Waring's problem for Beatty sequences and a local to global principle

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Abstract

In this paper, we investigate in various ways the representation of a large natural number N as a sum of s positive k-th powers of numbers from a fixed Beatty sequence. *Inter alia*, a very general form of the local to global principle is established in additive number theory. Although the proof is very short, it depends on a deep theorem of M. Kneser. There are numerous applications.

1 Introduction

The initial motivation for the work described in this memoir was the investigation of a variant of Waring's problem for Beatty sequences. In the process, however, a fundamental version of the local to global principle was established.

Given a set $\mathcal{A}$ of positive integers, the *lower asymptotic density of* $\mathcal{A}$ is the quantity

$$\underline{\mathbf{d}}(\mathcal{A}) = \liminf_{X\to\infty} \frac{\#\mathcal{A}(X)}{X},$$

where $\mathcal{A}(X) = \mathcal{A} \cap [1, X]$. For any natural number s, we denote the *s-fold sumset of* $\mathcal{A}$ by

$$s\mathcal{A} = \underbrace{\mathcal{A} + \cdots + \mathcal{A}}_{s \text{ copies}} = \big\{a_1 + \cdots + a_s : a_1, \ldots, a_s \in \mathcal{A}\big\}.$$

The following very general form of the local to global principle has many applications in additive number theory.

Theorem 1. *Suppose that there are numbers s_1, s_2 such that*

- (i) *For all $s \geqslant s_1$ and $m, n \in \mathbb{N}$, the sumset $s\mathcal{A}$ has at least one element in the arithmetic progression $n \bmod m$;*
- (ii) *The sumset $s_2\mathcal{A}$ has positive lower asymptotic density, i.e., $\underline{\mathbf{d}}(s_2\mathcal{A}) > 0$.*

Then, there is a number s_0 with the property that for any $s \geqslant s_0$ the sumset $s\mathcal{A}$ contains all but finitely many natural numbers.

Although the proof of Theorem 1 is very short (see §2 below), it relies on a deep and remarkable theorem of M. Kneser; see Halberstam and Roth [4, Chapter I, Theorem 18].

Theorem 1 has several interesting consequences. The following result (proved in §3) provides an affirmative answer in many instances to the question as to whether a given set of primes $\mathcal{P}$ is an asymptotic additive basis for $\mathbb{N}$.

Theorem 2. *Let $\mathcal{P}$ be a set of prime numbers with*

$$\liminf_{X\to\infty} \frac{\#\mathcal{P}(X)}{X/\log X} > 0.$$

Suppose that there is a number s_1 such that for all $s \geqslant s_1$ and $m, n \in \mathbb{N}$, the congruence

$$p_1 + \cdots + p_s \equiv n \pmod{m}$$

has a solution with $p_1, \ldots, p_s \in \mathcal{P}$. Then, there is a number s_0 with the property that for any $s \geqslant s_0$ the equation

$$p_1 + \cdots + p_s = N$$

has a solution with $p_1, \ldots, p_s \in \mathcal{P}$ for all but finitely many natural numbers N.

In 1770, Waring [17] asserted without proof that every natural number is the sum of at most four squares, nine cubes, nineteen biquadrates, and so on. In 1909, Hilbert [5] proved the existence of an $s_0(k)$ such that for all $s \geqslant s_0(k)$ every natural number is the sum of at most $s_0(k)$ positive k-th powers. The following result (proved in §3), which we deduce from Theorem 1, can be used to obtain many variants of the Hilbert–Waring theorem.

Theorem 3. *Let $k \in \mathbb{N}$, and let $\mathcal{B}$ be a set of natural numbers with $\underline{\mathbf{d}}(\mathcal{B}) > 0$. Suppose that there is a number s_1 such that for all $s \geqslant s_1$ and $m, n \in \mathbb{N}$, the congruence*

$$b_1^k + \cdots + b_s^k \equiv n \pmod{m}$$

has a solution with $b_1, \ldots, b_s \in \mathcal{B}$. Then, there is a number s_0 with the property that for any $s \geqslant s_0$ the equation

$$b_1^k + \cdots + b_s^k = N$$

has a solution with $b_1, \ldots, b_s \in \mathcal{B}$ for all but finitely many natural numbers N.

Our work in the present paper was originally motivated by a desire to establish a variant of the Hilbert–Waring theorem with numbers from a fixed Beatty sequence. More precisely, for fixed $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, we studied the problem of representing every sufficiently large natural number N as a sum of s positive k-th powers chosen from the *non-homogeneous Beatty sequence* defined by

$$\mathcal{B}_{\alpha,\beta} = \big\{ n \in \mathbb{N} : n = \lfloor \alpha m + \beta \rfloor \text{ for some } m \in \mathbb{Z} \big\}.$$

Beatty sequences appear in a variety of apparently unrelated mathematical settings, and the arithmetic properties of these sequences have been extensively explored in the literature. In the case that α is irrational, the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ is distributed evenly over the congruence classes of any fixed modulus. As the congruence

$$x_1^k + \cdots + x_s^k \equiv n \pmod{m}$$

admits an integer solution for all $m, n \in \mathbb{N}$ provided that s is large enough (this follows from the Hilbert–Waring theorem but can be proved directly using Lemmas 2.13 and 2.15 of Vaughan [11] and the Chinese Remainder Theorem; see also Davenport [2, Chapter 5]), it follows that the congruence condition of Theorem 3 is easily satisfied. Since we also have $\underline{\mathbf{d}}(\mathcal{B}_{\alpha,\beta}) = \alpha^{-1} > 0$, Theorem 3 yields the following corollary.

