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U. Muğan and A. Sakka

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Schlesinger transformations for Painlevé VI equation

U. Muğan and A. Sakka
Department of Mathematics, Bilkent University, 06533 Bilkent, Ankara, Turkey

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A method to obtain the Schlesinger transformations for Painlevé VI equation is given. The procedure involves formulating a Riemann–Hilbert problem for a transformation matrix which transforms the solution of the linear problem but leaves the associated monodromy data the same. © *1995 American Institute of Physics.*

I. INTRODUCTION

At the beginning of this century Painlevé[1,2] and his school[3] classified the equations of the form $y''=F(y',y,z)$, where F is rational in y', algebraic in y, and locally analytic in z, which have the Painlevé property, i.e., their solutions are free from movable critical points. Among fifty such equations, the six Painlevé equations are the most well-known nonlinear ordinary differential equations (ODE's), since they are irreducible and do not have the solutions in terms of the known functions. Besides the Painlevé property, these six Painlevé equations, PI–PVI, have mathematical and physical significance. Their mathematical importance originates from (a) They can be considered as the isomonodromic conditions for suitable linear system of ODE's with rational coefficients possessing both regular and irregular singular points.[4–7] (b) They can be obtained as the similarity reduction of the nonlinear partial differential equation (PDE's) solvable by the inverse scattering transform (IST).[8] For example, PI and PII can be obtained from the exact similarity reduction of the Korteweg–de Vries (KdV) equation. (c) For certain choice of parameters, PII–PVI admit a one parameter family of solutions which are either rational or can be expressed in terms of the classical transcendental functions. For example, PVI admit a one parameter family of solutions in terms of hypergeometric functions.[9,10] (d) There are transformations associated with PII–PVI, these transformations map the solutions of a given Painlevé equation to the solution of the same equation but with different values of parameters,[10–13] (e) PI–PV can be obtained from PVI by the process of contraction.[1] In a similar way, it is possible to obtain the associated transformations for PII–PIV from the transformation for PV. More over the initial value problem of the Painlevé equations (PI–PV) can be studied using the inverse monodromy problem (IMT) which is the extension of the inverse spectral method to ODE's.[14–17]

Here, we present a method to obtain the Schlesinger transformations for PVI. The same method was used to obtain the Schlesinger transformations for PII–PV in Ref. 18. These transformations lead to a new class of relations between the solutions of PVI when its parameters are changed. First non trivial transformation among the solutions of PVI was given by Fokas and Yortsos,[19] Fokas and Ablowitz.[10] This transformation has been obtained from the relation between PVI and a special equation which is second order and second degree possessing Painlevé property. Another type of transformation which can be considered as an analog of the quadratic transformations for hypergeometric functions was given by Kitaev.[20] However, the latter type of transformation is possible for only a special choice of the parameters of PVI.

Let $y(t)$ be the solution of PVI with the parameters $\alpha,\beta,\gamma,\delta$ (or θ_∞, θ_0, θ_1, θ_t). The associated monodromy problem for PVI is $\partial Y/\partial z=AY$ where z plays the role of spectral parameter. The analytic structure of $Y(z)$ in the complex z plane can be specified by the so-called monodromy data (MD). If we denote y, Y, and y', Y' for θ_i, θ_i', $i=0,1,t,\infty$, respectively, it is possible to find appropriate transformations of θ_i such that the MD are invariant. Then $Y'(z)=R(z)Y(z)$, and the Schlesinger transformation matrix $R(z)$, can be found in closed form, by solving a certain Riemann–Hilbert (RH) problem. The transformation matrix $R(z)$ leads to a new class of the transformations among the solutions of PVI.

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II. THE SIXTH PAINLEVÉ EQUATION

The sixth Painlevé equation,

$$\frac{d^2y}{dt^2}=\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\times\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right), \tag{2.1}$$

can be obtained as the compatibility condition of the following linear system of equations:[7]

$$\frac{\partial Y}{\partial z}=A(z)Y(z,t), \tag{2.2a}$$

$$\frac{\partial Y}{\partial t}=B(z)Y(z,t), \tag{2.2b}$$

where

$$A(z)=\frac{A_0}{z}+\frac{A_1}{z-1}+\frac{A_t}{z-t}=\begin{pmatrix} a_{11}(z) & a_{12}(z)\\ a_{21}(z) & a_{22}(z)\end{pmatrix},$$

$$A_0=\begin{pmatrix} u_0+\theta_0 & -w_0u_0\\ w_0^{-1}(u_0+\theta_0) & -u_0\end{pmatrix},\quad A_1=\begin{pmatrix} u_1+\theta_1 & -w_1u_1\\ w_1^{-1}(u_1+\theta_1) & -u_1\end{pmatrix}, \tag{2.3}$$

$$A_t=\begin{pmatrix} u_t+\theta_t & -w_tu_t\\ w_t^{-1}(u_t+\theta_t) & -u_t\end{pmatrix},\quad B(z)=-A_t\frac{1}{z-t}.$$

Setting

$$A_\infty=-(A_0+A_1+A_t)=\begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{pmatrix},\quad \kappa_1+\kappa_2=-(\theta_0+\theta_1+\theta_t), \tag{2.4a}$$

$$\kappa_1-\kappa_2=\theta_\infty, \tag{2.4b}$$

$$a_{12}(z)=-\frac{w_0u_0}{z}-\frac{w_1u_1}{z-1}-\frac{w_tu_t}{z-t}=\frac{k(z-y)}{z(z-1)(z-t)}, \tag{2.4c}$$

$$u=a_{11}(y)=\frac{u_0+\theta_0}{y}+\frac{u_1+\theta_1}{y-1}+\frac{u_t+\theta_t}{y-t}, \tag{2.4d}$$

$$\bar{u}=-a_{22}(y)=u-\frac{\theta_0}{y}-\frac{\theta_1}{y-1}-\frac{\theta_t}{y-t} \tag{2.4e}$$

then

$$u_0+u_1+u_t=\kappa_2, \tag{2.5a}$$

$$w_0u_0+w_1u_1+w_tu_t=0, \tag{2.5b}$$

$$\frac{u_0+\theta_0}{w_0}+\frac{u_1+\theta_1}{w_1}+\frac{u_t+\theta_t}{w_t}=0, \tag{2.5c}$$

$$(t+1)w_0u_0+tw_1u_1+w_tu_t=k, \tag{2.5d}$$

$$tw_0u_0=k(t)y, \tag{2.5e}$$

which are solved as

$$w_0=\frac{ky}{tu_0},\quad w_1=-\frac{k(y-1)}{u_1(t-1)},\quad w_t=\frac{k(y-t)}{t(t-1)u_t},$$

