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Mean semi-deviation from a target and robust portfolio choice under distribution and mean return ambiguity

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ABSTRACT

We consider the problem of optimal portfolio choice using the lower partial moments risk measure for a market consisting of n risky assets and a riskless asset. For when the mean return vector and variance/covariance matrix of the risky assets are specified without specifying a return distribution, we derive distributionally robust portfolio rules. We then address potential uncertainty (ambiguity) in the mean return vector as well, in addition to distribution ambiguity, and derive a closed-form portfolio rule for when the uncertainty in the return vector is modelled via an ellipsoidal uncertainty set. Our result also indicates a choice criterion for the radius of ambiguity of the ellipsoid. Using the adjustable robustness paradigm we extend the single-period results to multiple periods, and derive closed-form dynamic portfolio policies which mimic closely the single-period policy.

1. Introduction

The purpose of this paper is to give an explicit solution to the optimal portfolio choice problem by minimizing the lower partial moment risk measure of mean semi-deviation from a target return under distribution and mean return ambiguity using a robust optimization (RO) approach.

Portfolio optimization in single and multiple periods, using different criteria such as mean–variance and utility functions, has been studied extensively; see, e.g., [1–14]. In particular, Hakansson [5] treats correlations between time periods while Merton [8,9,15] concentrates on continuous-time problems. These references usually consider a stochastic model for the uncertain elements (asset returns) and study the properties of an optimal portfolio policy. An important tool here is stochastic dynamic programming.

The philosophy of robust optimization (RO) [16,17] is to treat the uncertain parameters in an optimization problem by confining their values to some uncertainty set without defining a stochastic model, and find a solution that satisfies the constraints of the problem regardless of the realization of the uncertain parameters in the uncertainty set. It has been applied with success to single-period portfolio optimization; see, e.g., [18–21]. The usual approach is to choose uncertainty sets that lead to tractable convex programming problems that are solved numerically. In the present paper, we instead find closed-form portfolio rules. In the case of multiple-period portfolio problems, RO was extended to adjustable robust optimization (ARO), an approach that does not resort to dynamic programming, and is more flexible than the classical RO for sequential problems, but may lead to more difficult optimization problem instances; see [22,23]. A related approach, which is data-driven with probabilistic guarantees and scenario generation, is explored in e.g. [24].

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The optimal portfolio choice problem using lower partial moments risk measures under distribution ambiguity was studied by Chen, He and Zhang in a recent paper [25] in the case of n risky assets. The authors assumed that the mean return vector μ and variance–covariance matrix Γ of risky assets are fixed, and compute portfolios that are distributionally robust in the sense that they minimize a worst-case lower partial moment risk measure over all distributions with fixed first-moment and second-moment information. They obtained closed-form distributionally robust optimal portfolio rules. In the present paper we first extend their result to the case where a riskless asset is also included in the asset universe, a case which is an integral part of optimal portfolio choice theory. The inclusion of the riskless asset in the asset universe simplifies considerably the optimal choice formula in some cases as we shall see below in Theorem 1. A criticism levelled against the distributionally robust portfolios of Chen et al. [25] is the sensitivity of these portfolios to uncertainties or estimation errors in the mean return data, a case that we refer to as *mean return ambiguity*; see [18]. To address this issue, we analyse the problem for when the mean return is subject to ellipsoidal uncertainty in addition to distribution ambiguity and derive a closed-form portfolio rule. Since the majority of contributions in robust portfolio optimization aim at providing convex optimization formulations our explicit portfolio rule constitutes a worthy addition to the literature. Our result is valid for choices of the ellipsoidal uncertainty (ambiguity radius) parameter ϵ not exceeding the optimal Sharpe ratio attainable in the market. Furthermore, the difference between the optimal mean semi-deviation risk under distribution ambiguity only and the same measure under joint uncertainty in distribution and mean return may also impose an optimal choice of ϵ, an observation which we illustrate numerically. For other related studies on portfolio optimization with distributional robustness, the reader is referred to [19,26,27]. We also obtain optimal dynamic portfolio rules using the adjustable robust optimization paradigm [22,23] for both cases of distribution ambiguity and expected return ambiguity combined with distribution ambiguity. The resulting portfolio rules are myopic replicas of the single-period results. The plan of the paper is as follows. In Section 2 we derive the optimal portfolio rules under distributional ambiguity for two measures of risk in the presence of a riskless asset. We study the multiple-period adjustable robust portfolio rules in Section 3. In Section 4, we derive the optimal portfolio rule for the mean squared semi-deviation from a target measure under distributional ambiguity and ellipsoidal mean return uncertainty. We also discuss the optimal choice of the uncertainty/ambiguity radius for the mean return. The multiple-period extension is given in Section 5.

2. Minimizing lower partial moments in the presence of a riskless asset: single period

The lower partial moment risk measure LPM_m for $m = 0, 1, 2$ is defined as

$$\mathbb{E}\,[r - X]_+^m$$

for a random variable X and target r. We assume, in addition to the n risky assets with given mean return μ and variance–covariance matrix Γ, that a riskless asset with return rate $R < r$ exists. If $R \geq r$, then the benchmark rate is attained without risk, i.e. the lower partial moment LPM_m is minimized taking value 0 by investing entirely in the riskless asset. Denote by y the variable for the riskless asset, for handling it separately, and by e the n-dimensional vector of entries 1; the LPM_m minimizing robust portfolio selection model under distribution ambiguity is

$$RPR_m = \min_{x,y} \sup_{\xi\sim(\mu,\Gamma)} \mathbb{E}\left[r - x^T\xi - yR\right]_+^m \tag{1}$$

$$\text{s.t}\; x^T e + y = 1. \tag{2}$$

We use the notation $\xi \sim (\mu, \Gamma)$ to mean that random vector ξ belongs to the set whose elements have mean μ and variance–covariance matrix Γ. Now, we provide the analytical solutions of the riskless asset counterpart of the problem for $m = 1, 2$ (expected shortfall and expected squared semi-deviation from a target, respectively) following a similar line to the proof of LPM_m solutions in [25]. The optimal portfolio choice for $m = 0$, which corresponds to minimizing the probability of falling short of the target, is uninteresting in the presence of a riskless asset in comparison to the case of risky assets only, since the optimal portfolio displays an extreme behaviour (the components vanish or go to infinity). Therefore, we exclude this case in the theorem below.

