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Dual π-Rickart Modules

Módulos π-Rickart duales

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Abstract. Let R be an arbitrary ring with identity and M a right R-module with $S = \mathrm{End}_R(M)$. In this paper we introduce dual π-Rickart modules as a generalization of π-regular rings as well as that of dual Rickart modules. The module M is said to be *dual π-Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $\mathrm{Im}\, f^n = eM$. We prove that some results of dual Rickart modules can be extended to dual π-Rickart modules for this general settings. We investigate relations between a dual π-Rickart module and its endomorphism ring.

Key words and phrases. π-Rickart modules, Dual π-Rickart modules, Fitting modules, Generalized left principally projective rings, π-regular rings.

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Resumen. Sea R un anillo arbitrario con identidad y M un R-modulo derecho con $S = \mathrm{End}_R(M)$. En este artículo introducimos los módulos π-Rickart duales como una generalización de los anillos π-regulares así como también de los módulos Rickart. El módulo M se dice *dual π-Rickart* si para cada $f \in S$, existe $e^2 = e \in S$ y un entero positivo n tales que $\mathrm{Im}\, f^n = eM$. Demostramos que algunos resultados de los módulos de Rickart pueden ser extendidos a los módulos π-Rickart duales para este marco general. Finalmente, investigamos las relaciones entre un módulo π-Rickart dual y su anillo de endomorfismos.

Palabras y frases clave. Módulos π-Rickart , módulos π-Rickart duales, módulos ajustados, anillos izquierdos principalmente proyectivos generalizados, anillos π-regulares.

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1. Introduction

Throughout this paper R denotes an associative ring with identity, and modules are unitary right R-modules. For a module M, $S = \mathrm{End}_R(M)$ is the ring of all right R-module endomorphisms of M. In this work, for the (S, R)-bimodule M, $l_S(\bullet)$ and $r_M(\bullet)$ are the left annihilator of a subset of M in S and the right annihilator of a subset of S in M, respectively. A ring is *reduced* if it has no nonzero nilpotent elements. *Baer rings* [8] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. Principally projective rings were introduced by Hattori [3] to study the torsion theory, that is, a ring R is called *left (right) principally projective* if every principal left (right) ideal is projective. The concept of left (right) principally projective rings (or left (right) Rickart rings) has been comprehensively studied in the literature. Regarding a generalization of Baer rings as well as principally projective rings, recall that a ring R is called *generalized left (right) principally projective* if for any $x \in R$, the left (right) annihilator of x^n is generated by an idempotent for some positive integer n. A number of papers have been written on generalized principally projective rings (see [4] and [7]). A ring R is *(von Neumann) regular* if for any $a \in R$ there exists $b \in R$ with $a = aba$. The ring R is called *π-regular* if for each $a \in R$ there exist a positive integer n and an element x in R such that $a^n = a^n x a^n$. Similarly, call a ring R *strongly π-regular* if for every element $a \in R$ there exist a positive integer n (depending on a) and an element $x \in R$ such that $a^n = a^{n+1}x$, equivalently, there exists $y \in R$ such that $a^n = ya^{n+1}$. Every regular ring is π-regular and every strongly π-regular ring is π-regular. There are regular or π-regular rings which are not strongly π-regular.

According to Rizvi and Roman, a module M is said to be *Rickart* [13] if for any $f \in S$, $r_M(f) = eM$ for some $e^2 = e \in S$. The class of Rickart modules is studied extensively by different authors (see [1] and [9]). Recently the concept of a Rickart module is generalized in [16] by the present authors. The module M is called *π-Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $r_M(f^n) = eM$. Dual Rickart modules are defined by Lee, Rizvi and Roman in [10]. The module M is called *dual Rickart* if for any $f \in S$, $\mathrm{Im}\, f = eM$ for some $e^2 = e \in S$.

In the second section, we investigate general properties of dual π-Rickart modules and Section 3 contains the results on the structure of endomorphism ring of a dual π-Rickart module. In what follows, we denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{Z}_n$, the ring of integers, rational numbers, real numbers and the ring of integers modulo n, respectively, and $J(R)$ denotes the Jacobson radical of a ring R.

2. Dual π-Rickart Modules

In this section, we introduce the concept of a dual π-Rickart module that generalizes the notion of a dual Rickart module as well as that of a π-regular ring. We prove that some properties of dual Rickart modules hold for this general setting. Although every direct summand of a dual π-Rickart module is dual π-Rickart, a direct sum of dual π-Rickart modules is not dual π-Rickart. We give an example to show that a direct sum of dual π-Rickart modules may not be dual π-Rickart. It is shown that the class of some abelian dual π-Rickart modules is closed under direct sums.