$$\begin{aligned}
u_0&=\frac{y}{t\theta_\infty}\{y(y-1)(y-t)\bar{u}^2+[\theta_1(y-t)+t\theta_t(y-1)-2\kappa_2(y-1)(y-t)]\\
&\quad\times\bar{u}+\kappa_2^2(y-t-1)-\kappa_2(\theta_1+t\theta_t)\},\\
u_1&=-\frac{y-1}{(t-1)\theta_\infty}\{y(y-1)(y-t)\bar{u}^2+[(\theta_1+\theta_\infty)(y-t)+t\theta_t(y-1)-2\kappa_2(y-1)(y-t)]\\
&\quad\times\bar{u}+\kappa_2^2(y-t)-\kappa_2(\theta_1+t\theta_t)-\kappa_1\kappa_2\},\\
u_t&=\frac{y-t}{t(t-1)\theta_\infty}\{y(y-1)(y-t)\bar{u}^2+[\theta_1(y-t)+t(\theta_t+\theta_\infty)(y-1)-2\kappa_2(y-1)(y-t)]\\
&\quad\times\bar{u}+\kappa_2^2(y-1)-\kappa_2(\theta_1+t\theta_t)-t\kappa_1\kappa_2\}.
\end{aligned} \tag{2.6}$$

The equation $Y_{zt}=Y_{tz}$ implies

$$\frac{dy}{dt}=\frac{y(y-1)(y-t)}{t(t-1)}\left(2u-\frac{\theta_0}{y}-\frac{\theta_1}{y-1}-\frac{\theta_t}{y-t}\right), \tag{2.7a}$$

$$\begin{aligned}
\frac{du}{dt}&=\frac{1}{t(t-1)}\{[-3y^2+2(1+t)y-t]u^2+[(2y-1-t)\theta_0+(2y-t)\theta_1+(2y-1)(\theta_t-1)]u\\
&\quad-\kappa_1(\kappa_2+1)\},
\end{aligned} \tag{2.7b}$$

$$\frac{1}{k}\frac{dk}{dt}=(\theta_\infty-1)\frac{y-t}{t(t-1)}. \tag{2.7c}$$

Thus y satisfies the sixth Painlevé equation (2.1), with the parameters

$$\alpha=\tfrac{1}{2}(\theta_\infty-1)^2,\quad \beta=-\tfrac{1}{2}\theta_0^2,\quad \gamma=\tfrac{1}{2}\theta_1^2,\quad \delta=\tfrac{1}{2}(1-\theta_t^2). \tag{2.8}$$

III. DIRECT PROBLEM

The essence of the direct problem is to establish the analytic structure of Y with respect to z, in the entire complex z plane. Since Eq. (2.2a) is a linear ODE in z, therefore the analytic structure is completely determined by its singular points. Equation (2.2a) has regular singular points at $z=0,1,t,\infty$.

A. Solution about $z=0$

It is well known that if the coefficient matrix of the linear ODE has an isolated singularity at $z=0$, then the solution in the neighborhood of $z=0$ can be obtained via a convergent power series. In this particular case the solution $Y_0(z)=(Y_0^{(1)}(z),Y_0^{(2)}(z))$, for $\theta_0 \neq n$, $n\in\mathbb{Z}$ has the form

$$Y_0(z)=\hat{Y}_0(z)z^{D_0}=G_0(I+Y_{01}z+Y_{02}z^2+\cdots)z^{D_0}, \tag{3.1}$$

where

$$G_0=\begin{pmatrix} 2k_0 & l_0w_0u_0 \\ 2\dfrac{k_0}{w_0} & l_0(u_0+\theta_0) \end{pmatrix}, \quad \det G_0=1, \quad D_0=\begin{pmatrix} \theta_0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$k_0=\tilde{k}_0e^{\sigma_0(t)}, \quad l_0=\tilde{l}_0e^{-\sigma_0(t)}, \quad \tilde{k}_0,\tilde{l}_0=\text{const}, \tag{3.2}$$

$$\sigma_0=\int^t \frac{1}{t'}\left[u_t+\theta_t-\frac{w_tu_t}{w_0}\right]dt'$$

and Y_{01} satisfies the following equation:

$$Y_{01}+[Y_{01},\, D_0]=-G_0^{-1}\left(A_1G_0-\frac{dG_0}{dt}\right). \tag{3.3}$$

If $\theta_0=n$, $n\in\mathbb{Z}$ then the solution $Y_0(z)$ may or may not have the $\log z$ term.

The monodromy matrix about $z=0$ is given as

$$Y_0(ze^{2i\pi})=Y_0(z)e^{2i\pi D_0}. \tag{3.4}$$

B. Solution about $z=1$

The solution $Y_1(z)=(Y_1^{(1)}(z),Y_1^{(2)}(z))$, of Eqs. (2.2) in the neighborhood of the regular singular point $z=1$ for $\theta_1 \neq n$, $n\in\mathbb{Z}$ has the form

$$Y_1(z)=\hat{Y}_1(z)(z-1)^{D_1}=G_1(I+Y_{11}(z-1)+Y_{12}(z-1)^2+\cdots)(z-1)^{D_1}, \tag{3.5}$$

where

$$G_1=\begin{pmatrix} 2k_1 & l_1w_1u_1 \\ 2\dfrac{k_1}{w_1} & l_1(u_1+\theta_1) \end{pmatrix}, \quad \det G_1=1, \quad D_1=\begin{pmatrix} \theta_1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$k_1=\tilde{k}_1e^{\sigma_1(t)}, \quad l_1=\tilde{l}_1e^{-\sigma_1(t)}, \quad \tilde{k}_1,\tilde{l}_1=\text{const}, \tag{3.6}$$

$$\sigma_1=\int^t \frac{1}{t'-1}\left[u_t+\theta_t-\frac{w_tu_t}{w_1}\right]dt'$$

and Y_{11} satisfies the following equation:

$$Y_{11}+[Y_{11},\, D_1]=G_1^{-1}\left(A_0G_1-\frac{dG_1}{dt}\right). \tag{3.7}$$

If $\theta_1=n$, $n\in\mathbb{Z}$, the solution $Y_1(z)$ may or may not contain the $\log(z-1)$ term.

