Thus, as the conjugation maps $c^g_{H^g}$ of M are bijections, from the basis theorem 2.1 we obtain

$$\begin{aligned}
(AT :_e 0)(X) &= \{x \in (AT)(X) : t^H_H \mu_{\mathbb{K}}(G) t^X_X x = 0\} \\
&= \{x \in (AT)(X) : c^g_{H^g} r^X_{H^g}(x) = 0,\ \forall g \in G,\ H^g \leq X\} \\
&= \{x \in (AT)(X) : r^X_{H^g}(x) = 0,\ \forall g \in G,\ H^g \leq X\} \\
&= (AT)(X) \bigcap \{x \in M(X) : r^X_{H^g}(x) = 0,\ \forall g \in G,\ H^g \leq X\}.
\end{aligned}$$

The result now follows from part (3). □

Theorem 8.4 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $e = t_H^H$ and $A = \mu_{\mathbb{K}}(G)$. Then, the maps*

$$S \to S(H) \qquad and \qquad AT \leftarrow T$$

define a bijective correspondence between the smallest elements of the set of all subfunctors S of M having H as a minimal subgroup and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $\underline{M}(H)$. In particular, $\underline{M}(H) = 0$ if and only if M has no subfunctor having H as a minimal subgroup.

Proof: We argue as in the proof of 7.4. Let

$$B = \mathbb{K}\overline{N}_G(H), \quad \mathcal{X} = \{X \leq G : X < H\}, \quad \text{and} \quad f = e_{\mathcal{X}}$$

be the idempotent of A defined as in 4.21.

We also define four sets $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ as follows:

$\mathcal{A}$ is the set of all subfunctors of M having H as a minimal subgroup,

$\mathcal{B}$ is the set of all A-submodules of M contained in $(M :_f 0)$ but not contained in $(M :_e 0)$,

$\mathcal{C}$ is the set of all eAe-submodules of eM contained in $e(M :_f 0)$, and

$\mathcal{D}$ is the set of all B-submodules of $\underline{M}(H)$.

It is easy to see that the sets $\mathcal{A}$ and $\mathcal{B}$ are equal. Moreover, it follows by 8.3 that the sets $\mathcal{C}$ and $\mathcal{D}$ are equal. Because, 8.3 implies that $e(M :_f 0) = \underline{M}(H)$ and that I_H annihilates $\underline{M}(H)$.

Now the result follows from part (1) of 8.2 which shows that the maps $S \to eS$ and $AT \leftarrow T$ define a bijective correspondence between the minimal elements of the sets $\mathcal{B}$ and $\mathcal{C}$. □

Given a $\mu_{\mathbb{K}}(G)$-module M and a subgroup H of G, the previous result implies that the number of simple $\mu_{\mathbb{K}}(G)$-submodules S of M isomorphic to $S_{H,V}^G$ for some simple $\mathbb{K}\overline{N}_G(H)$-module V is less than or equal to the number of simple

$\mathbb{K}\overline{N}_G(H)$-submodules of $\underline{M}(H)$. Indeed, we see by part (3) of 8.2 that the maps in 8.4 define a bijection between the simple $\mu_{\mathbb{K}}(G)$-submodules S of M having H as a minimal subgroup and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $\underline{M}(H)$ that satisfies $(AT :_e 0) = 0$, where $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$.

Remark 8.5 *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. If V is a semisimple A-module, then $(AT :_e 0) = 0$ for any eAe-submodule T of eV.*

Proof: As V is semisimple, $(AT :_e 0) \oplus W = AT$ for some A-submodule W of AT. Multiplying both sides with e we get $eW = T$, implying that

$$AT = AeW \subseteq W.$$

Hence, $(AT :_e 0) = 0$. □

Corollary 8.6 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$. Then:*

(1) *The maps*

$$S \to S(H) \qquad \text{and} \qquad AT \leftarrow T$$

define a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules S of M isomorphic to $S_{H,V}^G$ for some simple $\mathbb{K}\overline{N}_G(H)$-module V and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $\underline{M}(H)$ that satisfies

$$0 = \Big(\sum_{g\in G:{}^gH\le X} t_{{}^gH}^X c_H^g(T)\Big) \bigcap \Big(\bigcap_{g\in G:{}^gH\le X} \operatorname{Ker}\big(r_{{}^gH}^X : M(X) \to M({}^gH)\big)\Big)$$

for all $X \le G$ with $H < X$.

(2) *Let M be a semisimple $\mu_{\mathbb{K}}(G)$-module. Then, the maps*

$$S \to S(H) \qquad \text{and} \qquad AT \leftarrow T$$

define a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules S of M isomorphic to $S_{H,V}^G$ for some simple $\mathbb{K}\overline{N}_G(H)$-module V and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $\underline{M}(H)$.

Proof: Follows from 8.3, 8.5, and from the explanation given at the beginning of 8.5. □

Corollary 8.7 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put*

$$M' = \downarrow^G_{N_G(H)} M, \quad e = t^H_H, \quad A = \mu_{\mathbb{K}}(G), \quad \textit{and} \quad B = \mu_{\mathbb{K}}\big(N_G(H)\big).$$

Then:

(1) *The maps*

$$S \to BeS \qquad \textit{and} \qquad AeS' \leftarrow S'$$

define a bijective correspondence between the smallest elements of the set of all subfunctors S of M having H as a minimal subgroup and the smallest elements of the set of all subfunctors S' of M' having H as a minimal subgroup.

(2) *The map*

$$S \to BeS$$

define an injection from the set of all simple $\mu_{\mathbb{K}}(G)$-submodules S of M such that H is a minimal subgroup of S to the set of all simple $\mu_{\mathbb{K}}\big(N_G(H)\big)$-submodules S' of M' such that H is a minimal subgroup of S'.

(3) *For any simple $\mu_{\mathbb{K}}(G)$-submodule S of M such that H is a minimal subgroup of S, there is a simple $\mu_{\mathbb{K}}\big(N_G(H)\big)$-submodules S' of M' such that H is a minimal subgroup of S' and $S = AeS'$.*

Proof: (1) This can be deduced by arguing as in the proof of part (1) of 7.7.

(2) and (3) Let $K = N_G(H)$. Let S be a simple A-submodule of M having H as a minimal subgroup. S must be isomorphic to a simple functor of the form $S^G_{H,V}$. Using 4.15 we see that there is a simple B-submodule S' of

$$\downarrow^G_K S \subseteq \downarrow^G_K M$$

isomorphic to $S^K_{H,V}$. In particular, $eS = eS' \neq 0$. Moreover,

$$S = AeS \qquad \text{and} \qquad S' = BeS'$$

by the simplicity of the A-module S and the B-module S'. Using the equality $eS = eS'$, we obtain that $S = AeS'$ (proving part (3)) and that $S' = BeS$ (proving part (2)). □

The condition on T given in part (1) of 8.6 becomes simpler if we assume that H is normal in G. The results 8.1 and 8.6, and the Mackey axiom imply the following.

Lemma 8.8 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a normal subgroup of G. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t^H_H$. The following conditions are equivalent for any simple $\mathbb{K}(G/H)$-submodule T of $\underline{M}(H)$:*

(i) *AT is a simple $\mu_{\mathbb{K}}(G)$-submodule of M.*

(ii) $(AT :_e 0) = 0$.

(iii) $t^X_H(T) \cap \mathrm{Ker}\big(r^X_H : M(X) \to M(H)\big) = 0$ *for all $X \leq G$ with $H < X$.*

(iv) *For all $X \leq G$ with $H < X$,*

$$x \in T, \quad \big(\sum_{gH \subseteq X} c^g_H\big)x = 0 \quad \textit{implies} \quad t^X_H(x) = 0.$$

(v) *There is a simple $\mathbb{K}(G/H)$-module U and a nonzero $\beta \in \mathrm{Hom}_{\mathbb{K}(G/H)}(U, \underline{M}(H))$ with image equal to T and such that*

$$\{u \in U : \big(\sum_{gH \subseteq X} c^g_H\big)u = 0\} \subseteq \mathrm{Ker}\big(t^X_H \circ \iota_H \circ \beta\big)$$

for all $X \leq G$ with $H < X$, where $\iota_H : \underline{M}(H) \to M(H)$ is the inclusion.

A justification similar to the proof of 7.9 can be given for the following result.

Corollary 8.9 *Let M be a $\mu_{\mathbb{K}}(G)$-module, H be a subgroup of G, and U be a simple $\mathbb{K}\overline{N}_G(H)$-module. Then, $\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(S^G_{H,U}, M) \neq 0$ if and only if there is a nonzero element β of $\mathrm{Hom}_{\mathbb{K}\overline{N}_G(H)}(U, \underline{M}(H))$ such that*

$$\{u \in U : \big(\sum_{gH \subseteq X} c^g_H\big)u = 0\} \subseteq \mathrm{Ker}\big(t^X_H \circ \iota_H \circ \beta\big)$$

for all $X \leq G$ *with* $H < X \leq N_G(H)$, *where* $\iota_H : \underline{M}(H) \to M(H)$ *is the inclusion.*

Proof: By 4.15 we may assume that H is normal in G, because

$$\underline{M}(H) = \underline{(\downarrow_K^G M)}(H)$$

for any $H \leq K \leq G$.

Suppose that $\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(S_{H,U}^G, M) \neq 0$. There is a simple subfunctor S of M such that $S \cong S_{H,U}^G$. Moreover, $U \cong S(H)$ as $\mathbb{K}(G/H)$-modules. It follows by 8.6 that $S = AT$ and that T is a simple $\mathbb{K}(G/H)$-submodule of $\underline{M}(H)$ satisfying the equivalent conditions (in particular (v)) of 8.8. Thus there is a simple $\mathbb{K}(G/H)$-module U' and a (nonzero) $\mathbb{K}(G/H)$-module monomorphism $\beta' : U' \to \underline{M}(H)$ with image equal to T so that $U \cong U'$, and such that

$$\{u' \in U' : \big(\sum_{gH \subseteq X} c_H^g\big)u' = 0\} \subseteq \mathrm{Ker}\big(t_H^X \circ \iota_H \circ \beta'\big).$$

Let $f : U \to U'$ be a $\mathbb{K}(G/H)$-module isomorphism. Put $\beta = \beta' \circ f$ which is a nonzero element of $\mathrm{Hom}_{\mathbb{K}(G/H)}(U, \underline{M}(H))$.

Let $u \in U$ be such that

$$\big(\sum_{gH \subseteq X} c_H^g\big)u = 0.$$

We want to show that $u \in \mathrm{Ker}\big(t_H^X \circ \iota_H \circ \beta\big)$. As $f : U \to U'$ be a $\mathbb{K}(G/H)$-module isomorphism,

$$f(u) \in \{u' \in U' : \big(\sum_{gH \subseteq X} c_H^g\big)u' = 0\} \subseteq \mathrm{Ker}\big(t_H^X \circ \iota_H \circ \beta'\big).$$

Thus,

$$0 = t_H^X \circ \iota_H \circ \beta'(f(u)) = t_H^X \circ \iota_H \circ \beta' \circ f(u) = t_H^X \circ \iota_H \circ \beta(u).$$

Conversely, assume that there is a nonzero $\beta \in \mathrm{Hom}_{\mathbb{K}(G/H)}(U, \underline{M}(H))$ satisfying the required conditions. Letting T be the image of β, we see that T is a simple $\mathbb{K}(G/H)$-submodule of $\underline{M}(H)$ satisfying the condition (v) of 8.8 and such that

$T \cong U$. Thus $S = AT$ is a simple $\mu_{\mathbb{K}}(G)$-submodule of M, and H is a minimal subgroup of S, and $S(H) \cong U$ so that $S \cong S^G_{H,U}$. $\square$

The result 8.8 contains some equivalent conditions to be checked for all $X \leq G$ with $H < X$. We next observe that we do not need to check them for all such X.

Lemma 8.10 *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module and H is a normal subgroup of G. The following conditions are equivalent for any simple $\mathbb{K}(G/H)$-submodule T of $\underline{M}(H)$:*

(i) *The $\mu_{\mathbb{K}}(G)$-submodule of M generated by T is simple.*

(ii) *For any nontrivial p-subgroup X/H of G/H,*

$$x \in T, \quad \Big(\sum_{gH \subseteq X} c^g_H\Big)x = 0 \quad \textit{implies} \quad t^X_H(x) = 0.$$

Proof: The condition (i) is equivalent to the condition (iv) of 8.8. So, it suffices to see that part (ii) of the present result implies the part (iv) of 8.8. Let $Y \leq G$ with $H < Y$, and let $y \in T$ satisfying

$$\Big(\sum_{gH \subseteq Y} c^g_H\Big)y = 0.$$

We need to show that $t^Y_H(y) = 0$. Let X/H be a Sylow p-subgroup of Y/H. Using the the axioms in the definition of a Mackey functor we see that

$$0 = \Big(\sum_{gH \subseteq Y} c^g_H\Big)y = r^Y_H t^Y_H(y) = r^X_H r^Y_X t^Y_H(y) = \sum_{Xg \subseteq Y} r^X_H t^X_H c^g_H(y) = \Big(\sum_{gH \subseteq X} c^g_H\Big)x,$$

where

$$x = \sum_{Xg \subseteq Y} c^g_H(y) \in T.$$

As X/H is a (nontrivial) p-subgroup of G/H, we must have that $0 = t^X_H(x)$, which implies

$$0 = t^Y_H(x) = |Y : X| t^Y_H(y).$$

This gives that $t^Y_H(y) = 0$ because $|Y : X|$ is not divisible by p. $\square$

Proposition 8.11 *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module and H is a subgroup of G.*

(1) *The map*

$$S \to S(H)$$

define an injection from the set of all simple $\mu_{\mathbb{K}}(G)$-submodules S of M having H as a minimal subgroup to the set of all simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $\underline{M}(H)$ satisfying the following condition for any nontrivial p-subgroup X/H of $N_G(H)/H$:

$$x \in T, \quad (\sum_{gH \subseteq X} c_H^g)x = 0 \quad \text{implies} \quad t_H^X(x) = 0.$$

(2) *For any simple $\mu_{\mathbb{K}}(G)$-submodule S of M having H as a minimal subgroup, there is a simple $\mathbb{K}\overline{N}_G(H)$-submodule T of $\underline{M}(H)$ satisfying the condition given in the first part such that $S = AT$ where $A = \mu_{\mathbb{K}}(G)$.*

Proof: Follows by 8.7, 8.6, 8.8, and 8.10. □

Remark 8.12 *Let A be a finite dimensional $\mathbb{K}$-algebra and e be a nonzero idempotent of A. Let V be an A-module and let $W_1, W_2, ..., W_n$ be eAe-submodules of eV. Suppose that the A-submodules $AW_1, AW_2, ..., AW_n$ of V are all simple. If the sum of $W_1, W_2, ..., W_n$ is direct then the sum of $AW_1, AW_2, ..., AW_n$ is direct.*

Proof: Suppose that the sum of $AW_1, AW_2, ..., AW_n$ is not direct. Therefore one of these simple A-modules must be in the sum of the others, say

$$AW_i \subseteq \sum_{j:j \neq i} AW_j.$$

Multiplying by e we obtain that

$$W_i \subseteq \sum_{j:j \neq i} W_j,$$

which is not true because the sum of $W_1, W_2, ..., W_n$ is direct. □

Theorem 8.13 *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module. Then, the multiplicity of $S^G_{H,U}$ in*

$$\mathrm{Soc}(M)$$

is equal to the multiplicity of U in the socle of the following $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$:

$$\bigcap_{X/H} \{x \in \underline{M}(H) : (\sum_{gH \subseteq X} c^g_H)x = 0 \implies t^X_H(x) = 0\}$$

where X/H ranges over all nontrivial p-subgroups of $N_G(H)/H$.

Proof: It is easy to see that the subset of $\underline{M}(H)$ defined as the above intersection is indeed a $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$.

It follows from 4.15 that we may (and will do) assume that H is normal in G, because

$$\underline{M}(H) = \underline{(\downarrow^G_K M)}(H)$$

for any $H \leq K \leq G$. Let n be the multiplicity of $S^G_{H,U}$ in the socle of M, and let m be the multiplicity of U in the socle of the above given submodule of $\underline{M}(H)$ for which we use the notation $M_0(H)$ here.

Let $A = \mu_{\mathbb{K}}(G)$, $B = \mathbb{K}\overline{N}_G(H)$, and $e = t^H_H$. There are n simple A-submodules $S_1, S_2, ..., S_n$ of M whose sum is direct and all of them are isomorphic to $S^G_{H,U}$. Therefore the A-submodule

$$S_1 \oplus S_2 \oplus ... \oplus S_n$$

of M is a direct summand of $\mathrm{Soc}(M)$. By 8.6, 8.8 and 8.10 we know that each eS_i is a simple B-submodule of $M_0(H)$. As the multiplication by the idempotent e respects the direct sums we see that the B-submodule

$$eS_1 \oplus eS_2 \oplus ... \oplus eS_n$$

of $M_0(H)$ is a direct summand of $\mathrm{Soc}(M_0(H))$. As each $eS_i = S_i(H)$ is isomorphic to the simple B-module U, we conclude that $n \leq m$.

Conversely, there are m simple B-submodules $T_1, T_2, ..., T_m$ of $M_0(H) \subseteq \underline{M}(H)$ whose sum is direct and all of them are isomorphic to U. By part (1) of 8.3 we know that each T_i is also a simple eAe-submodule of $\underline{M}(H) \subseteq eM$. Moreover, it follows by 8.10 that each of the A-submodules AT_i of M is simple. Therefore we may apply 8.12 to deduce that the sum of the A-submodules $AT_1, AT_2, ..., AT_m$ of M is direct so that

$$AT_1 \oplus AT_2 \oplus ... \oplus AT_m$$

is a direct summand of $\mathrm{Soc}(M)$. By 8.4 each simple A-module AT_i has H as a minimal subgroup, and as $AT_i(H) = T_i \cong U$ all of them must be isomorphic to $S^G_{H,U}$. Consequently, $m \le n$. □

Corollary 8.14 *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module.*

(1) *There is a simple subfunctor of M having H as a minimal subgroup if and only if there is a simple $\mathbb{K}\overline{N}_G(H)$-submodule T of $\underline{M}(H)$ satisfying the following condition for any nontrivial p-subgroup X/H of $N_G(H)/H$:*

$$x \in T, \quad \Big(\sum_{gH \subseteq X} c^g_H\Big)x = 0 \quad \textit{implies} \quad t^X_H(x) = 0.$$

(2) *The multiplicity of $S^G_{H,U}$ in $\mathrm{Soc}(M)$ is less than or equal to the multiplicity of U in $\mathrm{Soc}\big(\underline{M}(H)\big)$.*

(3) *The multiplicity of $S^G_{H,U}$ in $\mathrm{Soc}(M)$ is greater than or equal to the multiplicity of U in the socle of the following $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$:*

$$\bigcap_{H<X\le N_G(H):|X:H|=p} \mathrm{Ker}\big(t^X_H : \underline{M}(H) \to M(X)\big).$$

(4) *Suppose that $\overline{N}_G(H)$ is a p'-group. Then, the multiplicity of $S^G_{H,U}$ in $\mathrm{Soc}(M)$ is equal to the multiplicity of U in $\underline{M}(H)$.*

Proof: (1) and (2) They are immediate from 8.13.

(3) By 8.13, it is enough to observe that the submodule of $\underline{M}(H)$ given in this part is in the submodule of $\underline{M}(H)$ given in 8.13: Let x be an element of the submodule of $\underline{M}(H)$ given in this part. It follows for any $X/H \leq N_G(H)/H$ with $|X : H| = p$ that $t_H^X(x) = 0$. Therefore, for any nontrivial p-subgroup Y/H of $N_G(H)/H$, it follows by the transitivity of trace maps on M that $t_H^Y(x) = 0$. Hence, x is in the submodule of $\underline{M}(H)$ given in 8.13.

(4) Follows from 8.13, because in this case the index set of the intersection defining the given submodule of $\underline{M}(H)$ is empty so that the intersection is equal to the semisimple $\mathbb{K}\overline{N}_G(H)$-module $\underline{M}(H)$. □

Part (2) of the previous result cannot be improved in general, because there may be two isomorphic simple $\mathbb{K}\overline{N}_G(H)$-submodules of $\underline{M}(H)$ such that the only one of them satisfies the condition given in part (1).

Letting $K = N_G(H)$, the adjointness of the pairs

$$(\uparrow_K^G, \downarrow_K^G) \qquad \text{and} \qquad (\mathrm{Inf}_{K/H}^K, L^-{}_{K/H})$$

and the isomorphism given in 2.10 and the result 4.13 imply that the multiplicity of a simple $\mu_{\mathbb{K}}(G)$-module $S_{H,U}^G$ in the socle of M is equal to the multiplicity of $S_{H/H,U}^{K/H}$ in the socle of the $\mu_{\mathbb{K}}(K/H)$-module

$$L^-{}_{K/H} \downarrow_K^G M.$$

Therefore, part (4) of 8.14 follows also from part (2) of 8.6 (because the Mackey algebra $\mu_{\mathbb{K}}(K/H)$ is semisimple in this case, see [TW]).

The next result indicates a case in which the multiplicities mentioned in part (3) of 8.14 become equal.