We start with our main definition.

Definition 1. Let M be an R-module with $S = \operatorname{End}_R(M)$. The module M is called *dual π-Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $\operatorname{Im} f^n = eM$.

For the sake of brevity, in the sequel, S will stand for the endomorphism ring of the module M considered. Dual π-Rickart modules are abundant. Every semisimple module, every injective module over a right hereditary ring and every module of finite length are dual π-Rickart. Also every quasi-projective strongly co-Hopfian module, every quasi-injective strongly Hopfian module, every Artinian and Noetherian module is dual π-Rickart (see Corollary 19). Every finitely generated module over a right Artinian ring is a dual π-Rickart module (see Proposition 20).

Proposition 2. *Let R be a ring. Then the right R-module R is a dual π-Rickart module if and only if R is a π-regular ring.*

Proof. If the right R-module R is a dual π-Rickart module and $f \in R$, then there exist $e^2 = e \in R$ and a positive integer n such that $\operatorname{Im} f^n = eR$. There exist $x, y \in R$ such that $e = f^n x$ and $f^n = ey$. Multiplying the first equation from the right by f^n, we have $f^n x f^n = ey = f^n$. Conversely, assume that R is a π-regular ring. Let $g \in R$. Then there exist a positive integer n and $x \in R$ such that $g^n = g^n x g^n$. Hence $e = g^n x$ is an idempotent of R. Since $e \in g^n R$ and $g^n = g^n x g^n = e g^n \in eR$, we have $\operatorname{Im} g^n = eR$. Therefore the right R-module R is dual π-Rickart. ☑

It is clear that every dual Rickart module is dual π-Rickart. The following example shows that every dual π-Rickart module need not be dual Rickart.

Example 3. Let R denote the ring $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ and M the right R-module $\begin{pmatrix} 0 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ with usual matrix operations. If $f \in S$, then there exist $a, b, c \in \mathbb{Z}_2$ such that

$$f \begin{pmatrix} 0 & x \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & ax \\ by & cx + bz \end{pmatrix}$$

By using this image of f, we prove that there exists a positive integer n such that $\operatorname{Im} f^n$ is a direct summand of M. Consider the following cases for $a, b, c \in \mathbb{Z}_2$.

Case 1. If $a = b = c = 1$, then f is an epimorphism.

Case 2. If $a = 0$, $b = 0$, $c = 1$, then $f^2 = 0$.

Case 3. If $a = 0$, $b = 1$, $c = 1$ or $a = 0$, $b = 1$, $c = 0$, then in either case $\operatorname{Im} f = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \middle| x, y \in \mathbb{Z}_2 \right\}$ is a direct summand of M.

Case 4. If $a = 1$, $b = 0$, $c = 1$, then $\operatorname{Im} f = \left\{ \begin{pmatrix} 0 & x \\ 0 & x \end{pmatrix} \middle| x \in \mathbb{Z}_2 \right\}$ is a direct summand of M.

Case 5. If $a = 1$, $b = 0$, $c = 0$, then $\operatorname{Im} f = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \middle| x \in \mathbb{Z}_2 \right\}$ is a direct summand of M.

Case 6. If $a = 1$, $b = 1$, $c = 0$, then f is an identity map.

Case 7. If $a = 0$, $b = 0$, $c = 0$, then f is a zero map.

In all cases there exists a positive integer n such that $\operatorname{Im} f^n$ is a direct summand of M and so M is a dual π-Rickart module. The module M is not dual Rickart, since $\operatorname{Im} f = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \middle| x \in \mathbb{Z}_2 \right\}$ in the second case.

Our next aim is to find conditions under which a dual π-Rickart module is dual Rickart.

Proposition 4. *Let M be a dual Rickart module. Then M is dual π-Rickart. The converse holds if S is a reduced ring.*

Proof. The first statement is clear. Suppose that S is a reduced ring and M is a dual π-Rickart module. Let $f \in S$. There exist a positive integer n and an idempotent $e \in S$ such that $\operatorname{Im} f^n = eM$. If $n = 1$, there is nothing to do. Assume that $n > 1$, then $(1-e)f^n M = 0$ and so $(1-e)f^n = 0$. Since S is a reduced ring, e is central and $\big((1-e)f\big)^n = 0$. Also it implies $(1-e)f = 0$ or $f = ef$. Thus $\operatorname{Im} f \leq eM$. The reverse inclusion $eM \leq \operatorname{Im} f$ follows from $eM = f^n M \leq f(f^{n-1})M \leq fM$. Therefore $eM = \operatorname{Im} f$ and M is a dual Rickart module. ☑

By using a different condition on an endomorphism ring of a module we show that a dual π-Rickart module is dual Rickart. To do this we need the following lemma.

