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Non-equilibrium phase transitions in the two-temperature Ising model with Kawasaki dynamics

Phase diagram from position space renormalization group transformation

B. Renklioglu and M.C. Yalabik[a]

Department of Physics, Bilkent University, Bilkent, 06800 Ankara, Turkey

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Abstract. Phase transitions of the two-finite temperature Ising model on a square lattice are investigated by using a position space renormalization group (PSRG) transformation. Different finite temperatures, T_x and T_y, and also different time-scale constants, α_x and α_y for spin exchanges in the x and y directions define the dynamics of the non-equilibrium system. The critical surface of the system is determined by RG flows as a function of these exchange parameters. The Onsager critical point (when the two temperatures are equal) and the critical temperature for the limit when the other temperature is infinite, previously studied by the Monte Carlo method, are obtained. In addition, two steady-state fixed points which correspond to the non-equilibrium phase transition are presented. These fixed points yield the different universality class properties of the non-equilibrium phase transitions.

1 Introduction

In recent years, work in the field of non-equilibrium phase transitions has provided a rich variety of knowledge about the critical behavior of stationary non-equilibrium systems [1,2]. Understanding the critical behavior of non-equilibrium systems is very important, as this phenomena appears in very different areas.

Research about a large class of non-equilibrium steady states is based on using uniformly or randomly driven lattices or the two temperature Ising model. Distinct long-range correlations and the universality properties of systems with an anisotropic conserved dynamics have attracted special interest. Long-range correlations occur at all temperatures above the critical temperature, while the universality properties of the system are quite different from the Ising universality class.

Katz et al. proposed a non-equilibrium model with particle-conserving hopping dynamics subjected to an external field [3,4]. Starting with this model, the driven lattice models (usually with attractive interactions) have provided a basis for studies in the field of non-equilibrium phenomena. Monte Carlo simulations (in two [3–10] and three [11] dimensions), mean-field solutions [12–14] and field theoretic renormalization analysis [15–18] have been used as main methods to investigate the critical behavior of these systems.

Long-range correlations with conserved anisotropic dynamics were studied by field-theoretic analysis (for driven lattices [19–21] and for the two temperature Ising model [22,23]), and confirmed by the Monte Carlo method (for driven lattices [19,20] and for the two temperature Ising model [22,23]).

The two temperature Ising model with conserved anisotropic dynamics has been extensively used to study non-equilibrium phase transitions. The two temperature Ising model with exchange dynamics, and in contact with two heat baths, one of them at an infinite temperature, has attracted special attention [12,13,22–25]. An interesting feature of this model is that criticality appears at a considerably elevated temperature (in comparison to the equilibrium system) of the finite-temperature heat bath.

Cheng et al. indicate that long-range correlations with the anisotropic two-temperature (one of them infinite) Kawasaki (exchange) dynamics occur at $T_c \approx 1.33T_o$ where T_o is the Onsager critical temperature [23]. Monte Carlo studies of this system yield the critical temperature at $T_c \approx 1.36T_o$ [25,26]. Note that for the driven lattice system, when the external field approaches to infinity, the spin exchange along the field direction becomes random. Consequently, the behavior of the system is analogous to the two temperature Ising model with one of the temperatures infinite. Therefore at this limit, one obtains the same critical temperature of the driven lattice from Monte Carlo studies [26], as that of the two temperature Ising model [25]. Præstgaard et al. also study the corresponding critical behavior of a non-equilibrium version of the time dependent Landau-Ginzburg model using renormalization group (RG) analysis [22]. They use a field-theoretic

[a] e-mail: yalabik@fen.bilkent.edu.tr

approach to construct an ϵ-expansion. Their study, as well as numerous others show that non-equilibrium systems have new universality classes [22,25–27].

In this work, we focus on the two-temperature Ising model with conserved dynamics. We analze the system for general values of temperature and exchange rate parameters and determine its steady state phase transition behavior. The results of the position-space renormalization group (PSRG) transformation explicitly show the different universality class property of the system. We report for the first time, the phase diagram for this system which includes a multitude of critical points investigated separately previously: the equilibrium, the steady state and the limits where one of the temperatures and/or exchange rates is infinite.