Proposition 8.15 *Let $\mathbb{K}$ be of characteristic $p > 0$. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module, H is a subgroup of G, and U is a simple $\mathbb{K}\overline{N}_G(H)$-module. If all the elements of $\overline{N}_G(H)$ of order p acts on U trivially, then the multiplicity of $S_{H,U}^G$ in*

$$\mathrm{Soc}(M)$$

is equal to the multiplicity of U in the socle of the following $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$:

$$\bigcap_{H<X\leq N_G(H):|X:H|=p} \mathrm{Ker}\big(t_H^X : \underline{M}(H) \to M(X)\big).$$

Proof: Let T be a simple $\mathbb{K}\overline{N}_G(H)$-submodule of $M_0(H)$ isomorphic to U where $M_0(H)$ denotes the submodule of $\underline{M}(H)$ defined in 8.13. If we show that T is in the $\mathbb{K}\overline{N}_G(H)$-submodule of $\underline{M}(H)$ defined in this result (which is a submodule of $M_0(H)$), then the result will follow by 8.13.

Take any X with $H < X \leq N_G(H)$ and $|X : H| = p$. As the $\mathbb{K}\overline{N}_G(H)$-modules T and U are isomorphic and as any element of $N_G(H)$ of order p acts on U trivially,

$$\Big(\sum_{gH\subseteq X} c_H^g\Big)T = 0.$$

This implies that $t_H^X(T) = 0$ because $T \subseteq M_0(H)$. □

If $\overline{N}_G(H)$ is a nilpotent group (or more generally, a group with normal Sylow p-subgroup), then (Clifford's theorem implies that) the hypothesis of 8.15 is satisfied for any simple $\mathbb{K}\overline{N}_G(H)$-module U. For another example, the hypothesis of 8.15 is satisfied for any group G and for any simple $\mathbb{K}\overline{N}_G(H)$-module U with $\dim_{\mathbb{K}} U = 1$.

We finish this chapter by giving some conditions on a $\mu_{\mathbb{K}}(G)$-module M equivalent to the condition $\underline{M}(H) \neq 0$ where H is a subgroup of G.

Remark 8.16 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Then the following conditions are equivalent:*

(i) $\underline{M}(H) \neq 0$.

(ii) *M has a subfunctor having H as a minimal subgroup.*

(iii) $\uparrow_H^G\downarrow_H^G M$ *has a simple subfunctor having H as a minimal subgroup.*

(iv) $\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(\uparrow_H^G S_{H,\mathbb{K}}^H, M) \neq 0$.

Proof: (i) equals to (ii): Follows by 8.4.

(ii) implies (iii): Let S be a subfunctor of M having H as a minimal subgroup. It is clear that

$$\downarrow_H^G S \cong nS_{H,\mathbb{K}}^H$$

where $n = \dim_{\mathbb{K}} S(H)$. Therefore, it follows by the exactness of the functors $\uparrow$ and $\downarrow$ that $\uparrow_H^G S_{H,\mathbb{K}}^H$ is isomorphic to a subfunctor of $\uparrow_H^G\downarrow_H^G M$. By 4.3 we know that the minimal subgroups of any nonzero subfunctor of $\uparrow_H^G S_{H,\mathbb{K}}^H$ is a G-conjugate of H. As a result, $\uparrow_H^G S_{H,\mathbb{K}}^H$ and hence $\uparrow_H^G\downarrow_H^G M$ has a simple subfunctor having H as a minimal subgroup.

(iii) implies (iv): Suppose that $\uparrow_H^G\downarrow_H^G M$ has a simple subfunctor having H as a minimal subgroup. Then there is a simple $\mathbb{K}\overline{N}_G(H)$-module V such that

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(S_{H,V}^G, \uparrow_H^G\downarrow_H^G M) \neq 0.$$

It is clear that $\downarrow_H^G S_{H,V}^G \cong nS_{H,\mathbb{K}}^H$ where $n = \dim_{\mathbb{K}} V$. Using the adjointness of the pairs $(\downarrow_H^G, \uparrow_H^G)$ and $(\uparrow_H^G, \downarrow_H^G)$ we see that

$$\begin{aligned}
0 &\neq \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(S_{H,V}^G, \uparrow_H^G\downarrow_H^G M)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}(\downarrow_H^G S_{H,V}^G, \downarrow_H^G M)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}(nS_{H,\mathbb{K}}^H, \downarrow_H^G M)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(\uparrow_H^G nS_{H,\mathbb{K}}^H, M)\\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(n\uparrow_H^G S_{H,\mathbb{K}}^H, M)\\
&\cong n\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(\uparrow_H^G S_{H,\mathbb{K}}^H, M)
\end{aligned}$$

(iv) implies (i): Firstly, using the adjointness of the pairs

$$(\uparrow_H^G, \downarrow_H^G) \qquad \text{and} \qquad (\mathrm{Inf}_{H/H}^H, L^-{}_{H/H})$$

and using the obvious isomorphisms

$$S_{H,\mathbb{K}}^H \cong \mathrm{Inf}_{H/H}^H S_{H/H,\mathbb{K}}^{H/H}, \quad \mu_{\mathbb{K}}(H/H) \cong \mathbb{K} \quad S_{H/H,\mathbb{K}}^{H/H} \cong \mathbb{K},$$

$$L^-{}_{H/H}\downarrow_H^G M = (L^-{}_{H/H}\downarrow_H^G M)(H/H) = \underline{M}(H),$$

we obtain that

$$\begin{aligned}
\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(\uparrow_H^G S_{H,\mathbb{K}}^H, M) &\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}(S_{H,\mathbb{K}}^H, \downarrow_H^G M) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}(\mathrm{Inf}_{H/H}^H S_{H/H,\mathbb{K}}^{H/H}, \downarrow_H^G M) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H/H)}(S_{H/H,\mathbb{K}}^{H/H}, L^-{}_{H/H} \downarrow_H^G M) \\
&\cong \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, \underline{M}(H))
\end{aligned}$$

□

Chapter 9

Composition factors

Almost all the materials in this chapter comes from [Yar5, Section 6].

Our aim in this chapter is to study $\mu_{\mathbb{K}}(G)$-modules M, especially their composition factors, satisfying some extreme conditions such as having a unique maximal or simple subfunctors, and being uniserial. For example, we refine some of the results in the previous two chapters, and we observe that the primordial subgroups of a uniserial $\mu_{\mathbb{K}}(G)$-module form a chain.

The result 7.8 contains some necessary and sufficient conditions for a $\mu_{\mathbb{K}}(G)$-module M to have a simple quotient functor of the form $S^G_{H,V}$. It is shown in [TW95, (15.7) Proposition] that if H is a maximal subgroup of G subject to the condition $\overline{M}(H) \neq 0$ and if we assume that H is normal in G, then for any maximal $\mathbb{K}\overline{N}_G(H)$-submodule $\overline{I}$ of $\overline{M}(H)$, the simple module $V = \overline{M}(H)/\overline{I}$ satisfies the condition (vi) of 7.8 so that M has a simple quotient functor of the form $S^G_{H,V}$. We first want to state this result in a slightly stronger form and then dualize it.

Lemma 9.1 *Let M be a $\mu_{\mathbb{K}}(G)$-module, and let Y and Z be subgroups of G.*

(1) *Assume that* $\underline{M}(Y) = 0$. *Then,*

$$\mathrm{Ker}\big(\alpha : M(Z) \to M(Y)\big) = \bigcap_{J<Y} \mathrm{Ker}\big(r_J^Y \circ \alpha : M(Z) \to M(J)\big)$$

for any $\mathbb{K}$*-space homomorphism* $\alpha : M(Z) \to M(Y)$.

(2) *Assume that* $\overline{M}(Y) = 0$. *Then,*

$$\beta(M(Y)) = \sum_{J<Y} \beta \circ t_J^Y(M(J))$$

for any $\mathbb{K}$*-space homomorphism* $\beta : M(Y) \to M(Z)$.

Proof: We only justify the first part. The second part may be justified similarly. As $\underline{M}(Y) = 0$, it follows by the definition of restriction kernels that the product of restriction maps

$$\varphi = \prod_{J<Y} r_J^Y : M(Y) \to \prod_{J<Y} M(J)$$

is injective. Thus, the kernels of the maps α and $\varphi \circ \alpha$ are equal, implying the result. □

Part (2) of 9.2 can be found in the proof of [TW95, (15.7) Proposition], whose dual version is part (1) of 9.2.

Lemma 9.2 *Let* M *be a* $\mu_{\mathbb{K}}(G)$*-module and* H *be a subgroup of* G.

(1) *If* H *is maximal subject to the condition* $\underline{M}(H) \neq 0$, *then,*

$$0 = \big(\mathrm{Ker} r_H^X\big) \bigcap \Big(\bigcap_{H \not\leq J < X} \mathrm{Ker} r_J^X \Big)$$

for any X *with* $H < X \leq G$.

(2) *If* H *is maximal subject to the condition* $\overline{M}(H) \neq 0$, *then,*

$$M(X) = t_H^X(M(H)) + \sum_{H \not\leq J < X} t_J^X(M(J))$$

for any X *with* $H < X \leq G$.

Proof: We only justify the first part by arguing as in the proof of [TW95, (15.7) Proposition]. Let $H < X \leq G$. By the maximality of H,

$$0 = \underline{M}(X) = \Big(\bigcap_{H \leq J < X} \mathrm{Ker} r_J^X\Big) \bigcap \Big(\bigcap_{H \nleq J < X} \mathrm{Ker} r_J^X\Big).$$

For any $H < J < X$, by the maximality of H we obtain that $\underline{M}(J) = 0$. Then, part (1) of 9.1 implies (by taking α to be the map r_J^X) that

$$\mathrm{Ker} r_J^X = \bigcap_{K < J} \mathrm{Ker} r_K^X.$$

Substituting this intersection for $\mathrm{Ker} r_J^X$ in the first intersection, and continuing to do this process we finally obtain that

$$0 = \big(\mathrm{Ker} r_H^X\big) \bigcap \Big(\bigcap_{H \nleq J < X} \mathrm{Ker} r_J^X\Big).$$

□

It is proved in [TW95, (15.7) Proposition] that for a $\mu_{\mathbb{K}}(G)$-module M, a subgroup H of G maximal subject to the condition $\overline{M}(H) \neq 0$, and a simple $\mathbb{K}\overline{N}_G(H)$-module U, the existence of a simple quotient of the functor M isomorphic to $S_{H,U}^G$ is equivalent to the existence of a simple quotient of $\overline{M}(H)$ isomorphic to U. We next show that not only existences but also their multiplicities in respective heads are equal.

Proposition 9.3 *Let M be a $\mu_{\mathbb{K}}(G)$-module, H be a subgroup of G, and U be a simple $\mathbb{K}\overline{N}_G(H)$-module.*

(1) *Suppose that H is maximal subject to the condition $\underline{M}(H) \neq 0$. Then, the multiplicity of $S_{H,U}^G$ in $\mathrm{Soc}(M)$ is equal to the multiplicity of U in $\mathrm{Soc}(\underline{M}(H))$.*

(2) *Suppose that H is maximal subject to the condition $\overline{M}(H) \neq 0$. Then, the multiplicity of $S_{H,U}^G$ in $M/\mathrm{Jac}(M)$ is equal to the multiplicity of U in $\overline{M}(H)/\mathrm{Jac}(\overline{M}(H))$.*

Proof: (1) Let X/H be a nontrivial p-subgroup of $N_G(H)/H$. We will show that $t_H^X(x) = 0$ for any $x \in \underline{M}(H)$ satisfying

$$\Big(\sum_{gH \subseteq X} c_H^g\Big)x = 0$$

(that is equivalent to the condition $r_H^X t_H^X(x) = 0$ by the Mackey axiom), from which the result follows by the virtue of 8.13.

Let $x \in \underline{M}(H)$ such that $r_H^X t_H^X(x) = 0$. Then

$$t_H^X(x) \in \mathrm{Ker} r_H^X.$$

For any $H \not\leq J < X$, it follows by the Mackey axiom that

$$r_J^X t_H^X(x) = \sum_{JgH \subseteq X} t_{J \cap H}^J c_{J^g \cap H}^g r_{J^g \cap H}^H(x).$$

We see that $J^g \cap H \neq H$ for any $g \in N_G(H)$ because $H \not\leq J$. This shows that $r_{J^g \cap H}^H(x) = 0$ because $x \in \underline{M}(H)$. So $r_J^X t_H^X(x) = 0$ for any J with $H \not\leq J < X$. Consequently,

$$t_H^X(x) \in \big(\mathrm{Ker} r_H^X\big) \bigcap \Big(\bigcap_{H \not\leq J < X} \mathrm{Ker} r_J^X\Big) = 0$$

where the last equality follows from part (1) of 9.2.

(2) We use the notations $\mathcal{A}_X$, V_X, and $\mathcal{M}_H$ defined in 7.14 and its proof. From 7.14, it suffices to show that $\mathcal{M}_H = 0$.

Let X/H be a nontrivial p-subgroup of $N_G(H)/H$. Part (2) of 9.2 implies that

$$r_H^X(M(X)) \subseteq r_H^X t_H^X(M(H)) + \sum_{H \not\leq J < X} r_H^X t_J^X(M(J)).$$

For any $H \not\leq J < X$, as in the first part $H \cap {}^g J \neq H$ if $g \in N_G(H)$, so the Mackey axiom implies that

$$r_H^X t_J^X(M(J)) \subseteq \sum_{HgJ \subseteq X} t_{H \cap {}^g J}^H r_{H \cap {}^g J}^{{}^g J} c_J^g M(J) \subseteq \sum_{HgJ \subseteq X} t_{H \cap {}^g J}^H M(H \cap {}^g J) \subseteq b_H(M).$$

Therefore,

$$r_H^X(M(X)) \subseteq r_H^X t_H^X(M(H)) + b_H(M)$$

implying that

$$r_H^X(\overline{M}(X)) \subseteq r_H^X t_H^X(\overline{M}(H)).$$

The last containment is equivalent by part (2) of 7.10 to the condition $V_X = 0$. Hence, $\mathcal{M}_H = 0$ as desired. $\square$

The following is an immediate consequence of 9.3.

Remark 9.4 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. If $\underline{M}(H) \neq 0$ (respectively, $\overline{M}(H) \neq 0$), then there is a subgroup K of G containing H such that M has a simple subfunctor (respectively, simple quotient functor) having K as a minimal subgroup.*

The next result shows that an example of a $\mu_{\mathbb{K}}(G)$-module M for which H is a maximal subgroup of G subject to the condition $\underline{M}(H) \neq 0$ occurs when $M = \uparrow_H^G T$ for some $\mu_{\mathbb{K}}(H)$-module T such that $\underline{T}(H) \neq 0$.

Proposition 9.5 *Let H be a subgroup of G and T be a $\mu_{\mathbb{K}}(H)$-module. For any subgroup K of G, we have the following $\mathbb{K}\overline{N}_G(K)$-module isomorphisms:*

(1)

$$\underline{(\uparrow_H^G T)}(K) \cong \bigoplus_{N_G(K)gH \subseteq G : K^g \leq H} \uparrow_{\overline{N}_{{}^gH}(K)}^{\overline{N}_G(K)} {}^g\big(\underline{T}(K^g)\big).$$

In particular, if $\underline{(\uparrow_H^G T)}(K)$ is nonzero, then $K \leq_G H$.

(2)

$$\overline{(\uparrow_H^G T)}(K) \cong \bigoplus_{N_G(K)gH \subseteq G : K^g \leq H} \uparrow_{\overline{N}_{{}^gH}(K)}^{\overline{N}_G(K)} {}^g\big(\overline{T}(K^g)\big).$$

In particular, if $\overline{(\uparrow_H^G T)}(K)$ is nonzero, then $K \leq_G H$.

Proof: We will prove only the first isomorphism. The second one can be proved similarly. The explicit description of induced functors given in 2.6 implies that

$$\underline{(\uparrow_H^G T)}(K) = \bigcap_{J<K} \mathrm{Ker} \tilde{r}_J^K = \bigoplus_{KgH \subseteq G} \Big(\bigcap_{J<K} \mathrm{Ker} r_{H \cap J^g}^{H \cap K^g} \Big)$$

where $\tilde{r}$ and r are restriction maps of $\uparrow_H^G T$ and T, respectively. Letting $J = {}^gH \cap K$, we see that $H \cap J^g = H \cap K^g$. So, if ${}^gH \cap K < K$ then

$$\bigcap_{J<K} \mathrm{Ker} r_{H\cap J^g}^{H\cap K^g} = 0.$$

Therefore, we must have that

$$\underline{(\uparrow_H^G T)}(K) = \bigoplus_{KgH\subseteq G: K^g \leq H} \Big(\bigcap_{J<K} \mathrm{Ker} r_{J^g}^{K^g} \Big) = \bigoplus_{KgH\subseteq G: K^g\leq H} \underline{T}(K^g).$$

Using the following obvious equality (which is also true if we replace $N_G(K)$ with any other subgroup L containing K)

$$N_G(K)gH = \biguplus_{Ku\left(N_G(K)\cap {}^gH\right)\subseteq N_G(K)} KugH,$$

and noting that the conditions $K^g \leq H$ and $K^{ug} \leq H$ are equivalent for any $g \in G$ and any $u \in N_G(K)$, we may write

$$\underline{(\uparrow_H^G T)}(K) = \bigoplus_{N_G(K)gH\subseteq G: K^g\leq H} \Big(\bigoplus_{uN_{{}^gH}(K)\subseteq N_G(K)} \underline{T}(K^g) \Big).$$

Writing $\mu_{\mathbb{K}}(G) \otimes_{\mu_{\mathbb{K}}(H)} T$ for $\uparrow_H^G T$, the last equality becomes

$$\underline{(\uparrow_H^G T)}(K) = \bigoplus_{N_G(K)gH\subseteq G: K^g\leq H} \Big(\bigoplus_{uN_{{}^gH}(K)\subseteq N_G(K)} c_K^u \big(c_{K^g}^g \otimes_{\mu_{\mathbb{K}}(H)} t_{K^g}^{K^g} \underline{T} \big) \Big),$$

see the explanation given at the beginning of 2.6. As the $\mathbb{K}\overline{N}_H(K^g)$-module structure on $t_{K^g}^{K^g}\underline{T}$ is given by left multiplications of elements $c_{K^g}^h$ of $\mu_{\mathbb{K}}(H)$, it is clear that the $\mathbb{K}\overline{N}_{{}^gH}(K)$-module ${}^g\big(\underline{T}(K^g)\big)$, which is the g-conjugate of $\underline{T}(K^g)$, is isomorphic to

$$c_{K^g}^g \otimes_{\mu_{\mathbb{K}}(H)} t_{K^g}^{K^g}\underline{T},$$

via the map, given for all x is in $\underline{T}(K^g)$, by $x \leftrightarrow c_{K^g}^g \otimes t_{K^g}^{K^g} x$. Now, the result is clear. □

The next result follows easily by 9.5.

Corollary 9.6 *Let H be a subgroup of G and T be a $\mu_{\mathbb{K}}(H)$-module. Then we have the following $\mathbb{K}\overline{N}_G(H)$-module isomorphisms:*

(1) $\underline{(\uparrow_H^G T)}(H) \cong \uparrow_1^{\overline{N}_G(H)} \underline{T}(H)$.

(2) $\overline{(\uparrow_H^G T)}(H) \cong \uparrow_1^{\overline{N}_G(H)} \overline{T}(H)$.

The previous result and 9.3 implies the following.

Proposition 9.7 *Let H be a subgroup of G and T be a $\mu_{\mathbb{K}}(H)$-module. Given a simple $\mathbb{K}\overline{N}_G(H)$-module U,*

(1) *The multiplicity of $S_{H,U}^G$ in $\mathrm{Soc}(\uparrow_H^G T)$ is equal to $\dim_{\mathbb{K}} \underline{T}(H)$.*

(2) *The multiplicity of $S_{H,U}^G$ in $(\uparrow_H^G T)/\mathrm{Jac}(\uparrow_H^G T)$ is equal to $\dim_{\mathbb{K}} \overline{T}(H)$.*

Proof: We will prove the first part only, the second part may be derived similarly.

Let n be the multiplicity of $S_{H,U}^G$ in $\mathrm{Soc}(\uparrow_H^G T)$. We may assume that $\underline{T}(H) \neq 0$, because otherwise 9.6 implies that $\underline{(\uparrow_H^G T)}(H) = 0$, which gives by 8.4 that $n = 0$.

Now it follows by 9.5 that H is a maximal subgroup of G subject to the condition $\underline{(\uparrow_H^G T)}(H) \neq 0$. Then it follows by 9.3 that n is equal to the multiplicity of U in

$$\mathrm{Soc}\big(\underline{(\uparrow_H^G T)}(H)\big) \cong \mathrm{Soc}\big(\uparrow_1^{\overline{N}_G(H)} \underline{T}(H)\big)$$

where the isomorphism follows by 9.6. Letting $r = \dim_{\mathbb{K}} U$ it is clear that

$$nr = \dim_{\mathbb{K}} \mathrm{Hom}_{\mathbb{K}\overline{N}_G(H)}\Big(U, \mathrm{Soc}\big(\uparrow_1^{\overline{N}_G(H)} \underline{T}(H)\big)\Big).$$

Using the adjointness of the pair $(\downarrow_1^{\overline{N}_G(H)}, \uparrow_1^{\overline{N}_G(H)})$ we obatin the following $\mathbb{K}$-space isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{K}\overline{N}_G(H)}\Big(U, \mathrm{Soc}\big(\uparrow_1^{\overline{N}_G(H)} \underline{T}(H)\big)\Big) &\cong \mathrm{Hom}_{\mathbb{K}\overline{N}_G(H)}\big(U, \uparrow_1^{\overline{N}_G(H)} \underline{T}(H)\big) \\
&\cong \mathrm{Hom}_{\mathbb{K}}\big(\downarrow_1^{\overline{N}_G(H)} U, \underline{T}(H)\big) \\
&\cong \mathrm{Hom}_{\mathbb{K}}\big(r\mathbb{K}, \underline{T}(H)\big) \\
&\cong \big(r \dim_{\mathbb{K}} \underline{T}(H)\big)\mathbb{K}.
\end{aligned}$$

This shows that $n = \dim_{\mathbb{K}} \underline{T}(H)$. □

The following is clear from the definitions.