Lemma 5. *Let M be a module. Then M is dual π-Rickart and S is a domain if and only if every nonzero element of S is an epimorphism.*

Proof. The sufficiency is clear. For the necessity, let M be a dual π-Rickart module and $0 \neq f \in S$. Then there exist a positive integer n and an idempotent $e \in S$ such that $\operatorname{Im} f^n = eM$. Hence $f^n = ef^n$. Since S is a domain and f^n is nonzero, we have $e = 1$ and so $\operatorname{Im} f^n = M$. This implies that $\operatorname{Im} f = M$. Thus f is an epimorphism. ☑

Recall that a module M has C_2 *condition* if any submodule N of M which is isomorphic to a direct summand of M is a direct summand, while a module M is said to have D_2 *condition* if any submodule N of M with M/N isomorphic to a direct summand of M, then N is a direct summand of M. In the next result we obtain relations between π-Rickart and dual π-Rickart modules by using C_2 and D_2 conditions. An endomorphism f of a module M is called *morphic* [12] if $M/fM \cong \operatorname{Ker} f$. The module M is called *morphic* if every endomorphism of M is morphic.

Theorem 6. *Let M be a module. Then we have the following.*

(1) If M is a dual π-Rickart module with D_2 condition, then it is π-Rickart.

(2) If M is a π-Rickart module with C_2 condition, then it is dual π-Rickart.

(3) If M is projective morphic, then it is π-Rickart if and only if it is dual π-Rickart.

Proof. Since $M/\operatorname{Ker} f^n \cong \operatorname{Im} f^n$ for any positive integer n, D_2 and C_2 conditions complete the proof of *(1)* and *(2)*. The proof of *(3)* is clear. ☑

The next result is an immediate consequence of Theorem 6.

Corollary 7. *Let M be a module with C_2 and D_2 conditions. Then M is a dual π-Rickart module if and only if it is π-Rickart.*

In [10, Proposition 2.6], it is shown that M is a dual Rickart module if and only if the short exact sequence $0 \to \operatorname{Im} f \to M \to M/\operatorname{Im} f \to 0$ splits for any $f \in S$. In this direction we can give a similar characterization for dual π-Rickart modules.

Lemma 8. *The following are equivalent for a module M.*

(1) M is a dual π-Rickart module.

(2) For every $f \in S$ there exists a positive integer n such that the short exact sequence $0 \to \operatorname{Im} f^n \to M \to M/\operatorname{Im} f^n \to 0$ splits.

Proof. For any $f \in S$ and any positive integer n consider the short exact sequence $0 \to \operatorname{Im} f^n \to M \to M/\operatorname{Im} f^n \to 0$. The short exact sequence splits in M if and only if $\operatorname{Im} f^n$ is a direct summand of M if and only if M is a dual π-Rickart module. ☑

One may suspect that every submodule of a dual π-Rickart module is dual π-Rickart. The following example shows that this is not the case.

Example 9. Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Then $S = \operatorname{End}_{\mathbb{Z}}(\mathbb{Q})$ is isomorphic to $\mathbb{Q}$. Since every element of S is an isomorphism or zero, $\mathbb{Q}$ is dual π-Rickart. Now consider the submodule $\mathbb{Z}$ and $f \in \operatorname{End}_{\mathbb{Z}}(\mathbb{Z})$ defined by $f(x) = 2x$, where $x \in \mathbb{Z}$. Since the image of any power of f can not be a direct summand of $\mathbb{Z}$, the submodule $\mathbb{Z}$ is not dual π-Rickart.

Although every submodule of a dual π-Rickart module need not be dual π-Rickart by Example 9, we now prove that every direct summand of dual π-Rickart modules is also dual π-Rickart.

Proposition 10. *Let M be a dual π-Rickart module. Then every direct summand of M is also dual π-Rickart.*

Proof. Let $M = N \oplus P$ with $S_N = \operatorname{End}_R(N)$. Define $g = f \oplus 0|_P$, for any $f \in S_N$ and so $g \in S$. By hypothesis, there exist a positive integer n and $e^2 = e \in S$ such that $\operatorname{Im} g^n = eM$ and $g^n = f^n \oplus 0|_P$. Hence $eM = \operatorname{Im} g^n = f^n N \leq N$. Let $M = eM \oplus Q$ for some submodule Q. Thus $N = eM \oplus (N \cap Q) = f^n N \oplus (N \cap Q)$. Therefore N is dual π-Rickart. ☑

Corollary 11. *Let R be a π-regular ring with $e = e^2 \in R$. Then eR is a dual π-Rickart R-module.*

Here we give the following result for π-regular rings.