At this point, we would like to comment on the suitability of using a position space renormalization procedure which is isotropic, when it is known that for the non-equilibrium phase transition under consideration the characteristic length scales in the x and y directions scale with different exponents. Our procedure constructs new block-spins which have a distance a factor $b = 2$ larger than the distance between the original spins. We also assign probabilities to configurations of these block spins consistent with the steady state probabilities of the original spin configurations. Therefore there is no doubt that the new system thus obtained corresponds to one in which the characteristic distances (such as the correlation lengths) have been scaled down by a factor $b = 2$. In particular, infinite (or zero) correlation length systems transform again into infinite (or zero) correlation length systems. Critical points flow eventually into fixed points. The procedure then allows for the determination of the phase diagram as usual. The limitation of the transformation appears in the determination of the correlation length exponent: the scaling relation for correlation length, $\xi(k)/b = \xi(kb^{\lambda_k})$, together with the behavior $\xi(k) \sim k^{-\nu}$ implies a unique correlation length exponent $\nu = 1/\lambda_k$ independent of direction (we have used the variable k to represent the small deviation from criticality instead of the customary variable t, which we use to denote the time). This is indeed a limitation of our approach, and we cannot report any anisotropy in this exponent. We do report the value corresponding to $1/\lambda_k$. We should also note that the dynamical critical exponent z is not obtained through a linearization of the RG transformation, but through a comparison of the time scales of the original and renormalized systems at a fixed point.

2 The model

In this study, the results of renormalization group study (RG) of a spin-1/2 Ising lattice gas on a square lattice in contact with two-finite temperature baths is presented. The energy of the system is written as

$$E = -\sum_{\langle ij \rangle} J s_i s_j, \tag{1}$$

where $\langle ij \rangle$ denotes a sum over nearest-neighbor pairs of sites, and J is the interaction energy constant. Variables s_i can take values ± 1. The dynamics of the system is taken to be driven by exchanges of the neighbor spins in the x and y directions. To constitute the dynamics of a non-equilibrium system, different temperatures for exchange in each direction are considered: along the x and y directions spin exchanges are influenced by different temperatures, T_x and T_y. If these temperatures are equal, $T_x = T_y$, then the system reduces to the equilibrium spin-1/2 Ising model.

If two neighboring spins are different from each other, an exchange in the x direction may occur with the transition rate

$$w_x = \alpha_x[1 - \tanh(\Delta E/2k_B T_x)] \tag{2}$$

and

$$w_y = \alpha_y[1 - \tanh(\Delta E/2k_B T_y)] \tag{3}$$

in the y direction. Here, ΔE is the change in the energy of the system upon exchange of the spins and k_B is the Boltzmann constant. We will use the unitless interaction constants K_x and K_y in place of T_x and T_y:

$$K_x = \frac{J}{k_B T_x} \quad \text{and} \quad K_y = \frac{J}{k_B T_y}. \tag{4}$$

The timescale constants for exchanges along the x and y directions are indicated by α_x and α_y respectively.

Critical behavior is expected when the total numbers of the ± 1 spins in the square lattice are equal. Our RG transformation described in the next section, maps a 4×4 system with periodic boundary conditions and with zero magnetization, to a scaled 2×2 system again with a zero magnetization.

3 The RG transformation

3.1 General remarks

PSRG transformations are appealing because they enable one to work directly on the lattice system of interest, in contrast to the field theoretical versions which represent some extreme limit of system parameters. The advantage of the field theoretical studies is the clear identification of the universality classes, and the availability of systematic series expansions for critical exponents, albeit usually around some dimensionality possibly not too close to that of the system under investigation. On the other hand, this approach is not as useful for obtaining the phase diagram as PSRG methods. The disadvantages of the use of a PSRG transformation are the loss of the systematic nature of the approximation, and the inaccuracies that normally need be introduced due to the truncations of the infinite lattice into finite, uncorrelated pieces, and due to the truncations of the interaction energies at some level of complexity.

In equilibrium PSRG, in order to construct a scaled version of the system, “block variables” which depend on

the "original variables" are formed. One can then associate the equilibrium probabilities of the original system with those of the scaled (renormalized) system. Interaction constants that will produce these probabilities are the renormalized versions of the interaction constants of the original system. One may need to introduce interaction constants that were not present in the original system in order to obtain a "fit" to the probabilities of the states of the renormalized system, indeed a truncation in the number of interaction constants may be necessary for practical reasons. In some other cases, a limited number of distinctly different probabilities of the states of the renormalized system may limit the number of renormalized interaction constants that may be assigned.

