STABILITY AND DWELL TIME ANALYSIS OF SWITCHED TIME DELAY SYSTEMS

A THESIS

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MASTER OF SCIENCE

By

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September 2007

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

STABILITY AND DWELL TIME ANALYSIS OF SWITCHED TIME DELAY SYSTEMS

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September 2007

In this thesis we deal with stability analysis of switched feedback system with time delays. We assume that, at any given time for each "candidate" system a controller is designed and a fixed feedback system is obtained until the next switching instant. We investigate the conservativeness of an LMI-based stability test for the time delay systems. This test is used for the dwell time analysis. After obtaining the limitations of this test, we find the exact bounds of allowable parameters appearing in the LMI-based test, in order to optimize the dwell time. For this purpose we consider simple first order systems and higher order systems separately. We also consider the LQR-based switched feedback controllers with time delays and investigate the effects of weighting matrices Q and R on the dwell time.

Keywords: Switched Time-Delay Systems, Dwell Time, Stability Analysis, Conservativeness Analysis

ÖZET

ANAHTARLAMALI ZAMAN GECİKMELİ SİSTEMLERİN KARARLILIK VE DURMA ZAMANI ANALİZİ

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Bu tez kapsamında zaman gecikmeli anahtarlamalı geribeslemeli kontrol sistemlerinin kararlılık analizinden bahsedilmiştir. Bilinen herhangi bir zamanda, her “aday” sistem için bir kontrol birimi tasarlandığı farz edilmiş ve bir sonraki anahtarlama anına kadar değişmez bir geribeslemeli sistem elde edilmiştir. Zaman gecikmeli sistemlerin kararlılığını test eden LMI tabanlı bir testin korunumluluğu incelenmiştir. Bu test, durma zamanı analizi için kullanılmaktadır. Bu testin sınırlamaları elde edildikten sonra, durma zamanını eniyileştirmek için testte geçen serbest bırakılabilir parametrelerin kesin sınırları bulunmuştur. Bu amaçla basit tek dereceli sistemler ve daha yüksek dereceli sistemler ayrı ayrı dikkate alınmıştır. Aynı zamanda, zaman gecikmeli LQR tabanlı anahtarlamalı geribeslemeli sistemler dikkate alınarak ağırlıklandırma matrisleri Q ve R’nin durma zamanı üzerindeki etkileri incelenmiştir.

Anahtar Kelimeler: Zaman Gecikmeli Anahtarlamalı Sistemler, Durma Zamanı, Kararlılık Analizi, Korunumluluk Analizi

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Dedicated to my father

Chapter 1

INTRODUCTION

This thesis deals with the stability analysis of switched time delay systems. In [21] by using an LMI-based stability approach for each candidate system a dwell time is obtained for stability of the switched system. That is, switched system is stable under arbitrary switching between these stable candidate systems provided that the smallest time interval between these switching instants is greater than a certain dwell time computed in [21]. We also investigate for simple systems how conservative the LMI-based test of [2], and how we can minimize the dwell time.

1.1 Literature Review

The analysis of time-delay systems have attracted attention especially in the last decade [2], [6], [9], [13], [16]. Delays appear in many engineering applications such as information network systems, process control, guidance and navigation. In the literature on time delay systems, stability is analyzed in two ways as delay-independent and delay dependent stability. Most popular approaches for the stability analysis of the delay systems are Lyapunov-like methods based either on the Razumikhin or the Krasovskii technique [2], [6], [9], [13], [16]. A numerical analysis also provided using the bifurcation theory and a toolbox for MATLAB "DDE-BIFTOOL" is presented in [3], [4], [5], [17]. Other numerical techniques also available [8], [14], [19], [20].

Switched systems are hybrid systems consisting of a family of continuous-time "candidate" systems and discrete-time logic, i.e. switching signal. Switching systems are used to improve the transient response and to achieve the stability when it is hard to achieve with a single system. Switching control has a various applications areas, e.g. mechanical systems, automotive industry and air traffic control. The stability of the candidate systems, does not always means the stability of the switched system [10]. In [7] it is shown that switching among stable systems results in a stable switched system, provided that the switching is slow on the average. Average dwell time introduced for degree of slowness for the switching process. In [11], it is stated that existence of common Lyapunov functions for each candidate system ensures the arbitrary switching between the candidate systems and a gradient algorithm is supplied to find common Lyapunov functions. Because it is usually hard to find common Lyapunov functions for each candidate systems, piecewise continuous Lyapunov functions are introduced in [15] and [22].

In this thesis, switched time delay systems are investigated regarding the improvement of the transient response of the modeled system. In [18] switched time delay system is investigated using an extension of common Lyapunov approach, whereas in [21] piecewise Lyapunov Razumikhin functionals are used along with the notion of minimum dwell time.

1.2 Problem Statement

In this thesis we deal with stability analysis of switched feedback systems with time delays. We assume that at any given time for each "candidate" system a controller is designed and a fixed feedback system is obtained until the next switching instant. On each fixed time intervals between switching times, system is assumed to be in the fixed form:

$$\dot{x} = Ax(t) + \bar{A}x(t-\tau) + Bu(t) \tag{1.1}$$

where $\tau > 0$, $u(t)$ is the input, $x(t)$ is the state variable, A, $\bar{A}$, B are appropriate size matrices. For this system, stability analysis is done using an LMI-based test form [2],[13]. Then using [21] we investigate the smallest dwell time which guarantees stability of the switching system.

1.3 Contribution and Organization

Our contributions can be summarized as follows:

- We investigate the conservativeness of an LMI-based stability test for the time delay systems. This test is used in [21] for dwell time analysis. Therefore we come up with the limitations of the dwell time analysis. For this purpose we consider simple first order systems to illustrate the level of conservatism.
- We find the exact bounds of allowable parameters appearing in the above mentioned LMI-based test, for a stable switched time delay system in order to optimize the dwell time.
- We also consider the LQR-based switched feedback controllers with time delays and investigate the effects of weighting matrices Q and R on the dwell time.

In Chapter 2, we first express feedback control problem with delays in terms of state feedback and state estimate models. We give preliminary results in this chapter. In Chapter 3 stability conditions of the LMI-based test mentioned above is investigated. The conservatism analysis for this test is provided. In Chapter 4, the results from previous chapters are processed to find a minimum dwell time for a first order control system and second order state estimation system. Furthermore, effects of the weighting matrices Q and R, on the minimum dwell time and $\bar{A}$ are investigated.

Chapter 2

PROBLEM DEFINITION AND PRELIMINARY RESULTS

We assume that the switched system consists of ℓ models. Between switching time instants the system is in the form:

$$\begin{aligned} \dot{x}_{\sigma(t)} &= A_{\sigma(t)}x_{\sigma(t)}(t) + B_{\sigma(t)}u_{\sigma(t)}(t) \\ y_{\sigma(t)}(t) &= C_{\sigma(t)}x_{\sigma(t)}(t - \tau_{\sigma(t)}) + D_{\sigma(t)}w_{\sigma(t)}(t) \end{aligned} \tag{2.1}$$

Between each consequent switching time instants t_i and t_{i+1}, switching signal selects one of ℓ models. Switching signal is described as follows.

$$\mathfrak{S} : i = \sigma(t) \in \{1, 2, ..., \ell\}$$

The switching signal causes an arbitrary selection between "candidate" systems. The switching signal design for control purposes is out of this thesis' scope. In other words we discuss what happens under arbitrary switching, which is determined externally or internally but out of our control. In particular, we will be interested in finding a dwell time for stability under arbitrary switching.

Assumptions:

1. There are ℓ candidate models in the form (2.1) where A_i, B_i, C_i and D_i are fixed matrices, u_i is the control input and w_i is the noise.
2. The delay, $\tau_i > 0$, is also assumed to be fixed and known.
3. Feedback system is formed by

   $$u_i(t) = -K_i y_i(t) + v_i(t)$$

   where $v_i(t)$ is the disturbance input. Then we can write this system as

   $$\dot{x}(t) = A_i x(t) + \bar{A}_i x(t-\tau) + B_i v_i(t) - K_i D_i w_i(t) \tag{2.2}$$

   where $\bar{A}_i = -B_i K_i C_i$.
4. In the scalar case, $A > 0$ and $\bar{A} < 0$, i.e. uncontrolled system is unstable, and we analyze the effect of K_i stability of each candidate systems.

We will return to this model later. In the rest of this chapter we assume that we have only one system and drop the subscripts.

2.1 Controller Model

Let us consider the simple first order plant model with transfer function for the plant shown in Figure 2.1:

$$P(s) = e^{-\tau s}(sI - A)^{-1}B \tag{2.3}$$

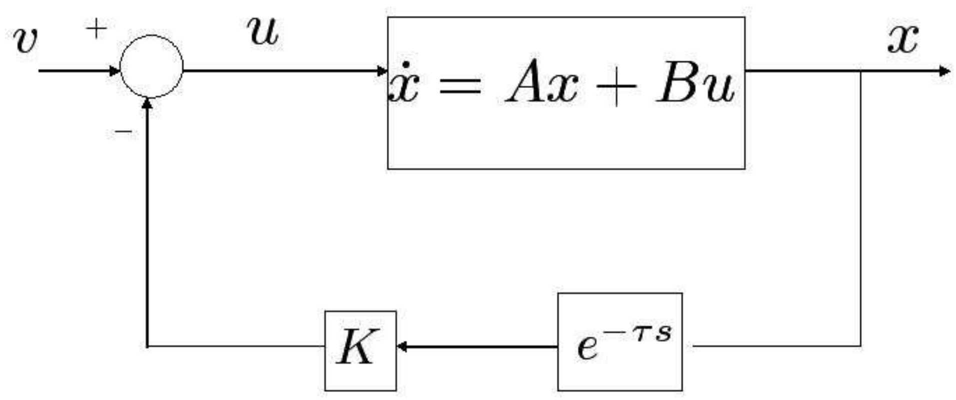

Figure 2.1: Plant Model for Feedback Control System

$\tau > 0$, $A > 0$.

Writing state-space realization for (2.3) in the form (2.2) gives

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
u(t) &= -Kx(t-\tau) + v(t)
\end{aligned}
\tag{2.4}
$$

The closed loop controlled state equation is

$$
\dot{x}(t) = Ax(t) - BKx(t-h) + Bv(t) \tag{2.5}
$$

Doing the transformation $-BK \rightarrow \bar{A}$, we can express (2.5) in the form of (2.2)

2.2 Observer Model (Dual Model for Controller)

The state-space model for the typical state estimation problem with delay is in the form:

$$
\dot{x}(t) = Ax(t) + Bv(t)
$$

$$y(t) = Cx(t-\tau) + Dw(t) \tag{2.6}$$

The observer equation is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(y(t) - C\hat{x}(t-\tau)) \tag{2.7}$$

where L is the Kalman gain matrix. Let the error function be

$$e(t) \triangleq x(t) - \hat{x}(t)$$

then

$$\dot{e}(t) = Ax(t) + Bv(t) - A\hat{x}(t) - L\left\{C\left[x(t-\tau) - \hat{x}(t-\tau) + Dw(t)\right]\right\} \tag{2.8}$$

$$\dot{e}(t) = Ae(t) - LCe(t-\tau) + Bv(t) - LDw(t) \tag{2.9}$$

Again with the transformation $-LC \rightarrow \bar{A}$, problem can be expressed in the form of (2.2).

