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Pricing and Revenue Management: The Value of Coordination

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The integration of systems for pricing and revenue management must trade off potential revenue gains against significant practical and technical challenges. This dilemma motivates us to investigate the value of coordinating decisions on prices and capacity allocation in a stylized setting. We propose two pairs of sequential policies for making static decisions—on pricing and revenue management—that differ in their degree of integration (hierarchical versus coordinated) and their pricing inputs (deterministic versus stochastic). For a large class of stochastic, price-dependent demand models, we prove that these four heuristics admit tractable solutions satisfying intuitive sensitivity properties. We further evaluate numerically the performance of these policies relative to a fully coordinated model, which is generally intractable. We find it interesting that near-optimal performance is usually achieved by a simple hierarchical policy that sets prices first, based on a nonnested stochastic model, and then uses these prices to optimize nested capacity allocation. This tractable policy largely outperforms its counterpart based on a deterministic pricing model. Jointly optimizing price and allocation decisions for the high-end segment improves performance, but the largest revenue benefits stem from adjusting prices to account for demand risk.

Keywords: revenue management; pricing; coordination; price-sensitive stochastic demand; hierarchical policies; lost sales rate elasticity
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1. Introduction

Revenue management is common in capacity-constrained service industries—including airlines, hotels, car rentals, event ticketing, and TV advertising—where demand is responsive to price changes. However, revenue management models and practice have traditionally focused on capacity allocation decisions while treating price and demand as exogenous. This focus is partly explained by rigid organizational structures that separate the functions of marketing (including pricing) and operations (revenue management) and also by the technical and operational difficulties inherent in implementing an integrated price–availability decision support system. Indeed, "departmental differences in personnel, expertise and decision-support systems make it difficult to coordinate . . . pricing and yield management decisions" (Jacobs et al. 2000). As a result, a sequential decision process is common in many industries (Talluri and van Ryzin 2004, Chap. 10; Kolisch and Zatta 2012).

Over the past decade, the importance of coordinating decisions on tactical pricing and revenue management has been widely acknowledged in the revenue management literature (McGill and van Ryzin 1999) and by practitioners (Garrow et al. 2006). In a wide-ranging review, Fleischmann et al. (2004) observe that pricing decisions have a direct effect on operations and vice versa. Yet, the systematic integration of operational and marketing functions remains in an emerging stage, both in academia and in business practice.

The need to learn more about the value of integrating pricing and revenue management motivates two broad types of research questions. First, from a modeling perspective, what are the technical challenges entailed by incorporating pricing decisions into a revenue management framework? In particular, what types of demand specifications lead to tractable problems, how should we model price-sensitive demand uncertainty, and when is it actually important to do so? Second, from the practical perspective of assessing benefits, when is it important to integrate pricing and availability decisions, and what is the financial impact of doing so—for example, as compared with a traditional sequential approach? In particular, given the practical limitations of coordination, are there simpler alternatives that can achieve comparable revenues?

Our research addresses these issues by studying four sequential policies that combine pricing with subsequent revenue management decisions and differ along two dimensions: the extent of coordination between price and allocation decisions and the firm's approach to pricing. These heuristics, which are modeled as two-stage stochastic programs, build price sensitivity and optimization into a stylized framework of static, two-fare-class revenue management (Belobaba 1987, Littlewood 1972). This standard building block model of revenue management theory and practice optimizes the nested allocation of limited capacity between two customer segments, where higher-paying customers arrive later in the horizon and where prices and demand are exogenously fixed.

The design of our study is simple but does capture the key elements of pricing and revenue management while allowing us to assess the value of coordinating these decisions under demand uncertainty. We focus on static two-fare-class pricing; preliminary analysis suggests that our main insights do extend to multiple classes. Static pricing is frequently observed in practice, for advertising, administrative, and competitive reasons (Talluri and van Ryzin 2004, p. 334) and it is supported theoretically by consumer behavior considerations (e.g., Besanko and Winston 1990, Nasiry and Popescu 2011). Finally, static models with few prices and independent demand can serve as good sources of approximation for more realistic dynamic problems (Gallego and van Ryzin 1994, Bitran and Caldentey 2003). It can be argued that modern dynamic pricing techniques remove the need for managing capacity allocation because a fare class can be closed by setting sufficiently high prices; however, as we have pointed out, there are many settings where dynamic pricing is not possible or practical, and actual implementations of fully dynamic pricing remain relatively rare.

This paper makes the following main contributions, as intimated by the questions raised at the outset. First, unlike a fully coordinated system, all the sequential policies studied here are proved to be tractable for a broad class of stochastic price-dependent demand models that capture increasing elasticity in the firm's lost sales rate (LSR). Examples include attraction models and additive-multiplicative specifications (e.g., with linear and isoelastic price dependence) with increasing failure rate (IFR). Our conditions on stochastic demand extend deterministic demand regularity conditions (Gallego and van Ryzin 1994, Ziya et al. 2004) as well as single-product newsvendor model assumptions (Kocabıyıkoğlu and Popescu 2011), and they allow for sensitivity results characterizing the interaction of price and capacity decisions. For example, we show that in a hierarchical environment (i.e., one where pricing decisions precede allocation decisions), an increase in the high-end price should be met with a lower protection level—that is, fewer reserved seats for this class—in contrast with implications of the standard revenue management model that does not capture price response. If LSR elasticity is increasing in price and quantity, then firms with expanding capacity should reserve more seats but offer lower prices for high-end customers because they will see lower revenue *rates*—for example, lower revenue per available seat (RAS) for airlines and lower revenue per available room (REVPAR) for hotels.

Second, we quantify the value of coordinating the decisions on pricing and capacity allocation. Throughout this paper, "coordination" refers to the full or partial integration of pricing and allocation decisions. Using extensive numerical simulations, we find that the revenue gains from full coordination can be large (typically 1%–10%) relative to a sequential policy that sets prices based on a deterministic demand model and subsequently optimizes booking limits. These gains increase when demand is large (compared to capacity) or more uncertain. On the other hand, we find it interesting that a similar policy, which adjusts prices to reflect demand risk (based on a tractable nonnested model) and then optimizes nested booking limits, achieves near-optimal performance in most of our simulations. Jointly optimizing price and allocation decisions for the high-end segment improves performance, but the largest revenue benefits typically stem from incorporating demand uncertainty in pricing decisions.

These insights have practical consequences for capacitated firms when one considers the organizational and implementation challenges posed by the integration of pricing and revenue management (Jacobs et al. 2000). Moreover, the financial consequences can be significant because small positive changes in revenue translate into spectacular profit gains for revenue management industries grappling with high fixed costs and extremely thin margins. For example, a 1% increase in revenue would have allowed the car rental company whose data inspired our numerical experiments—which in 2009 posted net profit margins of -1% on revenues of \$5 billion (U.S.)—to break even that year.

2. Relation to the Literature

Our work contributes to the vast literature on revenue management, for which the most comprehensive references to date are the books by Talluri and van Ryzin (2004) and Phillips (2005). McGill and van Ryzin (1999) review the earlier revenue management literature, and Elmaghraby and Keskinocak (2003) focus on dynamic pricing.

There is a growing body of work (reviewed by Bitran and Caldentey 2003) in the revenue management literature that addresses the problem of joint pricing and allocation. Several papers in this area use deterministic demand models to capture complex multiproduct, multiresource, or dynamic environments (e.g., Cote et al. 2003, Kachani and Perakis 2006, Kuyumcu and Popescu 2006). Ziya et al. (2004) analyze demand conditions that ensure regularity in deterministic models.

In contrast, we focus on stochastic demand models: we provide corresponding regularity conditions and assess the value of capturing price-sensitive demand uncertainty, relative to the value of coordination. Toward this end, we focus on a static, two-fare-class capacity allocation model (Belobaba 1987, Littlewood 1972) and extend it to manage and coordinate pricing decisions. A first step in this direction is due to Weatherford (1997), who evaluates numerically the revenue benefits—as a function of the requisite computational effort—from integrating allocation decisions and pricing in a static, single-resource environment with normally distributed additive-linear demand.

A few revenue management papers study joint pricing and allocation problems with aggregate demand uncertainty; they all use additive and/or multiplicative demand forms, which are special cases of our model. Bertsimas and de Boer (2005) provide regularity conditions for a static, partitioned allocation model and additive-multiplicative demand (similar to our model in §4.1) and then use that model to devise a heuristic for a multiperiod price–capacity allocation problem. In the context of nonprofit applications, de Vericourt and Lobo (2009) jointly optimize prices and allocations in a dynamic setting under a multiplicative demand model; their single-stage regularity condition is a special case of our LSR elasticity conditions. In a dynamic setting with competition, Mookherjee and Friesz (2008) assume increasing price elasticity in a multiplicative demand model with increasing generalized failure rate (IGFR) risk. These papers all rely on static regularity conditions to characterize more complex dynamic problems. Our results extend the static regularity conditions in these papers to more general demand models.

Several other approaches have been used for modeling price-sensitive demand uncertainty in revenue management. Dynamic pricing problems characterize price-sensitive stochastic demand as a Markov arrival process, which is typically described as being Poisson distributed with known price and time-dependent intensity (Feng and Xiao 2006, Gallego and van Ryzin 1994, Maglaras and Meissner 2006). Uncertainty about the arrival rate has been addressed in Bayesian learning frameworks (Aviv and Pazgal 2005) or by using robustness methods (Adida and Perakis 2010). Our modeling choice favors instead the simplest framework that allows us to explore the interplay of coordination and uncertainty about (price-sensitive) demand in a revenue management context.

Finally, our work is also related to a vast operations literature on coordinating pricing and inventory decisions, as reviewed by Chan et al. (2004) and Fleischmann et al. (2004). An important distinction is that models in this stream focus on storable goods rather than services. Our model can be viewed as a multiproduct extension of static newsvendor pricing models (for reviews, see Petruzzi and Dada 1999, Yano and Gilbert 2003). Most of this literature characterizes price-sensitive demand uncertainty in terms of additive and/or multiplicative models. Our general demand model and approach are based on Kocabıyıkoğlu and Popescu (2011), who use the concept of increasing LSR elasticity to provide general regularity conditions for the newsvendor pricing problem. Our analytical results in the first part of this paper show that similar demand regularity conditions are sufficient for several sequential pricing and revenue management problems. However, our primary concern differs from the concerns of this literature in that we aim to assess the value of coordinating pricing and capacity allocation decisions relative to a status quo hierarchical business process.

3. Hierarchical and Coordinated Revenue Management Models

In the standard revenue management model (Belobaba 1987, Littlewood 1972), a monopolistic firm optimizes the allocation of a fixed quantity of a flexible resource between two market segments with uncertain demands; the high-price segment arrives after the low-price segment, and prices are predetermined. In reality, firms have the ability to control prices, which in turn affect demand. In particular, the demand in major application areas of revenue management, such as airline travel and car rental, is sensitive to price changes (Talluri and van Ryzin 2004, Chap. 7). To capture price response, we model demand as a general stochastic function of price, $\mathbf{D}(p)$ (see §4) and extend the standard revenue management problem to optimize segment prices (§3.1). To assess the value of coordination, we introduce pricing models (§3.2) that provide input to sequential pricing and revenue management policies (§3.3).

3.1. Price-Sensitive Revenue Management

Let p and $\bar{p}$ denote the high- and low-fare prices (respectively); the corresponding random demands at these prices are $\mathbf{D}(p)$ and $\bar{\mathbf{D}}(\bar{p})$, which are assumed to be independent. Throughout this paper, the parameters pertaining to the low-fare class are denoted by a

bar (overline). Table A.1 in the appendix summarizes our notation.

The standard revenue management model allows for *nested* allocations of the firm's capacity K, which means that all capacity that is not sold to the low-fare class is made available for sale to the high-fare class.[1] Given a *protection level* $x \in [0, K]$ (i.e., the number of units reserved for the high-fare class), sales to the low-price segment are constrained by the *booking limit* $K - x$ and by low-fare demand $\bar{D}(\bar{p})$, so they amount to $\min\{\bar{\mathbf{D}}(\bar{p}), K - x\}$. Thus, the inventory available for sale to the high-fare class is ex ante uncertain and amounts to $\max\{x, K - \bar{\mathbf{D}}(\bar{p})\}$; in particular, it exceeds the protection level x if the low-fare demand falls short of the booking limit—that is, if $\bar{\mathbf{D}}(\bar{p}) \le K - x$. Since low-fare demand is realized before high-fare demand and the two are independent, it follows that expected sales to the high-fare class (conditional on the low-fare demand realization $\bar{\mathbf{D}}(\bar{p}) = \bar{D}$) can be calculated as $\mathbb{E}_{\mathbf{D}}[\min\{\mathbf{D}(p), \max\{x, K - \bar{D}\}\}]$. Taking sequential expectations, the firm's expected revenue from the two nested fare classes may be written as follows:

$$R(\bar{p}, p, x) = \bar{p}\,\mathbb{E}[\min\{\bar{\mathbf{D}}(\bar{p}), K - x\}] + p\,\mathbb{E}[\min\{\mathbf{D}(p), \max\{x, K - \bar{\mathbf{D}}(\bar{p})\}\}]. \quad (1)$$

A *fully coordinated* pricing and revenue management model, (F), simultaneously optimizes the prices p, $\bar{p} \ge 0$ and the protection level $x \in [0, K]$:

$$\text{(F)} \quad R^{***} = \max_{\bar{p}, p, x} R(\bar{p}, p, x). \quad (2)$$

Because model (F) is generally intractable, we study policies based on a partially *coordinated* model, (C), which jointly optimizes the price and allocation for the high-end market, given a low price $\bar{p}$:

$$\text{(C)} \quad R^{**}(\bar{p}) = \max_{p, x} R(\bar{p}, p, x). \quad (3)$$

By contrast, the standard revenue management model optimizes the protection level x, given fixed prices $\bar{p}$ and p. To reflect this *hierarchical* approach of optimizing allocation decisions after prices are set, we refer to this model as (H):

$$\text{(H)} \quad R^{*}(\bar{p}, p) = \max_{x} R(\bar{p}, p, x). \quad (4)$$

As broadly discussed in the introduction, our goal is to assess the value of coordinating decisions on pricing and capacity allocation and to provide tractable alternatives to the fully coordinated but generally intractable model (F). We study four sequential (hierarchical and partially coordinated) policies that employ, in a first stage, pricing heuristics (described in the next section) to provide segment prices, which are then used as input into model (H) or model (C) above.

3.2. Pricing Models

Depending on the industry, several pricing approaches are conceivable and used in practice; these include fixed prices, value-based and cost-plus methods, and matching the competition (Phillips 2005). In this paper we focus on normative, model-based pricing decisions (as opposed to descriptive, judgment-based approaches) and consider two demand-based pricing models that are common in the operations literature: the deterministic model (D) and the stochastic model (S).

The *deterministic pricing model* (Bitran and Caldentey 2003, Gallego and van Ryzin 1994) is a certainty-equivalent (or fluid) benchmark that replaces random demands with their means, $\mu(p) = \mathbb{E}[D(p)]$ and $\bar{\mu}(\bar{p}) = \mathbb{E}[\bar{D}(\bar{p})]$, to solve for optimal "deterministic" prices p^D and $\bar{p}^D$:

$$\text{(D)} \quad \mathcal{D} = \max_{p, \bar{p}} \; p\mu(p) + \bar{p}\bar{\mu}(\bar{p}) \quad \text{s.t.} \;\; \mu(p) + \bar{\mu}(\bar{p}) \le K. \quad (5)$$

The *stochastic pricing model* (Belobaba 1987, Bertsimas and de Boer 2005) is a nonnested version of model (F) that jointly optimizes prices p, $\bar{p}$ together with nonnested allocations for each segment; in other words, capacity is partitioned into separate blocks of size k and $K - k$ that can be sold only to the respective market segments. The optimal segment prices p^S and $\bar{p}^S$ solve the following:

$$\text{(S)} \quad \mathcal{S} = \max_{p, \bar{p};\, k \in [0, K]} p\,\mathbb{E}[\min\{\mathbf{D}(p), k\}] + \bar{p}\,\mathbb{E}[\min\{\bar{\mathbf{D}}(\bar{p}), K - k\}]. \quad (6)$$

This stochastic, nonnested (so-called partitioned allocation) model (S) has also been used to approximate nested or multiperiod revenue management models, which are typically more difficult to solve (Belobaba 1987, Bertsimas and de Boer 2005). In contrast with those papers, which use model (S) as a benchmark for making allocation decisions k, we will use model (S) to make pricing decisions. Unlike model (D), model (S) captures demand uncertainty in pricing decisions—in particular, the variance of both demand classes typically affects $(\bar{p}^S, p^S)$ but does not affect $(\bar{p}^D, p^D)$. Absent demand risk, the two models and corresponding prices coincide, so we can say that (S) adjusts deterministic prices set by (D) to account for (price-sensitive) demand uncertainty.