3.2 Previous work

Before we describe our particular RG analysis, we provide a historical (albeit necessarily incomplete) summary of dynamical PSRG methods that have been applied to equilibrium systems, most of them to systems with non-conserved order parameters. The dynamical version of the PSRG to obtain the equilibrium dynamical critical exponent z takes a number of forms: the earliest techniques utilized in position-space for obtaining the parameters of the renormalized dynamical equations employed the Monte Carlo method. Ma [28] has provided a first example to such an approach, carrying out a statistical analysis of the dynamics of block spins constructed out of spins driven by a Monte Carlo simulation with a non-conserved order parameter. Monte Carlo approaches to the problem have been extended by Swendsen [29] and by Tobochnik et al. [30]. Yalabik and Gunton [31] indicated that the relaxation times of various types of correlations may be used to parametrize the renormalized state, their discussion includes the two dimensional Ising model with Kawasaki dynamics.

Alternatively, one may attempt to renormalize the transition rates in the master equation directly [32–34] to obtain a scaled equation of a similar form, and the ratio of the constants that define the time scales at the fixed point will lead to b^z.

In both types of dynamical RG approaches, one needs to thin out the dynamical degrees of freedom as well as the spatial correlations of the system. The block spins in general contain the full dynamics of the original system: the probability functions associated with the original system, driven by the dynamics of the $N \times N$ Liouville operator, relax with the N time constants associated with the eigenvalues of this operator. Since the probabilities of the renormalized system are formed as linear combinations of those of the original system, the renormalized system also relaxes with the same time constants. In fact, the same eigenvalue structure can be preserved by relating the rate of change of block-spin probabilities to their higher order derivatives. This dynamics will no longer be Markovian (it now has "memory"): a block-spin which has just changed its direction will have a larger probability per unit time to reverse its direction because its state is determined by a very marginal majority. Once the state of the block-spin survives this "infant-mortality" stage, its transition rate will settle down to a smaller value. The smaller time-scale dynamics is thus assumed to contain the non-Markovian effects, and larger time-constants to better represent the Markovian effects (such an assumption of separation of time scales is also relevant to field-theoretic RG).

Neither can one claim detailed balance in the renormalized system, as the probability flow among states is now much more complicated. The complexity of the dynamics may be preserved to some degree in Monte Carlo RG studies, where one attempts to match time dependent correlation functions in the original and scaled systems [28,31,35,36]. More commonly though, one keeps only the large time-constant modes in the scaled system, usually connected with an assumption that a Markovian master equation for the renormalized system will be sufficient to describe its long-time behavior [31,33,37]. Zheng [36] in particular has found that there is no short-time scaling in the exchange dynamics of the two dimensional Ising Model relaxing from a random initial state.

It is not clear however how much of the experience gained from the scaling of the equilibrium critical dynamics will carry over to the steady-state problem.

Relaxation type of RG studies have also been used for non-equilibrium systems, for example those which have continuous growth mechanisms [38].

3.3 Our transformation

In the present study, we are applying a PSRG scheme to a non-equilibrium system at a steady state. To our knowledge, this is the first attempt in the use of a non-Monte Carlo dynamical PSRG method for a model with conserved dynamics. In order to calculate probabilities and relaxation rates in the original system, we approximate it with a 4×4 lattice with periodic boundary conditions. This leads to $N = 12\,870$ possible states with zero magnetization. In general, the rescaled 2×2 lattices have 16 possible spin configurations in the phase space. In our transformation, we only consider the small part of this phase space by using only the configurations with zero magnetization. The results would be more accurate by using all the phase space of the system. We however report the results of the 6 renormalized states with zero magnetization. As a result of this choice of small number of states, the equilibrium limit of the system becomes inaccurate. However the rest of the phase diagram and the phase transition points are consistent with the previous works on this problem. We determine the probabilities of the possible states of the block variables in our system based on the steady state probabilities of the original variables that constitute them. In the same spirit as equilibrium implementations, we then determine the interaction constants of the renormalized lattice that would produce these probabilities.

At this stage we assume that the renormalized system too obeys Markovian dynamics. The symmetry of the 2×2 lattice (in the presence of $x - y$ anisotropy) allows for only 3 distinct steady-state probability values for the

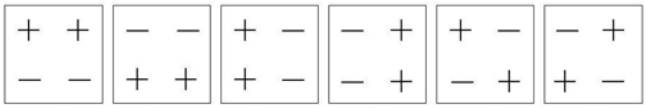

Fig. 1. Possible 6 rescaled states of the renormalized 2×2 system with $M' = 0$. Spin exchange can be accepted only for a single direction for the first four ordered states. The allowed exchange in (a) and (b) can occur along the y direction. Similarly, in (c) and (d) exchange can be seen along the x direction. However, spin exchange along both directions can be observed in (e) and (f).