Corollary 12. *Let $R = R_1 \oplus R_2$ be a π-regular ring with direct sum of the rings R_1 and R_2. Then the rings R_1 and R_2 are also π-regular.*

We now characterize π-regular rings in terms of dual π-Rickart modules.

Theorem 13. *Let R be a ring. Then R is π-regular if and only if every cyclic projective R-module is dual π-Rickart.*

Proof. The sufficiency is clear. For the necessity, let $M = mR$ be a projective module. Then $R = r_R(m) \oplus I$ for some right ideal I of R. Let $I \xrightarrow{\varphi} M$ denote the isomorphism and $f \in S$. By Proposition 2 and Proposition 10, $(\varphi^{-1} f \varphi)^n I = (\varphi^{-1} f^n \varphi) I$ is a direct summand of I for some positive integer n. Hence $I = (\varphi^{-1} f^n \varphi) I \oplus K$ for some right ideal K of I. Thus $\varphi I = (f^n \varphi) I \oplus \varphi K$, and so $M = f^n M \oplus \varphi K$. Therefore M is dual π-Rickart. ☑

Theorem 14. *Let R be a ring and consider the following conditions:*

(1) Every free R-module is dual π-Rickart,

(2) Every projective R-module is dual π-Rickart,

(3) Every flat R-module is dual π-Rickart.

Then (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1). Moreover (2) $\Rightarrow$ (3) holds for finitely presented modules.

Proof. *(3) $\Rightarrow$ (2) $\Rightarrow$ (1)* Clear. *(1) $\Rightarrow$ (2)* Let M be a projective R-module. Then M is a direct summand of a free R-module F. By *(1)*, F is dual π-Rickart, and so is M due to Proposition 10. *(2) $\Rightarrow$ (3)* is clear from the fact that finitely presented flat modules are projective. ☑

The next example reveals that a direct sum of dual π-Rickart modules need not be dual π-Rickart.

Example 15. Let R denote the ring $\begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$ and M the R-module $\begin{pmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$.

Let $f \in S$. Then there exist $a, c, u, t \in R$ such that

$$f\begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} ax+ur & ay+us \\ cx+tr & cy+ts \end{pmatrix}$$

where $\begin{pmatrix} x & y \\ r & s \end{pmatrix} \in M$.

Consider $f \in S$ defined by $a = c = 0$, $u = 3$ and $t = 2$. This implies that $f\begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} 3r & 3s \\ 2r & 2s \end{pmatrix}$ and for any positive integer n we obtain

$$f^n\begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} 3(2^{n-1})r & 3(2^{n-1})s \\ 2^n r & 2^n s \end{pmatrix}.$$

It follows that f^nM can not be a direct summand. On the other hand, consider the submodules $N = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$ of M. Then $\mathrm{End}_R(N)$ and $\mathrm{End}_R(K)$ are isomorphic to $\mathbb{R}$. Hence N and K are dual π-Rickart modules but M is not dual π-Rickart.

The following lemma is useful to show that a direct sum of some dual π-Rickart modules is a dual π-Rickart.

Lemma 16. *Let M be a module and $f \in S$. If $\mathrm{Im}\, f^n = eM$ for some central idempotent $e \in S$ and a positive integer n, then $\mathrm{Im}\, f^{n+1} = eM$.*

Proof. Let $f \in S$ and $\operatorname{Im} f^n = eM$ for some central idempotent $e \in S$ and a positive integer n. It is clear that $\operatorname{Im} f^{n+1} \subseteq \operatorname{Im} f^n$. Let $f^n(x) \in \operatorname{Im} f^n$, then $f^n(x) = ef^n(x) = f^n e(x)$. Since $e(x) \in \operatorname{Im} f^n$, $e(x) = f^n(y)$ for some $y \in M$. So $f^n(x) = f^n\big(f^n(y)\big) = f^{n+1}\big(f^{n-1}(y)\big) \in \operatorname{Im} f^{n+1}$. This completes the proof. ☑

A ring R is called *abelian* if every idempotent is central, that is, $ae = ea$ for any a, $e^2 = e \in R$. A module M is called *abelian* [14] if $fem = efm$ for any $f \in S$, $e^2 = e \in S$, $m \in M$. Note that M is an abelian module if and only if S is an abelian ring. We now prove that a direct sum of dual π-Rickart modules is dual π-Rickart for some abelian modules.

