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A CRITICAL REVIEW of the APPROACHES to OPTIMIZATION PROBLEMS under UNCERTAINTY

A THESIS

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MASTER OF SCIENCE

By

Filiz GÜRTUNA

September, 2001

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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ABSTRACT

A CRITICAL REVIEW of the APPROACHES to OPTIMIZATION

PROBLEMS under UNCERTAINTY

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M.S. in Industrial Engineering

Supervisor: Assoc. Prof. Barbaros Ç. Tansel

September 20001

In this study, the issue of uncertainty in optimization problems is studied. First of all, the meaning and sources of uncertainty are explained and then possible ways of its representation are analyzed.

About the modelling process, different approaches as sensitivity analysis, parametric programming, robust optimization, stochastic programming, fuzzy programming, multiobjective programming and imprecise optimization are presented with advantages and disadvantages from different perspectives. Some extensions of the concepts of imprecise optimization are also presented.

Key words: uncertainty, sensitivity analysis, parametric programming, robust optimization, stochastic programming, fuzzy programming, multiobjective optimization, imprecise optimization

ÖZET

BELİRSİZLİK DURUMUNDAKİ ENİYİLEME PROBLEMLERİNE YAKLAŞIMLARIN ELEŞTİREL İNCELENMESİ

Filiz Gürtuna

Endüstri Mühendisliği Bölümü Yüksek Lisans

Tez Yöneticisi: Doç. Dr. Barbaros Ç. Tansel

September 20001

Bu tezde, belirsizlik durumundaki eniyileme problemleri incelendi. Önce, belirsizliğin anlamı ve kaynakları üzerinde duruldu ve sonra belirsizliğin değişik sunum yolları analiz edildi.

Modelleme süreci ile ilgili olarak, duyarlılık analizi, parametrik programlama, sağlam eniyileme, rassal programlama, bulanık programlama, çokkriterli programlama ve belirsiz programlama, değişik açılardan avantajları ve dezavantajları ile birlikte anlatıldı. Belirsiz programlamada sunulan bazı kavramlar ilerletildi.

Anahtar Kelimeler: belirsizlik, duyarlılık analizi, parametrik programlama, gürbüz eniyileme, rassal programlama, bulanık programlama, çokkriterli eniyileme, belirsiz eniyileme

To me!

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I express my deepest feelings to each member of my family, especially to my mother and father for giving me their best genes. They were always with me, understanding and supporting me. I can not say much more since I can only feel the love I have for them and vice versa. Lastly, I would also thank to the God for giving me such a family.

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CHAPTER-1

INTRODUCTION

In most of the real world problems, one has imperfect information about the endogenous and/or exogenous parameters of a system. This fact, however, does not necessarily imply that uncertainty is an important aspect of all problems. If the level of uncertainty is low or the uncertain parameters have a minor impact on the system, then point estimates can be reasonable approximations and one does not have to worry about uncertainty that much. Since such instances are rare, we assume uncertainty is an important part of decision making in parallel with the saying in Ben-Tal and Nemirovski: "… one can not ignore the possibility that a small uncertainty in the data (intrinsic for most real-world problems) can make the usual optimal solution of the problem completely meaningless from a practical viewpoint." (see [9], page 416). This work, where 90 LPs from NETLIB were studied to see how much the constraints of the perturbed problem can be violated by the optimal solution to the nominal problem is a very good reference to see the effects of uncertainty.

In the literature, a number of terms have been used to describe non-deterministic situations like “imprecise”, “uncertain”, “inexact” and “risk”. For example, originally, decision-making situations were divided into three groups as certainty, risk and uncertainty by Luce and Raiffa [88]. For the certainty case, all the necessary information is available whereas in the risk situation one has incomplete information about the parameters but probabilities are known. For the uncertainty case, even the identification of probabilities is not possible. Additionally, some researchers made the distinction between imprecision and uncertainty as the former being related to a content of an item of information (value) and the latter to conformity to a reality (reliability) (for further discussion, see [37], page 2). Uncertainty can also be divided into two as controllable and uncontrollable. In the former case, decision maker has the ability to change or force some parameters to belong to, for example, some intervals whereas in the latter case, such an enforcement is not possible, which is also discussed in Demir [32] in a more detailed way. Understanding uncertainty in the sense of both incomplete and missing information, we eliminate such distinctions and use the above terms interchangeably to describe a situation with imperfect information though “uncertainty” will be preferred most of the time. According to the types of uncertain elements, Whalen [143] divides decision making situations as follows:

- Uncertainty about consequences
- Uncertainty about alternative courses of action
- Uncertainty about preferences

One can have any combination of these while modelling but in this study we exclude the third type of uncertainty, whereas the first and the second ones will correspond to uncertain objective function coefficients and uncertain feasible region, respectively.

There are mainly four situations leading to uncertainty:

- Some parameters of the system are not realizable at the time decisions must be made.
- Although parameters of the system are realizable, their values can not be determined exactly.
- Although the parameters of the system are known exactly, the decision made can not be implemented exactly.
- The abstraction of the real world into a model requires some simplifications, assumptions etc.

The third one has not been considered in the literature so far as we know before Ben-Tal and Nemirovski introduced such an understanding of uncertainty [7]. It is very interesting to see a certain model with a difficult-to-implement-solution (in terms of numerical precision) being modelled as an uncertain model.

In this thesis, we study the existing approaches to optimization problems under uncertainty of the input data pointing out the advantages and disadvantages of them and also establish relations among them. There are mainly seven types of approaches, namely, sensitivity analysis, parametric programming, robust optimization, stochastic programming, fuzzy programming, multi-objective optimization and imprecise optimization. These differ from each other in the input data requirement, notions of feasibility and optimality, computational requirements and so on. Among these approaches, sensitivity analysis does not handle uncertainty in the modelling phase but studies the effects of changes in the system parameters on the optimal solution, so it is a reactive approach. Moreover, it can also be applied to the models constructed by the other six approaches like stochastic programming [39]. Although parametric programming is not a reactive approach, it does not specify any ultimate solution for an uncertain program as the other approaches do. Because of these, we will first study, in chapter 2, sensitivity analysis and parametric programming which are studied together most of the time. In chapter 3, we define a general uncertain optimization problem and study different ways of uncertainty representation. Then, in chapter 4, we explain and analyze the general models of the existing approaches and construct a common framework for the models of these approaches. In chapter 5, the relations among the approaches will be given and the decision environments for which they are suitable are studied. Finally, we give concluding remarks and future perspectives in chapter 6.

CHAPTER-2

SENSITIVITY ANALYSIS and PARAMETRIC PROGRAMMING

One may have problems where for all possible values of the uncertain parameters, the same solution is optimal and also one may have problems as in the case of multiobjective programming, where the decision maker inputs weights, pairwise comparisons, strength of preference that are judgmental and carry a significant amount of uncertainty. Therefore, a mechanism is needed to tell if further investigation of the problem is required and this is where sensitivity analysis comes into play. The validation of a model is one of the reasons to perform sensitivity analysis and the other is to assess the worthiness of having better estimates for the parameters before making the final decision. There are also some indirect uses of sensitivity analysis. One example is its use as a solution tool in the determination of efficient frontiers in multi-objective programming as proposed by Gal [50] and another one is its use as a means of reducing the dimensionality of a certain class of transportation problems as proposed by Intrator and Engelberg [68].

As stated previously, focusing on the effects of changes in the problem parameters on the optimal solution, sensitivity analysis is a reactive approach and requires an optimal solution as input. Simply, for an LP, if uncertainty is related with the objective function coefficients, it gives a region of change over which the given solution is optimal and if uncertainty lies in the right hand side, it tells the region over which the given basis remains optimal.

Heller is said to be the first to use the term sensitivity analysis in 1954 [62]. Considering an LP, he studied the changes in the optimal objective value resulting from changes in the parameters. What he did is called differential sensitivity analysis today. Thereafter, there has been a large stream of research on this area, about which we refer the reader to Gal [53]. Also the recently published book of Gal and Greenberg [54] is an excellent reference. We also suggest the survey paper of Gal [52] and the critical paper by Wallace [136].

Traditional sensitivity analysis gives an interval for each coefficient under which the same basis remains optimal for an LP. Such a one-at-a-time approach is a limitation for real life situations. At this point, it should be noted that, if all of the corresponding variables are nonbasic, the above mentioned intervals can be used directly in case of simultaneous and independent changes in the cost coefficients or right hand side. On the other hand, if at least one of the basic variables' coefficients is altered, the region over which the same basis remains optimal becomes a convex polyhedron, the determination of which is not an easy task. In addition to the study of regions of optimality, the optimal value function and deviations from optimality are also studied [137]. There are some non-differential approaches to handle such situations, which will be investigated in the following part in terms of their applications to linear programming, but before doing so, two essential definitions will be given below and then parametric programming will be studied briefly. This is done since there are some relations to be stated between parametric programming and those methods.

Critical region: The critical region for a nominal data û of a parameter u, $CR_{û}$, is the set of values u can take without causing the optimal basis corresponding to û to change.

Optimal coefficient set: The optimal coefficient set for a nominal data û of a parameter u, $CS_{û}$, is the set of values u can take without causing the optimal solution of the problem with nominal data û to change.

Observe that, in case of degeneracy, $CR_{û} \subset CS_{û}$ and otherwise, they are equal.

Parametric programming represents uncertainty in a component of the model as a function of a parameter vector (single or multi dimensional) and aims to induce a subset of the decision space such that each element of this set is optimal for some problem instance. It dates back to1952 by Orchard-Hay's Master-Thesis as stated in [54] (page 1-2), where the right hand side of a linear program was perturbed parametrically so that the uncertain problem he studied is the following:

$$\begin{aligned} &\min c^T x \\ &\text{st} \qquad Ax = b^0 + \lambda b^1 \end{aligned}$$

where $\lambda \in T \subseteq R$ with T known.

He determined the so called critical regions, say CR_i, such that for any $\lambda \in CR_i$, i =1,…,I, the corresponding LP with the right hand side $b^0 + \lambda b^1$ has an optimal solution. In most publications, Mane is mentioned to be the first to deal with parametric programming with respect to the right hand side of an LP. Also, in 1954, Hoffmann and Jacobs [64] studied the LP with cost coefficients perturbed parametrically. They also considered two-parametric case, determining the critical regions, where the cost coefficients were perturbed as follows:

$$\begin{aligned} &\min (c^0 + \lambda_1 c^1 + \lambda_2 c^2)^T x \\ &\text{st} \qquad Ax \leq b \end{aligned}$$

where $(\lambda_1, \lambda_2) \in T \subseteq R^2$ with known T.

Multiparametric case involving right hand side or cost coefficients of an LP appeared in 1967 and then a group at the Humboldt University, in the second half of the sixties defined a (nonlinear) parametric mathematical programming problem as

$$(P_u) \qquad \min \{f(x,u) : x \in M(u)\}$$

where $u \in U$, $M(u) \subseteq X$, X and U are metric spaces, and f: $XxU \rightarrow R \cup \{-\infty, \infty\}$(see [54], page 1-5).

For more detailed information about parametric programming, the survey of Gal [51], Gal and Nedoma [55], who gave a simplex-based algorithm for determining the critical regions, Yu and Zeleny [149], Steuer [123] and Gal [53] may be very useful.

Parametric programs do not give an ultimate solution for an uncertain problem but specify the set of decisions which will be optimal at least for one realization. This is a very powerful information to use in the other uncertainty-related models where an ultimate solution is sought.

There are mainly four types of methods used in sensitivity analysis and parametric programming related with linear programming problems. These are as follows:

<u>One-Dimensional Cuts:</u> This method proposed by Saaty and Gass, in 1954, makes a one-dimensional cut through the region to be summarized and characterizes the end points of this cut [112]. Therefore, this method considers changes along a fixed direction, called a change vector using a single parameter to characterize points in that direction. Consequently, the nominal data û is perturbed as

$$u = \hat{u} + \gamma G$$

where G is a 1xn nonzero matrix. This method is suitable for simultaneous changes but not so for independent changes. Note that this representation corresponds to a single-parametric programming case. The special case of this, the most commonly used one, is where $G = e_j$, that is the jth unit vector in which case, one has the usual intervals of sensitivity analysis.

Higher-Dimensional Cuts: This method is an extension of the above one to handle simultaneous and independent changes and again proposed by Gass and Saaty, in 1955, redefining the perturbations as

$$u = \hat{u} + \gamma G$$

where G is an sxn matrix [56]. This method is difficult to implement when $s \geq 4$ (for a discussion, see [137], page 15).

The 100% Rule: This rule, a special case of the approximation region of Gal [53], proposed by Bradley, Hax and Magnanti [14], uses the one-dimensional cuts and requires a specification of directions of increase or decrease from each nominated value. In general, there are 2^n possible specifications (for cost coefficients) so this method is hard to apply.

Tolerance Approach: Wendell proposed this approach [138], [140] to handle simultaneous and independent perturbations having the general form

$$u = \hat{u} + \gamma G$$

where G is an sxn matrix but he gave primary emphasis to the case G is an nxn matrix with $G_{ij} = u_j'$ for $i = j$ and $G_{ij} = 0$ for $i \neq j$. For $u_j' = \hat{u}$, γ_j represents a multiplicative perturbation. In this special case, the tolerance approach gives the maximum tolerance percentage by which the coefficients can be simultaneously and independently perturbed within a priori bounds without causing the optimal basis to change. In the general setting, if τ denotes a finite, nonnegative number, called a tolerance then an allowable tolerance is defined to be a number τ if the same basis is optimal as long as

$$u = \hat{u} + \gamma G$$
$$\gamma \in T$$
$$\|\gamma\|_\infty \leq \tau$$

Then among the allowable tolerances, the maximum is selected. The main advantages of this method are the ease with which the solution can be interpreted and that information about allowable ranges of variation can be used to yield larger maximum tolerance percentages. On the other hand, there are some disadvantages with this method such as, for moderate or large size problems, the maximum tolerance may often be at or near zero. To solve this problem, Wendell proposed using bounds on the variations, which are shown to increase the maximum tolerance [139]. In chapter 5 of [54] written by Wendell, one can find further information about other attempts to expand this region.