[1] In contrast, nonnested models partition capacity into blocks designated to each fare class, and these cannot be offered for sale to another class; nonnested models are typically suboptimal but easier to solve.

3.3. Sequential Pricing and Revenue Management Policies

We are now ready to introduce two pairs of sequential policies that combine a (deterministic or stochastic) pricing model, (D) or (S), with a hierarchical or partially coordinated revenue management approach based on (H) or (C), respectively. Table 1 defines the four models of interest: (HD) and (HS) (respectively, (CD) and (CS)) are the hierarchical (respectively, coordinated) models with, respectively, deterministic and stochastic pricing. The operator $\mathscr{R}$ is used to denote the performance of a given policy evaluated by the nested objective R, as defined in (1); in particular, the fully coordinated model (F) achieves the maximum performance $\mathscr{R}[\mathrm{F}] = R^{***}$.

Each policy in Table 1 solves a two-stage stochastic program for making pricing and capacity allocation decisions. Specifically, (HS) sets prices equal to $(p^S, \bar{p}^S)$ determined by the nonnested stochastic pricing model (S) and subsequently optimizes the protection level x for these prices based on model (H), yielding $\mathscr{R}[\mathrm{HS}] = R^*(\bar{p}^S, p^S)$. By contrast, model (CS) *jointly* optimizes the high-end price p and the allocation x, using only the low-end price $\bar{p}^S$ from (S), so $\mathscr{R}[\mathrm{CS}] = R^{**}(\bar{p}^S)$. Models (HD) and (CD) are defined similarly, with (S) replaced by (D). Unlike hierarchical (H) policies, where pricing decisions are oblivious to subsequent allocation decisions, in coordinated (C) policies, the price and allocation decisions for the high-end market are integrated—in other words, high-end prices are set by anticipating optimal protection levels. Intuitively, these models aim to improve on the simple (HD) benchmark along two dimensions: coordination (CD), capturing demand stochasticity in pricing decisions (HS), or both (CS).

Preliminary results on the performance of various policies are summarized next, together with the usual bounds based on the value $\mathscr{D}$ of the deterministic model (D) (e.g., Bitran and Caldentey 2003, Proposition 6). We use the generic symbol (A) to refer to one of the pricing models (D) or (S) and cv^D and $\bar{\mathrm{cv}}^D$ to denote the coefficients of variation of demand at optimal deterministic prices, $\mathbf{D}(p^D)$ and $\bar{\mathbf{D}}(\bar{p}^D)$, respectively.

Table 1 Hierarchical vs. Coordinated Pricing and Revenue Management Policies

	Deterministic Pricing (D)	Stochastic Pricing (S)
(C) Coordinated	(CD) $\mathscr{R}[\mathrm{CD}] = \max_{x,p} R(\bar{p}^D, p, x)$	(CS) $\mathscr{R}[\mathrm{CS}] = \max_{x,p} R(\bar{p}^S, p, x)$
(H) Hierarchical	(HD) $\mathscr{R}[\mathrm{HD}] = \max_{x} R(\bar{p}^D, p^D, x)$	(HS) $\mathscr{R}[\mathrm{HS}] = \max_{x} R(\bar{p}^S, p^S, x)$
	$\{p^D, \bar{p}^D\} = \arg\max$ (D)	$\{p^S, \bar{p}^S\} = \arg\max$ (S)

Proposition 1. *For* $\mathrm{A} \in \{\mathrm{D}, \mathrm{S}\}$, $\mathscr{D} \geq R^{***} = \mathscr{R}[\mathrm{F}] \geq \mathscr{R}[\mathrm{CA}] \geq \mathscr{R}[\mathrm{HA}] \geq (1 - \frac{1}{2}\max\{\mathrm{cv}^D, \bar{\mathrm{cv}}^D\})\mathscr{D}$. *Moreover,* $\mathscr{R}[\mathrm{HS}] \geq \mathscr{S}$, *and both equal* $\mathscr{R}[\mathrm{F}]$ *if either* $\mathbf{D}(p)$ *or* $\bar{\mathbf{D}}(\bar{p})$ *is deterministic.*

Proofs are in the appendix. The result formalizes the intuition that coordination improves policy performance ($\mathscr{R}[\mathrm{CA}] \geq \mathscr{R}[\mathrm{HA}]$) and so does nesting ($\mathscr{R}[\mathrm{HS}] \geq \mathscr{S}$). Nesting is relevant when demand from both classes is uncertain; otherwise, policies based on stochastic pricing (S) are optimal. Although stronger analytical bounds are difficult to obtain, we complement Proposition 1 by assessing the performance of these policies numerically in §6. For a broad set of demand models, we show in the next two sections that the heuristics in Table 1 are indeed tractable.

4. Demand Model and Results for Hierarchical Processes

In this section we obtain conditions for the hierarchical pricing and revenue management models (HD) and (HS) presented in §3 to be tractable and then characterize sensitivity properties for the corresponding price and allocation decisions. Throughout this paper, we make the following assumption on the price-sensitive stochastic demand:[2]

Assumption 1. *Demand is given by* $\mathbf{D}(p) = d(p, \mathbf{Z}) \geq 0$ *a.e. such that* (a) *the random variable* $\mathbf{Z}$ *has finite mean and a continuous price-independent distribution* Φ *with density function* ϕ; (b) *the riskless demand function* $d(p, z)$ *is decreasing in price* p, *strictly increasing in* z, *and twice differentiable in* p *and* z; *and* (c) *the pathwise* (*riskless*) *unconstrained revenue* $\pi(p, z) = pd(p, z)$ *is strictly concave in* p (*i.e.*, $2d_p(p, z) + pd_{pp}(p, z) < 0$).

The random variable $\mathbf{Z}$ captures demand *risk*; in empirical estimation, this can be random noise or an independent variable in a regression model. Conceptually, $\mathbf{Z}$ can be any sales driver that is uncertain and not perfectly controlled by the firm; examples include market size, personal disposable income of the target market, brand awareness, and a reference price (see, e.g., Hanssens et al. 2001). Assumptions 1(a) and 1(b) ensure that the demand distribution $F(p, y) = \mathbb{P}[\mathbf{D}(p) \leq y]$ is continuous with a density $f(p, y)$; the survival (lost sale) function is denoted $L(p, y) = 1 - F(p, y)$.[3] Neither the concavity Assumption 1(c) or demand positivity are necessary for all our results, but they do simplify the analysis; in particular, the former ensures that the deterministic pricing

[2] We use the terms increasing (decreasing) and positive (negative) in their weak sense and denote partial derivatives by corresponding subscripts.

[3] This implies that the objective R in (1) is differentiable because $\mathbb{E}[\min\{\mathbf{D}(p), x\}] = \int_0^x L(p, y)\, dy$ is so.

model (D) has a unique solution, whereas the latter serves to characterize optimal prices for model (S).

For technical convenience, variables are restricted to positive compact intervals, in particular $x \in X = [0, K]$ and $p \in P = [p_{\min}, p_{\max}]$, where $p_{\max}$ is arbitrary, possibly infinite. We assume that $p_{\min} = \arg\max\{d(p, \Phi^{-1}(1-\bar{p}/p)) \mid p \geq \bar{p}\}$; this lower bound on price is used for regularity of coordinated (but not hierarchical) models and seems to be practically unrestrictive (see §4.3 and the appendix). Our results extend to any subintervals of P and X.

4.1. Regularity of Pricing Models and Lost Sales Rate Elasticity

We next characterize structural results for the pricing models introduced in §3.2. We begin by reviewing the well-known microeconomic results for model (D), which is concave owing to Assumption 1(c). An alternative regularity condition is that expected demands $\mu(p)$ and $\bar{\mu}(\bar{p})$ have increasing price elasticity; that is, $E(p) = -p\mu_p(p)/\mu(p)$ and $\bar{E}(\bar{p}) = -\bar{p}\bar{\mu}_{\bar{p}}(\bar{p})/\bar{\mu}(\bar{p})$ are increasing in p and $\bar{p}$, respectively. Define the optimal unconstrained prices $(p^o, \bar{p}^o)$, which solve $E(p^o) = \bar{E}(\bar{p}^o) = 1$.

Remark 1. The deterministic pricing model (D) has a unique solution $(\bar{p}^D, p^D)$ that equals $(p^o, \bar{p}^o)$ if $K \geq \mu(p^o) + \bar{\mu}(\bar{p}^o)$ and otherwise solves

$$\mu(p)(1 - E(p)) = \bar{\mu}(\bar{p})(1 - \bar{E}(\bar{p})) \geq 0, \tag{7}$$

$$\mu(p) = K - \bar{\mu}(\bar{p}). \tag{8}$$

In particular, expected demand at the optimal (D) prices is elastic: $E(p^D) \geq 1$, $\bar{E}(\bar{p}^D) \geq 1$.

To obtain regularity conditions for stochastic models, we rely on a different concept of elasticity: the price elasticity of the rate of lost sales—that is, the percentage change in the lost sales rate $L(p, x)$ with respect to the percentage change in price for a given capacity allocation x.

Definition 1 (Kocabiyikoğlu and Popescu 2011). The LSR elasticity corresponding to $\mathbf{D}(p)$ given a price p and allocation x is defined as $\mathscr{E}(p, x) = -pL_p(p, x)/L(p, x) = pF_p(p, x)/(1 - F(p, x))$.

The LSR elasticity $\bar{\mathscr{E}}(\bar{p}, x)$ for the low-fare class is defined similarly. The next proposition shows how the structural results for the deterministic model (D) extend to its stochastic counterpart (S) through the concept of LSR elasticity. In particular, the pricing problem (S) is tractable for stochastic demand models with LSR elasticity increasing in x. This condition is fairly general and satisfied by most demand specifications used in the literature (see §4.3).

Proposition 2. *Assume that $\mathscr{E}(p, x)$ and $\bar{\mathscr{E}}(\bar{p}, x)$ are increasing in x for all p and $\bar{p}$.*

(a) *The stochastic pricing model (S) has a unique solution $(p^S, \bar{p}^S, k^S)$ that solves*

$$\int_0^k L(p, y)(1 - \mathscr{E}(p, y))\, dy = \int_0^{K-k} \bar{L}(\bar{p}, y)(1 - \bar{\mathscr{E}}(\bar{p}, y))\, dy = 0, \tag{9}$$

$$pL(p, k) = \bar{p}\bar{L}(\bar{p}, K - k), \quad k \in [0, K]. \tag{10}$$

(b) *The optimal price for each product under model (S), keeping all other variables constant, is decreasing in its own allocation and is independent of the other product's price.*

Although the three-variable objective of model (S) is not jointly concave in general, the proof of Proposition 2 shows that it can be optimized as a concave univariate function along the optimal price paths for each segment, as determined by (9). Condition (10) states that capacity should be partitioned so as to balance the marginal expected revenue per inventory unit from each segment. These conditions resemble the deterministic marginal revenue condition in Remark 1.

The increasing LSR elasticity conditions thus extend the elasticity results for the deterministic model (D); in particular, from (9), the lost sales rate at the optimal solution is elastic, $\mathscr{E}(p^S, k^S) \geq 1$ and $\bar{\mathscr{E}}(\bar{p}^S, K - k^S) \geq 1$. The first part of Proposition 2 extends the single-product newsvendor results in Kocabıyıkoğlu and Popescu (2011) to the case of two products sharing a limited resource. A multiproduct extension of Proposition 2 follows along the same lines, generalizing the result obtained by Bertsimas and de Boer (2005) for additive-multiplicative demand models.

4.2. Structural Results for Hierarchical Models

A hierarchical process uses the prices determined by models such as (D) or (S) to make nested capacity allocation decisions based on the revenue management model (H). We next investigate how these protection levels should be set and how they respond to a change in prices. Suppose that in an uncoordinated environment, the marketing department announces a price cut for the high-end segment. Should the revenue management department respond by increasing or decreasing the allocation for this segment? The answer depends on the underlying price-sensitive demand uncertainty, and it helps also to establish structural properties for coordinated models in §5.

The objective function $R(\bar{p}, p, x)$ in (1) is quasiconcave in x, so for any $\bar{p} \leq p$, the optimal protection level $x^*(\bar{p}, p)$ for (H) is the unique solution of

$$L(p, x) = \mathbb{P}[\mathbf{D}(p) \geq x] = \bar{p}/p, \tag{11}$$

if less than K (i.e., if $L(p, K) \leq \bar{p}/p$) and equals K otherwise. Although we refer to p as the "high-end" price,

our models do not exclude the theoretical possibility that $p < \bar{p}$, in which case all policies based on (H), (C), and (F) prescribe $x^* = 0$, i.e., no availability control.[4]

Proposition 3 provides the optimal solution and sensitivity results for hierarchical models (HD) and (HS) based on results from §4.1 and existing comparative statics for the newsvendor with pricing problem (Kocabıyıkoğlu and Popescu 2011, Theorem 1(b)). We focus on sensitivity of $x^*(p) = x^*(p, \bar{p})$ to the high-end price p; the optimal protection level decreases in the low-fare price $\bar{p}$, regardless of price sensitivity, so we selectively omit functional dependence on $\bar{p}$ from notation.

Proposition 3. (a) *Model* (HD) *admits a unique optimal solution* $(\bar{p}^D, p^D, x^{HD} = x^*(\bar{p}^D, p^D))$ *that solves* (7), (8), *and* (11). *If* $\mathscr{E}(p, x)$ *and* $\bar{\mathscr{E}}(\bar{p}, x)$ *are increasing in* x, *then model* (HS) *admits a unique optimal solution* $(\bar{p}^S, p^S, x^{HS} = x^*(\bar{p}^S, p^S))$ *that solves* (9)–(11).

(b) *The optimal protection level* $x^*(p)$ *is decreasing in the high-end price* p *if and only if* $\mathscr{E}^*(p) = \mathscr{E}(p, x^*(p)) \geq 1$. *Moreover, the following alternative conditions are sufficient for* $x^*(p)$ *to be decreasing in* p: (i) $\mathscr{E}^*(p)$ *is increasing in* p, *and* (ii) $\mathscr{E}(p, x)$ *is increasing in* p *for all* x.

Part (a) shows how hierarchical models can be efficiently solved as a system of equations if demand is stochastically decreasing in price in hazard rate order. An example with additive-linear demand is solved explicitly in §4.3, showing that even for such simple models, the optimal (HD) and (HS) policies are generally not comparable.

Part (b) elucidates the relationship between price and protection level for the high-end segment; intuitively, this is determined by two effects that are typically opposed. On the one hand, a price hike increases the marginal return from protecting more capacity for this class, suggesting higher protection levels. On the other hand, increasing (high-end) prices implies a lower rate of lost sales (due to decreased demand) and hence a decrease in protection levels. Whichever effect dominates will determine the direction of change in $x^*(p)$. For example, when demand is not a function of price ($\mathbf{D}(p) \equiv \mathbf{D}$), price changes have no impact on the rate of lost sales ($\mathscr{E} \equiv 0$), and the protection level increases in p. This effect is reversed, however, when demand is sufficiently price sensitive—specifically, whenever the rate of lost sales is elastic with respect to changes in price (along the optimal allocation path; i.e., when $\mathscr{E}^*(p) \geq 1$). The pathwise bound on LSR elasticity fully characterizes this sensitivity result since it is both necessary and sufficient. Verifying the bound or condition (i) requires inverting the demand distribution to obtain $x^*(p)$. A sufficient condition that does not require calculating an inverse is that LSR elasticity be increasing in price.

[4] Alternatively, constraining $\bar{p} \leq p$ would instead prescribe no price discrimination; i.e., $\bar{p} = p$. All our results extend when this constraint is added to our models.

