Traveling Wave Solutions of Degenerate Coupled KdV Equation

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Abstract

We give a detailed study of the traveling wave solutions of ($\ell = 2$) Kaup-Boussinesq type of coupled KdV equations. Depending upon the zeros of a fourth degree polynomial, we have cases where there exist no nontrivial real solutions, cases where asymptotically decaying to a constant solitary wave solutions, and cases where there are periodic solutions. All such possible solutions are given explicitly in the form of Jacobi elliptic functions. Graphs of some exact solutions in solitary wave and periodic shapes are exhibited. Extension of our study to the cases $\ell = 3$ and $\ell = 4$ are also mentioned.

1 Introduction

Multi-component Kaup-Boussinesq(KB) equations can be obtained from the Lax operator

$$L = D^2 - \sum_{k=1}^{l} \lambda^{k-1} q^k(x,t), \tag{1.1}$$

where $q^k(x,t)$, $k = 1, 2, ..., l$ are the multi-KB fields [5]-[8]. Here $l \geq 2$ is a positive integer. The multi system of KB equation is given as

$$\begin{aligned} u_t &= \frac{3}{2} u u_x + q^2_x \\ q^2_t &= q^2 u_x + \frac{1}{2} u q^2_x + q^3_x \\ \vdots & \qquad \vdots \quad \vdots \quad \vdots \\ q^{l-1}_t &= q^{l-1} u_x + \frac{1}{2} u q^{l-1}_x + v_x \\ v_t &= -\frac{1}{4} u_{xxx} + v u_x + \frac{1}{2} u v_x, \end{aligned} \tag{1.2}$$

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where $q^1 = u$ and $q^l = v$. This system in (1.2) was shown to be also a degenerate KdV system of rank one [9], [12]. This system admits also recursion operator for all values of ℓ. In this work we shall investigate the traveling wave solutions of these coupled equations. For this purpose we start with the case $\ell = 2$. To find such solutions we use time and space translation symmetries of the coupled system.

The KB equation for $\ell = 2$ is

$$\begin{aligned} u_t &= \frac{3}{2}uu_x + v_x \\ v_t &= -\frac{1}{4}u_{xxx} + vu_x + \frac{1}{2}uv_x. \end{aligned} \tag{1.3}$$

In [17] the inverse problem of the above system was studied and soliton solutions which decay asymptotically were found. The $N = 1$ solution found in that work corresponds to the interaction of two solitary waves. It was also mentioned in [17] that there is no solution in the form of traveling wave. Here in this work we prove that there exists no asymptotically vanishing traveling wave solutions of system of equations for $\ell = 2$. This is consistent with the observation of [17]. We show that this is also valid for $\ell = 4$. We claim it to be true for all even positive integers. We show that it is possible to find solitary wave solutions of (1.3) which asymptotically decay to non-zero constants. Furthermore in addition to the solitary wave solutions of (1.3) we find all traveling wave solutions which are expressible in terms of Jacobi elliptic functions.

Traveling wave solutions of a system of equations can be obtained if the equations possess time and space translation symmetries. Such symmetries exist in our case. Hence letting $x - ct = \xi$ where c is a constant(the speed of the wave) and $u(x,t) = f(\xi)$, and $v(x,t) = g(\xi)$ from the first equation of (1.3) we have

$$-cf' = \frac{3}{2}ff' + g',$$

which gives

$$g(\xi) = -cf - \frac{3}{4}f^2 + d_1, \tag{1.4}$$

where d_1 is an integration constant. Using $g(\xi)$ in the second equation of (1.3) yields

$$-\frac{1}{4}f''' - 3cff' - \frac{3}{2}f^2f' + (d_1 - c^2)f' = 0.$$

Integrating above equation once we obtain

$$-\frac{1}{4}f'' - \frac{3}{2}cf^2 - \frac{1}{2}f^3 + (d_1 - c^2)f + d_2 = 0.$$

By using f' as an integrating factor, we can integrate once more. Finally we get

$$(f')^2 = -f^4 - 4cf^3 + 4(d_1 - c^2)f^2 + 8d_2f + 8d_3 = F(f), \tag{1.5}$$

where c, d_1, d_2, d_3 are constants. These constants can be determined from the initial conditions $f(0)$, $f'(0)$, $f''(0)$ and $g(0)$. If $F(f)$ has zeros, these zeros are related to these initial conditions. For asymptotically decaying solutions of ($\ell = 2$) KB equations f, f', f'' and g go to zero as $\xi \to \pm\infty$. Here in this work we shall find all possible solutions f of (1.5). Given a solution f one can find the corresponding solution $g(\xi)$ from (1.4).

In [14] and [15], a KB like system

$$h_t + (uh)_x + \frac{1}{4}\, u_{xxx} = 0,$$
$$u_t + uu_x + h_x = 0, \tag{1.6}$$

was considered. Traveling wave solutions of this system satisfy a differential equation like (1.5) but the corresponding polynomial $F_1(f)$ is asymptotically positive definite. This means that the above KB like system possesses asymptotically decaying traveling wave solutions. In [14] and [15] some solitary wave solutions were found. Since the fourth degree polynomial arising in traveling wave solutions of the system (1.6) is different than the one given in (1.5). Then the behavior of solutions here in this work and in [14], [15] are different.

In [16] a modified version of the system (1.6), i.e.

$$h_t + (uh)_x \pm \frac{1}{4}\, \varepsilon^2\, u_{xxx} = 0,$$
$$u_t + uu_x + h_x = 0, \tag{1.7}$$

was considered, where ε is a parameter which controls the dispersion effects. The upper sign is for the case when the gravity force dominates over the capillary one, and the lower sign is for the opposite case when capillary dominates over the gravity. The traveling wave solutions of the above system (1.7) were considered in [16]. The equation (1.5) becomes now $\varepsilon^2\,(f')^2 = \pm F_2(f)$. In both cases solitary wave solutions (dark and bright solitons) were found in [16]. The lower case (negative sign) resembles to our case. Hence our solution in section 3.1 can be considered as a dark soliton in the sense of [16]. This is the solution corresponding one double and two simple zeros of the polynomial $F(f)$. We have all other solutions corresponding to different combinations of the zeros of $F(f)$ in sections 3, 4, and 5.

The layout of our paper is as follows: In section 2, we study the behavior of the solutions in the neighborhood of the zeros of $F(f)$ and discuss all possible cases. We find all solitary wave solutions of the system (1.3) in section 3. These correspond to one double and two simple zeros of $F(f)$, and one triple and one simple zeros of $F(f)$. In section 4, we find all elliptic type of solutions starting from very special ones to the most general elliptic type of solutions. These solutions are given in terms of the zeros of the function $F(f)$. In section 5, we discuss $\ell = 3$ and $\ell = 4$ cases. In section 6, we give the graphs of the solutions corresponding to all cases considered in the text.