These methods can be used to summarize the critical regions or the optimal coefficient sets (see [52] and [59]). Attempts to summarize the optimal value function using one-dimensional and higher- dimensional cuts are computationally prohibitive. There are also two other methods, convex bounds and worst and best case bounds, to use in case of optimal coefficient sets (for detailed information, see [137], page 26).

The well studied perturbations are u = c, u = b, u = [b, c]. The case u = A is a difficult one but in the special case when perturbations occur in only a single row or column of B, the basis matrix, it is possible to give a mathematical expression for B^{-1} [53]. In more general cases, approximations are studied about which we refer the reader to [54], [58] and [49].

Sensitivity Analysis studies small perturbations whereas what parametric programming does is the study of the effects of large perturbations. For example, tolerance approach is a kind of generalization of the scalar parametric programming and it is a special case of multiparametric programming with independent parameters.

One chapter of [54], Qualitative Sensitivity Analysis, by Gautier, Granot and Granot is very interesting, since it is a pre-optimal analysis seeking to find out answers to questions such as "

- How does the magnitude of a change in an optimal value of a given variable depend on a change in a parameter associated with another variable? Where are the changes the strongest? The weakest? Are some variables unaffected? Less Affected than other?
- Is the optimal value of a given variable monotone in the parameter changed?

- Is the optimal value submodular (or supermodular) in the parameters changed? "

The approach was applied to network flow problems and monotropic problems.

For sensitivity analysis and parametric programming for nonlinear programming problems, the books by Bank et al. [2], Fiacco [47] and Gal and Greenberg [54] are very useful. Also the works of Fiacco and Ishizuka [45], Jenkins [71], Tanino [127], Fiacco and Ishizuka [46], Jongen and Weber [72], Kaul, Bhatia and Gupta [78] and Kyparisis [84] can be seen. For discrete optimization problems, the book [54] and the papers of Dawande and Hooker [28] and Yıldırım and Todd [145] and the references therein are suggested. Also the recent work of Thuan and Luc [133] on the sensitivity analysis in linear multiobjective programming can be seen.

CHAPTER-3

GENERAL UNCERTAIN OPTIMIZATION PROBLEM

3-1 The Model

Let P_u be a deterministic optimization problem defined as

$$(P_u) \quad \min \{ f_u(x) : x \in X_u \}$$

where

- x is the decision vector
- f_u: $X_D \subseteq R^n \rightarrow R^p$ is known for each fixed u
- $X_u := \{x : F_u(x) \in K \subseteq R^m\}$ and the mapping $F_u : X_D \rightarrow R^m$ is known for each fixed u
- the dimensions p, n, m and the sets X_D, K are known.

If u is not fixed but takes values from a known uncertainty set $U \subset R^M$, then we have a family of optimization problems, say P, so that each P_u is an instance of P. Therefore, we define an uncertain optimization problem as a set of problem instances, among which one will be realized so that we express it as:

$$P = \{ P_u : u \in U\}$$

Since we have a family of optimization problems, the notions of optimality and feasibility require different meanings from those in a deterministic case and this is why there are such a number of quite different approaches to handle such problems. Each such model transforms P into a single problem, say P', by transforming the sets $\{X_u : u \in U\}$ and $\{f_u : u \in U\}$ into a new feasible set X' and a new objective function f', respectively. P' has been named differently in the literature. For example, in the stochastic programming literature, it is called the deterministic equivalent and in robust optimization, it is called the robust. In this study, we call P' the *induced problem* and in parallel with this, we call f' and X' will be called the *induced objective function* and *induced feasible set*, respectively. As a last point, through the study, we will denote the optimum value of P_u as z_u^*.

Observe that for p = 1, we have a single objective optimization problem and the other case corresponds to the multiobjective case, where "min" requires a special meaning. In this case, any technique of multiobjective programming can be used and we will not focus on these techniques in this study. One more point is that if objective function and feasibility set are affected by different uncertain parameters, the decomposition of the uncertainty set U into O and F (O stands for objective and F for feasibility) such that U = OxF, and correspondingly the vector u into u_o and u_f to get the following model may increase computational efficiency and understanding of the model.

$$P = \{P_{(o,f)} : o \in O, f \in F\}$$

where $\quad P_{(o,f)} \quad \min \{f_o(x) : x \in X_f\}$

Observe that, the specification of P_u as

- $U = O x F$ with $O = [c]$ and $F = [A, b]$
- $f_o(x) = c^T x$
- $F_f(x) = Ax - b$
- $K = R_+^m$
- $X = R_+^n$

corresponds to the following LP:

$$P = P_u \ \min \{c^T x : Ax \geq b, x \geq 0\}$$

3-2 Uncertainty Set

In an uncertain problem, how the uncertainty is represented becomes a critical issue. Available information is a restriction in the modelling phase. Furthermore, the conformity of the form of uncertainty to the real situation affects the performance of the model. In the literature, there is a number of ways to represent uncertain data as will be explained below:

a- uncertain parameters are affine embeddings of a set of vectors
$U = \{u = u^0 + \lambda_1 u^1 + \ldots + \lambda_r u^r : (\lambda_1, \ldots, \lambda_r) \in T \subset R^r\}$, where T is a known set

b- uncertain parameters come from a convex set so that U is a bounded convex set

c- each uncertain parameter lies in an interval
$U = \{u : \underline{u}_j \leq u_j \leq \bar{u}_j, j = 1, \ldots, M\}$, corresponding to a closed multidimensional hyper-rectangle

d- uncertain parameters are represented by a number of scenarios
$U = \{u_s : s \in S\}$, where S is a known set of scenarios and u_s is the input data corresponding to scenario s

e- uncertain parameters are random variables
If w is a random variable with the support $\Omega \subset R^k$ and P is a probability

distribution function on R^k so that w is an element of the probability space (Ω, F, P), then

$U = \{g(w) : w \in \Omega)$ where $g : R^k \rightarrow R^M$

f- uncertain parameters come from fuzzy sets

$U = \{u: u_j \in \tilde{U}_j, j = 1,\ldots, M\}$ where each $\tilde{U}_j$ is a fuzzy set with the characteristic function c_j, which can be either a membership function μ_j or a possibility distribution π_j

g- uncertain parameters come from ellipsoidal sets

$U = \{\Pi(v) : \|Qv\| < 1\}$, where $v \rightarrow \Pi(v)$ is an affine embedding of certain R^L into R^M and Q is an KxL matrix

Each one of these types has its own advantages and disadvantages and also has some relations with the others, the specification of which will be useful in understanding them and in the numerical studies comparing different approaches assuming different types of uncertain data representations. Therefore, below, we study some of the representations from this perspective:

Intervals are easy to get, to agree on, especially in case of multiple decision makers and easy to interpret as the lower bound is understood as the pessimistic estimation and the upper bound as the optimistic one. In case of high level of uncertainty, when nothing more is known, this representation is one of the alternatives to use, the other one being the scenario representation. However, the difficulty to represent correlations is an important weakness. Another difficulty is that it may not be always possible to determine the bounds, especially when unpredictably rare events may occur. For example, inflation rate may be estimated to lie between 30% and 150%, which covers most realistic situations, but with a political crisis, as in Turkey, it may be realized as 250 %. Lastly, equal treatment of very extreme realizations and mostly expected realizations may not be suitable for all circumstances, which is also true for scenario representation.

First of all, it should be mentioned that, the term scenario does not mean discrete random variables as used sometimes. A scenario means a realization of the uncertain parameters with or without assigned probabilities. Scenario representation is also easy to understand and interpret. If there is a small number of important factors determining the rest as in the case of macroeconomic parameters, and if one has a dynamic environment, then scenario representation is useful in representing correlations. A weak point is the difficulty with the determination of a representative scenario set and the number of scenarios. One should be aware of the risk that if there is a small number of important factors leading to a small set of scenarios, the realization of an unspecified scenario will affect all of the system. Another disadvantage is the difficulty to model several endogenous linkages.

In both of these representations, computational difficulties arise as the number of uncertain elements and the number of scenarios increase.

For random variables, interpretation and understanding of probability distributions are easy as most of the time they are related to the frequency of occurrence. Different treatment of each element brings flexibility for some decision environments like the ones not involving risk averseness. They should be especially used if there is a theoretical foundation as in the case of queuing models. When enough information is not available, the determination of distributions and their parameters is very difficult. Computational burden is high especially in multivariable case with correlations and continuous random variables. This approach is applicable only if uncertainty comes from randomness.

Membership functions are not difficult to interpret as they are related to preferences within given tolerances but possibility distributions are very difficult to interpret and understand. Determination of them and their parameters are also problematic. These representations, usually, do not bring computational difficulties. Similar to the random case, different treatment of each element brings flexibility for some decision environments. Membership functions will be further discussed in fuzzy programming part of chapter 4.

To handle the difficulty arising with the increased number of scenarios with the use of scenario representation, intervals or "inequality-represented" polytopes, the ellipsoidal set representation was proposed. In case of ellipsoidal sets, the advantages can be stated as: being a wide family including polytopes, ability to approximate many cases of complicated convex sets and moderate size data requirement (for a further discussion, see [9] and [7]).

As a last point about the uncertainty representation, the relations among them may be discussed.

Intervals: From a scenario set, one can construct an interval for a parameter with the lower bound being the minimum element and the upper bound the maximum element. Also, the support of the probability distribution of or a α-confidence interval for a random variable and the support of the membership function of or a α-level cut for a fuzzy set can be taken to represent uncertainty with intervals.

Scenario: One may take the midpoints of a number of subintervals to represent an interval as a scenario set. Discrete random variables and fuzzy sets can be taken directly or indirectly (after some aggregation) as the scenario set. In case of continuity, discretization can be used.

Randomness: An interval can be taken as a uniform distribution and in the same way a scenario set as a discrete probability distribution with equal probabilities if probabilities are not assigned to scenarios. Otherwise, the probabilities can be used directly.

Fuzziness: An interval can be seen as a fuzzy set with equal grade of membership function and in the same way a scenario set as a discrete fuzzy set with equal grade of membership if probabilities are not assigned to scenarios. Otherwise, we can not say much about what the grade of memberships will be. To transform a random variable to a fuzzy number one can normalize the density function or use hybrid convolution proposed by Kaufmann [77].

It should be mentioned that how to represent randomness using a fuzzy set or vice versa is not clear since the relation between these is not explicit (see, for example, [63].

CHAPTER-4

APPROACHES to HANDLE UNCERTAINTY

In this chapter, we analyze the existing approaches to handle an uncertain optimization problem. Since our interest is to explain the main idea and logic of each approach but not a comprehensive survey, our models will be of general type not focusing on linear, nonlinear or discrete models. Due to the wide applicability of linear programming we will give a list of related works after the explanation of each type of model. For the discrete and nonlinear cases, some important and useful references will be given at the end of each approach.

4.1-Robust Optimization

Robustness approach aims to produce decisions that will have a reasonable (satisfactory) objective function value under any (or sometimes some) input data realization to the decision model. Whether this claim is justified or not or to what extent it is justified will be discussed at the end of this chapter. In this approach, the most inner solutions are sought, which are believed to be more stable than the boundary ones with respect to perturbations. On the other hand, selecting a non-extreme point instead of an extreme one may not be easily understood or interpreted. In fact, because of such an attempt, the ultimate solution may not be optimal for any P_u.

The specific models are given below in accordance with the types of uncertain parameters

1- Uncertainty about the consequences

Max-covering model was introduced by Gupta and Rosenhead, in 1968 [60] and in 1972, developed by Rosenhead, Elton and Gupta [111] in the field of strategic management. They defined robustness of a decision as the ability to achieve as many "'good' end states for expected external conditions which remain as open options" as possible. Then, in 1987, Rosenblatt and Lee [110] applied this idea to facilities design with uncertain demand (uncertain objective function), which is the first application of robustness approach in operations research. He considered as the index of robustness the number of times the solution lies within a prespecified percent of the optimal solution for the realizations of the scenario set (for each demand, a three point estimates have been assumed in the scenario set). Therefore P' is the following problem:

$$\max \quad \{|U_\alpha| / |U| : x \in X_D\}$$

where $\quad U_\alpha = \{u \in U : f_u(x) \leq (1+\alpha) z_u^*\}$

Min-max models take the well-known min-max criterion of game-theory for the induced objective function that we have the following induced problem:

$$f'(x) = \max \{f_u(x) : u \in U\}$$

$$P' \quad \min \{f'(x) : x \in X_D\}$$

With this approach, the case of uncertain cost coefficients was first studied by Falk, in 1975 [43], where he assumed that the cost vector comes from a convex set. Then, Sengupta [116] and Cai et al. [19] applied this approach to the case of random cost coefficients including in their formulation risk and utility.

Regret model uses the concept of regret proposed by Savage and defined, here, as the deviation from optimal objective value. In these models, the maximum regret is minimized so that the induced objective function and problem take the following forms:

$$f'(x) = \max \{f_u(x) - z_u^* : u \in U\}$$

$$P' \qquad \min \{f'(x) : x \in X_D\}$$

For a related work we refer the reader to Inuiguchi and Sakawa [69] and the references therein. They studied the uncertain cost coefficients assuming interval representation.