2.3 Maximum Allowable Delay in LQR Design

In order to obtain a stable system, roots of the equation

$$\det(sI - (A + \bar{A}e^{-\tau s})) = 0 \tag{2.10}$$

should be in $\mathbb{C}_-$, where $\bar{A} = -LC$ for observer model and $\bar{A} = -BK$ for the controller model.

For the numerical analysis given below we chose the observer design for the standard constant velocity vehicle model where
$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $C = I_{2\times 2}$

Design of L (or K) for given A and $\tau = 0$

For non-delayed system design of gain L is done as follows:

In LQR design define the cost function:

$$J(u) = \int_0^\infty \left(x^T Q x + u^T R u\right) dt \tag{2.11}$$

Where $u(t) = -Kx(t)$ and K minimizes J.

In order to find L, dual of the LQR problem is used as $A^T_{observer} \rightarrow A$, $C^T_{observer} \rightarrow B$, and $K^T \rightarrow L$.

After we find the observer gain L, we end up with a stable closed-loop system for $\tau = 0$. The next step is to find the largest allowable delay so that the feedback system is stable. In other words, we need to find the the minimum de-stabilizing delay.

Let $Q = qI_{2\times2}$ and $R = rI_{2\times2}$. The MATLAB program DDE-BIFTOOL is used to obtain the minimum de-stabilizing delay numerically. The allowable delay and the eigenvalues of the closed loop system changes with the choice of r and q. This change is illustrated for the observer model in Figure 2.2 and Figure 2.3

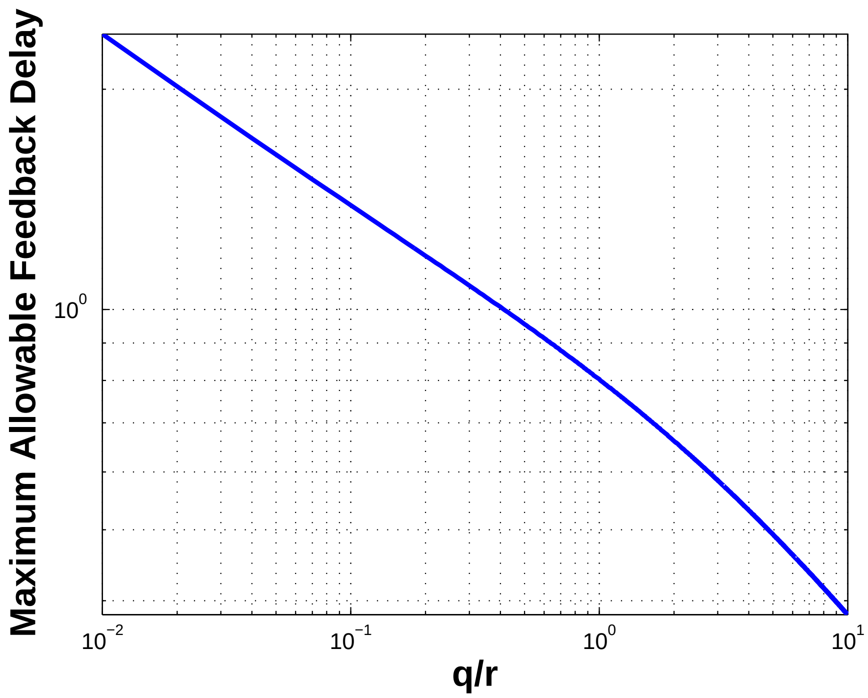

Figure 2.2: Allowable Delay for the choice of $\frac{q}{r}$

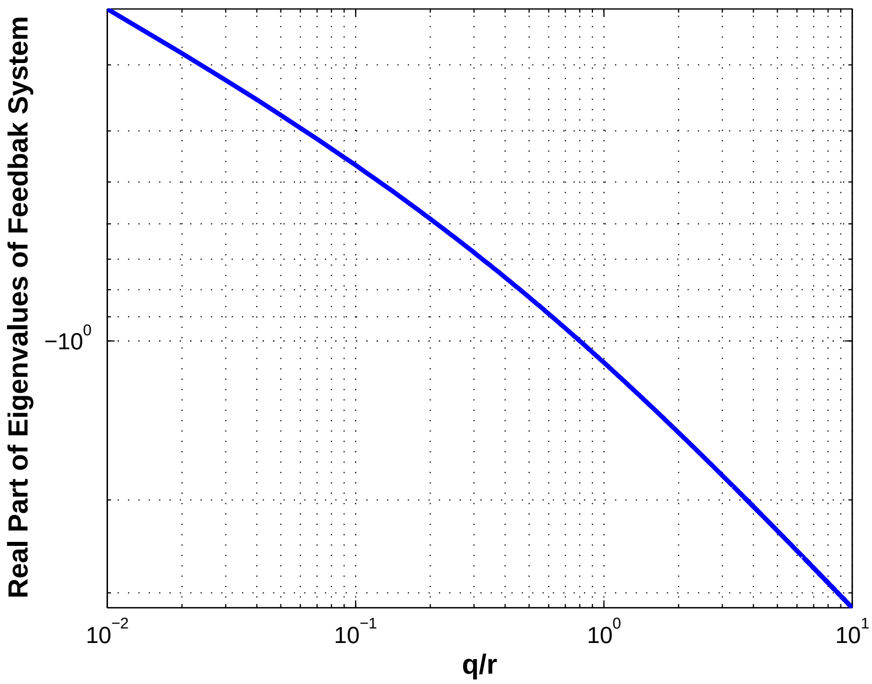

Figure 2.3: Placement of the real part of the eigenvalues for the choice of $\frac{q}{r}$

Observing Figure 2.2 and Figure 2.3 we come up with the following results:

1. As $\frac{q}{r}$ increases, eigenvalues of the closed loop system come closer to the imaginary axis

2. Large $\frac{q}{r}$ results in a faster transient response however the system becomes more aggressive and less robust to the delay. This conclusion is ensured with the Figure 2.2, as it is seen with increasing ratio of $\frac{q}{r}$ allowable delay decreases significantly.

If the weighting matrix R in (2.11) (here R is a scalar), is increased then the controller gain K is decreased. With a small K, the it is harder to make the closed loop system stable. This means the feedback control system can tolerate the smaller delays as given in Figure 2.2.

2.4 Maximum Allowable Delay for Observer Model Determined from the Small Gain Theorem

In order to have a stable closed loop system with delay τ, (2.10) must be satisfied. We can express (2.10) as

$$\det(sI - (A - LCe^{-\tau s})) = 0$$

$\Rightarrow$

$$\det\left(sI - (A - LC) - LC(1 - e^{-\tau s})\right) = 0$$

$\Rightarrow$

$$\det\left(sI - (A - LC)\right)\det\left(I - C(sI - (A - LC))^{-1}L(1 - e^{-\tau s})\right) = 0$$

The eigenvalues coming from the first part of the above equation are on the left-half plane because we have a stable closed loop system without delay as described earlier. Thus, now we are interested in the eigenvalues of the system shown in Figure 2.4 whose characteristic equation is

$$\det\left(I - C\left(sI - (A - LC)\right)^{-1} L\left(1 - e^{-\tau s}\right)\right) = 0 \tag{2.12}$$

Let

$$G(s) = C\left(sI - (A - LC)\right)^{-1} L\left(1 - e^{-\tau s}\right)$$

According to the *Small Gain Theorem* closed loop system is stable ***if***

$$\|G(s)\|_{\infty} < 1$$

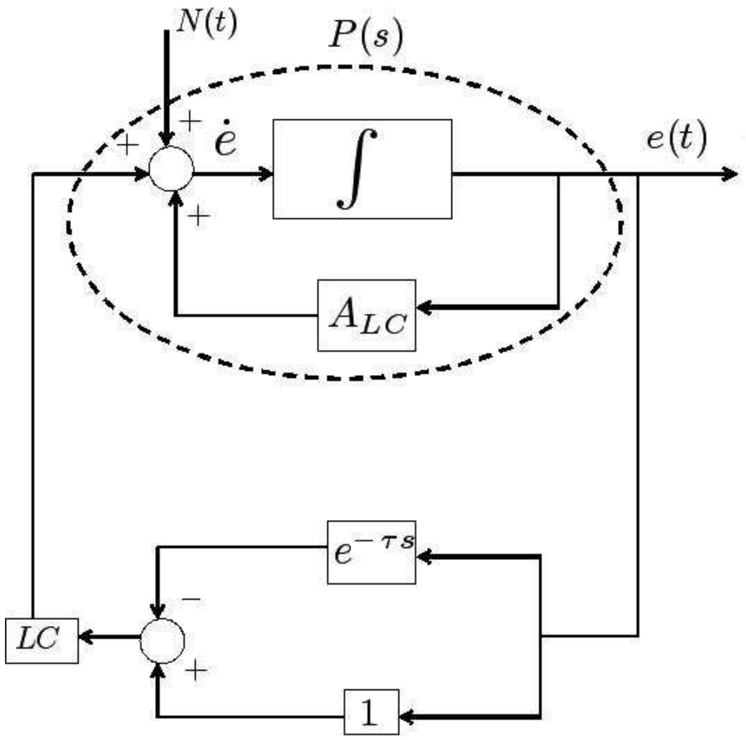

Figure 2.4: Conservative Analysis

If we rewrite $G(s)$ as

$$G(s) = C\left(sI - (A - LC)\right)^{-1} Ls \left(\frac{1 - e^{-\tau s}}{s}\right)$$

Since

$$\left\|\left(\frac{1 - e^{-\tau s}}{s}\right)\right\|_{\infty} \leq \tau$$

to guarantee stability using the Small Gain Theorem we need

$$\left\|sC\left(sI - (A - LC)\right)^{-1} L\right\|_{\infty} < \frac{1}{\tau}$$

which is equivalent to

$$\tau < \left\|sC\left(sI - (A - LC)\right)^{-1} L\right\|_{\infty}^{-1}. \tag{2.13}$$

Therefore, maximum allowable delay found from this analysis is the quantity on the right hand side of 2.13. Figure 2.5 shows the conservativeness of (2.13) with respect to the allowable delay found by using DDE-BIFTOOL toolbox.

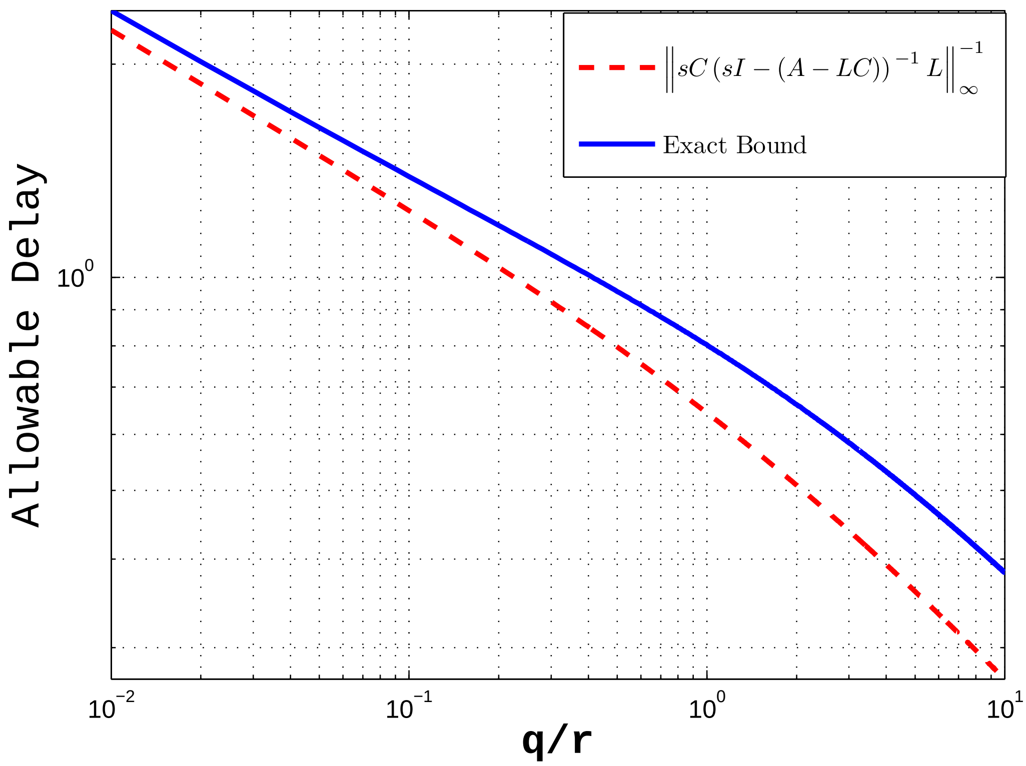

Figure 2.5: Conservatism Analysis for Allowable Delay found using the Small Gain Theorem

Chapter 3

STABILITY ANALYSIS FOR DELAY SYSTEMS

Let us begin with a review of some basic concepts from the Linear Algebra.