2 General waves of permanent form for $(\ell = 2)$

Proposition 2.1. *There is no real asymptotically vanishing traveling wave solution of the equation (1.3) in the form $u(x,t) = f(\xi)$ and $v(x,t) = g(\xi)$, where $\xi = x - ct$.*

Proof. If we apply the boundary conditions $f, f', f'', g \to 0$ as $\xi \to \pm\infty$ which describe the solitary wave, we get $d_1 = d_2 = d_3 = 0$. Hence we end up with

$$\begin{aligned}(f')^2 = -f^4 - 4cf^3 - 4c^2 f^2 &= -f^2(f^2 + 4cf + 4c^2) \\ &= -f^2(f + 2c)^2.\end{aligned}$$

Clearly, we do not have a real solution f. □

Now we will deal with the equation (1.5). In order to have real solutions, d_1, d_2, d_3 must take values so that the following inequality holds:

$$4d_1 f^2 + 8d_2 f + 8d_3 \geq f^2(f + 2c)^2.$$

2.1 Zeros of $F(f)$ and Types of Solutions

Here we will analyze the zeros of $F(f)$.

(i) If $f_1 = f(\xi_1)$ is a *simple zero* of $F(f)$ we have $F(f_1) = 0$. Taylor expansion of $F(f)$ gives

$$\begin{aligned}(f')^2 = F(f) &= F(f_1) + F'(f_1)(f - f_1) + O((f - f_1)^2) \\ &= F'(f_1)(f - f_1) + O((f - f_1)^2).\end{aligned}$$

From here we get $f'(\xi_1) = 0$ and $f''(\xi_1) = \frac{1}{2}F'(f_1)$. Hence we can write the function $f(\xi)$ as

$$\begin{aligned}f(\xi) &= f(\xi_1) + (\xi - \xi_1)f'(\xi_1) + \frac{1}{2}(\xi - \xi_1)^2 f''(\xi_1) + O((\xi - \xi_1)^3) \\ &= f_1 + \frac{1}{4}(\xi - \xi_1)^2 F'(f_1) + O((\xi - \xi_1)^3). \qquad (2.1)\end{aligned}$$

Thus, in the neighborhood of $\xi = \xi_1$, the function $f(\xi)$ has local minimum or maximum as $F'(f_1)$ is positive or negative respectively since $f''(\xi_1) = \frac{1}{2}F'(f_1)$.

(ii) If $f_1 = f(\xi_1)$ is a *double zero* of $F(f)$ we have $F(f_1) = F'(f_1) = 0$. Taylor expansion of $F(f)$ gives

$$\begin{aligned}(f')^2 = F(f) &= F(f_1) + F'(f_1)(f - f_1) + \frac{1}{2}(f - f_1)^2 F''(f_1) + O((f - f_1)^3) \\ &= \frac{1}{2}(f - f_1)^2 F''(f_1) + O((f - f_1)^3). \qquad (2.2)\end{aligned}$$

To have real solution f, we should have $F''(f_1) > 0$. From the equality (2.2) we get

$$f' \pm \frac{1}{\sqrt{2}} f \sqrt{F''(f_1)} \sim \pm \frac{1}{\sqrt{2}} f_1 \sqrt{F''(f_1)},$$

which gives

$$f(\xi) \sim f_1 + \alpha e^{\pm \frac{1}{\sqrt{2}} \sqrt{F''(f_1)} \xi}, \tag{2.3}$$

where α is a constant. Hence $f \to f_1$ as $\xi \to \mp\infty$. The solution f can have only one peak and the wave extends from $-\infty$ to ∞.

(iii) If $f_1 = f(\xi_1)$ is a *triple zero* of $F(f)$ we have $F(f_1) = F'(f_1) = F''(f_1) = 0$. Taylor expansion of $F(f)$ gives

$$\begin{aligned} (f')^2 &= F(f) \\ &= F(f_1) + F'(f_1)(f - f_1) + \frac{1}{2}(f - f_1)^2 F''(f_1) + \frac{1}{6}(f - f_1)^3 + O((f - f_1)^4) \\ &= \frac{1}{6}(f - f_1)^3 F'''(f_1) + O((f - f_1)^4). \end{aligned} \tag{2.4}$$

This is valid only if both signs of $(f - f_1)^3$ and $F'''(f_1)$ are same i.e. we have the following two possibilities to have real solution f:

1) $(f - f_1) > 0$ and $F'''(f_1) > 0$,

2) $(f - f_1) < 0$ and $F'''(f_1) < 0$.

Let us analyze these cases. If $(f - f_1) > 0$ and $F'''(f_1) > 0$ then we have

$$f' \sim \pm \frac{1}{\sqrt{6}} (f - f_1)^{3/2} \sqrt{F'''(f_1)},$$

which gives

$$f(\xi) \sim f_1 + \frac{4}{\left(\pm \frac{1}{\sqrt{6}} \sqrt{F'''(f_1)} \xi + \alpha_1 \right)^2}, \tag{2.5}$$

where α_1 is a constant. Thus $f \to f_1$ as $\xi \to \pm\infty$ if $F'''(f_1) > 0$.

Let $(f - f_1) < 0$ and $F'''(f_1) < 0$ hold. In this case, $(f_1 - f) > 0$ and $F'''(f_1) = -G(f_1)$, $G(f_1) > 0$. Then

$$f' \sim \pm \frac{1}{\sqrt{6}} (f_1 - f)^{3/2} \sqrt{G(f_1)},$$

which yields

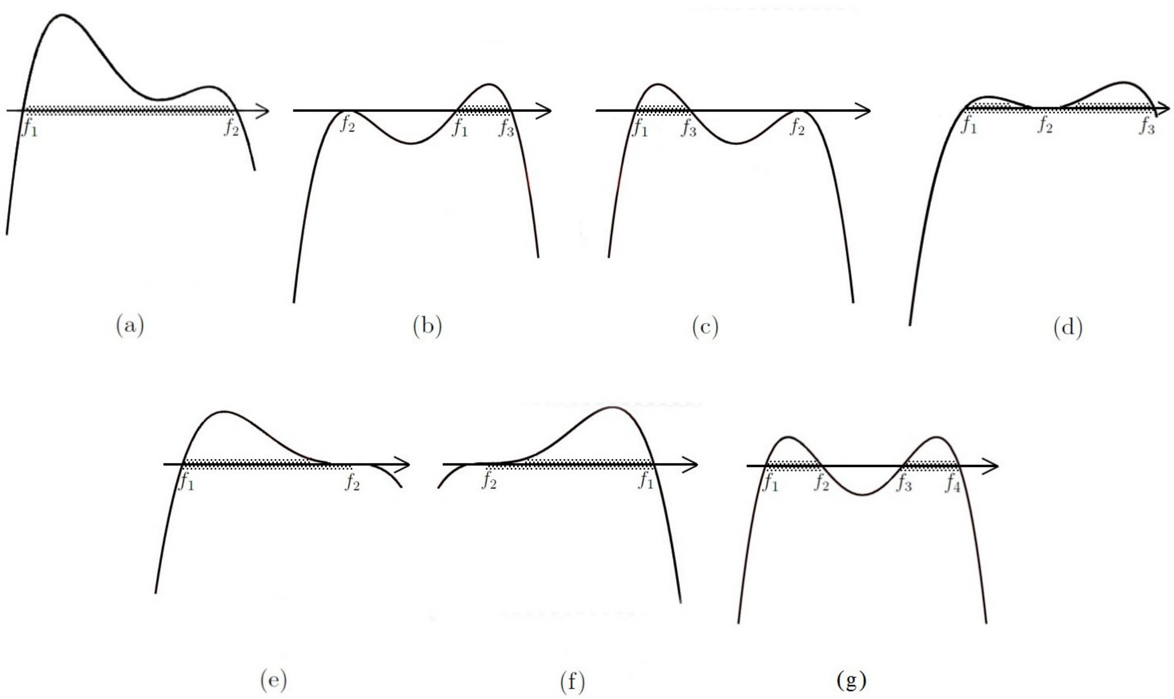
$$f(\xi) \sim f_1 - \frac{4}{\left(\pm \frac{1}{\sqrt{6}} \sqrt{G(f_1)} \xi + \alpha_2 \right)^2}, \tag{2.6}$$

where α_2 is a constant. Thus $f \to f_1$ as $\xi \to \pm\infty$ if $F'''(f_1) = -G(f_1) < 0$.