The monodromy matrix about $z=1$ is given as

$$Y_1(ze^{2i\pi})=Y_1(z)e^{2i\pi D_1}. \tag{3.8}$$

C. Solution about $z=t$

The solution $Y_t(z)=(Y_t^{(1)}(z),Y_t^{(2)}(z))$, of Eqs. (2.2) in the neighborhood of the regular singular point $z=t$ for $\theta_t\neq n$, $n\in\mathbf{Z}$ [if $\theta_t=n$, $n\in\mathbf{Z}$ the solution $Y_t(z)$ may or may not have the $\log(z-t)$ term] has the form

$$Y_t(z)=\hat{Y}_t(z)(z-t)^{D_t}=G_t(I+Y_{t1}(z-t)+Y_{t2}(z-t)^2+\cdots)(z-t)^{D_t}, \tag{3.9}$$

where

$$G_t=\begin{pmatrix} 2k_t & l_t w_t u_t \\ 2\dfrac{k_t}{w_t} & l_t(u_t+\theta_t \end{pmatrix},\quad \det G_t=1,\quad D_t=\begin{pmatrix} \theta_t & 0 \\ 0 & 0\end{pmatrix},$$

$$k_t=\tilde{k}_t e^{\sigma_t(t)},\quad l_t=\tilde{l}_t e^{-\sigma_t(t)},\quad \tilde{k}_t,\tilde{l}_t=\text{const}, \tag{3.10}$$

$$\sigma_t=\int^t\left[\frac{1}{t'}\left(u_0+\theta_0-\frac{w_0u_0}{w_t}\right)+\frac{1}{t'-1}\left(u_1+\theta_1-\frac{w_1u_1}{w_t}\right)\right]dt'$$

and Y_{11} satisfies the following equation:

$$Y_{t1}+[Y_{t1},D_t]=G_t^{-1}\frac{dG_t}{dt}. \tag{3.11}$$

The monodromy matrix about $z=t$ is given as

$$Y_t(ze^{2i\pi})=Y_t(z)e^{2i\pi D_t}. \tag{3.12}$$

D. Solution about $z=\infty$

The solution $Y_\infty(z)=(Y_\infty^{(1)}(z),Y_\infty^{(2)}(z))$, of Eqs. (2.2) in the neighborhood of the regular singular point $z=\infty$ for $\theta_\infty\neq n$, $n\in\mathbf{Z}$ (if $\theta_\infty=n$, $n\in\mathbf{Z}$, the solution may or may not have the $\log(1/z)$ term) has the form

$$Y_\infty(z)=\hat{Y}_\infty(z)\left(\frac{1}{z}\right)^{D_\infty}=\left(I+Y_{\infty1}\frac{1}{z}+Y_{\infty2}\left(\frac{1}{z}\right)^2+\cdots\right)\left(\frac{1}{z}\right)^{D_\infty}, \tag{3.13}$$

where

$$D_\infty=\begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{pmatrix},$$

$$\kappa_1=u_0+u_1+u_t,\quad \kappa_1-\kappa_2=\theta_\infty,\quad \kappa_1+\kappa_2=-(\theta_0+\theta_1+\theta_t) \tag{3.14}$$

and $Y_{\infty1}$ satisfies the following equation:

$$Y_{\infty1}+[Y_{\infty1},D_\infty]=-(A_1+tA_t). \tag{3.15}$$

The monodromy matrix about $z=\infty$ is given as

$$Y_\infty(ze^{2i\pi})=Y_\infty(z)e^{-2i\pi D_\infty}. \tag{3.16}$$

E. Monodromy data

The relations between the $Y_\infty(z)$ and $Y_i(z)$, $i=0$, 1, t are given by the connection matrices E_i

$$Y_\infty(z)=Y_i(z)E_i, \quad E_i=\begin{pmatrix}\mu_i & \nu_i\\ \zeta_i & \eta_i\end{pmatrix}, \quad \det E_i=1, \quad i=0,1,t. \tag{3.17}$$

The monodromy data MD=$\{\mu_0,\nu_0,\zeta_0,\eta_0,\mu_1,\nu_1,\zeta_1,\eta_1,\mu_t,\nu_t,\zeta_t,\eta_t\}$ satisfy the consistency condition

$$(E_0^{-1}e^{2i\pi D_0}E_0)(E_1^{-1}e^{2i\pi D_1}E_1)=e^{-2i\pi D_\infty}(E_t^{-1}e^{-2i\pi D_t}E_t) \tag{3.18}$$

in particular

$$\cos\pi(\theta_0-\theta_1)(\zeta_0\mu_0\eta_1\nu_1+\eta_0\nu_0\mu_1\zeta_1-\eta_0\mu_0\nu_1\zeta_1-\zeta_0\nu_0\eta_1\mu_1)+\cos\pi(\theta_0+\theta_1)(\nu_0\zeta_0\nu_1\zeta_1$$
$$+\eta_0\mu_0\eta_1\mu_1-\mu_0\zeta_0\nu_1\eta_1-\eta_0\nu_0\mu_1\zeta_1)=\mu_t\eta_t\cos\pi(\theta_\infty+\theta_t)-\nu_t\zeta_t\cos\pi(\theta_\infty-\theta_t). \tag{3.19}$$

IV. SCHLESINGER TRANSFORMATIONS

Let $R(z)$ be the transformation matrix which transforms the solution of the linear problem (2.2) as

$$Y'(z)=R(z)Y(z) \tag{4.1}$$

but leaves the monodromy data associated with $Y(z)$ the same. Let u_i', w_i', $\theta_i' = \theta_i + \lambda_i$ be the transformed quantities of u_i, w_i, θ_i, $i=0,1,t,\infty$. The consistency condition of the monodromy data (3.18) or (3.19) is invariant under the transformation if $\lambda_1+\lambda_0=k$, $\lambda_1-\lambda_0=l$, $\lambda_\infty+\lambda_t=m$, $\lambda_\infty-\lambda_t=n$, where k,l,m,n are either odd or even integers. It is enough to consider the following three cases:

$$a:\begin{cases}\theta_0'=\theta_0+\lambda_0\\ \theta_1'=\theta_1\\ \theta_t'=\theta_t\\ \theta_\infty'=\theta_\infty+\lambda_\infty,\end{cases}\qquad b:\begin{cases}\theta_0'=\theta_0\\ \theta_1'=\theta_1+\lambda_1\\ \theta_t'=\theta_t\\ \theta_\infty'=\theta_\infty+\lambda_\infty,\end{cases}\qquad c:\begin{cases}\theta_0'=\theta_0\\ \theta_1'=\theta_1\\ \theta_t'=\theta_t+\lambda_t\\ \theta_\infty'=\theta_\infty+\lambda_\infty.\end{cases} \tag{4.2}$$

Let the complex z plane be divided into two sectors $S^\pm$ by an infinite contour C passing through the points $z=0$, 1, t and let

$$R(z)=R^\pm(z), \quad \text{when} \quad z \quad \text{in} \quad S^\pm. \tag{4.3}$$