4.3. Examples and Implications for Modeling Demand

The results so far have shown that increasing LSR elasticity conditions, which emulate the well-known deterministic elasticity conditions for model (D), guarantee structural properties for both (HD) and (HS) models. We briefly argue that these demand conditions are intuitive, easy to verify, and relatively unrestrictive. It is natural to assume that demand is decreasing in price in a stochastic sense, and this is precisely what the increasing LSR elasticity condition means:

Remark 2 (Kocabiyikoğlu and Popescu 2011). $\mathscr{E}(p, x)$ is increasing in x if and only if $D(p)$ is stochastically decreasing in p with respect to the hazard rate order. In particular, this holds for additive-multiplicative models $d(p, \mathbf{Z}) = \alpha(p)\mathbf{Z} + \beta(p)$ if $\mathbf{Z}$ is IFR or if $\beta \equiv 0$ and $\mathbf{Z}$ is IGFR.[5]

The IFR assumption, which implies IGFR, is common in the operations literature and imposes mild restrictions on the demand distribution. A broad class of demand models have increasing LSR elasticity with respect to *both* x and p; these include additive-multiplicative and attraction models such as (i) additive-linear, logit, and exponential models with IFR risk $\mathbf{Z}$ and (ii) multiplicative-linear, isoelastic, and power models with IGFR risk $\mathbf{Z}$ (see Kocabıyıkoğlu and Popescu 2011, §5). In particular, our numerical experiments in §6 consider linear demand models with additive and multiplicative uncertainty, with normal (IFR) and gamma (IGFR) distributed risk, respectively; for these models, $\mathscr{E}(p, x)$ is increasing in x as well as in p.

Not all demand models take the form $d(p, \mathbf{Z})$ assumed in this paper. For example, the Poisson model with price-dependent demand rate $\lambda(p)$, which is commonly used in revenue management (Gallego and van Ryzin 1994), does not fit the $d(p, \mathbf{Z})$ form. However, its normal approximation $\mathbf{D}(p) = \lambda(p) + \sqrt{\lambda(p)}\mathbf{Z}$, with $\mathbf{Z} \sim N(0, 1)$ (hence IFR), is of additive-multiplicative form. For this model, $\mathscr{E}(p, x)$ increases in x; it also increases in p if $\lambda(p)$ is concave (or if $p\lambda'(p)$ is decreasing).

[5] By definition, $\mathbf{Z}$ is IFR if it has increasing failure rate $\phi(z)/(1 - \Phi(z))$ and $\mathbf{Z}$ is IGFR if it has increasing generalized failure rate $z\phi(z)/(1 - \Phi(z))$. A distribution $D(p)$ is said to be stochastically decreasing in p with respect to the hazard rate order if its hazard rate $f(p, x)/L(p, x)$ is decreasing in p; this stronger order is equivalent to first-order dominance for a large class of parametric families (Müller and Stoyan 2002, Table 1.1).

Table 2 $\mathscr{E}(p,x)$, $\mathscr{E}^*(p)$ **and** $p_{\min}$ **for Additive-Linear** $(a-bp+\mathbf{Z}, a, b>0)$ **and Multiplicative Isoelastic** $(ap^{-b}\mathbf{Z}, a>0, b>1)$ **Demand Models with Uniform** $(0,l)$ **and Mean-**l **Exponential Risk Z**

$d(p,\mathbf{Z})$	$\mathscr{E}(p,x)$		$\mathscr{E}^*(p)$		$p_{\min}$	
Z	Uniform	Exponential	Uniform	Exponential	Uniform	Exponential
$a-bp+\mathbf{Z}$	$\frac{bp}{l-x+a-bp}$	$\frac{bp}{l}$	$\frac{bp^2}{l\bar{p}}$	$\frac{bp}{l}$	$\max\left(\bar{p}, \sqrt{\frac{l\bar{p}}{b}}\right)$	$\max\left(\bar{p}, \frac{l}{b}\right)$
$ap^{-b}\mathbf{Z}$	$\frac{bx}{lap^{-b}-x}$	$\frac{bx}{lap^{-b}}$	$b\left(\frac{p}{\bar{p}}-1\right)$	$lb\ln\frac{p}{\bar{p}}$	$\bar{p}\left(1+\frac{1}{b}\right)$	$\bar{p}e^{1/b}$

Example 1. To further illustrate the assumptions underlying our results, Table 2 provides expressions for $\mathscr{E}(p,x)$, $\mathscr{E}^*(p)$, and $p_{\min}$ for the additive-linear and the multiplicative isoelastic demand models frequently used in the literature (see, e.g., Petruzzi and Dada 1999) with uniform $(0,l)$ or mean-l exponential risk $\mathbf{Z}$; both distributions are IFR. It is easy to verify that $\mathscr{E}(p,x)$ is increasing and that $\mathscr{E}^*(p)\geq 1$ whenever $p\geq p_{\min}$. Moreover, if $b>l/\bar{p}$ (i.e., if high-fare demand is sufficiently price sensitive), then $p_{\min}=\bar{p}$ for the additive model with exponential risk $\mathbf{Z}$, and so the lower bound $p_{\min}$ is unrestrictive.[6]

To this end, we illustrate how Propositions 2 and 3 serve to solve models (HD) and (HS) under linear-additive demand with exponential mean-l risk; we present this model because it yields closed form solutions. For (HD), we first compute prices by solving model (D) via Remark 1: $\bar{p}^D=(\bar{a}+\bar{l}+\kappa)/2\bar{b}$ and $p^D=(a+l+\kappa)/(2b)$, where $\kappa=((a+l+\bar{a}+\bar{l})/2-K)^+$. Then, from (11), we obtain the protection level for any price pair $(\bar{p},p)$, $x^*(\bar{p},p)=\min(K,(a-bp+l\log(p/\bar{p}))^+)$, which, in particular, for (HD) gives $x^{HD}=x^*(\bar{p}^D,p^D)=\min(K,((a-l-\kappa)/2+l\log((a+l+\kappa)/(\bar{a}+\bar{l}+\kappa))(\bar{b}/b))^+)$. Similarly, (HS) prices solve model (S) via Proposition 2(a): $\bar{p}^S=\bar{l}/\bar{b}$, and $p^S=l/b$, and the corresponding protection level is $x^{HS}=x^*(\bar{p}^S,p^S)=\min(K,(a-l+l\log(l\bar{b}/\bar{l}b))^+)$. Even for this simple model, no systematic ranking of (HS) and (HD) policies holds for all parameter values.

5. Structural Results for Coordinated Models

In a centralized environment, pricing and allocation decisions are made jointly by a single unit of the firm. Alternatively, coordination can be achieved if the marketing function makes pricing decisions while considering the subsequent optimal allocation decision to be made by the revenue management system. In this section we investigate the coordinated model (C), which optimizes expected revenue $R(p,x)$ as a function of high-end price p and allocation x; we omit for simplicity the functional dependence on the low-end price $\bar{p}$, which is kept fixed in this section. In contrast with the full recourse problem (F) which is generally nonconcave, model (C) is shown to be tractable under similar conditions as the hierarchical policies studied in §4.

Practical considerations endorse the relevance of managing the price and allocation decisions for the high-end segment for a given low-end price. In many revenue management settings, such as concerts and sporting events, low-end prices are kept fixed for brand image and for historical, fairness, or social considerations, whereas high-end prices are actively managed. There are also settings—such as airlines, hotels, car rentals, and advertising—in which the low-end market is highly competitive and with little degree of pricing power relative to the high-end segment (Zhang and Kallesen 2008). In fact, the first North American revenue management initiative, the American Airlines "Ultimate Super Saver" program, was purposely designed to conditionally match low-fare competitor People Express in the low-end segment while reserving capacity for higher-margin sales. Major airlines continue to offer low-fare products on a limited basis to compete against low-cost carriers such as Southwest, Ryanair, and EasyJet. In the high-end market, however, airline price dispersion is extremely high (up to 700%, according to Donofrio 2002) and competition less severe, suggesting that price is an important profit lever. These examples further motivate our focus on jointly optimizing allocation and pricing decisions for the high-end segment in this section.

5.1. Regularity Conditions for Model (C)

Model (C) is generally not jointly concave in the price and allocation decision for the high-end class. This coordinated problem can be viewed, equivalently, as a pricing model with recourse: the high-end price p is determined by anticipating that the protection level is optimally set in response to this price, $x=x^*(p)$, so the problem amounts to optimizing the univariate objective $R^*(p)=R(p,x^*(p))$. We show that this univariate objective is concave if the LSR elasticity is increasing in price or, alternatively, if it is larger than 1/2 along the optimal allocation path $x^*(p)$.

[6] This lower bound is used for sufficiency conditions in Proposition 3(b(i), (ii)). By definition, $p_{\min}\geq\bar{p}$ yields the largest possible protection level in (11).

Proposition 4. *Suppose that one of the following conditions holds*: (a) $\mathscr{E}^*(p) \geq 1/2$ *for all* p, (b) $\mathscr{E}^*(p)$ *is increasing in* p, *or* (c) $\mathscr{E}(p, x)$ *is increasing in* p *for all* x. *Then model* (C) *can be efficiently solved as a concave univariate problem and admits a unique optimal solution* (p^{**}, x^{**}).

In short, the conditions that guaranteed sensitivity results for hierarchical models (Proposition 3(b)) ensure regularity of the coordinated model (C). The conditions in Proposition 4 are satisfied by most demand functions of practical interest, as we argued in §4.3. This result also shows that regularity conditions in the revenue management context are no stronger than those that coordinate the simpler, price-setting newsvendor problem (Kocabıyıkoğlu and Popescu 2011, Theorem 2). In some cases, the lower bounds of 1/2 on LSR elasticity are not only sufficient but also necessary for concavity of the revenue function. For example, if d is linear in p (i.e., if $d(p, z) = \zeta(z) - p\xi(z)$), then it can be shown that $\mathscr{E}^* \geq 1/2$ is both necessary and sufficient for the concavity of $R^*(p)$. Therefore, no weaker constant bound can be expected to hold for all demand functions.

5.2. Extension: Substitution Effects

The coordinated model described so far assumes that demand for each class depends on its own fare price but not on the fare price of the other class since the market is perfectly segmented into low- and high-fare customers. Traditionally, airlines have achieved this segmentation by designing product fences (restrictions) such as booking more than 14 days prior to departure or staying over a Saturday night. However, in other practical settings (e.g., event ticketing) where perfect segmentation is more difficult to achieve, firms offer comparable products and the demand for a product may increase with the price of a substitute.

In this section we show that our results for model (C) extend when decisions on the high-end price p also affect low-fare demand, $\bar{\mathbf{D}}(p) = \bar{d}(p, \bar{\mathbf{Z}})$, where $\bar{d}(p, \bar{z})$ is increasing in p; we omit again the functional dependence on $\bar{p}$ for notational convenience. The effect of the high-end price p on *both* demand classes complicates our original model (C) as follows:

$$\max_{p,\,x} \mathbb{E}_{\bar{\mathbf{D}}}\big[\bar{p}\min\{\bar{\mathbf{D}}(p), K - x\} + p\,\mathbb{E}_{\mathbf{D}}[\min\{\mathbf{D}(p), \max\{x, K - \bar{\mathbf{D}}(p)\}\}]\big]. \quad (12)$$

Proposition 5. *Assume that* $\bar{d}_{pp} \leq 0$. *Then* (12) *has a unique price–allocation solution if either of the following conditions holds*: (a) $\mathscr{E}(p, x)$ *is increasing in* p *or* (b) $\mathscr{E}^*(p)$ *is increasing in* p.

This result shows that increasing LSR elasticity conditions continue to ensure structural properties even when the segmentation between classes is imperfect. The additional assumption of diminishing marginal impact of substitute high-end prices on low-fare demand holds for additive-linear demand systems $\mathbf{D}(p) = \mathbf{Z} - bp$, $\bar{\mathbf{D}}(p) = \bar{\mathbf{Z}} + \bar{b}p$ (e.g., Elmaghraby and Keskinocak 2003) as well as for multiplicative isoelastic models $\mathbf{D}(p) = p^{-b}\mathbf{Z}$, $\bar{\mathbf{D}}(p) = p^{\bar{b}}\bar{\mathbf{Z}}$, where $b, \bar{b} \geq 0$. For these models, $\mathscr{E}(p, x)$ increases in p if $\mathbf{Z}$ is IGFR (see Kocabıyıkoğlu and Popescu 2011, Table 2). Our model assumes independent risks $\mathbf{Z}, \bar{\mathbf{Z}}$ and captures substitution through price response; future research is needed to account for correlations between demand classes, in the spirit of Brumelle et al. (1990).

5.3. Summary and Sensitivity Results

We conclude our analytical investigation by providing sensitivity results that characterize the impact of capacity on joint pricing and allocation decisions as well as on optimal revenues.

In a hierarchical revenue management process, Littlewood's rule (11) implies that for a given price p, the optimal protection level is independent of capacity (or equal to it). However, this statement no longer holds when price and allocation decisions are made jointly. Our next result characterizes the effect of capacity on the optimal coordinated price–allocation solution. In particular, it confirms that optimal high-end prices decrease with capacity even when these prices are coordinated with allocation decisions. We shall further study the effect of capacity on the (marginal) revenues of model (C), $R^{**}(K) = R(p^{**}, x^{**}; K)$ and on the revenue rate per capacity unit $R^{**}(K)/K$.

Proposition 6. (a) *If* $\mathscr{E}(p, x)$ *is increasing in* p *and* x, *then* $p^{**}(K)$ *decreases with capacity* K *and* $x^{**}(K)$ *increases with capacity* K. (b) *The optimal revenue* $R^{**}(K)$ *from the coordinated model* (C) *is increasing and concave in capacity* K, *whereas the optimal revenue per unit of capacity,* $R^{**}(K)/K$, *is decreasing in* K.

In sum, firms that experience a freeing up or expansion of capacity should expect more revenue but lower revenue rates (e.g., lower RAS for airlines and lower REVPAR for hotels). Such firms should therefore set lower prices for the high-end segment but at the same time increase the protection level, if LSR elasticity is increasing in price and quantity. Our numerical results in the next section suggest that these sensitivity properties for model (C) mirror those for the fully coordinated model (F) and extend to all the sequential models described in Table 1.

To conclude, our analytical results suggest that increasing LSR elasticity is a unifying condition that enables us to solve efficiently the four pricing and revenue management models in Table 1 and also to characterize their sensitivity properties.

Table 3 Regularity Conditions for Hierarchical and Coordinated Models in Terms of LSR Elasticity

	Deterministic Pricing	Stochastic Pricing
Coordinated	(CD) $\mathscr{E}$ increasing in p	(CS) $\mathscr{E}$ increasing in p and x
Hierarchical	(HD) —	(HS) $\mathscr{E}$ increasing in x

Corollary 1. *Sufficient regularity conditions for the models* (HD), (HS), (CD), *and* (CS) *are summarized in Table* 3. *In particular, these models can all be solved as concave univariate problems for demand models that feature increasing LSR elasticity in p and x.*

6. Performance Assessment: Numerical Insights

In this section we provide a numerical analysis to evaluate the performance of the hierarchical and coordinated policies for pricing and revenue management described in Table 1. We quantify the benefits of coordinating decisions on pricing and allocation and of accounting for demand uncertainty in pricing. Motivated by existing literature (e.g., Weatherford 1997) and by our analysis of a booking data set for rental cars provided by Avis Europe in §6.2, our numerical experiments focus on linear demand models with either additive or multiplicative uncertainty. Our general insights appear to be robust to the specification of the demand function, the distributional assumptions on the risk variables $\mathbf{Z}$ and $\bar{\mathbf{Z}}$, and the choice of parameter values.

6.1. Random Parameter Sampling

As a first step toward assessing the relative performance of the various policies, we designed a simulation study (as in, e.g., Jain et al. 2011) to generate problem instances under linear demand with both additive and multiplicative uncertainty. The additive-linear demand model is given by $\mathbf{D}(p) = a - bp + \sigma\mathbf{Z}$ and $\bar{\mathbf{D}}(\bar{p}) = \bar{a} - \bar{b}\bar{p} + \bar{\sigma}\bar{\mathbf{Z}}$, where $\mathbf{Z}$ and $\bar{\mathbf{Z}}$ have independent standard normal distributions. The linear-multiplicative model is $\mathbf{D}(p) = (a - bp)\mathbf{Z}$ and $\bar{\mathbf{D}}(\bar{p}) = (\bar{a} - \bar{b}\bar{p})\bar{\mathbf{Z}}$, where $\mathbf{Z}$ and $\bar{\mathbf{Z}}$ have independent gamma distributions with unit mean. Under both models, the LSR elasticity is increasing in both p and x, and $\pi(p, z)$ is strictly concave in p. According to Corollary 1, all models in Table 1 can be solved efficiently and admit a unique solution.