Relative regret model uses the concept of regret again but maximum percent deviation from optimal objective value is minimized so that the objective function and the problem become:

$$f'(x) = \max \{(f_u(x) - z_u^*) / z_u^* : u \in U\}$$

$$P' \qquad \min \{f'(x) : x \in X_D\}$$

This approach has been applied, in 1994, by Gutierrez and Kouvelis [61] in the context of international sourcing. They assumed that uncertain cost coefficients are represented by a scenario set. In 1999, Mauser and Laguna [90] studied the same uncertain parameters assuming they are represented by intervals.

Min-max models are very conservative so that the solutions from this approach are likely to be very expensive but there are cases (for example, see [9] and [6]), where it is not so. Also, they are not always applicable as in the case of bridge building, where it is not possible to build a bridge that never falls down under any realizable scenario. This type is especially useful if some considerations of targets, budget limits or quotas exist that need to be met or it may also be appropriate for competitive environments. The regret and the relative regret models are less conservative and they provide a mechanism to capture the missed opportunities. On the other hand, the solutions of regret and relative regret models require knowing the optimal value for each realization so they may be harder to solve than the min-max models. If such a difficulty exists then it can be overcome if instead of optimal values some targeted values (may depend on u) are used in the formulas.

2- Uncertainty about alternative courses of action

Hard feasibility

In these models, infeasibility can not be tolerated so that only the decisions which are feasible for all problem instances are considered. Therefore, the induced feasible region becomes

$$X' := \{x: x \in X_u \ \forall u \in U\}$$

This notion of feasibility was first applied by Soyster in 1972 [120] and [121]. He studied the case where each column of the uncertain technology matrix comes from a convex set. Then, in 1999, Ben-Tal and Nemirovski studied the case where the uncertain parameters of technology matrix and the right hand side come from ellipsoidal sets though the cost coefficient uncertainty can also be handled [8].

Soft Feasibility

i- models where infeasibility is not reflected into cost

In [9], Ben-Tal and Nemirovski use the following model to allow some infeasibility. In fact, with a parameter ε representing the amount of uncertainty in the technology matrix coefficients and with a parameter δ for the amount of violation, which is determined constraintwise, he called the ultimate solution "(ε, δ)-reliable". Below, we present the model without specifying these parameters explicitly. Let $v_u^i(x)$ be the violation of the ith constraint as

$$v_u^i(x) = \min \{|k - F_u^i(x)| : k \in K^i\}$$

then, the induced feasible region can be defined as

$$X' = \{x: v_u^i(x) \leq g(u)\}$$

For more detailed discussion and the case for random uncertainty, the same paper can be seen.

ii- models where infeasibility is reflected into cost

This type of model was proposed by Mulvey, Vanderbei and Zenios, in 1994 [94] introducing a new concept, "model robust", where a decision is so if it remains "almost" feasible for all realizations of an uncertain parameter. They call a decision "solution robust" if it remains "close" to optimal for all realizations of the input data. Their model requires a discrete scenario set with assigned probabilities. In fact, goal programming formulations are involved to trade off the infeasibility and optimality. Their objective function is of the following form:

$$f' = g_o(f_u(x)) + wg_f(K, F_u(x))$$

The functions g_o and g_f account for optimality and feasibility and the trade off between them are represented with the weight w. For example, g can be defined as the worst case, mean value or higher moments. For equality constraints, g_f is suggested to be of a quadratic form and for inequality constraints to be of a maximum violation form. This model has the difficulty carried by the weight factor since selecting the right factor is a difficult task. In fact, it rates the amount of infeasibility with cost, which is usually difficult to assess. Afterwards, in 1995, Mulvey and Ruszcynski [93] and in 2000, Yu and Li further studied this type of models [147].

For robust optimization of discrete optimization problems, the book by Kouvelis and Yu [83] is an excellent reference, where one can find the main ideas of robust optimization and solvability properties of some discrete optimization problems.

Furthermore, one can benefit from the references therein. For a general approach for finding regret solutions for a class of combinatorial optimization problems where uncertainty comes from the objective function coefficients and represented by a scenario set, one can see Averbakh [1]. Additionally, Yu [148] studied regret or relative regret type discrete optimization problems providing pseudopolynomial algorithms under certain conditions.

For robust nonlinear programming one can see [7] and the recently published work of Indraneel with the references therein [66].

4.2-Stochastic Programming

In this approach, mathematical programming problems are handled where some of the parameters are random variables. It is said in the book of Prékopa [101, page viii] that "… either we study the statistical properties of the random optimum value or other random variables that come up with the problem or we formulate it into a decision type problem by taking into account the joint probability distribution of the random parameters.". For a comprehensive treatment of the subject, the books by Kall and Wallace [75], Birge and Louveaux [11] and Prékopa [101] are suggested.

In stochastic programming, two basic assumptions are made as uncertainty comes from random elements in the model and one has distributional knowledge (objective or subjective) about the random elements.

We divided the main robustness models in terms of their notions of feasibility and then subdivisions were given according to their notions of optimality. This is not an efficient way of determining the main types of models in stochastic programming since firstly, models with hard feasibility may be represented in the same way as those with soft feasibility as in the case of probabilistic constraints, and secondly, there is a variety of ways to allow constraint violation and to handle uncertain objective functions. Therefore, we determine two main streams of stochastic programming models in parallel with Ermoliev and Wets [41] as shown below.

Stochastic Programming Models

1-Adaptive Models	2-Recourse Models
1.1- Distribution Problem	2.1-Two-Stage
1.2- Anticipative Models	2.2- Multi-Stage
1.2.1- Probabilistic Models	
1.2.2- Moments-based Models	
1.2.3- Hybrid Models of 2.1&2.2	

1-Adaptive models:

In this type of models, optimization is seen as being made in a learning environment so that before making a decision, observations can be made. Let $B \subseteq F$ be a collection of sets that contain all the relevant information obtained from observations. Then, the decision must be determined on the basis of B, being a function of u whose values are B-dependent (or B-measurable). There are two important special cases of adaptive models, namely, distribution problem and anticipative models. But before going into details of them it should be mentioned that, in this work, we use the term adaptive just for the case where B is a nonempty proper subset of F. We will not focus on this explicitly since the only difference between anticipative models and adaptive models is that the former uses prior distributions whereas the latter uses posterior distributions. For example, if in an anticipative model, the induced objective function or the induced feasible region are defined as a function g and g' as shown below, then the same functions but conditioned on B would be used in an adaptive model as shown below.

	Anticipative	Adaptive
Objective:	$g\{f_u(x) : u \in U\}$	$g\{f_u(x) : u \in U \mid B\}$
Feasibility:	$g'\{S_u(x), K : u \in U\}$	$g'\{S_u(x), K : u \in U \mid B\}$

1.1. Distribution Problem: If B = F, one has the posterior distribution of u and solving for each P_u, one can obtain the probability distribution or some characteristics of random variables such as the probability distribution of the random optimum value or of the optimal solution in case of a random LP. One important special problem is finding basis stability, the probability that the basis remains unchanged. Also, finding the distribution induced on the recourse function, which will be explained below, is useful to find its expectation and to address other risk criteria that may not be given by the expectation functional. This type of problems can be seen as a generalization of sensitivity analysis or parametric programming. For the computation of characteristics of the random optimum value, simulation, discretization and Cartesian Integration are used. Dupačová and Wets [40], King and Rockafellar [80] and Shapiro [117] can be seen about the asymptotic distributions of optimal solutions in stochastic programming, which is another topic studied in these problems. Also, there are a number of papers studying laws of large numbers for random linear programs like Prékopa [98] and Kabe [73]. Interested reader can find more discussion of this subject in chapter 15 of [101].

1.2. Anticipative Models: If B= Ø, then one has nothing more than priori distributions of the parameters. Such models are called anticipative models in the literature. In each of these models, induced objective function and induced feasible set can be defined in terms of either probabilities or moments of the distribution function. We give some types of formulations below:

- 1.2.1. Probabilistic Models: Using probabilities, the induced objective function can be one of the following:

- max $P(f_u(x) \leq \check{z})$ (O.1.1)
- min $\check{z}$ where $\check{z}$ satisfies $P(f_u(x) \leq \check{z}) \geq \alpha$ (O.1.2)

Using probabilities, the induced feasibility can be defined as one of the following:

- $P(x \in S_u) \geq \alpha$ (F.1.1)
- $P(F_u^i(x) \in K^i) \geq \alpha_i$ where F_u^i is the ith left hand side and K^i is the ith right hand side (F.1.2)

Constraints of these types were first introduced by Charnes, Cooper and Symonds in 1958 with the formulation F.1.2 with random RHS, where they call their models *chance constrained programming* [26]. In 1963, Charnes and Cooper also suggested the use of F.1.1 and O.1.2 for the case of random RHS and cost coefficients and called this model the P-Model [25]. Then, in 1965, Miller and Wagner [92] studied the case of random RHS with independent components using the formulation F.1.1. The same case with dependent components was studied by Prékopa in 1970 [97]. O.1.2 was introduced by Kataoka, in 1963, to handle random cost coefficients and to increase safety [76]. All of these studies are related with linear programming problems. For more recent results for probabilistic programming see Dentcheva [34] and the references therein.

In probabilistic models, the levels z and α are arbitrarily chosen. Specifying z may be especially difficult without knowing anything about the solution. With the use of α, the effects of the tails of the distribution is ignored. This may affect the system abruptly so that, in a sense, one does not have any idea of the cost of violation or cost of being suboptimal.

- 1.2.2. Moments-based Models: Using moments, the induced objective function can be one of the following:

$$- \min \alpha E[f_u(x)] + \beta(\mathrm{var}[f_u(x)])^{1/2} \qquad (O.2.1)$$

$$- \min \alpha E[f_u(x)] + \beta \mathrm{var}[f_u(x)] \qquad (O.2.2)$$

$$- \min (E[f_u(x)], \mathrm{var}[f_u(x)]) \qquad (O.2.3)$$

Again, using moments, the induced feasibility can be defined as:

$$- E[g(F_u^i(x), K^i) | F_u(x) \notin K] \le d_i,\ i = 1,\ldots, m \qquad (F.2.1)$$

$$\text{where } g(F_u^i(x), K^i) = \min \{|k - F_u^i(x)| : k \in K^i\}$$

The early works related with linear programming problems are given here. The constraints involving conditional expectations, F.2.1, were studied first by Prékopa to ensure safety limiting the expected amount of violation constraintwise [97], [99]. There are also some formulations including conditional expectation and probabilities together for feasibility. These are called “induced chance constraints”. This type of constraints, used by Bloom [13] and Klein [81] seem to be useful especially for the case the technology matrix has randomness since they allow the convexity of the feasible region.

For formulas with expectations to be meaningful, the system has to repeat its performance independently, in a large number of cases, so that the average of the outcome is close enough to the expectation. Additionally, the magnitude of the variation of the outcome should not be large. This is why variance or standard deviation are included in the formulas.

• 1.2.3. Hybrid Models of 2.1&2.2: For an LP, Charnes and Cooper [25] suggested combining probabilistic constraints with moments-based objective functions. For the random RHS and cost coefficients, they suggested a model with F.1.2 feasibility with O.2.2 objective with $\beta = 0$ or $\alpha = 0$, which they call E- and V-Models respectively.

2- Recourse Models:

This type of models reflect a trade-off between anticipation and adaptation so that it is assumed that after observations are made, some corrective (recourse) actions can be taken to fill the gap between anticipated and realized values. So, in these models, infeasibilities are penalized. We can categorize recourse models as two-stage and multistage recourse models. The two-stage version of this model has been studied extensively and this is what we will study here mostly (for a further discussion, see Frauender [46]).

2.1. Two-Stage Models: In recourse models, uncertainty affecting only the feasible region is handled so that we assume a constant objective function, $f_u(x) = f(x)$, $\forall\, u \in U$. In these models, penalties coming from the violations of the constraints are added to the system cost. If one takes a decision x and if after uncertainty is resolved she/he has $F_u(x) \notin K$, it is assumed possible to take a corrective action y such that

$$F_u(x) + H_u(y) \in K$$

but this brings cost which is assumed to be linear and dependent on u in the general setting, say $q(u)^T x$. While choosing among the corrective actions, cost should be minimized so that a corrective action should solve the following optimization problem

$$\begin{aligned} P_u(x) \quad & \min\ q(u)^T y \\ \text{st} \quad & F_u(x) + H_u(y) \in K \end{aligned}$$

In this setting, one should consider all possible realizations of u and the corresponding possible corrective actions before making a decision. Let $Q_u^*(x)$ be the optimal value of $P_u(x)$. Then the two-stage recourse model can be defined as:

$$\begin{aligned} & \min\ f(x) + E[Q_u^*(x)] \\ \text{st} \quad & x \in X_D \end{aligned}$$

There are also some formulations adding probabilistic or moment type constraints to this model to increase safety but these also increase complexity of the model [41]. Furthermore, Evers [42] proposed to introduce an additional cost dependent on the probability of the constraint satisfaction or violation.

Two important assumptions are usually made about the feasibility of the $P_u(x)$, which is obviously dependent on x. The first being complete recourse, where $P_u(x)$ is feasible for any value of $x \in R^n$ and the second, relatively complete recourse where $P_u(x)$ is feasible for any value of $x \in X_D$.

Generally, q(u) is taken as constant and $H_u(y)$ as a linear mapping, e.g. $W_u y$. If also W_u is deterministic such a model is called fixed recourse.

One popular special case of fixed recourse models, called simple recourse proposed by the pioneers of stochastic programming, Dantzig [27] and Beale [3] corresponds to the case W = [I, -I], which implies constraintwise penalties for violations. Simple recourse models satisfy the complete recourse assumption. On the other hand, constraintwise penalties are not always legitimate as in the case of the existence of correlations in the random vector.