Then the transformation (4.1) can be written as

$$[Y^\pm(z)]'=R^\pm(z)Y^\pm(z), \quad \text{when} \quad z \quad \text{in} \quad S^\pm, \tag{4.4}$$

and the monodromy matrices (3.4), (3.8), (3.12), and (3.16) about $z=0$, 1, t, ∞ imply that the transformation matrix $R(z)$ satisfies the following RH problem:

$$a:\begin{cases} R^+(z)=R^-(z), & \text{on} \quad C_0^- \\ R^+(z)=R^-(ze^{2i\pi}), & \text{on} \quad C_0^+, \end{cases}$$

$$b:\begin{cases} R^+(z)=R^-(z), & \text{on} \quad C_1^- \\ R^+(z)=R^-(ze^{2i\pi}), & \text{on} \quad C_1^+, \end{cases} \tag{4.5}$$

$$c:\begin{cases} R^+(z)=R^-(z), & \text{on} \quad C_t^- \\ R^+(z)=R^-(ze^{2i\pi}), & \text{on} \quad C_t^+, \end{cases}$$

where $C_i^{\pm}$ are parts of the contour C joined at the point $z=0,1,t$ respectively. The boundary conditions for the RH problems are as follows:

$$a:\begin{cases} R^+(z)\sim \hat{Y}_0'(z)z^{\Lambda_0}\hat{Y}_{(0)}^{-1}(z), & \text{as} \quad z\to 0, \quad z \quad \text{in} \quad S^+ \\ R^+(z)\sim \hat{Y}_1'(z)\hat{Y}_1^{-1}(z), & \text{as} \quad z\to 1, \quad z \quad \text{in} \quad S^+ \\ R^+(z)\sim \hat{Y}_t'(z)\hat{Y}_t^{-1}(z), & \text{as} \quad z\to t, \quad z \quad \text{in} \quad S^+ \\ R^+(z)\sim \hat{Y}_\infty'(z)\left(\frac{1}{z}\right)^{\Sigma_0}\hat{Y}_\infty^{-1}(z), & \text{as} \quad |z|\to\infty, \quad z \quad \text{in} \quad S^+ \end{cases}, \tag{4.6}$$

$$b:\begin{cases} R^+(z)\sim \hat{Y}_0'(z)\hat{Y}_{(0)}^{-1}(z), & \text{as} \quad z\to 0, \quad z \quad \text{in} \quad S^+ \\ R^+(z)\sim \hat{Y}_1'(z)(z-1)^{\Lambda_1}\hat{Y}_1^{-1}(z), & \text{as} \quad z\to 1, \quad z \quad \text{in} \quad S^+ \\ R^+(z)\sim \hat{Y}_t'(z)\hat{Y}_t^{-1}(z), & \text{as} \quad z\to t, \quad z \quad \text{in} \quad S^+ \\ R^+(z)\sim \hat{Y}_\infty'(z)\left(\frac{1}{z}\right)^{\Sigma_1}\hat{Y}_\infty^{-1}(z), & \text{as} \quad |z|\to\infty, \quad z \quad \text{in} \quad S^+ \end{cases}, \tag{4.7}$$

$$c:\begin{cases} R^+(z)\sim \hat{Y}_0'(z)\hat{Y}_{(0)}^{-1}(z), & \text{as} \quad z\to 0, \quad z \quad \text{in} \quad S^+ \\ R^+(z)\sim \hat{Y}_1'(z)\hat{Y}_1^{-1}(z), & \text{as} \quad z\to 1, \quad z \quad \text{in} \quad S^+ \\ R^+(z)\sim \hat{Y}_t'(z)(z-t)^{\Lambda_t}\hat{Y}_t^{-1}(z), & \text{as} \quad z\to t, \quad z \quad \text{in} \quad S^+, \\ R^+(z)\sim \hat{Y}_\infty'(z)\left(\frac{1}{z}\right)^{\Sigma_t}\hat{Y}_\infty^{-1}(z), & \text{as} \quad |z|\to\infty, \quad z \quad \text{in} \quad S^+ \end{cases} \tag{4.8}$$

where

$$\Lambda_i=\begin{pmatrix} \lambda_i & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma_i=\begin{pmatrix} \frac{1}{2}(\lambda_\infty-\lambda_i) & 0 \\ 0 & -\frac{1}{2}(\lambda_\infty+\lambda_i) \end{pmatrix}, \quad i=0,1,t. \tag{4.9}$$

For each case a, b, and c there exists a function $R(z)$ which is analytic everywhere and the boundary conditions (4.6), (4.7), (4.8) specify $R(z)$ for each case, respectively.

All possible Schlesinger transformations admitted by the linear problem (2.2) may be generated by the following transformation matrices $R_{(k)}(z)$, $k=1,2,3,\ldots,12$:

$$\begin{cases} \theta_0'=\theta_0+1 \\ \theta_1'=\theta_1 \\ \theta_t'=\theta_t \\ \theta_\infty'=\theta_\infty+1, \end{cases} \quad R_{(1)}(z)=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}z+\begin{pmatrix} 1 & -w_0 \\ -r_1 & w_0r_1 \end{pmatrix}, \tag{4.10}$$

$$\begin{cases} \theta_0'=\theta_0-1 \\ \theta_1'=\theta_1 \\ \theta_t'=\theta_t \\ \theta_\infty'=\theta_\infty-1, \end{cases} \quad R_{(2)}(z)=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\dfrac{u_0+\theta_0}{u_0w_0}r_2 & -r_2\\ -\dfrac{u_0+\theta_0}{u_0w_0} & 1\end{pmatrix}\frac{1}{z}, \tag{4.11}$$

$$\begin{cases} \theta_0'=\theta_0-1 \\ \theta_1'=\theta_1 \\ \theta_t'=\theta_t \\ \theta_\infty'=\theta_\infty+1, \end{cases} \quad R_{(3)}(z)=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}1 & -\dfrac{u_0w_0}{u_0+\theta_0}\\ -r_1 & \dfrac{u_0w_0}{u_0+\theta_0}r_1\end{pmatrix}\frac{1}{z}, \tag{4.12}$$

$$\begin{cases} \theta_0'=\theta_0+1 \\ \theta_1'=\theta_1 \\ \theta_t'=\theta_t \\ \theta_\infty'=\theta_\infty-1, \end{cases} \quad R_{(4)}(z)=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}z+\begin{pmatrix}\dfrac{r_2}{w_0} & -r_2\\ -\dfrac{1}{w_0} & 1\end{pmatrix}, \tag{4.13}$$

$$\begin{cases} \theta_0'=\theta_0 \\ \theta_1'=\theta_1+1 \\ \theta_t'=\theta_t \\ \theta_\infty'=\theta_\infty+1, \end{cases} \quad R_{(5)}(z)=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}(z-1)+\begin{pmatrix}1 & -w_1\\ -r_1 & w_1r_1\end{pmatrix}, \tag{4.14}$$

$$\begin{cases} \theta_0'=\theta_0 \\ \theta_1'=\theta_1-1 \\ \theta_t'=\theta_t \\ \theta_\infty'=\theta_\infty-1, \end{cases} \quad R_{(6)}(z)=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\dfrac{u_1+\theta_1}{u_0w_0}r_2 & -r_2\\ -\dfrac{u_1+\theta_1}{u_1w_1} & 1\end{pmatrix}\frac{1}{z-1}, \tag{4.15}$$