(iv) If $f_1 = f(\xi_1)$ is a *quadruple zero* of $F(f)$ then there is only one possibility $F(f) = -(f - f_1)^4 = (f')^2$. It is clear that this case does not give a real solution except when $f = f_1$.

2.2 All Possible Cases

Here we present the sketches of the graphs of $F(f)$. Real solutions $(f')^2 = F(f) \geq 0$ occur in the shaded regions.



Now we analyze all possible cases about the zeros of $F(f)$ and above graphs.

(1) No real zero. If there is no real zeros of $F(f)$ then $F(f) < 0$. Hence there is no real solution of (1.5) in that case.

(2) Two simple real zeros. If there is a simple zero f_1 of $F(f)$, since the order of $F(f)$ is four, there should be another simple zero f_2 of $F(f)$. The corresponding graph to this case is given in (a). Here, the real solution occurs when f is between two simple zeros f_1 and f_2. At f_1, $F'(f_1) = f''(\xi_1) > 0$ so graph of the function f is concave up at ξ_1. At f_2, $F'(f_2) = f''(\xi_2) > 0$ hence graph of the function f is concave up at ξ_2. Thus it is clear that the solution is periodic.

(3) One double zero. If there is only one double zero f_1 then

$$(f')^2 = -(f - f_1)^2(f^2 + pf + q), \tag{2.7}$$

where f^2+pf+q has no real zero. This means $p^2-4q<0$ which yields that $f^2+pf+q>0$. Then $(f')^2=-(f-f_1)^2(f^2+pf+q)<0$ hence there is no real solution in that case except when $f=f_1$. Similarly, in the case when $F(f)$ has two double zeros f_1 and f_2, no real solutions exist since $(f')^2=-(f-f_1)^2(f-f_2)^2<0$ except when $f=f_1$ or $f=f_2$.

(4) One double and two simple zeros. The corresponding graphs for this case are (b), (c) and (d). In (b) and (c), there are two simple zeros f_1 and f_3 and one double zero f_2. We have $f_2<f_1<f_3$ in (b) and in the graph (c), $f_1<f_3<f_2$. In both cases, the real solution occurs when f is between two simple zeros f_1 and f_3. At f_1, $F'(f_1)=f''(\xi_1)>0$ so graph of the function f is concave up at ξ_1. At f_3, $F'(f_3)=f''(\xi_3)>0$ hence graph of the function f is concave up at ξ_3. It is clear that the solution is periodic in this case.

In (d), different than the graphs (b) and (c) we have $f_1<f_2<f_3$. The real solution occurs when f stays between f_1 and f_2 or f_2 and f_3. At f_1, $F'(f_1)=f''(\xi_1)>0$ hence graph of the function f is concave up at ξ_1. At double zero f_2, $f\to f_2$ as $\xi\to\pm\infty$. Hence we have a solitary wave solution with amplitude $f_1-f_2<0$.

Similarly at f_3, $F'(f_3)=f''(\xi_3)<0$, hence graph of the function f is concave down at ξ_3. Therefore, we also have a solitary wave solution with amplitude $f_3-f_2>0$. Explicit solitary wave solution for this case can be found in the next section.

(5) One triple and one simple zero. For this case, we can analyze the graphs (e) and (f). In (e), f_1 is simple and f_2 is triple zeros of $F(f)$. We see that $F'(f_1)=f''(\xi_1)>0$ hence graph of the function f is concave up at ξ_1. From the case (iii), we know that $f\to f_2$ as $\xi\to\pm\infty$ for $f-f_2<0$ and $F'''(f_2)<0$. Hence we have solitary wave solution with amplitude $f_1-f_2<0$.

Similarly, in (f) we have one triple zero f_1 and one simple zero f_2. For triple zero f_1 we have $f\to f_1$ as $\xi\to\pm\infty$ for $f-f_1>0$ and $F'''(f_2)>0$. For simple zero we have $F'(f_2)=f''(\xi_2)<0$ therefore graph of the function f is concave down at ξ_2. Clearly, we have a solitary wave solution with amplitude $f_2-f_1>0$. Explicit solitary wave solution for this case can be found in the next section.

(6) Four different simple zeros. The corresponding graph for this case is given in (g). Here, there are four simple zeros $f_1<f_2<f_3<f_4$. For f_1 and f_3, we have $F'(f_1)=f''(\xi_1)>0$ and $F'(f_3)=f''(\xi_3)>0$ thus graph of the function f is concave up at ξ_1 and ξ_3. For f_2 and f_4, we have $F'(f_2)=f''(\xi_2)<0$ and $F'(f_4)=f''(\xi_4)<0$ so graph of the function f is concave down at ξ_2 and ξ_4. Obviously, the solution is periodic.

As a summary we have the following results. By solution below, we mean non-constant solutions.

Proposition 2.2. *Equation (1.5) has no real solutions when the function F(f) has one the following properties: (i) it has no real zeros, (ii) it has only two real zeros, (iii) it has only one double zero, (iv) it has only two double zeros and (v) it has a quartic zero.*

Proposition 2.3. *Equation (1.5) admits solitary wave solutions when the function $F(f)$ admits (i) one double and two simple zeros and (ii) one triple and one simple zeros.*

From the proposition 2.2 we can conclude that the function $F(f)$ must have four zeros,

$$F(f) = -(f - f_1)(f - f_2)(f - f_3)(f - f_4).$$

The constants c, d_1, d_2, d_3 can be expressed in terms of the zeros of $F(f)$:

$$\begin{aligned} c &= -\frac{f_1 + f_2 + f_3 + f_4}{4} \\ d_1 &= \frac{(f_1 + f_2 + f_3 + f_4)^2}{16} - \frac{f_1 f_2 + f_2 f_4 + f_2 f_3 + f_1 f_4 + f_1 f_3 + f_3 f_4}{4} \\ d_2 &= \frac{f_1 f_2 f_4 + f_1 f_2 f_3 + f_2 f_3 f_4 + f_1 f_3 f_4}{8} \\ d_3 &= -\frac{f_1 f_2 f_3 f_4}{8}. \end{aligned} \tag{2.8}$$

In the next section we shall find the solitary wave solutions mentioned in the above proposition which correspond to special cases of the zeros f_1, f_2, f_3, f_4.