In addition to the criticism made about the use of expectations, recourse models can be criticized in that, the cost of violations of some constraints is not known always (in which case use of probabilistic constraints seems reasonable) and even if the costs are known, without any probabilistic constraints, the reliability of the system will be an open question in these models.

Different types of penalty functions have also been used. For example, Ben-Tal and Teboulle [10] penalized the violation of the feasible set using utility function and then Ben-Tal and Ben-Israil [5] incorporated value-risk function instead of utility function calling this model recourse certainty equivalent. In fact, the use of nonlinear utility functions especially considering their expected values and the Markowitz type mean/variance models as those given in the anticipative models are the ways to handle risk in stochastic programming. The use of nonlinear utility functions make the models more difficult to solve, in which case one can either include risk aversion but use simple second-stage description or use linear utility function but detailed second stage description or include risk aversion in a linear utility model under the form of a linear constraint called downside risk.

2.2. Multi-stage Models: In this type of models, there are a number of decisions and observations following each other. In addition to some computational difficulties encountered in two-stage models, in multistage models it is necessary to solve large system of linear or nonlinear equations to obtain a description of the evolution of the system. A recent work on this topic is due to Høyland and Wallace [65]. Multi-stage recourse models do not have separability properties so conventional recourse equations of dynamic programming can not be used here but these problems have a special structure called staircase, which allows some solution techniques like basic decomposition technique to be applicable. Other techniques are L-Shaped technique and scenario aggregation (for further information about these techniques see [41] and specifically about scenario aggregation, see Rockafellar and Wets [106], Robinson [105] and Dembo [31]).

In stochastic programming, the exact evaluations of the functions or of their subgradients, especially in multidimensional continuous random variable case can be an extremely demanding computational task. The existence of correlations even worsens the situation. In fact, much of the work of the theory is concerned with determining the properties of these integrals and devising suitable approximation schemes. There are some special cases where the computational aspect is not so bad as the following cases:

- For formulations with expectations, E[u] can be used if linearity exists.
- Probabilistic constraints with only random rhs can be reduced to a linear system of equations
- If all parameters of a random constraint are jointly normally distributed a linear system of equations can be obtained but knowledge of the covariance matrix is required
- Simple recourse problems especially with discrete random variables has a block angular structure and there exist special optimization techniques to solve these but the number of density points of the distribution should be small.

To handle the above mentioned computational difficulties some approximation schemes were proposed. Also, design of approximation schemes is not easy requiring convergence theory, error bounds, improvement schemes and so on.

One can use approximation techniques replacing the probability distribution with a simpler one especially with a discrete one so that one will have sums instead of integrals in the formulation. Also, stochastic quasigradient methods can be used where sampled realizations are used to get general statistical properties. This corresponds to replacing the function with the simpler ones. Additionally most of the time, independence assumption is made.

Some of the solution methods for probabilistic models are The SUMT (The Sequential Unconstrained Minimization Technique) [41], Supporting Hyperplane

Method studied by Prekopa and Szántai [102], GRG (General Reduced Gradient Method) proposed by Mayer [91] and the primal-dual algorithm of Komáromi [82].

Some of the solution methods for simple recourse problems are Primal Method proposed by Wets [142] and Dual Method of Prékopa [100].

About the solution procedures of two-stage recourse models one can see Basis Decomposition Technique by Strazicky [124], L-Shaped Method of Van Slyke and Wets [119] and [67]. For the methods, Discretization, Sublinear Upper Bounding Technique, Regularized Decomposition Method, Stochastic Decomposition and Conditional Stochastic Decomposition, Stochastic Quasigradient, one can see [41].

About multiobjective stochastic programming, the interactive approaches of Teghem [136] and Urli [134] are very popular for the linear case and for a general mathematical programming problem, one can see Stancu-Minasian [122] and Ringuest [104] for a method for generating nondominated solutions.

4.3-Fuzzy Programming

In 1965, Lotfi A. Zadeh introduced the concept of "Fuzzy Sets" and then, fuzzy approach was used extensively as a modelling tool especially as a way of modelling vague data. Vagueness is defined, by Fedrizzi, as a lack of clear-cut boundaries of the set of objects to which the meaning is applied [44]. So that by fuzziness it is meant a type of imprecision which is associated with classes in which there is no sharp transition from membership to nonmembership. In the representation of this concept, membership functions are used, which were defined by Zadeh [150] as:

"Let X be a space of points (objects), with a generic element of X denoted by x. Thus, X = {x}. A *fuzzy set* (*class*) A in X is characterized by a *membership* (*characteristic*) *function* $\mu_A(x)$ which associates with each point in X a real number in the interval [0, 1], with $\mu_A(x)$ representing the "grade of membership" of x in A."

The first proposal for fuzzy decision making comes from Bellman and Zadeh [4], in 1973, where fuzzy decision was defined as a fuzzy set resulting from the intersection of the fuzzy constraint and the fuzzy goal. If the fuzzy constraint and the fuzzy decision are characterized by the membership functions $\mu_C(x)$ and $\mu_G(x)$, respectively, then the fuzzy decision was said to be characterized by $\mu_D(x) = \min(\mu_C(x), \mu_G(x))$. Then the optimal solution is defined as the one maximizing $\mu_D(x)$. In this approach, the minimum grade between the grade of feasibility and goal satisfaction is maximized, from which it is apparent that, it is a conservative approach not allowing any trade off between the constraint satisfaction and goal satisfaction. Thereafter, a rich literature has been developed both in the theory of fuzzy sets and its application in operations research. We refer the reader to the books of Yager et.al [144], Kacprzyk and Orlovski [74], Dubois and Prade [37], Lai and Hwang [87] and Slowinski [118] for further information about fuzzy programming. In the first book, one can find the original papers of Zadeh whereas especially in the books of Lai and Slowinski, one can find comprehensive surveys of this area and lists of books, journals, application areas and papers. The study of different approaches of fuzzy linear programs and a survey of this area can be found in Delgado et.al. [30] and Rommelfanger [108], respectively.

A fuzzy programming model is not a uniquely defined type of model but depending on the assumptions or features of the real situation many variations are possible. Fuzzy programming was also proposed as a tool for solving vectormaximum problems by Zimmermann [152] and Ying-Jun [146], which enables a decision maker describe the efficient vectors to be preferred. In fuzzy models, violations of the constraints are tolerable and the goals do not have to be in a min or max form. Because of such flexibilities, as in the case of stochastic programming, the main streams of models will not be determined according to the notion of feasibility or optimality but according to the input data type, being either a membership function or a possibility distribution. In this study, we call the former case flexible programming and the latter possibilistic programming to distinguish them. Although, in the literature, the former is called fuzzy programming we will use this term to refer to both of them. Furthermore, flexible programming can be subdivided into two as symmetric and non-symmetric as shown below in the next page.

The distinction between flexible and possibilistic programming can be stated as follows: the grade of a membership function indicates a subjective degree of satisfaction within given tolerances so membership functions are constructed by eliciting the preference information from the decision maker whereas possibilistic programming handles imprecise numbers so that possibility distributions are constructed by considering the degree of the possibility of the occurrence of events. In this respect, the possibility distributions, where the involved fuzzy sets are assumed to be normal and convex, are often assumed to be triangular or trapezoidal functions whereas for membership functions this is not a requirement. In this study, we use the term "characteristic function" to refer to both membership functions and possibility distributions. One more distinction is the existence of fuzzy relations or goals. In flexible programming, one has fuzzy environments or preferences so that fuzzy maximization/minimization or fuzzy equality/inequality are allowed to be defined whereas in possibilistic programming, it is the existence of fuzzy numbers that cause imprecision like the case of random variables, so such concepts are not allowed to be used in possibilistic programming. Because of such distinctions, a solution of a flexible programming model has a degree of preference and a solution of a possibilistic programming model has a degree of possibility of occurrence. The solutions of the related models should be interpreted from this perspective.

<u>Fuzzy Programming Models</u>

<u>1-Flexible Programming Models</u> <u>2-Possibilistic Programming Models</u>

1.1- Symetric Models

2.1- Nonsymetric Models

There is a significant number of characteristic function types, which are listed in the next page in terms of applicability to flexible programming and possibilistic programming (see [118], page 183 for more detailed information).

An important point is how to get characteristic functions. Dishkant is one of the first to try to estimate the membership functions [35]. In the literature there exist some studies concerning this aspect, which can be divided into two as axiomatic approach and semantic approach (Giles, [57]) but the question is still not well answered [87, page 30]. In practice, even if the "true" shapes of membership functions are approximated well, to model realistically the part of a membership function belonging to small membership values is very difficult. A practical way of getting suitable membership functions is the procedure proposed by Rommelfanger [107]. For more information about the membership functions see Dombi [36]. There are also studies where the grade of a membership function or the support of a fuzzy set may be a fuzzy set, called generalized/extended fuzzy set (Buckley [15]). Buckley also suggests extending the distribution problem of stochastic programming to possibilistic programming case and defines the possibility distribution of the objective function [17].

<u>Types of Characteristic Functions</u>

<u>Flexible</u>	<u>Possibilistic</u>
• Linear	• Triangular
• Concave	• Trapezoid
* by exponential functions	
* by piecewise linear functions	
• s-shape	
* by piecewise linear functions	
* by hyperbolic functions	
* by hyperbolic inverse functions	
* by logistic functions	
* by cubical functions	

Since in a fuzzy programming one has fuzzy sets, the solutions are via intersections or unions of fuzzy sets and the resulting sets will also be fuzzy, whose characteristic functions are determined by defining some operators for union and intersection. As in the case of characteristic functions, there are also different types of operators for union and intersection and this also contributes to the high variety of fuzzy models. For example for the union and intersection, the originally proposed operators are max and min respectively. They have a pessimistic view and no attention has been paid to repetitive character of the information available giving solutions not acceptable (for a good example for this situation, see Hisdal [63]). Then six alternatives were suggested to use instead of max and six alternatives to use in place of min. The following list contains just the names of these operators without any formulation. We refer the reader to [87, page 54] for further information.

compensatory max operators	compensatory min operators
Algebraic Sum	Algebraic Product
Bounded Sum	Bounded Product
Hamacher's Max Operator	Hamacher's Min Operator
Yager's Max Operator	Yager's Min Operator
Dubois and Prade's Max Operator	Dubois and Prade's Min Operator
Werners's "Fuzzy Or" Operator	Werners's "Fuzzy Or" Operator

Therefore, while modelling, three points given below should be studied well since they have a significant impact on the model:

- type of characteristic function: membership function or possibility distribution
- the type of the membership function or possibility distribution
- the type of operators

Below, we present some type of models used in fuzzy programming in accordance with the type of the uncertain parameters. For different types of models or for more information about them, we refer the reader to [87].

Flexible programming

We assumed here, as Lai and Hwang [86], that the fuzzy equality or inequality relations can be incorporated into the fuzzy constraint especially to the fuzzy right hand side so that we do not include them in the models.

1- Uncertainty about the consequences

In this situation, each cost coefficient u_j, $j = 1,\ldots,n$ comes from a fuzzy with associated membership function $\mu_j : \tilde{U} \rightarrow [0,1]$, $j = 1,\ldots, n$. These fuzzy coefficients are aggregated with some of the operators from the previous list. If we let g_o represent such an aggregation, then the membership function of a cost vector becomes

$$\mu_o(u) := g_o\{\mu_j(u_j) : j = 1,\ldots,n\}$$

Then a parametric programming model is developed as

$$\min \{f_u(x): \mu_o(u) \geq \alpha, x \in X_D\} \text{ where } \alpha \in [0,1]$$

Each solution associated with the parameter α has a degree of preference α. One of the options to select an ultimate solution is seeking the one with the maximum degree of preference.

2- Uncertainty about alternative courses of action

Firstly, it is assumed that, the fuzzy parameters of the feasible region is somehow aggregated using some operators as in the previous situation so that the membership function of a solution for feasibility is

$$\mu_f(x) := g(F_u(x), c_j(u_j) : j = 1, \ldots, M)$$

Then the induced problem becomes

$$\max \{f(x) : \mu_f(x) \geq \alpha, x \in X_D\} \text{ where } \alpha \in [0,1]$$

Each solution associated with the parameter α has a degree of preference α for feasibility. There are also some works where the objective function is fuzzified although it has no uncertain parameters, for which we refer the reader to [87].

Possibilistic programming

1- Uncertainty about the consequences

Here we give two examples of the models used in this approach. The first one is similar to a stochastic programming model where all of the random variables are replaced with their expected values. The model is as follows:

$$\min \{f_{\bar{u}}(x) : x \in X_D\} \text{ where } \bar{u} := (\bar{u}_1, \ldots \bar{u}_n) \text{ and } \bar{u}_i := w_i u_i^p + (1- w_i) u_i^o \;\; i = 1, \ldots, n$$

Here u_i^p and u_i^o represent the pessimistic (min) and optimistic (max) values of the uncertain parameters and the solution corresponding to the most possible problem instance are sought. The determination of weights is very questionable.

In the second model, a multiobjective approach is used to get the following model:

$$\min \{(f_{\tilde{u}}^{1}(x), f_{\tilde{u}}^{2}(x), f_{\tilde{u}}^{3}(x)) : x \in X_D\}$$

where $f_{\tilde{u}}^{1}(x)$ is the problem instance corresponding to the pessimistic instance, $f_{\tilde{u}}^{2}(x)$ corresponds to the optimistic instance and $f_{\tilde{u}}^{3}(x)$ corresponds to the most possible instance. This model carries the difficulty a multiobjective programming problem has as to the determination of the final solution. There are also alternative models to this where the left and right spreads of the possibility distribution of the objective function are maximized and minimized, respectively and these models have the same idea as a stochastic programming model where the expected value and the variance are considered as two objectives.