At each of 200 iterations, we randomly chose the parameters of these demand models, computed the optimal revenues from all policies A $\in$ {HD, HS, CD, CS}, and assessed their performance relative to the optimal revenue from the fully coordinated policy (F), $\mathscr{R}$[A]/$\mathscr{R}$[F]. We find the optimal (F) solution via a search algorithm; preliminary analysis suggests that our demand conditions may not be sufficient for (F) to be (pathwise) quasi-concave.

6.1.1. Additive Demand Models. To reduce the number of parameters (from seven to five) for the simulation scenarios, we used the following reparametrization. Without loss of generality, we take the total capacity $K = 1$. We denote the total market size by M (effectively measured in multiples of K) and the fraction of high-end customers in the market by $f \in [0, 0.5]$; with this notation, $a = fM$ and $\bar{a} = (1-f)M$ in the original demand models. We also rescale prices so that without loss of generality $E[\mathbf{D}(p)] = Mf(1-p)$ and $E[\bar{\mathbf{D}}(\bar{p})] = M(1-f)(1-\bar{p}/\gamma)$, where $\gamma \in [0, 1]$ (i.e., the high-end demand has a higher maximum willingness to pay). To ensure a high probability of positive demand, we set an upper bound of 1.2 on the coefficient of variation of demand $\mathrm{cv} = \mathrm{cv}(\mathbf{D}(p^0)) = 2\sigma/a$ at unconstrained prices $p^0 = a/2b$ and similarly for $\bar{\mathbf{D}}(\bar{p})$.[7] Then, at each iteration we randomly and independently generated five parameters: M from the uniform distribution on $[0, 12]$ (i.e., the overall market size can reach up to 12 times capacity); f from the uniform distribution on $[0, 0.5]$; γ from the uniform distribution on $[0, 1]$; and cv and $\bar{\mathrm{cv}}$ from the uniform distribution on $[0, 1.2]$.

Figure 1 plots the histograms over all iterations of $\mathscr{R}$[A]/$\mathscr{R}$[F], the performance of each policy A relative to the optimum F;[8] the optimality gap $1 - \mathscr{R}$[A]/$\mathscr{R}$[F] illustrates the value of full coordination. It is apparent that the policies based on stochastic pricing (HS and CS) generally lead to revenues closer to the optimal revenue $\mathscr{R}$[F] than do the policies based on deterministic pricing (HD and CD). Moreover, the similarity of histograms for (CD) and (HD), as well as that of histograms for (CS) and (HS), suggests that for any given pricing strategy, the benefits of coordination over a hierarchical approach are generally marginal.

We examine these insights in more detail by separately assessing the differential value of stochastic pricing and of coordination, thereby confirming and complementing the insights from Proposition 1. To assess the value of stochastic pricing, Figure 2 plots the performance difference between the stochastic pricing policies (CS) and (HS) and their respective deterministic pricing counterparts (CD) and (HD), relative to the optimum revenue $\mathscr{R}$[F]. The benefit of using (S) over (D) to set prices is prevalent and can be substantial for both hierarchical and partially coordinated heuristics. Figure 3 illustrates the value of (partial) coordination, as measured by the

[7] Indeed, this implies $\mathbb{P}(\mathbf{D}(p = 0) > 0) = \mathbb{P}(\mathbf{Z} \geq -a/\sigma) \geq \mathbb{P}(\mathbf{Z} \geq 2/1.2) \geq 0.95$.

[8] For simplicity, in the figure captions we omit the operator $\mathscr{R}$ from notation.

Figure 1 **Histograms of Policies' Performance Relative to Optimum (F), Additive Demand Model**

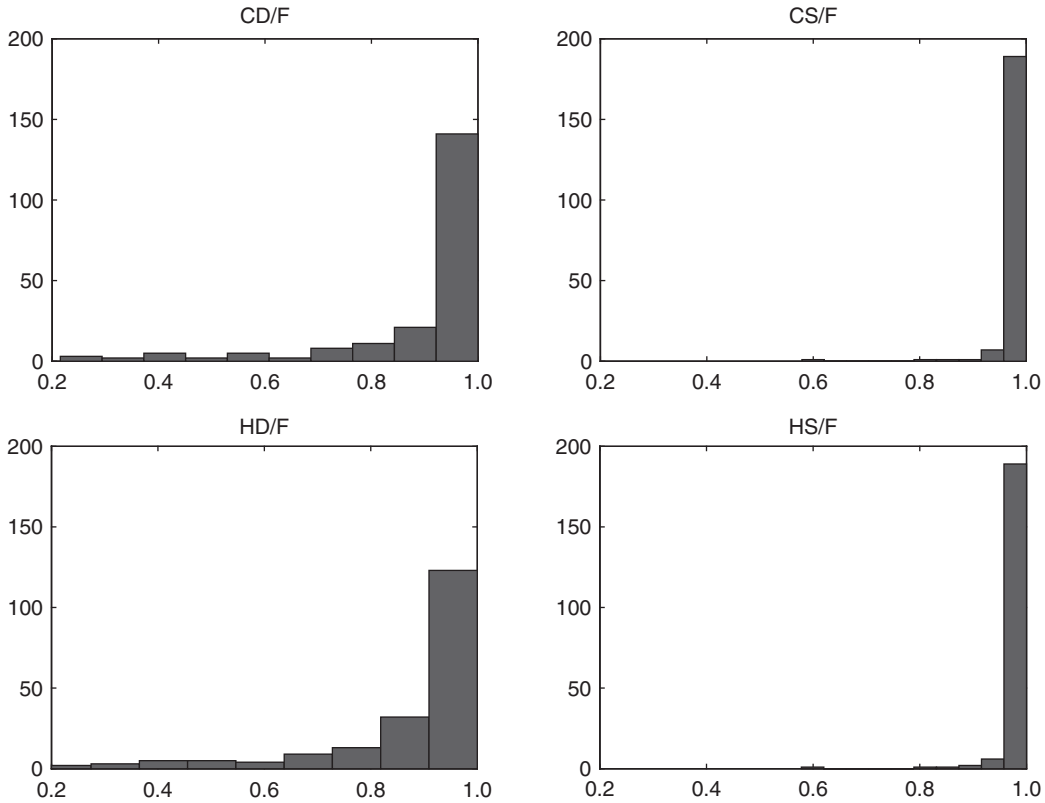


Figure 2 **Histograms of Value of Stochastic Pricing, Additive Demand Model**

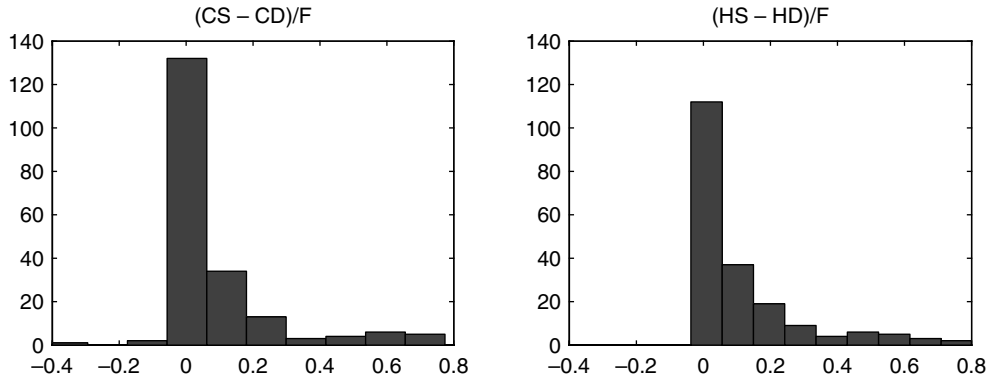


Figure 3 **Histograms of Value of Coordination, Additive Demand Model**

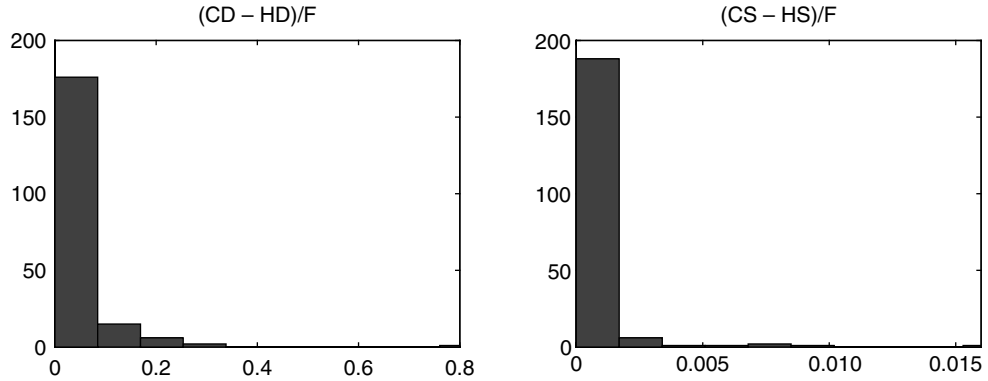
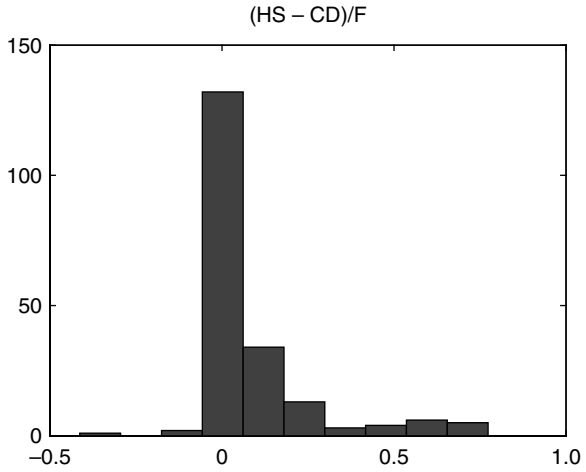


Figure 4 **Relative Performance of (HS) and (CD), Additive Demand Model**



performance difference between the coordinated policies (CD) and (CS) and their respective hierarchical counterparts (HD) and (HS), relative to the optimum revenue $\mathcal{R}[F]$. This value is always positive, consistent with Proposition 1, but relatively small, especially for models based on (S) prices. Confirming the insights from Figure 1, these histograms show that policies (HS) and (CS) have almost identical performance and dominate (HD).

Figure 4 assesses the relative performance of (HS) and (CD) as the difference of their revenues relative to the optimal revenue $\mathcal{R}$[F]. Typically (HS) outperforms (CD) in 85% of iterations; otherwise, the difference tends to be small, although in a very few cases it can be substantial (up to 41% of $\mathcal{R}$[F]). These insights are confirmed throughout our numerical experiments.

6.1.2. Multiplicative Demand Models. We replicated the simulation study under the assumption that demand follows a multiplicative model of the form $\mathbf{D}(p, \mathbf{Z}) = (a - bp)\mathbf{Z}$ and $\bar{\mathbf{D}}(\bar{p}, \bar{\mathbf{Z}}) = (\bar{a} - \bar{b}\bar{p})\bar{\mathbf{Z}}$, where demand risks $\mathbf{Z}$ and $\bar{\mathbf{Z}}$ have unit mean gamma distributions.[9] Using a similar parametrization and parameter ranges as in §6.1.1, we assessed the relative performance of all policies over 200 iterations of randomly generated parameters. Figures 5–8 plot the histograms of policies' performance for multiplicative demand, similar to the histograms for the additive model from §6.1.1. These graphs suggest that our insights about the superior performance of (HS) versus (CD), and about the relative benefits of stochastic pricing versus coordination, are robust for multiplicative demand models across a wide range of parameter scenarios.

[9] The coefficients of variation of high-fare and low-fare demand are then equal to the standard deviations of $\mathbf{Z}$ and $\bar{\mathbf{Z}}$, respectively; these are then sampled randomly from the uniform distribution on $[0, 1.2]$, like for the additive model.

6.2. Sensitivity Analysis: Factors Affecting Policy Performance

In this section we conduct sensitivity analysis to better understand which factors affect the performance of the policies defined in Table 1. We focus on a linear-additive demand model and anchor our experiments on the following set of parameters inspired by the analysis of a car rental data set obtained from Avis: $a = 30$, $b = 0.25$, $\sigma = 2$ for the high fare class and $\bar{a} = 80$, $\bar{b} = 2.00$, $\bar{\sigma} = 12$ for the low-fare class.[10] We further vary these parameters, as well as capacity levels K, to provide sensitivity results. Extensive numerical experiments with a wide range of parameters suggest that the insights illustrated here are robust (see also §6.3). In particular, we repeated these experiments with parameters anchored on Weatherford (1997) and confirmed the same insights under both additive and multiplicative demand uncertainty.

6.2.1. The Effect of Capacity. In revenue management, the load in the market is measured ex ante by the demand factor, which is the ratio of expected demand to capacity. In our set-up, expected demand is a function of selling prices that are not determined a priori, so the demand factor is policy specific. The results in this section are obtained by varying the capacity K via the (unconstrained) *demand factor*, $\Lambda = \Lambda(p^o, \bar{p}^o) = (\mu(p^o) + \bar{\mu}(\bar{p}^o))/K = \frac{1}{2}(a + \bar{a})/K$, corresponding to the optimal unconstrained prices ($p^o = p^D(K = \infty) = a/2b$ and $\bar{p}^o = \bar{p}^D(K = \infty) = \bar{a}/2\bar{b}$) as defined in §4.1.[11] Revenue management is most relevant when capacity is binding yet ample enough to serve both segments ($\Lambda \in [1, 5]$ for the fluid model); for completeness, we report results for $\Lambda \in [0.5, 5]$.

The upper-left panel of Figure 9 shows how the performance of each policy as a percentage from optimum revenue, $\mathcal{R}$[F], varies with capacity, as reflected in the demand factor Λ. Confirming our insights from §6.1, the performance of (HS) is very close to the upper bound of (F) and practically undistinguishable from its partially coordinated counterpart (CS). In contrast, (CD) and (HD) typically exhibit

[10] The data consisted of prices and car rentals by individual customers at four major European airports between January 1, 2008, and March 31, 2008. Demand for car rentals is highly heterogeneous and has complex dynamics driven by regional and socioeconomic factors, so from our price-only data it was not possible to provide an exhaustive analysis of the price–demand relationship for car rentals in the absence of other factors. Instead, we used these data to derive an anchor set of parameter values for the additive demand model with normal risk (this model fit our data better than other, e.g., multiplicative, specifications). For this model, the lower bound on price $p_{\min}$ introduced in §4 is practically unconstraining, as illustrated in the appendix.

[11] We emphasize that Λ is different from (and typically much larger than) the actual demand factor, which depends on the firm's pricing policy. In fact, as long as $\Lambda \geq 1$, the demand factor at deterministic prices is $\Lambda(p^D, \bar{p}^D) = (\mu(p^D) + \bar{\mu}(\bar{p}^D))/K = 1$; i.e., capacity is binding in the fluid model (Remark 1).