3 Exact Solitary Wave Solutions

3.1 One double zero and two simple zeros

Let f_1 and f_3 be simple zeros and f_2 be a double zero of $F(f)$. Thus we have

$$(f')^2 = F(f) = -(f - f_2)^2 (f - f_1)(f - f_3).$$

Let $f - f_2 = u$ and so $f - f_1 = u - u_1$, where $u_1 = f_1 - f_2$ and $f - f_2 = u - u_3$, where $u_3 = f_3 - f_2$. Hence the above equation becomes

$$(u')^2 = -u^2 (u - u_1)(u - u_3).$$

Using the substitution $u = \dfrac{1}{y}$

$$\begin{aligned} (y')^2 = -y^2 \Big(\frac{1}{y} - u_1\Big)\Big(\frac{1}{y} - u_3\Big) &= -(1 - y u_1)(1 - y u_3) \\ &= -u_1 u_3 \Big(\frac{1}{u_1} - y\Big)\Big(\frac{1}{u_3} - y\Big). \end{aligned}$$

After some arrangements we have

$$(y')^2 = -u_1 u_3 \Big\{ \Big[y - \frac{1}{2}\Big(\frac{1}{u_1} + \frac{1}{u_3}\Big)\Big]^2 - \frac{1}{4}\Big(\frac{1}{u_1} - \frac{1}{u_3}\Big)^2 \Big\}. \tag{3.1}$$

Using the trigonometric substitution

$$y - \frac{1}{2}\Big(\frac{1}{u_1} + \frac{1}{u_3}\Big) = \frac{1}{2}\Big(\frac{1}{u_1} - \frac{1}{u_3}\Big)\cosh\theta$$

the equation (3.1) becomes

$$(\theta')^2 = -u_1 u_3.$$

Note that in the case when $F(f)$ has two simple zeros and one double zero, the solitary wave solution occurs only when we have $f_1 < f_2 < f_3$ and this makes $u_1 u_3 < 0$ or $-u_1 u_3 > 0$. So from the above equation we get $\theta' = \pm\sqrt{-u_1 u_3}$ which yields

$$\theta = \pm\sqrt{-u_1 u_3}(\xi - \xi_0),$$

where ξ_0 is an integration constant. Hence the solution f is

$$f = f_2 + \frac{2}{c_1 + c_2 \cosh(\sqrt{(f_2 - f_1)(f_3 - f_2)}(\xi - \xi_0))}, \tag{3.2}$$

where $c_1 = \Big(\frac{1}{f_1 - f_2} + \frac{1}{f_3 - f_2}\Big)$ and $c_2 = \Big(\frac{1}{f_1 - f_2} - \frac{1}{f_3 - f_2}\Big)$. It is clear that $f \to f_2$ as $\xi \to \pm\infty$.

Note that when $u_1 u_3 > 0$ which means $f_1 < f_3 < f_2$ or $f_2 < f_1 < f_3$ we have the following solution which is not a solitary wave solution:

$$f = f_2 + \frac{2}{c_1 \pm c_2 \sin(\sqrt{(f_2 - f_1)(f_3 - f_2)}(\xi - \xi_0))}, \tag{3.3}$$

with the same c_1 and c_2 stated above.

3.2 One triple and one simple zeros

Let f_1 be simple and f_2 be triple zeros of $F(f)$. Hence

$$(f')^2 = F(f) = -(f - f_1)^3 (f - f_2). \tag{3.4}$$

The relations between the zeros of $F(f)$ and the parameters are

$$c = -\frac{f_2 + 3f_1}{4},\ d_1 = \frac{f_2^2 - 6f_1 f_2 - 3f_1^2}{16},\ d_2 = \frac{3f_1^2 f_2 + f_1^3}{8},\ d_3 = -\frac{f_1^3 f_2}{8}. \tag{3.5}$$

Let us solve the equation (3.4). Let $f - f_1 = u$ so above equation becomes

$$(u')^2 = -u^3(u - u_0), \quad u_0 = f_2 - f_1.$$

We have

$$\frac{du}{u^{3/2}\sqrt{u_0 - u}} = \frac{\sqrt{u_0 - u}\,du}{(u_0 - u)u\sqrt{u}} = d\xi.$$

By making the substitution $t = \dfrac{\sqrt{u_0 - u}}{\sqrt{u}}$, the above equality can be solved as

$$\frac{-2}{u_0}\sqrt{\frac{u_0 - u}{u}} = \xi - \xi_0,$$

where ξ_0 is an integration constant. Hence we find

$$u = \frac{u_0}{1 + \frac{u_0^2}{4}(\xi - \xi_0)^2},$$

and inserting $u = f - f_1$ and $u_0 = f_2 - f_1$ we get the solution

$$f = f_1 + \frac{f_2 - f_1}{1 + \frac{1}{4}(f_2 - f_1)^2(\xi - \xi_0)^2}.$$

It is clear that $f \to f_1$ as $\xi \to \pm\infty$.

3.3 Limiting Cases

Here we will analyze the solution (3.2) which corresponds to the case when $F(f)$ has one double zero f_2 and two simple zeros f_1 and f_3.

(a) When $f_1 + f_3 = 2f_2$, the solution (3.2) reduces to

$$f = f_2 + \frac{2(f_2 - f_1)(f_3 - f_2)}{(f_3 - f_1)}\text{sech}(\sqrt{(f_2 - f_1)(f_3 - f_2)}(\xi - \xi_0)). \tag{3.6}$$

(b) When $2f_1f_3 = f_2(f_1 + f_3)$, the solution (3.2) reduces to

$$f = \frac{c_2\cosh(\sqrt{(f_2 - f_1)(f_3 - f_2)}(\xi - \xi_0))}{c_1 + c_2\cosh(\sqrt{(f_2 - f_1)(f_3 - f_2)}(\xi - \xi_0))}, \tag{3.7}$$

which can be converted to

$$f = \frac{c_2}{c_2 + c_1\text{sech}(\sqrt{(f_2 - f_1)(f_3 - f_2)}(\xi - \xi_0))}, \tag{3.8}$$

where $c_1 = \left(\dfrac{1}{f_1 - f_2} + \dfrac{1}{f_3 - f_2}\right)$ and $c_2 = \left(\dfrac{1}{f_1 - f_2} - \dfrac{1}{f_3 - f_2}\right)$.

