



A Global Error Bound for Quadratic Perturbation of Linear Programs

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Abstract—We prove a global error bound result on the quadratic perturbation of linear programs. The error bound is stated in terms of function values. © 2002 Elsevier Science Ltd. All rights reserved.

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The quadratic perturbation of linear programs is intimately related to the quadratic penalty functions applied to the linear program. More precisely, the quadratic perturbation is obtained in the primal problem if a quadratic penalty function is applied to the dual. The above observation was made and pursued in a series of papers by Mangasarian [1,2] and Li [3]. The application of quadratic penalty functions to linear programs was also studied by Pinar [4]. In a recent paper [5], Tseng derived a local error bound result for perturbation of linear programs. In the present note, we give a simple, global error bound result in Theorem 1 for the quadratic perturbation of linear programs. The result is inspired by early work of Güler [6] on the global convergence estimates of augmented Lagrangian algorithms on linear programs. It is given in terms of function values.

Consider the linear program

$$\min_x \{c^\top x \mid Ax = b, x \geq 0\} \quad (1)$$

with its dual

$$\max_y \{-b^\top y \mid A^\top y + c \geq 0\}. \quad (2)$$

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We use the quadratic penalty function

$$\min_y \Phi(y, t) \equiv tb^\top y + \frac{1}{2} r^\top(y) W(y) r(y), \quad (3)$$

where t is a positive scalar, $W(y)$ is a diagonal matrix with diagonal entries defined as

$$W_{ii}(y) = \begin{cases} 0, & \text{if } r_i(y) \geq 0, \\ 1, & \text{if } r_i(y) < 0, \end{cases}$$

and $r(y) = A^\top y + c$. The dual of this problem is precisely the perturbation problem

$$\min_x \left\{ c^\top x + \frac{t}{2} x^\top x \mid Ax = b, x \geq 0 \right\}. \quad (4)$$

A solution y_t of the quadratic penalty problem (3) satisfies the following identity:

$$(AW(y_t)A^\top) y_t = -AW(y_t)c - tb. \quad (5)$$

Now, it was shown in [4] that

$$x_t \equiv -\frac{W(y_t)r(y_t)}{t} = -\frac{W(y_t)(A^\top y_t + c)}{t}$$

solves the perturbation problem (4) if y_t solves the quadratic penalty problem (3), and that $W(y_t)$ is constant for any y_t which is a minimizer of $\Phi(\cdot, t)$. Pinar [4] also shows that $W(y_t)$ behaves as a piecewise linear function of t and there exists $t^* > 0$ such that $W(y_t)$ remains constant for $0 < t \leq t^*$.

The main result of this note is the following theorem.

THEOREM 1. *Let $t^* > 0$ be such that $W(y_t)$ remains constant for $0 < t \leq t^*$. For any $t > 0$ such that $t > t^*$, the following bound holds:*

$$c^\top x_t - \omega^* = \mathcal{O}\left(\frac{1}{t^*} - \frac{1}{t}\right), \quad (6)$$

where x_t solves the perturbation problem (4), and ω^* is the optimal value of problem (1).

We will give the proof of the theorem after establishing some useful facts. We use $N(B)$ and $R(B)$ to denote the null space and range of a matrix B , respectively.

LEMMA 1. *If the system $Bx = b$ is consistent, then $b \in R(BB^\top)$.*

PROOF. Consider the problem $\min \{\|x\|^2 : Bx = b\}$. The optimal solution satisfies $x^* = B^\top y$ for some y , and thus, $b = Bx^* = BB^\top y \in R(BB^\top)$. ■

Incidentally, Lemma 1 proves that $R(B) = R(BB^\top)$ for any matrix B .

LEMMA 2. *If $u = v - w$ such that v and w are orthogonal, then $u^\top w = -\|w\|^2$.*

The proof of this lemma is trivial, and is therefore omitted. The following lemma is a standard result in penalty methods, which we include for completeness.

LEMMA 3. *$c^\top x_t$ is decreasing, that is if $0 < t_1 < t_2$, then $c^\top x_{t_1} \leq c^\top x_{t_2}$.*

PROOF. Suppose $t_2 > t_1 > 0$. One has

$$\begin{aligned} c^\top x_{t_2} + \frac{t_2}{2} \|x_{t_2}\|^2 &\leq c^\top x_{t_1} + \frac{t_2}{2} \|x_{t_1}\|^2, \\ c^\top x_{t_1} + \frac{t_1}{2} \|x_{t_1}\|^2 &\leq c^\top x_{t_2} + \frac{t_1}{2} \|x_{t_2}\|^2. \end{aligned}$$

Divide the inequalities by $1/t_2$ and $1/t_1$, respectively. The proof is completed by adding the resulting inequalities and simplifying the results. \blacksquare