Remark 9.8 *Let N be a normal subgroup of G and K be a subgroup of G with $K \geq N$. For any $\mu_{\mathbb{K}}(G)$-module M, we have the following $\mathbb{K}\overline{N}_G(K)$-module isomorphisms:*

(1) $\overline{(\mathrm{Inf}_{G/N}^{G} L^{+}{}_{G/N} M)}(K) \cong \overline{M}(K)$.

(2) $\underline{(\mathrm{Inf}_{G/N}^{G} L^{-}{}_{G/N} M)}(K) \cong \underline{M}(K)$.

(3) $(L^{+}{}_{\overline{N}_G(K)/K} \downarrow^{G}_{\overline{N}_G(K)} M)(K) \cong \overline{M}(K)$.

(4) $(L^{-}{}_{\overline{N}_G(K)/K} \downarrow^{G}_{\overline{N}_G(K)} M)(K) \cong \underline{M}(K)$.

Given a subgroup H of G and a $\mu_{\mathbb{K}}(G)$-module M, a smallest element of the set of all subfunctors of M having H as a minimal subgroup may not be a simple functor (but it is indecomposable by the explanation given after 9.10). However, it possesses some properties of simple functors.

Remark 9.9 *Let H be a subgroup of G and M be a $\mu_{\mathbb{K}}(G)$-module. Suppose that H is a minimal subgroup of M. Then, M has no proper subfunctor having H as a minimal subgroup if and only if the following conditions hold:*

(i) *M is generated as a $\mu_{\mathbb{K}}(G)$-module by its value $M(H)$.*

(ii) *H is the unique, up to G-conjugacy, minimal subgroup of M.*

(iii) *$M(H)$ is a simple $\mathbb{K}\overline{N}_G(H)$-module.*

Proof: For any $\mu_{\mathbb{K}}(G)$-module M, it is clear by the definition of restriction kernels that if H is a minimal subgroup of M then $\underline{M}(H) = M(H) \neq 0$. The result follows by part (3) of 8.3 and by the bijective correspondence given in 8.4. □

Any simple $\mu_{\mathbb{K}}(G)$-module M having H as a minimal subgroup satisfies the conditions (i)-(iii) of the previous result so that the previous result explains what happens in the converse situation of [TW, (2.3) Proposition].

Proposition 9.10 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$. Then:*

(1) *For any $\mathbb{K}\overline{N}_G(H)$-submodule T of $\underline{M}(H)$, the maps*

$$J \to J(H) \qquad \text{and} \qquad (AT :_e I) \leftarrow I$$

define a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules J of AT and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules I of T. Moreover, any simple quotient functor of AT has H as a minimal subgroup.

(2) *For any $\mathbb{K}\overline{N}_G(H)$-submodule $\overline{I}$ of $\overline{M}(H)$ where $\overline{I} = I/b_H(M)$, the maps*

$$S \to S(H) \qquad \text{and} \qquad AT \leftarrow T$$

define a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules S of

$$\widetilde{M} = M/(M :_e I)$$

and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of

$$\widetilde{M}(H) \cong \overline{M}(H)/\overline{I}.$$

Moreover, any simple subfunctor of $\widetilde{M}$ has H as a minimal subgroup.

Proof: Put $B = \mathbb{K}\overline{N}_G(H)$.

(1) Part (1) of 8.3 implies that eAe-submodules and B-submodules of T are the same. As $eAT = T$ and $AT = Ae(AT)$, the required bijection follows from 4.8. It follows by this bijection that any simple quotient of AT is of the form

$$AT/(AT :_e I)$$

for some maximal B-submodule I of T. The value of

$$AT/(AT :_e I)$$

at H is isomorphic to T/I which is nonzero. For any $X < H$, part (3) of 8.3 implies that $(AT)(X) = 0$. Consequently, H is a minimal subgroup of

$$AT/(AT :_e I).$$

(2) Firstly, using part (5) of 7.1 we see that $(\widetilde{M} :_e 0) = 0$. Moreover, it follows by part (2) of 7.5 that $\widetilde{M}(X) = 0$ for all $X < H$. Now, by using 4.4 and part (1) of 7.5, and by arguing as in the first part, one may prove the results. $\square$

Let M a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Let M' be any smallest element of the set of all subfunctors of M having H as a minimal subgroup. It follows by 8.4 that $M' = AT$ for some simple $\mathbb{K}\overline{N}_G(H)$-submodule T of $\underline{M}(H)$, where $A = \mu_{\mathbb{K}}(G)$. We see by using part (1) of 9.10 that M' has a unique maximal subfunctor implying that M' is indecomposable. In particular, any $\mu_{\mathbb{K}}(G)$-module satisfying the conditions of 9.9 has a unique maximal subfunctor (and so it is indecomposable).

Let M a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. Given any composition series

$$0 = T_0 \subset T_1 \subset ... \subset T_{n-1} \subset T_n = \underline{M}(H)$$

of the $\mathbb{K}\overline{N}_G(H)$-module $\underline{M}(H)$, letting $A = \mu_{\mathbb{K}}(G)$ we obtain the series

$$0 = AT_0 \subset AT_1 \subset ... \subset AT_{n-1} \subset AT_n = A\underline{M}(H)$$

of $\mu_{\mathbb{K}}(G)$-submodules of M. The inclusions

$$AT_{i-1} \subseteq AT_i$$

are strict because $eAT_i = T_i$ where $e = t_H^H$. Part (1) of 9.10 implies that

$$(AT_i :_e T_{i-1})$$

is a maximal $\mu_{\mathbb{K}}(G)$-submodule of AT_i whose quotient

$$AT_i/(AT_i :_e T_{i-1})$$

is isomorphic to S^G_{H,V_i}, where V_i is isomorphic to T_i/T_{i-1}. Moreover, we see by part (1) of 7.1 that

$$AT_{i-1} \subseteq (AT_i :_e T_{i-1}).$$

Consequently, we have proved for any simple $\mathbb{K}\overline{N}_G(H)$-module V that the multiplicity of the simple $\mu_{\mathbb{K}}(G)$-module $S^G_{H,V}$ as a composition factor of M (indeed, of $A\underline{M}(H)$) is greater than or equal to the multiplicity of V as a composition factor of $\underline{M}(H)$. This is the dual version of [TW95, (6.2) Proposition]. Moreover, as the evaluation of

$$(AT_i :_e T_{i-1})/AT_{i-1}$$

at subgroups of H are 0, we see that the multiplicity of V as a composition factor of $\underline{M}(H)$ is equal to the the multiplicity of $S^G_{H,V}$ as a composition factor of $A\underline{M}(H)$. This can also be deduced by using the next result.

Proposition 9.11 *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a minimal subgroup of M. Then, for any simple $\mathbb{K}\overline{N}_G(H)$-module V, the multiplicity of $S^G_{H,V}$ as a composition factor of M is equal to the multiplicity of V as a composition factor of $M(H)$.*

Proof: Let $0 = M_0 \subset M_1 \subset ... \subset M_{n-1} \subset M_n = M$ be a composition series of M. Evaluating at H yields a series

$$0 = M_0(H) \subseteq M_1(H) \subseteq ... \subseteq M_{n-1}(H) \subseteq M_n(H) = M(H)$$

of $\mathbb{K}\overline{N}_G(H)$-submodules of $M(H)$. Each M_i/M_{i-1} is isomorphic to a simple $\mu_{\mathbb{K}}(G)$-module of the form $S_i = S^G_{H_i,V_i}$ for some H_i and V_i. We will show that $M_{i-1}(H) \neq M_i(H)$ if and only if $H_i =_G H$. This clearly finishes the proof, because the isomorphism of two simple functors of the form $S^G_{A,U}$ and $S^G_{B,W}$ is equivalent to the existence of a $g \in G$ satisfying $B = {}^gA$ and $W \cong {}^gU$ and because $S^G_{A,U}(A) \cong U$ for any simple functor $S^G_{A,U}$ (see 2.5).

As $S_i(H_i) \neq 0$, we see that $M_{i-1}(H_i) \neq M_i(H_i)$ and that $M_i(H_i) \neq 0$. From $0 \neq M_i(H_i) \subseteq M(H_i)$ we obtain that $H_i \not<_G H$ because H is a minimal subgroup of M. On the other hand, if $M_{i-1}(H) \neq M_i(H)$ then $S_i(H) \neq 0$ implying that $H_i \leq_G H$. Consequently, $M_{i-1}(H) \neq M_i(H)$ if and only if $H_i =_G H$. □

It is clear that a $\mu_{\mathbb{K}}(G)$-module M has a composition factor having 1 as a minimal subgroup if and only if $M(1) \neq 0$. Therefore, taking $H = 1$ in 9.11 one obtains [TW95, (6.3) Proposition].

Corollary 9.12 *Let M be a $\mu_{\mathbb{K}}(G)$-module, H be a subgroup of G, and V be a simple $\mathbb{K}\overline{N}_G(H)$-module. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$. Then:*

(1) *For any $\mathbb{K}\overline{N}_G(H)$-submodule T of $\underline{M}(H)$, the multiplicity of $S_{H,V}^G$ as a composition factor of AT is equal to the multiplicity of V as a composition factor of T.*

(2) *For any $\mathbb{K}\overline{N}_G(H)$-submodule $\overline{I} = I/b_H(M)$ of $\overline{M}(H)$, the multiplicity of $S_{H,V}^G$ as a composition factor of $M/(M :_e I)$ is equal to the multiplicity of V as a composition factor of $\overline{M}(H)/\overline{I}$.*

Proof: (1) Using part (3) of 8.3 we see that H is a minimal subgroup of the functor AT. Then, the result follows from 9.11, because $(AT)(H) = T$.

(2) Using part (2) of 7.5 we see that H is a minimal subgroup of the functor

$$M/(M :_e I).$$

Then, the result follows from 9.11, because the evaluation of

$$M/(M :_e I)$$

at H is isomorphic to $\overline{M}(H)/\overline{I}$. □

The following special case of the previous result explains the precise version of the situation about multiplicities explained at the beginning of 9.11.

Theorem 9.13 *Let M be a $\mu_{\mathbb{K}}(G)$-module, H be a subgroup of G, and V be a simple $\mathbb{K}\overline{N}_G(H)$-module. Put $A = \mu_{\mathbb{K}}(G)$ and $e = t_H^H$. Then:*

(1) *The multiplicity of V as a composition factor of $\underline{M}(H)$ is equal to the multiplicity of $S_{H,V}^G$ as a composition factor of $A\underline{M}(H)$.*

(2) *The multiplicity of V as a composition factor of $\overline{M}(H)$ is equal to the multiplicity of $S^G_{H,V}$ as a composition factor of $M/(M :_e b_H(M))$.*

If a $\mu_{\mathbb{K}}(G)$-module M has a unique maximal submodule whose simple quotient has H as a minimal subgroup, then it follows by 9.3 that H is the unique, up to G-conjugacy, maximal subgroup of G subject to the condition $\overline{M}(H) \neq 0$. We next want to study such $\mu_{\mathbb{K}}(G)$-modules including the uniserial ones. A finite dimensional module of an algebra is said to be uniserial if its submodule lattice is a chain, equivalently if it has a unique composition series.

Lemma 9.14 *Let M be a $\mu_{\mathbb{K}}(G)$-module, and let H and K be subgroups of G. Put $A = \mu_{\mathbb{K}}(G)$.*

(1) *Suppose that $\overline{M}(H) \neq 0$. If $AM(H) \subseteq AM(K)$ then $H \leq_G K$.*

(2) *Suppose that $\underline{M}(H) \neq 0$. If $A\underline{M}(H) \subseteq A\underline{M}(K)$ then $K \leq_G H$.*

Proof: (1) Evaluation at H gives that

$$M(H) \subseteq t^H_H \mu_{\mathbb{K}}(G) t^K_K M(K).$$

Using the basis theorem 2.1 we see that

$$M(H) \subseteq t^H_H \mu_{\mathbb{K}}(G) t^K_K M(K) = \sum_{g \in G, J \leq H^g \cap K} t^H_{{}^gJ} c^g_J r^K_J (M(K)).$$

If ${}^gJ < H$ for any g and J appearing in the above sum, then the sum is in $b_H(M)$ so that $M(H) \subseteq b_H(M)$ contradicting the assumption $\overline{M}(H) \neq 0$. So there is a $g \in G$ and $J \leq H^g \cap K$ satisfying ${}^gJ = H$. This shows that $H \leq_G K$.

(2) We obtain by evaluation at H that

$$0 \neq \underline{M}(H) \subseteq t^H_H \mu_{\mathbb{K}}(G) t^K_K \underline{M}(K).$$

As $r^K_J(\underline{M}(K)) = 0$ for any $J < K$, arguing as in the first part we see by using the basis theorem 2.1 that $J = K$ for some $g \in G$ and $J \leq H^g \cap K$. This shows that $K \leq_G H$. $\square$

In the next result we observe that the primordial subgroups of a uniserial $\mu_{\mathbb{K}}(G)$-module M (i.e., subgroups X of G for which $\overline{M}(X) \neq 0$) form a chain with respect to the subgroup conjugacy relation $\leq_G$.

Proposition 9.15 *Let M be a uniserial $\mu_{\mathbb{K}}(G)$-module, and let H and K be subgroups of G.*

(1) *If $\overline{M}(H) \neq 0$ and $\overline{M}(K) \neq 0$, then $H \leq_G K$ or $K \leq_G H$.*

(2) *If $\underline{M}(H) \neq 0$ and $\underline{M}(K) \neq 0$, then $H \leq_G K$ or $K \leq_G H$.*

Proof: As the justifications of both parts are similar, we only justify the first part. Since M is uniserial, we must have that

$$AM(H) \subseteq AM(K) \qquad \text{or} \qquad AM(K) \subseteq AM(H)$$

where $A = \mu_{\mathbb{K}}(G)$. Part (1) of 9.14 implies that $H \leq_G K$ or $K \leq_G H$. □

Lemma 9.16 *Let M be a $\mu_{\mathbb{K}}(G)$-module for which there is a unique, up to G-conjugacy, subgroup H of G maximal subject to the condition $\overline{M}(H) \neq 0$. If*

$$M_2 \subseteq M_1$$

are $\mu_{\mathbb{K}}(G)$-submodules of M such that

$$M/M_1 \cong S^G_{H,V} \qquad \textit{and} \qquad M_1/M_2 \cong S^G_{K,W}$$

for some simple $\mu_{\mathbb{K}}(G)$-modules $S^G_{H,V}$ and $S^G_{K,W}$, then $H \leq_G K$ or $K \leq_G H$.

Proof: Assume that $H \not\leq_G K$. Take any $X \leq K$. Then $H \not\leq_G X$. Evaluation of

$$M/M_1 \cong S^G_{H,V}$$

at X is 0 implying that $M(X) = M_1(X)$. Thus,

$$b_K(M) = b_K(M_1) \qquad \text{and} \qquad M(K) = M_1(K)$$

so that $\overline{M}(K) = \overline{M_1}(K)$. As $M_1/M_2 \cong S^G_{K,W}$, it follows from 7.4 that $\overline{M_1}(K) \neq 0$. Hence $\overline{M}(K) \neq 0$, and the maximality of H implies that $K \leq_G H$. $\square$

We now observe that the minimal subgroups of any two successive simple factors of the composition series of a uniserial $\mu_{\mathbb{K}}(G)$-module can be compared with respect to the subgroup conjugacy relation $\leq_G$.

Proposition 9.17 *Let M be a uniserial $\mu_{\mathbb{K}}(G)$-module with the composition series*

$$0 = M_0 \subset M_1 \subset M_2 \subset ... \subset M_{n-1} \subset M_n = M \quad \textit{where} \quad M_i/M_{i-1} \cong S^G_{H_i,V_i}$$

for each i. Then, $H_i \leq_G H_{i-1}$ or $H_{i-1} \leq_G H_i$ for each i.

Proof: The $\mu_{\mathbb{K}}(G)$-module M_i is uniserial for each i. In particular, M_{i-1} is the unique maximal $\mu_{\mathbb{K}}(G)$-submodule of M_i. So, 9.3 implies that H_i is the unique, up to G-conjugacy, maximal subgroup of G subject to the condition $\overline{M}(H_i) \neq 0$. Now the result follows from 9.16 applied to the submodules $M_{i-2} \subseteq M_{i-1}$ of M_i. $\square$

The previous result may also be deduced as an immediate consequence of [TW95, (14.3) Theorem] involving a condition for Ext groups of simple functors to be 0. Indeed, in the case of 9.17, one has a non-split exact sequence

$$0 \to S^G_{H_{i-1},V_{i-1}} \to M_i/M_{i-2} \to S^G_{H_i,V_i} \to 0$$

so that

$$\mathrm{Ext}^1_{\mu_{\mathbb{K}}(G)}\big(S^G_{H_i,V_i}, S^G_{H_{i-1},V_{i-1}}\big) \neq 0,$$

implying by the above mentioned result of [TW95] that $H_i \leq_G H_{i-1}$ or $H_{i-1} \leq_G H_i$. Moreover, by using [TW95, (14,6) Theorem] one conclude more that $H_i \trianglelefteq {}^gH_{i-1}$ or $H_{i-1} \trianglelefteq {}^gH_i$ for some $g \in G$.

Proposition 9.18 *Let $\mathbb{K}$ be of characteristic $p > 0$. Let M be a $\mu_{\mathbb{K}}(G)$-module having a unique maximal $\mu_{\mathbb{K}}(G)$-submodule, say $M/\mathrm{Jac}(M) \cong S^G_{H,V}$, and let K be a subgroup of G such that $\overline{M}(K) \neq 0$, and let X be a subgroup of G such that $S^G_{H,V}(X) \neq 0$. Then:*

(1) *$\overline{M}(H)$ has a unique maximal $\mathbb{K}\overline{N}_G(H)$-submodule, and the simple head of the $\mathbb{K}\overline{N}_G(H)$-module $\overline{M}(H)$ is isomorphic to V.*

(2) *$K \leq_G H$, and if $K \neq_G H$ then p divides $|N_G(K) : K|$.*

(3) *M is generated as a $\mu_{\mathbb{K}}(G)$-module by its value $M(X)$ at X.*

(4) *For any $K <_G H$,*

$$\overline{M}(K) = \sum_{K<Y \leq N_G(K):|Y:K|=p} r_K^Y\big(\overline{M}(Y)\big).$$

Proof: (1) Put $J = \mathrm{Jac}(M)$. We see that J is the unique largest element of the set of all subfunctors J' of M whose quotient M/J' has H as a minimal subgroup. Then 7.4 implies that $\overline{M}(H)$ has a unique maximal $\mathbb{K}\overline{N}_G(H)$-submodule, which is $J(H) = \mathrm{Jac}(\overline{M}(H))$. Moreover, evaluating the isomorphic functors M/J and $S_{H,V}^G$ at H we see that the head of $\overline{M}(H)$ is isomorphic to V.

(2) Choose a maximal subgroup L of G containing K subject to the condition $\overline{M}(L) \neq 0$. It follows from 9.3 that M has a maximal $\mu_{\mathbb{K}}(G)$-submodule whose simple quotient has L as a minimal subgroup. As M has a unique maximal $\mu_{\mathbb{K}}(G)$-submodule, $L =_G H$ so that $K \leq_G H$. Moreover, if $K \neq_G H$ then part (4) of 7.15 implies that p divides $\overline{N}_G(K)$.

(3) Put $J = \mathrm{Jac}(M)$, $A = \mu_{\mathbb{K}}(G)$ and $e = t_X^X$. The idempotent $e \in A$ does not annihilate the simple A-module M/J. Then part (7) of 7.1 implies that

$$AeM + J = M.$$

If $AeM \neq M$ then, as J contains every proper A-submodule of M, it follows that $J = M$, which is not the case. Hence $AeM = M$.

(4) This follows by part (3) of 7.15. □

The following dual version of the previous result may be justified similarly.