2- Uncertainty about alternative courses of action

Most of the models require the use of a fuzzy ranking procedure to define the feasible region and with that definition reduces an inequality relation, for example, to a number of inequalities, which we do not discuss here explicitly. If some ranking procedure is assumed to be performed $g(\pi_i(u_i)\ i = 1,\ldots, M)$ and the possibility distribution of the feasible region $\pi(x)$ is determined than the following model can be given as an example.

$$\min \{f(x) : \pi (x) \geq \alpha\}$$

For both programming types, if uncertainty affects objective function and feasible region simultaneously, then any combination of the existing formulations can be used. There are also some interactive approaches not mentioned here, which can also be used to handle this situation.

Fuzzy programming models are reduced to classical LP, goal programming, parametric programming or nonlinear programming problems. Especially, models with linear membership functions and max-min operators can be solved efficiently by standard LP methods.

There are some important criticisms about fuzzy programming and some about the theory behind the fuzzy set theory, which should be mentioned here. Firstly, the meaning of grades of membership functions is questionable especially for possibility distributions. This is so since the basic assumption made is that randomness is not equal to fuzziness. On the other hand, determining possibility distributions require evaluating the possibility of the occurrence of an event, which has some relation with the frequency of the occurrence of that event. The exact relation between those is an open question although there are a number of attempts studying this area. One of the main differences between those is related with the consistency. That is, in general, the union of a fuzzy set and its complement not equal to the attribute universe. This inconsistency is seen as a serious weakness since in probability theory one can say the degree to satisfy some condition knowing the degree not to satisfy but can not do this in case of fuzzy sets. Another major criticism is the lack of standard definitions for fundamental concepts like negation, probability of fuzzy events, union or intersection of fuzzy sets… In Kerre [79], several proofs of important properties are shown to be incorrect. Therefore, while modelling with fuzzy sets, one should very carefully study the theory behind it.

In the next page, we give some literature related with flexible and possibilistic linear programming respectively.

For the fuzzy nonlinear problems, Sakawa and Yano [114] and [115] proposed an interactive method for multiobjective nonlinear programming with fuzzy parameters using augmented minimax problems. Also, Dumitru and Luban [38] and the survey of Sakawa [113] and the references therein are useful.

For discrete optimization problems, we refer the reader to Chanas and Kuchta [24], where they present selected problems and algorithms of fuzzy discrete optimization.

Some Works Related with Flexible Programming

Parameter	Approach	Year
[b]	Tanaka et.al. [87, p 6]	(1974)
	Verdegay [87, p 6]	(1982)
	Werners [141]	(1987)
[c]	Verdegay [135]	(1984)
[b, z]	Zimmermann [151]	(1976)
	Chanas [21]	(1983)
	Chanas, Kolodziejczyk [22]	(1986)
	Lai, Hwang [86]	(1992)
[A] and/or [b] and/or [c]	Carlsson Korhonen [20]	(1986)
[A,z] or [z,A,b]	Lai, Hwang [85]	(1992)

Some Works Related with Possibilistic Programming

Parameter type	Approach	Year
[c]	Luhandjula [89]	(1987)
[A, b]	Tanaka et al. [125]	(1984)
	Ramik, Rimanek [103]	(1985)
	Dubois [87, p 6]	(1987)
[b] or [c]	Rommelfanger et.al. [109]	(1989)
	Delgado et al. [30]	(1990)
[A] or [b,c]	Fuller [87, p 6]	(1986)
	Buckley [16]	(1988)
	Negi [95]	(1989)
[b], [c], [A], [A, c] or [b,c]	Lai Hwang [85]	(1992)
[A,b] and fuzzy <	Delgado et al. [29]	(1989)
[A] and/or [b] and/or [c]	Buckley [18]	(1990)

4.4-Multiobjective Optimization:

Though there are uncertainties inherent for the multiobjective optimization problems related with the preference-structure, here our concern is not to study them since the general uncertain model given in this study does not recognize such type of uncertainties. Instead we will consider multiobjective optimization as a tool for solving the uncertain optimization problems as defined previously. The idea was proposed by Ishibuchi and Tanaka in 1990 [70], where they define two objective functions on $\{f_u : u \in U\}$ to take into account the worst case and the average case. The work of Rommelfanger is also related to this type of modelling [109]. In 1996, Chanas and Kuchta generalized the original proposal [23]. A more general model with more than two objective functions can be given as follows:

Let f_j' j = 1,…, r be defined on $\{f_u : u \in U\}$ and assuming X_u is not constant. With the notion of hard feasibility, the problem to be solved becomes

$$\min \{f_1'(x), \ldots, f_r'(x)\}$$
$$\text{st} \qquad x \in \cap X_u$$

This model can be applied to the case original problem has single objective function but in the other case, may not work well. It can be useful to handle different risk levels with the aim to perform well in all of them.

4.5- Imprecise Optimization

In this approach, new solution concepts based on the relationship between the decision space and the uncertainty set have been introduced. The initial work on this that introduces the concepts of weak, permanent and strong solutions dates back to 1988 with an unpublished research report by Tansel and Scheuenstuhl [131]. The gest of the approach relies on associating a subset of the uncertainty set with each point in the decision space. The associated subset with a given point of the decision space is referred to as the *optimality-region* of that point and includes the set of all data realizations for which the given point is an optimal solution. This key concept leads to various kinds of solutions that give different meanings to what we understand from the word “robust”. These solution concepts have been developed by Tansel [128], Tansel and Demir [130], Demir, Tansel and Scheuenstuhl [33] and Tansel [129]. For applications of these solution concepts in the context of a facility location problem we refer the reader to [131], [130], [33] and particularly [32].

The approach induces a subset of the decision set $X_w \subset X_D$ such that each element of this set is optimal for at least one realization of the uncertain parameter and each element of this set is called a weak solution. So the following defines the set of weak solutions:

$$X_w := \{x : x \in \operatorname{argmin} \{f_u(x) : x \in X_D\} \text{ for some } u \in U\}$$

Then for each element of X_w, they induce a subset of U, say U(x) such that x is an optimal solution for every element of U(x), which is called optimality-region of x, so we define U(x) as:

$$U(x) := \{u \in U : x \in \operatorname{argmin} \{f_u(x) : x \in X_D\}\}$$

Observe that if U(x) = U for some $x \in X_w$, then such decisions would be optimal under any realization of u, which is what is really sought in an uncertain problem providing total protection against the unknown. Such x is called a *permanent solution*, which is the first type of solution defined in this approach. A permanent solution (or set of permanent solutions) does not exist most of the time and in order to handle this situation the second type of solution concept, *approximate permanent solution*, is proposed as the solution of the following auxiliary problem.

$$\min_{x \in X_w} \quad g(x)$$

where $g: X \rightarrow R$ is a function indicating the performance of x to cover the uncertainty set.

Two types of performance functions have been suggested as l_∞ norm of the distance to the uncertainty set and volume of coverage of the uncertainty set, which are defined in the following ways respectively.

$$g(x) = \max \{ \min \{d(u, v): v \in U(x)\}: u \in U\}$$
$$g(x) = - vol(U(x))/ vol(U)$$

where d(u, v) is a distance function and vol(.) is assumed to be defined, both of which use the optimality-region concept in their formulations.

The interpretation of the functions defined so is very meaningful as stated also in Demir (2001). For instance, in a decision making situation with repetitive character like a dynamic system where the realizations from the uncertainty set occur in a number of times through time, then the volume maximizing solutions guarantee optimality for the largest possible portion of time. For the other case, where the decision making situation is of a unique type (nonrepetitive) with the realization of the uncertain parameter occurring once, the volume maximizing solution guarantees the maximum likelihood of being optimal. Additionally, a minimum distance solution has the ability to have a smooth optimality-region, which, we believe, decreases the risk of being suboptimal with a small perturbation of the parameter.

The use of approximate permanent solution provides very important information about the way to decrease the uncertainty in case of controllable uncertainty since if one has the ability to reduce the uncertainty set somehow or at some cost, then with the information of approximate permanent solution a permanent solution can be found for the reduced uncertainty situation. Its power increases as one has higher coverage or lower distance performance. For example, 1 minus the volume coverage rate can be used as an indicator of the risk of being suboptimal.

One extension of this model may be the introduction of a new concept, α-optimality-region, defined as:

$$U_\alpha(x) = \{u: f_u(x) \leq (1+\alpha)z_u^*\}$$

Then using $U_\alpha(x)$ instead of U(x) in the above formulations increases the possibility of finding decisions with greater coverage and with a near optimal objective value performance.

The last type of solution concept, called unionwise permanent solution, is introduced by defining a minimal subset of X_w such that their region of optimality cover the uncertainty set together.

$$X_{uw} := \text{argmin } \{|Z|: Z = \{x \in X_w: \cup U(x) = U\}\}$$

In this situation, how to find an ultimate solution depends on the situation on hand. For instance, such a set may have very small number of solutions, all of which can be implemented if possible. While doing so, one should consider the solution set providing the minimum total cost if there exists a number of unionwise permanent solutions but this case may not be frequently encountered. Another thing may be the use of regret concept as suggested by Demir as "… minimize the maximum regret of having made a suboptimal choice.", from which we understand that the ultimate solution will be chosen from the unionwise permanent solutions. Because of this, the minimum regret of this model will be greater than the minimum regret of a usual regret model.

The optimality-region concept is one of the most important concepts introduced in this approach not because it is used in the definition of a new type of solution but it can be extended, as in the case of α-optimality-region, to construct a common framework in a way that all of the models mentioned handling uncertainty in the objective function coefficients can be represented using it. Afterwards, we will consider the case of uncertain feasibility and introduce new concepts of feasibility, which have the potential to be used to form such a common framework.

1- uncertainty about the consequences

In this situation, corresponding to uncertainty in the objective function coefficients, one has a deterministic decision space X_D, the uncertainty set U and a set of objective function values, $\{f_u(x): x \in X_D, u \in U\}$. In order to represent the existing models in a uniform way, we introduce the extended-optimaliy-region of a decision, $U_\beta(x)$, defined as follows:

$$U_\beta(x) = \{u: g(f_u(x)) \leq \beta(x)\}$$

where $\beta(x) \in R$ for each $x \in X_D$. This extension allows one to use some other criteria for the objective function performance like expected values, utility functions etc. Below, we transform each previously studied model into a new one with the extended-optimality-region concept.

Min-max: Let $U_\beta(x) = \{u: f_u(x) \leq \beta(x)\}$, then a min-max problem becomes

$$\min \{\beta(x) : U_\beta(x) = U, x \in X_D\}$$

Max covering: Let $g(f_u(x)) = (f_u(x) - z_u) / z_u$ and $\beta(x) = \alpha$, $\forall x \in X_D$ so that $U_\beta(x) = \{u: f_u(x) \leq (1+\alpha)z_u\}$, then the problem becomes

$$\max \{|U\alpha(x)| / |U| : x \in X_D\}$$

Regret: Let $g(f_u(x)) = f_u(x) - z_u$ so that $U_\beta(x) = \{u: f_u(x) \leq z_u + \beta(x)\}$, then the problem becomes, then the problem becomes

$$\min \{\beta(x) : U_\beta(x) = U, x \in X_D\}$$

Relative Regret: Let $g(f_u(x)) = (f_u(x) – z_u)/ z_u$, then a relative regret problem becomes the following problem with $U_\beta(x)=\{u: f_u(x) \leq (1 + \beta(x))z_u\}$

$$\min \{\beta(x) : U_\beta(x) = U, x \in X_D\}$$

Probabilistic formulation

The O.1.1 formulation of the objective function as max $P(f_u(x) \leq z)$ corresponds to the following problem:

$\max \{P(U_\beta(x)) : x \in X_D\}$ where $U_\beta(x) = \{u: f_u(x) \leq \beta(x)\}$ with $\beta(x) = z$, $\forall x \in X_D$

The O.1.2 formulation of the objective function as min z st $P(f_u(x) \leq z) \geq \alpha$ corresponds to the following problem

$\min \{\beta(x) : P(U_\beta(x)) \geq \alpha, x \in X_D\}$ where $U_\beta(x) = \{u: f_u(x) \leq \beta(x)\}$ with $\beta(x) = z$, $\forall x \in X_D$

Moments based formulation

For the formulations O.2.1 and O.2.2, the following can be used

$\min \{\beta(x) : U_\beta(x) = U, x \in X_D\}$ where $U_\beta(x) = \{u: g(f_u(x)) \leq \beta(x)$ and
$g(f_u(x)) = \alpha E[f_u(x)] + \beta(var[f_u(x)])^{1/2}$ for O.2.1 model
$g(f_u(x)) = \alpha E[f_u(x)] + \beta var[f_u(x)]$ for O.2.2 model

For O.2.3 formulation we introduce two sets of extended-optimality-regions and get the following transformation

$\min \{[\beta(x), \mu(x)] : U_\beta(x) = U, U_\mu(x) = U, x \in X_D\}$ where
$U_\beta(x) = \{u: E[f_u(x)] \leq \beta(x)\}$
$U_\mu(x) = \{u: var[f_u(x)] \leq \mu(x)\}$

Fuzzy Programming Formulation

For the flexible programming formulation $\min \{f_u(x): \mu_o(u) \geq \alpha, x \in X_D\}$ where $\alpha \in [0,1]$, we can define $U_\beta^\alpha(x) := \{u: f_u(x) \leq \beta(x), \mu(u) \geq \alpha\}$ to get the following equivalent

$\min \{\beta(x): U_\beta^\alpha(x) = U\}$

where $\alpha \in [0,1]$ and U can be considered as the cartesian product of the supports of the fuzzy sets from which components of the parameter vector come.

- *uncertainty about the alternative courses of action*

In this situation corresponding to uncertainty in the feasible set, one has a deterministic decision space X_D, the uncertainty set U and a set of feasible sets $\{X_u : u \in U\}$.