6 renormalized states (Fig. 1). The symmetry also reduces the number of possible nearest-neighbor spin-exchange rates connecting the 6 states to 4 distinct values (one could also assume the presence of next-nearest neighbor and 4-spin-exchange processes in the renormalized lattice, however the rates for these processes would be expected to be very small, due to the nature of the mechanisms that drive the block-spin dynamics). The steady-state probabilities depend on the ratio of a pair of such rates, resulting in a form which may be described using the detailed balance condition. The 2×2 lattice with periodic boundary conditions then, together with the Markovian assumption, may be interpreted as a system with exchange dynamics driven at two different temperatures (defined through detailed balance) in the two orthogonal directions with isotropic nearest neighbor coupling. Note that even within this assumption determination of the coupling constants carry some arbitraryness, as the detailed balance condition may be achieved by a variety of spin-exchange rates. However, the set of coupling constants thus obtained may be seen as just a re-parametrization of the spin-exchange rates in terms of the two coupling constants. The constants that define the time-scales of the renormalized dynamics are determined from the time-constants of the two slowest decaying modes of the system, lending some justification to the assumption of Markovian dynamics.

Allowing for non-Markovian dynamics and/or more complicated spin-exchange processes for the block-spins would enable the introduction of more parameters into the dynamics (this is also a necessity if the detailed balance assumption for the renormalized dynamics is to be lifted). In fact, determination of additional of eigenvectors (and corresponding eigenvalues) of the original system would allow the introduction of two more independent coupling constants and an additional time scale to the dynamics (of course, this dynamics must be implemented in the original system as well so that recursions can be constructed). We doubt however, if the introduction this next level of complexity to the analysis would lead to any significant improvements in our results. One would need to follow faster processes in the original and the scaled systems, and identification of the corresponding modes in the two systems could be a challenge. There is even some evidence from the equilibrium case which suggests that shorter time-scale processes (for conserved dynamics) may best be treated separately for dynamical scaling [36]. In any case, we truncate the complexity in the dynamics with the assumption that the form displayed in equations (2) and (3) are valid for the renormalized system as well.

We first construct a $6 \times 12\,870$ transformation matrix T which implements the block spin transformation. This matrix transforms the probabilities of the states of the original system to those of the rescaled system as

$$P'(i) = \sum_j T_{ij} P(j). \tag{5}$$

The square lattice of 4×4 sites is partitioned into 4 individual blocks. Each block generates the new spin on a site of the rescaled square lattice. The sign of the sum of spins in each block determines the possible states of the rescaled spin. If the sum is equal to zero, the sign of the rescaled spin is decided based on the constraint that the total magnetization of the rescaled system is zero. If there is more than one such possibility, the probability is equally shared between these possible states (note that this transformation will not preserve the symmetry of the ordered states if the energy coupling constant J is negative, corresponding to an antiferromagnetic system). Conservation of the probability implies

$$\sum_i T_{ij} = 1, \tag{6}$$

for all j. On the other hand, we expect a totally random original system (with equal probabilities $P(i)$) should map to a totally random renormalized state. This in turn implies as a second condition that $\sum_j T_{ij}$ should be independent of i. Although our transformation satisfies equation (6) exactly, the second condition is satisfied approximately, within 0.5%.

The transformation from the original system to the rescaled system involves the determination of the scaled interaction constants, K'_x and K'_y and the scaled transition rates α'_x and α'_y in terms of the original values K_x, K_y, α_x and α_y.