Proposition 17. *Let M_1 and M_2 be abelian R-modules. If M_1 and M_2 are dual π-Rickart with $\operatorname{Hom}_R(M_i, M_j) = 0$ for $i \neq j$, then $M_1 \oplus M_2$ is a dual π-Rickart module.*

Proof. Let $S_1 = \operatorname{End}_R(M_1)$, $S_2 = \operatorname{End}_R(M_2)$ and $M = M_1 \oplus M_2$. We may describe S as $\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$. Let $\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \in S$, where $f_1 \in S_1$ and $f_2 \in S_2$. Then there exist positive integers n, m and $e_1^2 = e_1 \in S_1$ and $e_2^2 = e_2 \in S_2$ such that $\operatorname{Im} f_1^n = e_1 M_1$ and $\operatorname{Im} f_2^m = e_2 M_2$. Consider the following cases:

i.) Let $n = m$. Obviously, $\operatorname{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^n = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M$.

ii.) Let $n < m$. By Lemma 16, we have $\operatorname{Im} f_1^n = \operatorname{Im} f_1^m = e_1 M_1$. Clearly, $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M \leq \operatorname{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m$. Now let $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m$. Then $m_1 \in \operatorname{Im} f_1^m = e_1 M_1$ and $m_2 \in \operatorname{Im} f_2^m = e_2 M_2$. Hence $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$. Thus $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M$. Therefore $\operatorname{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m \leq \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M$.

iii.) Let $m < n$. Since M_2 is abelian, the proof is similar to case ii. ☑

We close this section with the relations among strongly co-Hopfian modules, Fitting modules and dual π-Rickart modules.

Recall that a module M is called *co-Hopfian* if every injective endomorphism of M is an automorphism, while M is called *strongly co-Hopfian* [5], if for any endomorphism f of M the descending chain

$$\operatorname{Im} f \supseteq \operatorname{Im} f^2 \supseteq \cdots \supseteq \operatorname{Im} f^n \supseteq \cdots$$

stabilizes.

We now give a relation between abelian and strongly co-Hopfian modules by using dual π-Rickart modules.

Corollary 18. *Let M be a dual π-Rickart module and S an abelian ring. Then M is strongly co-Hopfian.*

Proof. It follows from Lemma 16 and [5, Proposition 2.6]. ☑

A module M is said to be *a Fitting module* [5] if for any $f \in S$, there exists an integer $n \geq 1$ such that $M = \operatorname{Ker} f^n \oplus \operatorname{Im} f^n$. Due to Armendariz, Fisher and Snider [2] or [15, Proposition 5.7], the module M is Fitting if and only if S is strongly π-regular.

We now give the following relation between Fitting modules and dual π-Rickart modules.

Corollary 19. *Every Fitting module is a dual π-Rickart module.*

Then we have the following result.

Proposition 20. *Let R be an Artinian ring. Then every finitely generated R-module is dual π-Rickart.*

Proof. Let M be a finitely generated R-module. Then M is an Artinian and Noetherian module. Hence M is a Fitting module and so it is dual π-Rickart. ☑

Proposition 21. *Let R be a ring and n a positive integer. If the matrix ring $M_n(R)$ is strongly π-regular, then R^n is a dual π-Rickart R-module.*

Proof. Let $M_n(R)$ be a strongly π-regular ring. Then by [5, Corollay 3.6], R^n is a Fitting R-module and so it is dual π-Rickart. ☑

3. The Endomorphism Ring of a Dual π-Rickart Module

In this section we study relations between a dual π-Rickart module and its endomorphism ring. We prove that the endomorphism ring of a dual π-Rickart module is always a generalized left principally projective ring. The converse holds if the module is self-cogenerator. The modules whose endomorphism rings are π-regular are characterized. It is shown that if the module satisfies D_2 condition, then it is dual π-Rickart if and only if the endomorphism ring of the module is a π-regular ring.

Lemma 22. *If M is a dual π-Rickart module, then S is a generalized left principally projective ring.*

Proof. Let $f \in S$. By assumption, there exist $e^2 = e \in S$ and a positive integer n such that $\operatorname{Im} f^n = eM$. Hence $l_S(f^n M) = S(1-e) = l_S(f^n)$. Thus S is a generalized left principally projective ring. ☑

The next result is a consequence of Theorem 10 and Lemma 22.

Corollary 23. *If R is a π-regular ring, then eRe is a generalized left principal projective ring for any $e^2 = e \in R$.*

Corollary 24. *Let M be a dual π-Rickart module and $f \in S$. Then Sf^n is a projective left S-module for some positive integer n.*

Proof. Clear from Lemma 22, since $Sf^n \cong S/l_S(f^n)$. ☑

Recall that a module is called *self-cogenerator* if it cogenerates all its factor modules. The following result shows that the converse of Lemma 22 is true for self-cogenerator modules. On the other hand, Theorem 25 generalizes the result [17, 39.11].