Theorem 1. *Suppose $\Gamma \succ 0$ and $R < r$. The optimal portfolio in (1)–(2) is obtained in the two different cases as follows.*

1. *For the case $m = 1$ the optimal portfolio rule is*

$$x^* = \frac{2\tilde{r}}{1+H}\Gamma^{-1}\tilde{\mu}.$$

2. *For the case $m = 2$ the optimal portfolio rule is*

$$x^* = \frac{\tilde{r}}{1+H}\Gamma^{-1}\tilde{\mu},$$

where $H = \tilde{\mu}^T\Gamma^{-1}\tilde{\mu}$, $\tilde{\mu} = \mu - Re$ and $\tilde{r} = r - R$.

Proof. The equality constraint (2) can be dropped letting $y = 1 - x^T e$:

$$RPR_m = \min_x \sup_{\xi\sim(\mu,\Gamma)} \mathbb{E}\left[(r-R) - x^T(\xi - eR)\right]_+^m.$$

One-to-one correspondence between the sets of distributions

$$D = \{\pi \,|\, \mathbb{E}_\pi[\xi] = \mu, \mathrm{Cov}_\pi[\xi] = \Gamma \succ 0\}$$

and

$$\tilde{D} = \{\pi \,|\, \mathbb{E}_\pi[\xi] = \mu - eR, \mathrm{Cov}_\pi[\xi] = \Gamma \succ 0\}$$

can be easily established. Hence, the model can be written as

$$RPR_m = \min_x \sup_{\xi\sim(\tilde{\mu},\Gamma)} \mathbb{E}\left[\tilde{r} - x^T\xi\right]_+^m$$

where $\tilde{r} = r - R$ and $\tilde{\mu} = \mu - eR$. To be able to use the bounds derived for LPM_m, the equivalent single-variable optimization model should be noted:

$$RPR_m = \min_x \sup_{\zeta\sim(x^T\tilde{\mu}, x^T\Gamma x)} \mathbb{E}\left[\tilde{r} - \zeta\right]_+^m.$$

The equivalence of the single-variable and multi-variable optimization models is based on the one-to-one correspondence of the sets of distributions that ξ and ζ may assume (a proof of this fact can be found in [25]).

We define objective functions with respect to mean return and variance, using the tight bounds provided for LPM_m, $m = 1, 2$ in [25]:

$$f_1(s,t) := \sup_{X\sim(s,t^2)} E\left[\left(\tilde{r} - X\right)_+^1\right] = \frac{\tilde{r} - s + \sqrt{t^2 + \left(\tilde{r} - s\right)^2}}{2},$$

$$f_2(s,t) := \sup_{X\sim(s,t^2)} E\left[\left(\tilde{r} - X\right)_+^2\right] = \left[\left(\tilde{r} - s\right)_+\right]^2 + t^2.$$

Then, ν being the optimal value, the problem becomes

$$\begin{aligned} \nu(RPR_m) &= \min_x \left\{f_m\left(x^T\tilde{\mu}, \sqrt{x^T\Gamma x}\right)\right\} \\ &= \min_{s\in\mathbb{R}} \min_x \left\{f_m\left(s, \sqrt{x^T\Gamma x}\right) \mid x^T\tilde{\mu} = s\right\}. \end{aligned} \tag{3}$$

Noting that f_m is non-decreasing in variance (t^2) for $m = 1, 2$, the inner optimization in (3) is solved by minimizing the variance:

$$\min_x x^T\Gamma x$$
$$\text{s.t } x^T\tilde{\mu} = s.$$

Thus, we obtain

$$2\Gamma x - u\tilde{\mu} = 0,$$
$$x^T\tilde{\mu} = s,$$
$$u \in \mathbb{R}.$$

Hence we have the optimal solution for the inner optimization:

$$x = \frac{u\Gamma^{-1}\tilde{\mu}}{2},$$
$$\frac{u\tilde{\mu}^T\Gamma^{-1}\tilde{\mu}}{2} = s,$$

which gives

$$u = \frac{2s}{\tilde{\mu}^T\Gamma^{-1}\tilde{\mu}}$$
$$x_s^* = \frac{s}{\tilde{\mu}^T\Gamma^{-1}\tilde{\mu}}\Gamma^{-1}\tilde{\mu}.$$

Having found the optimal value for x given a fixed value of s, we can now define the objective function as a function of s only:

$$\begin{aligned}\phi_m(s) &:= f_m\left(\tilde{\mu}^T x_s^*, \sqrt{x_s^{*T}\Gamma x_s^*}\right)\\ &= f_m\left(s, \sqrt{x_s^{*T}\Gamma x_s^*}\right).\end{aligned}$$

Following the notation $H = \tilde{\mu}^T\Gamma^{-1}\tilde{\mu}$, we have

$$\begin{aligned}x_s^{*T}\Gamma x_s^* &= \frac{s^2}{H^2}\tilde{\mu}^T\Gamma^{-1}\Gamma\Gamma^{-1}\tilde{\mu}\\ &= \frac{s^2}{H^2}\tilde{\mu}^T\Gamma^{-1}\tilde{\mu}\\ &= \frac{s^2}{H},\end{aligned}$$

and thus we write

$$\begin{aligned}&\phi_m(s) = f_m\left(s, \sqrt{\frac{s^2}{H}}\right),\\ &\nu(RPR_m) = \min_s\{\phi_m(s)\}.\end{aligned}$$