Corollary 1. *Fix $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and suppose that α is irrational. Then, there is a number s_0 with the property that for any $s \geqslant s_0$ the equation*

$$b_1^k + \cdots + b_s^k = N$$

has a solution with $b_1, \ldots, b_s \in \mathcal{B}_{\alpha,\beta}$ for all but finitely many natural numbers N.

Of course, the value of s_0 depends on α and *a priori* could be inordinately large for general α. However, by utilising the power of the Hardy–Littlewood method we obtain the asymptotic formula for the number of solutions and show the existence of some solutions for a reasonably small value of s_0 that depends only on k.

Theorem 4. *Fix $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and suppose that α is irrational. Suppose further that $k \geqslant 2$ and that*

$$s \geqslant \begin{cases} 2^k + 1 & \text{if } 2 \leqslant k \leqslant 5, \\ 57 & \text{if } k = 6, \\ 2k^2 + 2k - 1 & \text{if } k \geqslant 7. \end{cases}$$

Then, the number $R(N)$ of representations of N as a sum of s positive k-th powers of members of the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ satisfies

$$R(N) \sim \alpha^{-s}\Gamma(1 + 1/k)^s\Gamma(s/k)^{-1}\mathfrak{S}(N)N^{s/k-1} \qquad (N \to \infty),$$

where $\mathfrak{S}(N)$ is the singular series in the classical Waring's problem.

By [11, Theorems 4.3 and 4.6] the singular series $\mathfrak{S}$ satisfies $\mathfrak{S}(N) \asymp 1$ for the permissible values of s in the theorem.

The lower bound demands on s can be significantly reduced by asking only for the existence of solutions for all large N.

Theorem 5. *Fix $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and suppose that α is irrational. Then, there is a function $H(k)$ which satisfies*

$$H(k) \sim k \log k \qquad (k \to \infty)$$

such that if $k \geqslant 2$ and $s \geqslant H(k)$, then every sufficiently large N can be represented as a sum of s positive k-th powers of members of the Beatty sequence $\mathcal{B}_{\alpha,\beta}$.

In the interests of clarity of exposition, we have made no effort to optimise the methods employed. Certainly many refinements are possible. For instance, in the range $5 \leqslant k \leqslant 20$ it would be possible to give explicit values for the function $H(k)$ by extracting the relevant bounds for Lemma 2 below from Vaughan and Wooley [13, 14, 15, 16], and doubtless the exponent $4k$ of $S(\vartheta)$ can be replaced by 2 with some reasonable effort.

1.1 Notation

The notation $\|x\|$ is used to denote the distance from the real number x to the nearest integer, that is,

$$\|x\| = \min_{n \in \mathbb{Z}} |x - n| \qquad (x \in \mathbb{R}).$$

We denote by $\{x\}$ the fractional part of x. We put $\mathbf{e}(x) = e^{2\pi i x}$ for all $x \in \mathbb{R}$. Throughout the paper, we assume that k and n are natural numbers with $k \geqslant 2$.

For any finite set S, we denote by $\#S$ the number of elements in S.

In what follows, any implied constants in the symbols $\ll$ and O may depend on the parameters $\alpha, \beta, k, s, \varepsilon, \eta$ but are absolute otherwise. We recall that for functions F and G with $G \geqslant 0$ the notations $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \leqslant cG$ holds for some constant $c > 0$. If $F \geqslant 0$ also, then $F \gg G$ is equivalent to $G \ll F$. We also write $F \asymp G$ to indicate that $F \ll G$ and $G \ll F$.

2 The proof of Theorem 1

Let $\delta_s = \underline{\mathbf{d}}(s\mathcal{A})$ for each s. Note that hypothesis (ii) implies that $\delta_s > 0$ for all $s \geqslant s_2$. We now suppose that $s = \max(s_1, s_2)$ and appeal to Kneser's theorem in the form given in [4, §1, Theorem 18]; we conclude that for each $t = 1, 2, \ldots,$ either (case 1) $\delta_{ts} \geqslant t\,\delta_s$ or (case 2) there is a set of integers $\mathcal{A}'$ which is worse than $\mathcal{A}_{ts}$ and degenerate mod g' for some positive integer g' (here, *worse* means that $\mathcal{A}_{ts} \subset \mathcal{A}'$ and that the sets $\mathcal{A}_{ts}$ and $\mathcal{A}'$ coincide from some point onwards, and *degenerate mod* g' means that $\mathcal{A}'$ is a union of residue classes to some modulus g'). Since $\delta_s > 0$ and $\delta_{ts} \leqslant 1$ it follows that case 2 must occur if t is large enough. Let t be fixed with this property. As $ts \geqslant ts_1 \geqslant s_1$, from the definition of s_1 we see that for arbitrary h, m and n the residue class $h + mg'$ mod ng' intersects $\mathcal{A}_{ts}$. By a judicious choice of m and n there will be a sufficiently large element of $\mathcal{A}_{ts}$ in the residue class $h + mg'$ mod ng', and this element will also lie in $\mathcal{A}'$. Clearly, this element also lies in the residue class h mod g'. Since h is arbitrary and $\mathcal{A}'$ is degenerate mod g', it follows that $\mathcal{A}' = \mathbb{Z}$. But $\mathcal{A}_{ts}$ and $\mathcal{A}'$ coincide from some point onwards, and therefore, $\mathcal{A}_{ts}$ contains every sufficiently large positive integer.

3 The proofs of Theorems 2 and 3

For any set $\mathcal{S} \subset \mathbb{N}$, let $R_s(n; \mathcal{S})$ be the number of s-tuples $(a_1, \ldots, a_s)$ with entries in $\mathcal{S}$ for which $a_1 + \cdots + a_s = n$.