Figure 5 **Histograms of Policies' Performance Relative to Optimum (F), Multiplicative Demand Model**

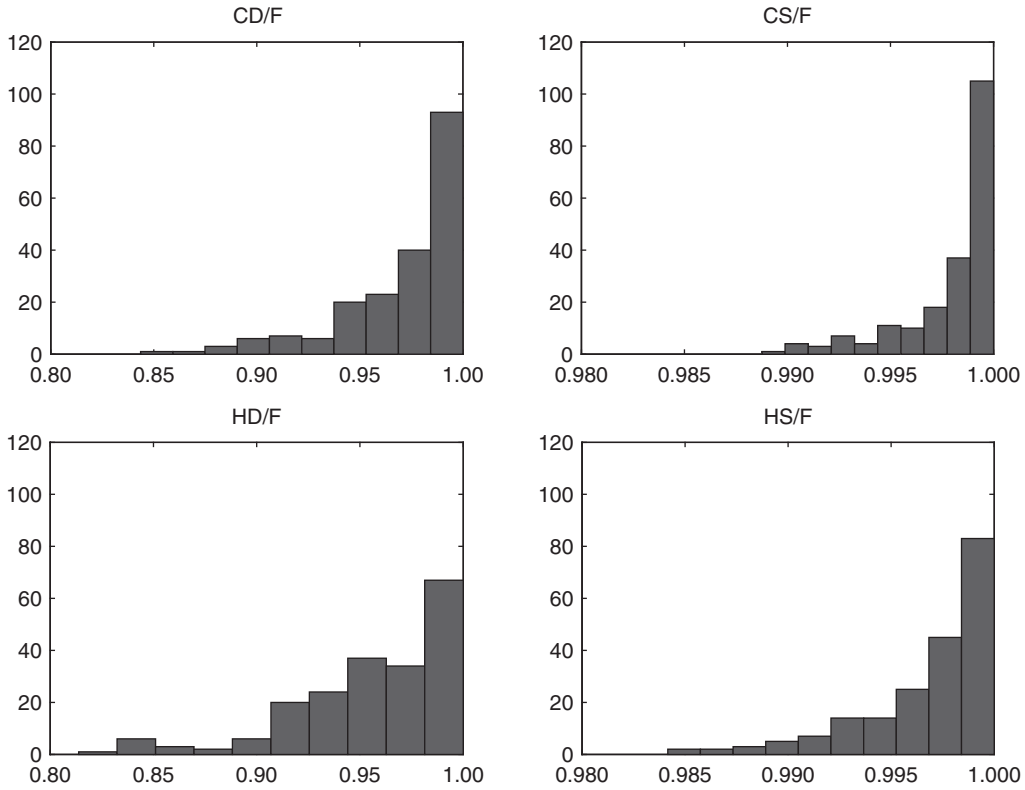


Figure 6 **Histograms of Value of Stochastic Pricing, Multiplicative Demand Model**

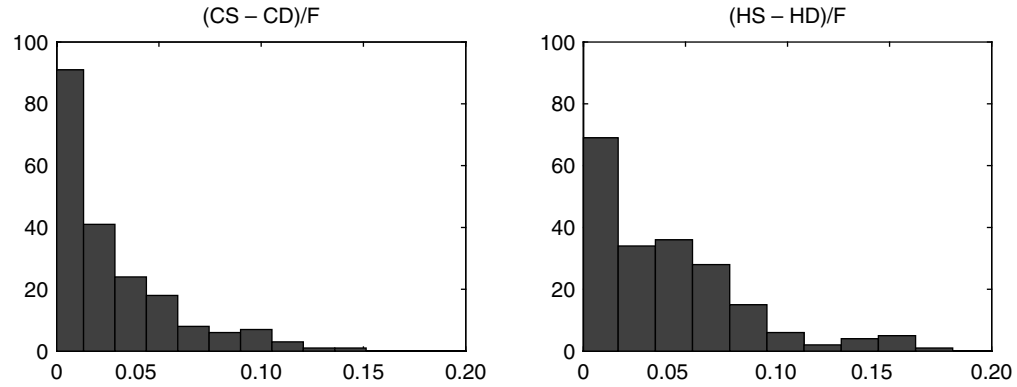


Figure 7 **Histograms of Value of Coordination, Multiplicative Demand Model**

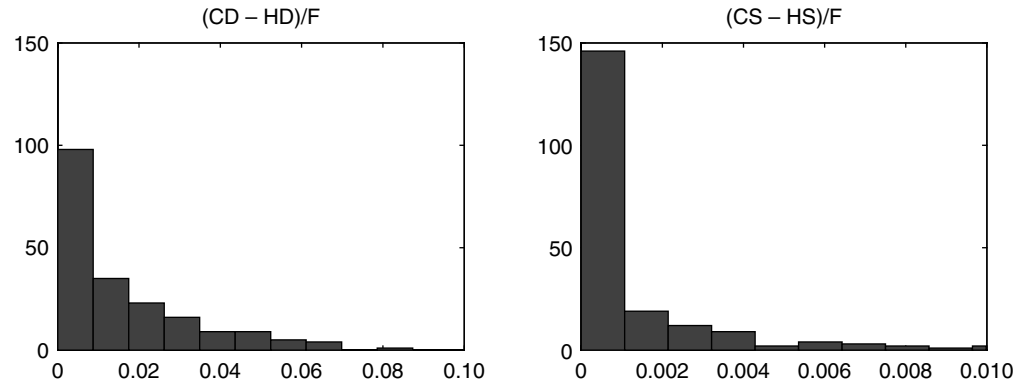
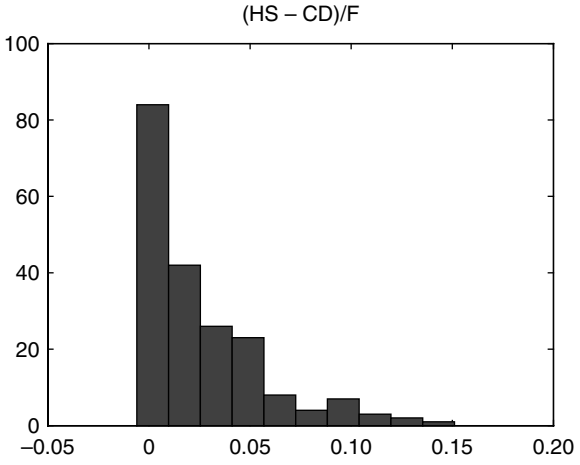


Figure 8 **Relative Performance of (HS) and (CD), Multiplicative Demand Model**



significantly larger optimality gaps. The panels on the right show that (HS) systematically sets nearly optimal prices, which can be significantly higher or lower than those set by (HD) or (CD); this explains the superior performance of (HS) relative to these policies.

The value of coordination is small for low demand factors because revenues from all policies tend to be the same. Intuitively, when capacity is ample, the value of protecting capacity diminishes and demand uncertainty becomes less relevant for pricing as prices converge to the unconstrained optima ($p^o = 60$; $\bar{p}^o = 20$). Figure 9 further suggests that capacity has a nonmonotone effect on policy performance. In particular, there appears to be an intermediate capacity level (here, $\Lambda \simeq 1.5$) where pricing policies determined by (D) and (S) single-cross the optimal pricing policy, so all heuristics perform near optimally.[12] Confirming Proposition 6, the absolute revenue per capacity unit (not reported here) decreases with capacity for all policies, as do the optimal prices (right panels of Figure 9).

In summary, relative to the hierarchical model with deterministic prices (HD), the value of full coordination is typically substantial, particularly when capacity is scarce. Relative to (HS), however, the value of full coordination is substantially lower, suggesting that most coordination benefits actually stem from adjusting prices (up or down) to reflect demand risk, consistent with the insights from §6.1.

6.2.2. The Effect of Demand Variability. We next investigate the effect of demand variability on policy performance, complementing the theoretical bounds in Proposition 1. We keep the same parameters as in the previous section ($a = 30$, $b = 0.25$, $\bar{a} = 80$, $\bar{b} = 2.00$, $\sigma = 2$, $\bar{\sigma} = 12$) and fix $\Lambda = 2$—a choice that is explained and expanded by our analysis in the next section.

To study the impact of overall demand variability, we first scale σ and $\bar{\sigma}$ proportionally by a factor $\theta \in [0, 1]$. We then plot, in the left panel of Figure 10, the percentage revenues relative to the optimal policy (F) as a function of θ. As the left panel confirms, all policies converge as demand becomes more predictable ($\theta \to 0$). The relative value of full coordination increases with overall demand variability—in other words, as the system becomes more difficult to control. As before, the (HS) policy outperforms (HD) and (CD) and is close to the fully coordinated upper bound (F). As variability increases, prices set with (HD) and (CD) are increasingly distant from the (F) optimal ones, which are closely replicated by (HS) (Figure 10, right panel). Low-end prices, not reported here, exhibit similar patterns. This disparity in prices appears to drive the trend in the value of coordination, illustrating the high cost of ignoring demand uncertainty when deciding on prices.

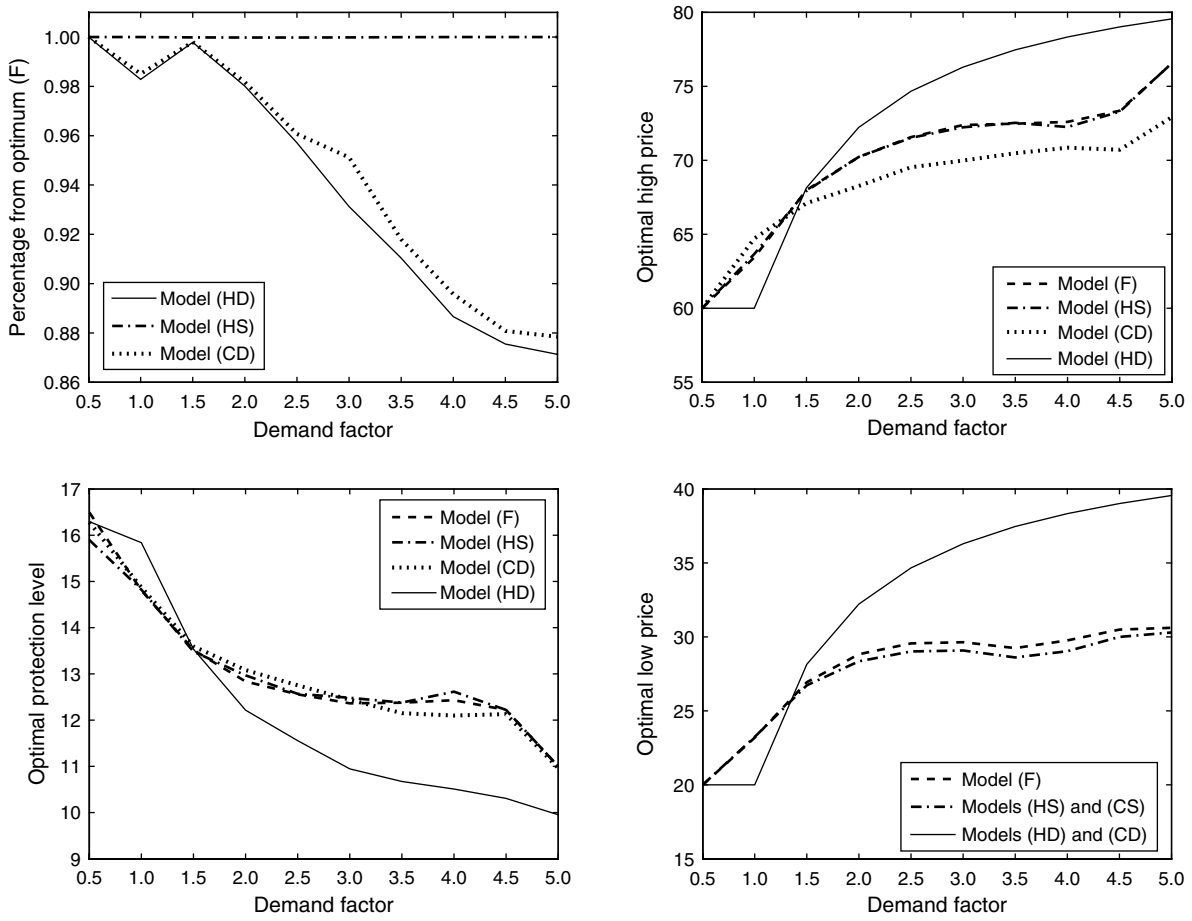
We also study the revenue impact of unilaterally increasing either high-end or low-end demand variability as measured by the corresponding coefficients of variation. We separately vary the values of the standard deviations σ and $\bar{\sigma}$ of $\mathbf{Z}$ and $\bar{\mathbf{Z}}$ so that the coefficients of variation of the base demand $\mathbf{D}(p^o)$ and $\bar{\mathbf{D}}(\bar{p}^o)$, $\mathrm{cv} = 2\sigma/a$, and $\bar{\mathrm{cv}} = 2\bar{\sigma}/\bar{a}$ range between 0.1 and 1.0. This corresponds to a range of $(1.5, 15.0)$ for σ and of $(4.0, 40.0)$ for $\bar{\sigma}$. For consistency with the values in the rest of this section, when σ varies we fix $\bar{\sigma} = 12$ and when $\bar{\sigma}$ varies we fix $\sigma = 2$.

Figure 11 plots the percentage revenues relative to the optimal policy (F) as a function of variability in the high- and low-end demand, respectively. The value of full coordination is greater for all policies when low-end demand becomes more variable, confirming our previous insights. For high-end demand, however, this effect reverses for policies (HD) and (CD) based on deterministic prices, as the left panel of Figure 11 illustrates. In particular, this figure captures a situation where (CD) modestly dominates (HS). This occurs when high-end demand is highly variable ($\mathrm{cv} \geq 0.65 \simeq 2\bar{\mathrm{cv}}$); in this case, intuitively, coordinating decisions on high-end price and allocation becomes more important. Even so, the next section suggests that the situation depicted in Figure 11 is not typical and it is contingent on the value of the demand factor.[13]

[12] This is similar to a 0.5 critical fractile in newsvendor models with symmetric demand distribution, where the deterministic model policy is optimal.

[13] We emphasize that all measures reported here are relative; the *absolute* expected revenues from all policies (not reported here) decrease with variability in both demands and with overall variability (θ) because the value of information increases.

Figure 9 **Demand Factor (Capacity) vs. Percentage from Optimum Revenue $\mathcal{R}[F]$ and Optimal Prices and Protection Levels**



6.2.3. Capacity and Demand Variability: Joint Analysis. To better understand what drives the relative performance of policies (HS) and (CD), we jointly analyze the impact of capacity and high-end demand variability. We computed the optimal revenues from (HS), (CD), and (F) for a grid of values of the demand factor ranging from 0.5 to 3 and values of the coefficient of variation of high-end demand ranging from 0.1 to 1.2. Figure 12 presents a two-dimensional comparison between the performance of (HS) and (CD) relative to the optimal policy (F); the vertical axis represents the demand factor Λ, and the horizontal axis gives the coefficient of variation of high-end demand at optimal unconstrained prices, cv.

Figure 10 **Percentage Revenue from Optimum (F) and High-End Price vs. Overall Demand Variability (θ)**

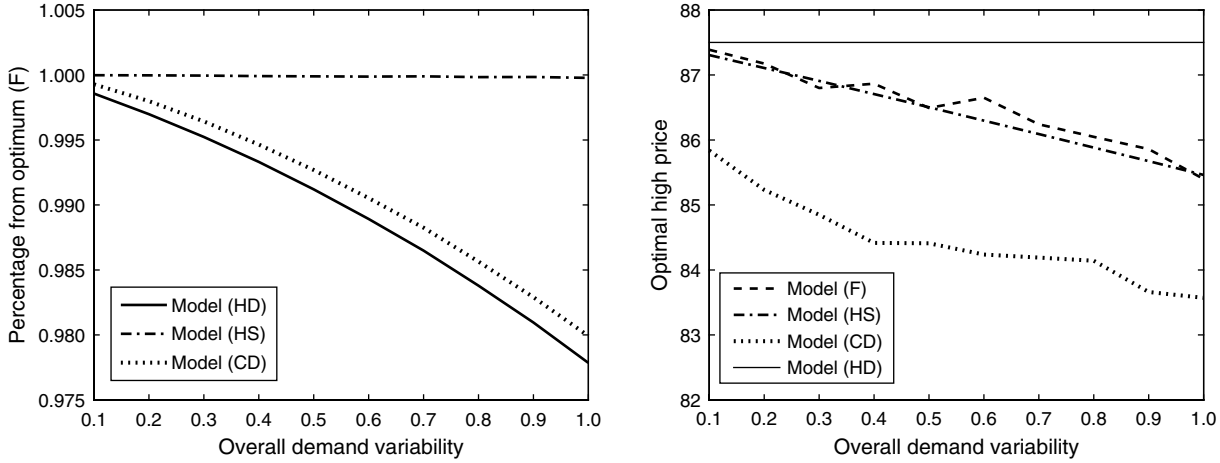
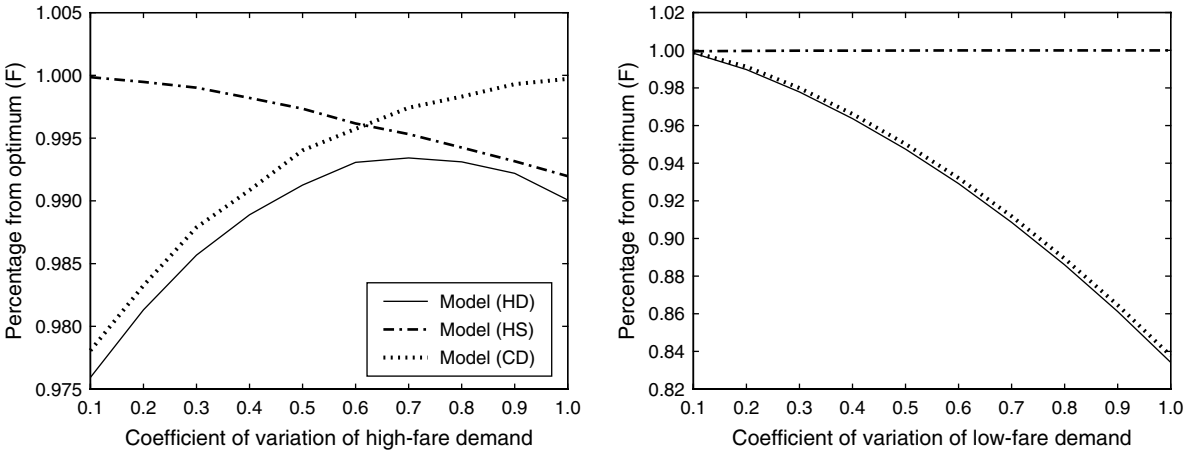


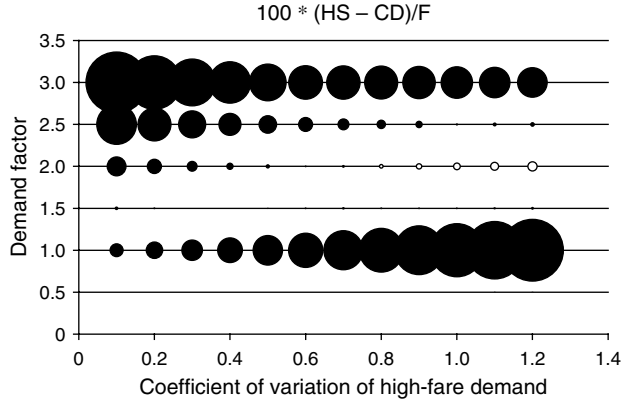
Figure 11 **Percentage Revenue from Optimum (F) vs. High-End and Low-End Demand Variability; $\Lambda = 2$**



The bubble size is proportional to the magnitude of the percentage difference between the revenues of (HS) and (CD) relative to (F); empty bubbles correspond to negative values, and filled ones stand for positive values. Consistent with Figure 11, (HS) outperforms (CD), unless the coefficient of variation of high-end demand is very high (above 0.7). Moreover, the effect is limited to a particular range of demand factors $\Lambda \approx 2$.