Now we can seek s that minimizes $\phi_m(s)$, for cases $m = 1$ and 2 separately. For $m = 1$ we have

$$\phi_1(s) = \frac{\tilde{r} - s + \sqrt{\frac{s^2}{H} + \left(\tilde{r} - s\right)^2}}{2}.$$

We are minimizing $\gamma_1(s) = \sqrt{\frac{s^2}{H} + \left(\tilde{r} - s\right)^2} - s$. The first-order condition gives

$$\gamma_1{}'(s) = \frac{\frac{s}{H} - \tilde{r} + s}{\sqrt{\frac{s^2}{H} + \left(\tilde{r} - s\right)^2}} - 1 = 0,$$

which is equivalent to

$$\left(\frac{s}{H} - \tilde{r} + s\right)^2 = \frac{s^2}{H} + \left(\tilde{r} - s\right)^2 \quad \text{and} \tag{4}$$

$$\frac{s}{H} - \tilde{r} + s \geq 0. \tag{5}$$

Eq. (4) has two roots, one of which is 0, not satisfying (5). The other root, $s = \frac{2\tilde{r}}{1+\frac{1}{H}}$, satisfies (5) and is the minimizer of $\gamma_1(s)$, since $\gamma_1'(s)$ is negative to the left and positive to the right of this value. To see this, we let $a := 1 + \frac{1}{H}$, and write

$$\begin{aligned}\gamma_1'(s) &= \frac{\frac{s}{H} - \tilde{r} + s}{\sqrt{\frac{s^2}{H} + \left(\tilde{r} - s\right)^2}} - 1\\ &= \frac{s\left(1 + \frac{1}{H}\right) - \tilde{r}}{\sqrt{s^2\left(1 + \frac{1}{H}\right) + \tilde{r}^2 - 2\tilde{r}s}} - 1\\ &= \frac{as - \tilde{r}}{\sqrt{as^2 + \tilde{r}^2 - 2\tilde{r}s}} - 1,\end{aligned} \tag{6}$$

observing $\gamma_1{}'(s) \leq -1$ if $s \leq \frac{\tilde{r}}{a}$. If $s \in \left(\frac{\tilde{r}}{a}, \frac{2\tilde{r}}{a}\right)$, then the nominator in (6) is positive, but

$$\begin{aligned}\left(as - \tilde{r}\right)^2 &= a^2s^2 - 2as\tilde{r} + \tilde{r}^2\\ &= as\left(as - 2\tilde{r}\right) + \tilde{r}^2\end{aligned}$$

$$\begin{aligned} &< s\left(as - 2\tilde{r}\right) + \tilde{r}^2 \\ &= as^2 - 2\tilde{r}s + \tilde{r}^2 \\ &= \left(\sqrt{as^2 - 2\tilde{r}s + \tilde{r}^2}\right)^2. \end{aligned} \tag{7}$$

We have $as - \tilde{r} < \sqrt{as^2 + \tilde{r}^2 - 2\tilde{r}s}$, and thus $\gamma_1{}'(s) < 0$. In inequality (7), note that $\tilde{r} > 0, a > 1, s > 0$ and $as - 2\tilde{r} < 2\tilde{r} - 2\tilde{r} = 0$. If $s > \frac{2\tilde{r}}{a}$, then $as - 2\tilde{r} > 0$; and inequality (7) is in the opposite direction. It follows that $\gamma_1{}'(s) > 0$ if $s > \frac{2\tilde{r}}{a}$, and $\gamma_1{}'(s) < 0$ if $s < \frac{2\tilde{r}}{a}$; hence $s_1^* = \frac{2\tilde{r}}{a}$ is the unique minimizer of $\gamma_1(s)$ and $\phi_1(s)$.

Finally, for $m = 2$, the minimizer of $\phi_2(s)$ is $s_2^* = \frac{\tilde{r}}{1+\frac{1}{H}}$. $\phi_2(s)$ can be defined in piecewise form:

$$\phi_2(s) = \left[\left(\tilde{r} - s\right)_+\right]^2 + \frac{s^2}{H} = \begin{cases} \left(\tilde{r} - s\right)^2 + \dfrac{s^2}{H} & \text{if } s < \tilde{r} \\ \dfrac{s^2}{H} & \text{if } s \geq \tilde{r}, \end{cases}$$

and has continuous first derivative:

$$\phi_2'(s) = \begin{cases} 2s\left(1 + \dfrac{1}{H}\right) - 2\tilde{r} & \text{if } s < \tilde{r} \\ \dfrac{2s}{H} & \text{if } s \geq \tilde{r}. \end{cases}$$

$\phi_2'(s)$ is positive if $s \geq \tilde{r}$, and $2s\left(1 + \frac{1}{H}\right) - 2\tilde{r}$ is an affine function of s with positive slope that takes value 0 at $s_2^* = \frac{\tilde{r}}{1+\frac{1}{H}} < \tilde{r}$. Therefore s_2^* is the unique minimizer of $\phi_2(s)$, with negative first derivative to the left and positive to the right side. □

The constant H that appears in the optimal portfolio rules is the highest attainable Sharpe ratio in the market; see e.g. [28]. This constant plays an important role in Theorem 3 in Section 4.

Comparing our results for $m = 1, 2$ to the corresponding result (Theorem 2.5) of [25] we notice that the optimal portfolio rules look much simpler. In fact, the optimal portfolio rule in the case where $m = 1$ is exactly twice the optimal portfolio rule in the case where $m = 2$. This simple relationship between the two rules can be attributed to the fact that the case $m = 2$ is more conservative in that it punishes more severely the deviations from target compared to the case $m = 1$. As the two optimal portfolios are almost identical up to a constant multiplicative factor, and it is easier to deal with the case $m = 2$, we shall concentrate on that case in the next section.