Minor: The $i \times j$ minor of an $n \times n$ matrix, X, denoted $|M_{ij}|$, is the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column of X.

Leading Principal Minor: The k^{th} order principal leading minor of $n \times n$ matrix X, denoted by $|M_k|$, is the determinant of the first k rows and columns of X

Theorem: $n \times n$ symmetric matrix X is negative definite if and only if

$$(-1)^k |M_k| > 0, k \in \{1, 2, ..., n\}$$

Now consider the stability test used in [21] taken from [2]

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \bar{A}x(t-\tau) \\ x(\theta) &= \phi(\theta), \forall \theta \in [-\tau, 0] \end{aligned} \tag{3.1}$$

The triplet $\Sigma := (A, \bar{A}, \tau) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{+}$ is *asymptotically stable dependent of delay* if the following lemma holds:

$$\begin{bmatrix} \Omega & P\bar{A}M \\ M^T \bar{A}^T P & -R \end{bmatrix} < 0 \tag{3.2}$$

where

$\Omega = \tau^{-1}\left[(A+\bar{A})^T P + P(A+\bar{A})\right] + p(\alpha+\beta)P$,

$M = \left[\begin{array}{cc} A & \bar{A} \end{array}\right]$,

$R = diag(\alpha P, \beta P)$,

and $\alpha > 0,\ \beta > 0$ and $p > 1$ are scalars

$P \in \mathbb{R}^{n\times n}$ is symmetric positive definite matrix.

If we assume that α, β and p are to be fixed, as $\alpha = \alpha^*$, $\beta = \beta^*$ and $p = p^*$, then (3.2) becomes an LMI whose decision variable is the matrix P. For dwell time analysis given in [21], we need to find feasible solution set for $(P,\ \alpha,\ \beta,\ p)$ satisfying (3.2). In order to find a feasible set, random and linear searches are done assuming fixed values for α, β and p, searching for positive definite P matrix using LMI-toolbox ([12]) developed for MATLAB. This tests, especially for n^{th} order systems where $n > 1$, show us it is very difficult to find a feasible set for (3.2). This lead us to the need for analysis of the conservativeness of the test given in (3.2).

In this section we test the conservatism of the test given in (3.2) on a simple first order system where A and $\bar{A}$ are scalars.

Thus, applying the theorem to (3.2) using the first order controller model described in Section 1.1, we obtain the following matrix inequality (because all of the variables are scalar and P multiplies each non-zero element, we can erase P from each element):

$$\left[\begin{array}{ccc} \tau^{-1}\left[2(A+\bar{A})\right] + p\,(\alpha+\beta) & A\bar{A} & \bar{A}^2 \\ A\bar{A} & -\alpha & 0 \\ \bar{A}^2 & 0 & -\beta \end{array}\right] < 0 \tag{3.3}$$

As before assume that $A > 0$ and $\bar{A} < 0$

3.1 Feasibility Analysis of the LMI-Based Test in (3.3)

Preliminaries:

Consider the second order polynomial, with coefficients a, b and c,
$P(x) = ax^2 + bx + c$

1. $\frac{c}{a}$ is the multiplication of the roots of $P(x) = 0$
2. $\frac{-b}{a}$ is the summation of the roots of $P(x) = 0$
3. If the discriminant ($\Delta = b^2 - 4ac$) is negative and $a > 0$, then the polynomial is always positive for all x
4. If the discriminant is negative and $a < 0$, then the polynomial is always negative for all x

First Leading Principal Minor

According to the theorem the following inequalities must hold first:

$$2\tau^{-1}(A + \bar{A}) + p(\alpha + \beta) < 0$$

or

$$p(\alpha + \beta) < -2\tau^{-1}(A + \bar{A}) \tag{3.4}$$

According to (3.4), because p, α, β and τ are positive, $(A+\bar{A})$ should be negative. This means:

$$|\bar{A}| > A \tag{3.5}$$

Second Leading Principal Minor

Now checking the second leading principal minor, we should have

$$-\alpha\left[2\tau^{-1}(A + \bar{A}) + p(\alpha + \beta)\right] - (A\bar{A})^2 > 0 \tag{3.6}$$

Rewriting (3.6) as treating α as the variable of the polynomial,

$$p\alpha^2 + \left[2\tau^{-1}(A+\bar{A}) + p\beta\right]\alpha + (A\bar{A})^2 < 0 \tag{3.7}$$

Let us investigate the properties of the inequality given in (3.7):

- Because the coefficient of the α^2 term is positive, the discriminant of the polynomial in (3.7) should be greater than 0 (if the discriminant is negative, then the polynomial takes positive values for all α). Thus, the polynomial has two real roots, namely α_1 and α_2.

- Both the constant term and the coefficient of the term α^2 of the polynomial in (3.7) is positive, so is the multiplication of the roots of the polynomial. This means, the roots α_1 and α_2 are either both negative or both positive.

- By definition, α is positive and according to (3.7) solution set of α lies between the roots α_1 and α_2 on the real axis (see Figure 3.1).

Using the above information, we conclude that both of the roots must be positive. To obtain positive roots for the polynomial, the coefficient of the term α should be negative. It is easily verified that the term

$$\left[2\tau^{-1}(A+\overline{A}) + p\beta\right]$$

is negative using (3.4).

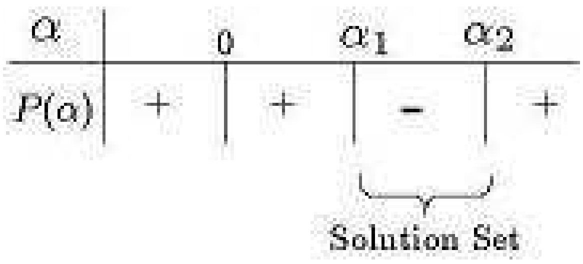

Figure 3.1: Solution set for the inequality given in (3.7)

Using the inequality given in (3.7) we obtain an upper bound and lower bound for α, as it is depicted in Figure 3.1. Let $\alpha_{lower_1} = \alpha_1$ and $\alpha_{upper_1} = \alpha_2$.

Let us go back to the requirement that the discriminant of (3.7) should be positive. We come to the inequality below:

$$\left[2\tau^{-1}(A+\bar{A}) + p\beta\right]^2 - 4(A\bar{A})^2 p > 0 \tag{3.8}$$

We can express (3.8) as

$$\begin{aligned}&\left(-\left[2\tau^{-1}(A+\bar{A}) + p\beta\right] + 2A\bar{A}\sqrt{p}\right) \times \\ &\left(-\left[2\tau^{-1}(A+\bar{A}) + p\beta\right] - 2A\bar{A}\sqrt{p}\right) > 0\end{aligned} \tag{3.9}$$

Because the second multiplier is always positive, the first one should also be positive

$\Rightarrow$

$$\left(-\left[2\tau^{-1}(A+\overline{A}) + p\beta\right] + 2A\bar{A}\sqrt{p}\right) > 0$$

$\Rightarrow$

$$\beta < \frac{2A\bar{A}\sqrt{p} - 2\tau^{-1}(A+\bar{A})}{p}. \tag{3.10}$$

This inequality defines an upper bound for β, namely β_{upper_1}. Because β is positive by definition, this bound should also be positive.

$$2A\bar{A}\sqrt{p} - 2\tau^{-1}(A+\bar{A}) > 0$$

$\Rightarrow$

$$\sqrt{p} < \frac{\tau^{-1}(A+\bar{A})}{A\bar{A}}. \tag{3.11}$$

By definition $p > 1$, then the upper bound found in the previous step should be also greater than 1. This gives us the following bound on τ:

$$\frac{(A+\bar{A})}{A\bar{A}} > \tau. \tag{3.12}$$

Third Leading Principal Minor

Let us now check the Third Leading Principal Minor:

$$\left[2\tau^{-1}(A+\bar{A}) + p(\alpha+\beta)\right]\alpha\beta - \left[-\bar{A}^4\alpha - \beta(A\bar{A})^2\right] < 0 \tag{3.13}$$

Rewriting (3.13) as treating β as the variable of the polynomial

$$p\alpha\beta^2 + \left[2\tau^{-1}\alpha(A+\bar{A}) + p\alpha^2 + (A\bar{A})^2\right]\beta + \alpha\bar{A}^4 < 0 \tag{3.14}$$

Let us investigate the properties of the inequality given in (3.14):

- Because the coefficient of the β^2 term is positive, the discriminant of the polynomial in (3.14) should be greater than 0 (if the discriminant is negative, then the polynomial takes positive values for all β). Thus, the polynomial has two real roots, namely β_1 and β_2.
- Both the constant term and the coefficient of the term β^2 of the polynomial in (3.14) is positive, so is the multiplication of the roots of the polynomial. This means, the roots β_1 and β_2 are either both negative or both positive.
- By definition, β is positive and according to (3.14) solution set of β lies between the roots β_1 and β_2 on the real axis (see Figure 3.2).

Using the above information, we conclude that both of the roots must be positive. To obtain positive roots for the polynomial, the coefficient of the term β should be negative.

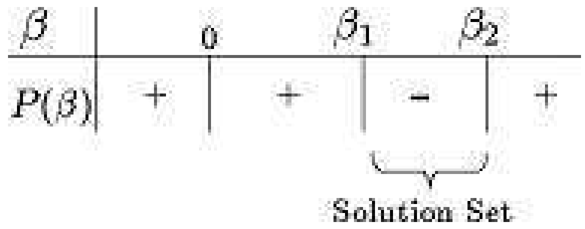

Figure 3.2: Solution set for the inequality given in (3.14)

Using the inequality given in (3.14) we obtain an extra upper bound and a lower bound for β, as it is depicted in Figure 3.2. Let $\beta_{lower_1} = \beta_1$ and $\beta_{upper_2} = \beta_2$.

The condition for (3.14) to have positive roots is:

$$p\alpha^2 + 2\tau^{-1}\alpha(A + \bar{A}) + (A\bar{A})^2 < 0 \tag{3.15}$$

- Because the coefficient of the α^2 term is positive, the discriminant of the polynomial in (3.15) should be greater than 0 (if the discriminant is negative, then the polynomial takes positive values for all α). Thus, the polynomial has two real roots, namely α_3 and α_4.
- It is easily verified that both of the roots of the polynomial in (3.15) are positive.