$$\begin{cases} \theta_0'=\theta_0 \\ \theta_1'=\theta_1-1 \\ \theta_t'=\theta_t \\ \theta_\infty'=\theta_\infty+1, \end{cases} \quad R_{(7)}(z)=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}1 & -\dfrac{u_1w_1}{u_1+\theta_1}\\ -r_1 & \dfrac{u_1w_1}{u_1+\theta_1}r_1\end{pmatrix}\frac{1}{z-1}, \tag{4.16}$$

$$\begin{cases} \theta_0'=\theta_0 \\ \theta_1'=\theta_1+1 \\ \theta_t'=\theta_t \\ \theta_\infty'=\theta_\infty-1, \end{cases} \quad R_{(8)}(z)=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}(z-1)+\begin{pmatrix}\dfrac{r_2}{w_1} & -r_2\\ -\dfrac{1}{w_1} & 1\end{pmatrix}, \tag{4.17}$$

$$\begin{cases} \theta_0'=\theta_0 \\ \theta_1'=\theta_1 \\ \theta_t'=\theta_t+1 \\ \theta_\infty'=\theta_\infty+1, \end{cases} \quad R_{(9)}(z)=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}(z-t)+\begin{pmatrix}1 & -w_t\\ -r_1 & w_tr_1\end{pmatrix}, \tag{4.18}$$

$$\begin{cases} \theta_0'=\theta_0 \\ \theta_1'=\theta_1 \\ \theta_t'=\theta_t-1 \\ \theta_\infty'=\theta_\infty-1, \end{cases} \quad R_{(10)}(z)=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\dfrac{u_t+\theta_t}{u_tw_t}r_2 & -r_2\\ -\dfrac{u_t+\theta_t}{u_tw_t} & 1\end{pmatrix}\frac{1}{z-t}, \tag{4.19}$$

$$\begin{cases} \theta_0'=\theta_0 \\ \theta_1'=\theta_1 \\ \theta_t'=\theta_t-1 \\ \theta_\infty'=\theta_\infty+1, \end{cases} \quad R_{(11)}(z)=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}+\begin{pmatrix} 1 & -\dfrac{u_t w_t}{u_t+\theta_t} \\ -r_1 & \dfrac{u_t w_t}{u_t+\theta_t} r_1 \end{pmatrix}\frac{1}{z-t}, \tag{4.20}$$

$$\begin{cases} \theta_0'=\theta_0 \\ \theta_1'=\theta_1 \\ \theta_t'=\theta_t+1 \\ \theta_\infty'=\theta_\infty-1, \end{cases} \quad R_{(12)}(z)=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(z-t)+\begin{pmatrix} \dfrac{r_2}{w_t} & -r_2 \\ -\dfrac{1}{w_t} & 1 \end{pmatrix}, \tag{4.21}$$

where

$$r_1=-\frac{1}{1+\theta_\infty}\left(\frac{u_1+\theta_1}{w_1}+\frac{u_t+\theta_t}{w_t}t\right), \quad r_2=\frac{1}{1-\theta_\infty}(w_1u_1+tw_tu_t) \tag{4.22}$$

and u_i, w_i, $i=0,1,t$ are given in Eq. (2.6). The transformation matrices $R_{(k)}(z)$, $k=1,2,\ldots,12$ are sufficient to obtain the transformation matrix $R(z)$ which shifts the exponents θ_0, θ_1, θ_t, θ_∞ to θ_0', θ_1', θ_t', θ_∞' with any integer differences. If

$$Y'(z,t;u_0',u_1',u_t',w_0',w_1',w_t',\theta_0',\theta_1,\theta_t',\theta_\infty')=R_{(j)}(z,t;u_0,\ldots,\theta_\infty)Y(z,t;u_0,\ldots,\theta_\infty) \tag{4.23}$$

and

$$Y''(z,t;u_0'',u_1'',u_t'',w_0'',w_1'',w_t'',\theta_0'',\theta_1'',\theta_t'',\theta_\infty'')=R_{(k)}(z,t;u_0',\ldots,\theta_\infty')Y(z,t;u_0',\ldots,\theta_\infty') \tag{4.24}$$

then

$$R_{(k)}(z,t;u_0'(u_0,\ldots,\theta_\infty),\ldots)R_{(j)}(z,t;u_0,\ldots,\theta_\infty)=I \tag{4.25}$$

for $k=j+1$, $j=1,3,5,7,9,11$.

Also, $R_{(3)}(z)R_{(6)}(z)=R_{(3,6)}(z)$ shifts the exponents as $\theta_0' = \theta_0 - 1$, $\theta_1' = \theta_1 + 1$, $\theta_t' = \theta_t$, $\theta_\infty' = \theta_\infty$, $R_{(4)}(z)R_{(8)}(z)=R_{(4,8)}(z)$ shifts the exponents as, $\theta_0' = \theta_0 + 1$, $\theta_1' = \theta_1 - 1$, $\theta_t' = \theta_t$, $\theta_\infty' = \theta_\infty$, and $R_{(1)}(z)R_{(7)}(z)=R_{(1,7)}(z)$ shifts the exponents as, $\theta_0' = \theta_0 + 1$, $\theta_1' = \theta_1 + 1$, $\theta_t' = \theta_t$, $\theta_\infty' = \theta_\infty$. The explicit forms of $R_{(3,6)}$, $R_{(4,8)}$, and $R_{(1,7)}$ are

$$\begin{cases} \theta_0'=\theta_0-1 \\ \theta_1'=\theta_1+1 \\ \theta_t'=\theta_t \\ \theta_\infty'=\theta_\infty, \end{cases} \quad R_{(3,6)}(z)=I+\frac{1}{w_1(u_0+\theta_0)-u_0w_0}\begin{pmatrix} -w_1(u_0+\theta_0) & w_1w_0u_0 \\ -(u_0+\theta_0) & u_0w_0 \end{pmatrix}\frac{1}{z}, \tag{4.26}$$

$$\begin{cases} \theta_0'=\theta_0+1 \\ \theta_1'=\theta_1-1 \\ \theta_t'=\theta_t \\ \theta_\infty'=\theta_\infty, \end{cases} \quad R_{(4,8)}(z)=I+\frac{1}{w_0(u_1+\theta_1)-u_1w_1}\begin{pmatrix} -w_0(u_1+\theta_1) & w_1w_0u_1 \\ -(u_1+\theta_1) & u_1w_1 \end{pmatrix}\frac{1}{z-1}, \tag{4.27}$$

$$\begin{cases} \theta_0' = \theta_0 + 1 \\ \theta_1' = \theta_1 + 1 \\ \theta_t' = \theta_t \\ \theta_\infty' = \theta_\infty , \end{cases} \quad R_{(1,7)}(z) = Iz + \frac{1}{w_1 - w_0} \begin{pmatrix} -w_1 & w_1 w_0 \\ -1 & w_0 \end{pmatrix}. \tag{4.28}$$