To prove Theorem 3 we specialise the set $\mathcal{A}$ in Theorem 1 to be the set of k-th powers of elements of $\mathcal{B}$. Let $\mathcal{A}^*$ denote the set of k-th powers of *all* natural numbers, and suppose that $s > 2^k$. Using Theorem 2.6 and (2.19) of [11] we have

$$R_s(n; \mathcal{A}) \leqslant R_s(n; \mathcal{A}^*) \ll n^{s/k-1}.$$

Also, the hypothesis $\underline{\mathbf{d}}(\mathcal{B}) > 0$ implies that

$$\#\mathcal{A}(N/s) = \#\mathcal{B}((N/s)^{1/k}) \gg (N/s)^{1/k} \gg N^{1/k}$$

provided that $(N/s)^{1/k}$ is no smaller than the least element of $\mathcal{B}$. Thus, if we write $A_s(N) = \#(s\mathcal{A} \cap [1,N])$, then for such N we have

$$N^{s/k} \ll (\#\mathcal{A}(N/s))^s \leqslant \sum_{n=1}^{N} R_s(n;\mathcal{A}) \ll A_s(N) N^{s/k-1}.$$

We can conclude the proof by observing that the congruence condition in Theorem 1 is immediate from that in Theorem 3.

Theorem 2 can be established in the same way. It suffices to show that if $\mathcal{P}^*$ is the set of *all* primes, then for some s we have

$$R_s(n;\mathcal{P}^*) \ll n^{s-1}(\log 2n)^{-s} \qquad (n \in \mathbb{N}).$$

When $s = 3$ this is immediate from Theorem 3 and (3.15) in Chapter 3 of [11], and it would also follow rather easily from a standard application of sieve theory, although none of the standard texts establish the required result explicitly. Alternatively, the standard sieve bound

$$R_2(n;\mathcal{P}^*) \ll \frac{n^2}{\varphi(n)(\log 2n)^2} \qquad (n \in \mathbb{N})$$

(which follows from Halberstam and Richert [3, Corollary 2.3.5], for example) and a simple application of Cauchy's inequality show that $\underline{\mathbf{d}}(2\,\mathcal{P}) > 0$.

4 The generating functions

The rest of this memoir is devoted to the study of the special case of sums of k-th powers of members of a Beatty sequence *via* the Hardy–Littlewood method. Let

$$\mathcal{B}(P) = \big\{n \in \mathcal{B}_{\alpha,\beta} : n \leqslant P\big\} \qquad \text{and} \qquad \mathcal{A}(P,R) = \big\{n \leqslant P : p \mid n \implies p \leqslant R\big\},$$

and put

$$S(\vartheta) = \sum_{n \in \mathcal{B}(P)} e(\vartheta n^k), \qquad T(\vartheta) = \sum_{n \leqslant P} e(\vartheta n^k),$$
$$U(\vartheta) = \sum_{n \in \mathcal{A}(P,R)\cap\mathcal{B}(P)} e(\vartheta n^k), \qquad V(\vartheta) = \sum_{n \in \mathcal{A}(P,R)} e(\vartheta n^k),$$

Lemma 1. *Suppose that t satisfies*

$$t \geqslant \begin{cases} 3 & \text{if } k = 2, \\ 2^{k-1} & \text{if } 3 \leqslant k \leqslant 5, \\ 56 & \text{if } k = 6, \\ 2k^2 + 2k - 2 & \text{if } k \geqslant 7. \end{cases}$$

If F is one of S, U or V, then

$$\int_0^1 |F(\vartheta)|^{2t}\, d\vartheta \leqslant \int_0^1 |T(\vartheta)|^{2t}\, d\vartheta \ll P^{2t-k}.$$

Proof. When $k=2$ the bound on $\int_0^1 |T(\vartheta)|^{2t}\, d\vartheta$ follows from a standard application of the Hardy–Littlewood method, when $k=3$ from Vaughan [8, Theorem 2], when $k=4$ or 5 from Vaughan [9], when $k=6$ from Boklan [1], and when $k \geqslant 7$ from Wooley [18, Corollary 4] and a routine application of the Hardy–Littlewood method. The proof is completed by interpreting each integral as the number of solutions of the diophantine equation

$$x_1^k + \cdots + x_t^k = x_{t+1}^k + \cdots + x_{2t}^k$$

with the x_j lying in $\mathcal{B}(P)$, $\mathbb{N} \cap [1,P]$, $\mathcal{A}(P,R) \cap \mathcal{B}(P)$ or $\mathcal{A}(P,R)$, respectively. □

Lemma 2. *There is a number $\eta > 0$ and a function $H_1(k)$ such that*

$$H_1(k) \sim k \log k \qquad (k \to \infty)$$

with the property that whenever $2t \geqslant H_1(k)$ and $R = P^\eta$ we have

$$\int_0^1 |S(\vartheta)^{4k} U(\vartheta)^{2t}|\, d\vartheta \leqslant \int_0^1 |T(\vartheta)^{4k} V(\vartheta)^{2t}|\, d\vartheta \ll P^{2t+3k}.$$