(c) When $f_2 = 0$, then the solution (3.2) reduces to

$$f = \frac{2f_1f_3}{(f_1 + f_3) + (f_3 - f_1)\cosh(\sqrt{-f_1f_3}(\xi - \xi_0))}, \tag{3.9}$$

which can also be written as

$$f = \frac{2f_1f_3\text{sech}(\sqrt{-f_1f_3}(\xi - \xi_0))}{(f_3 - f_1) + (f_1 + f_3)\text{sech}(\sqrt{-f_1f_3}(\xi - \xi_0))}, \quad f_1f_3 < 0. \tag{3.10}$$

(d) When $f_2 \to f_1$ or $f_2 \to f_3$, then the case turns to the case when $F(f)$ has one triple zero and one simple zero. If $f_2 \to f_1$, the solution (3.2) reduces to

$$f = f_1,$$

and if $f_2 \to f_3$, the solution (3.2) reduces to

$$f = f_3 + \frac{f_1 - f_3}{1 + \frac{1}{4}(f_1 - f_3)^2(\xi - \xi_0)^2}. \tag{3.11}$$

4 Exact Solutions in Terms of Elliptic Functions

In this section we will find exact solutions of (1.3) by using the Jacobi elliptic functions. Let us give the list of the Jacobi elliptic functions and first order differential equations satisfied by them.

4.1 Jacobi Elliptic Functions

$$y = \mathrm{sn}v \quad (y')^2 = (1 - y^2)(1 - k^2y^2), \tag{4.1}$$
$$y = \mathrm{cn}v \quad (y')^2 = (1 - y^2)(1 - k^2 + k^2y^2), \tag{4.2}$$
$$y = \mathrm{dn}v \quad (y')^2 = (1 - y^2)(y^2 - 1 + k^2), \tag{4.3}$$
$$y = \mathrm{tn}v \quad (y')^2 = (1 + y^2)[1 + (1 - k^2)y^2], \tag{4.4}$$
$$y = \frac{1}{\mathrm{sn}v} \quad (y')^2 = (y^2 - 1)(y^2 - k^2), \tag{4.5}$$
$$y = \frac{1}{\mathrm{cn}v} \quad (y')^2 = (y^2 - 1)[(1 - k^2)y^2 + k^2], \tag{4.6}$$
$$y = \mathrm{dn}v\mathrm{tn}v \quad (y')^2 = (1 + y^2)^2 - 4k^2y^2, \tag{4.7}$$

and for the squares of these functions we have cubic equations

$$y = \mathrm{sn}^2v \quad (y')^2 = 4y(1 - y)(1 - k^2y), \tag{4.8}$$
$$y = \mathrm{cn}^2v \quad (y')^2 = 4y(1 - y)(1 - k^2 + k^2y), \tag{4.9}$$
$$y = \mathrm{dn}^2v \quad (y')^2 = 4y(1 - y)(y - 1 + k^2), \tag{4.10}$$
$$y = \mathrm{tn}^2v \quad (y')^2 = 4y(1 + y)[1 + (1 - k^2)y], \tag{4.11}$$
$$y = \frac{1}{\mathrm{cn}^2v} \quad (y')^2 = 4y(y - 1)[(1 - k^2)y + k^2], \tag{4.12}$$
$$y = \frac{1}{\mathrm{sn}^2v} \quad (y')^2 = 4y(y - 1)[y - k^2], \tag{4.13}$$
$$y = \mathrm{dn}^2v\mathrm{tn}^2v \quad (y')^2 = 4y[(1 + y)^2 - 4k^2y]. \tag{4.14}$$

We will also make analysis at the limiting points $k = 0$ and $k = 1$. Remind that

$$\begin{aligned} &k = 0 \quad \mathrm{sn}v = \sin v \quad , \quad \mathrm{cn}v = \cos v \quad , \quad \mathrm{dn}v = 1, \\ &k = 1 \quad \mathrm{sn}v = \tanh v \quad , \quad \mathrm{cn}v = \mathrm{dn}v = \mathrm{sech}v. \end{aligned} \tag{4.15}$$

4.2 Special Solutions of (1.3) in Terms of Elliptic Functions

For some special values of c, d_1, d_2, d_3, we have solutions of (1.3) in terms of Jacobi elliptic functions. Here we will present two such types of solutions.

Case 1. Solutions of the form $u(x,t) = f(\xi) = \gamma + \alpha y(\beta\xi)$

Here we shall find the solutions of (1.3) having the form $u(x,t) = f(\xi) = \gamma + \alpha y(\beta\xi)$, where γ, α, β are constants, $\xi = x - ct$ and y is one of the Jacobi elliptic functions. When we use this form in (1.5) we get the following equation:

$$\begin{aligned}(y')^2 &= -\frac{\alpha^2}{\beta^2}y^4 - \frac{4\alpha}{\beta^2}(c+\gamma)y^3 + \frac{2}{\beta^2}(2d_1 - 6c\gamma - 3\gamma^2 - 2c^2)y^2 \\ &\quad + \frac{4}{\alpha\beta^2}(2d_2 + 2d_1\gamma - 2c^2\gamma - 3c\gamma^2 - \gamma^3)y \\ &\quad + \frac{1}{\alpha^2\beta^2}(-\gamma^4 - 4c^2\gamma^2 + 8d_2\gamma - 4c\gamma^3 + 8d_3 + 4d_1\gamma^2). \qquad (4.16)\end{aligned}$$

Since the parameters are real, we have $\alpha^2/\beta^2 > 0$. Hence the coefficient of the term y^4 is negative. Thus there are two possibilities: $\alpha^2 = k^2\beta^2$ which corresponds to Jacobi elliptic function $\mathrm{cn}v$ and $\alpha^2 = \beta^2$ corresponding to $\mathrm{dn}v$. Comparing the differential equations for $\mathrm{cn}v$ and $\mathrm{dn}v$ with (4.16), we note that the coefficients of the terms y^3 and y should be zero. That gives

$$\begin{aligned}\gamma &= -c = \frac{f_1 + f_2 + f_3 + f_4}{4} \\ d_2 &= cd_1 \\ &= \frac{f_1 + f_2 + f_3 + f_4}{64}\Big[4(f_1f_2 + f_1f_3 + f_1f_4 + f_2f_3 + f_2f_4 + f_3f_4) - (f_1 + f_2 + f_3 + f_4)^2\Big],\end{aligned}$$

where d_1 is given in (2.8). Note that the equality $d_2 = cd_1$ yields a relation between the zeros of $F(f)$:

$$(f_1 + f_2 - f_3 - f_4)(f_1 + f_3 - f_2 - f_4)(f_1 + f_4 - f_2 - f_3) = 0. \qquad (4.17)$$

The equation (4.16) is simplified as

$$(y')^2 = -\frac{\alpha^2}{\beta^2}y^4 + \frac{\mu_2}{\beta^2}y^2 + \frac{\mu_0}{\alpha^2\beta^2}, \qquad (4.18)$$

where

$$\begin{aligned}\mu_2 &= \frac{3}{8}(f_1 + f_2 + f_3 + f_4)^2 - (f_1f_2 + f_1f_3 + f_1f_4 + f_2f_3 + f_2f_4 + f_3f_4) \\ \mu_0 &= \frac{(f_1 + f_2 + f_3 + f_4)^2}{16}(f_1f_2 + f_1f_3 + f_1f_4 + f_2f_3 + f_2f_4 + f_3f_4) \\ &\quad - \frac{5}{256}(f_1 + f_2 + f_3 + f_4)^4 - f_1f_2f_3f_4.\end{aligned}$$