V. TRANSFORMATIONS FOR PVI

The linear equation (2.2a) is transformed under the Schlesinger transformations defined by the transformation matrices $R_{(k)}(z)$, $k=1,2,\ldots,12$ as follows:

$$\frac{\partial Y'}{\partial z} = A'(z)Y, \tag{5.1a}$$

$$A'(z) = \left[R_{(k)}(z)A(z) + \frac{\partial R_{(k)}}{\partial z} \right] R_{(k)}^{-1}(z). \tag{5.1b}$$

Equation (5.1b) gives the relation between u_i, w_i and the transformed quantities u_i', w_i', $i=0,1,t$. From these relations the transformation between the solution $y(t)$ for the parameters $\alpha,\beta,\gamma,\delta$ and the solution $y'(t)$ for the parameters $\alpha',\beta',\gamma',\delta'$ of PVI can be obtained using Eq. (2.5e)

$$y' = \frac{t u_0' w_0'}{k'}. \tag{5.2}$$

The transformations between the solutions of PVI obtained via the Schlesinger transformation matrices $R_k(z)$, $k=1,2,\ldots,12$ may be listed as follows:

$$R_{(1)}(z) \quad : u_0' w_0' = w_0 \left[(w_1 - w_0) \left(\frac{u_1 + \theta_1}{w_1} - \frac{w_1}{w_0} \right) + \frac{1}{t} (w_t - w_0) \left(\frac{u_t + \theta_t}{w_t} - \frac{u_t}{w_0} \right) \right],$$

$$k' = -\theta_\infty w_0, \tag{5.3}$$

$$\alpha' = \tfrac{1}{2}[(2\alpha)^{1/2} + 1]^2, \quad \beta' = -\tfrac{1}{2}[(-2\beta)^{1/2} + 1]^2, \quad \gamma' = \gamma, \quad \delta' = \delta.$$

$$R_{(2)}(z): \quad u_0' w_0' = (\theta_0 - 1) r_2 + \left[u_1 w_1 \left(\frac{u_1 + \theta_1}{u_1 w_1} - \frac{u_0 + \theta_0}{u_0 w_0} \right) \left(\frac{u_0 + \theta_0}{u_0 w_0} - \frac{1}{w_1} \right) \right.$$
$$\left. + \frac{u_t w_t}{t} \left(\frac{u_t + \theta_t}{u_t w_t} - \frac{u_0 + \theta_0}{u_0 w_0} \right) \left(\frac{u_0 + \theta_0}{u_0 w_0} - \frac{1}{w_t} \right) \right] r_2^2,$$

$$k' = (t-1) u_1 w_1 + \left[\theta_1 + t(\theta_0 - \theta_1 - 1) + 2(t-1) u_1 w_1 \left(\frac{u_0 + \theta_0}{u_0 w_0} - \frac{1}{w_1} \right) \right] r_2 - \theta_\infty \frac{u_0 + \theta_0}{u_0 w_0} r_2^2, \tag{5.4}$$

$$\alpha' = \tfrac{1}{2}[(2\alpha)^{1/2} - 1]^2, \quad \beta' = -\tfrac{1}{2}[(-2\beta)^{1/2} - 1]^2, \quad \gamma' = \gamma, \quad \delta' = \delta.$$

$$R_{(3)}(z): \quad u_0' w_0' = \frac{u_0 w_0}{w_1} \left(\frac{u_0 w_0}{u_0 + \theta_0} - w_1 \right) \left(\frac{u_1 w_1}{u_0 w_0} - \frac{u_1 + \theta_1}{u_0 + \theta_0} \right)$$
$$+ \frac{1}{t} \frac{u_0 w_0}{w_t} \left(\frac{u_0 w_0}{u_0 + \theta_0} - w_t \right) \left(\frac{u_t w_t}{u_0 w_0} - \frac{u_t + \theta_t}{u_0 + \theta_0} \right),$$

$$k'=-\theta_\infty\frac{u_0w_0}{u_0+\theta_0}, \tag{5.5}$$

$$\alpha'=\tfrac{1}{2}[(2\alpha)^{1/2}+1]^2,\quad \beta'=-\tfrac{1}{2}[(-2\beta)^{1/2}-1]^2,\quad \gamma'=\gamma,\quad \delta'=\delta.$$

$$R_{(4)}(z):\quad u_0'w_0'=-(\theta_0+1)r_2+\left[\left(\frac{u_1+\theta_1}{w_1}-\frac{u_1}{w_0}\right)\left(\frac{w_1}{w_0}-1\right)+\frac{1}{t}\left(\frac{u_t+\theta_t}{w_t}-\frac{u_t}{w_0}\right)\left(\frac{w_t}{w_0}-1\right)\right]r_2^2,$$

$$k'=-tu_0w_0-\left[\theta_0+\theta_t+1+(\theta_0+\theta_1+1)t+2\frac{u_t}{w_0}(w_0-w_t)+2t\frac{u_1}{w_0}(w_0-w_1)\right]r_2-\theta_\infty\frac{1}{w_0}r_2^2, \tag{5.6}$$

$$\alpha'=\tfrac{1}{2}[(2\alpha)^{1/2}-1]^2,\quad \beta'=-\tfrac{1}{2}[(-2\beta)^{1/2}+1]^2,\quad \gamma'=\gamma,\quad \delta'=\delta.$$

$$R_{(5)}(z):\quad u_0'w_0'=w_1(w_0-w_1)\left(\frac{u_0+\theta_0}{w_0}-\frac{u_0}{w_1}\right),$$

$$k'=-\theta_\infty w_1, \tag{5.7}$$

$$\alpha'=\tfrac{1}{2}[(2\alpha)^{1/2}+1]^2,\quad \beta'=\beta,\quad \gamma'=\tfrac{1}{2}[(2\gamma)^{1/2}+1]^2,\quad \delta'=\delta.$$

$$R_{(6)}(z):\quad u_0'w_0'=-u_0w_0-\left[\theta_0-2u_0w_0\left(\frac{u_1+\theta_1}{u_1w_1}-\frac{1}{w_0}\right)\right]r_2+u_0w_0\times\left(\frac{u_0+\theta_0}{u_0w_0}-\frac{u_1+\theta_1}{u_1w_1}\right)\left(\frac{u_1+\theta_1}{u_1w_1}-\frac{1}{w_0}\right)r_2^2,$$

$$k'=-tu_0w_0-\left[\theta_1-1+(1-\theta_0+\theta_1)t-2tu_0w_0\left(\frac{u_1+\theta_1}{u_1w_1}-\frac{1}{w_0}\right)\right]r_2-\theta_\infty\frac{u_1+\theta_1}{u_1w_1}r_2^2, \tag{5.8}$$

$$\alpha'=\tfrac{1}{2}[(2\alpha)^{1/2}-1]^2,\quad \beta'=\beta,\quad \gamma'=\tfrac{1}{2}[(2\gamma)^{1/2}-1]^2,\quad \delta'=\delta.$$