Proof. In view of Lemma 1, it can be supposed that $k \geqslant k_0$ for a suitable k_0. According to [11, Theorem 12.4] we have

$$\int_0^1 |V(\vartheta)|^{2s}\, d\vartheta \ll P^{\lambda_s + \varepsilon},$$

where

$$\lambda_s = 2s - k + k\exp(1 - 2s/k).$$

Let $\mathfrak{m}$ denote the set of real numbers $\vartheta \in [0,1]$ such that if $|\vartheta - a/q| \leqslant q^{-1}P^{3/4-k}$ with $(a,q)=1$, then $q > P^{3/4}$, and let $\mathfrak{M} = [0,1] \setminus \mathfrak{m}$. Then, by Vaughan [10, Theorem 1.8] we have

$$\sup_{\vartheta \in \mathfrak{m}} |V(\vartheta)| \ll P^{1-\sigma_k+\varepsilon},$$

where

$$\sigma_k = \max_{\substack{n \in \mathbb{N} \\ n \geqslant 2}} \frac{1}{4n}\bigl(1 - (k-2)(1-1/k)^{n-2}\bigr).$$

Note that

$$\sigma_k \sim \frac{1}{4k\log k} \qquad (k \to \infty).$$

We now put

$$s = \lfloor \tfrac{1}{2}k\log k + k\log\log k \rfloor + 1 \qquad \text{and} \qquad t = s + k.$$

Then,

$$\int_{\mathfrak{m}} |V(\vartheta)|^{2t}\, d\vartheta \ll P^{2t-k+\mu_k+\varepsilon},$$

where

$$\mu_k = k\exp(1-2s/k) - 2k\sigma_k < e(\log k)^{-2} - 2k\sigma_k < 0$$

provided that $k > k_0$. Hence

$$\int_{\mathfrak{m}} |T(\vartheta)^{4k} V(\vartheta)^{2t}|\, d\vartheta \ll P^{2t+3k}.$$

By the methods of [11, Chapter 4] we also have

$$\int_{\mathfrak{M}} |T(\vartheta)^{4k} V(\vartheta)^{2t}|\, d\vartheta \ll P^{2t} \int_{\mathfrak{M}} |T(\vartheta)|^{4k}\, d\vartheta \ll P^{2t+3k},$$

and the lemma is proved. □

In what follows, we denote

$$S(q,a) = \sum_{m=1}^{q} e(am^k/q) \qquad \text{and} \qquad I(\phi) = \int_0^P e(\phi x^k)\, dx.$$

Lemma 3. *Suppose that α is irrational. Then, for every real number $P \geqslant 1$ there is a number $Q = Q(P)$ such that*

(i) $Q \leqslant P^{1/2}$;

(ii) $Q \to \infty$ *as* $P \to \infty$;

(iii) Let $\mathfrak{m}$ denote the set of real numbers ϑ with the property that $q > Q$ whenever the inequality $|\vartheta - a/q| \leqslant Qq^{-1}P^{-k}$ holds with $(a,q) = 1$. Then,

$$S(\vartheta) \ll PQ^{-1/k} \qquad (\vartheta \in \mathfrak{m});$$

(iv) If $q \leqslant Q$, $|\vartheta - a/q| \leqslant Qq^{-1}P^{-k}$, and $(a,q) = 1$, then

$$S(\vartheta) = \alpha^{-1} q^{-1} S(q,a) I(\vartheta - a/q) + O(PQ^{-1/k}).$$

Proof. Since $\alpha \notin \mathbb{Q}$, there is at most one pair of integers m, n such that $n = \alpha m + \beta$ and at most one pair such that $n = \alpha m + \beta - 1$. For any other value of n we have

$$n = \lfloor \alpha m + \beta \rfloor \text{ for some } m \qquad \Longleftrightarrow \qquad 1 - \alpha^{-1} < \{\alpha^{-1}(n-\beta)\} < 1.$$

Let $\Psi(x) = x - \lfloor x \rfloor - \frac{1}{2}$ for all $x \in \mathbb{R}$; then Ψ is periodic with period one, and for $x \in [0,1)$ we have

$$\alpha^{-1} + \Psi(x) - \Psi(x+\alpha^{-1}) = \begin{cases} 1 & \text{if } 1-\alpha^{-1} < x < 1, \\ 0 & \text{if } 0 < x < 1-\alpha^{-1}, \\ \frac{1}{2} & \text{if } x = 0 \text{ or } x = 1-\alpha^{-1}. \end{cases}$$

Consequently,

$$S(\vartheta) = \alpha^{-1}T(\vartheta) + \sum_{n \leqslant P} \big(\Psi(\alpha^{-1}(n-\beta)) - \Psi(\alpha^{-1}(n-\beta+1))\big)\, e(\vartheta n^k) + O(1).$$

Now let

$$T(\vartheta,\phi) = \sum_{n \leqslant P} e(\vartheta n^k + \phi n) \tag{4.1}$$

and

$$W(\phi) = \sum_{n \leqslant P} \min\big\{1, H^{-1}\|\alpha^{-1}n - \phi\|^{-1}\big\},$$

where H is a positive parameter to be determined below. By Montgomery and Vaughan [6, Lemma D.1] we have

$$S(\vartheta) = \alpha^{-1}T(\vartheta) - \sum_{0<|h|\leqslant H} \frac{e(\alpha^{-1}(1-\beta)h) - e(-\alpha^{-1}\beta h)}{2\pi i h} T(\vartheta, \alpha^{-1}h)$$
$$+ O\left(1 + W(\alpha^{-1}\beta) + W(\alpha^{-1}(\beta-1))\right).$$

Choose $r = r(P)$ maximal and b so that

$$(b,r) = 1, \qquad |\alpha^{-1} - b/r| \leqslant r^{-2} \qquad \text{and} \qquad r^2|\alpha^{-1} - b/r|^{-1} \leqslant P^{1/4}. \tag{4.2}$$