Theorem 25. *Let M be a module and $f \in S$.*

i.) If Sf^n is a projective left S-module for some positive integer n, then the submodule $N = \bigcap\{\operatorname{Ker} g : g \in S, \operatorname{Im} f^n \leq \operatorname{Ker} g\}$ is a direct summand of M.

ii.) If M is self-cogenerator and S is a generalized left principally projective ring, then M is a dual π-Rickart module.

Proof. i.) Let Sf^n be a projective left S-module for some positive integer n. Since $Sf^n \cong S/l_S(f^n)$, $l_S(f^n) = Se$ for some $e^2 = e \in S$. We prove $(1-e)M = N$. Due to $ef^nM = 0$, we have $f^nM \leq (1-e)M$. By definition of N we have $N \leq (1-e)M$. Let $g \in S$ with $\operatorname{Im} f^n \leq \operatorname{Ker} g$. Then $gf^nM = 0$ or $gf^n = 0$. Hence $g \in l_S(f^n) = Se$ and $ge = g$. So $g(1-e)M = 0$ from which we have $(1-e)M \leq \operatorname{Ker} g$ for all g with $\operatorname{Im} f^n \leq \operatorname{Ker} g$. Thus $(1-e)M \leq N$. Therefore $(1-e)M = N$.

ii.) Assume that M is self-cogenerator and S is generalized left principally projective. There exist $e^2 = e \in S$, a positive integer n such that $l_S(f^n) = Se$ and $M/\operatorname{Im} f^n$ is cogenerated by M. By [17, 14.5],

$$\bigcap \big\{ \operatorname{Ker} g : g \in \operatorname{Hom}(M/\operatorname{Im} f^n, M) \big\} = 0.$$

Hence

$$\operatorname{Im} f^n = \bigcap \big\{ \operatorname{Ker} g : g \in S, \operatorname{Im} f^n \leq \operatorname{Ker} g \big\}.$$

Thus conditions of i.) are satisfied and so $\operatorname{Im} f^n$ is a direct summand. ☑

For an R-module M, it is shown that, if S is a von Neumann regular ring, then M is a dual Rickart module (see [10, Proposition 3.8]). We obtain a similar result for dual π-Rickart modules.

Lemma 26. *Let M be a module. If S is a π-regular ring, then M is dual π-Rickart.*

***Proof*.** Let $f \in S$. Since S is π-regular, there exist a positive integer n and $g \in S$ such that $f^n = f^n g f^n$. Then $e = f^n g$ is an idempotent of S. Now we show that $\operatorname{Im} f^n = f^n gM$. It is clear that $f^n M = ef^n M \leq eM$. For the other inclusion, let $m \in M$. Hence $em = f^n gm \in f^n M$. Thus $\operatorname{Im} f^n = eM$. ☑

Since every strongly π-regular ring is π-regular, we have the next result.

Corollary 27. *Let M be a module. If S is a strongly π-regular ring, then M is dual π-Rickart.*

The converse statement of Corollary 27 does not hold in general, that is, there exists a dual π-Rickart module having not a strongly π-regular endomorphism ring.

Example 28. Let D be a division ring, M a vector space over D with an infinite basis $\{e_i \in M : i = 1, 2, \ldots\}$ and $S = \operatorname{End}_D(M)$. As a semisimple right D-module, M is dual π-Rickart, and by [17, 3.9] S is a regular and so π-regular ring. Assume that S is a strongly π-regular ring and we reach a contradiction. Let $f \in S$ defined by $f(e_i) = e_{i+1}$ for all $i = 1, 2, 3, \ldots$. By assumption, there is a positive integer n such that $f^n = f^{n+1}g$ for $g \in S$. Then $f^n = f^{n+1}g$ implies $f^n S = f^{n+1}S$ and so $f^n M = f^{n+1}M$. Since $f^n(e_i) = e_{i+n}$ for all i, we have $f^n M = \sum_{i>n} e_i D \neq f^{n+1}M$. This is a contradiction. Hence S is not a strongly π-regular ring (see also [15, 5.5]).

The proof of Lemma 29 may be in the context.

Lemma 29. *Let M be a module. Then S is a π-regular ring if and only if there exists a positive integer n such that $\operatorname{Ker} f^n$ and $\operatorname{Im} f^n$ are direct summands of M for any $f \in S$.*

Now we recall some known facts about π-regular rings that will be needed.