$$R_{(7)}(z):\quad u_0'w_0'=\frac{u_1w_1}{w_0}\left(\frac{u_0+\theta_0}{u_1+\theta_1}-\frac{u_0w_0}{u_1w_1}\right)\left(w_0-\frac{u_1w_1}{u_1+\theta_1}\right),$$

$$k'=-\theta_\infty\frac{u_1w_1}{u_1+\theta_1}, \tag{5.9}$$

$$\alpha'=\tfrac{1}{2}[(2\alpha)^{1/2}+1]^2,\quad \beta'=\beta,\quad \gamma'=\tfrac{1}{2}[(2\gamma)^{1/2}-1]^2,\quad \delta'=\delta.$$

$$R_{(8)}(z):\quad u_0'w_0'=-u_0w_0-\left[\theta_0-2u_0\left(\frac{w_0}{w_1}-1\right)\right]r_2+\left(\frac{u_0+\theta_0}{w_0}-\frac{u_0}{w_1}\right)\left(\frac{w_0}{w_1}-1\right)r_2^2,$$

$$k'=-tu_0w_0+\left[\theta_1+1-t(\theta_0+\theta_1+1)+2tu_0\left(\frac{w_0}{w_1}-1\right)\right]r_2-\frac{\theta_\infty}{w_1}r_2^2, \tag{5.10}$$

$$\alpha'=\tfrac{1}{2}[(2\alpha)^{1/2}-1]^2,\quad \beta'=\beta,\quad \gamma'=\tfrac{1}{2}[(2\gamma)^{1/2}+1]^2,\quad \delta'=\delta.$$

$$R_{(9)}(z):\quad u_0'w_0'=w_t(w_0-w_t)\left(\frac{u_0+\theta_0}{w_0}-\frac{u_0}{w_t}\right),$$

$$k' = -\theta_\infty w_t, \tag{5.11}$$

$$\alpha' = \tfrac{1}{2}[(2\alpha)^{1/2}+1]^2, \quad \beta'=\beta, \quad \gamma'=\gamma, \quad \delta' = \tfrac{1}{2} - \tfrac{1}{2}[(1-2\delta)^{1/2}+1]^2.$$

$$R_{(10)}(z):\quad u_0'w_0' = -tu_0w_0 - \left[\theta_0 - 2u_0w_0\left(\frac{u_t+\theta_t}{u_tw_t} - \frac{1}{w_0}\right)\right]r_2 + \frac{u_0w_0}{t}\left(\frac{u_t+\theta_t}{u_tw_t} - \frac{1}{w_0}\right)$$
$$\times\left(\frac{u_0+\theta_0}{u_0w_0} - \frac{u_t+\theta_t}{u_tw_t}\right)r_2^2,$$

$$k' = (t-1)u_1w_1 + \left[\theta_1 - (\theta_t - 1)t - 2u_1w_1\left(\frac{u_t+\theta_t}{u_tw_t} - \frac{1}{w_1}\right)\right]r_2 - \theta_\infty\frac{u_t+\theta_t}{u_tw_t}r_2^2, \tag{5.12}$$

$$\alpha' = \tfrac{1}{2}[(2\alpha)^{1/2}-1]^2, \quad \beta'=\beta, \quad \gamma'=\gamma, \quad \delta' = \tfrac{1}{2} - \tfrac{1}{2}[(1-2\delta)^{1/2}-1]^2.$$

$$R_{(11)}(z):\quad u_0'w_0' = \frac{1}{t}\left[\frac{u_tw_t}{u_t+\theta_t}(2u_0+\theta_0) - \frac{1}{w_0}\left(\frac{u_tw_t}{u_t+\theta_t}\right)^2(u_0+\theta_0) - u_0w_0\right],$$

$$k' = -\theta_\infty\frac{u_tw_t}{u_t+\theta_t}, \tag{5.13}$$

$$\alpha' = \tfrac{1}{2}[(2\alpha)^{1/2}+1]^2, \quad \beta'=\beta, \quad \gamma'=\gamma, \quad \delta' = \tfrac{1}{2} - \tfrac{1}{2}[(1-2\delta)^{1/2}-1]^2,$$

$$R_{(12)}(z):\quad u_0'w_0' = -tu_0w_0 - \left[\theta_0 - 2u_0\left(\frac{w_0}{w_t} - 1\right)\right]r_2 + \frac{1}{t}\left(\frac{w_0}{w_t} - 1\right)\left(\frac{u_0+\theta_0}{w_0} - \frac{u_0}{w_t}\right)r_2^2,$$

$$k' = -tu_0w_0 - \left[\theta_0 + \theta_t + 1 - (\theta_t+1)t - 2u_0\left(\frac{w_0}{w_t} - 1\right)\right]r_2 - \frac{\theta_\infty}{w_t}r_2^2, \tag{5.14}$$

$$\alpha' = \tfrac{1}{2}[(2\alpha)^{1/2}-1]^2, \quad \beta'=\beta, \quad \gamma'=\gamma, \quad \delta' = \tfrac{1}{2} - \tfrac{1}{2}[(1-2\delta)^{1/2}+1]^2,$$

where u_i, w_i, $i=0,1,t$ and r_1, r_2 are given in the Eqs. (2.6) and (4.22), respectively.

It is well known[9,10] that PVI admit one parameter family of solutions characterized by the Riccati type equation which can be reduced to hypergeometric equation via a suitable transformation. It is possible to obtain the Riccati type equation associated with PVI from all transformations (5.3) and (5.14). For example, the transformation between the solutions y and y' of PVI for the parameters $\alpha,\beta,\gamma,\delta$ and α', β', γ', δ', respectively, obtained from $R_{(9)}(z)$ [Eq. (5.11)], for $u_0 \neq 0$, $u_t \neq 0$ is as follows:

$$y' = \frac{y-t}{\theta_\infty(t-1)u_t}[(y-1)\bar{u} - \kappa_2][y(y-1)\bar{u} - \kappa_2 y + \theta_\infty];$$
$$\alpha' = \tfrac{1}{2}[(2\alpha)^{1/2}+1]^2, \quad \beta'=\beta, \quad \gamma'=\gamma, \quad \delta' = \tfrac{1}{2} - \tfrac{1}{2}[(1-2\delta)^{1/2}+1]^2, \tag{5.15}$$

where $\bar{u}$ and κ_2 are given in Eq. (2.4e) and (2.4b), respectively. The transformation (5.15) breaks down iff $u_t=0$, then one should also require that $\bar{u}=0$ and $\kappa_2=0$. Hence, setting $\bar{u}=0$ in Eq. (2.4e) and using Eq. (2.7a) gives

$$t(t-1)\frac{dy}{dt} = (1-\theta_\infty)y^2 - [\theta_0 + \theta_t + 1 + (\theta_0+\theta_1)t]y + \theta_0 t \tag{5.16}$$