This is always possible if P is large enough. Indeed, by Dirichlet's theorem on diophantine approximation, or by the theory of continued fractions, there are infinitely many coprime pairs b, r that satisfy the first inequality, and at least one of the pairs will satisfy the second inequality if P is sufficiently large. Moreover, the two inequalities together imply that $r \leqslant P^{1/16}$, so the maximal r exists. Note that $r = r(P)$ tends to infinity as $P \to \infty$ since α is irrational. Let $\xi = \alpha^{-1}r^2 - br$, choose c so that $|\phi r - c| \leqslant \frac{1}{2}$, put $\eta = \phi r - c$, and for every $n \leqslant P$ write $n = ur + v$ with $-r/2 < v \leqslant r/2$ and $0 \leqslant u \leqslant 1 + P/r$. For any given u, let w be an integer closest to $u\xi$, and put $\kappa = u\xi - w$. Then,

$$W(\phi) = \sum_{u,v} \min\big\{1, H^{-1}\|\alpha^{-1}(ur+v) - \phi\|^{-1}\big\}.$$

Moreover,

$$\alpha^{-1}(ur+v) - \phi = ub + \frac{vb + w - c}{r} + \frac{\kappa}{r} + \frac{v\xi}{r^2} - \frac{\eta}{r},$$

and for any given u we have

$$\left\|\alpha^{-1}(ur+v)-\phi\right\| \geqslant \left\|\frac{vb+w-c}{r}\right\| - \frac{3}{2r}.$$

Hence the contribution to W from any fixed u is

$$\ll 1 + H^{-1} r \log r,$$

and so summing over all u we derive the bound

$$W(\phi) \ll Pr^{-1} + PH^{-1}\log r.$$

The choice $H = r^{1/3}$ gives

$$S(\vartheta) = \alpha^{-1}T(\vartheta) - \sum_{0<|h|\leqslant r^{1/3}} \frac{e(\alpha^{-1}(1-\beta)h) - e(-\alpha^{-1}\beta h)}{2\pi i h} T(\vartheta, \alpha^{-1}h) + O\big(Pr^{-1/4}\big). \quad (4.3)$$

The error term here is acceptable provided that $Q \leqslant r^{1/4}$.

Next, we show that the sum over h is also $\ll PQ^{-1}$ provided that $Q = Q(P)$ grows sufficiently slowly. Choose a, q with $(a, q) = 1$ such that $|\vartheta - a/q| \leqslant q^{-1}P^{\frac{1}{2}-k}$ and $q \leqslant P^{k-\frac{1}{2}}$. Then, by [11, Lemma 2.4], when $q > P^{1/2}$ there is a $\delta = \delta(k) > 0$ such that

$$T(\vartheta, \phi) \ll P^{1-\delta} \qquad (\phi \in \mathbb{R}).$$

Since $T(\vartheta) = T(\vartheta, 0)$ and $r \leqslant P^{1/16}$, we derive the bound

$$S(\vartheta) \ll P^{1-\delta}\log P + Pr^{-1/4} \ll PQ^{-1}$$

provided that $Q \leqslant \min\big\{P^{\delta}/\log P, r^{1/4}\big\}$, and we are done in this case.

Now suppose that $q \leqslant P^{1/2}$. We have

$$\begin{aligned} T(\vartheta, \alpha^{-1}h) &= \sum_{m=1}^{q} e(am^k/q) \sum_{\substack{n\leqslant P\\ n\equiv m \,(\mathrm{mod}\ q)}} e((\vartheta - a/q)n^k + \alpha^{-1}hn) \\ &= q^{-1} \sum_{\frac{hq}{\alpha}-\frac{q}{2}<\ell\leqslant\frac{hq}{\alpha}+\frac{q}{2}} S(q, a, \ell) \sum_{n\leqslant P} e\big((\vartheta - a/q)n^k + (\alpha^{-1}h - \ell/q)n\big), \end{aligned}$$

where

$$S(q, a, \ell) = \sum_{m=1}^{q} e(am^k/q + \ell m/q).$$

Let g be the polynomial

$$g(x) = (\vartheta - a/q)x^k + (\alpha^{-1}h - \ell/q)x.$$

For $0 \leqslant x \leqslant P$ and $\frac{hq}{\alpha} - \frac{q}{2} < \ell \leqslant \frac{hq}{\alpha} + \frac{q}{2}$ it is easy to verify that

$$|g'(x)| \leqslant kq^{-1}P^{-1/2} + \tfrac{1}{2} < \tfrac{3}{4}$$

if P is large enough. Hence, by Titchmarsh [7, Lemma 4.8] we see that

$$\sum_{n \leqslant P} e\big((\vartheta - a/q)n^k + (\alpha^{-1}h - \ell/q)n\big) = \int_0^P e(g(x))dx + O(1). \tag{4.4}$$

In the case that $|\alpha^{-1}h - \ell/q| \geqslant 1/(2q)$, we have

$$|g'(x)| \geqslant |\alpha^{-1}h - \ell/q| - kq^{-1}P^{-1/2} \gg |\alpha^{-1}h - \ell/q|,$$

and therefore by [7, Lemma 4.2] the integral in (4.4) is

$$\ll |\alpha^{-1}h - \ell/q|^{-1}.$$

Also, we have trivially $|S(q, a, \ell)| \leqslant q$. Thus, the total contribution to $T(\vartheta, \alpha^{-1}h)$ from the numbers ℓ with $|\alpha^{-1}h - \ell/q| \geqslant 1/(2q)$ is

$$\ll \sum_{\substack{\ell \\ |\alpha^{-1}h - \ell/q| \geqslant 1/(2q)}} |\alpha^{-1}h - \ell/q|^{-1} \ll q \log q,$$

and summing over h with $0 < |h| \leqslant r^{1/3}$ the overall contribution to the sum in (4.3) is

$$\ll q \log q \cdot \log r \ll P^{3/4},$$

which is acceptable.