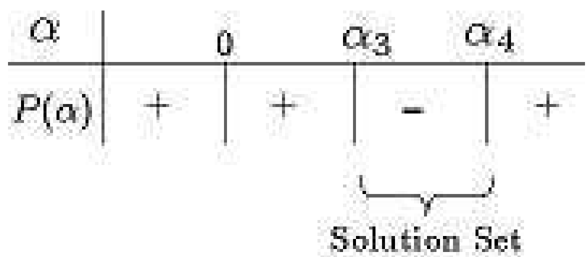

Figure 3.3: Solution set for the inequality given in (3.15)

Using the inequality given in (3.15) we obtain an extra upper bound and an extra lower bound for α, as it is depicted in Figure 3.3. Let $\alpha_{lower_2} = \alpha_3$ and

$\alpha_{upper_2} = \alpha_4$.

Let us go back to the requirement that the discriminant of the polynomial in (3.15) should be positive.

$$4\tau^{-2}(A+\bar{A})^2 - 4p(A\bar{A})^2 > 0$$

$\Rightarrow$

$$p < \frac{(A+\bar{A})^2}{\tau^2(A\bar{A})^2} \tag{3.16}$$

Notice that we conclude with the same upper bound for p in (3.11)

Let us go back to the requirement that the discriminant of (3.14) should be positive, we come with the inequality below:

$$\left[2\tau^{-1}\alpha(A+\bar{A}) + p\alpha^2 + (A\bar{A})^2\right]^2 - 4\alpha^2 p\bar{A}^4 > 0$$

$\Rightarrow$

$$\begin{aligned}\left(-\left[2\alpha\tau^{-1}(A+\bar{A}) + p\alpha^2 + (A\bar{A})^2\right] - 2\alpha\bar{A}^2\sqrt{p}\right) \times \\ \left(-\left[2\alpha\tau^{-1}(A+\bar{A}) + p\alpha^2 + (A\bar{A})^2\right] + 2\alpha\bar{A}^2\sqrt{p}\right) > 0\end{aligned} \tag{3.17}$$

Because the second multiplier is always positive, the first one should also be positive

$\Rightarrow$

$$p\alpha^2 + \left[2\bar{A}^2\sqrt{p} + 2\tau^{-1}(A+\bar{A})\right]\alpha + (A\bar{A})^2 < 0 \tag{3.18}$$

Let us investigate the properties of the inequality given in (3.18):

- Because the coefficient of the α^2 term is positive, the discriminant of the polynomial in (3.18) should be greater than 0 (if the discriminant is negative, then the polynomial takes positive values for all α). Thus, the polynomial has two real roots, namely α_5 and α_6.

- Both the constant term and the coefficient of the term α^2 of the polynomial in (3.18) is positive, so is the multiplication of the roots of the polynomial. This means, the roots α_5 and α_6 are either both negative or both positive.
- By definition, α is positive and according to (3.18) solution set of α lies between the roots α_5 and α_6 on the real axis (see Figure 3.4).

Using the above information, we conclude that both of the roots must be positive. To obtain positive roots for the polynomial, the coefficient of the term α should be negative.

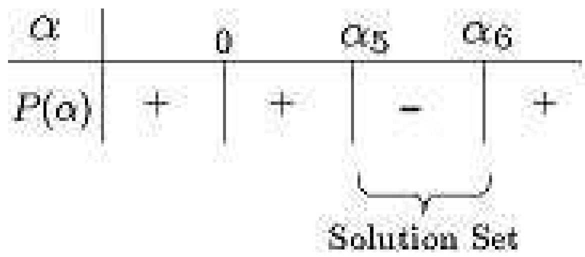

Figure 3.4: Solution set for the inequality given in (3.18)

Using the inequality given in (3.18) we obtain an extra upper bound and an extra lower bound for α, as it is depicted in Figure 3.4. Let $\alpha_{lower_3} = \alpha_5$ and $\alpha_{upper_3} = \alpha_6$.

The condition for (3.18) to have positive roots is:

$$2\tau^{-1}(A + \bar{A}) + 2\sqrt{p}\bar{A}^2 < 0$$

$\Rightarrow$

$$\sqrt{p} < \frac{-(A + \bar{A})}{\tau \bar{A}^2}. \tag{3.19}$$

By definition $p > 1$, then the upper bound found in the previous step should be also greater than 1. This gives us the following bound on τ:

$$\frac{(A+\bar{A})}{\bar{A}^2} > \tau. \tag{3.20}$$

Let us go back to the requirement that the discriminant of (3.18) should be positive, we come to the inequality below:

$$\left[2\bar{A}^2\sqrt{p} + 2\tau^{-1}(A+\bar{A})\right]^2 - 4p(A\bar{A})^2 > 0$$

$\Rightarrow$

$$\begin{aligned}\left(-\left[2\bar{A}^2\sqrt{p} + 2\tau^{-1}(A+\bar{A})\right] - 2A\bar{A}\sqrt{p}\right) \times \\ \left(-\left[2\bar{A}^2\sqrt{p} + 2\tau^{-1}(A+\bar{A})\right] + 2A\bar{A}\sqrt{p}\right)\end{aligned} \tag{3.21}$$

Because the first multiplier is always positive, the second one should also be positive

$\Rightarrow$

$$-2\bar{A}^2\sqrt{p} - 2\tau^{-1}(A+\bar{A}) + 2A\bar{A}\sqrt{p} > 0 \tag{3.22}$$

$\Rightarrow$

$$\sqrt{p} < \frac{-(A+\bar{A})}{\tau(\bar{A}^2 - A\bar{A})} \tag{3.23}$$

By definition $p > 1$, then the upper bound found in the previous step should be also greater than 1. This gives us the following bound on τ:

$$\frac{-(A+\bar{A})}{(\bar{A}^2 - A\bar{A})} > \tau \tag{3.24}$$

Rewriting (3.13) treating α as the variable of the polynomial, we find new bounds for β, p and α:

$$p\beta\alpha^2 + \left[2\tau^{-1}\beta(A+\bar{A}) + p\beta^2 + (\bar{A})^4\right]\alpha + \beta(A\bar{A})^2 < 0 \tag{3.25}$$

Let us investigate the properties of the inequality given in (3.25):

- Because the coefficient of the α^2 term is positive, the discriminant of the polynomial in (3.25) should be greater than 0 (if the discriminant is negative, then the polynomial takes positive values for all α). Thus, the polynomial has two real roots, namely α_7 and α_8.

- Both the constant term and the coefficient of the term α^2 of the polynomial in (3.25) are positive, so is the multiplication of the roots of the polynomial. This means, the roots α_7 and α_8 are either both negative or both positive.

- By definition, α is positive and according to (3.25) solution set of α lies between the roots α_7 and α_8 on the real axis (see Figure 3.5).

Using the above information, we conclude that both of the roots must be positive. To obtain positive roots for the polynomial, the coefficient of the term α should be negative.

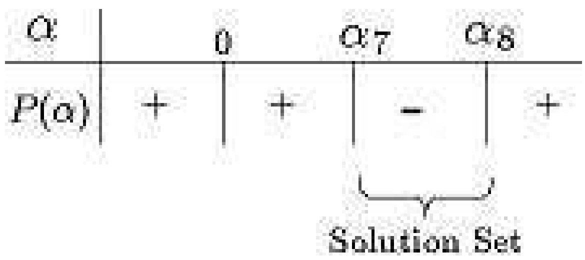

Figure 3.5: Solution set for the inequality given in (3.25)

Using the inequality given in (3.25) we obtain an extra upper bound and an extra lower bound for α, as it is depicted in Figure 3.5. Let $\alpha_{lower_4} = \alpha_7$ and $\alpha_{upper_4} = \alpha_8$.

The condition for (3.25) to have positive roots is:

$$p\beta^2 + 2\tau^{-1}\beta(A + \bar{A}) + \bar{A}^4 < 0 \tag{3.26}$$

- Because the coefficient of the β^2 term is positive, the discriminant of the polynomial in (3.26) should be greater than 0 (if the discriminant is negative, then the polynomial takes positive values for all β). Thus, the polynomial has two real roots, namely β_3 and β_4.
- It is easily verified that both of the roots of the polynomial in (3.26) are positive.

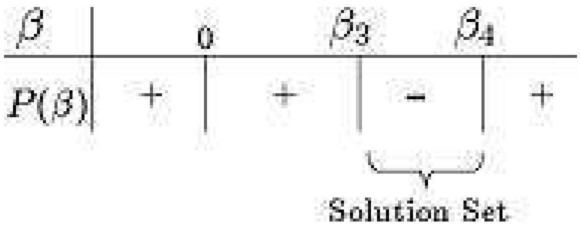

Figure 3.6: Solution set for the inequality given in (3.26)

Using the inequality given in (3.26) we obtain an extra upper bound and an extra lower bound for β, as it is depicted in Figure 3.6. Let $\beta_{lower_2} = \beta_3$ and $\beta_{upper_3} = \beta_4$.

Let us go back to the requirement that the discriminant of (3.26).

$$4\tau^{-2}(A + \bar{A})^2 - 4p\bar{A}^4 > 0$$

$\Rightarrow$

$$p < \frac{(A + \bar{A})^2}{\tau^2 \bar{A}^4} \tag{3.27}$$

Notice that we conclude with the same upper bound for p in (3.19).

Let us go back to the requirement that the discriminant of (3.25) should be positive, we come with the inequality below:

$$\left[2\tau^{-1}\beta(A + \bar{A}) + p\beta^2 + \bar{A}^4\right]^2 - 4\beta^2 p(A\bar{A})^2 > 0$$

$\Rightarrow$

$$\left(-\left[2\beta\tau^{-1}(A+\bar{A})+p\beta^2+\bar{A}^4\right]-2\beta(A\bar{A})\sqrt{p}\right)\times$$
$$\left(-\left[2\beta\tau^{-1}(A+\bar{A})+p\beta^2+\bar{A}^4\right]+2\beta(A\bar{A})\sqrt{p}\right)>0 \tag{3.28}$$

Because the first multiplier is always positive, the second one should also be positive.

$\Rightarrow$

$$p\beta^2+\left[2(A\bar{A})\sqrt{p}+2\tau^{-1}(A+\bar{A})\right]\beta+\bar{A}^4<0 \tag{3.29}$$

Let us investigate the properties of the inequality given in (3.29):

- Because the coefficient of the β^2 term is positive, the discriminant of the polynomial in (3.29) should be greater than 0 (if the discriminant is negative, then the polynomial takes positive values for all β). Thus, the polynomial has two real roots, namely β_5 and β_6.
- Both the constant term and the coefficient of the term β^2 of the polynomial in (3.29) are positive, so is the multiplication of the roots of the polynomial. This means, the roots β_5 and β_6 are either both negative or both positive.
- By definition, β is positive and according to (3.29) solution set of β lies between the roots β_5 and β_6 on the real axis (see Figure 3.7).

Using the above information, we conclude that both of the roots must be positive. To obtain positive roots for the polynomial, the coefficient of the term β should be negative.