Proposition 9.19 *Let $\mathbb{K}$ be of characteristic $p > 0$. Let M be a $\mu_{\mathbb{K}}(G)$-module having a unique simple $\mu_{\mathbb{K}}(G)$-submodule, say $\mathrm{Soc}(M) \cong S_{H,V}^G$, and let K be a* *subgroup of G such that $\underline{M}(K) \neq 0$, and let X be a subgroup of G such that $S^G_{H,V}(X) \neq 0$. Then:*

(1) *$\underline{M}(H)$ has a unique simple $\mathbb{K}\overline{N}_G(H)$-submodule, and the simple socle of the $\mathbb{K}\overline{N}_G(H)$-module $\underline{M}(H)$ is isomorphic to V.*

(2) *$K \leq_G H$, and if $K \neq_G H$ then p divides $|N_G(K) : K|$.*

(3) *$(M :_e 0) = 0$ where $e = t^X_X$.*

(4) *For any $K <_G H$,*

$$0 = \bigcap_{K<Y\leq N_G(K):|Y:K|=p} \operatorname{Ker}\big(t^Y_K : \underline{M}(K) \to M(Y)\big).$$

Let A be a finite dimensional $\mathbb{K}$-algebra and V be an A-module. If V is isomorphic to a nonzero quotient module of a projective indecomposable A-module P then it is clear that the heads of P and V are isomorphic so that the head of V is a simple A-module. Conversely, if the head of V is isomorphic to a simple A-module S then there are A-module epimorphisms $\pi : V \to S$ and $f : P(S) \to S$ where $P(S)$ is the projective cover of S. By the projectivity of $P(S)$ we may find an A-module homomorphisms $\gamma : P(S) \to V$ satisfying $\pi \circ \gamma = f$. Using the relation $\pi \circ \gamma = f$ one sees that $\gamma : P(S) \to V$ is an epimorphism. Hence, an A-module has unique maximal submodule if and only if it is isomorphic to a nonzero quotient of a projective indecomposable A-module. In a similar way, one sees that a module has unique simple submodule if and only if it is isomorphic to a submodule of an injective indecomposable module.

As in [TW95] we denote by $P^G_{H,V}$ the projective cover of a simple $\mu_{\mathbb{K}}(G)$-module of the form $S^G_{H,V}$. Thus, 9.18 applies to $P^G_{H,V}$ and its nonzero quotients.

Remark 9.20 *Let M be a uniserial $\mu_{\mathbb{K}}(G)$-module. Then, for any subgroup H of G, the $\mathbb{K}\overline{N}_G(H)$-modules $\underline{M}(H)$ and $\overline{M}(H)$ are uniserial.*

Proof: Let T_1 and T_2 be $\mathbb{K}\overline{N}_G(H)$-submodules of $\underline{M}(H)$. By part (1) of 8.3 they are also eAe-submodules of $M(H)$ where $A = \mu_{\mathbb{K}}(G)$ and $e = t^H_H$. Therefore,

$eAT_i = T_i$ for each i. As M is a uniserial A-module, its A-submodules AT_1 and AT_2 must be comparable, say $AT_1 \subseteq AT_2$. Multiplying this containment by the idempotent e we get $T_1 \subseteq T_2$. Hence, $\underline{M}(H)$ is uniserial. Similar arguments may be used to justify the result for $\overline{M}(H)$. □

As an easy consequence of 9.18 and 4.8 we obtain the following criterion for a $\mu_{\mathbb{K}}(G)$-module to have a unique maximal submodule.

Remark 9.21 *Let M be a $\mu_{\mathbb{K}}(G)$-module. Then, M has a unique maximal $\mu_{\mathbb{K}}(G)$-submodule if and only if there is a subgroup H of G satisfying the following conditions:*

(i) *M is generated as a $\mu_{\mathbb{K}}(G)$-module by its value $M(H)$ at H.*

(ii) *$M(H)$ has a unique maximal $t_H^H \mu_{\mathbb{K}}(G) t_H^H$-submodule.*

Using 9.19 and 4.4 we obtain the following dual version of the previous result.

Remark 9.22 *Let M be a $\mu_{\mathbb{K}}(G)$-module. Then, M has a unique simple $\mu_{\mathbb{K}}(G)$-submodule if and only if there is a subgroup H of G satisfying the following conditions:*

(i) *$(M :_e 0) = 0$ where $e = t_H^H$.*

(ii) *$M(H)$ has a unique simple $t_H^H \mu_{\mathbb{K}}(G) t_H^H$-submodule.*

It is desirable to replace the second condition of 9.21 with a condition involving $\overline{M}(H)$ and $\mathbb{K}\overline{N}_G(H)$. This can be done if $\mathbb{K}$ is of characteristic $p > 0$ and G is a p-group, because in this case it follows from [TW95, (15.1) Lemma] that $S_{K,\mathbb{K}}^G(X) \neq 0$ implies $X =_G K$.

The next result is a slight general form of [TW95, (15.1) Lemma].

Lemma 9.23 *Let $\mathbb{K}$ be of characteristic $p > 0$ and $S^G_{H,V}$ be a simple $\mu_{\mathbb{K}}(G)$-module. Let K be a subgroup of G. Assume that either K is a normal subgroup of G or $\dim_{\mathbb{K}} V = 1$. Then, $S^G_{H,V}(K) \neq 0$ if and only if there is a $g \in G$ with ${}^gH \leq K$ satisfying the following conditions:*

(i) *$\overline{N}_K({}^gH)$ acts on ${}^gV$ trivially.*

(ii) *p does not divide $|N_K({}^gH) : {}^gH|$.*

Proof: We first try to find conditions equivalent to the condition $S^G_{H,V}(G) \neq 0:$ Using the isomorphism given in 2.10 and using the explicit description of induced functors given in 2.6 we see that

$$S^G_{H,V}(G) \cong S^{\overline{N}_G(H)}_{1,V}(\overline{N}_G(H)) = tr_1^{\overline{N}_G(H)}(V) \subseteq V^{\overline{N}_G(H)}$$

where tr denotes the relative trace map, because $S^{\overline{N}_G(H)}_{1,V}$ is the (unique simple) subfunctor of the fixed point functor $FP_V^{\overline{N}_G(H)}$ generated by

$$FP_V^{\overline{N}_G(H)}(1) = V,$$

see [TW] for more details about the fixed point functors. Note that $V^{\overline{N}_G(H)}$ is a submodule of the simple $\mathbb{K}\overline{N}_G(H)$-module V. Thus, if $S^G_{H,V}(G) \neq 0$ then

$$V = V^{\overline{N}_G(H)}$$

implying that $\overline{N}_G(H)$ acts on V trivially (i.e., V is the trivial module). Moreover, if V is the trivial module then we see that

$$S^G_{H,V}(G) \cong |N_G(H) : H|V.$$

Consequently, $S^G_{H,V}(G) \neq 0$ if and only if $\overline{N}_G(H)$ acts on V trivially and p does not divide $|N_G(H) : H|$.

Let K and V satisfy the conditions of the hypothesis. If K is normal or if $\dim_{\mathbb{K}} V = 1$, then Clifford's theorem for Mackey algebras [Yar1] or 4.20 implies respectively that $\downarrow^G_K S^G_{H,V}$ is semisimple. Thus,

$$0 \neq S^G_{H,V}(K) = (\downarrow^G_K S^G_{H,V})(K)$$

if and only if there is a simple $\mu_{\mathbb{K}}(K)$-module S direct summand of the semisimple $\mu_{\mathbb{K}}(K)$-module $\downarrow^G_K S^G_{H,V}$ such that $S(K) \neq 0$. It follows by 4.17 that simple direct summands of the semisimple $\mu_{\mathbb{K}}(K)$-module

$$\downarrow^G_K S^G_{H,V}$$

are precisely of the form $S^K_{{}^gH,W}$ where $g \in G$ with ${}^gH \leq K$ and W is a simple $\mathbb{K}\overline{N}_K({}^gH)$-submodule of ${}^gV$. Thus, $S^G_{H,V}(K) \neq 0$ if and only if $S^K_{{}^gH,W}(K) \neq 0$ for some $g \in G$ with ${}^gH \leq K$ and for some simple $\mathbb{K}\overline{N}_K({}^gH)$-submodule W of ${}^gV$. This is, by what we have proved in the first paragraph, equivalent to the requirements that W is the trivial $\mathbb{K}\overline{N}_K({}^gH)$-module and that p does not divide $|N_K({}^gH) : {}^gH|$. If $\dim_{\mathbb{K}} V = 1$ then $W = {}^gV$ so that the result follows.

Assume that K is normal in G. Then $N_K({}^gH)$ is normal in $N_G({}^gH)$. Take any simple $\mathbb{K}\overline{N}_K({}^gH)$-submodule U of ${}^gV$. Then Clifford's theorem for group algebras implies that any simple direct summand of the semisimple $\mathbb{K}\overline{N}_K({}^gH)$-module ${}^gV$ is an $\overline{N}_G({}^gH)$-conjugate of U. Therefore, $\mathbb{K}\overline{N}_K({}^gH)$ acts on U trivially if and only if it acts on ${}^gV$ trivially. □

Proposition 9.24 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group. Let M be a $\mu_{\mathbb{K}}(G)$-module. Then:*

(1) *M has a unique simple $\mu_{\mathbb{K}}(G)$-submodule if and only if there is a subgroup H of G satisfying the following conditions:*

- (i) *$(M :_e 0) = 0$ where $e = t^H_H$.*
- (ii) *$\underline{M}(H)$ has a unique simple $\mathbb{K}\overline{N}_G(H)$-submodule.*

(2) *M has a unique maximal $\mu_{\mathbb{K}}(G)$-submodule if and only if there is a subgroup H of G satisfying the following conditions:*

- (i) *M is generated as a $\mu_{\mathbb{K}}(G)$-module by its value $M(H)$ at H.*
- (ii) *$\overline{M}(H)$ has a unique maximal $\mathbb{K}\overline{N}_G(H)$-submodule.*

Proof: We only prove the first part. If M has a unique simple subfunctor, say of the form $S^G_{H,\mathbb{K}}$, then it follows from 9.19 that the subgroup H satisfies the desired conditions. Suppose that there is a subgroup H of G satisfying the given conditions. It follows from $(M :_e 0) = 0$ that M has no nonzero subfunctor whose evaluation at H is 0. Thus, if M has a simple subfunctor of the form $S^G_{K,\mathbb{K}}$ then $S^G_{K.\mathbb{K}}(H) \neq 0$ implying by 9.23 that $K =_G H$. Consequently, any simple subfunctor of M has H as a minimal subgroup. Now 8.4 implies that M has a unique simple subfunctor. □

Proposition 9.25 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group. Let H be a subgroup of G, and let M be a $\mu_{\mathbb{K}}(G)$-module. Put $e = t^H_H$ and*

$$b^0_H(M) = \sum_{H<K:|K:H|=p} r^K_H(M(K)) + b_H(M),$$

which is a $\mathbb{K}\overline{N}_G(H)$-module. Then, the maps

$$J \to J(H) \qquad \textit{and} \qquad (M :_e I) \leftarrow I$$

define a bijective correspondence between the maximal $\mu_{\mathbb{K}}(G)$-submodules J of M such that $M/J \cong S^G_{H,\mathbb{K}}$ and the maximal $\mathbb{K}\overline{N}_G(H)$-submodules I of $M(H)$ containing $b^0_H(M)$. Moreover,

$$b^0_H(M) \subseteq \mathrm{Jac}(M)(H)$$

and the $\mathbb{K}\overline{N}_G(H)$-module

$$\mathrm{Jac}(M)(H)/b^0_H(M)$$

is the radical of $M(H)/b^0_H(M)$.

Proof: Let J be a subfunctor of M such that $M/J \cong S^G_{H,\mathbb{K}}$. For any $K > H$, it follows from 9.23 that r^K_H annihilates M/J so that $r^K_H(M(K)) \subseteq J(H)$. We also know from 7.4 that $b_H(M) \subseteq J(H)$. Therefore, $J(H)$ contains $b^0_H(M)$.

Let I be a $\mathbb{K}\overline{N}_G(H)$-submodule of $M(H)$ containing $b^0_H(M)$. Take any $X > H$. By the transitivity of restriction maps (i.e., $r^B_A r^C_B = r^C_A$ for $A \leq B \leq C$) we see that $r^K_H(M(K)) \subseteq I$ for any $K > H$. Therefore,

$$\{x \in M(X) : c^g_{H^g} r^X_{H^g}(x) \in I, \forall g \in G, H^g \leq X\} = M(X)$$

so that we can deduce the maximality of the subfunctor $(M :_e I)$ from part (1) of 7.6. Now, the required bijection follows from 7.4.

For any maximal subfunctor J' of M with $M/J' \cong S^G_{K,\mathbb{K}}$, if $K \neq_G H$ then 9.23 implies that $J'(H) = M(H)$. Thus, $\mathrm{Jac}(M)(H)$ is the intersection of all $J(H)$ where J ranges over all maximal subfunctors of M with $M/J \cong S^G_{H,\mathbb{K}}$. By the bijective correspondence proved above, we see that $b^0_H(M) \subseteq \mathrm{Jac}(M)(H)$ and the quotient is the radical of $M(H)/b^0_H(M)$. □

Regarding simple subfunctors, one may prove the following similar to 9.25.

Proposition 9.26 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group. Let H be a subgroup of G, and let M be a $\mu_{\mathbb{K}}(G)$-module. Put $A = \mu_{\mathbb{K}}(G)$ and*

$$k^0_H(M) = \bigcap_{H<K:|K:H|=p} \mathrm{Ker}\big(t^K_H : \underline{M}(H) \to M(K)\big),$$

which is a $\mathbb{K}\overline{N}_G(H)$-module. Then, the maps

$$S \to S(H) \qquad and \qquad AT \leftarrow T$$

define a bijective correspondence between the simple $\mu_{\mathbb{K}}(G)$-submodules S of M such that $S \cong S^G_{H,\mathbb{K}}$ and the simple $\mathbb{K}\overline{N}_G(H)$-submodules T of $M(H)$ contained in $k^0_H(M)$. Moreover,

$$\mathrm{Soc}(M)(H) \subseteq k^0_H(M)$$

and the $\mathbb{K}\overline{N}_G(H)$-module $\mathrm{Soc}(M)(H)$ is the socle of $k^0_H(M)$.

Let V be a finite dimensional module of an algebra. For any natural number $i \geq 1$ we put

$$\mathrm{Jac}^i(V) = \mathrm{Jac}\big(\mathrm{Jac}^{i-1}(V)\big) \qquad \text{and} \qquad \mathrm{Soc}^i(V)/\mathrm{Soc}^{i-1}(V) = \mathrm{Soc}\big(V/\mathrm{Soc}^{i-1}(V)\big)$$

where $\mathrm{Jac}^0(V) = V$ and $\mathrm{Soc}^0(V) = 0$. One has the radical series

$$V = \mathrm{Jac}^0(V) \supset \mathrm{Jac}^1(V) \supset ... \supset \mathrm{Jac}^n(V) = 0$$

of V, and the socle series

$$0 = \mathrm{Soc}^0(V) \subset \mathrm{Soc}^1(V) \subset ... \subset \mathrm{Soc}^m(V) = V$$

of V. The lengths of the radical series and the socle series of V are equal (i.e., $n = m$), and it is called the Loewy length of V.

We next state a result giving a lower bound for Loewy lengths.

Proposition 9.27 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group. Let M be a $\mu_{\mathbb{K}}(G)$-module, and $H \leq K$ be subgroups of G with $|K : H| = p^n$. If*

$$t_H^K(M(H)) \neq 0 \qquad \text{or} \qquad r_H^K(M(K)) \neq 0,$$

then the Loewy length of M is greater than or equal to $n + 1$.

Proof: For any natural number k let $J_k = \mathrm{Jac}^k(M)$. If $X \leq Y$ are subgroups of G with $|Y : X| = p$, then it follows by 9.23 that both of the elements t_X^Y and r_X^Y of $\mu_{\mathbb{K}}(G)$ annihilate the semisimple $\mu_{\mathbb{K}}(G)$-modules, in particular J_k/J_{k+1}. This gives that

$$t_X^Y(J_k(X)) \subseteq J_{k+1}(Y) \qquad \text{and} \qquad r_X^Y(J_k(Y)) \subseteq J_{k+1}(X).$$

Using the transitivity of trace and restriction maps on a Mackey functor, the above argument can be used repeatedly to obtain that

$$t_A^B(J_k(A)) \subseteq J_{k+m}(B) \qquad \text{and} \qquad r_A^B(J_k(B)) \subseteq J_{k+m}(A)$$

where $|B : A| = p^m$. Therefore,

$$0 \neq t_H^K(M(H)) = t_H^K(J_0(H)) \subseteq J_n(H).$$

This shows that the Loewy length of M is at least $n + 1$. □

If $\mathbb{K}$ is of characteristic $p > 0$ and G is a p-group, then one may see that the Loewy length of the fixed point functor $FP_{\mathbb{K}}^G$ is $n + 1$ where $|G| = p^n$, see 14.1. As the restriction maps of $FP_{\mathbb{K}}^G$ are all injective, $r_1^G(FP_{\mathbb{K}}^G) \neq 0$ so that the lower bound obtained by 9.27 is attained by the Loewy length of $FP_{\mathbb{K}}^G$.

Chapter 10

Maximal subfunctors of Burnside functor

All the materials in this chapter comes from [Yar5, Section 7].

In this chapter we want to study the maximal subfunctors of the Burnside functor $\mathrm{B}_{\mathbb{K}}^G$ for G over $\mathbb{K}$.

We begin with recalling the maps between Burnside algebras of subgroups of G making $\mathrm{B}_{\mathbb{K}}^G$ a Mackey functor for G, see [Dr, Bo, TW95]. Let H be a subgroup of G. The set of isomorphism classes of finite H-sets form a commutative semiring under the operations disjoint union and cartesian product. The associated Grothendieck ring $\mathrm{B}_{\mathbb{Z}}(H)$ is called the Burnside ring of H. The Burnside algebra of H over $\mathbb{K}$ is the $\mathbb{K}$-algebra $\mathrm{B}_{\mathbb{K}}^G(H) = \mathbb{K} \otimes_{\mathbb{Z}} \mathrm{B}_{\mathbb{Z}}(H)$. Therefore, letting V runs over representatives of the conjugacy classes of subgroups of H, then $[H/V]$ comprise (without repetition) a $\mathbb{K}$-basis of $\mathrm{B}_{\mathbb{K}}^G(H)$, where the notation $[H/V]$ denotes the isomorphism class of transitive H-sets whose stabilizers are H-conjugates of V. The maps on $\mathrm{B}_{\mathbb{K}}^G$ are given as follows:

$$t_H^K([H/V]) = [K/V],$$
$$r_H^K([K/W]) = \sum_{HgW \subseteq K} [H/H \cap {}^gW],$$
$$c_H^g([H/U]) = [{}^gH/{}^gU].$$

For any prime number p and any natural number n we write n_p to denote the p-part of n.

Theorem 10.1 *Let $M = \mathrm{B}_{\mathbb{K}}^{G}$, and let H and K be subgroups of G. For any subgroup L of G we put*

$$M_L = \big(M :_{e_L} b_L(M)\big)$$

where $e_L = t_L^L$. Then:

(1) *Any maximal $\mu_{\mathbb{K}}(G)$-submodule of M is of the form M_L for some subgroup L of G.*

(2) *If M_H is a maximal $\mu_{\mathbb{K}}(G)$-submodule of M then $M/M_H \cong S_{H,\mathbb{K}}^{G}$.*

(3) *$M_H = M_K$ if and only if $H =_G K$.*

(4) *If $\mathbb{K}$ is of characteristic 0 then M_H is a maximal $\mu_{\mathbb{K}}(G)$-submodules of M.*

(5) *Let $\mathbb{K}$ be of characteristic $p > 0$. Then:*

(i) *M_H is a maximal $\mu_{\mathbb{K}}(G)$-submodules of M if and only if*

$$|N_G(H) : H|_p = 1.$$

(ii)

$$M/\mathrm{Jac}(M) \cong \bigoplus_{L \leq_G G : |N_G(L):L|_p = 1} S_{L,\mathbb{K}}^{G}.$$

(iii)

$$\mathrm{Jac}(M)(K) = \bigcap_{X \leq K : |N_G(X):X|_p = 1} \{x \in M(X) : r_X^K(x) \in b_X(M)\}.$$

Proof: (1) and (2) It follows from the relations

$$[L/V] = t_V^L([V/V]) \qquad \text{and} \qquad c_H^g([L/V]) = [{}^gL/{}^gV]$$

that $\overline{M}(L) \cong \mathbb{K}$, as $\mathbb{K}\overline{N}_G(L)$-modules, for any subgroup L of G. The result follows by 7.4.

(3) It follows by part (2) of 7.5

(4) In this case the Mackey algebra is semisimple by [TW], and so the result follows by part (2) of 7.6.

(5) Using the first three parts we see that M_H is maximal if and only if $S^G_{H,\mathbb{K}}$ appears in the head of M. For any $X > H$, as $r^X_H([X/X]) = [H/H]$ we see that

$$r^X_H(M(X)) + b_H(M) = M(H).$$

Thus, if p divides $|N_G(H) : H|$ then 7.16 implies that $S^G_{H,\mathbb{K}}$ does not appear in the head of M. On the other hand, if p does not divide $|N_G(H) : H|$ then part (4) of 7.15 implies that the multiplicity of $S^G_{H,\mathbb{K}}$ in the head of M is 1. These finish the proofs of parts (i) and (ii).

Jac(M) is the intersection of subfunctors M_X where X ranges over all subgroups X of G such that p does not divide $|N_G(X) : X|$. Therefore,

$$x \in \mathrm{Jac}(M)(X)$$

if and only $x \in M_X$ for any such subgroup X. The desired result follows by part (2) of 7.5. □

If we assume that $\mathbb{K}$ is algebraically closed then part (5)(ii) of 10.1 follows also by [TW95, (8.9) Corollary] which express $\mathrm{B}^G_{\mathbb{K}}$ as a direct sum of principal indecomposable $\mu_{\mathbb{K}}(G)$-modules.