Next, let ℓ be a number for which $|\alpha^{-1}h - \ell/q| < 1/(2q)$; note that there is at most one such ℓ for each h. Since $(a, q) = 1$, by [11, Theorem 7.1] we have that $S(q, a, \ell) \ll q^{1-1/k+\varepsilon}$. Hence the total contribution to the sum in (4.3) from such an ℓ is $\ll q^{-1/k+\varepsilon} P \log r$. When $q > r^{1/3}$ this is sufficient provided that $Q \leqslant r^{1/4}$. Now suppose that $q \leqslant r^{1/3}$. Since α is irrational and r is large, we have $b \neq 0$ by (4.2), and we claim that $hb/r \neq \ell/q$. Indeed, suppose on the contrary that $hbq = r\ell$. Then $b \mid \ell$, and we can write $\ell = mb$, and $hq = rm$. Since $h \neq 0$, it follows that $m \neq 0$. But this is impossible since $|h|q \leqslant r^{2/3}$, and the claim is proved. Therefore, using (4.2) again, we have

$$|\alpha^{-1}h - \ell/q| = \big|hb/r - \ell/q + h(\alpha^{-1} - b/r)\big| \geqslant \big|hb/r - \ell/q\big| - |h|r^{-2} \geqslant (rq)^{-1} - r^{-5/3} \gg (rq)^{-1}.$$

Arguing as before, we see that $|g'(x)| \gg (rq)^{-1}$, the integral in (4.4) is $\ll rq$, and therefore $T(\vartheta, \alpha^{-1}h) \ll q^{1-1/k+\varepsilon} r$ for each h associated with such an ℓ; hence the total contribution to the sum in (4.3) is

$$\ll q^{1-1/k+\varepsilon} r \log r \ll r^{4/3} \leqslant P^{1/12}.$$

It remains only to deal with the single term

$$\alpha^{-1}T(\vartheta).$$

By [11, Theorem 4.1] we have

$$\alpha^{-1}T(\vartheta) = \alpha^{-1}q^{-1}S(q,a)I(\vartheta - a/q) + O(q),$$

and since $q \leqslant P^{1/2}$ the error term here is acceptable. By [11, Lemma 2.8],

$$I(\vartheta - a/q) \ll \min(P, |\vartheta - a/q|^{-1/k})$$

and by [11, Theorem 4.2] we have

$$S(q,a) \ll q^{1-1/k}.$$

Hence, if $q > Q$ or $|\vartheta - a/q| > Q/(qP^k)$ we see that

$$\alpha^{-1}T(\vartheta) \ll PQ^{-1/k}.$$

The only remaining ϑ to be considered are those for which there exist coprime integers a, q with $q \leqslant Q$ and $|\vartheta - a/q| \leqslant Qq^{-1}P^{-k}$. Thus, we have shown that for all ϑ in $\mathfrak{m}$ the desired bound holds. For the remaining ϑ, we have established that (iv) holds as required. □

For $\varphi \in \mathbb{R}$ and a parameter $A > 1$ at our disposal which will eventually be chosen as a function of ε (only), define

$$f_-(\varphi) = \max\left\{0, (A+1)(1 - 2\alpha\|1 - \tfrac{1}{2\alpha} - \varphi\|)\right\} - \max\left\{0, A - 2\alpha(A+1)\|1 - \tfrac{1}{2\alpha} - \varphi\|\right\},$$
$$f_+(\varphi) = \max\left\{0, A + 1 - 2\alpha A\|1 - \tfrac{1}{2\alpha} - \varphi\|\right\} - \max\left\{0, A(1 - 2\alpha\|1 - \tfrac{1}{2\alpha} - \varphi\|)\right\}.$$

Let

$$S_\pm(\vartheta) = \sum_{n \leqslant P} f_\pm((n - \beta)/\alpha)e(\vartheta n^2). \tag{4.5}$$

The functions $f_\pm$ respectively minorize and majorize the characteristic function of the set $[1 - 1/\alpha, 1] \bmod 1$. Thus, following the discussion in the first paragraph of the proof of Lemma 3, with the choice $P = N^{1/2}$ we have

$$\int_0^1 S_-(\vartheta)^s e(-\vartheta N)d\vartheta \leqslant R(N) \leqslant \int_0^1 S_+(\vartheta)^s e(-\vartheta N)d\vartheta \tag{4.6}$$

in the case that $k = 2$. The functions $f_\pm$ have Fourier expansions

$$f_\pm(\varphi) = \sum_{h=-\infty}^{\infty} c_\pm(h)e(h\varphi) \tag{4.7}$$

whose coefficients are given by

$$c_-(0) = \alpha^{-1}\left(1 - \frac{1}{2(A+1)}\right), \qquad c_+(0) = \alpha^{-1}\left(1 + \frac{1}{2A}\right), \tag{4.8}$$

and for any $h \neq 0$,

$$c_-(h) = \frac{e(\frac{1}{2}\alpha^{-1}h)(A+1)\alpha}{\pi^2 h^2}\left(\cos\frac{\pi\alpha^{-1}hA}{A+1} - \cos\pi\alpha^{-1}h\right),$$
$$c_+(h) = \frac{e(\frac{1}{2}\alpha^{-1}h)A\alpha}{\pi^2 h^2}\left(\cos\pi\alpha^{-1}h - \cos\frac{\pi\alpha^{-1}h(A+1)}{A}\right).$$