To further illustrate the drivers and magnitude of these effects, Figure 13 presents in several graphs the relative performance of (HS) and (CD) policies as a function of the demand factor and of the variability of high-end demand. The top two panels are plots of the value of full coordination for (HS) and (CD), as captured by the percentage optimality gaps $100 * (1 - \mathscr{R}[\mathrm{HS}]/\mathscr{R}[\mathrm{F}])$ and $100 * (1 - \mathscr{R}[\mathrm{CD}]/\mathscr{R}[\mathrm{F}])$, respectively. It is apparent that the optimality gaps are generally nonmonotone with the demand factor. The performance of (HS) deteriorates with increasing variability of high-end demand, whereas that of (CD) improves, at least for sufficiently high demand factors. Nevertheless, the optimality gaps for (CD) are in most cases up to an order of magnitude larger than the optimality gaps for (HS). Consistent with the insights from Figure 12, the bottom two panels of Figure 13 give three-dimensional and contour plots of $100 * (\mathscr{R}[\mathrm{HS}] - \mathscr{R}[\mathrm{CD}])/\mathscr{R}[\mathrm{F}]$, the percentage revenue difference between (HS) and (CD) relative to the optimum revenue of (F).

Figure 12 **Relative Performance Difference of (HS) and (CD) as Percentage of (F)**

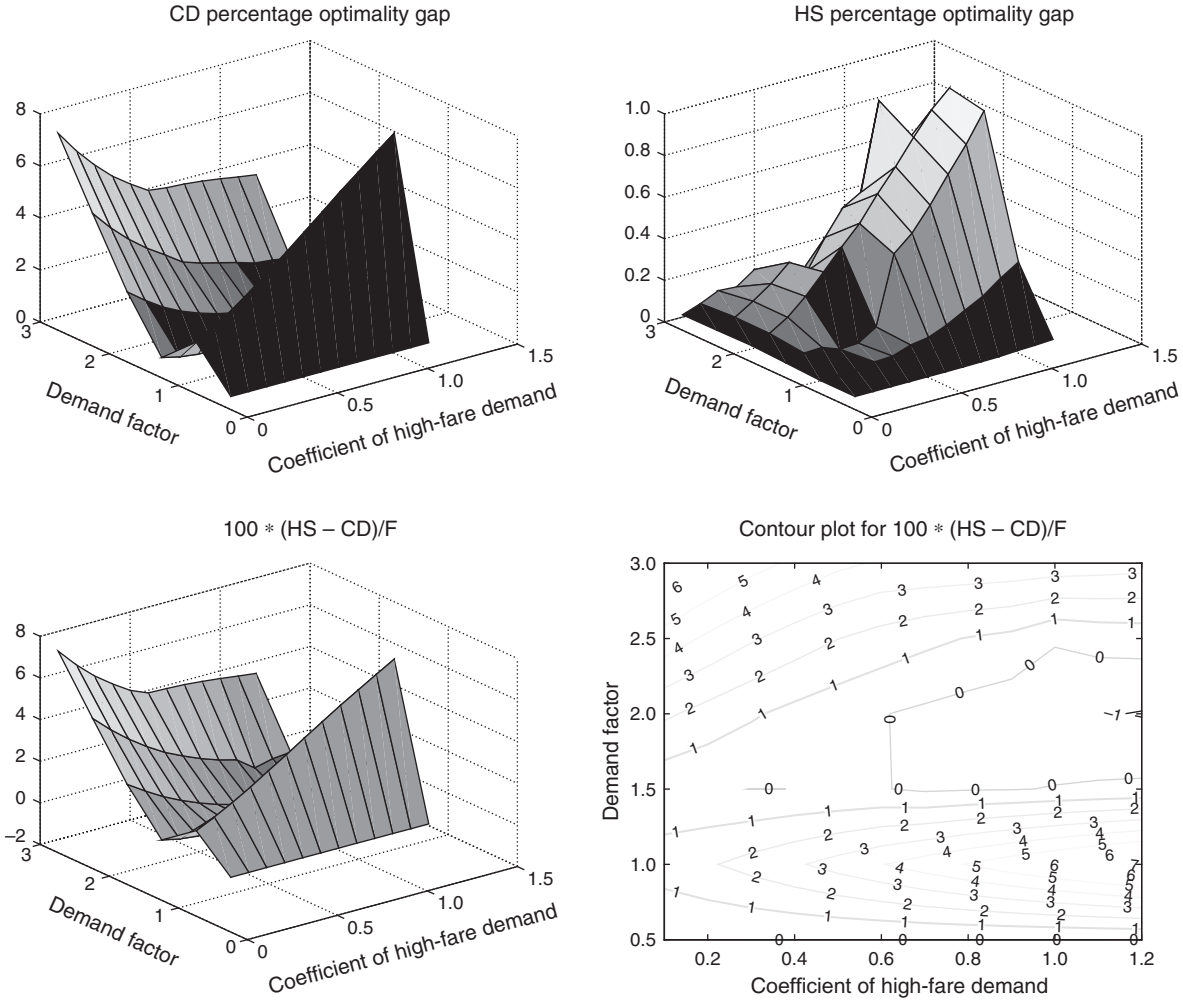


6.3. Summary of Insights and Robustness

To summarize, our numerical analysis generated the following insights. (1) The value of fully integrating pricing and revenue management (F) is high, relative to sequential heuristics based on deterministic prices (HD, CD). This value increases with system variability and when capacity becomes very scarce. (2) This value of coordination can be captured to a large extent by adjusting prices to reflect demand risk, based on the stochastic model (S). Indeed, in most practically relevant demand scenarios, the hierarchical heuristic (HS) achieves near-optimal performance because it sets near-optimal prices (so does CS, unlike HD and CD). (3) The cost of ignoring demand uncertainty when making pricing decisions is significant and may not be effectively mitigated by improving coordination. In particular, the hierarchical policy (HS) typically dominates the coordinated policy (CD) for most cases of practical interest; exceptions do occur when high-end demand is extremely volatile under additive (but not multiplicative) models, but only around a critical capacity level.

These insights suggest that capturing market uncertainty when deciding on static prices can be particularly useful to mitigate the lack of coordination with revenue management. Extensive simulations with a wide range of parameters and distribution classes indicate that these insights are robust, as also illustrated in the next subsection. Finally, we remark that

Figure 13 Relative Performance of (HS) and (CD) Policies as a Function of the Demand Factor and of the Variability of High-End Demand



revenue figures can have a strong influence on profitability: given the industry's notoriously thin margins, a 1% increase in revenue could actually double profits.

6.3.1. Robustness. Distributional Assumptions and Miscalibration. Our insights in this paper emphasize the importance of modeling demand uncertainty for pricing and revenue management decisions. We conclude this section by assessing the sensitivity of optimal revenues to assumptions about the distribution of demand risks Z and $\bar{Z}$. We briefly study the impact on optimal revenue of incorrectly assuming a certain distribution (i.e., when another distribution fits the data better). How does the potential revenue loss from such mis-estimation compare with the revenue impact of using different pricing policies with the correct demand distribution? Is the robustness of the demand assumption more or less important than the approach to pricing?

Although demand estimation is a broad topic that goes beyond our scope, we provide some preliminary answers to the above questions by focusing on three distributions (normal, gamma, and uniform) for the demand risks Z and $\bar{Z}$; for each of these distributions, we generate one data set of 100 demand realizations.[14] Then, for each of the three demand data sets and each pricing policy, we compute the optimal revenues separately under the assumption that the demand risks have normal, gamma, or uniform distributions. Table 4 gives the robustness gaps, which for each case are computed as the percentage difference between the expected revenues under the assumed demand model and the true model for all pricing policies. The robustness gaps are small unless the uniform distribution is incorrectly assumed for the demand

[14] For consistency with the previous section, the model parameters are set at the Avis values ($a = 30$, $\bar{a} = 80$, $b = 0.25$, $\bar{b} = 2$, $\Lambda = 2$) and the normal distributions have mean 0 and standard deviations $\sigma = 2$ and $\bar{\sigma} = 12$. The gamma distributions have scale parameters $c = 1.30$ and $\bar{c} = 2.30$ and shape parameters $d = 1.30$ and $\bar{d} = 2.30$; and the uniform distributions are defined on the intervals $(-h, h)$ and $(-\bar{h}, \bar{h})$, where $h = 2.50$ and $\bar{h} = 6.00$.

Table 4 Robustness Tests: Percentage Gaps in Revenues Under Different True and Assumed Demand Models

True model	Fitted model	(F)	(HS)	(CD)	(HD)
Normal	Gamma	1.09	1.13	1.94	2.04
	Uniform	3.63	5.63	21.74	33.11
Gamma	Normal	0.30	0.26	0.22	0.14
	Uniform	7.50	7.47	10.10	10.81
Uniform	Normal	0.58	0.55	1.08	1.44
	Gamma	1.05	1.10	1.32	1.45

risks. Assuming a normal distribution when the true model is not normal has a small negative impact on optimal revenue across all pricing policies; in most cases, the robustness gaps for the normal distribution are less than 1%. We thus conclude that if a normal distribution is assumed for the demand risks, then the potential violation of this assumption has less effect on optimal revenues than does the choice of pricing policy.

Our insights imply that forecasting and estimating the distribution of demand is important not only for revenue management but also for pricing decisions. In particular, more research is needed to quantify how ignoring the demand censoring resulting from capacity controls would bias prices and revenues; the importance of such effects was demonstrated in Cooper et al. (2006). Although demand estimation goes beyond our focus in this paper, we have also undertaken some preliminary numerical experiments that suggest our insights concerning the superior performance of (HS) are robust when demand is censored; the value of demand uncensoring can be significant. In practice, any analysis of booking data should start with demand untruncation (Queenan et al. 2007) before determining the specific impact of prices.

7. Summary and Conclusions

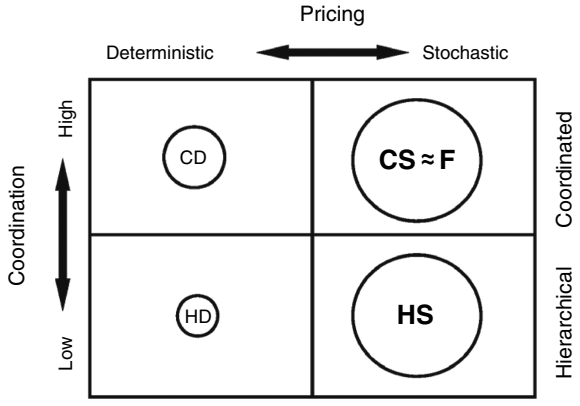
We have investigated the value of coordinating price–allocation decisions in a framework of static, two-fare-class revenue management. In this context, we considered two pairs of pricing and revenue management models that differ in their approach to pricing and degree of coordination. The first part of this paper characterized the associated demand conditions under which these models admit unique solutions with natural sensitivity properties. The second part of this paper relied on numerical experiments to assess the performance of these heuristics and the benefits of integrating the decisions on price and revenue management. Here we summarize our main findings and insights along these two dimensions.

First, from a methodological perspective, we have identified a broad class of demand models for which the hierarchical and coordinated pricing and revenue management models (HD), (HS), (CD), and (CS) can be solved efficiently as concave univariate problems. Our approach is valid if LSR elasticity is increasing in both x and p, a condition satisfied by most demand models of practical interest. This condition also led to sensitivity results—for instance, that the optimal protection level is decreasing in price for sequential heuristics and that the joint optimal price–allocation solution is monotonic with respect to capacity.

Second, in terms of assessing benefits, our numerical experiments suggest that a fully integrated pricing and revenue management model (F) can yield significant value relative to sequential approaches based on deterministic pricing (HD, CD). Although the fully coordinated model (F) remains intractable, we find it interesting that in our numerical experiments its performance is closely approximated by a simple two-stage heuristic (HS), which adjusts prices to capture demand uncertainty (based on a nonnested model) and then optimizes corresponding booking limits. This policy can be operationalized via Proposition 3(a) as long as demand is stochastically decreasing in price in the hazard rate order (i.e., if LSR elasticity is increasing in x). Figure 14 illustrates the relative performance ranking of the four policies summarized in Table 1 in the context of our numerical experiments. In particular, it appears that accounting for demand uncertainty in pricing decisions (i.e., stochastic pricing) can be particularly valuable in this context, more so than improving the integration of price and allocation decisions at the segment level.

Our model and results have several limitations. First, the analytical results presented here do not optimize the low-end price, a modeling choice motivated by the practical considerations discussed in the introduction and in §4. From a technical standpoint, preliminary analysis suggests that the general demand conditions used in this paper would need to be strengthened in order for the fully coordinated model (F) to be tractable. However, our numerical

Figure 14 Relative Performance of the Heuristics from Table 1 in the Context of Our Data



experiments indicate that the practical benefits of such full coordination may be negligible because the average difference in performance between (F) and the tractable sequential heuristic (HS) was less than 0.02% in our simulation study. Although these values are compelling, they are based on numerical experiments; obtaining analytical performance bounds remains an open challenge.

The results we present pertain to a static, two-fare-class monopolistic model. This stylized model is limited because it ignores dynamic, multidimensional, and competitive aspects of the revenue management context; whether our results extend in such settings remains to be investigated. Nevertheless, static pricing is common in practice, and it is supported by theoretical considerations, as discussed in the introduction. Preliminary analysis suggests that our two-class insights extend to three or more fare classes. A fully coordinated system, which is clearly intractable in this case, is closely approximated by a (provably tractable) hierarchical policy that makes static pricing decisions based on a multiproduct extension of the stochastic model (S) and then uses those prices to make nested allocation decisions recursively (cf. Brumelle and McGill 1993). This hierarchical process with stochastic pricing is a natural extension of our two-fare-class model (HS) and appears to perform well in our numerical experiments.

In sum, our insights from this research emphasize the importance of modeling price-sensitive demand uncertainty in pricing decisions. In a static environment, we propose that heuristics based on stochastic pricing could provide capacitated firms with a technically tractable and practically compelling alternative to full coordination.

Acknowledgments

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Appendix

First we review briefly the notation, most of which is summarized in Table A.1. Functions related to low-fare demand are denoted with an overline. We use $\mathfrak{f}^*(p) = \mathfrak{f}(p, x)|_{x=x^*(p)}$ to denote the evaluation of any generic function $\mathfrak{f}$ along the optimal solution path $x^*(p)$, and we use $\mathfrak{f}_x^*(p) = \mathfrak{f}_x(p, x)|_{x=x^*(p)}$ to denote the derivative of $\mathfrak{f}(p, x)$ with respect to x evaluated at the optimal quantity. In this notation, which is used throughout the paper, the derivative always precedes functional evaluation. We employ the standard joint expectation notation: $\mathbb{E}[A; B] = \mathbb{E}[A \mid B]\mathbb{P}[B]$.