Using the inequality given in (3.29) we obtain an extra upper bound and an extra lower bound for β, as it is depicted in Figure 3.7. Let $\beta_{lower_3}=\beta_5$ and $\beta_{upper_4}=\beta_6$. The condition for (3.29) to have positive roots is:

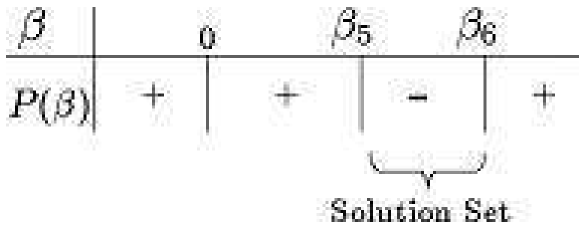

Figure 3.7: Solution set for the inequality given in (3.29)

$$2\tau^{-1}(A+\bar{A}) + 2\sqrt{p}(A\bar{A}) < 0$$

⇒

$$\sqrt{p} < \frac{(A+\bar{A})}{\tau(A\bar{A})} \tag{3.30}$$

Notice that we conclude with the same upper bound for p in (3.11).

Let us go back to the requirement that the discriminant of (3.29) should be positive, we come to the inequality below:

$$\left[2(A\bar{A})\sqrt{p} + 2\tau^{-1}(A+\bar{A})\right]^2 - 4p\bar{A}^4 > 0$$

⇒

$$\begin{aligned}&\left(-\left[2(A\bar{A})\sqrt{p} + 2\tau^{-1}(A+\bar{A})\right] - 2\bar{A}^2\sqrt{p}\right) \times \\ &\quad \left(-\left[2(A\bar{A})\sqrt{p} + 2\tau^{-1}(A+\bar{A})\right] + 2\bar{A}^2\sqrt{p}\right)\end{aligned} \tag{3.31}$$

Because the second multiplier is always positive, the first one should also be positive

⇒

$$-2A\bar{A}\sqrt{p} - 2\tau^{-1}(A+\bar{A}) - 2\bar{A}^2\sqrt{p} > 0 \tag{3.32}$$

⇒

$$\sqrt{p} < \frac{-(A+\bar{A})}{\tau(\bar{A}^2 + A\bar{A})} \tag{3.33}$$

Note that, using (3.5) it is found that $(\bar{A}^2 + A\bar{A}) > 0$.

By definition $p > 1$, then the upper bound found in the previous step should be also greater than 1. This gives us the following bound on τ:

$$\frac{-(A+\bar{A})}{(\bar{A}^2 + A\bar{A})} > \tau \tag{3.34}$$

3.2 Conservatism Analysis of the LMI-based Test For First Order Systems

Let's define the delay system as

$$\dot{x}(t) = Ax(t) + \bar{A}x(t-\tau) \tag{3.35}$$

where τ is the delay introduced to the system.
Assume that; $A > 0$ and $\bar{A} < 0$
Let's define $-\bar{A} = kA$, $k > 1$ according to the stability condition given in(3.5).
We can represent (3.35) in *Laplace* domain as

$$s - A + kAe^{-\tau s} = 0$$

$\Rightarrow$

$$1 + \frac{kAe^{-\tau s}}{s - A} = 0 \tag{3.36}$$

Applying Nyquist Criteria, we require a diagram similar to Figure 3.8 (i.e. the point $(-1 + j0)$ should be encircled once in the *counter clock-wise direction*).

To achieve this, at the cross-over frequency ω_c, the following phase condition should be met:

$$-\pi < -\tau\omega_c - \left(\pi - tan^{-1}\left(\frac{\omega_c}{A}\right)\right)$$

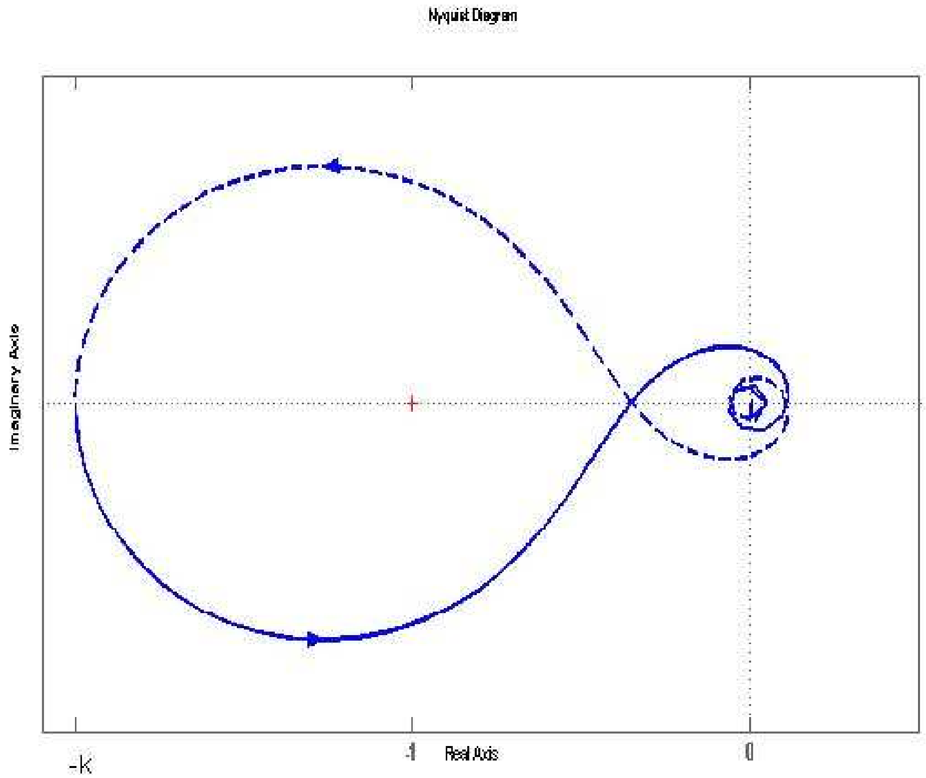

Figure 3.8: Nyquist diagram for stable first order delay system

$\Rightarrow$

$$tan^{-1}\left(\frac{\omega_c}{A}\right) > \tau\omega_c \tag{3.37}$$

where

$$\left|\frac{-kA}{j\omega_c - A}\right| = 1 \Rightarrow \omega_c = \sqrt{k^2A^2 - A^2} = A\sqrt{k^2-1}$$

In order to define a bound on τA, (3.37) can be expressed as

$$\frac{tan^{-1}\left(\sqrt{k^2-1}\right)}{\sqrt{k^2-1}} > \tau A\ , \tag{3.38}$$

which is shown in Figure 3.9 as the exact bound.

Using (3.12) we found a similar bound on τA. If we express (3.12) as

$$\frac{1+\frac{\bar{A}}{A}}{\frac{\bar{A}}{A}} > \tau A,$$

and use the same definition for k, where $k > 1$, we find a bound on τA as

$$\frac{k-1}{k} > \tau A. \tag{3.39}$$

Using (3.20) we found another bound on τA with the same definition for k:

$$\frac{k-1}{k^2} > \tau A \tag{3.40}$$

Using 3.24 we find the last bound on τA:

$$\frac{k-1}{k^2(1+\frac{1}{k})} > \tau A \tag{3.41}$$

Finally, using 3.34 we find the last bound on τA:

$$\frac{k-1}{k^2(1-\frac{1}{k})} > \tau A \tag{3.42}$$

Among all of the four bounds given in (3.39), (3.40), (3.41) and (3.42), the one in (3.41) is most conservative one. It is shown in Figure 3.9 as the conservative bound. The level of conservativeness of the LMI-based test given in (3.1) can be viewed in Figure 3.9.

In particular, Figure 3.9 shows that, for example when $\tau A = 0.2$ we cannot find a solution using (3.2) (yet for this case there exists an $\bar{A} = -kA$ with $1 < \sqrt{k^2-1} < 7.2$ such that the feedback system is stable. For the values of $\tau A < 0.17$ it is possible to find $\bar{A} = -kA$, for which (3.2) gives a solution.

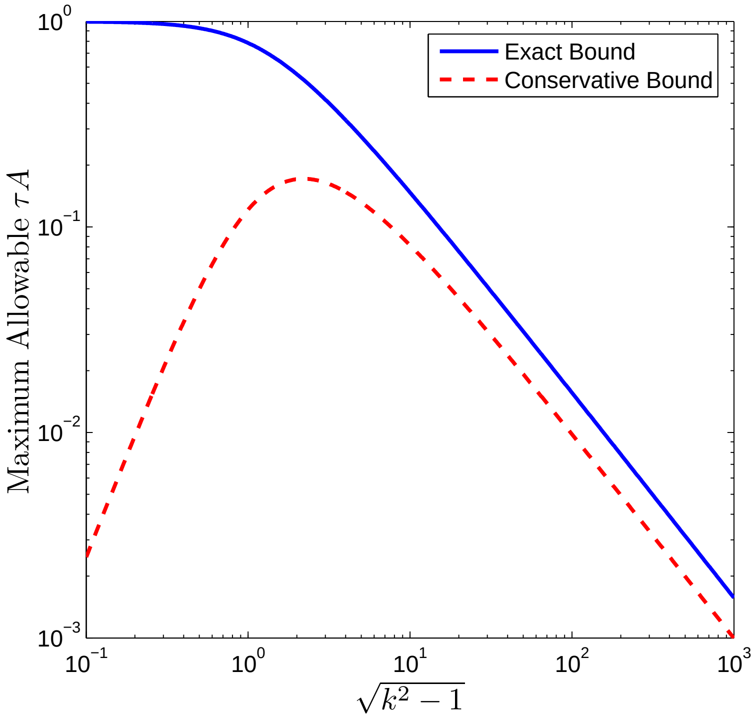

Figure 3.9: Conservativeness Analysis of LMI Based Test for First Order Systems

Chapter 4

DWELL TIME ANALYSIS

Let us begin with the results on dwell time obtained in [21]. The switched delay system consists of ℓ triplets as $\Sigma_i := (A_i, \bar{A}_i, \tau_i)$, where $i \in \{1, 2, ..., \ell\}$. The switched time delay system is asymptotically stable if all triplets are asymptotically stable. The following definitions are provided:

$$\begin{aligned} S_i := -\{&P_i(A_i + \bar{A}_i) + (A_i + \bar{A}_i)^T P_i \\ &+\tau_i \alpha^{-1} P_i \bar{A}_i A_i P_i^{-1} A_i^T \bar{A}_i^T P_i \\ &+\tau_i \beta^{-1} P_i (\bar{A}_i)^2 P^{-1} (\bar{A}_i^T)^2 P_i + \tau_i p_i(\alpha_i + \beta_i) P_i\} \end{aligned} \tag{4.1}$$

$\kappa_i := \sigma_{min}[P_i]$

$\bar{\kappa}_i := \sigma_{max}[P_i]$

$w_i := \sigma_{min}[S_i]$

$\lambda := \max_i \frac{\bar{\kappa}_i}{\kappa_i}$

$\mu := \max_i \frac{\bar{\kappa}_i}{w_i}$

Then the dwell time τ_D is defined as

$$\tau_D := T^* + 2\tau_{max} \tag{4.2}$$

where

$$T^* = \lambda\mu\lfloor\frac{\lambda - 1}{\bar{p} - 1} + 1\rfloor \tag{4.3}$$

and $\bar{p} := \min_i\{p_i\}$, $\tau_{max} = \max_i \tau_i$

In (4.1), the scalars α, β, p and the matrix P are chosen to satisfy the test given in (3.2).

Lemma[21]: Switched time delay system is stable under arbitrary switching if the difference between consecutive switching time instants is strictly grater than the dwell time τ_D.

4.1 Dwell Time Analysis for First Order Systems

Advantages of the first order system :

1. Simple model
2. Dwell time analysis is reduced to analyze μ parameter
3. Because the problem is reduced to finding an optimum (in this case the minimum) μ, analysis can be done for each candidate system separately, the one with maximum dwell time will dominate the overall dwell time of the system.