1.a cn solution

Let $y = \mathrm{cn}(\beta\xi)$ with $\xi = x - ct$ where the function y satisfies the first order differential equation (4.2). Hence when we compare the coefficients of (4.16) and (4.2), we get

$$\beta^2 = \frac{\mu_2}{2k^2-1} \qquad \alpha^2 = \frac{k^2\mu_2}{2k^2-1} \qquad k^2 = \frac{1}{2} + \frac{\mu_2}{2\sqrt{4\mu_0+\mu_2^2}}.$$

Note that there is also one more condition that should be satisfied given in (4.17). Assume that $f_1+f_3-f_2-f_4=0$ or $f_4=f_1+f_3-f_2$. In addition, without loss of generality let $f_1 \le f_2 \le f_3$. Then we have

$$\beta = \pm\sqrt{(f_2-f_3)(f_1-f_2)} \qquad \alpha = \pm\frac{(f_1-f_3)}{2} \qquad k^2 = \frac{(f_1-f_3)^2}{4(f_1-f_2)(f_2-f_3)}. \tag{4.19}$$

Hence the solution is

$$u(x,t) = \pm\frac{(f_1-f_3)}{2}\mathrm{cn}\Big[\sqrt{(f_2-f_3)(f_1-f_2)}\Big(x+\frac{f_1+f_3}{2}t\Big)\Big] + \frac{f_1+f_3}{2}. \tag{4.20}$$

Let us check the limiting points. For $k=0$, we have $f_1=f_3$ and the solution becomes $u(x,t)=f_1$. For $k=1$, we get the relation

$$2f_2 = f_1+f_3. \tag{4.21}$$

Hence the solution is

$$u(x,t) = \pm\frac{(f_1-f_3)}{2}\mathrm{sech}\Big[\frac{f_1-f_3}{2}\Big(x+\frac{f_1+f_3}{2}t\Big)\Big] + \frac{f_1+f_3}{2}. \tag{4.22}$$

1.b dn solution

Let $y=\mathrm{dn}(\beta\xi)$ with $\xi=x-ct$ where the function satisfies the differential equation (4.3). If we compare the coefficients of (4.16) and (4.3), we get

$$\beta^2 = \alpha^2 = \frac{2\mu_0}{-\mu_2+\sqrt{4\mu_0+\mu_2^2}} \qquad k^2 = 2 + \frac{\mu_2^2}{2\mu_0} - \frac{\mu_2}{2\mu_0}\sqrt{4\mu_0+\mu_2^2}.$$

If we assume that $f_4=f_1+f_3-f_2$ and $f_1\le f_2\le f_3$ then we have

$$\beta = \pm\alpha = \pm\frac{f_1-f_3}{2} \qquad k = \pm 2\frac{\sqrt{(f_1-f_2)(f_2-f_3)}}{f_1-f_3}. \tag{4.23}$$

Hence the solution is

$$u(x,t) = \pm\frac{f_1-f_3}{2}\mathrm{dn}\Big[\frac{f_1-f_3}{2}\Big(x+\frac{f_1+f_3}{2}t\Big)\Big] + \frac{f_1+f_3}{2}. \tag{4.24}$$

For $k=0$, we have either $f_1=f_2$ or $f_2=f_3$. For both cases, depending on the sign of β, we have constant solutions: $u(x,t)=f_1$ or $u(x,t)=f_3$. For $k=1$, from (4.23) we get the relation $2f_2=f_1+f_3$. Thus the corresponding solution is

$$u(x,t) = \pm\frac{f_1-f_3}{2}\mathrm{sech}\Big[\frac{f_1-f_3}{2}\Big(x+\frac{f_1+f_3}{2}t\Big)\Big] + \frac{f_1+f_3}{2}. \tag{4.25}$$

Case 2. Solutions of the form $u(x,t) = f(\xi) = \dfrac{a_1}{a_2 + b_2 y(\beta\xi)}$

Here we shall find solutions of (1.3) having the form $u(x,t) = f(\xi) = \dfrac{a_1}{a_2 + b_2 y(\beta\xi)}$, where a_1, a_2, b_2, β are constants and $\xi = x - ct$. If we use this form in the equation (1.5) we get the following equation:

$$\begin{aligned}
(y')^2 &= \frac{8d_3 b_2^2}{\beta^2 a_1^2} y^4 + \frac{8}{\beta^2 a_1^2}(d_2 a_1 b_2 + 4d_3 a_2 b_2) y^3 \\
&+ \frac{4}{\beta^2 a_1^2}(12 d_3 a_2^2 + a_1^2 d_1 + 6 d_2 a_1 a_2 - a_1^2 c^2) y^2 \\
&+ \frac{4}{\beta^2 a_1^2 b_2}(2a_1^2 d_1 a_2 - c a_1^3 - 2a_1^2 c^2 a_2 + 6 d_2 a_1 a_2^2 + 8 d_3 a_2^3) y \\
&+ \frac{1}{\beta^2 a_1^2 b_2^2}(8 d_3 a_2^4 + 8 d_2 a_1 a_2^3 + 4 a_1^2 d_1 a_2^2 - 4 a_1^2 c^2 a_2^2 - 4 c a_1^3 a_2 - a_1^4), \qquad (4.26)
\end{aligned}$$

where $a_1, b_2, \beta \neq 0$. As we did in the previous case we shall again use Jacobi elliptic functions (4.1)-(4.5) and study the special cases for $k = 0$ and $k = 1$. The differential equations satisfied by these elliptic functions do not have terms with y^3 and y. Hence the coefficients of y^3 and y should be zero in (4.26). Let also $a = \dfrac{a_2}{a_1}$ and $b = \dfrac{b_2}{a_1}$, $a_1 \neq 0$. Then we get

$$\begin{aligned}
d_1 &= \frac{c}{2a} + c^2 + 8a^2 d_3 \\
&= \frac{1}{16a}[a(f_1^2 + f_2^2 + f_3^2 + f_4^2) + 2a(f_1 f_2 + f_1 f_3 + f_1 f_4 + f_2 f_3 + f_2 f_4 + f_3 f_4) \\
&\quad -2(f_1 + f_2 + f_3 + f_4) - 16a^3 f_1 f_2 f_3 f_4] \\
d_2 &= -4 d_3 a = \frac{1}{2} a f_1 f_2 f_3 f_4 \\
a &= \frac{f_1 f_2 f_4 + f_1 f_2 f_3 + f_2 f_3 f_4 + f_1 f_3 f_4}{4 f_1 f_2 f_3 f_4}
\end{aligned}$$

with a relation between the zeros of $F(f)$:

$$\begin{aligned}
&(f_1 f_2 f_3 - f_2 f_3 f_4 - f_1 f_2 f_4 + f_1 f_3 f_4)(f_1 f_2 f_3 - f_2 f_3 f_4 + f_1 f_2 f_4 - f_1 f_3 f_4) \\
&\qquad \times (f_1 f_2 f_3 + f_2 f_3 f_4 - f_1 f_2 f_4 - f_1 f_3 f_4) = 0. \qquad (4.27)
\end{aligned}$$