Here, the optimality-region concept becomes irrelevant but a similar concept, feasibility-region which has been initially introduced by Tansel [128] can be used as defined below.

$$Fs(x) = \{u: x \in X_u, u \in U\}$$

In parallel with Demir's solution definitions with optimality-region, we propose the following feasibility definitions. The first type of solutions, *weak feasible solutions* are those, which are feasible for at least one problem instance. So that a solution is so iff $Fe(x) \neq Ø$. In the second type, those solutions are included being feasible for all input data realizations. We call such a solution *permanent feasible solution* implying a solution is so iff Fe(x) = U. Observe that, the hard feasibility concept and the probabilistic constraints with reliability level 1 correspond to this definition. Lastly, *unionwise permanent feasible* solutions are introduced by defining a minimal subset of X_D such that their feasibility region cover the uncertainty set together.

$$X_{uw} := \text{argmin } \{|Z| : Z = \{x \in X_D : \cup Fs(x) = U\}\}$$

If one has only one set satisfying this condition and if infeasibility can not be tolerated, then the implementation of the solutions, if possible, belonging to this set together gives total feasibility. On the other had, if there exist a number of such sets and still infeasibility can not be tolerated, the set of solutions with the minimum total cost should be implemented if possible. Lastly, if feasibility is not a hard requirement, one can select a solution from such a set with some other criterion, for example with consideration of maximum violation.

Permanent feasible solutions may not exit most of the time and even if they exist, they may form a very small set or one may tolerate some amount of infeasibility for benefiting from other attributes of a solution such as objective function performance.

In such situations, some function of feasibility violation can be introduced (like the distance measure or volume criterion of approximate permanent solutions) and limited from above as will be proposed by extended feasibility region concept below.

$$Fs_\beta(x) = \{u: g(F_u(x), K) \leq \beta(x)\} \text{ where } \beta(x) \in R \text{ for each } x \in X_D.$$

Therefore we introduce a new type of solution as those giving a performance of violation within the limits. We call such a solution *approximate permanent feasible solution* implying a solution is so iff

$$h(Fs_\beta(x), U) \leq d$$

Here observe that the induced feasibility regions of probabilistic models can easily be represented with this concept as:

For $P(x \in S_u) \geq \alpha$, $h(Fs_\beta(x), U) = -P(Fs(x))$

For $P(F_u^i(x) \in K^i) \geq \alpha_i$ $i=1,\ldots,m$, $h^i(F^is_\beta(x), U) = -P(Fs^i(x))$ with $Fs^i(x)$ is defined as:

$Fs^i(x) = \{u: F_u^i(x) \in K^i\} i=1,\ldots,m$, where F_u^i is the ith left hand side and K^i is the ith right hand side so that

Another possibility in parallel with this formulation is to consider the volume of the feasibility region so that one can define feasibility as

$$Vol(Fs(x)) / Vol(U) \geq \alpha$$

where α is a prespecified value and if one can tolerate more violation for some not-so-important constraints, the same criterion can be applied componentwise as

$$Vol(Fs^i(x)) / Vol(U) \geq \alpha_i$$

In the same way, one can define the performance function $g(Fs(x), K)$ componentwise, say $g_i(Fs(x), K)$ as the maximum violation of each constraint or the expected violation, where expectation is taken over the violated cases as follows, respectively:

$$g_i(Fs^i(x), K^i) = \max \{\min \{|k^i - F_u^i(x)| : k^i \in K^I\} : u \in U\}$$
$$g_i(Fs^i(x), K^i) = E[\min \{|k^i - F_u^i(x)| : k^i \in K^I\} | u \notin Fs^i(x)]$$

For the possibilistic programming formulation min $\{f(x) : \pi(x) \geq \alpha, x \in X_D\}$, we can define the induced feasibility region as in the case of above probabilistic formulation but using possibility distribution instead of probability distribution. In fact, this approach was introduced by Luhandjula in 1987 [89], where he defined the concept of β-possibly optimal and extend it to the α-possibly feasible and β-possibly efficient concepts and we believe that similar concepts can be used in flexible programming with the name of β-preferability and etc.

- *a combination of these*

If one has feasibility uncertainty and objective uncertainty in the same problem, then any combination of the concepts suggested above can be used to find an ultimate solution. Possible combinations and some existing models belonging to some of these are the followings:

Permanent feasible solution & permanent solution

Permanent feasible solutions & approximate permanent solution

Hard feasibility – max-covering, min-max, regret, relative regret

Permanent feasible solutions & unionwise permanent solution

Approximate permanent feasible solution & permanent solution

Approximate permanent feasible solution & approximate permanent solution

Probabilistic or moments-based objective with probabilistic and/or moments-based feasibility and also robust models with soft feasibility

Approximate permanent feasible solution & unionwise permanent solution

Unionwise permanent feasible solution & permanent solution

Unionwise permanent feasible solution & approximate permanent solution

Unionwise permanent feasible solution & unionwise permanent solution

For the multiobjective case, new concepts of efficiency (or proper and super proper efficiency) can also be introduced in parallel with the above proposed concepts of solution and feasibility. Therefore, a solution to a multiobjective problem will be called *weak efficient* if it is efficient for at least one realization of the uncertain parameter and *permanent efficient* if it is efficient for all realizations from U. A set of solutions will be called *unionwise permanent efficient* if they are together efficient for all realizable problem instances. Again, introducing some performance function, one can define an *approximate permanent efficient* solution having a good performance function like those being efficient with a probability level α or with a volume level α. There is an analogy between permanent efficient and the solutions Bitran seeks in [12].

From all of these it becomes clear that, each of the different types of models is a play between some of the following pairs of sets: U(x)&U, Fe(x)&U and $\{f_u(x)\}\&\{z_u\}$. From this view, there exist some similarities between the existing models. For example considering the interplay between U(x)&U, probabilistic formulation based stochastic programming, max-covering model and optimality region based approach are similar, which try to solve the problem instances collectively. On the other hand, focusing on the play between $\{f_u(x)\}\&\{z_u\}$ but not the others, moments formulation based stochastic programming and robust optimization models (except the max covering) are similar to each other trying to solve a single problem. After this explanation, a natural question comes related to the meaning of these plays. First of all, the play between Fe(x) and U, indicates the power of a solution being feasible under all circumstances. The play between U(x) and U indicates the power of being optimal under all circumstances, related to which we define as done in the sensitivity analysis literature, the stability concept as:

The stability of a solution is the ability to be an optimal solution for all parameter realizations. This is a very powerful attribute of a solution.

Lastly, the play between $\{f_u(x)\}\&\{z_u\}$ indicates the degree of having good objective value for all parameter realizations. This is also a useful attribute of a solution, especially the kinds as min-max, regret and relative regret used in robustness models.

In the literature, for a long time, robustness has been related with the good objective function performance under any realization. On the other hand, originally, the concept was related with one more attribute of a solution, that is stability. For example, Rosenblatt, one of the first applicants of the robustness approach in optimization proposed by Rosenhead and Gupta defined the robustness of a solution as: 'the number of times a solution is within a prespecified percentage of the optimal solution for all realizations of the uncertain situation". We think that, such kind of a robustness concept gives the power that the word actually is worth.

Imprecise optimization solutions have this property very well. There is one more thing to be explained. As previously stated, if we consider the more-likely-to-encounter type of solutions, approximate permanent solutions, it becomes obvious that the effects of the uncovered region on the system that is the magnitude of being suboptimal, is an open issue. In this respect, the incorporation of a performance function to measure the distance to the worst optimum value of the uncovered problem instances may help to handle this weakness. One suggestion can be the minimization of the maximum regret so we introduce a new objective function as:

$$g'(x) = \max \{f_u(x) - z_u^* : u \in U\}$$

Then we have a vector optimization problem as

$$\min\ [g(x), g'(x)]$$
$$\text{st} \quad x \in X_w$$

Observe that, in case this vector optimization problem is solved by aggregating the two objectives with weights w_1 and $(1-w_1)$ to see the trade-off between coverage (stability) and robustness, $w_1 = 1$ corresponds to the approximate permanent solution and $w_1 = 0$ corresponds to the usual regret problem with the difference of giving always a solution which will be optimal for some instances.

Because of the difficulties that may arise with multiple objectives, one may proceed as looking at the maximum regret of the approximate permanent solution and then according to her/his satisfaction, gives the final decision.

To summarize, to define a solution as the best, three criteria should be considered as its stability, its cost of being suboptimal and its feasibility. While measuring stability, one can consider the problem instances for which the decision is optimal or near optimal. To measure the cost of being suboptimal, one can focus on the worst case performance or worst case deviation from optimality over all problem instances and to measure feasibility, the problem instances for which the decision is feasible should be considered. According to the type of uncertainty, probability distributions or characteristic functions can be used in the measurements and depending on the type of decision situation, one can give more emphasis to some of these three criteria.

CHAPTER-5

RELATIONS and COMPARISONS

5.1- Relations

Imprecise optimization and max-covering are based on the same idea of covering the uncertainty set as much as possible with the difference the latter uses discrete sets and the concept of near optimality and uses the cardinality as the performance function whereas imprecise optimization does not restrict the uncertainty set and considers volume of the coverage or l_∞ norm of the distance to the uncertainty set as the performance function. In the same manner, the probabilistic formulations behave with the difference of weighting each uncertain parameter by its probability. These three methods have the same attitude toward solving an uncertain problem collectively.

Imprecise optimization seeks to minimize the distance between region of optimality and U whereas the robust optimization models minimize the distance between the objective value and the optimal value.

Robust optimization problems are similar to those moments based stochastic programming problems, since they reduce a lot of information from the uncertainty set to a single one, just to the regret or just to the expectation. At this point, there is one more similarity between those, since expectation is also a kind of regret, total regret.

The soft robust model is identical to a two-stage stochastic programming problem with recourse but considering the whole set U, not a subset of it as sometimes done in the stochastic programming case. This property introduces additional power to deal with uncertainty. If there is no feasible corrective action, two-stage recourse models declare infeasibility but soft robust models always give a solution.

The two objective formulation, O.2.3. of moments-based stochastic programming formulation has the same idea with that introduced by multiobjective optimization approach.

Lastly, it can be said that what parametric programming does is to determine the set of weak solutions introduced by imprecise optimization.

5.2-Strengths&Weaknesses

Here the mentioned approaches will be evaluated and we only state those if one approach or a specific model of it is very powerful or weak for that criterion.

5.2.1 Modeling Artifacts

- *Input data:*

Stochastic programming has severe modelling problems with continuous distributions especially when correlations exist for the random variables. Fuzzy programming is very questionable in terms of the meaning of possibility distributions.

- *Modeling Assumptions:*

In the stochastic programming models, the complete recourse assumption, as usually made, may not always be satisfied. This can be handled but the complexity of the model increases.

- *Validation of the Model&Value of Information:*

One of the most powerful tolls of validating a model is to perform sensitivity analysis. We believe that, as is mentioned in [54], to perform such an analysis, at least in the usual way is not likely for fuzzy programming although there are some such studies in fuzzy programming literature. In stochastic programming, the value of information is a well studied subject [11] whereas it is not so for fuzzy programming although there are some attempts like [126] and [96]. This is a kind of weakness, since such a concept may be very helpful in narrowing down the uncertainty.

- *Suggestions to Control Uncertainty:*

Imprecise optimization is very and perhaps the most powerful to suggest how to decrease uncertainty in case one can control uncertainty. Like this case, tolerance approach, as stated previously, has such kind of an advantage.

5.2.2 Solution Procedures

- *Computability:*

With the huge amount of literature on the solution algorithms, stochastic programming problems are very hard to solve. The use of approximations is suggested but one can not understand what type of a problem she/he is solving. Fuzzy programming has a power in this respect, since it can handle most situations in an efficient way, at least for some characteristic function types.

5.2.3 Applicability to Decision Environments

- *Nature of decision:*

Especially for future external events of nonrepetitive variety, where the assignment of probabilities is not possible, like strategic decisions robust optimization problems will perform well, considering the worst case. On the other hand they are too conservative for environments of repetitive nature, like tactical problems, particularly those concerned with operating systems, in which case, stochastic programming or imprecise optimization may perform well. If the decision environment has actually vagueness in it, when only verbal construction of values, constraints etc. are possible or when human being related models are being handled, then there is not any other modelling way of handling this situation but fuzzy programming.

- *Handling risk:*

Stochastic programming does not handle risk-aversion in a direct fashion, especially those including expectation types formulations not suitable for risk averse case since they imply risk neutral behaviour. On the other hand, robust optimization models are powerful in handling risk aversion. Also, we believe that the incorporations of any type of risk measures will be easily handled in the definition of extended-optimality-region in imprecise optimization.

- *Time of Evaluation of Decision:*

This is a serious weakness for stochastic programming problems with moments-type based approaches since the time of evaluation is immediate, then waiting for the values to converge the expectation will be useless.

- *Structure of the Original System:*

Especially, for the systems requiring hard feasibility, one can not apply any violation-allowed approach. In the other case around, there may not be serious problems but the problem of cost. In this respect the appropriate approach should be chosen for the environment.

- *The level of uncertainty:*

In case of significant data uncertainty stochastic programming and fuzzy optimization are not suitable. In fact, they are only suitable for the cases of randomness and vagueness, which restrict their applicibality.

CHAPTER-6

CONCLUSION AND FUTURE PERSPECTIVE

In this study, we look at the different ways of modelling under uncertainty of input data. We first analyzed the ways of representing such data and emphasized its effect on the model performance. In fact, there is a strong need to study uncertainty with different types of representation since in real world, most of the time, one can not have situations where all uncertain parameters are random variables or come from intervals. There are only a few studies with this subject focusing only on randomness and fuzziness. We believe that, imprecise optimization can handle such combinations easily since it does not focus on any type of uncertainty representation.