An immediate but slight generalization is to allow a budget W_0 instead of 1 in (2). This has the effect of redefining $\tilde{r}$ as $r - W_0 R$.

3. A multi-period portfolio rule under distribution ambiguity with a riskless asset

In the present section we shall extend the result of the previous section for the case $m = 2$ to a multiple-period adjustable robustness setting. The reason that we limit ourselves to $m = 2$ is the fact that we shall deal exclusively with that case in the rest of the paper when we consider ambiguity in mean return.

Consider now, for the sake of illustration, a multiple-period problem with three periods, i.e., $T = 3$. The situation is the following. At the beginning of time period $t = 1$, the investor has a capital W_0 which he allocates among n risky assets with mean return vector μ_1 and variance/covariance matrix Γ_1 and riskless rate R (for the sake of simplicity, assumed constant throughout the entire horizon) according to the expected semi-deviation from a target risk measure. His endowment is W_1 at the beginning of period $t = 2$ where he faces expected return vector μ_2, and variance/covariance matrix Γ_2 where he allocates his wealth again to obtain at the end of period $t = 2$ a wealth W_2. This wealth is again invested into risky assets with expected return vector μ_3 and matrix Γ_3. It is assumed that all matrices $\Gamma_i i = 1, 2, 3$ are invertible.

Let the portfolio positions be represented by vectors $x_t \in \mathbb{R}^n$ for $t = 1, 2, 3$ (risky assets), and by scalars y_t, for $t = 1, 2, 3$ (riskless asset). For a chosen end-of-horizon target wealth r, the adjustable robust portfolio selection problem is defined recursively as follows:

$$V_3 = \min_{x_3, y_3} \max_{\xi_3 \sim (\mu_3, \Gamma_3)} \mathbb{E}[r - \xi_3^T x_3 - R y_3]_+^2$$

subject to

$$e^T x_3 + y_3 = W_2$$

$$V_2 = \min_{x_2, y_2} \max_{\xi_2 \sim (\mu_2, \Gamma_2)} \mathbb{E}[V_3]$$

subject to

$$e^T x_2 + y_2 = W_1$$
$$V_1 = \min_{x_1, y_1} \max_{\xi_1 \sim (\mu_1, \Gamma_1)} \mathbb{E}[V_2]$$

subject to

$$e^T x_1 + y_1 = W_0.$$

The idea is that while for an observer at the beginning of period 1, the wealths W_1 and W_2 are random quantities, the realized wealth, $\tilde{W}_1$ say, is a known quantity at the beginning of period 2. The same is true of realized wealth, $\tilde{W}_2$ say, at the beginning of period 3. These observations allow us to adjust the portfolio according to realized random information instead of selecting all portfolios for all periods at the very beginning.

We begin solving the problem above from period $t = 3$. Using Theorem 1, we have that

$$x_3^* = \left(\frac{r - W_2 R}{1 + H_3}\right) \Gamma_3^{-1} \tilde{\mu}_3$$

where $\tilde{\mu}_3 = \mu_3 - Re$ and $H_3 = \tilde{\mu}_3^T \Gamma_3^{-1} \tilde{\mu}_3$. We substitute this quantity into the objective function and obtain the expression

$$V_3 = \frac{1}{(1 + H_3)^2} [(r - W_2 R)_+^2 + (r - W_2 R)^2 H_3].$$

Now we need to find the supremum of the expectation of V_3 over all random variables $\xi_2 \sim (\mu_2, \Gamma_2)$, i.e., we need to solve the problem

$$\sup_{\xi_2 \sim (\mu_2, \Gamma_2)} \frac{1}{(1 + H_3)^2} \mathbb{E}[(r - R^2 W_1 - R(\xi_2 - Re)^T x_2)_+^2 + H_3 (r - R^2 W_1 - R(\xi_2 - Re)^T x_2)^2]$$

after substituting for y_2. This maximization problem is solved using a simple extension of Lemma 1 of [25] (its proof is a verbatim repetition of the proof of Lemma 1 of [25], and thus omitted):

Lemma 1. *Let the random variable X have mean and variance (μ, σ^2). Then we have for any $\alpha, \beta \in \mathbb{R}$*

$$\sup_{X \sim (\mu, \sigma^2)} \mathbb{E}[\alpha (r - X)_+^2 + \beta (r - X)^2] = (\alpha + \beta)\sigma^2 + \beta (r - \mu)^2 + \alpha (r - \mu)_+^2.$$

Applying the above result gives the function

$$\frac{1}{(1 + H_3)^2} \left[R^2 x_2^T \Gamma_2 x_2 + H_3 (r - R^2 W_1 - R(\mu_2 - Re)^T x_2)^2 + (r - R^2 W_1 - R(\mu_2 - Re)^T x_2)_+^2\right]$$

to be minimized over x_2 using the techniques in the proof of Theorem 1. This results in the solution

$$x_2^* = \left(\frac{r - W_1 R^2}{R(1 + H_2)}\right) \Gamma_2^{-1} \tilde{\mu}_2,$$

where $\tilde{\mu}_2 = \mu_2 - Re$ and $H_2 = \tilde{\mu}_2^T \Gamma_2^{-1} \tilde{\mu}_2$. Repeating the above steps for V_2 (the details are left as an exercise) we obtain the solution x_1^* as

$$x_1^* = \left(\frac{r - W_0 R^3}{R^2 (1 + H_1)}\right) \Gamma_1^{-1} \tilde{\mu}_1,$$

with $\tilde{\mu}_1 = \mu_1 - Re$ and $H_1 = \tilde{\mu}_1^T \Gamma_1^{-1} \tilde{\mu}_1$. The above process can be routinely generalized to arbitrary integer T time periods. Thus we have the following theorem.