Lemma 30. *Let R be a ring. Then*

i.) If R is π-regular, then eRe is also π-regular for any $e^2 = e \in R$.

ii.) If $M_n(R)$ is π-regular for any positive integer n, then so is R.

iii.) If R is a commutative ring, then R is π-regular if and only if $M_n(R)$ is π-regular for any positive integer n.

Proposition 31. *Let R be a commutative π-regular ring. Then every finitely generated projective R-module is dual π-Rickart.*

Proof. Let M be a finitely generated projective R-module. So the endomorphism ring of M is $eM_n(R)e$ with some positive integer n and an idempotent e in $M_n(R)$. Since R is commutative π-regular, $M_n(R)$ is also π-regular, and so is $eM_n(R)e$ by Lemma 30. Hence M is dual π-Rickart by Lemma 26. ☑

Theorem 32. *Let M be a module with D_2 condition. Then M is dual π-Rickart if and only if S is π-regular.*

Proof. The necessity holds by Lemma 26. For the sufficiency, let $0 \neq f \in S$. Since M is dual π-Rickart, $\operatorname{Im} f^n$ is a direct summand of M for some positive integer n. Because of $M/\operatorname{Ker} f^n \cong \operatorname{Im} f^n$, D_2 condition implies that $\operatorname{Ker} f^n$ is a direct summand of M. The rest is obvious from Lemma 29. ☑

The following is a consequence of Proposition 31 and Theorem 32.

Corollary 33. *Let R be a commutative ring and satisfy D_2 condition. Then the following are equivalent.*

i.) R is a π-regular ring.

ii.) Every finitely generated projective R-module is dual π-Rickart.

Recall that a module M is called *quasi-projective* if it is M-projective. Since every quasi-projective module has D_2 condition, we have the following.

Corollary 34. *If M is a quasi-projective dual π-Rickart module, then the endomorphism ring of M is a π-regular ring.*

Theorem 35. *The following are equivalent for a ring R.*

i.) $M_n(R)$ is π-regular for every positive integer n.

ii.) Every finitely generated projective R-module is dual π-Rickart.

Proof. i.) ⇒ ii.) Let M be a finitely generated projective R-module. Then $M \cong eR^n$ for some positive integer n and $e^2 = e \in M_n(R)$. Hence S is isomorphic to $eM_n(R)e$. By i.), S is π-regular. Thus, due to Lemma 26, M is π-Rickart.

ii.) ⇒ i.) $M_n(R)$ can be viewed as the endomorphism ring of a projective R-module R^n for any positive integer n. By ii.), R^n is dual π-Rickart. Then, by Corollary 34, $M_n(R)$ is π-regular. ☑

Recall that an R-module M is called *duo* if every submodule of M is fully invariant, i.e., for any submodule N of M, $fN \leq N$ for each $f \in S$. Equivalently, every right R-submodule of M is also left S-submodule. Our next aim is to determine conditions under which any factor module of a dual π-Rickart module is also dual π-Rickart.

Corollary 36. *Let M be a quasi-projective module and N a fully invariant submodule of M. If M is dual π-Rickart, then so is M/N.*

Proof. Let $f \in S$ and π denote the natural epimorphism from M to M/N. Consider the following diagram.

$$\begin{array}{ccc} M & \xrightarrow{\ \pi\ } & M/N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^*} \\ M & \xrightarrow[\ \pi\ ]{} & M/N \end{array}$$

Since N is fully invariant, we have $\operatorname{Ker}\pi \subseteq \operatorname{Ker}\pi f$. By the Factor Theorem, there exists a unique homomorphism f^* such that $f^*\pi = \pi f$. Hence we define a homomorphism $\varphi : S \to \operatorname{End}_R(M/N)$ with $\varphi(f) = f^*$ for any $f \in S$. As M is quasi-projective, φ is an epimorphism. Thus $\operatorname{End}_R(M/N) \cong S/\operatorname{Ker}\varphi$. By Corollary 34, S is π-regular, and so is $S/\operatorname{Ker}\varphi$. Therefore, due to Lemma 26 M/N is dual π-Rickart. ☑

Corollary 37. *Let M be a quasi-projective duo module. If M is dual π-Rickart, then M/N is also dual π-Rickart for every submodule N of M.*

Corollary 38. *If M be a quasi-projective dual π-Rickart module, then so is $M/\operatorname{Rad}(M)$ and $M/\operatorname{Soc}(M)$.*

Proposition 39. *Let M be a dual π-Rickart module. Then every endomorphism of M with a small image in M is nilpotent.*

Proof. Let $f \in S$ with $\operatorname{Im} f$ small in M. Then $\operatorname{Im} f^n$ is a direct summand of M for some positive integer n. Also $\operatorname{Im} f^n$ is small in M. Hence $f^n = 0$. ☑