Since the dynamics of the 2×2 system is simple, we calculate it exactly. When we have ordered the states in the sequence shown in Figure 1, the Liouville matrix is obtained as

$$L' = \begin{pmatrix} -2\Omega_y & 0 & 0 & 0 & \omega_y & \omega_y \\ 0 & -2\Omega_y & 0 & 0 & \omega_y & \omega_y \\ 0 & 0 & -2\Omega_x & 0 & \omega_x & \omega_x \\ 0 & 0 & 0 & -2\Omega_x & \omega_x & \omega_x \\ \Omega_y & \Omega_y & \Omega_x & \Omega_x & -2(\omega_x+\omega_y) & 0 \\ \Omega_y & \Omega_y & \Omega_x & \Omega_x & 0 & -2(\omega_x+\omega_y) \end{pmatrix} \tag{7}$$

where the transition rates may be expressed in terms of the detailed-balance condition:

$$\frac{\omega_x}{\Omega_x} = \exp(8K'_x) \qquad \text{and} \qquad \frac{\omega_y}{\Omega_y} = \exp(8K'_y). \tag{8}$$

The factor 8 appears because we assume periodic boundary conditions in both directions.

The three largest eigenvalues and corresponding eigenvectors of the Liouville matrix are sufficient to calculate the scaled interaction constants, K'_x and K'_y, and the ratio of the scaled transition rates α'_x/α'_y. The largest eigenvalue $\lambda_{\max} = 0$ corresponds to the steady state. The corresponding eigenvector yields the steady state probabilities which have the following form consistent with the symmetries of the system:

$$\Psi^{(0)} = \begin{pmatrix} a \\ a \\ b \\ b \\ c \\ c \end{pmatrix} \tag{9}$$

with

$$\left(\frac{b}{c}\right) = \left(\frac{\omega_x}{\Omega_x}\right) = \exp(8K'_x) \tag{10}$$

and

$$\left(\frac{a}{c}\right) = \left(\frac{\omega_y}{\Omega_y}\right) = \exp(8K'_y). \tag{11}$$

We also determine the eigenvalues corresponding to slowest relaxation with the same symmetry as the eigenvectors shown in below:

$$\lambda_1 = -2\Omega_y \quad \text{with} \quad \Psi^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

and

$$\lambda_2 = -2\Omega_x \quad \text{with} \quad \Psi^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{13}$$

Note that $\Psi^{(1)}$ and $\Psi^{(2)}$ correspond to relaxation of magnetization waves in y and x directions, respectively. We assume that transition rates of the rescaled system are also in the form given by equations (2) and (3). This then implies that the steady state probabilities and relaxation time which are to be obtained from the scaling of the 4×4 system may be used to calculate the rescaled parameters as follows:

$$K'_x = \frac{1}{8} \ln\left(\frac{b}{c}\right) \tag{14}$$

$$K'_y = \frac{1}{8} \ln\left(\frac{a}{c}\right) \tag{15}$$

$$\alpha'_x = -\frac{\lambda_2}{4}\left(\frac{b}{c} + 1\right) \tag{16}$$

$$\alpha'_y = -\frac{\lambda_1}{4}\left(\frac{a}{c} + 1\right). \tag{17}$$

Our RG procedure is then as follows: we first construct a $12\,870 \times 12\,870$ Liouville operator Ł for the 4×4 original system. The eigenvalues λ and the corresponding eigenvectors Φ of the original system can be calculated as

$$Ł\Phi^{(i)} = \lambda_i \Phi^{(i)}. \tag{18}$$

We need to determine only three of the eigenstates, the one corresponding to the steady state probabilities, $\Phi^{(0)}$, and those corresponding to the slowest relaxation of the system with symmetries given in equations (12) and (13), $\Phi^{(1)}$ and $\Phi^{(2)}$. Converting the 12 870 states of the original system into the rescaled system with 6 possible states, the transformation matrix T transforms the eigenvectors of the original system to the ones of the rescaled system. Transformation of the steady state probabilities to their rescaled version is accomplished by

$$\Psi_i^{(0)} = \sum_j T_{ij} \Phi_j^{(0)}. \tag{19}$$

The eigenvalues λ_1 and λ_2 yield the parameters used in equations (16) and (17). This formulation provides all the quantities necessary to get the rescaled parameters as in equations (14)–(17).

4 Results and conclusion

We obtain the critical behavior of the system by studying the RG flows in the space of interaction parameters. Transformation of parameters from the original system to the rescaled one constitute RG flows in parameter space. The critical surface is determined by RG flows that extend into critical fixed points.

The critical RG flow obtained from the transformation is shown schematically in Figure 2. The critical surface is plotted as a function of the parameters K_x, K_y, and r, where $r = (\alpha_x - \alpha_y)/(\alpha_x + \alpha_y)$. Note that since one can scale the time arbitrarily, it is the ratio α_x/α_y which plays a role in the determination of the steady state. In addition, the system is symmetric under the simultaneous transformation $K_x \longleftrightarrow K_y$ and $r \longleftrightarrow -r$. In Figure 2, we show the RG flows only for $K_x > K_y$.