Theorem 2. *Let $r - W_{t-1} R^{T-t+1} > 0$ for $t = 1, \ldots, T$. The adjustable robust multi-period portfolio rule using the expected squared semi-deviation from a target wealth r risk measure in a setting of T periods is*

$$x_t^* = \left(\frac{r - W_{t-1} R^{T-t+1}}{R^{T-t}(1 + H_t)}\right) \Gamma_t^{-1} \tilde{\mu}_t,$$

for $t = 1, 2, \ldots, T$ where $\tilde{\mu}_t = \mu_t - Re$ and $H_t = \tilde{\mu}_t^T \Gamma_t^{-1} \tilde{\mu}_t$.

Compared to Theorem 3.1 of [25] our result is very much simpler, and gives a myopic dynamic portfolio policy in the following sense. The single-period optimal portfolio policy consists in setting a target excess wealth beyond that which could be obtained by putting all the present wealth in the riskless asset: $r - W_0R$. Dividing this excess target wealth by the optimal Sharpe ratio H plus 1, one obtains the coefficient in the optimal rule. A similar formula is given in the previous theorem for the multi-period case. Note that each term $\frac{r-W_{t-1}R^{T-t+1}}{R^{T-t}}$ has the following economic meaning: the investor looks at the end of the current period t and sets the excess wealth target equal to

$$\frac{r}{R^{T-t}} - W_{t-1}R$$

which is exactly the discounted target wealth value at time $t+1$ minus the wealth that would be obtained if the current wealth W_t was kept in the riskless account for one period. If this number is equal to zero or is negative, then the final target can simply be achieved by investing the current wealth into the riskless asset for the rest of the horizon; hence the optimal position in risky assets would be zero for the remaining periods. If this excess target remains positive for all periods t, divided by the optimal period t Sharpe ratio H_t plus 1, we have the optimal rule for each period. In other words, it is as if the investor is solving at each time period the following problem:

$$\min_{x_t,y_t} \max_{\xi_t\sim(\mu_t,\Gamma_t)} \mathbb{E}\left[\frac{r}{R^{T-t}} - \xi_t^T x_t - Ry_t\right]_+^2$$

subject to

$$e^T x_t + y_t = W_{t-1}.$$

4. Distribution and expected return ambiguity: single period

It is well-documented that the optimal portfolios may be quite sensitive to inaccuracies in the mean return vector; see e.g. [29,30,18]. To address this issue we consider now the problem

$$RPRR_2 = \min_{x,y} \sup_{\xi\sim(\bar\mu,\Gamma),\bar\mu\in U_{\bar\mu}} \mathbb{E}\left[r - x^T\xi - yR\right]_+^2 \tag{8}$$

$$\text{s.t } x^T e + y = 1 \tag{9}$$

where we define the ellipsoidal uncertainty set $U_{\bar\mu} = \{\bar\mu | \|\Gamma^{-1/2}(\bar\mu - \mu_{\text{nom}})\|_2 \le \sqrt{\epsilon}\}$ for the mean return denoted $\bar\mu$, where μ_{nom} denotes a nominal mean return vector which can be taken as the available estimate of the mean return. The ellipsoidal representation of uncertain parameters is now a well-established choice in the robust optimization literature, and in particular in portfolio optimization; see e.g. [16,17,31–33] for discussion and motivations for the choice of an ellipsoidal set. We refer to the positive parameter ϵ as the *radius of ambiguity*.

Theorem 3. *Let $\tilde r > 0$ and $\mu^* = \mu_{\text{nom}} - Re$. Then the optimal portfolio rule x^* for (8)–(9) is given by*

$$x^* = \frac{\tilde r(\sqrt{H}+\sqrt{\epsilon})(H-\epsilon)}{\sqrt{H}[(\sqrt{H}+\sqrt{\epsilon})^2 + (H-\epsilon)^2]}\Gamma^{-1}\mu^*$$

provided that $\epsilon < H$. If $\epsilon \ge H$, all wealth is invested into the riskless asset.

Proof. Using Remark 2.8 of [25] we can pose the problem (8)–(9) as

$$\min_{x,s} (\tilde r - s)_+^2 + x^T\Gamma x$$

subject to

$$s \le \min_{\bar\mu\in U_{\bar\mu}} x^T(\bar\mu - Re)$$

after elimination of the variable y as in the proof of Theorem 1. Writing the constraint explicitly we get the problem

$$\min_{x,s} (\tilde r - s)_+^2 + x^T\Gamma x$$

subject to the conic constraint

$$s \le x^T\mu^* - \sqrt{\epsilon}\sqrt{x^T\Gamma x},$$

where $\mu^* = \mu_{\text{nom}} - Re$. Assuming $\tilde r > s$ and the constraint to be active, the stationarity equations of the KKT necessary and sufficient optimality conditions give (note that Slater's condition holds trivially)

$$s = \frac{\lambda}{2} + \tilde r$$

where λ is a non-negative Lagrange multiplier, and

$$2\Gamma x - \lambda\left(\mu^* - \frac{\sqrt{\epsilon}}{\sqrt{x^T\Gamma x}}\Gamma x\right) = 0,$$

under the hypothesis that $x \neq 0$. Defining $\sigma = \sqrt{x^T\Gamma x}$ we rewrite the last equation as

$$2\Gamma x - \lambda\left(\mu^* - \frac{\sqrt{\epsilon}}{\sigma}\Gamma x\right) = 0,$$

which yields

$$x^* = \frac{\lambda\sigma}{2\sigma + \lambda\sqrt{\epsilon}}\Gamma^{-1}\mu^*.$$

We have two equations that allow us to solve for λ and σ: the first equation comes from the definition of σ and gives

$$\frac{\lambda^2\sigma^2}{(2\sigma + \lambda\sqrt{\epsilon})^2}H = \sigma^2,$$

and the second equation comes from the conic constraint

$$\tilde{r} - \frac{\lambda}{2} = \frac{\lambda\sigma}{2\sigma + \lambda\sqrt{\epsilon}}H - \sqrt{\epsilon}\frac{\lambda\sigma}{2\sigma + \lambda\sqrt{\epsilon}}\sqrt{H}.$$