Corollary 40. *Let M be a dual π-Rickart discrete module. Then $J(S)$ is nil and $S/J(S)$ is von Neumann regular.*

Proof. Since M is discrete, by [11, Theorem 5.4], $J(S)$ consists of endomorphisms with small image. By Proposition 39, $J(S)$ is nil and again by [11, Theorem 5.4], $S/J(S)$ is von Neumann regular. ☑

Theorem 41. *The following are equivalent for a module M.*

i.) M is a dual π-Rickart module.

ii.) S is a generalized left principally projective ring and $f^nM = r_M\big(l_S(f^nM)\big)$ for all $f \in S$ and a positive integer n.

Proof. i.) $\Rightarrow$ ii.) By Lemma 22, we only need to show that $f^nM = r_M\big(l_S(f^nM)\big)$ for all $f \in S$. Since M is dual π-Rickart, for any $f \in S$, $f^nM = eM$ for some $e^2 = e \in S$ and a positive integer n. Thus $r_M\big(l_S(f^nM)\big) = r_M\big(l_S(eM)\big) = eM = f^nM$.

ii.) $\Rightarrow$ i.) Let $f \in S$. Since S is a generalized left principally projective ring, $l_S(f^nM) = Se$ for some $e^2 = e \in S$ and a positive integer n. By hypothesis, $f^nM = r_M\big(l_S(f^nM)\big) = r_M(Se) = (1-e)M$. Thus M is dual π-Rickart. ☑

Corollary 42. *Let M be a module. Then M is dual π-Rickart if and only if $f^nM = r_M\big(l_S(f^nM)\big)$ and $r_M\big(l_S(f^nM)\big)$ is a direct summand of M.*

Theorem 43. *Let M be a dual π-Rickart module. Then the left singular ideal $Z_l(S)$ of S is nil and $Z_l(S) \subseteq J(S)$.*

Proof. Let $f \in Z_l(S)$. Since M is dual π-Rickart, $\mathrm{Im}(f^n) = eM$ for some positive integer n and $e = e^2 \in S$. Then, by Lemma 22, $l_S(f^n) = S(1-e)$. Since $l_S(f^n)$ is essential in S as a left ideal, we have $l_S(f^n) = S$. This implies that $f^n = 0$ and so $Z_l(S)$ is nil. On the other hand, for any $g \in S$ and $f \in Z_l(S)$, according to previous discussion, $(gf)^n = 0$ for some positive integer n. Hence $1 - gf$ is invertible. Thus $f \in J(S)$. Therefore $Z_l(S) \subseteq J(S)$. ☑

Proposition 44. *The following are equivalent for a module M.*

i.) M is an indecomposable dual π-Rickart module.

ii.) Each element of S is either an epimorphism or nilpotent.

Proof. i.) $\Rightarrow$ ii.) Let $f \in S$. Then f^nM is a direct summand of M for some positive integer n. As M is indecomposable, we see that $f^nM = 0$ or $f^nM = M$. This implies that f is an epimorphism or nilpotent.

ii.) $\Rightarrow$ i.) Let $e = e^2 \in S$. If e is nilpotent, then $e = 0$. If e is an epimorphism, then $e = 1$. Hence M is indecomposable. Also for any $f \in S$, $fM = M$ or $f^nM = 0$ for some positive integer n. Therefore M is dual π-Rickart. ☑

Theorem 45. *Consider the following conditions for a module M.*

i.) S is a local ring with nil Jacobson radical.

ii.) M is an indecomposable dual π-Rickart module.

Then i.) $\Rightarrow$ ii.). If M is a morphic module, then ii.) $\Rightarrow$ i.).

Proof. i.) $\Rightarrow$ ii.) Clearly, each element of S is either an epimorphism or nilpotent. Then, due to Proposition 44, M is indecomposable dual π-Rickart.

ii.) $\Rightarrow$ i.) Let $f \in S$. Then $f^n M = eM$ for some positive integer n and an idempotent e in S. If $e = 1$, then f is an epimorphism. Since M is morphic, f is invertible by [12, Corollary 2]. If $e = 0$, then $f^n = 0$. Hence $1 - f$ is invertible. This implies that S is a local ring. Now let $0 \neq f \in J(S)$. Since f is not invertible and M is morphic, f is nilpotent by Proposition 44. Therefore $J(S)$ is nil. ☑

The next result can be obtained from Theorem 45 and [6, Lemma 2.11].

Corollary 46. *Let M be an indecomposable dual π-Rickart module. If M is morphic, then S is a left and right π-morphic ring.*

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