Because P will be scalar with a first order system, $\lambda = 1$. Design parameters are only included in T^* we can focus on this parameter. With the fact that $\lambda = 1$, T^* is reduced to

$$T^* = \mu$$

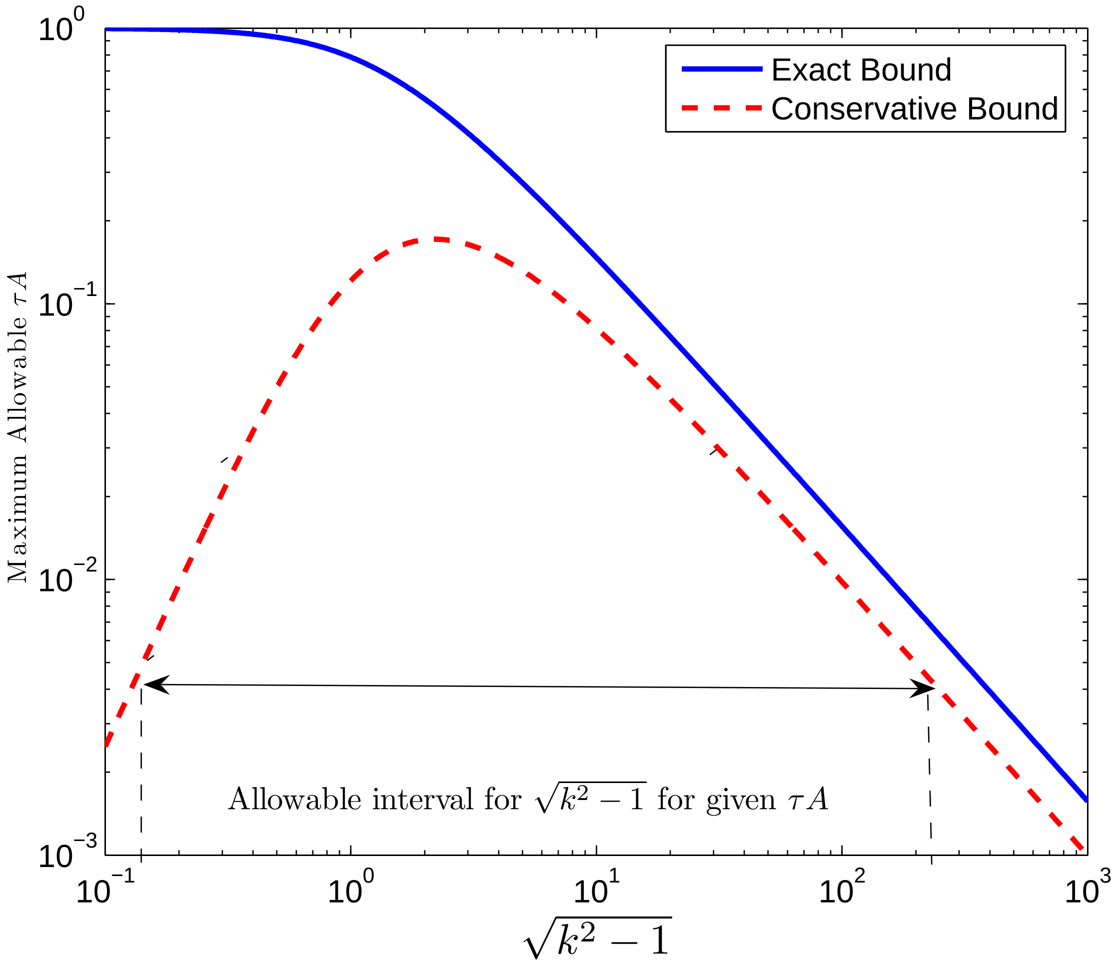

Figure 4.1: Allowable Range for $\sqrt{k^2-1}$

$\Rightarrow$

$$\mu = \frac{1}{\left|2(A+\overline{A})+\tau\alpha^{-1}(A\overline{A})^2+\tau\beta^{-1}\overline{A}^4+\tau p(\alpha+\beta)\right|}$$

According to (3.2) S is always negative. Thus

$$T^* = \frac{-1}{2(A+\overline{A})+\tau\alpha^{-1}(A\overline{A})^2+\tau\beta^{-1}\overline{A}^4+\tau p(\alpha+\beta)} \tag{4.4}$$

For the numerical analysis A is chosen as 0.1. For each fixed (τA) there exists a range $\sqrt{k^2-1}$, i.e. $\bar{A} = -kA$, feedback system is stable as shown in Figure 4.1.

Now assume that $\bar{A}$ is obtained from LQR design. For different values of $\frac{q}{r}$ we obtain different $\bar{A}$. For each $\bar{A}$, maximum allowable delay (τ_{max}) is obtained using Figure 4.1. Then we select $\tau = \frac{\tau_{max}}{10}$. The boundaries for α, β, p and $\bar{A}$ are used to find the minimum dwell time with the help of MATLAB *Optimization Toolbox*. The results are given with Figure 4.2.

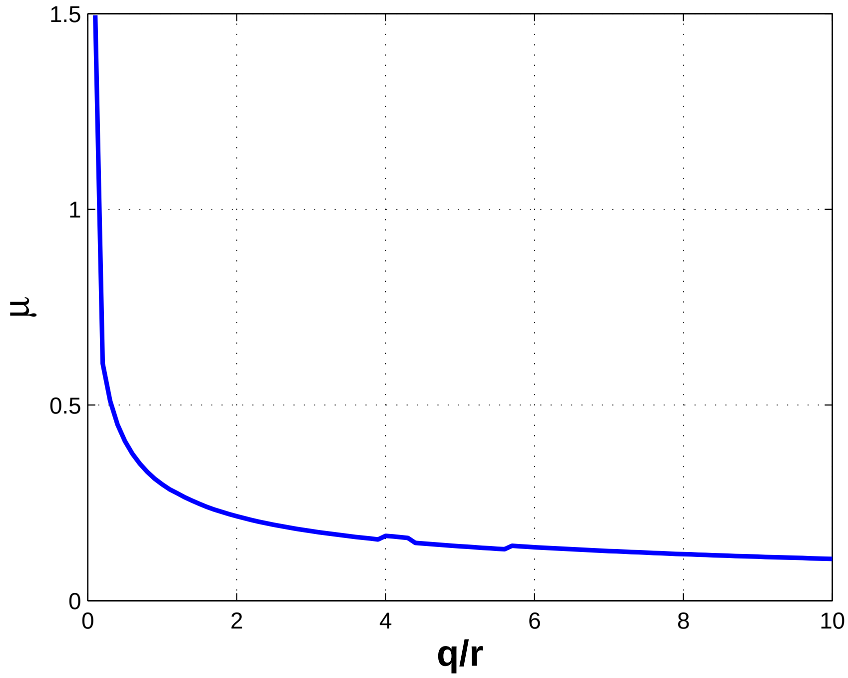

Figure 4.2: Minimum μ versus $\frac{q}{r}$ for minimum dwell time

In Figure 4.2, we observe that with increasing ratio of $\frac{q}{r}$ the minimum dwell time is decreased. Recall that in Figure 2.2, the increasing ratio of $\frac{q}{r}$ causes increased maximum allowable delay due to the increased robustness of the system to the delay type disturbance. Here, we can assume that the decreased dwell time shows us the degree of robustness of the switched time delay system.

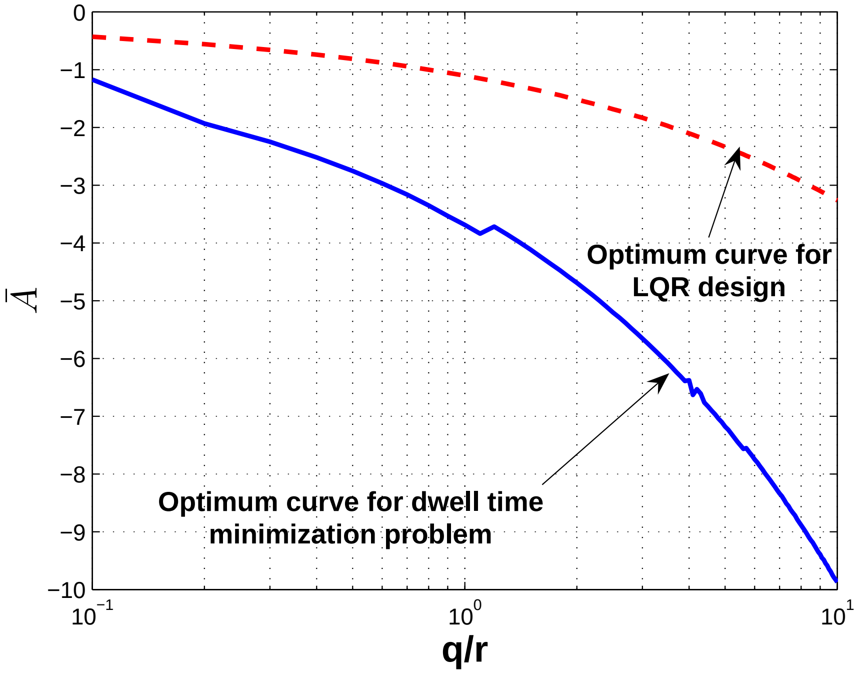

Figure 4.3: $\bar{A}$ versus $\frac{q}{r}$ for minimum dwell time

In Figure 4.3, we observe that the optimum value of $\bar{A}$ to minimize the dwell time is different from $\bar{A}$ found initially to minimize the cost function of LQR design in (2.11).

4.2 Minimum Dwell Time For n^{th} Order Systems

In Chapter 3 we illustrated how conservative the LMI-test given in (3.2) for some choice of τA. Furthermore the conservativeness analysis requires complicated calculations even for first order systems. Thus, we regard some extra assumptions in order to analyze the dwell time characteristics for stable switched time delay systems.

Let us assume that $P = \gamma I_{n\times n}$ and p is a constant, say p^*. Rewriting 3.2 results in:

$$X : \begin{bmatrix} \tau^{-1}\gamma\left[(A+\bar{A})+(A+\bar{A})^T\right]+p\gamma\,(\alpha+\beta) & \gamma\bar{A}A & \gamma\bar{A}^2 \\ \gamma A^T\bar{A}^T & -\gamma\alpha I_{n\times n} & 0_{n\times n} \\ \gamma(\bar{A}^T)^2 & 0_{n\times n} & -\gamma\beta I_{n\times n} \end{bmatrix} < 0 \tag{4.5}$$

With this selection of P and p^*,

- $\lambda = 1$. As in the scalar case, the problem of finding minimum dwell time for given system is reduced to find the minimum μ problem.
- 3.2 becomes an LMI.
- Because S in (4.1) is positive definite matrix, singular values σ_k, $k \in \{1, 2, ..., n\}$ of S are equal to the eigenvalues λ_k, μ is independent of γ
- Because γ multiplies each non-zero term in (4.7), γ can be canceled out and the overall problem becomes independent of choice of γ.