Table A.1 Summary of Notation

K	Capacity
x	Protection level
p and $\bar{p}$	High-end and low-end price
$p_{\min}$	Lower bound on high-end price
$\mathbf{D}(p) = d(p, \mathbf{Z})$	High-fare stochastic demand function
$z(p, x)$	Inverse of the deterministic demand $d(p, z(p, x)) = x$
$\pi(p, z) = pd(p, z)$	Riskless profit for a given price p and realization z of $\mathbf{Z}$
$L(p, x) = \mathbb{P}[\mathbf{D}(p) \geq x]$	Lost sales rate (LSR); high-fare demand survival function
$f(p, x)$ and $F(p, x)$	Density and cumulative distribution function of $\mathbf{D}(p)$
$\phi(z)$ and $\Phi(z)$	Density and cumulative distribution function of $\mathbf{Z}$
$\mathscr{E}(p, x) = -pL_p(p, x)/L(p, x)$	LSR elasticity (elasticity of the rate of lost sales)
$\bar{\mathbf{D}}(\bar{p}) = \bar{d}(\bar{p}, \bar{\mathbf{Z}})$	Low-fare stochastic demand function
$\bar{L}(\bar{p}, x) = \mathbb{P}[\bar{\mathbf{D}}(\bar{p}) \geq x]$	Survival function of $\bar{\mathbf{D}}(\bar{p})$
$\bar{f}(\bar{p}, x)$ and $\bar{F}(\bar{p}, x)$	Density and cumulative distribution function of $\bar{\mathbf{D}}(\bar{p})$
$\mu(p) = \mathbb{E}[\mathbf{D}(p)]$, $\bar{\mu}(\bar{p}) = \mathbb{E}[\bar{\mathbf{D}}(\bar{p})]$	Expected high-fare and low-fare demand
$\boldsymbol{\Omega} = (\mathbf{D}(p) \leq x)$	Event that high-fare demand does not exceed x
$\bar{\boldsymbol{\Omega}} = (\bar{\mathbf{D}}(\bar{p}) \leq K - x)$	Event that low-fare demand does not exceed $K - x$
$\bar{\mathbf{K}} = K - \bar{\mathbf{D}}$	Excess capacity after all low-fare demand has been served

Proof of Proposition 1. First,

$$\begin{aligned}\mathscr{R}[\mathrm{F}] &= \max_{\bar{p}, p, x} R(\bar{p}, p, x) \geq \mathscr{R}[\mathrm{CA}] \\ &= \max_{p, x} R(\bar{p}^A, p, x) \\ &\geq \max_{x} R(\bar{p}^A, p^A, x) = \mathscr{R}[\mathrm{HA}].\end{aligned}$$

To prove $\mathscr{R}[\mathrm{F}] \leq \mathscr{D}$, consider the sample path revenue for a given feasible policy $p, \bar{p}, x \in [0, K]$ and demand realization $(q, \bar{q})$:

$$\begin{aligned}\hat{R}(p, \bar{p}, x, q, \bar{q}) = \bar{p}[\min\{\bar{q}, K - x\}] \\ + p[\min\{q, \max\{K - \bar{q}, x\}\}].\end{aligned} \tag{13}$$

Using Q and $\bar{Q}$ to denote (respectively) the number of units actually sold to high- and low-fare customers, so that $\bar{Q} = \min\{\bar{q}, K - x\}$ and $Q = \min\{q, \max\{K - \bar{q}, x\}\} = \min\{q, K - \bar{Q}\}$, we obtain

$$\begin{aligned}\hat{R}(p, \bar{p}, x, q, \bar{q}) = \max_{Q, \bar{Q}} \; & pQ + \bar{p}\bar{Q} \\ \text{s.t.} \; & Q \leq q, \quad \bar{Q} \leq \bar{q}, \\ & Q + \bar{Q} \leq K, \\ & \bar{Q} \leq K - x.\end{aligned} \tag{14}$$

The right-hand side is a linear program, so $\hat{R}$ is concave in q and $\bar{q}$. This allows us to apply Jensen's inequality to show that $\mathscr{R}[\mathrm{F}] = \max_{\bar{p}, p, x} \mathbb{E}[\hat{R}(p, \bar{p}, x, \mathbf{D}(p), \bar{\mathbf{D}}(\bar{p}))] \leq \max_{\bar{p}, p, x} \hat{R}(p, \bar{p}, x, \mathbb{E}[\mathbf{D}(p)], \mathbb{E}[\bar{\mathbf{D}}(\bar{p})]) = \mathscr{D}$.

For any feasible $p, \bar{p}, x \in [0, K]$, the objective in (S) can be equivalently written as

$$V(p, \bar{p}, x) := r(p, x) + \bar{r}(\bar{p}, x), \tag{15}$$

where

$$r(p, x) = p\,\mathbb{E}[\min\{\mathbf{D}(p), x\}] \text{ and}$$
$$\bar{r}(\bar{p}, x) = \bar{p}\,\mathbb{E}[\min\{\bar{\mathbf{D}}(\bar{p}), K - x\}]$$

are the respective expected revenues from the high and low segments. Because $r(p, x)$ is increasing in x, we obtain $\mathscr{S} = r(p^S, k^S) + \bar{r}(\bar{p}^S, k^S) \le \mathbb{E}[r(p^S, \max\{K - \bar{\mathbf{D}}(\bar{p}^S), k^S\})] + \bar{r}(\bar{p}^S, k^S) = R(p^S, \bar{p}^S, k^S) \le \max_x R(p^S, \bar{p}^S, x) = \mathscr{R}[\text{HS}]$.

For the lower bound, observe that $\mathscr{R}[\text{HS}] \ge \mathscr{S} \ge \max_x V(p^D, \bar{p}^D, x)$, which also bounds $\mathscr{R}[\text{HD}] = \max_x R(p^D, \bar{p}^D, x) \ge \max_x V(p^D, \bar{p}^D, x)$. Moreover, $\max_x V(p^D, \bar{p}^D, x) \ge r(p^D, \mu(p^D)) + \bar{r}(\bar{p}^D, \mu(p^D)) \ge r(p^D, \mu(p^D)) + \bar{r}(\bar{p}^D, K - \bar{\mu}(\bar{p}^D)) \ge p^D\mu(p^D)(1 - (\text{cv}^D)/2) + \bar{p}^D\bar{\mu}(\bar{p}^D)(1 - (\bar{\text{cv}}^D)/2) \ge (1 - \frac{1}{2}\max\{\text{cv}^D, \bar{\text{cv}}^D\})\mathscr{D}$, where the third inequality follows, e.g., from Bitran and Caldentey (2003, Proposition 6), for each segment. Finally, it is easy to see that when either demand is deterministic, nesting becomes superfluous, so models (S) and (F) are equivalent. □

Proofs of the Results in §4

The proofs in this section rely on results for the price-setting newsvendor problem with objective $\Pi(p, x) = \Pi(p, x; c, \mathbf{D}) = p\,\mathbb{E}[\min\{\mathbf{D}(p), x\}] - cx = r(p, x) - cx$. The newsvendor-optimal price path and quantity path are defined as $p^N(x) = \arg\max_{p \in P} \Pi(p, x)$ and $x^N(p) = \arg\max_{x \in X} \Pi(p, x)$, respectively. The next lemma combines results from Theorems 1 and 2 of Kocabıyıkoğlu and Popescu (2011) under our demand model and assumptions. These results are used to obtain structural results for the more complex models (S) and (H).

Lemma 1. (Kocabiyikoğlu and Popescu 2011). (a) *If $\mathscr{E}(p, x)$ is increasing in x, then $\Pi(p^N(x), x)$ is concave in x and $p^N(x)$ is decreasing in x. The latter holds if and only if $\mathscr{E}(p^N(x), x) \ge 1$.* (b) *If $\mathscr{E}(p, x)$ is increasing in p, then $\Pi(p, x^N(p))$ is concave in p and $x^N(p)$ is decreasing in x. The latter holds if and only if $\mathscr{E}(p, x^N(p)) \ge 1$.*

Proof of Proposition 2. Use (15) to define the optimal price paths for each product as $p^S(x) = \arg\max_p V(p, \bar{p}, x) = \arg\max_p r(p, x)$ and $\bar{p}^S(x) = \arg\max_{\bar{p}} V(p, \bar{p}, x) = \arg\max_{\bar{p}} \bar{r}(\bar{p}, x)$ for any $x \in X = [0, K]$. In particular, these are independent of the price of the other product, proving the last part of (b). The first-order conditions can be written as $r_p(p, x) = \bar{r}_{\bar{p}}(\bar{p}, x) = 0$ and $r_x(p, x) = \bar{r}_x(\bar{p}, x)$. Specifically, (9) and (10) follow by writing $r(p, x) = p\,\mathbb{E}[\min\{\mathbf{D}(p), x\}] = p\int_0^x L(p, y)\,dy$, so $r_x(p, x) = pL(p, x)$ and $r_p(p, x) = \int_0^x L(p, y) + pL_p(p, y)\,dy = \int_0^x L(p, y)(1 - \mathscr{E}(p, y))\,dy$. Analogous reasoning applies to $\bar{p}$. The resulting solution of (9) and (10) is either interior $x \in (0, K)$ or else it is optimal to allocate all the capacity to one segment; i.e., $x \in \{0, K\}$.

We next argue that $r(p^S(x), x)$ and $\bar{r}(\bar{p}^S(x), x)$ are both concave in x under the assumptions of the proposition, implying that $V(p^S(x), \bar{p}^S(x), x) = r(p^S(x), x) + \bar{r}(\bar{p}^S(x), x)$ is concave in x. Indeed, both $r(p, x)$ and $\bar{r}(\bar{p}, x)$ can be viewed as newsvendor pricing problems with zero cost. Formally, $r(p, x) = \Pi(p, x; c = 0)$ and $p^S(x) = p^N(x)$. Hence, by Lemma 1(a), $r(p^S(x), x) = \Pi(p^N(x), x; c = 0)$ is concave in x if $\mathscr{E}(p, x)$ is increasing in x and similarly for $\bar{r}(\bar{p}, x) = \bar{\Pi}(\bar{p}, K - x; c = 0)$; this proves part (a). Finally, Lemma 1(a) also implies that $p^S(x) = p^N(x)$ is decreasing in x and that $\bar{p}^S(x) = \bar{p}^N(K - x)$ is increasing in x, which proves part (b). □

Proof of Proposition 3. Part (a) follows by combining the results in §4.1, in particular Proposition 2, and Littlewood's rule (11). The latter implies that the optimal protection level of problem (H) coincides with the optimal quantity decision of a (price-sensitive) newsvendor facing unit cost $c = \bar{p}$ because both satisfy the same critical fractile condition (even though their objectives are quite different). Formally, $x^*(p) = x^N(p; c = \bar{p})$. By Lemma 1(b), this decreases in p whenever $1 \le \mathscr{E}(p, x^N(p)) = \mathscr{E}(p, x^*(p)) = \mathscr{E}^*(p)$ and, in particular, if $\mathscr{E}(p, x)$ is increasing in p. Finally, condition (ii) in part (b) is sufficient because $\mathscr{E}^*(p_{\min}) \ge 1$ by the definition of $p_{\min}$, so $\mathscr{E}^*(p)$ increasing implies that $\mathscr{E}^*(p) \ge 1$ on P. For an alternative, direct proof that computes the marginal effect of price on allocation, see Equation (20) to follow. □

Proofs of the Results in §5

Note that to simplify notation, *in all remaining proofs we set $\bar{p} = 1$*, so p can be interpreted as a percentage markup over the low fare price $\bar{p}$, which is fixed in this section. Further, we work with the unconstrained protection level $x^*(p)$ given by (11); i.e., $L^*(p) = L(p, x^*(p)) = 1/p$. All this is without loss of generality.[15] Define $z(p, x)$ so that $d(p, z(p, x)) = x$; the inverse is unique because $d(p, z)$ is monotone in z. Denote $\mathbf{\Omega} = \mathbf{\Omega}(p, x) = (\mathbf{D}(p) \le x) = (\mathbf{Z} \le z(p, x))$ and $\bar{\mathbf{\Omega}} = \bar{\mathbf{\Omega}}(x) = (\bar{\mathbf{D}} \le K - x) = (\bar{\mathbf{K}} \ge x)$ the events that high- and low-fare demand fall short of their allocations; here $\bar{\mathbf{K}} = K - \bar{\mathbf{D}}$ is the uncertain excess capacity after all low-fare demand has been served.

Proof of Proposition 4. We write $R(p) = \mathbb{E}[\min\{\bar{\mathbf{D}}, K - x\}] + \mathbb{E}[r(p, \max\{x, \bar{\mathbf{K}}\})]$, which is concave in p because so is $r(p, x) = p\,\mathbb{E}[\min\{\mathbf{D}(p), x\}] = \mathbb{E}[\min\{\pi(p, \mathbf{Z}), px\}]$, by concavity of π (the low-fare class revenue is independent of p). Because $x^*(p)$ is unique and differentiable, we can apply the envelope theorem:

$$\frac{\partial R^*(p)}{\partial p} = R_p^*(p) = R_p(p, x)|_{x = x^*(p)} = \mathbb{E}[r_p(p, \max\{x^*(p), \bar{\mathbf{K}}\})]$$
$$= \bar{L}^*(p) r_p^*(p) + \mathbb{E}[r_p(p, \bar{\mathbf{K}}); \bar{\mathbf{\Omega}}^*(p)]. \tag{16}$$

Here the second term is the marginal revenue in the event $\bar{\mathbf{\Omega}}^* = \bar{\mathbf{\Omega}}^*(p) = (\bar{\mathbf{D}} \le K - x^*(p)) = (\bar{\mathbf{K}} \ge x^*(p))$ that low-fare demand is not constrained by the optimal booking limit, and $\bar{L}^*(p) = \bar{L}(K - x^*(p)) = 1 - \mathbb{P}[\bar{\mathbf{\Omega}}^*]$ is the lost sales rate for the low-fare class. If we put $\bar{f}^*(p) = \bar{f}(K - x^*(p))$, then the derivative of (16) is

$$\frac{\partial R_p^*(p)}{\partial p} = \frac{\partial \bar{L}^*(p)}{\partial p} r_p^*(p) + \bar{L}^*(p) \frac{\partial r_p^*(p)}{\partial p} + \mathbb{E}[r_{pp}(p, \bar{\mathbf{K}}); \bar{\mathbf{\Omega}}^*]$$
$$- r_p^*(p) \bar{f}^*(p) \frac{\partial x^*(p)}{\partial p}. \tag{17}$$

[15] Indeed, the argument that marginal revenues decrease along the unconstrained path $x^*(p)$ extend when this (decreasing) path is capped at K.