and

$$\theta_0+\theta_1+\theta_t+\theta_\infty=0. \tag{5.17}$$

Equation (5.16) can be transformed to a hypergeometric equation; if

$$y=\frac{t(t-1)}{\theta_\infty-1}\frac{dv/dt}{v} \tag{5.18}$$

then $v(t)$ satisfies a certain hypergeometric equation. It should be noted that, this is not the only choice to obtain the Riccati type equation which gives the one parameter family of solutions of PVI. Since, if one removes both restrictions $u_0 \neq 0$, $u_t \neq 0$, then Eq. (2.5b) implies either $u_1=0$ or $w_1=0$. If $u_1=0$ one obtains Eqs. (5.16) and (5.17). When $w_1=0$, one should require $u_1+\theta_1=0$ [see A_1 in Eq. (2.3)]. Thus substituting $u_0=u_t=w_1=0$ and $u_1+\theta_1=0$ in Eqs. (2.4e), (2.7e), and (2.5a) yields

$$t(t-1)\frac{dy}{dt}=(1-\theta_\infty)y^2-[\theta_0+\theta_t+1+(\theta_0-\theta_1)t]y+\theta_0 t \tag{5.19}$$

and

$$\theta_0+\theta_t+\theta_\infty-\theta_1=0. \tag{5.20}$$

If one removes the restriction on u_0 only, i.e., $u_0=0$, $u_t \neq 0$, then Eqs. (2.5d) and (2.5e) imply that either $u_1=0$ or $w_1=0$. For the case of $u_0=u_1=0$, $u_t \neq 0$, Eq. (2.5b) implies $w_t=0$; then one should require $u_t+\theta_t=0$ or from Eq. (2.5a) $\kappa_2+\theta_t=0$. Hence, by using these in Eqs. (2.4d), (2.7), and (2.4e), one gets

$$t(t-1)\frac{dy}{dt}=(1-\theta_\infty)y^2-[\theta_0-\theta_t+1+(\theta_0+\theta_1)t]y+\theta_0 t \tag{5.21}$$

and

$$\theta_0+\theta_1+\theta_\infty-\theta_t=0. \tag{5.22}$$

When $u_0=w_1=w_t=0$, $u_1+\theta_1=0$, $u_t+\theta_t=0$ one obtains

$$t(t-1)\frac{dy}{dt}=(1-\theta_\infty)y^2-[\theta_0-\theta_t+1+(\theta_0-\theta_1)t]y+\theta_0 t \tag{5.23}$$

and

$$\theta_0-\theta_1-\theta_t+\theta_\infty=0. \tag{5.24}$$

Similarly, for $u_0 \neq 0$, $u_t=0$

$$\begin{gathered} t(t-1)\frac{dy}{dt}=(1-\theta_\infty)y^2-[\theta_t-\theta_0+1-(\theta_0-\theta_1)t]y-\theta_0 t, \\ \theta_1+\theta_t+\theta_\infty-\theta_0=0, \end{gathered} \tag{5.25}$$

which follows from $u_1=u_t=w_0=0$, $u_0+\theta_0=0$, and

$$t(t-1)\frac{dy}{dt}=(1-\theta_\infty)y^2-[\theta_t-\theta_0+1-(\theta_0+\theta_1)t]y-\theta_0 t,$$
$$\theta_0+\theta_1-\theta_t-\theta_\infty=0, \tag{5.26}$$

which follows from $u_0=u_t=w_1=0$, $u_1+\theta_1=0$.

One can obtain infinite hierarchies of elementary solutions of PVI by using the transformations (5.3) and (5.14). But it should be noticed that one should start with the solution $y(t)$ of PVI for the parameters $\alpha,\beta,\gamma,\delta$ $(\theta_\infty,\theta_0,\theta_1,\theta_t)$ such that θ_j, $j=0,1,t,\infty$ should not satisfy certain conditions under which PVI can be reduced to a Riccati type equation, since, under these restrictions on θ_j, $j=0,1,t,\infty$ the transformations break down. One can avoid these restrictions, first by using the Lie-point discrete symmetries

$$y'(t;\alpha',\beta',\gamma',\delta')=ty\left(\frac{1}{t};\alpha,\beta,\gamma,\delta\right),$$
$$\alpha'=\alpha,\quad \beta'=\beta,\quad \gamma'=-\delta+\tfrac{1}{2},\quad \delta'=-\gamma+\tfrac{1}{2}, \tag{5.27}$$

$$y'(t;\alpha',\beta',\gamma',\delta')=1-y(1-t;\alpha,\beta,\gamma,\gamma,\delta),$$
$$\alpha'=\alpha,\quad \beta'=-\gamma,\quad \gamma'=-\beta,\quad \delta'=\delta, \tag{5.28}$$

$$y'(t;\alpha',\beta',\gamma',\delta')=1-(1-t)y\left(\frac{1}{1-t};\alpha,\beta,\gamma,\delta\right),$$
$$\alpha'=\alpha,\quad \beta'=\delta-\tfrac{1}{2},\quad \gamma'=-\beta,\quad \delta'=-\gamma+\tfrac{1}{2} \tag{5.29}$$

or the transformation given in Ref. 10 to obtain the new solution and then use the transformations (5.3) and (5.14). For example, if one starts with the solution[10]

$$y(t)=\frac{t(ct^2-2ct+c-1)}{2ct^3-3ct^2+c-1},\quad c \quad \text{is an arbitrary constant},$$
$$\alpha=\tfrac{9}{2},\quad \beta=-\tfrac{1}{2},\quad \gamma=12,\quad \delta=\tfrac{1}{2} \tag{5.30}$$

then the transformation (5.11) yields

$$y'(t)=\frac{t(ct^3-3ct^2+3ct-3t-c+1)}{2(ct^4-2ct^3+2ct-2t-c+1)},$$
$$\alpha'=8,\quad \beta'=-\tfrac{1}{2},\quad \gamma'=\tfrac{1}{2},\quad \delta'=0. \tag{5.31}$$

Using (5.31) in transformation (5.11) gives

$$y''(t)=\frac{t(ct^4-4ct^3+6ct^2-6t^2-4ct+4t-1)}{2ct^5-5ct^4+10ct^2-10t^2-10ct+10t+3c-3};$$
$$\alpha''=\tfrac{25}{2},\quad \beta''=-\tfrac{1}{2},\quad \gamma''=\tfrac{1}{2},\quad \delta''=-\tfrac{3}{2}. \tag{5.32}$$

It can be verified that $y'(t)$ and $y''(t)$ satisfy PVI. Hence, one can generate infinitely many distinct exact solutions of PVI by using the transformations (5.3) and (5.14). Also, it should be noticed that the consecutive application of the transformations generated by $R_{(k)}$ and $R_{(j)}$, $k=j+1$, $j=1,3,5,7,9,11$ yields the identity.

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