Chapter 11

Radical series of Burnside functor

Almost all the materials in this chapter comes from [Yar5, Section 7].

In this chapter we study the radical series of the Burnside functor, mainly for a (an abelian) p-group over a field of prime characteristic $p > 0$. For example, we show that if $\mathbb{K}$ is of characteristic $p > 0$ then the simple $\mu_{\mathbb{K}}(G)$-module $S_{1,\mathbb{K}}^G$ appears (only) in J_m/J_{m+1} where $|G|_p = p^m$ and $J_k = \mathrm{Jac}^k(\mathrm{B}_{\mathbb{K}}^G)$.

Proposition 11.1 *Let $\mathbb{K}$ be of characteristic $p > 0$ and $M = \mathrm{B}_{\mathbb{K}}^G$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$. Let n be a natural number. Then:*

(1) *$J_n(H) = M(H)$ for any p-subgroup H of G with $|G : H|_p \geq p^n$.*

(2) *$J_n(H) = b_H(M)$ for any p-subgroup H of G with $|G : H|_p = p^{n-1}$, where $n \geq 1$.*

(3) *$J_n(H) = b_H(M)$ for any p-subgroup H of G with $|G : H|_p = p^{n-2}$, where $n \geq 2$.*

(4) *$J_{n+1}(H) = J_n(H)$ for any p-subgroup H of G with $|G : H|_p = p^{n-1}$, where $n \geq 1$.*

Proof: (1) Part (5)(iii) of 10.1 shows that the result is true for $n = 1$.

Assume that the result is true for n. Take any p-subgroup K of G with

$$|G : K|_p \geq p^{n+1}.$$

Our aim is to show that $J_{n+1}(K) = M(K)$.

As $J_n(K) = M(K)$, we see that $J_{n+1}(K) = M(K)$ if and only if evaluation of any simple summand of J_n/J_{n+1} at K is 0. Let $S^G_{L,U}$ be a simple summand of J_n/J_{n+1}. If $S^G_{L,U}(K) \neq 0$ then $L \leq_G K$ so that L is a p-subgroup of G with

$$|G : L|_p \geq p^{n+1}.$$

We will finish the proof by showing that there is no simple functor in the head of J_n that has X as a minimal subgroup where X is a p-subgroup of G with $|G : X|_p \geq p^{n+1}$. Let X be such a subgroup. It is clear that

$$J_n(X) = M(X) \qquad \text{and} \qquad b_X(J_n) = b_X(M),$$

and that $J_n(Y) = M(Y)$ for any $Y > X$ with $|Y : X| = p$. As $\overline{J_n}(X) \cong \mathbb{K}$, we see by using 7.4 that if $S^G_{X,V}$ appears in the head of J_n for some simple $\mathbb{K}\overline{N}_G(X)$-module V, then $V = \mathbb{K}$. Thus, it follows by 7.16 that the multiplicities of $S^G_{X,\mathbb{K}}$ in the heads of J_n and M are equal. But p divides $|N_G(X) : X|$, and so by 10.1 we see that $S^G_{X,\mathbb{K}}$ does not appear in the head of M.

(2) The result is true for $n = 1$ by part (5)(iii) of 10.1.

Assume that the result is true for n. Take any p-subgroup K of G with

$$|G : K|_p = p^n.$$

We want to show that $J_{n+1}(K) = b_K(M)$.

Using part (1) we see that $J_n(K) = M(K)$ and $b_K(J_n) = b_K(M)$. Let $X > K$ with $|X : K| = p$. Then $J_n(X) = b_X(M)$ by the assumption of the result for n. We calculate easily that

$$r^X_K(J_n(X)) = r^X_K(b_X(M)) \subseteq b_K(M),$$

and so $r^X_K(\overline{J_n}(X)) = 0$. Thus, 7.16 implies that $S^G_{K,\mathbb{K}}$ appears in the head of J_n. As $\overline{J_n}(K) = \overline{M}(K) \cong \mathbb{K}$ and as $b_K(J_n) = b_K(M)$, we deduce by 7.4 that J_n

has a unique maximal subfunctor I whose simple quotient has K as a minimal subgroup, and that I satisfies $I(K) = b_K(M)$.

For any p-subgroup Y of G with $|G : Y|_p \geq p^{n+1}$ it follows by part (1) that $\overline{J_n}(Y) \cong \mathbb{K}$ so that any simple functor having Y as a minimal subgroup and appearing in the head of J_n must be of the form $S^G_{Y,\mathbb{K}}$. Now 7.16 implies that the multiplicity of $S^G_{Y,\mathbb{K}}$ in the heads of J_n and M are equal. Thus, 10.1 gives that J_n has no simple functor in its heads with a minimal subgroup Y satisfying $|G : Y|_p \geq p^{n+1}$. Consequently, if J is a maximal subfunctor of J_n whose simple quotient J_n/J is nonzero at K, then J must be equal to I. Hence,

$$J_{n+1}(K) = I(K) = b_K(M)$$

because J_{n+1} is the intersection of maximal subfunctors of J_n.

(3) We first show that the result is true for $n = 2$: Let H be a p-subgroup of G with $|G : H|_p = 1$. By part (2) we obtain that $J_1(H) = b_H(M)$. For any p-subgroup X of G such that $|G : X|_p \geq p$, part (1) gives that $J_1(X) = M(X)$, in particular, $\overline{J_1}(H) = 0$ and $\overline{J_1}(X) \cong \mathbb{K}$. Thus 7.4 implies that if a simple functor whose minimal subgroup is a p-group appears in the head of J_n then it must be of the form $S^G_{X,\mathbb{K}}$ where X is a p-subgroup with $|G : X|_p \geq p$. Using 7.16 we see easily that the simple functors in the head of J_1 whose minimal subgroups are p-groups are precisely of the form $S^G_{K,\mathbb{K}}$ where K ranges over all subgroups of G with $|G : K|_p = p$. Now 9.23 implies that the evaluation of J_1/J_2 at H is 0. Hence, $J_2(H) = J_1(H) = b_H(M)$.

Assume that the result is true for n. Take a subgroup K of G with

$$|G : K|_p = p^{n-1}.$$

We want to justify that $J_{n+1}(K) = b_K(M)$.

Then $J_n(K) = b_K(M)$ by part (2), and $b_Z(J_n) = b_Z(M)$ for any p-subgroup Z with $|G : Z|_p \geq p^n$ by part (1). As in the first paragraph proving the result for $n = 2$, we may see that the simple functors in the head of J_n whose minimal subgroups are p-groups in some conjugate of K are of the form $S^G_{A,\mathbb{K}}$ where A are some subgroups of G with $|G:A|_p = p^n$. Thus, applying 9.23 again we see that the value of J_n/J_{n+1} at K is zero. Therefore,

$$J_{n+1}(K) = J_n(K) = b_K(M)$$

where the last equality follows from part (2).

(4) It follows by parts (2) and (3) that they both equal to $b_H(M)$. $\square$

Theorem 11.2 *Let $\mathbb{K}$ be of characteristic $p > 0$ and $M = \mathrm{B}_{\mathbb{K}}^G$. Let H be a p-subgroup of G, and V be a simple $\mathbb{K}\overline{N}_G(H)$-module, and let n be a natural number with $p^n \leq |G|_p$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$. Then:*

(1) *If $S_{H,V}^G$ appears in J_n/J_{n+1} then $|G:H|_p \leq p^n$ and $|G:H|_p \neq p^{n-1}$.*

(2) *If $|G:H|_p = p^n$ and $S_{H,V}^G$ appears in J_n/J_{n+1} then $V = \mathbb{K}$.*

(3) *If $|G:H|_p = p^n$ then the multiplicity of $S_{H,\mathbb{K}}^G$ in J_n/J_{n+1} is 1.*

(4) *The multiplicity of $S_{1,\mathbb{K}}^G$ in M is 1, and it appears in J_m/J_{m+1} where $p^m = |G|_p$.*

(5) *The Loewy length of M is greater than or equal to $m+1$.*

Proof: (1) If $|G:H|_p \geq p^{n+1}$ or $|G:H|_p = p^{n-1}$ then by 11.1 we obtain that $J_n(H) = J_{n+1}(H)$. Thus the result follows.

(2) It follows by 11.1 that $\overline{J_n}(H) \cong \mathbb{K}$. The conclusion $V = \mathbb{K}$ follows from 7.4.

(3) Let $|G:H|_p = p^n$ and let $X > H$ with $|X:H| = p$. Then 11.1 gives that

$$J_n(X) = b_X(M), \quad J_n(H) = M(H), \quad \text{and} \quad b_H(J_n) = b_H(M).$$

It is easy to see that

$$r_H^X(J_n(X)) = r_H^X(b_X(M)) \subseteq b_H(M).$$

Now, 7.16 shows that the multiplicity of $S^G_{H,\mathbb{K}}$ in J_n/J_{n+1} is 1.

(4) As $\dim_{\mathbb{K}} M(1) = 1$, it is clear that the multiplicity of $S^G_{1,\mathbb{K}}$ in M is 1 (see also 9.11). Moreover, we see by part (3) that $S^G_{1,\mathbb{K}}$ appears in J_m/J_{m+1} where $p^m = |G|_p$.

(5) This follows by part (4). □

The following result may be obtained by using the previous two results and their proofs.

Corollary 11.3 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group with $|G| \geq p^3$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then:*

(1)

$$J_2/J_3 \cong \Big(\bigoplus_{H \leq_G G : |G:H| = p^2} S^G_{H,\mathbb{K}} \Big) \bigoplus \lambda S^G_{G,\mathbb{K}},$$

where λ is the number of elements of the set

$$\{V \leq_G G : |G : V| = p\}.$$

(2)

$$J_3/J_4 \cong \Big(\bigoplus_{H \leq_G G : |G:H| = p^3} S^G_{H,\mathbb{K}} \Big) \bigoplus \Big(\bigoplus_{H \leq_G G : |G:H| = p} \lambda_H S^G_{H,\mathbb{K}} \Big),$$

where λ_H is the number of $\overline{N}_G(H)$-orbits of the set

$$\{V \leq_H H : |H : V| = p\}$$

on which $\overline{N}_G(H)$ acts by conjugation.

Proof: This can be justified by using 9.23 and 9.25 and by arguing as in (proofs of) the previous two results. Details left to the reader we give only some information about evaluations of radical terms J_i.

$$J_1(X) = \begin{cases} b_X(M) & ; \quad X = G \\ M(X) & ; \quad X \neq G \end{cases}$$

$$J_2(X) = \begin{cases} b_X(J_1) & ; \quad |G:X| \leq p \\ M(X) & ; \quad |G:X| \geq p^2 \end{cases}$$

$$J_3(X) = \begin{cases} b_X(J_2) & ; \quad |G:X| \leq p^2 \\ M(X) & ; \quad |G:X| \geq p^3 \end{cases}$$

$$J_4(X) = \begin{cases} b_X(J_3) & ; \quad X = G \\ * & ; \quad |G:X| = p \\ b_X(J_3) & ; \quad p^2 \leq |G:X| \leq p^3 \\ M(X) & ; \quad |G:X| \geq p^4 \end{cases}$$

The module * may not be $b_X(J_3)$ and may be complicated in general which defer us to find radicals J_k with $k \geq 5$. Indeed, let $H \leq G$ with $|G:H| = p$. Then, $J_4(H)/b^0_H(J_3)$ is equal by 9.25 to the radical of the $\mathbb{K}\overline{N}_G(H)$-module $J_3(H)/b^0_H(J_3)$. Thus, if $J_4(H) = b_H(J_3)$ then $\overline{J_3}(H)$ must be a semisimple $\mathbb{K}\overline{N}_G(H)$-module. We may see easily that

$$\overline{J_3}(H) = \bigoplus_{V \leq_H H : |H:V| = p} \mathbb{K}\big([H/V] + b_H(J_3)\big)$$

which is not necessarily semisimple. Moreover, if $J_4(H) = b_H(J_3)$ then we get by 9.25 that $b^0_H(J_3) = b_H(J_3)$. As

$$b_H(J_3) = \bigoplus_{V \leq_H H : |H:V| \geq p^2} \mathbb{K}[H/V] \quad \text{and} \quad J_3(G) = \bigoplus_{V \leq_G G : |G:V| \geq p^2} \mathbb{K}[G/V],$$

we compute that

$$\begin{aligned} b^0_H(J_3) &= r^G_H(J_3(G)) + b_H(J_3) \\ &= b_H(J_3) \bigoplus \underbrace{\Big(\bigoplus_{V \leq_G H : |H:V| = p, \ N_G(V) = H} \mathbb{K}\big(\sum_{gH \subseteq G} [H/{}^gV] \big) \Big)}_{b'_H(J_3)} . \end{aligned}$$

Consequently, if $J_4(H) = b_H(J_3)$ then $b'_H(J_3) = 0$ so that every subgroup of H whose index in G is p^2 must be normal in G. □

In the case of the previous result, one sees that $J_{k+1}(X) = b_X(J_k)$ for any $k \in \{0, 1, 2\}$ and any $X \leq G$ with $|G:X| \leq p^k$. However, this may not be true for $k \geq 3$ unless G is abelian.

Lemma 11.4 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| \geq p^3$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^G$. Then, $J_{n+1}(H) = b_H(J_n)$ for any n and any $H \leq G$ with $|G : H| \leq p^n \leq |G|$.*

Proof: Let X be a subgroup of G. As G is abelian, $\overline{N}_G(X)$ acts on $M(X)$ trivially so that $M(X)$, and hence each nonzero quotient of each $J_k(X)$, is a semisimple $\mathbb{K}\overline{N}_G(X)$-module. Then, 9.25 shows that $J_{k+1}(X) = b_X^0(J_k)$ for any k and any X.

We will prove the result by induction on n. It may be seen easily by using 11.1 that the result is true for $n = 0, 1$. Assume that the result is true for n. Take any subgroup K of G with $|G : K| \leq p^{n+1}$. We want to obtain that

$$J_{n+2}(K) = b_K(J_{n+1}).$$

By the above, $J_{n+2}(K) = b_K^0(J_{n+1})$. Let $Y > K$ with $|Y : K| = p$. Then,

$$|G : Y| \leq p^n$$

implying by the assumption of the result for n that $J_{n+1}(Y) = b_Y(J_n)$. Using the Mackey axiom we see that

$$r_K^Y(J_{n+1}(Y)) = \sum_{Z<Y:Y=KZ} t_{K\cap Z}^K r_{K\cap Z}^Z(J_n(Z)).$$

From the condition $Y = KZ$ it follows that $K \cap Z < Z$ and $K \cap Z < K$. As J_n/J_{n+1} is semisimple, 9.23 implies that the element $r_{K\cap Z}^Z$ of $\mu_{\mathbb{K}}(G)$ annihilates J_n/J_{n+1}. This gives that

$$r_{K\cap Z}^Z(J_n(Z)) \subseteq J_{n+1}(K \cap Z).$$

Therefore,

$$r_K^Y(J_{n+1}(Y)) \subseteq \sum_{Z<Y:Y=KZ} t_{K\cap Z}^K\big(J_{n+1}(K \cap Z)\big) \subseteq b_K(J_{n+1}).$$

Consequently, $b_K^0(J_{n+1}) = b_K(J_{n+1})$ proving that $J_{n+2}(K) = b_K(J_{n+1})$. □

For any rational number r we denote by $\lfloor r \rfloor$ the largest integer which is less than or equal to r.

Theorem 11.5 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. Let H be a subgroup of G with $|G : H| = p^m$ and n be a natural number with $m \leq n-1$ and $p^n \leq |G|$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^G$. Then:*

(1)

$$J_n(H) = \bigoplus_{V \leq H : |H:V| \geq p^{s+1}} \mathbb{K}[H/V],$$

where $s = \lfloor (n-m-1)/2 \rfloor$.

(2) *$S_{H,\mathbb{K}}^G$ does not appear in J_n/J_{n+1} if and only if $n-m$ is an odd number.*

(3) *Suppose that $n-m$ is an even number. Then, the multiplicity of $S_{H,\mathbb{K}}^G$ in J_n/J_{n+1} is equal to the number of elements of the set*

$$\{V \leq H : |H : V| = p^{(n-m)/2}\}.$$

Proof: (1) For any nonnegative integer i, we see that $i \leq s$ if and only if $m+i \leq n-i-1$. Thus, if $i \leq s$ then we get by 11.4 that

$$J_{n-i}(X) = b_X(J_{n-i-1})$$

for any $X \leq G$ with $|G : X| \leq p^{n-i-1}$. Moreover, by the transitivity of trace maps on a Mackey functor (i.e, $t_B^A t_C^B = t_C^A$ for $C \leq B \leq A$) we see that $b_K(M)$ is the sum of $\mathbb{K}$-subspaces of $M(K)$ of the form $t_L^K(M(L))$ where L ranges over all subgroups of K satisfying $|L : K| = p$.

The result will follow by repeated applications of 11.4. To illustrate it, assuming $s \geq 2$, we see that

$$\begin{aligned} J_n(H) &= b_H(J_{n-1}) \\ &= \sum_{X_1 \leq H : |H:X_1| = p} t_{X_1}^H (J_{n-1}(X_1)) \\ &= \sum_{X_1 \leq H : |H:X_1| = p} t_{X_1}^H (b_{X_1}(J_{n-2})) \\ &= \sum_{X_1 \leq H : |H:X_1| = p} t_{X_1}^H \sum_{X_2 \leq X_1 : |X_1:X_2| = p} t_{X_2}^{X_1} (J_{n-2}(X_2)) \\ &= \sum_{X_2 \leq H : |H:X_2| = p^2} t_{X_2}^H (J_{n-2}(X_2)). \end{aligned}$$

By the explanation given in the first paragraph of the proof we can apply 11.4 to $J_n(H)$ as above s-times to obtain

$$J_n(H) = \sum_{Y \leq H : |H:Y| = p^s} t_Y^H(J_{n-s}(Y)).$$

It is clear that

$$n - m - 2 \leq 2s \leq n - m - 1,$$

and so

$$(n - s) - 2 \leq m + s \leq (n - s) - 1.$$

As $|G : Y| = p^{m+s}$ we must have by 11.1 that $J_{n-s}(Y) = b_Y(M)$. Hence, the result follows.

(2) It is a consequence of 9.23 that $S^G_{H,\mathbb{K}}$ does not appear in J_n/J_{n+1} if and only if $J_n(H) = J_{n+1}(H)$, which is, by part (1), equivalent to the requirement that

$$\lfloor (n - m - 1)/2 \rfloor = \lfloor (n - m)/2 \rfloor.$$

The result is clear now.

(3) Suppose that $n - m$ is even. Then, 9.23 implies that the multiplicity of $S^G_{H,\mathbb{K}}$ in J_n/J_{n+1} is equal to the dimension of $J_n(H)/J_{n+1}(H)$. The result now follows by part (1). □

By part (1) of 11.5 we know the evaluations $J_n(H)$ where G is an abelian p-group, n is a natural number with $p^n \leq |G|$, and H is a subgroup of G with $|G : H| \leq p^{n-1}$. For a subgroup H of G with $|G : H| \geq p^n$ we already knew by part (1) of 11.1 that $J_n(H) = M(H)$. Moreover, if $|G : H| \geq p^n$ then the integer s in part (1) of 11.5 is a negative integer so that every subgroup V of H satisfies $|H : V| \geq p^{s+1}$. The conclusion is that we can drop the condition $m \leq n - 1$ from the hypothesis of part (1) of 11.5.

The following is an immediate consequence of 11.2 and 11.5.

Corollary 11.6 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then, for any* *natural number n with $p^n \leq |G|$ we have:*

$$J_n/J_{n+1} \cong \bigoplus_{l=0}^{\lfloor n/2 \rfloor} \Big(\bigoplus_{H \leq G : |G:H| = p^{n-2l}} \lambda_H^l S_{H,\mathbb{K}}^G \Big)$$

where λ_H^l is the number of elements of the set $\{V \leq H : |H : V| = p^l\}$.

Let G be an abelian p-group with $|G| = p^n$. To study the the radical factors J_{n+r}/J_{n+r+1} of $\mathrm{B}_{\mathbb{K}}^G$, where $r \geq 1$, we first extend 11.4 to other cases.

Lemma 11.7 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| = p^n$. Let $r \geq 1$ be a natural number. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^G$. Then, $J_{n+r}(H) = b_H(J_{n+r-1})$ for any subgroup H of G.*

Proof: The result is true for $r = 1$ by 11.4.

Assume that the result is true for r.

As each $M(H)$ is a semisimple $\mathbb{K}\overline{N}_G(H)$-module, it follows by 9.25 that

$$J_{n+r+1}(H) = b_H^0(J_{n+r}).$$

It can be seen by arguing as in the proof of 11.4 that

$$b_H^0(J_{n+r}) = b_H(J_{n+r}).$$

□

The radical factors of $\mathrm{B}_{\mathbb{K}}^G$ not covered in 11.5 is the content of the next result.

Theorem 11.8 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| = p^n$. Let H be a subgroup of G with $|G : H| = p^m$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^G$. Then:*

(1)

$$J_k(H) = \bigoplus_{V \le H : |H:V| \ge p^{s+1}} \mathbb{K}[H/V],$$

where $s = \lfloor (k-m-1)/2 \rfloor$.