Now, equipped with these facts, we can give the proof of Theorem 1. We can write $W_t = W(y_t)$, without ambiguity. Since $x_t = -W_t(A^\top y_t + c)/t$ and $Ax_t = b$,

$$AW_t(A^\top y_t + c) = -tb. \quad (7)$$

Thus, $b \in R(AW_t)$, and Lemma 1 implies that $AW_t(AW_t)^\top d = b$ for some d . Since $W_t^2 = W_t$, we have $AW_t A^\top d = b$. Substituting this in equation (7) and setting

$$\bar{y}_t = y_t + t d \quad (8)$$

gives

$$AW_t(W_t A^\top \bar{y}_t + W_t c) = 0, \quad (9)$$

that is, $W_t A^\top \bar{y}_t + W_t c \in N(AW_t)$. Since $W_t A^\top \bar{y}_t \in R(W_t A^\top) = N(AW_t)^\perp$, we see that

$$W_t c = (W_t A^\top \bar{y}_t + W_t c) - W_t A^\top \bar{y}_t \quad (10)$$

is an orthogonal decomposition of $W_t c$ onto $N(AW_t)$ and its orthogonal complement $R(W_t A^\top)$.

Suppose now that $t_2 > t_1 > 0$ such that $W_{t_2} = W_{t_1} := W$. Then, using the notation $u^\top v = \langle u, v \rangle$, we have

$$\begin{aligned} 0 \leq \langle c, x_{t_2} - x_{t_1} \rangle &= -\frac{1}{t_2} \langle c, W(A^\top y_{t_2} + c) \rangle + \frac{1}{t_1} \langle c, W(A^\top y_{t_1} + c) \rangle \\ &= \left(\frac{1}{t_1} - \frac{1}{t_2} \right) \|Wc\|^2 - \frac{1}{t_2} \langle c, W A^\top y_{t_2} \rangle + \frac{1}{t_1} \langle c, W A^\top y_{t_1} \rangle. \end{aligned} \quad (11)$$

Now, we have

$$\begin{aligned} -\frac{1}{t_2} \langle Wc, W A^\top y_{t_2} \rangle + \frac{1}{t_1} \langle Wc, W A^\top y_{t_1} \rangle &= -\frac{1}{t_2} \langle Wc, W A^\top \bar{y}_{t_2} \rangle + \frac{1}{t_1} \langle Wc, W A^\top \bar{y}_{t_1} \rangle \\ &= \frac{1}{t_2} \|W A^\top \bar{y}_{t_2}\|^2 - \frac{1}{t_1} \|W A^\top \bar{y}_{t_1}\|^2, \end{aligned} \quad (12)$$

where the first equality comes from (8), and the second one from (10) and Lemma 2.

Note that (9) implies that $u := \bar{y}_{t_2} - \bar{y}_{t_1}$ satisfies $AWA^\top u = 0$. But, then $0 = u^\top AWA^\top u = \|WA^\top u\|^2$. Thus, $WA^\top u = 0$. Then, $WA^\top \bar{y}_{t_2} = WA^\top \bar{y}_{t_1}$. This shows that the quantity in (12) is nonpositive. Thus, (11) reduces to

$$0 \leq c^\top x_{t_2} - c^\top x_{t_1} \leq \left(\frac{1}{t_1} - \frac{1}{t_2} \right) \|c\|_2. \quad (13)$$

If $t_2^* > t_2 > t_1 > t_1^*$ where t_2^* and t_1^* are consecutive breakpoints, then $W_{t_2} = W_{t_1}$, and inequality (13) applies. Now, x_t is a minimizer of the optimization problem (4). It is easy to verify that x_t is the projection of the vector $-c/t$ on the feasible set $F := \{x : Ax = b, x \geq 0\}$, that is, x_t is the solution to the problem $\min \{\|x + (c/t)\| : x \in F\}$. The projection operator onto a convex set is nonexpansive, so that $\|x_{t_2} - x_{t_1}\| \leq (1/t_1 - 1/t_2)\|c\|$. This proves that x_t is a continuous function of t when $t > 0$. Consequently, we see that (13) also holds when t_2 and t_1 are replaced by t_2^* and t_1^* , respectively. Since $c^\top x_{t^*} = \omega^*$ (in fact $c^\top x_t = \omega^*$ for all $t \in (0, t^*], [1, 2, 4]$), where ω^* is the optimal value of the original linear program, the proof is completed.

Interestingly Güler [6] first gives a global convergence rate estimate of $\mathcal{O}(1/\sum_{i=0}^{k-1} \lambda_i)$ for the augmented Lagrangian algorithm, where λ_i is the penalty parameter. Then he modifies the multiplier iteration and sharpens the bound to $\mathcal{O}((1/\sum_{i=0}^{k-1} \sqrt{\lambda_i})^2)$. It is interesting that the bound we obtain in the theorem is also linear in the inverse of t .

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