Note that

$$c_\pm(h) \ll h^{-2}A\alpha \qquad (h \neq 0). \tag{4.9}$$

Lemma 4. *Suppose that $(a,q) = 1$ and $|\vartheta q - a| \leqslant P^{-1}$. Then*

$$S_\pm(\vartheta) \ll A\alpha\left(\frac{P}{(q + P^2|\vartheta q - a|)^{1/2}} + q^{1/2}\right).$$

Proof. By (4.1), (4.5) and (4.7),

$$S_\pm(\vartheta) = \sum_{h=-\infty}^{\infty} c_\pm(h)e(-h\beta/\alpha)T(\vartheta, h/\alpha).$$

The conclusion then follows from (4.9) and Vaughan [12, Theorem 5]. □

Lemma 5. *Suppose that α is irrational. Then, for every real number $P \geqslant 1$ there is a number $Q = Q(P)$ such that*

(i) $Q \leqslant P^{1/2}$;

(ii) $Q \to \infty$ *as* $P \to \infty$;

(iii) For any coprime integers a, q with $q \leqslant Q$ and $|\vartheta - a/q| \leqslant Qq^{-1}P^{-2}$ we have

$$S_\pm(\vartheta) = c_\pm(0)q^{-1}S(q,a)I(\vartheta - a/q) + O(PQ^{-1/2}).$$

Proof. This can be established in the same way as Lemma 3. □

5 The proofs of Theorems 4 and 5

When $k > 2$, Theorem 4 follows from Lemmas 1 and 3 by a routine application of the Hardy–Littlewood method.

When $k = 2$, let Q be as in Lemma 5. Now define

$$\mathfrak{M}(q,a) = \{\vartheta : |\vartheta - a/q| \leqslant Qq^{-1}P^{-2}\}$$

and let $\mathfrak{M}$ denote the union of the $\mathfrak{M}(q,a)$ with $1 \leqslant a \leqslant q \leqslant Q$ and $(a,q) = 1$. Put $\mathfrak{m} = [QP^{-2}, 1 + QP^{-2}] \setminus \mathfrak{M}$, so that $\mathfrak{m} \subset [QP^{-2}, 1 - QP^{-2})$. Now for any $\vartheta \in \mathfrak{m}$ we choose coprime integers a, q with $1 \leqslant a \leqslant q \leqslant P$ and $|\vartheta - a/q| \leqslant q^{-1}P^{-1}$. Note that, by the definition of $\mathfrak{m}$, we have $|\vartheta - a/q| > q^{-1}P^{-1}$ when $q \leqslant Q$. By Lemma 4, whenever $s \geqslant 5$ we have

$$\begin{aligned}\int_{\mathfrak{m}} |S_\pm(\vartheta)|^s d\vartheta \ll{}& \sum_{q\leqslant Q} q \int_{Qq^{-1}P^{-2}}^{1/(qP)} (A\alpha)^s \left(q^{-s/2}\varphi^{-s/2} + q^{s/2}\right) d\varphi \\ &+ \sum_{Q<q\leqslant P} q \int_0^{1/(qP)} (A\alpha)^s \left(P^s(q + P^2 q\varphi)^{-s/2} + q^{s/2}\right) d\varphi \\ \ll{}& (A\alpha)^s \sum_{q\leqslant Q} \left(P^{s-2}Q^{1-s/2} + P^{-1}q^{s/2}\right) + (A\alpha)^s \sum_{Q<q\leqslant P} \left(q^{1-s/2}P^{s-2} + P^{-1}q^{s/2}\right) \\ \ll{}& (A\alpha)^s \left(Q^{-1/2}P^{s-2} + P^{s/2}\right) \ll \alpha^{-s}P^{s-2}Q^{-1/4}.\end{aligned}$$

Choosing $P = N^{1/2}$, a routine application of Lemma 5 shows that

$$\int_{\mathfrak{M}} S_\pm(\vartheta)^s e(-N\vartheta) d\vartheta = c_\pm(0)\Gamma(3/2)^s\Gamma(s/2)^{-1}\mathfrak{S}(N)N^{s/2-1} + O(N^{s/2-1}Q^{-1/4}).$$

Now suppose that $A = 1/\varepsilon$, where ε is positive but small. Then, by (4.6) and (4.8) it follows that

$$R(N) = \alpha^{-s}\Gamma(3/2)^s\Gamma(s/2)^{-1}\mathfrak{S}(N)N^{s/2-1} + O(\varepsilon N^{s/2-1}) \qquad (N > N_0(\varepsilon)),$$

and this completes the proof of Theorem 4.

To prove Theorem 5 we take $P = N^{1/k}$, R and t as in Lemma 2 and consider the number $R(N)$ of representations of N in the form

$$N = x_1^k + \cdots + x_{4k+1}^k + y_1^k + \cdots + y_{2t}^k$$

with $x_1, \ldots, x_{4k+1} \in \mathcal{B}(P)$ and $y_1, \ldots, y_{2t} \in \mathcal{A}(P,R) \cap \mathcal{B}(P)$. Clearly,

$$R(N) = \int_0^1 S(\vartheta)^{4k+1} U(\vartheta)^{2t} e(-N\vartheta)\, d\vartheta.$$

Let $\mathfrak{M}(q,a)$ denote the set of ϑ with $|\vartheta - a/q| \leqslant Qq^{-1}P^{-k}$, let $\mathfrak{M}$ be the union of all such intervals with $1 \leqslant a \leqslant q \leqslant Q$ and $(a,q)=1$, and put $\mathfrak{m} = (QP^{-k}, 1+QP^{-k}] \setminus \mathfrak{M}$. By Lemmas 2 and 3 we have

$$\int_{\mathfrak{m}} |S(\vartheta)^{4k+1}U(\vartheta)^{2t}|\, d\vartheta \ll P^{3k+2t+1}Q^{-1/k}.$$