(2) *Assume that* $k \ge n+1$. *Then,* $S^G_{H,\mathbb{K}}$ *appears in*

$$J_k/J_{k+1}$$

if and only if $k-m$ *is an even number satisfying*

$$(k-m)/2 \le (n-m).$$

Moreover, in this case, the multiplicity of $S^G_{H,\mathbb{K}}$ *in*

$$J_k/J_{k+1}$$

is equal to the number of elements of the set

$$\{V \le H : |H:V| = p^{(k-m)/2}\}.$$

Proof: (1) We may assume that $k = n+r$ where $r \ge 1$ is a natural number, because the result is true for $k \le n$ by the virtue of part (1) of 11.5. It follows by repeated applications of 11.7 that

$$J_{n+r}(H) = \sum_{X \le H : |H:X| = p^r} t^H_X(J_n(X)).$$

Then, part (1) of 11.5 implies that

$$J_{n+r}(H) = \bigoplus_{V \le H : |H:V| = p^{s'+r+1}} \mathbb{K}[H/V]$$

where $s' = \lfloor (n-m-r-1)/2 \rfloor$. The result follows because

$$s' + r = \lfloor (n+r-m-1)/2 \rfloor.$$

(2) It follows by 9.23 that $S^G_{H,\mathbb{K}}$ appears in J_k/J_{k+1} if and only if

$$J_k(H) \ne J_{k+1}(H).$$

Note also that if $J_k(H) \neq J_{k+1}(H)$ then $J_k(H) \neq 0$ so that $|H| \geq p^{s+1}$ by part (1). Therefore, part (1) gives the equivalency of $J_k(H) \neq J_{k+1}(H)$ to the conditions

$$\lfloor (k-m-1)/2 \rfloor \neq \lfloor (k-m)/2 \rfloor \qquad \text{and} \qquad (n-m) \geq \lfloor (k-m-1)/2 \rfloor .$$

The result now follows easily. □

The following obvious consequence of 11.8 deals with the cases not contained in 11.6.

Corollary 11.9 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| = p^n$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^G$. Then, for any $k \geq n+1$,*

$$J_k/J_{k+1} \cong \bigoplus_{l=k-n}^{\lfloor k/2 \rfloor} \Big(\bigoplus_{H \leq G : |G:H| = p^{k-2l}} \lambda_H^l S_{H,\mathbb{K}}^G \Big)$$

where λ_H^l is the number of elements of the set

$$\{V \leq H : |H : V| = p^l\}.$$

In particular, the Loewy length of M is $2n+1$.

Remark 11.10 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group with $|G| = p^n$. Then, the Loewy length of $\mathrm{B}_{\mathbb{K}}^G$ is greater than or equal to $2n+1$.*

Proof: As $t_1^G r_1^G([G/G]) = [G/1]$,

$$t_1^G r_1^G(\mathrm{B}_{\mathbb{K}}^G(G)) \neq 0.$$

The proof of 9.27 shows that the Loewy length of $\mathrm{B}_{\mathbb{K}}^G$ is greater than or equal to $2n+1$. □

For any subgroup H of G it is obvious that $\downarrow_H^G \mathrm{B}_{\mathbb{K}}^G = \mathrm{B}_{\mathbb{K}}^H$.

Remark 11.11 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. Let H be a subgroup of G with $|G : H| = p^m$. Then, for any natural number k with $k \geq m$ we have:*

$$\downarrow_H^G \mathrm{Jac}^k(\mathrm{B}_{\mathbb{K}}^G) = \mathrm{Jac}^{k-m}(\mathrm{B}_{\mathbb{K}}^H).$$

Proof: It can be obtained easily by using part (1) of 11.8. □

As an example obtained from 11.6 and 11.8 we next record

Example 11.12 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a cylic group of order p^4. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^G$. Let $1 < H_1 < H_2 < H_3 < G$ be the lattice of the subgroups of G so that $|H_i| = p^i$ for each i. Then:*

$$M = J_0 \supset J_1 \supset J_2 \supset J_3 \supset J_4 \supset J_5 \supset J_6 \supset J_7 \supset J_8 \supset J_9 = 0,$$

where

$$\begin{array}{rcc} J_0/J_1 & \cong & S_{G,\mathbb{K}}^G \\ J_1/J_2 & \cong & S_{H_3,\mathbb{K}}^G \\ J_2/J_3 & \cong & S_{H_2,\mathbb{K}}^G \oplus S_{G,\mathbb{K}}^G \\ J_3/J_4 & \cong & S_{H_1,\mathbb{K}}^G \oplus S_{H_3,\mathbb{K}}^G \\ J_4/J_5 & \cong & S_{1,\mathbb{K}}^G \oplus S_{H_2,\mathbb{K}}^G \oplus S_{G,\mathbb{K}}^G \\ J_5/J_6 & \cong & S_{H_1,\mathbb{K}}^G \oplus S_{H_3,\mathbb{K}}^G \\ J_6/J_7 & \cong & S_{H_2,\mathbb{K}}^G \oplus S_{G,\mathbb{K}}^G \\ J_7/J_8 & \cong & S_{H_3,\mathbb{K}}^G \\ J_8/J_9 & \cong & S_{G,\mathbb{K}}^G \end{array}$$

One may see the symmetry of the diagram showing the radical layers of the functor in the previous example. Indeed, up to multiplicities of simple functors in radical layers, the shape of the diagram showing the radical layers of $\mathrm{B}_{\mathbb{K}}^G$, where G is an abelian p-group, is still symmetric.

Remark 11.13 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| = p^n$. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^G$. Let r be a natural number with $1 \leq r \leq n$ and let H be a subgroup of G. Then, $S_{H,\mathbb{K}}^G$ appears in*

$$J_{n-r}/J_{n-r+1}$$

if and only if $S_{H,\mathbb{K}}^G$ appears in

$$J_{n+r}/J_{n+r+1}.$$

However, the multiplicities may be different.

Proof: Let $|G : H| = p^m$.

Suppose that $S^G_{H,\mathbb{K}}$ appears in J_{n-r}/J_{n-r+1}. It follows by 11.2 and 11.5 that $m \leq n-r$ and $n-r-m$ is an even number. Then, $n+r \geq n+1$, the number $n+r-m$ is even, and $(n+r-m)/2 \leq (n-m)$. Thus, we see by part (2) of 11.8 that $S^G_{H,\mathbb{K}}$ appears in J_{n+r}/J_{n+r+1}.

Converse part may be proved similarly. □

Chapter 12

Minimal subfunctors of Burnside functor

All the materials in this chapter comes from [Yar5, Section 7].

Here we want to study the minimal subfunctors of $\mathrm{B}_{\mathbb{K}}^{G}$ where $\mathbb{K}$ is of characteristic $p > 0$ and G is a (abelian) p-group. This turns out to be harder than the study of the radical series we presented in this section because determination of restriction kernels of $\mathrm{B}_{\mathbb{K}}^{G}$ is much harder than determination of Brauer quotients of $\mathrm{B}_{\mathbb{K}}^{G}$, all of which were isomorphic to trivial modules.

For any finite group H we use the notation $\Phi(H)$ to denote the Frattini subgroup of H which is the intersection of all maximal subgroups of H. It is the set of all nongenerators of H so that $\Phi(H)X \neq H$ for any proper subgroup X of H.

Lemma 12.1 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be a p-group. Put $M = \mathrm{B}_{\mathbb{K}}^{G}$. For any subgroups K and L of G we have:*

(1) *If $K \leq L$ with $|L : K| = p$ then*

$$\mathrm{Ker}(r_K^L : M(L) \to M(K)) \subseteq \Big(\bigoplus_{V \leq_L L : N_L(V) \nleq K} \mathbb{K}[L/V] \Big).$$

(2)

$$\Big(\bigoplus_{V \trianglelefteq L : V \leq \Phi(L)} \mathbb{K}[L/V] \Big) \subseteq \underline{M}(L) \subseteq \Big(\bigoplus_{V \trianglelefteq L} \mathbb{K}[L/V] \Big).$$

(3)

$$\big(k_L^0(M) \big)^{\overline{N}_G(L)} \subseteq \Big(\bigoplus_{V \leq_L L : N_G(V) = L} \mathbb{K}[L/V] \Big).$$

Proof: (1) Let $K \trianglelefteq L$ with $|L : K| = p$. Any subgroup V of L satisfies exactly one of the three conditions:

$$N_L(V) \leq K; \quad V \leq K \not\geq N_L(V); \quad V \not\leq K.$$

As these conditions closed under taking L-conjugates of V, we can write the set of L-conjugacy classes of subgroups of L as a disjoint union of the three sets:

$$\mathcal{B}_1 = \{V \leq_L L : N_L(V) \leq K\},$$

$$\mathcal{B}_2 = \{V \leq_L L : V \leq K \not\geq N_L(V)\},$$

$$\mathcal{B}_3 = \{V \leq_L L : V \not\leq K\}.$$

Thus, letting

$$B_i = \bigoplus_{V \in \mathcal{B}_i} \mathbb{K}[L/V],$$

we may write

$$M(L) = B_1 \oplus B_2 \oplus B_3$$

as $\mathbb{K}$-spaces. Using the definitions of restriction maps on M it is easy to verify the three properties:

$$r_K^L : B_1 \to M(K) \text{ is injective}; \;\; r_K^L(B_2) = 0; \;\; r_K^L(B_1) \cap r_K^L(B_3) = 0.$$

Now, let $x \in M(L)$ and write

$$x = x_1 + x_2 + x_3$$

where $x_i \in B_i$ for each i. If $r_K^L(x) = 0$ then it follows by the above properties that $x_1 = 0$. This completes the proof.

(2) Let $x \in \underline{M}(L)$. Assume that there is a nonnormal subgroup V of L such that $[L/V]$ appears in x with nonzero coefficient. We can choose a maximal subgroup K of L containing $N_L(V)$. Then $|L : K| = p$ and $x \in \mathrm{Ker} r_K^L$. But this is impossible by part (1). The other inclusion is obvious.

(3) Let

$$x \in \left(k_L^0(M)\right)^{\overline{N}_G(L)}.$$

Take a subgroup $\overline{X} = X/L$ of $\overline{N}_G(L)$ of order p. Then,

$$x \in \mathrm{Ker}(t_L^X : \underline{M}(L)^{\overline{X}} \to M(X))$$

(see 9.26). It follows by part (2) that

$$x \in \mathrm{Ker}(t_L^X : U^{\overline{X}} \to M(X))$$

where

$$U = \bigoplus_{V \trianglelefteq L} \mathbb{K}[L/V].$$

The $\mathbb{K}\overline{X}$-module U is a permutation module with a permutation basis

$$S = \{[L/V] : V \trianglelefteq L\}.$$

The $\overline{X}$-orbit sums of S form a $\mathbb{K}$-basis of $U^{\overline{X}}$. As the order of $\overline{X}$ is p, the sizes of $\overline{X}$-orbits of S are 1 or p. It is is obvious that the image under t_L^X of any orbit sum of size p is 0. Furthermore, if V and W are normal subgroups of L such that

$$N_X(V) = X = N_X(W)$$

(equivalently, the sizes of orbits containing each are both equal to 1) then

$$t_L^X([L/V]) = [X/V] \qquad \text{and} \qquad t_L^X([L/W]) = [X/W]$$

are distinct basis elements of $M(X)$. If we write x as a linear combination of $\overline{X}$-orbit sums of S then we see that the coefficient of any orbit sum of size 1 must be 0. Therefore, x can be written as a linear combination of elements of $M(L)$ of the form $[L/V]$ with $N_X(V) = L$.

To finish, if $[L/V]$ with $V \trianglelefteq L$ and with $N_G(V) \neq L$ appears in x, then we may choose a subgroup of Y/L of $N_G(V)/L$ of order p. Then $N_Y(V) = Y$, which is impossible, because what we have observed above implies that $N_Y(V) = L$. $\square$

Proposition 12.2 *Let $\mathbb{K}$ be of characteristic $p > 0$. Let G be a p-group and H be a subgroup of G. Put $M = \mathrm{B}_{\mathbb{K}}^{G}$. Then:*

(1) *If $S_{H,\mathbb{K}}^{G}$ appears in $\mathrm{Soc}(M)$ then $H = N_G(V)$ for some subgroup V of H.*

(2) *The multiplicity of $S_{G,\mathbb{K}}^{G}$ in $\mathrm{Soc}(M)$ is equal to $\dim_{\mathbb{K}} \underline{M}(G)$, which is nonzero.*

(3) *$S_{N_G(\Phi(H)),\mathbb{K}}^{G}$ appears in $\mathrm{Soc}(M)$.*

(4) *If G is abelian, then $\mathrm{Soc}(M)(G) = \underline{M}(G)$ and $\mathrm{Soc}(M)(X) = 0$ for any proper subgroup X of G.*

Proof: (1) Let S be a simple subfunctor of M such that S is isomorphic to $S_{H,\mathbb{K}}^{G}$. It follows by 9.26 that $S(H) \subseteq k_H^0(M)$. As $\overline{N}_G(H)$ acts on $S(H) \cong \mathbb{K}$ trivially, we must have that

$$S(H) \subseteq \left(k_H^0(M)\right)^{\overline{N}_G(H)}.$$

In particular, $\left(k_H^0(M)\right)^{\overline{N}_G(H)} \neq 0$. The result follows by part (3) of 12.1.

(2) It follows by part (2) of 12.1 and by 9.3.

(3) By part (2) we may assume that $N_G(\Phi(H)) \neq G$. For any subgroup V of G with $N_G(V) \neq G$ we put

$$x_V = \sum_{gN_G(V) \subseteq N_G(N_G(V))} [N_G(V)/{}^gV].$$

It is easy to see that an element $g \in N_G(N_G(V))$ satisfies

$$[N_G(V)/V] = [N_G(V)/{}^gV]$$

if and only if $g \in N_G(V)$. This shows that

$$x_V \in M(N_G(V))^{N_G(N_G(V))}.$$

Take any $K \geq N_G(V)$ with $|K : N_G(V)| = p$. Then,

$$N_G(V) \trianglelefteq K \leq N_G(N_G(V))$$

so that

$$x_V = \sum_{N_G(V)a\subseteq K} c^a_{N_G(V)}\Big(\sum_{Kb\subseteq N_G(N_G(V))}[N_G(V)/{}^bV]\Big),$$

implying that

$$t^K_{N_G(V)}(x_V) = |K : N_G(V)| \sum_{Kb\subseteq N_G(N_G(V))}[K/{}^bV] = 0.$$

Letting now $V = \Phi(H)$ and $L = N_G(V)$ we see by the above and by part (2) of 12.1 that

$$x_V = k^0_L(M)^{N_G(L)}.$$

Thus, $\mathbb{K}x_V$ is a $\mathbb{K}\overline{N}_G(L)$-submodule of $k^0_L(M)$ isomorphic to the trivial module $\mathbb{K}$, in particular it is simple. Hence, 9.26 implies that $S^G_{N_G(V),\mathbb{K}}$ appears in $\mathrm{Soc}(M)$.

(4) If $S^G_{H,\mathbb{K}}$ appears in $\mathrm{Soc}(M)$ then part (1) implies that $H = G$. The result follows by 9.26. □

Part (1) of 12.2 is a special case of [Ni, Proposition 2.4], that can also be obtained by using it. Moreover, calculating the dimension of $\underline{M}(G)$, where $M = \mathrm{B}^G_{\mathbb{K}}$, is not easy even for small abelian p-groups. See [Ni, Section 3] where this dimension is calculated for some abelian p-groups.

As the Mackey algebra $\mu_{\mathbb{K}}(G)$ is not self-injective unless p^2 does not divide $|G|$ (see [TW95, (19.2) Theorem]), the socle of a principal indecomposable $\mu_{\mathbb{K}}(G)$-module $P^G_{H,V}$ may not be isomorphic to $S^G_{H,V}$. Thus, determination of the socle of a $\mu_{\mathbb{K}}(G)$-module of the form $P^G_{H,V}$ is not out of interest and studied in [Ni]. In particular, letting $\mathbb{K}$ be algebraically closed and G be a p-group, it is shown in [Ni, Proposition 2.4] by using a filtration of projective functors described in [We2] that if $S^G_{K,\mathbb{K}}$ appears in $\mathrm{Soc}(P^G_{H,\mathbb{K}})$ then $K = N_H(L)$ for some $L \le H$. In the general case, by the category equivalence described in [TW95, Section 10], finding

$$\mathrm{Soc}(P^G_{H,V})$$

is equivalent to finding

$$\mathrm{Soc}\big(P^{\overline{N}_G(J)}_{H/J,V}\big)$$

where $J = O^p(H)$. Thus, to understand socles of principal indecomposable functors one has to find the socle of a $\mu_{\mathbb{K}}(G)$-module of the form $P^G_{H,V}$ where H is a

p-group. Moreover, letting $\mathbb{K}$ be algebraically closed, we have by [TW95, (8.6) Theorem] that $P^G_{H,V}$ is a direct summand of $\uparrow^G_H \mathrm{B}^H_{\mathbb{K}}$. Therefore, studying the socle of the Burnside functor $\mathrm{B}^H_{\mathbb{K}}$ for a p-subgroup of H of G is important for the determination of the socle of $P^G_{H,V}$. Regarding this problem we only state the following.

Proposition 12.3 *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$, and let K and H be subgroups of G. Suppose that W is a simple $\mathbb{K}\overline{N}_G(H)$-module and V is a simple $\mathbb{K}\overline{N}_G(K)$-module. Then:*

(1) *Assume that H is a p-subgroup of G. If a simple $\mu_{\mathbb{K}}(G)$-module S appears in the socle of $P^G_{H,W}$, then $S(N_H(L)) \neq 0$ for some $L \leq H$.*

(2) *Assume that H is a p-subgroup of G and $\dim_{\mathbb{K}} V = 1$. If $S^G_{K,V}$ appears in the socle of $P^G_{H,W}$, then $K =_G N_H(L)$ for some $L \leq H$.*

(3) *Assume that H is a normal p-subgroup of G. If $S^G_{K,V}$ appears in the socle of $P^G_{H,W}$, then $K = N_H(L)$ for some $L \leq H$.*

(4) *If $\overline{N}_G(H)$ is a p-group, then $S^G_{H,\mathbb{K}}$ appears in the socle of $P^G_{H,\mathbb{K}}$ with multiplicity equal to $\dim_{\mathbb{K}} \underline{T}(H)$, where $T = \mathrm{B}^H_{\mathbb{K}}$.*

(5) *Assume that H is a p-subgroup of G. Then, for any simple $\mathbb{K}\overline{N}_G(H)$-module U there is a simple $\mathbb{K}\overline{N}_G(H)$-module U' such that $S^G_{H,U}$ appears in the socle of $P^G_{H,U'}$.*

Proof: (1) Let $B = \mu_{\mathbb{K}}(H)$ and $T = \mathrm{B}^H_{\mathbb{K}}$. Suppose that S appears in the socle of $P^G_{H,W}$. As $P^G_{H,W}$ is a direct summand of $\uparrow^G_H T$, it follows by the adjointness of the pair $(\downarrow^G_H, \uparrow^G_H)$ that

$$\mathrm{Hom}_B(\downarrow^G_H S, T) \neq 0.$$

Let $\mathcal{X} = \{N_H(X) : X \leq H\}$ and $e = e_{\mathcal{X}}$ be the idempotent of B defined as in 4.21 by

$$e_{\mathcal{X}} = \sum_{X \in \mathcal{X}} t^X_X.$$

Part (1) of 12.2 implies that T has no nonzero B-submodule annihilated by e. Then, by part (1) of 4.10, we see that $\mathrm{Hom}_{eBe}(eS, eT) \neq 0$. In particular $eS \neq 0$, implying the result.

(2) and (3) They follow by part (1) and by 9.23.

(4) Let $T = \mathrm{B}_{\mathbb{K}}^{H}$. Using 9.5 we see that the $\mathbb{K}\overline{N}_G(H)$-modules

$$\underline{(\uparrow_H^G T)}(H) \qquad \text{and} \qquad n\mathbb{K}\overline{N}_G(H)$$

are isomorphic where $n = \dim_{\mathbb{K}} \underline{T}(H)$.

It is clear that taking restriction kernels respects finite direct sums. Indeed, for any Mackey functor M for G, we have by part (2) of 8.3 that

$$\underline{M}(H) = (M :_f 0)(H)$$

for some idempotent f. So, part (6) of 7.1 implies that taking restriction kernels respects finite direct sums. This fact is also immediate from the isomorphism

$$(L^-{}_{N_G(H)/H} \downarrow_{N_G(H)}^G M)(H/H) \cong \underline{M}(H)$$

of $\mathbb{K}\overline{N}_G(H)$-modules, because the functors L^- and $\downarrow$ respect finite direct sums.

For a principal indecomposable $\mu_{\mathbb{K}}(G)$-module $P = P_{Y,U}^G$ it follows by 9.5 that if $\underline{P}(H) \neq 0$ then $H \leq_G Y$. Thus, using the formula [TW95, (8.6) Theorem] expressing $\uparrow_H^G T$ as a direct sum of principal indecomposable $\mu_{\mathbb{K}}(G)$-modules, we see that

$$\underline{(\uparrow_H^G T)}(H) \cong \underline{P_{H,\mathbb{K}}^G}(H).$$

Hence, the multiplicity of $S_{H,\mathbb{K}}^G$ in the socle of $P_{H,\mathbb{K}}^G$ is equal by 9.3 to n.