We solve the first equation for λ holding σ fixed, and obtain the roots

$$\frac{2(\sqrt{\epsilon} + \sqrt{H})\sigma}{H - \epsilon}, \frac{2(\sqrt{\epsilon} - \sqrt{H})\sigma}{H - \epsilon}.$$

The second root is always negative while the first root is positive for $\epsilon < H$ provided σ is positive, which we assume to be the case. Solving for σ from the second equation above, we obtain after some straightforward simplification

$$\sigma = \frac{\tilde{r}(H - \epsilon)(\sqrt{H} + \sqrt{\epsilon})}{(\sqrt{H} + \sqrt{\epsilon})^2 + (H - \epsilon)^2}$$

which is positive provided $\epsilon < H$. Substituting back into the expression for x^* and simplifying we get the desired expression. If $H \leq \epsilon$ then our hypothesis that $\sigma > 0$ is false; hence the KKT conditions do not yield an optimal solution, except when $H = \epsilon$ in which case the optimal choice is $x^* = 0$.

On the other hand, assuming $\tilde{r} \leq s$, we obtain $\lambda = 0$ and $x^* = 0$ which gives $s \leq 0$. But this is a contradiction since $\tilde{r} > 0$, so this case is impossible. □

The optimal portfolio is a mean–variance efficient portfolio. The optimal Sharpe ratio H serves as an upper bound for the radius of ambiguity. We note that when $\epsilon = 0$ we recover exactly the optimal portfolio rule for case 3 of Theorem 1 with $\mu_{\text{nom}} = \mu$ where μ was defined in Section 2. We define for ease of notation

$$\kappa(\epsilon) = \frac{(\sqrt{H} + \sqrt{\epsilon})(H - \epsilon)}{\sqrt{H}[(\sqrt{H} + \sqrt{\epsilon})^2 + (H - \epsilon)^2]},$$

which is the critical factor introduced by robustness against ambiguity in the mean return vector. This quantity $\kappa(\epsilon)$ is a decreasing function in ϵ as illustrated in Fig. 1.

Consider now the difference $RPR_2 - RPRR_2$ in the risk measures. The respective values are given as

$$RPR_2 = \left[(r - R)\left(1 - \frac{H}{1 + H}\right)\right]_+^2 - (r - R)^2\frac{H}{1 + H},$$

and

$$RPRR_2 = \left[(r - R)(1 - \kappa(\epsilon)H - \kappa(\epsilon)H\sqrt{\epsilon})\right]_+^2 - (r - R)^2\kappa(\epsilon)H.$$

We expect the difference $RPR_2 - RPRR_2$ to be positive at least for a range of values of ϵ. Since the function $RPRR_2$ is complicated to analyse, we provide a numerical example with $H = 0.24$, $r = 1.05$ and $R = 1.03$ in Fig. 2. The gain in mean squared semi-deviation risk reaches a peak for some value ϵ^* of ϵ and then starts to fall. This behaviour could guide the choice for an appropriate value of the radius of ambiguity ϵ. We also note that as H increases, the maximizer ϵ^* shifts to the right as well. This can be seen by comparing with Fig. 3 where we used $H = 0.54$, all other parameters being equal.

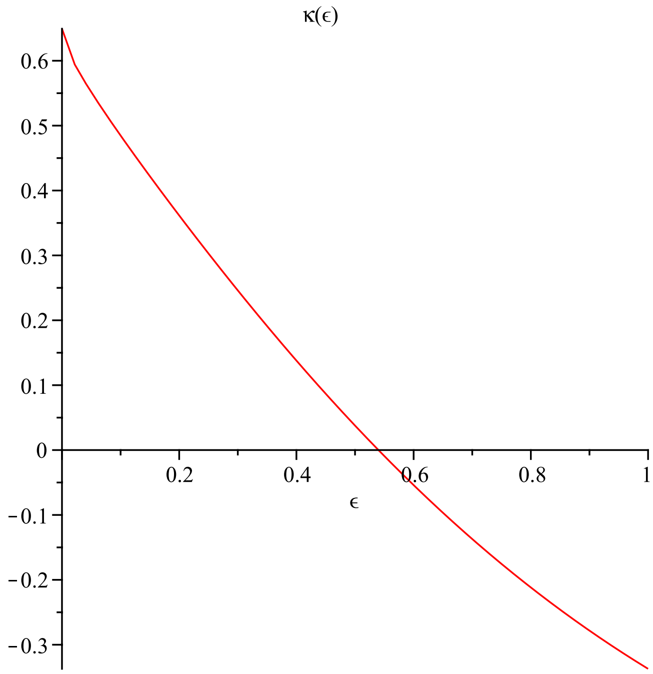

Fig. 1. κ as a function of the ellipsoidal uncertainty radius ϵ with $H = 0.54$.