Let

$$Z(\vartheta) = \begin{cases} \alpha^{-1}q^{-1}S(q,a)I(\vartheta - a/q) & \text{if } \vartheta \in \mathfrak{M}(q,a), \\ 0 & \text{if } \vartheta \in \mathfrak{m}. \end{cases}$$

Then, by (iv) of Lemma 3 and a routine argument we have

$$\int_{\mathfrak{M}} S(\vartheta)^{4k+1}U(\vartheta)^{2t}e(-N\vartheta)\, d\vartheta = \int_{QP^{-k}}^{1+QP^{-k}} Z(\vartheta)^{4k+1}U(\vartheta)^{2t}e(-N\vartheta)\, d\vartheta + O(P^{3k+2t+1}Q^{-1/k}).$$

By the methods of [11, Chapter 4] we have

$$\int_{QP^{-k}}^{1+QP^{-k}} Z(\vartheta)^{4k+1}e(-m\vartheta)\, d\vartheta = \alpha^{-4k-1}\frac{\Gamma(1+1/k)^{4k+1}}{\Gamma(4+1/k)}m^{3+1/k}\mathfrak{S}(m) + O(P^{3k+1}Q^{-1/k})$$

uniformly for $1 \leqslant m \leqslant N$, and

$$\int_{QP^{-k}}^{1+QP^{-k}} Z(\vartheta)^{4k+1}e(-m\vartheta)\, d\vartheta \ll P^{3k+1}Q^{-1/k}$$

uniformly for $m \leqslant 0$. Here $\mathfrak{S}$ is the usual singular series associated with Waring's problem; note that $\mathfrak{S}(m) \asymp 1$. Therefore,

$$\begin{aligned} &\int_{QP^{-k}}^{1+QP^{-k}} Z(\vartheta)^{4k+1}U(\vartheta)^{2t}e(-N\vartheta)\, d\vartheta \\ &\qquad = \sum_{y_1,\dots,y_{2t}} \alpha^{-4k-1}\frac{\Gamma(1+1/k)^{4k+1}}{\Gamma(4+1/k)}(N - y_1^k - \cdots - y_{2t}^k)^{3+1/k}\mathfrak{S}(N - y_1^k - \cdots - y_{2t}^k) \\ &\qquad\quad + \quad O(P^{3k+2t+1}Q^{-1/k}), \end{aligned}$$

where the sum is taken over those $y_1,\dots,y_{2t} \in \mathcal{B}(P)$ with $(N - y_1^k - \cdots - y_{2t}^k)^{3+1/k} > 0$. By restricting to those $y_1,\dots,y_{2t}$ that do not exceed $P/(4t)$, one sees that

$$R(N) \gg N^{3+1/k+2t/k}$$

if N is sufficiently large, and this completes the proof of Theorem 5.

References

[1] K. D. Boklan, 'The asymptotic formula in Waring's problem', *Mathematika* **41** (1994), no. 2, 329–347.

[2] H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*. Second edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2005.

[3] H. Halberstam and H.-E. Richert, *Sieve methods*. London Mathematical Society Monographs, No. 4. Academic Press, London-New York, 1974.

[4] H. Halberstam and K. F. Roth, *Sequences*. Second edition. Springer-Verlag, New York-Berlin, 1983.

[5] D. Hilbert, 'Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n^{ter} Potenzen (Waringsches Problem)', *Math. Ann.* **67** (1909), no. 3, 281–300.

[6] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory. I. Classical theory*. Cambridge Studies in Advanced Mathematics, **97**. Cambridge University Press, Cambridge, 2007.

[7] E. C. Titchmarsh, *The theory of the Riemann zeta-function*. Second edition. The Clarendon Press, Oxford University Press, New York, 1986.

[8] R. C. Vaughan, 'On Waring's problem for cubes', *J. Reine Angew. Math.* **365** (1986), 122–170.

[9] R. C. Vaughan, 'On Waring's problem for smaller exponents II', *Mathematika* **33** (1986), no. 1, 6–22.

[10] R. C. Vaughan, 'A new iterative method in Waring's problem', *Acta Math.* **162** (1989), no. 1-2, 1–71.

[11] R. C. Vaughan, *The Hardy-Littlewood method*. Second edition. Cambridge Tracts in Mathematics, **125**. Cambridge University Press, Cambridge, 1997.

[12] R. C. Vaughan, 'On generating functions in additive number theory, I', *Analytic Number Theory, Essays in Honour of Klaus Roth*, 436–448, Cambridge Univ. Press, Cambridge, 2009.

[13] R. C. Vaughan and T. D. Wooley, 'Further improvements in Waring's problem, III: Eighth powers', *Philos. Trans. Roy. Soc. London Ser. A* **345** (1993), no. 1676, 385–396.

[14] R. C. Vaughan and T. D. Wooley, 'Further improvements in Waring's problem, II: Sixth powers', *Duke Math. J.* **76** (1994), no. 3, 683–710.

[15] R. C. Vaughan and T. D. Wooley, ‘Further improvements in Waring’s problem’, *Acta Math.* **174** (1995), no. 2, 147–240.

[16] R. C. Vaughan and T. D. Wooley, ‘Further improvements in Waring’s problem, IV: Higher powers’, *Acta Arith.* **94** (2000), no. 3, 203–285.

[17] E. Waring, *Meditationes algebraicæ*. Cambridge, England, 1770.

[18] T. D. Wooley, ‘Vinogradov’s mean value theorem via efficient congruencing’, *Ann. Math.*, to appear.