(5) As

$$\downarrow_H^G S_{H,U}^G \cong (\dim_{\mathbb{K}} U) S_{H,\mathbb{K}}^H$$

and as $S_{H,\mathbb{K}}^H$ appears in the socle of $\mathrm{B}_{\mathbb{K}}^H$ (by part (2) of 12.2), we see by using the adjointness of the pair $(\downarrow_H^G, \uparrow_H^G)$ that $S_{H,U}^G$ appears in the socle of $\uparrow_H^G \mathrm{B}_{\mathbb{K}}^H$. The result follows by using the formula [TW95, (8.6) Theorem] expressing $\uparrow_H^G \mathrm{B}_{\mathbb{K}}^H$ as a direct sum of principal indecomposable and by arguing as in part (4). $\square$

Chapter 13

Socle series of Burnside functor

Almost all the materials in this chapter comes from [Yar5, Section 7].

We want to study the socle series of $\mathrm{B}_{\mathbb{K}}^{G}$ and obtain results similar to the ones in Chapter 11. However, because of the difficulty arisen in the computation of restriction kernels, here we required to assume that G is abelian.

Lemma 13.1 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. Let H be a subgroup of G and n be a natural number. For any natural number k we put $S_k = \mathrm{Soc}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^{G}$. Then:*

(1) $S_{n+1}(H)/S_n(H) = k_H^0(M/S_n)$.

(2) *If* $|G : H| \geq p^n$ *then* $S_n(H) = 0$.

(3) *If* $|G : H| = p^{n-1}$ *and* $n \geq 1$ *then* $S_n(H) = \underline{M}(H)$.

(4) *If* $|G : H| = p^{n-2}$ *and* $n \geq 2$ *then* $S_n(H) = \underline{M}(H)$.

(5) *If* $|G : H| \leq p^n \leq |G|$ *then* $S_{n+1}(H)/S_n(H) = \underline{(M/S_n)}(H)$.

Proof: (1) As G is abelian, $\overline{N}_G(H)$ acts on $M(H)$ trivially so that each submodule of each quotient of $M(H)$, in particular $k_H^0(M/S_n)$, is semisimple. The result follows by 9.26.

(2) The result is true for $n = 0, 1$ by 12.2.

Assuming that the result is true for n, take a subgroup K of G such that $|G : K| \geq p^{n+1}$. We want to show that $S_{n+1}(K) = 0$.

Let $H \geq K$ with $|H : K| = p$. Then,

$$S_n(H) = 0 = S_n(K)$$

by the assumption of the result for n. As G is abelian, the map t_K^H on M, and hence on M/S_n, is injective. This means by 9.23 that $S_{K,\mathbb{K}}$ does not occur in S_{n+1}/S_n so that, by 9.23 again,

$$S_{n+1}(K) = S_n(K) = 0.$$

(3) The result is true for $n = 1$ by 12.2.

Assume that the result is true for n. Take a subgroup K of G with

$$|G : K| = p^n.$$

We want to show that $S_{n+1}(K) = \underline{M}(K)$. We will achieve this by first calculating $k_K^0(M/S_n)$ and then by using part (1).

Part (2) implies that

$$\underline{(M/S_n)}(K) = \{x \in M(K) : r_J^K(x) \in S_n(J), \forall J < K\}/S_n(K) = \underline{M}(K)/0.$$

Let $H \geq K$ with $|H : K| = p$. For any $x \in \underline{M}(K)$, we see by using the Mackey axiom that $r_J^H t_K^H(x) = 0$ for any $J < H$ so that

$$t_K^H(x) \in \underline{M}(H) = S_n(H).$$

Hence,

$$k_K^0(M/S_n) = \underline{(M/S_n)}(K) = \underline{M}(K)/0.$$

As $S_n(K) = 0$, the result follows by part (1).

(4) Using the first three parts we see that

$$k_G^0(M/S_1) = \underline{(M/S_1)}(G) = \underline{M}(G)/\underline{M}(G) = 0$$

implying by 9.26 that $S^G_{G,\mathbb{K}}$ does not appear in S_2/S_1, and so

$$S_2(G) = S_1(G) = \underline{M}(G)$$

by 9.23. Hence, the result is true for $n = 2$. An easy induction argument on n finishes the proof.

(5) The result is true for $n = 0$ because $S_1(G) = \underline{M}(G)$ by 12.2.

Assume that the result is true for n. Take a subgroup K of G with

$$|G : K| \leq p^{n+1}.$$

Our aim is to obtain that $S_{n+2}(K)/S_{n+1}(K) = \underline{(M/S_{n+1})}(K)$.

We have by part (1) that

$$S_{n+2}(K)/S_{n+1}(K) = k^0_K(M/S_{n+1}).$$

Let $x \in M(K)$ be such that

$$\begin{aligned} x + S_{n+1}(K) &\in \underline{(M/S_{n+1})}(K) \\ &= \{y \in M(K) : r^K_J(y) \in S_{n+1}(J), \forall J < K\}/S_{n+1}(K). \end{aligned}$$

Then, $r^K_J(x) \in S_{n+1}(J)$ for any $J < K$. Take any $H \geq K$ with $|H : K| = p$. Then, for any $I < H$, it follows by the Mackey axiom that

$$r^H_I t^H_K(x) = |H : IK| t^I_{I\cap K} r^K_{I\cap K}(x).$$

If $r^H_I t^H_K(x) \neq 0$, then $H = IK$ implying that $I \cap K < I$ and $I \cap K < K$. It follows by 9.23 that the element $t^I_{I\cap K}$ of $\mu_{\mathbb{K}}(G)$ annihilates the semisimple functor S_{n+1}/S_n. This gives that $r^H_I t^H_K(x) \in S_n(I)$, because $r^K_{I\cap K}(x) \in S_{n+1}(I \cap K)$. Therefore,

$$r^H_I t^H_K(x) \in S_n(I)$$

for every $I < H$, that means

$$\begin{aligned} t^H_K(x) + S_n(H) &\in \{z \in M(H) : r^H_J(z) \in S_n(J), \forall J < H\}/S_n(H) \\ &= \underline{(M/S_n)}(H). \end{aligned}$$

Now, the assumption of the result for n gives that $t_K^H(x) \in S_{n+1}(H)$. Consequently, any element $x + S_{n+1}(K)$ of $\underline{(M/S_{n+1})}(K)$ is mapped by t_K^H to the zero element of $M(H)/S_{n+1}(H)$. This yields that

$$k_K^0(M/S_{n+1}) = \underline{(M/S_{n+1})}(K),$$

as desired. □

Theorem 13.2 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. Let H be a subgroup of G and n be a natural number with $p^n \leq |G|$. For any natural number k we put $S_k = \mathrm{Soc}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^G$. Then:*

(1) *If $S_{H,\mathbb{K}}^G$ appears in S_{n+1}/S_n then $|G : H| \leq p^n$.*

(2) *If $|G : H| = p^{n-1}$ then $S_{H,\mathbb{K}}^G$ does not appear in S_{n+1}/S_n.*

(3) *If $|G : H| = p^n$ then the multiplicity of $S_{H,\mathbb{K}}^G$ in S_{n+1}/S_n is $\dim_{\mathbb{K}} \underline{M}(H)$.*

(4) *$S_{1,\mathbb{K}}^G$ appears in S_{m+1}/S_m where $p^m = |G|$.*

Proof: (1) and (2) They follow by parts (2)-(4) of 13.1.

(3) The multiplicity of $S_{H,\mathbb{K}}^G$ in S_{n+1}/S_n is equal by 9.23 to the dimension of $S_{n+1}(H)/S_n(H)$, that is isomorphic by 13.1 to $\underline{M}(H)$.

(4) Follows by part (3). □

Theorem 13.3 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. Let H be a subgroup of G with $|G : H| = p^m$ and n be a natural number with $m \leq n - 1$ and $p^n \leq |G|$. For any natural number k we put $S_k = \mathrm{Soc}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^G$. Then:*

(1)

$$S_n(H) = \bigcap_{X \leq H : |H:X| = p^{s+1}} \mathrm{Ker}(r_X^H : M(H) \to M(X)),$$

where $s = \lfloor (n - m - 1)/2 \rfloor$.

(2) *If $n-m$ is an odd number then $S^G_{H,\mathbb{K}}$ does not appear in S_{n+1}/S_n.*

Proof: (1) For any subgroup K of G with $|G:K| \leq p^n$ it follows by part (5) of 13.1 that

$$\begin{aligned} S_{n+1}(K) &= \{x \in M(K) : r^K_J(x) \in S_n(J), \forall J < K\} \\ &= \bigcap_{J \leq K : |K:J| = p} \{x \in M(K) : r^K_J(x) \in S_n(J)\}. \end{aligned}$$

We will use this equality repeatedly to obtain the result. Arguing as in the proof of part (1) of 11.5, we apply the above equality s-times to $S_n(H)$ and obtain that

$$S_n(H) = \bigcap_{Y \leq H : |H:Y| = p^s} \{x \in M(H) : r^H_Y(x) \in S_{n-s}(Y)\}.$$

As $|G:Y| = p^{m+s}$ and as

$$(n-s) - 2 \leq m+s \leq (n-s) - 1,$$

we see by parts (3) and (4) of 13.1 that $S_{n-s}(Y) = \underline{M}(Y)$. Thus, the result follows.

(2) It follows by the first part, because if $n-m$ is an odd number then

$$\lfloor (n-m-1)/2 \rfloor = \lfloor (n-m)/2 \rfloor.$$

□

The following is immediate from 13.3.

Corollary 13.4 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. For any natural number k we put $S_k = \mathrm{Soc}^k(M)$ where $M = \mathrm{B}^G_{\mathbb{K}}$. Then, for any natural number n with $p^n \leq |G|$ we have:*

$$S_{n+1}/S_n \cong \bigoplus_{l=0}^{\lfloor n/2 \rfloor} \Big(\bigoplus_{H \leq G : |G:H| = p^{n-2l}} \lambda^l_H S^G_{H,\mathbb{K}} \Big)$$

for some nonnegative integers λ^l_H.

Some of the numbers λ_H^l in 13.4 may be 0. For instance, letting G be the cyclic group of order p^4, one may calculate that $S_3/S_2 \cong 2S_{H,\mathbb{K}}^G$ where $|G:H| = p^2$, in particular, $S_{G,\mathbb{K}}^G$ does not appear in S_3/S_2. See Example 13.7.

Imitating the proofs of 11.7 and 11.8 one may obtain the following.

Theorem 13.5 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group with $|G| = p^n$. Let H be a subgroup of G with $|G : H| = p^m$. For any natural number k we put $S_k = \mathrm{Soc}^k(M)$ where $M = \mathrm{B}_{\mathbb{K}}^G$. Then:*

(1)

$$S_k(H) = \bigoplus_{X \leq H : |H:X| = p^{s+1}} \mathrm{Ker}(r_X^H : M(H) \to M(X)),$$

where $s = \lfloor (k - m - 1)/2 \rfloor$.

(2) *Assume that $k \geq n+1$. If $k-m$ is an odd number, then $S_{H,\mathbb{K}}^G$ does not appear in S_{k+1}/S_k.*

(3) *If $k \geq n+1$, then*

$$S_{k+1}/S_k \cong \bigoplus_{l=k-n}^{\lfloor k/2 \rfloor} \Big(\bigoplus_{K \leq G : |G:K| = p^{k-2l}} \lambda_K^l S_{K,\mathbb{K}}^G \Big)$$

for some nonnegative integers λ_K^l.

The following consequence of the previous result may be used to derive some results about the Burnside functor of an abelian group G by using induction on the order of G.

Corollary 13.6 *Let $\mathbb{K}$ be of characteristic $p > 0$ and G be an abelian p-group. Let H be a subgroup of G with $|G : H| = p^m$. Then, for any natural number k with $k \geq m$ we have:*

$$\downarrow_H^G \mathrm{Soc}^k(\mathrm{B}_{\mathbb{K}}^G) = \mathrm{Soc}^{k-m}(\mathrm{B}_{\mathbb{K}}^H).$$

Proof: It can be obtained easily by using part (1) of 13.5. □

Using part (1) of 13.5 one may get the following example.

Example 13.7 *Let* $\mathbb{K}$ *be of characteristic* $p > 0$ *and* G *be a cylic group of order* p^4. *For any natural number* k *we put* $S_k = \mathrm{Soc}^k(M)$ *where* $M = \mathrm{B}_{\mathbb{K}}^G$. *Let* $1 < H_1 < H_2 < H_3 < G$ *be the lattice of the subgroups of* G *so that* $|H_i| = p^i$ *for each* i. *Then:*

$$0 = S_0 \subset S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \subset S_6 \subset S_7 \subset S_8 \subset S_9 = M,$$

where

$$\begin{array}{ccccccccc} 4S^G_{G,\mathbb{K}}; & 3S^G_{H_3,\mathbb{K}}; & 2S^G_{H_2,\mathbb{K}}; & S^G_{H_1,\mathbb{K}}; & S^G_{1,\mathbb{K}}; & S^G_{H_1,\mathbb{K}}; & S^G_{H_2,\mathbb{K}}; & S^G_{H_3,\mathbb{K}}; & S^G_{G,\mathbb{K}} \\ \cong & \cong & \cong & \cong & \cong & \cong & \cong & \cong & \cong \\ S_1/S_0 & S_2/S_1 & S_3/S_2 & S_4/S_3 & S_5/S_4 & S_6/S_5 & S_7/S_6 & S_8/S_7 & S_9/S_8 \end{array}$$

Chapter 14

Series of fixed point functor

All the materials in this chapter comes from [Yar5, Section 7].

To give more applications of general results we obtained in previous chapters, we study in this short chapter the fixed point functor FP_V^G where V is a one dimensional $\mathbb{K}G$-module and $\mathbb{K}$ is of characteristic $p > 0$. As V is one dimensional, the $\mathbb{K}K$-module V is simple for any subgroup K of G, and if H is a p-subgroup of G then $V^H = V \neq 0$. Therefore, the image of the (relative) trace map t_H^K is 0 if $H < K$ are p-subgroups of G. Moreover, restrictions maps on a fixed point functor are all inclusions (so that injective), and in the case $\dim_{\mathbb{K}} V = 1$ we see if we assume $V^K \neq 0$ that the (relative) trace map t_H^K on FP_V^G is surjective if and only if p does not divide $|K : H|$.

Lemma 14.1 *Let $\mathbb{K}$ be of characteristic $p > 0$ and V be a one dimensional $\mathbb{K}G$-module. Let H be a subgroup of G and W be a simple $\mathbb{K}\overline{N}_G(H)$-module. Let J and S be $\mu_{\mathbb{K}}(G)$-submodules of M where $M = FP_V^G$. Then:*

(1) *$\overline{J}(H) \neq 0$ if and only if $J(H) = M(H)$ and H is a p-subgroup of G.*

(2) *$S_{H,W}^G$ appears in the head of J if and only if H is a maximal subgroup of G subject to the condition $\overline{J}(H) \neq 0$ and the $\mathbb{K}\overline{N}_G(H)$-module W is isomorphic to $V^H = V$.*

(3) *Any minimal subgroup of M/S is a p-subgroup of G.*

(4) $\underline{(M/S)}(H) \neq 0$ *if and only if* H *is a minimal subgroup of* M/S.

Proof: As $\dim_{\mathbb{K}} M(X) \leq 1$ for any subgroup X of G, we see that $J(X) \neq 0$ if and only if $J(X) = M(X) \neq 0$. We will use this trivial observation in the proof.

(1) This is trivial by the explanation given before 14.1.

(2) Suppose that $S^G_{H,W}$ appears in the head of J. By 7.4 the module W is isomorphic to a simple quotient module of the $\mathbb{K}\overline{N}_G(H)$-module $\overline{J}(H)$. As $\dim_{\mathbb{K}} M(Y) \leq 1$ for any $Y \leq G$, it is clear that if $\overline{J}(H) \neq 0$ then

$$\overline{J}(H) \cong M(H) = V^H = V.$$

In particular, $\dim_{\mathbb{K}} W = 1$ so that we may use 7.16. Assume that H is not maximal subject to the required condition. Then there is a $K > H$ satisfying $\overline{J}(K) \neq 0$. Using part (1) we can find a subgroup X with $H < X \leq K$ with $|X : H| = p$. Now

$$0 \neq r^K_H(J(K)) \subseteq r^X_H(J(X))$$

implying that $r^X_H(J(X)) = J(H)$. But then 7.16 implies that $S^G_{H,W}$ does not appear in the head of J.

The converse implication follows by 9.3.

(3) Let X be a minimal subgroup of M/S. Then $M(X) \neq 0$, $S(X) = 0$ and $S(Y) = M(Y)$ for any $Y < X$. If X is not a p-group then

$$M(X) = t^X_Z(S(Z)) \subseteq S(X)$$

where Z is a Sylow p-subgroup of X.

(4) Suppose that $\underline{(M/S)}(H) \neq 0$. Then

$$0 \neq r^H_X(M(H)) \subseteq S(X)$$

for any $X < H$. Thus, $M(X) = S(X)$ for any $X < H$ implying that H is a minimal subgroup of M/S. □

Theorem 14.2 *Let $\mathbb{K}$ be of characteristic $p > 0$ and V be a one dimensional $\mathbb{K}G$-module. For any natural number k we put $J_k = \mathrm{Jac}^k(M)$ and $S_k = \mathrm{Soc}^k(M)$ where $M = FP_V^G$. Let n be the natural number satisfying $p^n = |G|_p$. Then:*

(1)

$$J_k/J_{k+1} \cong \bigoplus_{H \leq_G G : |H| = p^{n-k}} S_{H,V}^G.$$

(2)

$$S_{k+1}/S_k \cong \bigoplus_{H \leq_G G : |H| = p^k} S_{H,V}^G.$$

(3) *The Loewy length of M is $n+1$.*

(4) *Let X be a p-subgroup of G. Then, $J_k(X) = 0$ if and only if $|X| \geq p^{n+1-k}$.*

(5) *Let X be a p-subgroup of G. Then, $S_k(X) = 0$ if and only if $|X| \geq p^k$.*

(6) *If G is a p-group then the socle and the radical series of M coincide.*

Proof: Firstly, as $\dim_{\mathbb{K}} M(X) \leq 1$ for any $X \leq G$, the multiplicity of any composition factor of M is 1.

(1) Parts (1) and (2) of 14.1 imply that $J_0/J_1 \cong S_{H,V}^G$ where $|H| = p^n$.

Assume that the result is true for $k = 1, 2, ..., r$.

Let K be a p-subgroup of G. Then, it follows by 9.23 that the evaluation of M/J_{r+1} at K is nonzero if and only if $|K| \geq p^{n-r}$. As $\dim_{\mathbb{K}} M(K) = 1$, we conclude that $J_{r+1}(K) = M(K)$ if and only if

$$|K| \leq p^{n-(r+1)}.$$

Therefore, parts (1) and (2) of 14.1 imply that the result is true for $k = r+1$.

(2) It may be justified as in part (1).

(3) It follows by part (1) or by part (2).

(4) As $\dim_{\mathbb{K}} M(X) = 1$, we see that $J_k(X) = 0$ if and only if the evaluation of M/J_k at X is nonzero. This is equivalent to the requirement that the evaluation of

$$J_{k-m-1}/J_{k-m}$$

at X is nonzero for some $m \geq 0$. Using part (1) and 9.23 we now conclude that $J_k(X) = 0$ if and only if

$$|X| = p^{n-(k-m-1)} \geq p^{n+1-k}.$$

(5) It may be justified as in part (4).

(6) It follows by parts (4) and (5). □

Chapter 15

Adjoints of restriction and inflation

The main concern of this last chapter is to study socles and heads of Mackey functors obtained by applying adjoints of restriction and inflation to a Mackey functor. As the results here depend on functorial properties of restriction and inflation, almost every thing in this chapter can be done for modules of group algebras.

We begin by investigating possible relations between $\mathrm{Soc}(T)$ and $\mathrm{Soc}(\uparrow_H^G T)$, where H is a subgroup of G and T is a $\mu_{\mathbb{K}}(H)$-module.

Proposition 15.1 *Let H be a subgroup of G, and let T_1 and T_2 be $\mu_{\mathbb{K}}(H)$-modules with $T_1 \subseteq T_2$. If $\mathrm{Soc}(\uparrow_H^G T_1) = \mathrm{Soc}(\uparrow_H^G T_2)$ then $\mathrm{Soc}(T_1) = \mathrm{Soc}(T_2)$.*

Proof: It is enough to see that every nonzero $\mu_{\mathbb{K}}(H)$-submodule T of T_2 intersects T_1 nontrivially. This follows from the exactness of the functor $\uparrow_H^G$ (which is a consequence of 2.8), implying by the condition $\mathrm{Soc}(\uparrow_H^G T_1) = \mathrm{Soc}(\uparrow_H^G T_2)$ that

$$0 \neq (\uparrow_H^G T) \cap (\uparrow_H^G T_1) = \uparrow_H^G (T \cap T_1).$$

□

The containment condition $T_1 \subseteq T_2$ in 15.1 is necessary. For instance, assuming the existence of a simple $\mu_{\mathbb{K}}(H)$-module T such that, for some $g \in N_G(H)$, its conjugate ${}^gT$ is not isomorphic to T, we see that $\uparrow_H^G T \cong \uparrow_H^G {}^gT$, and hence their socles are isomorphic.