The case $r = \pm 1$ demands special consideration: when this limit is reached by setting one of the rates α equal to zero, exchange in the corresponding direction is inhibited, the total magnetization along the other direction is conserved, the steady state properties become dependent on the initial condition. To avoid such ergodicity problems, we assume that this limit is reached by speeding up the exchange (corresponding α approaching ∞) in one direction, while the exchange in the other direction is kept at a finite rate. Two interesting special cases appear at this limit. When the temperature corresponding to the infinitely fast process itself is also infinite, the corresponding column or row effectively becomes completely randomized, and the effect of the finite exchange rate in the other direction can then be treated exactly [12,13,24]. One expects mean-field like behavior for this condition [13]. We indicate this case as R_1 in Figure 2.

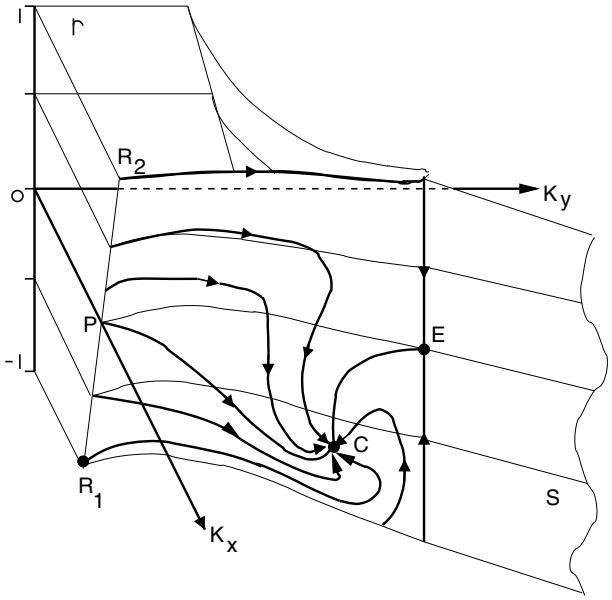

Fig. 2. A schematic drawing of the critical surface of the system. C and E indicate the fixed points for the steady-state and the equilibrium, respectively. R_1, R_2 and P denote the critical points for certain limits. Thick lines indicate the RG flows. Thin lines refer to the cross sections at certain values of the variable r. Surface S (at $K_x = K_y$) corresponds to the first-order phase transition between the ordered states at low temperatures.

Alternatively, if the temperature corresponding to the fast process is finite, but the temperature corresponding to the finite exchange is infinite, we conjecture that the system is effectively equivalent to the equilibrium Ising system. It is driven by the fast process, the slow process acting to randomize the magnetization of the fast columns or rows as the case may be. Since the equilibrium condition corresponds to equal magnetization of rows and columns, the randomization acts to maintain that condition. One then would expect equilibrium type behavior at this limit. This critical point is indicated by R_2 in Figure 2. Note that the RG flow from R_2 is expected to flow into the equilibrium fixed point (our numerical calculation cannot implement the RG transformation for very small or very large values of α. We therefore use extrapolations to $r = \pm 1$ case from calculations at $r \approx \pm 0.82$, for which $(\alpha_x/\alpha_y)^{\pm 1} = 0.1$).

We first list known or previously studied points about this system. When the two temperatures T_x and T_y are equal, one obtains the equilibrium Ising model. For this case, while $K_x = K_y$, the critical behavior is independent of α_x and α_y. The exact solution of the two-dimensional Ising model gives a critical coupling $K_o = J/k_B T_o = 0.4407\ldots$ [40].

The case of $\alpha_x/\alpha_y = 1$ and $K_x = 0$ as a function of K_y was studied extensively by Præstgaard et al. [22]. They report the critical point at $K_P = 0.322 = 0.732 K_o$ (point P in Fig. 2). Another limit of the model was studied by Krug et al. with $\alpha_x/\alpha_y = 0$ and $K_y = 0$ [13]. Using this analysis, Sanli has reported an exact critical coupling value of $K_{R1} = 0.59 K_o$ for the type of exchange we are using (point R_1 in Fig. 2) [24]. Again, we expect that the critical coupling value K_{R2} for K_x when $K_y = 0$ and $r = 1$ to be equal to K_o (point R_2 in Fig. 2).