Then, in chapter 3, we studied sensitivity analysis and parametric programming, where a promising area is qualitative sensitivity analysis as briefly mentioned in that part. Such a pre-optimal study may be very helpful in other approaches to uncertain optimization. For example, detecting the uncertain elements having the least effect on the solution, one may have the chance to reduce the uncertainty set, or detecting the elements having the weakest relation with a specific uncertain element may be helpful in decreasing the number of correlated elements especially for the models where the representation of correlations bring significant computational burden. Because of such uses, we believe that qualitative sensitivity analysis should be further studied, extended to other kinds of problems and incorporated into the models mentioned in this study.

In chapter 5, the analysis of different models was given and two general types of approaches have been recognized, those handling the set of problem instances collectively and the other inducing that set to a single problem. The type of models belonging to the same approach have also been compared with each other with respect to the applicability of decision situations, computability, etc. Then, the concept of stability and robustness were introduced, which we believe are very important. We believe that a promising type of modelling approach should focus on stability, robustness and, sometimes, worst case performance. All of the suggested concepts especially those related with feasibility-region and formulations made as an extension of imprecise optimization should be studied from computational tractability perspective and further applied to real life problems.

In the previous chapter, we stated the strongest and the weakest points of different approaches or models concluding that none of them is totally promising for all types of criteria. Such a study can be extended focusing on just one problem type, solving each type of models presented here and then analysing each solution from these perspectives.

As a last point, we believe that the study of the uncertainty set may be very useful or with the sensitivity of a model to the uncertainty set may make the decision maker feel more comfortable (or less according to the results).

Bibliography

[1] Averbakh I., 2000, "Minmax Regret Solutions for Minimax Optimization Problems with Uncertainty", Operations Research Letters, vol 27, 57-65

[2] Bank, B., Guddat, J., Klatte, D., Kummer, B. and Tammer, K., 1982, Nonlinear Parametric Optimization, Akademic Verlag, Berlin

[3] Beale, E.M.L., 1955, "Linear Programming under Uncertainty", Management Science, vol 1, 197-206

[4] Bellman, R.E. and Zadeh, L.A., 1973, "Decision Making in a Fuzzy Environment", Management Science, vol 17, 149-156

[5] Ben-Tal, A. and Ben-Israel, A., 1991, "A Recourse Certainty Equivalent for Decisions under Uncertainty", Annals of Operations Research, vol 30, 3-44

[6] Ben-Tal, A. and Nemirovski, A., 1997, "Robust Truss Topology Design via Semidefinite Programming", SIAM Journal on Optimization, vol 7, no 4, 991-1016

[7] Ben-Tal, A. and Nemirovski, A., 1998, "Robust Convex Optimization", Mathematics of Operations Research, vol 23, issue 4, 769-805

[8] Ben-Tal A. and Nemirovski A., 1999, "Robust Solutions of Uncertain Linear Programs", Operations Research Letters, vol 25, 1-13

[9] Ben-Tal A. and Nemirovski A., 2000, "Robust Solutions of Linear Programming Problems Contaminated with Uncertain Data", Mathematical Programming, Series A, vol 88, 411-424

[10] Ben-Tal, A. and Teboulle, M., 1986, "Expected Utility, Penalty Functions, and Duality in Stochastic Nonlinear Programming", Management Science, vol 32, 1445-1466

[11] Birge, J.R., 1995, "Models and Model Value in Stochastic Programming", Annals of Operations Research, vol 59, 1-18

[12] Bitran, G.R., 1980, "Linear Multiple Objective Problems with Interval Coefficients", Management Science, vol 26, no 7, 694-706

[13] Bloom, J.A., 1983, "Solving an Electricity Generating Capacity Expansion Planning Problem by Generating Benders' Decomposition", Operations Research, vol 31, 84-100

[14] Bradly, S.P., Hax, A.C. and Magnanti, T.L., 1977, Applied Mathematical Programming, Addison-Wesley

[15] Buckley, J.J., 1988, "Generalized and Extended Fuzzy Sets with Applications", Fuzzy Sets and Systems, vol 25, 159-174

[16] Buckley, J.J., 1988, "Possibilistic Linear Programming with Triangular Fuzzy Numbers", Fuzzy Sets and Systems, vol 26, 135-138

[17] Buckley, J.J., 1989, "Solving Possibilistic Linear Programming Problems", Fuzzy Sets and Systems, vol 31, 329-341

[18] Buckley, J.J., 1990, "Multiobjective Possibilistic Linear Programming", Fuzzy Sets and Systems, vol 35, 23-28

[19] Cai, X., Teo, K., Yang, X. and Zhou Y.X., 2000, "Portfolio Optimization under a Minimax Rule", Management Science, vol 46, no 7, 957-972

[20] Carlsson, C. and Korhonen, P., 1986, "A Parametric Approach to Fuzzy Linear Programming", Fuzzy Sets and Systems, vol 20, 17-30

[21] Chanas, S., 1983, "The Use of Parametric Programming in Fuzzy Linear Programming", Fuzzy Sets and Systems, vol 11, 243-251

[22] Chanas, S. and Kolodziejczyk, W., 1986, "Integer Flows in Network with Fuzzy Capacity Constraints", Networks, vol 16, 17-31

[23] Chanas, S. and Kuchta, D., 1996, "Multiobjective Programming in Optimization of Interval Objective Functions - A Generalized Approach", European Journal of Operational Research, vol 94, issue 3, 594-598

[24] Chanas, S. and Kuchta, D., 1998, "Discrete Fuzzy Optimization", In: Fuzzy Sets in Decision Analysis, Operations Research and Statistics, Kluwer, London, 249-280

[25] Charnes, A. and Cooper, W.W., 1963, "Deterministic Equivalents for Optimizing and Satisfying under Chance Constraint", Operations Research, vol 11, 18-39

[26] Charnes, A., Cooper, W.W. and Symonds, G.H., 1958, "Cost Horizons and Certainty Equivalents: An Approach to Stochastic Programming of Heating Oil", Management Science, vol 4, 235-263

[27] Dantzig B.G., 1955, "Linear Programming under Uncertainty", Management Science, vol 1, no 384, 197-206

[28] Dawande, M.W. and Hooker, J.N., 2000, "Inference-based Sensitivity Analysis for Mixed Integer/Linear Programming", Operations Research, vol 48, no 4, 623-634

[29] Delgado, M., Verdegay, J.L. and Villa, M.A., 1989, "A General Model for Fuzzy Linear Programming", Fuzzy Sets and Systems, vol 29, 21-29

[30] Delgado, M., Verdegay, J.L. and Vila, M.A., 1990, "Relating Different Approaches to Solve Linear Programming Problems with Imprecise Costs", Fuzzy Sets and Systems, vol 37, 33-42

[31] Dembo S.R., 1991, "Scenario Optimization", Annals of Operations Research, vol 30, 63-80

[32] Demir, M.H., 2001, "Concepts and Analysis in Facility Location under Uncertainty: Applications to 1-Median Problem", Ph.D. Dissertation, Dept. of. I.E., Bilkent University

[33] Demir, M.H., Tansel, B.Ç. and Scheuenstuhl, G.F., 2000, "Tree Network 1-Median Location with Interval Data: A Parameter Space Based Approach", to appear in IIE Transactions

[34] Dentcheva, D., Prékopa, A. and Ruszczyński, A., 2000, "Concavity and Efficient Points of Discrete Distributions in Probabilistic Programming", Mathematical Programming, Series A 89, 55-77

[35] Dishkant, H., 1981, "About Membership Functions Estimation", Fuzzy Sets and Systems, vol 5, 141-147

[36] Dombi, J., 1990, "Membership Function as an Evaluation", Fuzzy Sets and Systems, vol 35, 1-21

[37] Dubois, D. and Prade, H., 1988, Possibility Theory, Plenum, New York

[38] Dumitru, V. and Luban, F., 1986, "On Some Optimization Problems under Uncertainty", Fuzzy Sets and Systems, vol 18, 257-272

[39] Dupačová, J., 1990, "Stability and Sensitivity Analysis for Stochastic Programming", Annals of Operations Research, vol 27, 115-142

[40] Dupačová, J. and Wets, R.J.-B., 1988, "Asymptotic Behavior of Statistical Estimators and of Optimal Solutions of Stochastic Optimization Problems", Annals of Statistics, vol 16, 1517-1549

[41] Ermoliev, Y. and Wets, R.J.-B. (eds.), 1988, Numerical Techniques for Stochastic Optimization, Springer-Verlag, New York

[42] Evers, W.H., 1967, "New Model for Stochastic Linear Programming", Management Science, vol 13, 680-693

[43] Falk, J.E., 1976, "Exact Solutions to Inexact Linear Programs", Operations Research, 783-787

[44] Fedrizzi, M., 1987, "Introduction to Fuzzy Sets and Possibility Theory", In: Optimization Models Using Fuzzy Sets and Possibility Theory, Kacprzyk, J. and Orlovski, S.-A. (eds.), 1987, Kluwer, London, 13-26

[45] Fiacco, A.V. and Ishizuka, Y., 1990, "Sensitivity and Stability Analysis for Nonlinear Programming", Annals of Operations Research, vol 27, 215-236

[46] Fiacco, A.V. and Ishizuka, Y., 1990, "Suggested Research Topics in Sensitivity and Stability Analysis for Semi-Infinite Programming Problems", Annals of Operations Research, vol 27, 65-76

[47] Fiacco, A.V., 1983, Introduction to Sensitivity and Stability Analysis in Nonlinear Programming, Academic Press, New York

[48] Frauendorfer, K., 1992, Stochastic Two-Stage Programming, Lecture Notes in Economics and Mathematical Systems, vol 392, Springer-Verlag

[49] Freund, R.M., 1985, "Postoptimal Analysis of a Linear Program under Simultaneous Changes in Matrix Coefficients", Mathematical Programming Study, vol 24, 1-13

[50] Gal, T., 1977, "A General Method for Determining the Set of Optimal Efficient Solutions to a Linear Vectormaximum Problem", European Journal of Operational Research, vol 1, 307-322

[51] Gal, T., 1984, "Linear Programming-A Brief Survey", Mathematical Programming Study, vol 21, 43-68

[52] Gal, T., 1986, "Shadow Prices and Sensitivity Analysis in Linear Programming under Degeneracy: A State of the Art Survey", Operations Research Spektrum, vol 8, 59-71

[53] Gal, T., 1995, Postoptimal Analyses, Parametric Programming, and Related Topics, W. de Gruyter, Berlin, New York

[54] Gal, T. and Greenberg, H.J., 1997, Advances in Sensitivity Analysis and Parametric Programming, Kluwer Academic Publishers, London

[55] Gal, T. and Nedoma, J., 1972, "Multiparametric Linear Programming", Management Science, vol 18, 406-422

[56] Gass, S.I. and Saaty, T.L., 1955, "Parametric Objective Function (Part 2)-Generalization", Operations Research, vol 3, no 4, 395-401

[57] Giles, R., 1988, "The Concept of Grade of Membership", Fuzzy Sets and Systems, vol 25, 297-323

[58] Gollmer, R., 1985, "On Linear Multiparametric Optimization with Parameter-dependent Constraint Matrix", Optimization, vol 16, 15-28

[59] Greenberg, H. J., 1986, "An Analysis of Degeneracy", Naval Research Logistics Quarterly, vol 33, 635-655

[60] Gupta, S.K. and Rosenhead, J., 1968, "Robustness in Sequential Investment Decisions", Management Science, vol 15, no 2, B-18-B-29

[61] Gutierrez, G. J. and Kouvelis, P., 1995, "A Robustness Approach to International Sourcing", Annals of Operations Research, vol 59, 165-193

[62] Heller, I., 1954, "Sensitivity Analysis in Linear Programming", L.R.P. Seminar, Logistics Research Project (The George Washington University)

[63] Hisdal, E., 1988, "Are Grades of Membership Probabilities?", Fuzzy Sets and Systems, vol 25, 325-348

[64] Hoffmann, H.J. and Jacobs, W., 1954, "Smooth Patterns of Production", Management Science, vol 1, 86-91

[65] Høyland, K. and Wallace, S.W., 2001, "Generating Scenario Trees for Multistage Decision Problems", Management Science, vol 47, no 2, 295-307

[66] Indraneel, D., 2000, "Robustness Optimization for Constrained Nonlinear Programming Problems", Engineering Optimization, vol 32, issue 5, 585-618

[67] Infanger, G., 1992, "Monte Carlo (Importance) Sampling within a Benders Decomposition Algorithm for Stochastic Linear Programs", Annals of Operations Research, vol 39, 69-95

[68] Intrator, J. and Engelberg, A., 1980, "Sensitivity Analysis as a Means of Reducing the Dimensionality of a Certain Class of Transportation Problems", Naval Research Logistics Quarterly, 297-313

[69] Inuiguchi, M. and Sakawa, M., 1995, "Minimax Regret Solution to Linear Programming Problems with an Interval Objective Function", European Journal of Operational Research, vol 86, issue 3, 526-536

[70] Ishibuchi, H. and Tanaka, H., 1990, "Multiobjective Programming in Optimization of the Interval Objective Function", European Journal of Operational Research, vol 48, 219-225

[71] Jenkins, L., 1990, "Parametric Methods in Integer Linear Programming", Annals of Operations Research, vol 27, 77-96

[72] Jongen, H.T. and Weber, G.-W., 1990, "On Parametric Nonlinear Programming", Annals of Operations Research, vol 27, 253-283

[73] Kabe, D.G., 1983, "Laws of Large Numbers for Random Payoff Games", The Journal of the Industrial Mathematics Society, vol 33, part 1, 73-86