Now, the problem of ”minimizing dwell time” can be expressed as ”maximize the smallest singular value (in this case the minimum eigenvalue) of the matrix S”. Considering the variables α and β as the decision variables, we can state the problem as,

$$maximize\ z$$

subject to

$$S - zI_{n\times n} > 0$$

$$X < 0 \tag{4.6}$$

$$\alpha > 0, \beta > 0$$

We can express the nonlinear constraint $S - zI$ in terms of LMI using Schur Complement property ([1]) as:

$$\begin{bmatrix} \tau^{-1}\gamma\left[(A+\bar{A})+(A+\bar{A})^T - zI\right] + p\gamma(\alpha+\beta) & \gamma\bar{A}A & \gamma\bar{A}^2 \\ \gamma A^T\bar{A}^T & -\gamma\alpha I_{n\times n} & 0_{n\times n} \\ \gamma(\bar{A}^T)^2 & 0_{n\times n} & -\gamma\beta I_{n\times n} \end{bmatrix} < 0 \tag{4.7}$$

For the numerical analysis, we chose the observer design for the standard constant velocity vehicle model where
$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $C = I_{2\times 2}$.
$\bar{A}$ is obtained using dual of LQR problem along with the transformation $-LC \rightarrow \bar{A}$. Using DDE-BIFTOOL maximum allowable delay (τ_{max}) is computed. Tests for finding minimum dwell time, showed that for only smaller delays than the allowable maximum delay, a feasible solution for the problem given in (4.7) exists. This reminds us the conservativeness of the test in (3.2).

In Figure 4.4, change of the parameter μ with respect to the parameter p is depicted. For values greater than approximately 3.7, the solution becomes infeasible which means we have upper bound for p as in first order system. Note that, as p decreases, μ also decreases.

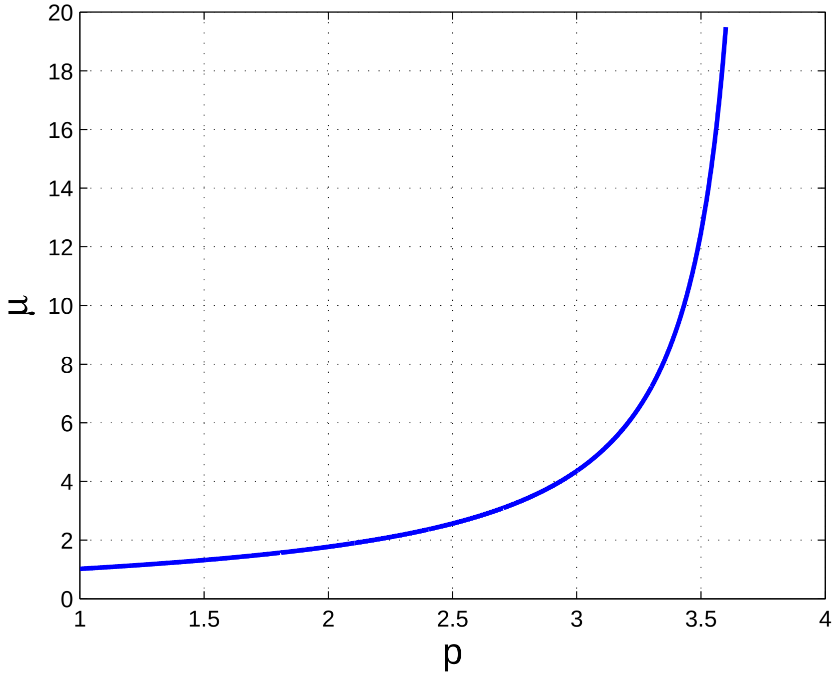

Figure 4.4: μ versus p for minimum dwell time

Chapter 5

CONCLUSIONS

In this thesis, we investigate the stability requirements of switched time delay systems. The stability analysis is basically done in terms of dwell-time.

First, we analyze linear delay systems. For this analysis, we used first order feedback control systems and second order constant velocity vehicle models. At first step we determine the conservativeness of our techniques on the stability with respect to the maximum allowable delays found using DDE-BIFTOOL toolbox.

Second, we investigated the limitations of the LMI-based stability test given in (3.2) for a first order feedback control system. We found the bounds for the stability in terms of α, β and p. Using the upper bounds found for p, we built the relationship with the stability analysis of first order delay system using Nyquist criteria and outlined the conservativeness of the LMI-based test. We found that for small τA values the test is quite conservative.

In the final chapter we gave numerical results for dwell time using the results of the previous chapters for first order controller model and second order constant velocity vehicle observer model. Due to the lack of stability analysis for n^{th} order systems where $n > 1$, we could give the minimum dwell time analysis with respect to the scalar p. The tests showed us that for maximum delay found by DDE-BIFTOOL the minimization problem is infeasible as in first order system. For a feasible problem, the maximum allowable delay was decreased, which led us to interrogate the conservativeness of the test given in (3.2) as an open problem.

For first order systems, dwell time minimization problem was solved as assuming $\bar{A}$ as a variable just like α, β and p. In Figure 4.3, it is shown that the optimum $\bar{A}$ for minimum dwell time is different from the value of $\bar{A}$ which minimizes the cost function of LQR design given in (2.11). This bring us to a trade off between optimum minimum dwell time problem and LQR problem, which is again an open problem to be tackled down. The same situation is valid for n^{th} order systems as well.

APPENDIX A

MATLAB CODE - SECOND ORDER SYSTEM

search.m

```
%This script searches for the maximum allowable delay
%iteratively using bisection method
%Searches until the maximum real part of the roots
%becomes closest possible to the imaginary axis on the
%left half plane
lqrDes; %First construct the matrices from LQR design
mx=3;
mn=0;
nm=0.5;
while mx-mn > 0.05
    roots_=dri(A,L,C,nm);
    if roots_<0
        mn=nm;
    else
        mx=nm;
```

```
    end
    nm=(mx+mn)/2;
end
nm=nm*1000;
nm=ceil(nm);
nm=nm/1000;
roots_=dri(A,L,C,nm);
if roots_<0
    nm=nm+0.001;
    while roots_<0
        roots_=dri(A,L,C,nm);
        nm=nm+0.001;
    end
    roots_=nm-0.002
else
    nm=nm-0.001;
    while roots_>=0
        roots_=dri(A,L,C,nm);
        nm=nm-0.001;
    end
    roots_=nm+0.001
end
```

lqrDes.m

```
%Finds L (or K matrix) using LQR design
A=[ 0 , 1; ...
    0 , 0];
C=eye(2);
R=r*eye(2);
Q=q*eye(2);
sys_=ss(A',C',zeros(2),0);
```

```
L=lqr(sys_,Q,R);
L=L';
```

dri.m

```
%For given system parameters and delay(tau), returns
%the maximum real part of the roots of the delayed system

function [roots_]=dri(A,L,C,tau) LC=L*C; stst.kind='stst';
stst.parameter=[A(1,1) A(1,2) LC(1,1) LC(1,2)...
                A(2,1) A(2,2) LC(2,1) LC(2,2) tau];
stst.x=[0 0]';
method=df_mthod('stst');
[stst,success]=p_correc(stst,[],[],method.point);
stst.x;
stst.stability=p_stabil(stst,method.stability);
roots_=max(real(stst.stability.l0));
figure(1);
clf;
p_splot(stst);
```

```
sys_init.m

%Initialize the delayed system in order to use the
%DDE_BIFTOOL toolbox, declare the name and the dimensions
function [name,dim]=sys_init()
name='max_delay';
dim=2;
%path for the DDE_BIFTOOL toolbox files, i.e. .../ddebiftool
path(path,'C:\Documents and Settings\...');
return;
```

sys_rhs.m

```
%The right hand side of the delayed system
%PAR contains the parameters including delay, XX contains the present and the
%past states (here the states are the error driven from
%observer and state equations)
function f=sys_rhs(xx,par)
%   PAR: [ A11 A12 LC11 LC12 A21 A22 LC21 LC22 tau ]
%   XX : [ e1(t) e1(t-tau) ; e2(t) e2(t-tau) ]
f(1,1)= par(1) * xx(1,1) + par(2) * xx(2,1) ...
- par(3) * xx(1,2)-par(4) * xx(2,2);
f(2,1)= par(5) * xx(1,1) + par(6) * xx(2,1) ...
-par(7) * xx(1,2) - par(8) * xx(2,2);
return;
```

sys_deri.m

```
%Defines the first order partial derivatives wrt parameters
function J=sys_deri(xx,par,nx,np,v)
%   PAR: [ A11 A12 LC11 LC12 A21 A22 LC21 LC22 tau ]
%   XX : [ e1(t) e1(t-tau) ; e2(t) e2(t-tau) ]
J=[];
if length(nx)==1 & length(np)==0 & isempty(v)
  % first order derivatives wrt state variables
  if nx==0 % derivative wrt x(t)
    J(1,1)=par(1);
    J(1,2)=par(2);
    J(2,1)=par(5);
```

```
    J(2,2)=par(6);
  elseif nx==1 % derivative wrt x(t-tau1)
    J(1,1)=-par(3);
    J(1,2)=-par(4);
    J(2,1)=-par(7);
    J(2,2)=-par(8);
  end;
end;
if isempty(J)
  err=[nx np size(v)]
  error('SYS_DERI: requested derivative could not be computed!');
end;
return;
```

sys_tau.m

```
%Declares the order of the  delay term in parameters vector
function tau=sys_tau()
%   PAR: [ A11 A12 LC11 LC12 A21 A22 LC21 LC22 tau ]
tau=[9];
return;
```

minimize_mu.m

```
%This script minimize the mu parameter, assuming alpha
%and beta parameter are the decision variables, by maximizing
%minimum singular value of the S matrix. Searches minimum mu parameter
%for different p values
gamma=1;
```

```
P=gamma*eye(2);
counter=1;
for p=1.001:0.001:10
    [alpha,beta,z]=minc(A,Abar,p,tau);
    S = (A+Abar)'*P + P*(A+Abar) + tau ...
        * ( 1/alpha * P * Abar * A * inv(P) * A' * Abar' * P...
        + 1/beta * P * Abar^2 * inv(P) * (Abar')^2 * P + p ...
        * (alpha+beta) * P );
    S=-S;
    svd1=svd(P);
    kappa=min(svd1);
    kappa_bar=max(svd1);
    w=min(svd(S));
    mu=kappa_bar/w;
    result(counter,1)=p;
    result(counter,2)=mu;
    counter=counter+1;
end
```

minc.m

```
%Used by the driver script minimize_mu.m script
%Maximize the objective function for the constraints outlined
%in Chapter 4 of the thesis.
function [alpha,beta,z]=minc(A,Abar,p,tau);
setlmis([]);
alpha=lmivar(1,[1 1]);
beta=lmivar(1,[1 1]);
z=lmivar(1,[1 1]); %Parameter to be maximized
lmiterm([1 1 1 0], (1/tau)*(A+Abar)'+(1/tau)*(A+Abar));
```

```
lmiterm([1 1 1 alpha],p*eye(2),eye(2));
lmiterm([1 1 1 beta],p*eye(2),eye(2));
lmiterm([1 1 1 z],(1/tau)*eye(2),eye(2));
lmiterm([1 2 1 0],-A'*Abar');
lmiterm([1 3 1 0], -Abar'*Abar');
lmiterm([1 2 2 alpha],-eye(2),eye(2));
lmiterm([1 3 2 0], zeros(2,2));
lmiterm([1 3 3 beta],-eye(2),eye(2));
lmiterm([-2,1,1,alpha],1,1); %0<alpha
lmiterm([-3,1,1,beta],1,1); %0<beta
lmiterm([4 1 1 0], (1/tau)*(A+Abar)'+(1/tau)*(A+Abar));
lmiterm([4 1 1 alpha],p*eye(2),eye(2));
lmiterm([4 1 1 beta],p*eye(2),eye(2));
lmiterm([4 2 1 0], A'*Abar');
lmiterm([4 3 1 0], Abar'*Abar');
lmiterm([4 2 2 alpha],-eye(2),eye(2));
lmiterm([4 3 2 0], zeros(2,2));
lmiterm([4 3 3 beta],-eye(2),eye(2));
lmis = getlmis;
c=[0;0;-1]; [copt,xopt]=mincx(lmis,c);
alpha = dec2mat(lmis,xopt,alpha);
beta = dec2mat(lmis,xopt,beta);
z = dec2mat(lmis,xopt,z);
```