Hence (4.26) is simplified as

$$(y')^2 = \frac{1}{\beta^2}\nu_4 y^4 + \frac{1}{\beta^2}\nu_2 y^2 + \frac{1}{\beta^2 b^2}\nu_0, \qquad (4.28)$$

where

$$\begin{aligned}
\nu_4 &= -b^2 f_1 f_2 f_3 f_4 \\
\nu_2 &= \frac{(f_1 f_2 f_3 + f_1 f_2 f_4 + f_1 f_3 f_4 + f_2 f_3 f_4)^2}{8 f_1 f_2 f_3 f_4} - \frac{2 f_1 f_2 f_3 f_4 (f_1 + f_2 + f_3 + f_4)}{f_1 f_2 f_3 + f_1 f_2 f_4 + f_1 f_3 f_4 + f_2 f_3 f_4} \\
\nu_0 &= \frac{(f_1 f_2 f_3 + f_1 f_2 f_4 + f_1 f_3 f_4 + f_2 f_3 f_4)(f_1 + f_2 + f_3 + f_4)}{8 f_1 f_2 f_3 f_4} \\
&\quad - \frac{(f_1 f_2 f_3 + f_1 f_2 f_4 + f_1 f_3 f_4 + f_2 f_3 f_4)^4}{256 (f_1 f_2 f_3 f_4)^3} - 1.
\end{aligned}$$

Now let us study the elliptic functions satisfying (4.28).

2.a sn solution

Let $y = \mathrm{sn}(\beta\xi)$ with $\xi = x - ct$ where the function y satisfies the first order differential equation (4.1). Then when we compare the coefficients of (4.28) and (4.1), we get

$$\beta^2 = -\nu_4 - \nu_2 \qquad k^2 = -\frac{\nu_4}{\nu_4 + \nu_2} \qquad b^2 = -\frac{\nu_0}{\nu_4 + \nu_2}. \tag{4.29}$$

By the relation (4.27) let us take $f_4 = \dfrac{f_1 f_2 f_3}{f_2 f_3 + f_1 f_2 - f_1 f_3}$. Hence (4.29) becomes

$$\begin{aligned} \beta^2 &= \frac{2b^2 f_1^2 f_2^2 f_3^2 - f_1^2 f_2^2 + 2f_1^2 f_2 f_3 - 2f_1^2 f_3^2 + 2f_2 f_3^2 f_1 - f_2^2 f_3^2}{f_2 f_3 + f_1 f_2 - f_1 f_3} \\ k^2 &= \frac{2b^2 f_1^2 f_2^2 f_3^2}{2b^2 f_1^2 f_2^2 f_3^2 - f_1^2 f_2^2 + 2f_1^2 f_2 f_3 - 2f_1^2 f_3^2 + 2f_2 f_3^2 f_1 - f_2^2 f_3^2}, \end{aligned} \tag{4.30}$$

and we have four choices for the value b: $\pm\dfrac{f_1 - f_3}{2f_1 f_3}$, $\pm\dfrac{f_1 f_2 - 2f_1 f_3 + f_2 f_3}{2f_1 f_2 f_3}$. Taking $b = \dfrac{f_1 - f_3}{2f_1 f_3}$ yields

$$\begin{aligned} \beta^2 &= -\frac{f_2^2 (f_1 + f_3)^2 - 4f_1 f_3 (f_1 f_2 + f_2 f_3 - f_1 f_3)}{4(f_1 f_2 + f_2 f_3 - f_1 f_3)} \\ k^2 &= \frac{f_2^2 (f_1 - f_3)^2}{f_2^2 (f_1 + f_3)^2 - 4f_1 f_3 (f_1 f_2 + f_2 f_3 - f_1 f_3)}. \end{aligned} \tag{4.31}$$

Hence the solution is

$$u(x,t) = \frac{2f_1 f_3}{(f_1 + f_3) + (f_1 - f_3)\mathrm{sn}[\beta(x - ct)]}, \tag{4.32}$$

where

$$c = -\frac{f_1 + f_2 + f_3}{4} - \frac{f_1 f_2 f_3}{4(f_2 f_3 + f_1 f_2 - f_1 f_3)}.$$

Let us study the limiting cases. For $k = 0$, there are two possibilities: $f_2 = 0$ or $f_1 = f_3$. If $f_2 = 0$ then $\beta = \pm\sqrt{f_1 f_3}$, $f_1 f_3 > 0$ and the corresponding solution is

$$u(x,t) = \frac{2f_1 f_3}{(f_1 + f_3) \pm (f_1 - f_3)\sin\left[\sqrt{f_1 f_3}(x + \frac{f_1 + f_3}{4} t)\right]}. \tag{4.33}$$

If $f_1 = f_3$ then $a = \dfrac{1}{f_1}$ and $b = 0$ so we have constant solution $u(x,t) = f_1$. For $k = 1$ then from (4.31) we have

$$4f_1 f_3 (f_2 - f_1)(f_2 - f_3) = 0.$$

It is not possible to have $f_1 = 0$ or $f_3 = 0$ because of the definition of b. If $f_1 = f_2$ or $f_2 = f_3$ we have $\beta^2 \leq 0$. Hence we do not have real solution for $k = 1$.

2.b cn solution

Let $y = \mathrm{cn}(\beta\xi)$ with $\xi = x - ct$ where the function y satisfies the first order differential equation (4.2). If we compare the coefficients of (4.28) and (4.2), we get

$$\beta^2 = -2\nu_4 - \nu_2 \qquad k^2 = \frac{\nu_4}{2\nu_4 + \nu_2} \qquad b^2 = -\frac{\nu_0}{\nu_4 + \nu_2}. \tag{4.34}$$

By (4.27) let us take $f_4 = \dfrac{f_1 f_2 f_3}{f_2 f_3 + f_1 f_2 - f_1 f_3}$. Since we have the same relation for b as in the Case 2.a, we may also take $b = \dfrac{f_1 - f_3}{2 f_1 f_3}$. Hence (4.34) becomes

$$\beta^2 = \frac{f_1 f_3 (f_1 - f_2)(f_2 - f_3)}{f_2 f_3 + f_1 f_2 - f_1 f_3} \qquad k^2 = \frac{f_2^2 (f_1 - f_3)^2}{4 f_1 f_3 (f_1 - f_2)(f_2 - f_3)}. \tag{4.35}$$

Thus the solution is

$$u(x,t) = \frac{2 f_1 f_3}{(f_1 + f_3) + (f_1 - f_3)\mathrm{cn}[\beta(x - ct)]}, \tag{4.36}$$

where

$$c = -\frac{f_1 + f_2 + f_3}{4} - \frac{f_1 f_2 f_3}{4(f_2 f_3 + f_1 f_2 - f_1 f_3)}.$$

For $k = 0$, there are two possibilities: $f_2 = 0$ or $f_1 = f_3$. If $f_2 = 0$ then $\beta = \pm\sqrt{f_1 f_3}$, $f_1 f_3 > 0$ and the corresponding solution is

$$u(x,t) = \frac{2 f_1 f_3}{(f_1 + f_3) + (f_1 - f_3)\cos[\sqrt{f_1 f_3}(x + \frac{f_1 + f_3}{4} t)]}. \tag{4.37}$$

If $f_1 = f_3$ then $a = \dfrac{1}{f_1}$ and $b = 0$ so we have a constant solution $u(x,t) = f_1$. For $k = 1$, we have the following relation from (4.35):

$$2 f_1 f_3 = f_2 (f_1 + f_3). \tag{4.38}$$

Hence the solution is

$$u(x,t) = \frac{2 f_1 f_3}{(f_1 + f_3) + (f_1 - f_3)\mathrm{sech}\left[\frac{f_1 - f_3}{f_1 + f_3}\sqrt{f_1 f_3}(x - ct)\right]}, \tag{4.39}$$

where $c = -\dfrac{f_1^2 + 6 f_1 f_3 + f_3^2}{4(f_1 + f_3)}$.