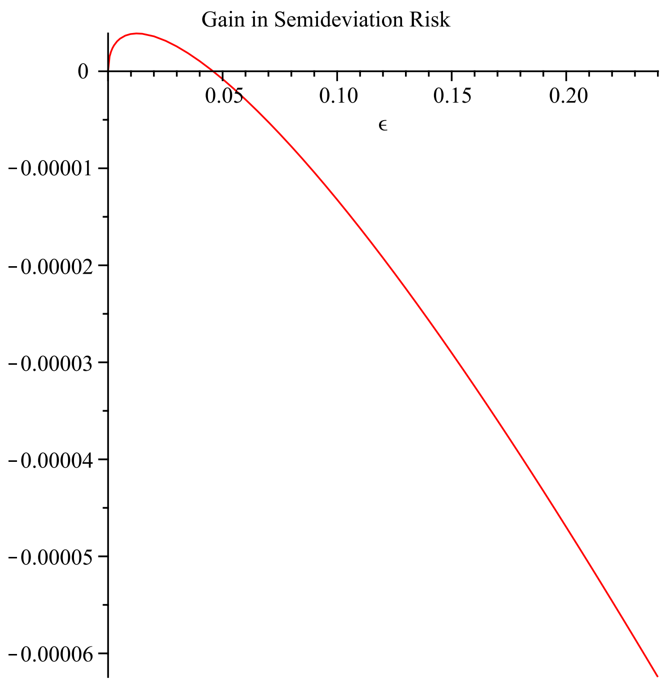

Fig. 2. Gain in mean semi-deviation risk as a function of the ellipsoidal uncertainty radius ϵ with $H = 0.24$, $r = 1.05$, $R = 1.03$.

5. A multi-period case under distribution and mean return ambiguity with a riskless asset

In the present section we shall extend the result of the previous section to a multiple-period adjustable robustness setting.

Consider again a multiple-period problem with three periods, i.e., $T = 3$. The situation is the following. At the beginning of time period $t = 1$, the investor has a capital W_0 which she allocates among n risky assets with mean return vector $\bar{\mu}_1$ and variance/covariance matrix Γ_1 and riskless rate R (for the sake of simplicity, assumed constant throughout the entire horizon) according to the expected semi-deviation from a target risk measure. Her wealth is W_1 at the beginning of period

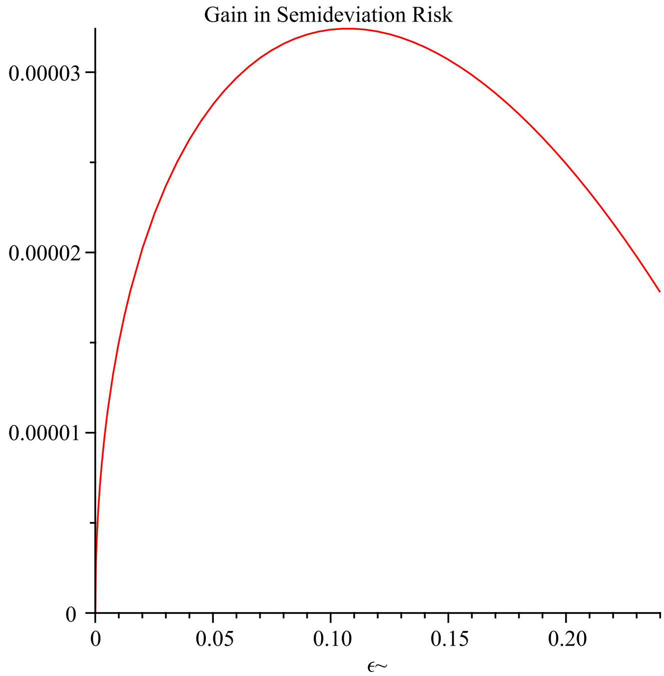

Fig. 3. Gain in mean semi-deviation risk as a function of the ellipsoidal uncertainty radius ϵ with $H = 0.54$, $r = 1.05$, $R = 1.03$.

$t = 2$ where she faces expected return vector $\bar{\mu}_2$, and variance/covariance matrix Γ_2 where she allocates her wealth again to obtain at the end of period $t = 2$ a wealth W_2. This wealth is again invested into risky assets with expected return vector $\bar{\mu}_3$ and matrix Γ_3. It is assumed that all matrices Γ_i $i = 1, 2, 3$ are positive definite. The vectors $\bar{\mu}_1$, $\bar{\mu}_2$ and $\bar{\mu}_3$ are assumed uncertain, and to belong to respective ellipsoids

$$U^t_{\bar{\mu}} = \{\bar{\mu} | \|\Gamma^{-1/2}(\bar{\mu} - \mu^t_{\text{nom}})\|_2 \leq \sqrt{\epsilon_t}\}$$

for the mean return denoted as $\bar{\mu}$, where μ^t_{nom} denotes a nominal mean return vector and the ϵ_t are positive constants for $t = 1, 2, 3$. Let the portfolio positions be represented by vectors $x_t \in \mathbb{R}^n$ for $t = 1, 2, 3$ (risky assets), and by scalars y_t for $t = 1, 2, 3$ (riskless asset). For a chosen end-of-horizon target wealth r, the adjustable robust portfolio selection problem is defined recursively as follows:

$$V_3 = \min_{x_3, y_3} \max_{\xi_3 \sim (\bar{\mu}_3, \Gamma_3), \bar{\mu}_3 \in U^3_{\bar{\mu}}} \mathbb{E}[r - \xi_3^T x_3 - R y_3]^2_+$$

subject to

$$e^T x_3 + y_3 = W_2$$
$$V_2 = \min_{x_2, y_2} \max_{\xi_2 \sim (\bar{\mu}_2, \Gamma_2), \bar{\mu}_2 \in U^2_{\bar{\mu}}} \mathbb{E}[V_3]$$

subject to

$$e^T x_2 + y_2 = W_1$$
$$V_1 = \min_{x_1, y_1} \max_{\xi_1 \sim (\bar{\mu}_1, \Gamma_1), \bar{\mu}_1 \in U^1_{\bar{\mu}}} \mathbb{E}[V_2]$$

subject to

$$e^T x_1 + y_1 = W_0.$$

The computations are tedious but similar to those in Section 3 with the exception that one has to use the proof technique of Theorem 3 in solving the period minimization problems. By way of illustration we look at the periods $t = 3$ and $t = 2$. For $t = 3$, we have immediately the optimal portfolio rule from Theorem 3:

$$x^*_3 = \frac{(r - W_2 R)(\sqrt{H_3} + \sqrt{\epsilon_3})(H_3 - \epsilon_3)}{\sqrt{H_3}[(\sqrt{H_3} + \sqrt{\epsilon_3})^2 + (H_3 - \epsilon_3)^2]} \Gamma_3^{-1} \mu^*_3$$

where $\mu_3^* = \mu_{\text{nom}}^3 - Re$ provided that $\epsilon_3 < H_3$ and $r > W_2R$. Substituting this solution back to the objective function and using Lemma 1 of Section 3 to evaluate the sup over $\xi_2 \sim (\bar{\mu}_2, \Gamma_2)$ we obtain the function

$$(A\tilde{r}_2 - AR(\bar{\mu}_2 - Re)^T x_2)_+^2 + A^2R^2x_2^T\Gamma_2x_2 + B^2R^2x_2^T\Gamma_2x_2 + B^2(\tilde{r}_2 - R(\bar{\mu}_2 - Re)^Tx_2)^2,$$

where $\tilde{r}_2 = r - W_1R^2$, $A = 1 - \kappa_3H_3 + \kappa_3\sqrt{H_3\epsilon_3}$, $B = \kappa_3\sqrt{H_3}$, and

$$\kappa_3 = \frac{(\sqrt{H_3} + \sqrt{\epsilon_3})(H_3 - \epsilon_3)}{\sqrt{H_3}[(\sqrt{H_3} + \sqrt{\epsilon_3})^2 + (H_3 - \epsilon_3)^2]}.$$

Now we evaluate the sup of the above expression over $\bar{\mu}_2 \in U_{\bar{\mu}}^2$ and obtain the optimization problem

$$\min_{x_2,s}(A\tilde{r}_2 - ARs)_+^2 + A^2R^2x_2^T\Gamma_2x_2 + B^2R^2x_2^T\Gamma_2x_2 + B^2(\tilde{r}_2 - Rs)^2$$

subject to

$$s \le \mu_2^{*T}x_2 - \sqrt{\epsilon_2}\sqrt{x_2^T\Gamma_2x_2}$$

where $\mu_2^* = \mu_{\text{nom}}^2 - Re$. We solve this problem using the KKT conditions exactly as in the proof of Theorem 3 and obtain

$$x_2^* = \frac{(r - W_1R^2)(\sqrt{H_2} + \sqrt{\epsilon_2})(H_2 - \epsilon_2)}{R\sqrt{H_2}[(\sqrt{H_2} + \sqrt{\epsilon_2})^2 + (H_2 - \epsilon_2)^2]}\Gamma_1^{-1}\mu_2^*,$$

under the condition $\epsilon_2 < H_2$ and $r > W_1R^2$. Repeating the above steps one more time for $t = 1$ we arrive at

$$x_1^* = \frac{(r - W_0R^3)(\sqrt{H_1} + \sqrt{\epsilon_1})(H_1 - \epsilon_1)}{R^2\sqrt{H_1}[(\sqrt{H_1} + \sqrt{\epsilon_1})^2 + (H_1 - \epsilon_1)^2]}\Gamma_1^{-1}\mu_1^*$$

under the conditions $\epsilon_1 < H_1$ and $r > W_0R^3$. Hence, generalizing the previous derivation to an arbitrary number of periods we obtain the following theorem.

Theorem 4. *Let $r - W_{t-1}R^{T-t+1} > 0$ and $\mu_t^* = \mu_{\text{nom}}^t - Re$ and $H_t = (\mu_t^*)^T\Gamma_t^{-1}\mu_t^*$ for $t = 1, \dots, T$. Then the adjustable robust multi-period optimal portfolio rule under distribution and mean return ambiguity in a setting of T periods is given by*

$$x_t^* = \frac{(r - W_{t-1}R^{T-t+1})(\sqrt{H_t} + \sqrt{\epsilon_t})(H_t - \epsilon_t)}{R^{T-t}\sqrt{H_t}[(\sqrt{H_t} + \sqrt{\epsilon_t})^2 + (H_t - \epsilon_t)^2]}\Gamma_t^{-1}\mu_t^*,$$

provided that $\epsilon_t < H_t$ for $t = 1, \dots, T$.

Notice that for $\epsilon_t = 0$ and $\mu_{\text{nom}}^t = \mu_t$ as defined in Section 3, we obtain the dynamic portfolio rule of Theorem 2. Remarks similar to those made after Theorem 2 in Section 3 also hold for the dynamic portfolio rule of Theorem 4, i.e., the excess target wealth is chosen exactly as described at the end of Section 3. The remaining part of the portfolio rule is identical to the single-period rule. In other words, it is as if the investor is solving at each time period the following problem:

$$\min_{x_t,y_t} \max_{\xi_t\sim(\bar{\mu}_t,\Gamma_t),\bar{\mu}_t\in U_{\bar{\mu}}^t} \mathbb{E}\left[\frac{r}{R^{T-t}} - \xi_t^Tx_t - Ry_t\right]_+^2$$

subject to

$$e^Tx_t + y_t = W_{t-1}.$$

The conditions $r - W_{t-1}R^{T-t+1} > 0$ also make economic sense because if at any time point the condition fails to hold it means that we have achieved a wealth figure that can equal or exceed the target wealth r by staying in the riskless asset for the remaining portion of the horizon until the end of period T.

6. Conclusion

In this paper we derived explicit optimal portfolio rules in single-period and multiple-period investment environments using the risk measure of expected squared semi-deviation from a target under both distribution ambiguity of asset returns and ambiguity of mean returns. We incorporated a riskless asset into the asset universe, which considerably simplifies the portfolio rules. In multiple periods, the optimal portfolio rule is a myopic replica of the single-period rule in the following sense. If the target has not been reached, it is as if the investor is solving at every period a single-period problem with some adjustments to the target. The case of expected shortfall (i.e., $m = 1$) under distribution and mean return ambiguity remains a challenge. It will be addressed in a future work.

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