APPENDIX B

MATLAB CODE - FIRST ORDER SYSTEM

minimize_dwellTime.m

```
%This script is used to investigate the effect of Q and
%R in LQR design. Finds initial Abar value and fixed delay,
%then minimizes the dwell time
global A_ global tau counter=1; for k=0.1:0.1:10
    counter
    if counter==5
        counter;
    end
    lqr_des
    search
    Abar=-L*C;
    A_=A;
    tau=delay/10;
    driver_min
    result(counter,1)=k;
```

```
    result(counter,2)=-L;
    result(counter,3:6)=sol_';
    result(counter,7)=fval;
    result(counter,8)=tau;
    counter=counter+1;
end
```

lqr_des.m

```
%LQR design for first order controller
A_=0.1;
C=1;
R=1;
Q=k;
L=lqr(A_',C',Q,R);
```

sys_rhs.m

```
%The right hand side of the delayed system
%PAR contains the parameters including delay, XX contains the present and the
%past states (here the states are the error driven from
%observer and state equations)
function f=sys_rhs(xx,par)
%   PAR: [ A11 LC11 tau ]
%   XX : [ e1(t) e1(t-tau) ]
f(1,1)= par(1) * xx(1,1) - par(2) * xx(1,2);
return;
```

sys_deri.m

```
%First order partial derivatives are defined
function J=sys_deri(xx,par,nx,np,v)
%   PAR: [ A11 LC11 tau ]
%   XX : [ e1(t) e1(t-tau) ]
J=[];
if length(nx)==1 & length(np)==0 & isempty(v)
  % first order derivatives wrt state variables
  if nx==0 % derivative wrt x(t)
    J(1,1)=par(1);
  elseif nx==1 % derivative wrt x(t-tau1)
    J(1,1)=-par(2);
  end;
end;
if isempty(J)
  err=[nx np size(v)]
  error('SYS_DERI: requested derivative could not be computed!');
end;
return;
```

driver_min.m

```
%Minimize the mu parameter using the constraints for
%alpha, beta, p and Abar defined in nonlcon1.m
global A_
global tau
A_=A;
clear A
options = optimset('Display','iter','MaxFunEvals',1000000,...
```

```
'MaxIter',1000000,'TolCon',0.00001);
bounds_Abar
Abar_ust=min([Abar1,Abar3,Abar5,Abar6]);
Abar_alt=max([Abar2,Abar4,Abar7]);
Abar_init=(Abar_alt+Abar_ust)/2;
Abar=Abar_init;
p_bounds
p_ust=min([p1,p2,p3,p4]);
p_alt=1;
p_init=(p_ust+p_alt)/2;
p=p_init;
bounds_beta
beta_ust=min([beta1,beta3,beta5,beta7]);
beta_alt=max(beta2,beta4);
beta_init=(beta_alt+beta_ust)/2;
beta=beta_init;
bounds_alpha
alpha_ust= min([alpha1,alpha3,alpha5,alpha7,alpha9]);
alpha_alt=max([alpha2,alpha4,alpha6,alpha8]);
alpha_init=(alpha_ust+alpha_alt)/2;
x0=[p_init alpha_init beta_init Abar_init];
[sol_,fval,exitflag,output] = fmincon(@myfun,x0,...
[],[],[],[],[],[],@nonlcon1,options);
p=sol_(1);
alpha=sol_(2);
beta=sol_(3);
Abar=sol_(4);
%Checks the feasibility of the found parameters treating
%P as the decision variable.
[P,flag,tmin,lhs1,rhs1]=findDelay_func_delayDepend...
(A_,Abar,p,alpha,beta,tau);
```

myfun.m

```
%Objective function
function [mu]=myfun(x)
global A_
global tau
%x:[p;alpha;beta;Abar]
p=x(1);
alpha=x(2);
beta=x(3);
Abar=x(4);
mu=-(2*(A_+Abar)+tau/alpha*(Abar*A_)^2+tau/beta*(Abar)^4...
 + tau*p*(alpha+beta));
mu=1/mu;
```

nonlcon1.m

```
%nonlinear constraint function
function [c,ce] = nonlcon1(x)
ce=[];
global A_
global tau
%x:[p;alpha;beta;Abar]
p=x(1);
beta=x(3);
alpha=x(2);
Abar=x(4);
%Bounds on p, alpha, beta and Abar are found
```

```
p_bounds
p_ust=min([p1,p2,p3,p4]);
p_alt=1;
p_step=(p_ust-p_alt)/100;
p_ust=p_ust-p_step;
p_alt=p_alt+p_step;
bounds_beta
beta_ust=min([beta1,beta3,beta5,beta7,beta8]);
beta_alt=max([beta2,beta4,beta9]);
beta_step=(beta_ust-beta_alt)/100;
beta_ust=beta_ust-beta_step;
beta_alt=beta_alt+beta_step;
bounds_alpha
alpha_ust=min([alpha1,alpha3,alpha5,alpha7,alpha9]);
alpha_alt=max([alpha2,alpha4,alpha6,alpha8]);
alpha_step=(alpha_ust-alpha_alt)/100;
alpha_ust=alpha_ust-alpha_step;
alpha_alt=alpha_alt+alpha_step;
bounds_Abar
Abar_ust=min([Abar1,Abar3,Abar5,Abar6]);
Abar_alt=max([Abar2,Abar4,Abar7]);
Abar_step=(Abar_ust-Abar_alt)/100;
Abar_ust=Abar_ust-Abar_step;
Abar_alt=Abar_alt+Abar_step;
%Constraints stored in c vector
c(1)=p_alt-x(1);
c(2)=-p_ust+x(1);
c(3)=alpha_alt-x(2);
c(4)=-alpha_ust+x(2);
c(5)=beta_alt-x(3);
c(6)=-beta_ust+x(3);
```

```
c(7)=Abar_alt-x(4);
c(8)=-Abar_ust+x(4);
```

p_bounds.m

```
p1=(Abar+A_)^2/tau^2/(Abar*A_)^2;
p2=(Abar+A_)^2/tau^2/(Abar)^4 ;
p3=(Abar+A_)^2/(tau^2*(Abar^2-A_*Abar));
p4=(Abar+A_)^2/(tau^2*(Abar^2+A_*Abar));
```

bounds_alpha.m

```
disc1=((2*(A_+Abar)/tau+p*beta))^2- 4*(A_*Abar)^2*p;
alpha1=(-((2*(A_+Abar)/tau+p*beta))+sqrt(disc1))/(2*p);
alpha2=(-((2*(A_+Abar)/tau+p*beta))-sqrt(disc1))/(2*p);
disc2=(2*(A_+Abar)/tau)^2- 4*(A_*Abar)^2*p;
alpha3=(-((2*(A_+Abar))/tau)+sqrt(disc2))/(2*p);
alpha4=(-((2*(A_+Abar))/tau)-sqrt(disc2))/(2*p);
disc3=(2*(A_+Abar)*beta/tau+p*beta^2+(Abar)^4)^2 -4...
* (Abar*A_)^2* beta^2*p;
alpha5=(-(2*(A_+Abar)*beta/tau+p*beta^2+(Abar)^4)...
+sqrt(disc3))/(2* p*beta);
alpha6=(-(2*(A_+Abar)*beta/tau+p*beta^2+(Abar)^4)...
-sqrt(disc3))/(2* p*beta); disc4=(
2*Abar^2*sqrt(p)+2/tau*(A_+Abar))^2 - 4* p*(A_*Abar)^2;
alpha7= (-(2*Abar^2*sqrt(p)+2/tau*(A_+Abar) )+sqrt(disc4)) / (2*p);
alpha8= (-(2*Abar^2*sqrt(p)+2/tau*(A_+Abar) )-sqrt(disc4)) / (2*p);
alpha9=-2*(Abar+A_)/tau/p-beta;
```

bounds_beta.m

```
disc_beta1=(2*(A_+Abar)/tau)^2- 4* Abar^4*p;
beta1=(-((2*(A_+Abar))/tau)+sqrt(disc_beta1))/(2*p);
beta2=(-((2*(A_+Abar))/tau)-sqrt(disc_beta1))/(2*p);
disc_beta2=(2*(A_*Abar)*sqrt(p)+2/tau*(A_+Abar))^2-...
 4 * p*Abar^4;
beta3=(-(2*(A_*Abar)*sqrt(p)+2/tau*(A_+Abar))+...
sqrt(disc_beta2))/(2*p);
beta4=(-(2*(A_*Abar)*sqrt(p)+2/tau*(A_+Abar))-...
sqrt(disc_beta2))/(2* p);
beta5=(2*A_*Abar*sqrt(p)-2/tau*(A_+Abar))/p;
beta7=-2*(A_+Abar)/tau/p;
if exist('alpha','var')==1
    disc_beta3=(2*(A_+Abar)*alpha/tau+p*alpha^2+(Abar*A_)^2)^2...
     -4* (Abar)^4 * alpha^2*p;
    beta8= (-(2*(A_+Abar)*alpha/tau+p*alpha^2+(Abar*A_)^2)+...
    sqrt(disc_beta3))/ (2 * p*alpha);
    beta9= (-(2*(A_+Abar)*alpha/tau+p*alpha^2+(Abar*A_)^2)-...
    sqrt(disc_beta3))/ (2 * p*alpha);
end
```

bounds_Abar.m

```
delta_Abar1=(1-A_*tau)^2-4*A_*tau;
Abar1=(-(1-A_*tau)+sqrt(delta_Abar1))/(2*tau);
Abar2=(-(1-A_*tau)-sqrt(delta_Abar1))/(2*tau);
delta_Abar2=(1+A_*tau)^2-4*A_*tau;
Abar3=(-(1+A_*tau)+sqrt(delta_Abar2))/(2*tau);
```

```
Abar4=(-(1+A_*tau)-sqrt(delta_Abar2))/(2*tau); Abar5=-A_/(1-tau*A_);
delta_Abar3=1-4*tau*A_;
Abar6=(-1+sqrt(delta_Abar3))/(2*tau);
Abar7=(-1-sqrt(delta_Abar3))/(2*tau);
```

findDelay_func_delayDepend.m

```
%Checks the feasibility of the given parameters, treating P
%as the decision variable
function [P,flag,tmin,lhs1,rhs1]=findDelay_func_delayDepend...
(A_,Abar,p,alpha,beta,tau);
setlmis([]);
P=lmivar(1,[1 1]);
lmiterm([1 1 1 P], (A+Abar)', (1/tau),'s');
lmiterm([1 1 1 P],p*(alpha+beta),1);
lmiterm([1 2 1 P], A'*Abar', 1);
lmiterm([1 3 1 P], Abar'*Abar', 1);
lmiterm([1 2 2 P], (-1)*alpha, 1);
lmiterm([1 3 2 0], 0);
lmiterm([1 3 3 P], (-1)*beta,1);
lmiterm([-2,1,1,P],1,1); %0<P
lmis=getlmis;
[tmin,xfeas] = feasp(lmis,[0,0,-1,0,0]);
if tmin<0 %feasible solution exist
    P = dec2mat(lmis,xfeas,P);
    evals = evallmi(lmis,xfeas);
    [lhs1,rhs1] = showlmi(evals,1);
else
    flag=0;
    P=0;
```

```
end
```

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