We want to show that expression (17) is negative. Because $\partial\bar{L}^*(p)/\partial p = \bar{f}^*(p)(\partial x^*(p)/\partial p)$, the first and last terms cancel. The third term (and hence r) is negative by concavity of π, as argued previously. We next show that the second term is also negative. Evaluated at $x^*(p)$, the derivative of $r(p,x) = pxL(p,x) + \mathbb{E}[\pi(p,\mathbf{Z});\boldsymbol{\Omega}]$ is

$$\begin{aligned} r_p^*(p) &= r_p(p,x^*(p)) = \mathbb{E}[\pi_p(p,\mathbf{Z});\boldsymbol{\Omega}^*(p)] + x^*(p)L^*(p) \\ &= \mathbb{E}[\pi_p(p,\mathbf{Z});\boldsymbol{\Omega}^*(p)] + \frac{x^*(p)}{p}, \end{aligned} \tag{18}$$

where $\boldsymbol{\Omega}^* = \boldsymbol{\Omega}^*(p) = (\mathbf{D}(p) \le x^*(p))$ and the second equality holds because $L^*(p) = 1/p$. We obtain

$$\begin{aligned} \frac{\partial}{\partial p} r_p^*(p) &= \mathbb{E}[\pi_{pp}(p,\mathbf{Z});\boldsymbol{\Omega}^*] - \pi_p^*(p)\frac{\partial}{\partial p}L^*(p) + \frac{\partial}{\partial p}\left(\frac{x^*(p)}{p}\right) \\ &= \mathbb{E}[\pi_{pp};\boldsymbol{\Omega}^*] + \left[\frac{\pi_p^*(p)}{p^2} - \frac{x^*(p)}{p^2} + \frac{1}{p}\frac{\partial x^*(p)}{\partial p}\right]. \end{aligned} \tag{19}$$

The first term is negative because π is concave. Since $\pi_p^*(p) = pd_p^*(p) + d^*(p)$ and $d^*(p) = d(p,z(p,x^*(p))) = x^*(p)$, the term in brackets equals $d_p^*(p) + \partial x^*(p)/\partial p$. This value is negative if $x^*(p)$ is decreasing—in particular, under the conditions of Proposition 3(b)—which proves parts (b) and (c). To prove part (a), we argue that

$$\begin{aligned} \frac{\partial x^*(p)}{\partial p} &= -\frac{L_p^*(p)}{L_x^*(p)}\left(1 + \frac{1}{p^2 L_p^*(p)}\right) = -\frac{L_p^*(p)}{L_x^*(p)}\left(1 - \frac{1}{\mathscr{E}^*(p)}\right) \\ &= d_p^*(p)\left(1 - \frac{1}{\mathscr{E}^*(p)}\right). \end{aligned} \tag{20}$$

Indeed, the first equality obtains from differentiating both sides of $L(p,x^*(p)) = 1/p$ with respect to p. The second equality is derived from $\mathscr{E}^*(p) = -pL_p^*(p)/L^*(p) = -p^2 L_p^*(p)$ by (11). Finally, the last equality follows by differentiating $L(p,d(p,z)) = 1 - \Phi(z)$ with respect to p to obtain $L_p/L_x = -d_p$. This proves (20) and so provides an alternative, direct proof of Proposition (3b). It also implies that $d_p^*(p) + \partial x^*(p)/\partial p = d_p^*(p)(2 - 1/\mathscr{E}^*(p)) \le 0$ whenever $\mathscr{E}^*(p) \ge 1/2$, which completes the proof. □

Proof of Proposition 5. Because here $\bar{\mathbf{D}}(p)$ is a function of p, the expressions involving low-fare demand are different from those in the previous proofs. By abuse of notation, we keep the same letters for the same concepts even though their mathematical expressions are changed. In particular, we use R to denote the more complex objective in (12); we use $\bar{r}(p,x) = \bar{r}(p,x;K) = \mathbb{E}[\min\{K-x, \bar{\mathbf{D}}(p)\}]$ for the low-class expected revenue, $\bar{\pi}(p,\bar{z}) = p\bar{d}(p,\bar{z})$ for the pathwise revenue, $\bar{L}(p,x) = \mathbb{P}[\bar{\mathbf{D}}(p) \ge K - x]$ for the lost sales rate, $\bar{\mathbf{K}}(p) = K - \bar{\mathbf{D}}(p)$, and so on. We next show that $R^*(p) = R(p,x^*(p))$ is concave in p.

By the envelope theorem, we have $\partial R^*(p)/\partial p = R_p^*(p) = \bar{r}_p^*(p) + \bar{L}^*(p)r_p^*(p) + \mathbb{E}[r_p(p,\bar{\mathbf{K}}(p));\bar{\boldsymbol{\Omega}}^*]$. The first term is the marginal revenue from the low-fare class, and the other two terms give the marginal revenues from the high-fare class depending on whether low-fare demand does or does not exceed the optimal booking limit; the latter event is $\bar{\boldsymbol{\Omega}}^* = \bar{\boldsymbol{\Omega}}^*(p) = (\bar{\mathbf{D}}(p) \le K - x^*(p)) = (\bar{\mathbf{K}}(p) \ge x^*(p))$. The derivative of each term is

$$\begin{aligned} \frac{\partial \bar{r}_p^*(p)}{\partial p} &= \frac{\partial}{\partial p}\mathbb{E}[\bar{d}_p(p,\bar{\mathbf{Z}});\bar{\boldsymbol{\Omega}}^*] \\ &= \mathbb{E}[\bar{d}_{pp}(p,\bar{\mathbf{Z}});\bar{\boldsymbol{\Omega}}^*] - \bar{d}_p^*(p)\bar{f}^*(p)\frac{\partial x^*(p)}{\partial p}; \end{aligned} \tag{21}$$

$$\begin{aligned} \frac{\partial}{\partial p}\bar{L}^*(p)r_p^*(p) &= \bar{L}^*(p)\frac{\partial r_p^*(p)}{\partial p} + \frac{\partial \bar{L}^*(p)}{\partial p}r_p^*(p) \\ &= \bar{L}^*(p)\frac{\partial r_p^*(p)}{\partial p} + \bar{f}^*(p)\frac{\partial x^*(p)}{\partial p}r_p^*(p). \end{aligned} \tag{22}$$

We write the third term of $\partial R^*(p)/\partial p$ as $\mathbb{E}[r_p(p,\bar{\mathbf{K}}(p));\bar{\boldsymbol{\Omega}}^*] = A + B$, where $A = \mathbb{E}[\pi_p(p,\mathbf{Z});\mathbf{D}(p) \le \bar{\mathbf{K}}(p);\bar{\boldsymbol{\Omega}}^*]$ and $B = \mathbb{E}[(K - \bar{\pi}_p(p,\bar{\mathbf{Z}}))L(p,\bar{\mathbf{K}}(p));\bar{\boldsymbol{\Omega}}^*]$. Here A corresponds to the unconstrained case in which there is excess capacity and the booking limit is nonbinding; B is the same except capacity K is binding. The respective derivatives of these terms are (we omit some functional arguments for readability)

$$\begin{aligned} A_p &= \mathbb{E}[\pi_{pp};\mathbf{D}(p) \le \bar{\mathbf{K}}(p);\bar{\boldsymbol{\Omega}}^*] - \mathbb{E}[(K - \bar{d} + pd_p)(d_p + \bar{d}_p)f;\bar{\boldsymbol{\Omega}}^*] \\ &\quad - \mathbb{E}[\pi_p;\boldsymbol{\Omega}^*]\bar{f}^*(p)\frac{\partial x^*(p)}{\partial p}; \\ B_p &= -\mathbb{E}[\bar{\pi}_{pp}L(p,\bar{\mathbf{K}}(p));\bar{\boldsymbol{\Omega}}^*] + \mathbb{E}[(K - \bar{d} + p\bar{d}_p)(d_p + \bar{d}_p)f;\bar{\boldsymbol{\Omega}}^*] \\ &\quad - (x^*(p) - p\bar{d}_p^*(p))L^*(p)\bar{f}^*(p)\frac{\partial x^*(p)}{\partial p}. \end{aligned}$$

Combining the second terms of each expression, regrouping the last terms, and using (18), we obtain

$$\begin{aligned} &\frac{\partial}{\partial p}\mathbb{E}[r_p(p,\bar{\mathbf{K}}(p));\bar{\boldsymbol{\Omega}}^*] \\ &\quad = \mathbb{E}[\pi_{pp};\mathbf{D}(p) \le \bar{\mathbf{K}}(p);\bar{\boldsymbol{\Omega}}^*] - \mathbb{E}[\bar{\pi}_{pp}L(p,\bar{\mathbf{K}}(p));\bar{\boldsymbol{\Omega}}^*] \\ &\qquad - p\,\mathbb{E}[(d_p + \bar{d}_p)^2 f;\bar{\boldsymbol{\Omega}}^*] - \bar{f}^*(p)r_p^*(p)\frac{\partial x^*(p)}{\partial p} \\ &\qquad + \bar{f}^*(p)\bar{d}_p^*(p)pL^*(p)\frac{\partial x^*(p)}{\partial p}. \end{aligned} \tag{23}$$

If we combine (21)–(23), then the last term of (22) and the fourth term of (23) cancel; the last terms of (21) and (23) also cancel because $p^*L^*(p) = 1$. Furthermore, we can use (19) to obtain

$$\begin{aligned} \frac{\partial R_p^*(p)}{\partial p} &= \mathbb{E}[\pi_{pp};\mathbf{D}(p) \le \bar{\mathbf{K}}(p);\bar{\boldsymbol{\Omega}}^*] - 2\mathbb{E}[\bar{d}_p L(p,\bar{\mathbf{K}}(p));\bar{\boldsymbol{\Omega}}^*] \\ &\quad - \mathbb{E}[\bar{d}_{pp}(pL(p,\bar{\mathbf{K}}(p)) - 1);\bar{\boldsymbol{\Omega}}^*] \\ &\quad - \mathbb{E}[p(d_p + \bar{d}_p)^2 f;\bar{\boldsymbol{\Omega}}^*] + \bar{L}^*(p)\frac{\partial r_p^*(p)}{\partial p}. \end{aligned} \tag{24}$$

The second and third terms are obtained by writing $\bar{\pi}_{pp} = 2\bar{d}_p + p\bar{d}_{pp}$. The first term is negative by concavity of $\pi(p,z)$ and the second because $\bar{d}_p(p,z) \ge 0$. Negativity of the third term follows because $\bar{d}_{pp}(p,\bar{z}) \le 0$ and $L(p,\bar{\mathbf{K}}(p)) \le L^*(p) = 1/p$ on $\bar{\boldsymbol{\Omega}}^* = (\bar{\mathbf{K}}(p) \ge x^*(p))$. The fourth term is obviously negative. From the proof of Proposition 4, the last term (which is unchanged because it does not involve $\bar{\mathbf{D}}$) is also

negative under the assumptions of this proposition. It follows that $R_p^*(p)$ is decreasing, so $R^*(p)$ is concave in p. □

Proof of Proposition 6. (a) For the first part, by definition we have $p^{**}(K) = \arg\max_p R(p, x^*(p; K); K) = \arg\max_p R^*(p; K)$. By the envelope theorem, $\partial R^*(p; K)/\partial p = R_p(p, x^*(p; K); K)$. From (11), the (unconstrained) optimal protection level for a given price, $x^*(p; K) \equiv x^*(p)$ solves $L(p, x) = 1/p$, so it is independent of capacity. This allows us to write $(\partial/\partial K)(\partial R^*(p; K)/\partial p) = (\partial/\partial K)R_p(p, x^*(p); K) = R_{pK}(p, x^*(p); K)$. Furthermore,

$$R_{pK}(p, x; K) = \mathbb{E}[r_{pK}(p, \bar{\mathbf{K}}); \bar{\mathbf{\Omega}}] = \mathbb{E}[(1 - \mathscr{C}(p, \bar{\mathbf{K}}))L(p, \bar{\mathbf{K}}); \bar{\mathbf{\Omega}}]$$
$$\leq (1 - \mathscr{C}(p, x))\,\mathbb{E}[L(p, \bar{\mathbf{K}})); \bar{\mathbf{\Omega}}]$$

because $\mathscr{C}(p, x)$ is increasing in x and $\bar{\mathbf{\Omega}} = (\bar{\mathbf{K}} \geq x)$. We thus obtain $R_{pK}(p, x^*(p)) \leq (1 - \mathscr{C}^*(p))\,\mathbb{E}[L(p, \bar{\mathbf{K}}); \bar{\mathbf{\Omega}}] \leq 0$ whenever $\mathscr{C}^*(p) = \mathscr{C}(p, x^*(p)) \geq 1$, which follows by Proposition 3, since $\mathscr{C}(p, x)$ is increasing in p. This shows that $p^{**}(K)$ is decreasing in K as long as the protection level is interior, i.e., if $L(p, K) \leq 1/p$. Otherwise, $x^* \equiv K$, so the objective R reduces to a newsvendor model, and $p^{**}(K) = p^N(K)$ is decreasing in K via Lemma 1; the result follows by continuity.

For the second part, $x^{**}(K) = x^*(p^{**}(K)))$ given by (11) is increasing in K because $p^{**}(K)$ is decreasing in K (from part (a)) and because $x^*(p)$ is decreasing in p whenever $\mathscr{C}(p, x)$ is increasing in p (by Proposition 3).

(b) We show by a sample path argument that $R(p, x; K)$, and hence $R^{**}(K)$, is increasing and concave in K. It is sufficient that these properties hold for each sample path revenue for a given policy and demand realization—in other words, for $\bar{R}(p, \bar{p}, x, q, \bar{q})$ as defined in (13). Based on (14), this can be formulated as a linear program parametrized by K, so it is indeed increasing and concave in K.

For the second part, we show that $(\partial/\partial K)(R^{**}(K)/K) = (KR_K^{**}(K) - R^{**}(K))/K^2 \leq 0$. We write

$$R(K) = \bar{r}(x; K) + \mathbb{E}[r(p, \max(x, K - \bar{\mathbf{D}})]$$
$$= \bar{r}(x; K) + \bar{L}(K - x)r(p, x) + \mathbb{E}[r(p, K - \bar{\mathbf{D}}); \bar{\mathbf{\Omega}}]. \quad (25)$$

The derivative of the low-fare revenue $\bar{r}(x; K) = \mathbb{E}[\min\{\bar{\mathbf{D}}, K - x\}] = \int_0^{K-x} \bar{L}(y)\,dy$ is $\partial\bar{r}(x; K)/\partial K = \bar{L}(K - x)$. The derivative of the high-fare revenue with respect to K is $p\,\mathbb{E}[L(p, K - \bar{\mathbf{D}}); \bar{\mathbf{\Omega}}]$, as we show next.

Indeed, the derivative of the second term of Equation (25) is $(\partial/\partial K)(\bar{L}(K - x)r(p, x)) = -\bar{f}(K - x)r(p, x)$. The last term of (25) can be written as $\mathbb{E}[r(p, \bar{\mathbf{K}}); \bar{\mathbf{\Omega}}] = \mathbb{E}[p(K - \bar{\mathbf{D}})L(p, \bar{\mathbf{K}}); \bar{\mathbf{\Omega}}] + \mathbb{E}[\pi(p, \mathbf{Z}); \mathbf{D}(p) \leq \bar{\mathbf{K}}; \bar{\mathbf{\Omega}}]$, and its derivative (after canceling boundary terms) is $(\partial/\partial K)\,\mathbb{E}[r(p, \bar{\mathbf{K}})\bar{\mathbf{\Omega}}] = \mathbb{E}[pL(p, \bar{\mathbf{K}}); \bar{\mathbf{\Omega}}] + \bar{f}(K - x)r(p, x)$.

We obtain $R_K(K) = \bar{L}(K - x) + \mathbb{E}[pL(p, \bar{\mathbf{K}}); \bar{\mathbf{\Omega}}]$. From the envelope theorem, $KR_K^{**}(K) - R^{**}(K) = -\bar{L}(K - x)x(pL(p, x) - 1) - \bar{L}(K - x)\,\mathbb{E}[\pi(p, \mathbf{Z}); \mathbf{\Omega}] + \mathbb{E}[(pL(p, \bar{\mathbf{K}}) - 1)\bar{\mathbf{D}}; \bar{\mathbf{\Omega}}] - \mathbb{E}[\pi(p, \mathbf{Z}); \mathbf{D}(p) \leq \bar{\mathbf{K}}; \bar{\mathbf{\Omega}}]|_{x=x^{**}, p=p^{**}}$. The first term is equal to 0 because $p^{**}L(p^{**}, x^{**}) = 1$, and negativity of the third term follows because $L(p^{**}, \bar{\mathbf{K}}) \leq 1/p^{**}$ on $\bar{\mathbf{\Omega}}^{**} = (\bar{\mathbf{K}} \geq x^{**})$. Hence $KR_K^{**}(K) - R^{**}(K) \leq 0$, concluding the proof. □

Table A.2 Lower Bound on Price $p_{\min}$ for Different Demand Distributions

	$\bar{p}$								
Z distribution	20	25	30	35	40	45	50	55	60
Normal	28	33	38	42	47	52	57	62	67
Gamma	20	25	30	35	40	45	50	55	60
Uniform	26	29	32	35	40	45	50	55	60

Numerical Assessment of the Lower Bound $p_{\min}$

The regularity conditions derived for partially coordinated models (CS) and (CD) rely on a lower bound $p_{\min}$ on high-end prices, defined in §4. We conclude by evaluating $p_{\min}$ for the three demand models presented in §6.3 and for a wider range of low-end prices $\bar{p}$ than would be suggested by the data and models. Recall that the linear model implies $\bar{p} \leq \bar{a}/\bar{b} = 40$. The values for $p_{\min}$ reported in Table A.2 are consistently close to the low-end prices $\bar{p}$, which suggests that the technical assumption $p \geq p_{\min}$ for optimizing the (C) model is practically unrestrictive. The largest difference for normally distributed $\mathbf{Z}$ amounts to a markup of approximately 1.4 on the low-fare price; for $\mathbf{Z}$ with a gamma distribution we have $p_{\min} = \bar{p}$, which (in effect) imposes no additional constraints on the high-fare price.

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