[74] Kacprzyk, J. and Orlovski, S.-A. (eds.), 1987, Optimization Models Using Fuzzy Sets and Possibility Theory, Kluwer Academic Publishers, London

[75] Kall, P. and Wallace, S. W., 1994, Stochastic Programming, Wiley

[76] Kataoka, S., 1963, "A Stochastic Programming Model", Econometrica, vol 31, 181-196

[77] Kaufmann, A., 1986, "On the Relevance of Fuzzy Sets for Operations Research", European Journal of Operational Research, vol 25, 330-335

[78] Kaul, R.N., Bhatia, D. and Gupta, P., 1999, "Tolerance Approach to Sensitivity Analysis in Quadratic Programming Problems", Operational Research Society of India, vol 36, no 1, 1-9

[79] Kerre, E., 1988, "A Call for Crispness in Fuzzy Set Theory", Fuzzy Sets and Systems, vol 29, 57-65

[80] King, A. and Rockafellar, R.T., 1993, "Asymptotic Theory for Solutions in Statistical Estimation and Stochastic Programming", Mathematics of Operations Research, vol 18, 148-162

[81] Klein Haneveld, W. K., 1986, "Duality in Stochastic Linear and Dynamic Programming", Lecture Notes in Economics and Mathematical Systems, 274, Springer-Verlag, New York

[82] Komáromi, É., 1986, "A Dual Method of Probabilistic Constrained Linear Programming Problem and Its Dual", Journal of Optimization Theory and Applications, vol 55, 337-390

[83] Kouvelis, P. and Yu, G., 1997, Robust Discrete Optimization and Its Applications, Kluwer Academic Publishers, Boston

[84] Kyparisis, J., 1990, "Sensitivity Analysis for Nonlinear Programs and Variational Inequalities with Nonunique Multipliers", Mathematics of Operations Research, vol 15, no 2, 286-298

[85] Lai, Y.J and Hwang, C.L., 1992, "A New Approach to Some Possibilistic Linear Programming Problem", Fuzzy Sets and Systems, vol 49

[86] Lai, Y.J. and Hwang, C.L., 1992, "Interactive Fuzzy Linear Programming", Fuzzy Sets and Systems, vol 45, 169-183

[87] Lai, Y.J. and Hwang, C.L., 1992, Fuzzy Mathematical Programming, Methods and Applications, Lecture Notes in Economics and Mathematical Systems, vol 394, Springer-Verlag

[88] Luce, D.R. and Raiffa, H., 1954, Games and Decisions: Introduction and Critical Survey, Wiley, New York

[89] Luhandjula, M.K., 1987, "Linear Programming with a Possibilistic Objective Function", European Journal of Operational Research. vol 31, 110-117

[90] Mausser, H.E. and Laguna M., 1999, "Minimising the Maximum Relative Regret for Linear Programmes with Interval Objective Function Coefficients", Journal of the Operational Research Society, vol 50, 1063-1070

[91] Mayer, J., 1979, "A Nonlinear Programming Method for the Solution of a Stochastic Programming Model of A. Prekopa", In: Survey of Mathematical Programming, A. Prekopa (eds.), North-Holland, vol 2, 129-139

[92] Miller, L.B. and Wagner, H., 1965, "Chance-Constrained Programming with Joint Constraints", Operations Research, vol 13, 930-945

[93] Mulvey, J.M. and Ruszczynski, A., 1995, "A New Scenario Decomposition Method for Large-scale Stochastic Optimization", Operations Research, vol 43, no 3, 477-490

[94] Mulvey, J.M., Vanderbei, J.R. and Zenios, A.S., 1994, " Robust Optimization of Large Scale Systems", Operations Research, vol 43, no 2, 264-281

[95] Negi, D.S., 1989, "Fuzzy Analysis and Optimization", Ph.D. Dissertation, Dept. of. I.E., Kansas State University

[96] Pedrycz, W., 1989, "Fuzzy Modelling: Fundamentals, Construction and Evalution", in: Dubois, D. and Prade, H., 1988, Possibility Theory, Plenum, New York

[97] Prékopa, A., 1970, "On Probabilistic Constrained Programming", Mathematical Programming Study, vol 28, 113-138

[98] Prékopa, A., 1972, "Laws of Large Numbers for Random Linear Programs", Mathematical Systems Theory, vol 6, 277-288

[99] Prékopa, A., 1973, "Contributions to the Theory of Stochastic Programming", Mathematical Programming, vol 4, 202-221

[100] Prékopa, A., 1990, "Dual Method for a One-Stage Stochastic Programming Problem with Random RHS Obeying a Discrete Probability Distribution", ZOR-Methods and Models of Operations Research, vol 34, 441-461

[101] Prékopa, A., 1995, Stochastic Programming, Kluwer Academic Publishers,

[102] Prékopa, A. and Szántai, T., 1978, "Flood Control Reservoir System Design Using Stochastic Programming", Mathematical Programming Study, vol 9, 138-151

[103] Ramík, J. and Římánek, J., 1985, "Inequality Relation Between Fuzzy Numbers and Its Use in Fuzzy Optimization", Fuzzy Sets and Systems, vol 16, 123-138

[104] Ringuest, J.L. and Graves, S.B., 2000, "A Sampling-based Method for Generating Nondominated Solutions in Stochastic MOMP Problems", European Journal of Operational Research, vol 126, 651-661

[105] Robinson M.S., 1991, "Extended Scenario Analysis", Annals of Operations Research, vol 31, 385-398

[106] Rockafellar, R.T. and Wets, J.B., 1991, "Scenarios and Policy Aggregation in Optimization under Uncertainty", Mathematics of Operations Research, vol 16, no 1, 119-147

[107] Rommelfanger, H., 1989, "Interactive Decision Making in Fuzzy Linear Optimization Problems", European Journal of Operational Research, vol 41, 210-217

[108] Rommelfanger, H., 1996, "Fuzzy Linear Programming and Applications", European Journal of Operational Research, vol 92, 512-527

[109] Rommelfanger, H., Hanuscheck, R. and Wolf, J., 1989, "Linear Programming with Fuzzy Objectives", Fuzzy Sets and Systems, vol 29, 31-48

[110] Rosenblatt, J.M. and Lee, L.H., 1987, "A Robustness Approach to Facilities Design", International Journal of Production Research, vol 25, no 4, 479-486

[111] Rosenhead, J., Elton, M. and Gupta, K.S., 1972, "Robustness and Optimality as Criteria for Strategic Decisions", Operational Research Quarterly, vol 23, 413-430

[112] Saaty, T.L. and Gass, S.I., 1954, "The Parametric Objective Function", Operations Research, vol 2, 316-319

[113] Sakawa, M., 1998, "Fuzzy Nonlinear Programming with Single or Multiple Objective Functions", In: Fuzzy Sets in Decision Analysis, Operations Research and Statistics, Kluwer, London, 215-248

[114] Sakawa, M. and Yano, H., 1986, "Interactive Fuzzy Decision Making for Multiobjective Nonlinear Programming Using Augmented Minimax Problems", Fuzzy Sets and Systems, vol 20, 31-43

[115] Sakawa, M. and Yano, H., 1989, "Interactive Decision Making for Multiobjective Nonlinear Programming Problems with Fuzzy Parameters", Fuzzy Sets and Systems, vol 29, 315-326

[116] Sengupta, K.J., 1991, "Robust Solutions in Stochastic Linear Programming", Journal of the Operational Research Society, vol 42, no 10, 857-870

[117] Shapiro, A., 1993, "Asymptotic Behavior of Optimal Solutions in Stochastic Programming", Mathematics of Operations Research, vol 18, 829-845

[118] Slowinski, R., 1998, Fuzzy Sets in Decision Analysis, Operations Research and Statistics, Kluwer Academic Publishers, London

[119] Slyke, R. van and Wets, R.J.-B., 1969, "L-Shaped Linear Program with Application to Optimal Control and Stochastic Linear Programs", SIAM Journal on Applied Mathematics, vol 17, 638-663

[120] Soyster, A.L., 1973, "Convex Programming with Set-Inclusive Constraints and Applications to Inexact Linear Programming", Operations Research, vol 21, 1154-1157

[121] Soyster, A.L., 1974, "A Duality Theory for Convex Programming with Set-Inclusive Constraints", Operations Research, vol 22, 892-898

[122] Stancu-Minasian, I.M., 1984, Stochastic Programming with Multiple Objective Functions, D. Reidel, Boston

[123] Steuer R.E., 1981, "Algorithms for Linear Programming Problems with Interval Objective Function Coefficients", Mathematics of Operations Research, vol 6, no 3, 333-348

[124] Strazicky, B., 1974, "On an Algorithm for Solution of the Two-Stage Stochastic Programming Problem", Methods of Operations Research, vol 19, 300-302

[125] Tanaka, H., Ichihashi, H. and Asai, K., 1984, "A Formulation of Fuzzy Linear Programming Problems Based on Comparison of Fuzzy Numbers", Fuzzy Sets and Systems, vol 13, 1-10

[126] Tanaka, H., Ichihashi, H. and Asai, K., 1986, "A Value of Information in FLP Problems via Sensitivity Analysis", Fuzzy Sets and Systems, vol 18, 119-129

[127] Tanino, T., 1990, "Stability Analysis in Multiobjective Nonlinear Programming", Annals of Operations Research, vol 27, 97-114

[128] Tansel, B.Ç., 1992, Unpublished Research Notes, Department of Industrial Engineering, Bilkent University, Ankara, Turkey

[129] Tansel, B.Ç., 2000, in personal communication, Department of Industrial Engineering, Bilkent University, Ankara, Turkey

[130] Tansel, B.Ç. and Demir, M.H., 1996, "modelling and Analysis of Optimality Concepts in Multimedian Location with Interval Data", Research Report: IEOR-9614, Department of Industrial Engineering, Bilkent University, Ankara, Turkey

[131] Tansel, B.Ç. and Scheuenstuhl, G.H., 1988, "Facility Location on Tree Networks with Imprecise Data", Research Report: IEOR-8819, Department of Industrial Engineering, Bilkent University, Ankara, Turkey

[132] Teghem Jr, J., Dufrasne, D., Thauvoye, M. and Kunsch, P., 1986, "STRANGE-An Interactive Method for Multiobjective Linear Programming under Uncertainty", European Journal of Operational Research, vol 26, 65-82

[133] Thuan, L.V. and Luc, D.T., 2000, "On Sensitivity in Linear Multiobjective Programming", Journal of Optimization Theory and Applications, vol 107, no 3, 615-626

[134] Urli, B. and Nadeau, R., 1990, "Stochastic MOLP with Incomplete Information: An Interactive Approach with Recourse", Operational Research Society, vol 41, no 12, 1143-1152

[135] Verdegay, J.L., 1984, "A Dual Approach to Solve the Fuzzy Linear Programming Problem", Fuzzy Sets and Systems, vol 14, 131-141

[136] Wallace, S.W., 2000, "Decision Making under Uncertainty: Is Sensitivity Analysis of any Use?", Operations Research, vol 48, no 1, 20-25

[137] Ward, J.E. and Wendell, R.E., 1990, "Approaches to Sensitivity Analysis in Linear Programming", Annals of Operations Research, vol 27, 3-38

[138] Wendell, R.E., 1982, "A Preview of a Tolerance Approach to Sensitivity Analysis in Linear Programming", Discrete Mathematics, vol 38, 121-124

[139] Wendell, R. E., 1984, "Using Bounds on the Data in Linear Programming: The Tolerance Approach to Sensitivity Analysis", Mathematical Programming, vol 29, 304-322

[140] Wendell, R. E., 1985, "The Tolerance Approach to Sensitivity Analysis in Linear Programming", Management Science, vol 31, no 5, 564-578

[141] Werners, B., 1987, "Interactive Multiple Objective Programming Subject to Flexible Constraints", European Journal of Operational Research, vol 31, 342-349

[142] Wets, R.J.-B., 1983, "Solving Stochastic Programs with Simple Recourse", Stochastics, vol 10, 219-242

[143] Whalen, T., 1987, “Introduction to Decision Making under Various Kinds of Uncertainty”, In: Optimization Models Using Fuzzy Sets and Possibility Theory, Kluwer Academic Publishers, London, 27-49

[144] Yager, R.R., Ovchinnikov, S., Tong, R.M. and Nguyen, H.T. (eds), 1987, Fuzzy Sets and Applications, Wiley

[145] Yıldırım, E.A. and Todd, M.J., 2001, “Sensitivity Analysis in Linear Programming and Semidefinite Programming Using Interior-point Methods”, Mathematical Programming, Series A 90, 229-261

[146] Ying-jun, F., 1983, “A Method Using Fuzzy Mathematics to Solve the Vectormaximum Problem”, Fuzzy Sets and Systems, vol 9, 137-151

[147] Yu, C. and Li, H., 2000, “A Robust Optimization Model for Stochastic Logistic Problems”, International Journal of Production Economics, vol 64, 385-397

[148] Yu, G., 1998, “Min-Max Optimization of Several Classical Discrete Optimization Problems”, Journal of Optimization Theory and Applications, vol 98, no 1, 221-242

[149] Yu, P.L. and Zeleny, M., 1976, “Linear Multiparametric Programming by Multicriteria Simplex Method”, Management Science, vol 23, 159-170

[150] Zadeh, L.A., 1965, “Fuzzy Sets”, Information and Control, vol 8, 338-353

[151] Zimmermann, H.-J., 1976, “Description and Optimization of Fuzzy System”, International Journal of General System, vol 2, 209-216

[152] Zimmermann, H.-J., 1978, “Fuzzy Programming and Linear Programming with Several Objective Functions”, Fuzzy Sets and Systems, vol 1, 45-55