Note that our transformation depends on the three largest eigenvalues and the corresponding eigenvectors of the Liouville matrix. Consequently, the dynamical critical exponent of the system is observed in the long time scaling regime. For the steady state fixed point, dynamical scaling relations imply $z = 4 - \eta$ while the field theoretic RG method using the ϵ-expansion yields the critical exponents $\eta = (4/243)\epsilon^2$, $\nu = (1/2) + (1/12)\epsilon + O(\epsilon^2)$ [22]. Monte Carlo simulations of the two temperature lattice gas are consistent with these theoretical results and the critical exponents are given as $\nu = 0.60(5)$, $\eta = 0.20(8)$, $\beta = 0.33(6)$ and $\gamma = 1.08(8)$ [25]. This implies that the dynamical critical exponent is approximately $z \approx 3.80$.

Our transformation produces an equilibrium critical value of $K_E = 0.8789$ corresponding to the fixed point E in Figure 2. This high value is a consequence of the strong finite size effects resulting from the conserved order parameter dynamics. Our calculations result in the critical values $K_P = 0.73 K_E$, $K_{R1} = 0.68 K_E$ and $K_{R2} = 0.83 K_E$.

The full RG flow is demonstrated in Figure 2. Note that the flow has the symmetry indicated above. The equilibrium state corresponds to a fixed point at $\alpha_x/\alpha_y = 1$, $K_x = K_y = K_E$. Critical couplings on the line with $K_x = K_y = K_E$ with $\alpha_x/\alpha_y \neq 1$ flow into this fixed point. There is a steady state fixed point, shown as C in Figure 2 (with $r_C = 0.099$, $K_{Cx} = 0.791$, $K_{Cy} = 0.693$), which corresponds to the non-equilibrium phase transition. This fixed point generates the universality class for the non-equilibrium transition, different from the equilibrium universality class associated with fixed point E. Using the eigenvalues of the linearized transformation around the fixed points, we report the values of the correlation length exponent ν for both universality classes. We also report the time-scale exponent $z = \log_2(\alpha/\alpha')$ for these fixed points, as well as the exponent λ_c corresponding to the critical crossover from the equilibrium to the steady state.

Figure 3 shows the phase diagram for various values of r. Note that the coupling constants have been scaled by the equilibrium critical coupling. The filled points are where critical RG trajectories cross the planes with a particular r (the irregularities at point positions arise due to the inaccuracies in estimating the path of the trajectory from recursion points). The corresponding lines are Bezier curve smoothed fits to these points. The dashed extensions to the lines are extrapolations to values on the axes, represented by open circles. These extrapolated values on the K_y axis are carried into the plot in the inset. This plot gives estimates for the critical values of K_y when $K_x = 0$ at $r = -1$ and $r = +1$ reported in Table 1. The variation in the phase diagram with respect to the relative exchange time scales in the x and y directions is not much, but noticeable. It is remarkable that our PSRG can produce this very untypical phase diagram.

Table 1. Quantitative results for various phase transition points studied in this work. Results from other studies are also included for comparison. Critical points P, R_1, and R_2 belong to steady state, mean-field, and equilibrium universality classes respectively.

Phase point (K_x, K_y, r)	Quantity	This work	Previous studies
Equilibrium E:(K_E, K_E, 1)	K_E	0.8789	$K_o = 0.4407\ldots$ [40]
	ν	1.74	1 (exact)
	z	3.72	3.75 [41,42]
	λ_c	0.36	–
Steady state C:(0.791, 0.693, 0.099)	ν	0.65	0.60(5) [25]
	z	3.1	≈3.80 [22,25]
P (K_P, 0, 0)	K_P	$0.73K_E$	$0.732K_o$ [22]
R_1 (K_{R1}, 0, −1)	K_{R1}	$0.68K_E$	$0.59K_o$ [24]
R_2 (K_{R2}, 0, 1)	K_{R2}	$0.83K_E$	K_o (our conjecture)