Proposition 15.2 *Let $\mathbb{K}$ be of characteristic $p > 0$, and let H be either a p'-subgroup of G or a normal subgroup of G. For any $\mu_{\mathbb{K}}(H)$-modules T_1 and T_2, if* $\mathrm{Soc}(T_1) \cong \mathrm{Soc}(T_2)$ *then* $\mathrm{Soc}(\uparrow_H^G T_1) \cong \mathrm{Soc}(\uparrow_H^G T_2)$.

Proof: Let S be any simple $\mu_{\mathbb{K}}(G)$-module such that $\downarrow_H^G S \neq 0$. We first note that $\downarrow_H^G S$ is a semisimple $\mu_{\mathbb{K}}(H)$-module. Indeed, if H is a p'-subgroup then $\mu_{\mathbb{K}}(H)$ is a semisimple algebra by [TW] so that $\downarrow_H^G S$ is a semisimple $\mu_{\mathbb{K}}(H)$-module. If H is a normal subgroup of G, then it follows by Clifford's theorem for Mackey functors [Yar1, Theorem 3.10] that $\downarrow_H^G S$ is semisimple.

The result now follows from the adjointness of the pair $(\uparrow_H^G, \downarrow_H^G)$ which implies the following $\mathbb{K}$-space isomorphisms for any simple $\mu_{\mathbb{K}}(G)$-module S :

$$\begin{aligned}
\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(S, \mathrm{Soc}(\uparrow_H^G T_1)\big) &\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(S, \uparrow_H^G T_1\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}\big(\downarrow_H^G S, T_1\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}\big(\downarrow_H^G S, \mathrm{Soc}(T_1)\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}\big(\downarrow_H^G S, \mathrm{Soc}(T_2)\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(H)}\big(\downarrow_H^G S, T_2\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(S, \uparrow_H^G T_2\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(S, \mathrm{Soc}(\uparrow_H^G T_2)\big).
\end{aligned}$$

□

Corollary 15.3 *Let $\mathbb{K}$ be of characteristic $p > 0$, and let H be either a p'-subgroup of G or a normal subgroup of G. For any $\mu_{\mathbb{K}}(H)$-module T,*

(1) $\mathrm{Soc}(\uparrow_H^G T) = \mathrm{Soc}\big(\uparrow_H^G \mathrm{Soc}(T)\big)$.

(2) *If p does not divide $|G:H|$ then* $\mathrm{Soc}(\uparrow_H^G T) = \uparrow_H^G \mathrm{Soc}(T)$.

Proof: (1) Follows from 15.2 because the socles of $\mu_{\mathbb{K}}(H)$-modules T and $\mathrm{Soc}(T)$ are equal.

(2) This follows from part (1). Indeed, either G is a p'-group in which case the result is trivial by the semisimplicity of the algebra $\mu_{\mathbb{K}}(G)$ [TW], or H is a normal subgroup of G whose index is not divisible by p. In the latter case, it follows by [Yar1, Corollary 3.8] that $\uparrow_H^G T'$ is a semisimple $\mu_{\mathbb{K}}(G)$-module for any semisimple $\mu_{\mathbb{K}}(H)$-module T'. This clearly implies the result. □

Given a subgroup H of G and a $\mu_{\mathbb{K}}(H)$-module T, we next want to obtain some results about heads of Mackey functors T and $\uparrow_H^G T$.

Proposition 15.4 *Let H be a subgroup of G, and let T_1 and T_2 be $\mu_{\mathbb{K}}(H)$-modules with a $\mu_{\mathbb{K}}(H)$-module epimorphism $T_1 \to T_2$. If*

$$(\uparrow_H^G T_1)/\mathrm{Jac}(\uparrow_H^G T_1) \cong (\uparrow_H^G T_2)/\mathrm{Jac}(\uparrow_H^G T_2)$$

then $T_1/\mathrm{Jac}(T_1) \cong T_2/\mathrm{Jac}(T_2)$.

Proof: Let $f : T_1 \to T_2$ be a $\mu_{\mathbb{K}}(H)$-module epimorphism. For any $\mu_{\mathbb{K}}(H)$-submodule T of T_1 satisfying $\mathrm{Ker}f + T = T_1$ we must show that $T = T_1$. This follows from the exactness of the functor $\uparrow_H^G$, inducing a $\mu_{\mathbb{K}}(G)$-module epimorphism $\uparrow_H^G T_1 \to \uparrow_H^G T_2$ whose kernel is equal to $\uparrow_H^G \mathrm{Ker}f$. Indeed, $\mathrm{Ker}f + T = T_1$ implies that

$$\uparrow_H^G \mathrm{Ker}f + \uparrow_H^G T = \uparrow_H^G T_1.$$

As the heads of $\uparrow_H^G T_1$ and $\uparrow_H^G T_2$ are isomorphic, we deduce that $\uparrow_H^G T = \uparrow_H^G T_1$. Now it follows by the containment $T \subseteq T_1$ and by the exactness of the functor $\uparrow_H^G$ that $\uparrow_H^G (T_1/T) = 0$, from which $T = T_1$ is obtained. □

The example given after 15.1 indicates the necessity of the surjectivity assumption of a $\mu_{\mathbb{K}}(H)$-module homomorphism $T_1 \to T_2$ given in 15.4.

The same arguments of the proofs of 15.2 and 15.3 can be used to deduce the next two results.

Proposition 15.5 *Let $\mathbb{K}$ be of characteristic $p > 0$, and let H be either a p'-subgroup of G or a normal subgroup of G. For any $\mu_{\mathbb{K}}(H)$-modules T_1 and T_2, if $T_1/\mathrm{Jac}(T_1) \cong T_2/\mathrm{Jac}(T_2)$ then*

$$(\uparrow_H^G T_1)/\mathrm{Jac}(\uparrow_H^G T_1) \cong (\uparrow_H^G T_2)/\mathrm{Jac}(\uparrow_H^G T_2).$$

Proposition 15.6 *Let $\mathbb{K}$ be of characteristic $p > 0$, and let H be either a p'-subgroup of G or a normal subgroup of G. For any $\mu_{\mathbb{K}}(H)$-module T,*

(1) *The heads of the $\mu_{\mathbb{K}}(G)$-modules $\uparrow_H^G T$ and $\uparrow_H^G \big(T/\mathrm{Jac}(T)\big)$ are isomorphic.*

(2) *If p does not divide $|G : H|$ then $\uparrow_H^G \mathrm{Jac}(T) = \mathrm{Jac}(\uparrow_H^G T)$.*

As the restriction of a Mackey functor for G to a proper subgroup of G may be the zero Mackey functor, if we replace inductions with restrictions then the results 15.1 and 15.4 will be no longer true. Nevertheless, we want to give some similar results for this case also.

For any $\mu_{\mathbb{K}}(G)$-module M and any subgroup H of G we denote by $\mathrm{Soc}^H(M)$ the sum of all simple subfunctors of M having a minimal subgroup contained in H. In a dual way, we denote by $\mathrm{Jac}^H(M)$ the intersection of all maximal $\mu_{\mathbb{K}}(G)$-submodules J of M whose quotient M/J has a minimal subgroup contained in H.

The following result follows from the definitions.

Remark 15.7 *Let H be a subgroup of G, and let M be a $\mu_{\mathbb{K}}(G)$-module.*

(1) $\mathrm{Soc}^H(M)$ *is the $\mu_{\mathbb{K}}(G)$-submodule of M generated by $\downarrow_H^G \mathrm{Soc}(M)$. In other words,* $\mathrm{Soc}^H(M)$ *is the smallest $\mu_{\mathbb{K}}(G)$-submodule of M satisfying*

$$\downarrow_H^G \mathrm{Soc}^H(M) = \downarrow_H^G \mathrm{Soc}(M).$$

(2) $\mathrm{Jac}^H(M)$ *is the largest $\mu_{\mathbb{K}}(G)$-submodule of M satisfying*

$$\downarrow_H^G \mathrm{Jac}^H(M) = \downarrow_H^G \mathrm{Jac}(M).$$

(3) *$M/\mathrm{Jac}^H(M)$ is isomorphic to the sum of all simple $\mu_{\mathbb{K}}(G)$-submodules of $M/\mathrm{Jac}(M)$ with a minimal subgroup contained in H. In particular, $M/\mathrm{Jac}^H(M)$ is a semisimple $\mu_{\mathbb{K}}(G)$-module.*

One may imitate the proofs of 15.1 and 15.4 to obtain the following result.

Proposition 15.8 *Let H be a subgroup of G, and let M_1 and M_2 be $\mu_{\mathbb{K}}(G)$-modules.*

(1) *Suppose that $M_1 \subseteq M_2$. If $\mathrm{Soc}(\downarrow^G_H M_1) = \mathrm{Soc}(\downarrow^G_H M_2)$, then*

$$\mathrm{Soc}^H(M_1) = \mathrm{Soc}^H(M_2) \quad \textit{so that} \quad \downarrow^G_H \mathrm{Soc}(M_1) = \downarrow^G_H \mathrm{Soc}(M_2).$$

(2) *Suppose that there is a $\mu_{\mathbb{K}}(H)$-module epimorphism $M_1 \to M_2$. If the heads of the $\mu_{\mathbb{K}}(H)$-modules $\downarrow^G_H M_1$ and $\downarrow^G_H M_2$ are isomorphic, then the $\mu_{\mathbb{K}}(G)$-modules*

$$M_1/\mathrm{Jac}^H(M_1) \quad \textit{and} \quad M_2/\mathrm{Jac}^H(M_2)$$

are isomorphic, which implies that

$$(\downarrow^G_H M_1)/\big(\downarrow^G_H \mathrm{Jac}(M_1)\big) \quad \textit{and} \quad (\downarrow^G_H M_2)/\big(\downarrow^G_H \mathrm{Jac}(M_2)\big)$$

are isomorphic $\mu_{\mathbb{K}}(H)$-modules .

The conditions on H given in the next result guarantees that the induced $\mu_{\mathbb{K}}(G)$-module $\uparrow^G_H T$ is semisimple for any simple $\mu_{\mathbb{K}}(H)$-module T, (see the proof of 15.3). Thus, the next result may be justified by using the adjointness properties of induction and restriction, (see 2.8).

Proposition 15.9 *Let $\mathbb{K}$ be of characteristic $p > 0$, and let H be a normal subgroup of G such that $|G : H|$ is not divisible by p. For any $\mu_{\mathbb{K}}(G)$-modules M_1 and M_2,*

(1) *If $\mathrm{Soc}^H(M_1) \cong \mathrm{Soc}^H(M_2)$ then $\mathrm{Soc}(\downarrow^G_H M_1) \cong \mathrm{Soc}(\downarrow^G_H M_2)$.*

(2) *If the $\mu_{\mathbb{K}}(G)$-modules*

$$M_1/\mathrm{Jac}^H(M_1) \quad and \quad M_2/\mathrm{Jac}^H(M_2)$$

are isomorphic, then the heads of the $\mu_{\mathbb{K}}(H)$-modules

$$\downarrow^G_H M_1 \quad and \quad \downarrow^G_H M_2$$

are isomorphic.

The following result is an easy consequence of 15.9.

Corollary 15.10 *Let $\mathbb{K}$ be of characteristic $p > 0$, and let H be a normal subgroup of G such that $|G : H|$ is not divisible by p. For any $\mu_{\mathbb{K}}(G)$-modules M,*

(1) $\downarrow^G_H \mathrm{Soc}(M) = \mathrm{Soc}(\downarrow^G_H M)$.

(2) $\downarrow^G_H \mathrm{Jac}(M) = \mathrm{Jac}(\downarrow^G_H M)$.

We now want to study socles and heads of Mackey functors obtained by applying inflation and its adjoints to a Mackey functor. We begin with the following which can be obtained by using the definitions and adjointness properties of functors involved, see [Yar4, Section 3].

Remark 15.11 *Let N be a normal subgroup of G, and T be a $\mu_{\mathbb{K}}(G/N)$-module.*

(1) *Let $S^G_{H,V}$ be a simple $\mu_{\mathbb{K}}(G)$-module and $\mathfrak{F}$ be any of the functors ${L^+}_{G/N}$ and ${L^-}_{G/N}$. Then,*

$$\mathfrak{F}S^G_{H,V} \cong S^{G/N}_{H/N,V} \;\; if \;\; N \leq H, \;\; and \;\; \mathfrak{F}S^G_{H,V} = 0 \;\; if \;\; N \nleq H.$$

(2) ${L^+}_{G/N}\mathrm{Inf}^G_{G/N}T \cong T \cong {L^-}_{G/N}\mathrm{Inf}^G_{G/N}T$.

(3) *T is a semisimple $\mu_{\mathbb{K}}(G/N)$-module if and only if $\mathrm{Inf}^G_{G/N}T$ is a semismple $\mu_{\mathbb{K}}(G)$-module.*

Proposition 15.12 *Let N be a normal subgroup of G. Suppose that M is a $\mu_{\mathbb{K}}(G)$-module and T be a $\mu_{\mathbb{K}}(G/N)$-module.*

(1) $\mathrm{Soc}(\mathrm{Inf}_{G/N}^{G}T) = \mathrm{Inf}_{G/N}^{G}\mathrm{Soc}(T)$ *and* $\mathrm{Jac}(\mathrm{Inf}_{G/N}^{G}T) = \mathrm{Inf}_{G/N}^{G}\mathrm{Jac}(T)$.

(2) $L^{-}{}_{G/N}\mathrm{Soc}(M) = \mathrm{Soc}(L^{-}{}_{G/N}M)$.

(3) *The head of the $\mu_{\mathbb{K}}(G/N)$-module $L^{+}{}_{G/N}M$ is isomorphic to*

$$L^{+}{}_{G/N}\big(M/\mathrm{Jac}(M)\big).$$

Proof: Follows by 15.11 and by the adjointness of the pairs (L^{+}, Inf) and (Inf, L^{-}). □

The next result concerns the cases not covered in 15.12.

Corollary 15.13 *Let N be a normal subgroup of G, and let M be a $\mu_{\mathbb{K}}(G)$-module and T be a $\mu_{\mathbb{K}}(G/N)$-module.*

(1) $\mathrm{Soc}(L^{+}{}_{G/N}M) \cong L^{+}{}_{G/N}\mathrm{Soc}(\mathrm{Inf}_{G/N}^{G}L^{+}{}_{G/N}M)$.

(2) $\mathrm{Jac}(L^{-}{}_{G/N}M) \cong L^{-}{}_{G/N}\mathrm{Jac}(\mathrm{Inf}_{G/N}^{G}L^{-}{}_{G/N}M)$.

(3) $\mathrm{Jac}(L^{+}{}_{G/N}M) \cong L^{+}{}_{G/N}\mathrm{Jac}(\mathrm{Inf}_{G/N}^{G}L^{+}{}_{G/N}M)$.

Proof: Follows by part (2) of 15.11 and part (1) of 15.12. □

The $\mu_{\mathbb{K}}(G)$-module $\mathrm{Inf}_{G/N}^{G}L^{+}{}_{G/N}M$ (respectively, $\mathrm{Inf}_{G/N}^{G}L^{-}{}_{G/N}M$) appeared in 15.13 is isomorphic to the largest quotient functor (respectively, subfunctor) of M that can be inflated from a Mackey functor for the quotient group G/N, see [Yar4, Section 5].

We next want to construct two functors connecting $\mu_{\mathbb{K}}(G)$-modules with $\mathbb{K}\overline{N}_G(H)$-modules where H is a subgroup. Similar functors appears in [TW] and [We2]. We know by [TW] that the evaluation of Mackey functors at trivial subgroup is a left adjoint of the fixed point functor. For another example, we know by [We2] that the functors $\Delta^G_{H,-}$ and $\nabla^G_{H,-}$ form left and right adjoints of taking restriction kernel and Brauer quotients at H. These facts about the functors Δ and ∇ can be derived by using the adjoints of restriction, inflation, and evaluation at the trivial subgroup. To be more precise, we have the following functors for a subgroup H of G:

$H\_ : \mu_{\mathbb{K}}(G)\text{-mod} \to \mathbb{K}\overline{N}_G(H)\text{-mod}, \quad M \mapsto \underline{M}(H).$

$H^{-} : \mu_{\mathbb{K}}(G)\text{-mod} \to \mathbb{K}\overline{N}_G(H)\text{-mod}, \quad M \mapsto \overline{M}(H).$

$\Delta^G_{H,-} : \mathbb{K}\overline{N}_G(H)\text{-mod} \to \mu_{\mathbb{K}}(G)\text{-mod},$

$$V \mapsto \Delta^G_{H,V} = \uparrow^G_{N_G(H)} \operatorname{Inf}^{N_G(H)}_{N_G(H)/H} FQ_U^{N_G(H)/H}$$

where FQ_U is the fixed quotient functor.

$\nabla^G_{H,-} : \mathbb{K}\overline{N}_G(H)\text{-mod} \to \mu_{\mathbb{K}}(G)\text{-mod},$

$$V \mapsto \Delta^G_{H,V} = \uparrow^G_{N_G(H)} \operatorname{Inf}^{N_G(H)}_{N_G(H)/H} FP_U^{N_G(H)/H}$$

where FP_U is the fixed point functor.

The pairs $(\Delta^G_{H,-}, H\_)$ and $(H^{-}, \nabla^G_{H,-})$ are adjoint pairs, see [We2] for more details. Some of our results may also obtained by using the functors Δ and ∇.

Here we want to construct another two functors from the module category of the algebra $\mu_{\mathbb{K}}(G)$ to the module category of the algebra $\mathbb{K}\overline{N}_G(H)$.

Let A be a finite dimensional $\mathbb{K}$-algebra and I be a two sided ideal of A. The canonical A-module epimorphism $A \to A/I$ induces three functors (namely, restriction, induction, and coinduction):

$$\operatorname{inf}_I := {}_A(A/I) \otimes_{A/I} -, \qquad \operatorname{def}_I := {}_{A/I}(A/I) \otimes_A -,$$

$$\operatorname{codef}_I := {}_{A/I}\operatorname{Hom}_A\big({}_A(A/I)_{A/I}, -\big).$$

In particular, $(\operatorname{def}_I, \operatorname{inf}_I)$ and $(\operatorname{inf}_I, \operatorname{codef}_I)$ are adjoint pairs.

Remark 15.14 *Assume the notations of the above paragraph. For any A-module V we have the following A/I-module isomorphisms:*

(1) $\mathrm{def}_I V = (A/I) \otimes_A V \cong V/IV, \qquad (a+I) \otimes v \leftrightarrow av + IV.$

(2) $\mathrm{codef}_I V = \mathrm{Hom}_A(A/I, V) \cong \{v \in V : Iv = 0\}, \quad f \to f(1+I)$ *and* $f_v \leftarrow v$ *where* $f_v(a+I) = av$.

The next result is easy to derive.

Remark 15.15 *Everything in Remark 15.11, Proposition 15.12, and Corollary 15.13 remain true if we replace the terms*

$$\mu_{\mathbb{K}}(G), \quad \mu_{\mathbb{K}}(G/N), \quad \mathrm{Inf}^G_{G/N}, \quad {L^+}_{G/N}, \quad {L^-}_{G/N}$$

with the following respective terms

$$A, \quad A/I, \quad \mathrm{inf}_I, \quad \mathrm{def}_I, \quad \mathrm{codef}_I.$$

We now need to recall the functors given in [Gr2, pp. 83-87] (see, Chapter 3). Let B be a finite dimensional algebra and e be a nonzero idempotent of B. We have the following functors:

$$R_e : \mathrm{Mod}(B) \to \mathrm{Mod}(eBe) \qquad \text{and} \qquad C_e, I_e : \mathrm{Mod}(eBe) \to \mathrm{Mod}(A)$$

given on the objects by

$$R_e(V) = eV, \quad C_e(W) = \mathrm{Hom}_{eBe}(eB, W) \; \text{ and } \; I_e(W) = Be \otimes_{eBe} W.$$

Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G. We want to find adjoints of the functors that map M to the $\mathbb{K}\overline{N}_G(H)$-modules

$$M(H)/I_H M(H) \qquad \text{and} \qquad \{x \in M(H) : I_H x = 0\}$$

where I_H is the two sided ideal of $t^H_H \mu_{\mathbb{K}}(G) t^H_H$ given in 4.1. Note that the first module has $\overline{M}(H)$ as a quotient, and the second module has $\underline{M}(H)$ as a submodule. We define two functors:

$$e_H^+ : \mu_{\mathbb{K}}(G)\text{-mod} \to \mathbb{K}\overline{N}_G(H)\text{-mod}, \qquad M \mapsto M(H)\big/I_H M(H).$$

$$e_H^- : \mu_{\mathbb{K}}(G)\text{-mod} \to \mathbb{K}\overline{N}_G(H)\text{-mod}, \qquad M \mapsto \{x \in M(H) : I_H x = 0\}.$$

Now letting

$$B = \mu_{\mathbb{K}}(G), \qquad e = t_H^H, \qquad A = eBe, \qquad I = I_H$$

we see that

$$e_H^+ = \mathrm{def}_I \circ R_e \qquad \text{and} \qquad e_H^- = \mathrm{codef}_I \circ R_e.$$

Therefore the following pairs are adjoint pairs:

$$(e_H^+, C_e \circ \inf{}_I) \qquad \text{and} \qquad (I_e \circ \inf{}_I, e_H^-).$$

For instance, if a $\mu_{\mathbb{K}}(G)$-module has no subfunctor whose evaluation at H is 0 then one may see that $e_H^- \mathrm{Soc}(M) = \mathrm{Soc}(e_H^- M)$.

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