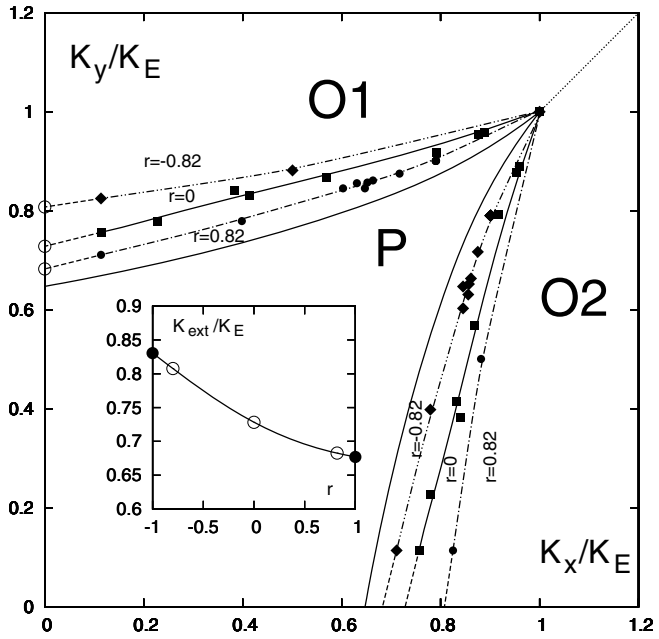

Fig. 3. The phase diagram for various values of the parameter r. P represents the disordered paramagnetic phase, while $O1$ and $O2$ are the two symmetric ordered phases separated from one another by the first order transition line at the upper right corner. The inner most phase boundary is the result of Monte Carlo work reported in reference [39] for $r = 0$. The diagram and the inset are further explained in the text.

Major flaw of our calculations is the large values obtained for K_E and the equilibrium correlation length exponent. We believe that this arises due to the very strong finite-size effects associated with conserved dynamics in the original and the scaled lattices (indeed, an exact evaluation of the specific heat for a 4×4 system yields a peak at approximately $K = 2.6$ when the order parameter is conserved, while this value is approximately $K = 1.5$ for the non-conserved system). The dynamical critical exponents, which are determined from the ratios of the time scales associated with the original and renormalized lattices, do turn out to be reasonably accurate. We note that these time scales vary slowly for each of the lattices, and that the ratio is relatively insensitive to the precise value of the fixed point parameters. We believe that the inaccuracy introduced by finite size effects is a systematic one, as evident in the very atypical phase diagram of Figure 3 (consistent with previous Monte Carlo studies) which we obtain when all interactions are scaled by the critical equilibrium coupling.

We report the full phase diagram of the system, displaying the interrelation of the various critical points studied in previous work. Crossover of critical behavior (from equilibrium to steady state) would be apparent for transitions when T_x and T_y are not too different. Monte Carlo studies in this regime could provide a check for the value of λ_c we report.

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Appendix: Finite size and truncation effects for the non-conserved RG transformation

We provide for comparison, the performance of our RG transformation (near the equilibrium critical point) when the order parameter is not conserved. We keep the form of the transformation and the assumption of periodic boundary conditions. Again, the 4×4 lattice is transformed into a 2×2 lattice. Since the renormalized lattice now allows for a larger number (16) of final states, a larger number of interactions may be incorporated into the Hamiltonian. In Table 2, we provide a summary of the results one obtains for various levels of truncations of the interactions in the system. Each line in the table shows a progressively lower level of the truncation of interaction constants, keeping nearest-neighbor K_{nn}, next-nearest-neighbor K_{nnn}, and four-spin K_4 product terms. The critical nearest neighbor interaction K_c (which flows into the fixed point under repeated renormalizations), the fixed point values of the interaction constants K^*, and ν, the critical exponent for the correlation length has been listed. The level of truncation is apparent from the values listed for the fixed point values of the interaction constants.

These results show that our transformation produces relatively good results (for equilibrium systems in which the order parameter is not conserved) if interactions of

Table 2. PSRG results of the Ising model with non-conserved order parameter near the equilibrium point via the transformation method presented in this paper. Different type of interactions are considered in each case.

Type of interactions	K_c	K^*_{nn}	K^*_{nnn}	K^*_4	ν
Nearest neighbor	0.704	0.704	–	–	0.3797
Next nearest neighbor	0.4189	0.2989	0.08657	–	0.947
Four spin	0.4184	0.2999	0.08704	−0.001172	0.9256

order higher than the nearest neighbor interaction (especially the next nearest neighbor interaction) may be kept. Unfortunately, when the number of allowed states in the small size renormalized lattice is restricted due to conservation of the order parameter, only the nearest-neighbor interaction may be kept in the transformation, leading to an accuracy comparable to that of the non-conserved system with the same level of truncation.

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