2.c dn solution

Let $y = \mathrm{dn}(\beta\xi)$ with $\xi = x - ct$ where the function y satisfies the first order differential equation (4.3). When we compare the coefficients of (4.28) and (4.3), we get

$$\beta^2 = -\nu_4 \qquad k^2 = \frac{2\nu_4 + \nu_2}{\nu_4} \qquad b^2 = -\frac{\nu_0}{\nu_4 + \nu_2}. \tag{4.40}$$

Same as before let us take $f_4 = \dfrac{f_1f_2f_3}{f_2f_3 + f_1f_2 - f_1f_3}$ and $b = \dfrac{f_1 - f_3}{2f_1f_3}$. Hence (4.40) becomes

$$\beta^2 = \frac{f_2^2(f_1 - f_3)^2}{4(f_2f_3 + f_1f_2 - f_1f_3)} \qquad k^2 = \frac{4f_1f_3(f_1 - f_2)(f_2 - f_3)}{f_2^2(f_1 - f_3)^2}. \tag{4.41}$$

Thus the solution is

$$u(x,t) = \frac{2f_1f_3}{(f_1 + f_3) + (f_1 - f_3)\mathrm{dn}[\beta(x - ct)]}, \tag{4.42}$$

where

$$c = -\frac{f_1 + f_2 + f_3}{4} - \frac{f_1f_2f_3}{4(f_2f_3 + f_1f_2 - f_1f_3)}.$$

For $k = 0$, there are four possibilities: $f_1 = 0$, $f_3 = 0$, $f_1 = f_2$ or $f_2 = f_3$. We cannot have $f_1 = 0$ or $f_3 = 0$ because of the definition of b. If $f_1 = f_2$ or $f_2 = f_3$, the solution is $u(x,t) = f_3$. For $k = 1$, we have $2f_1f_3 = f_2(f_1 + f_3)$. So the corresponding solution is

$$u(x,t) = \frac{2f_1f_3}{(f_1 + f_3) + (f_1 - f_3)\mathrm{sech}\left[\frac{f_1 - f_3}{f_1 + f_3}\sqrt{f_1f_3}(x - ct)\right]}, \quad f_1f_3 > 0, \tag{4.43}$$

where $c = -\dfrac{f_1^2 + 6f_1f_3 + f_3^2}{4(f_1 + f_3)}$.

2.d tn solution

Let $y = \mathrm{tn}(\beta\xi)$ with $\xi = x - ct$ where the function y satisfies the first order differential equation (4.4). Hence when we compare the coefficients of (4.28) and (4.4), we get

$$\beta^2 = \nu_2 - \nu_4 \qquad k^2 = \frac{\nu_2 - 2\nu_4}{\nu_2 - \nu_4} \qquad b^2 = \frac{\nu_0}{\nu_2 - \nu_4}. \tag{4.44}$$

Let us take $f_4 = \dfrac{f_1f_2f_3}{f_2f_3 + f_1f_2 - f_1f_3}$. Here we notice that third equality of (4.44) reveals that b is not real for any values of k. Hence for all values of $k^2 \in [0, 1]$ we do not have real solution.

2.e 1/sn solution

Let $y = \dfrac{1}{\mathrm{sn}(\beta\xi)}$ with $\xi = x - ct$ where the function y satisfies the first order differential equation (4.5). Hence when we compare the coefficients of (4.28) and (4.5), we get

$$\beta^2 = \nu_4 \qquad k^2 = \frac{-\nu_2 - \nu_4}{\nu_4} \qquad b^2 = \frac{-\nu_0}{\nu_2 + \nu_4}. \tag{4.45}$$

If we take $f_4 = \dfrac{f_1f_2f_3}{f_2f_3 + f_1f_2 - f_1f_3}$ and $b = \dfrac{f_1 - f_3}{2f_1f_3}$, (4.45) becomes

$$\beta^2 = -\frac{f_2^2(f_1 - f_3)^2}{4(f_2f_3 + f_1f_2 - f_1f_3)} \qquad k = \pm\frac{f_2(f_1 + f_3) - 2f_1f_3}{f_2(f_1 - f_3)}. \tag{4.46}$$

The corresponding solution is

$$u(x,t) = \frac{2f_1f_3\text{sn}[\beta(x-ct)]}{(f_1+f_3)\text{sn}[\beta(x-ct)]+(f_1-f_3)}, \tag{4.47}$$

where

$$c = -\frac{f_1+f_2+f_3}{4} - \frac{f_1f_2f_3}{4(f_2f_3+f_1f_2-f_1f_3)}.$$

For $k=0$, we have the relation $f_2(f_1+f_3)=2f_1f_3$ and the solution becomes

$$u(x,t) = \frac{2f_1f_3\sin\left[\frac{f_1-f_3}{f_1+f_3}\sqrt{-f_1f_3}(x-ct)\right]}{(f_1+f_3)\sin\left[\frac{f_1-f_3}{f_1+f_3}\sqrt{-f_1f_3}(x-ct)\right]\pm(f_1-f_3)}, \quad -f_1f_3>0, \tag{4.48}$$

where $c = -\dfrac{f_1^2+6f_1f_3+f_3^2}{4(f_1+f_3)}$. For $k=1$ then from (4.46) we have

$$4f_1f_3(f_2-f_1)(f_2-f_3)=0.$$

It is not possible to have $f_1=0$ or $f_3=0$ because of the definition of b. If $f_1=f_2$ or $f_2=f_3$ we have $\beta^2<0$. Hence we do not have real solution for $k=1$.

2.f 1/cn solution

Let $y = \dfrac{1}{\text{cn}(\beta\xi)}$ with $\xi = x-ct$ where the function y satisfies the first order differential equation (4.6). If we compare the coefficients of (4.28) and (4.6), we get

$$\beta^2 = \nu_2+2\nu_4 \qquad k^2 = \frac{\nu_2+\nu_4}{\nu_2+2\nu_4} \qquad b^2 = \frac{-\nu_0}{\nu_2+\nu_4}. \tag{4.49}$$

Since we take $f_4 = \dfrac{f_1f_2f_3}{f_2f_3+f_1f_2-f_1f_3}$ and $b = \dfrac{f_1-f_3}{2f_1f_3}$, (4.49) becomes

$$\beta^2 = -\frac{f_1f_3(f_1-f_2)(f_2-f_3)}{f_2f_3+f_1f_2-f_1f_3} \qquad k^2 = -\frac{[f_2(f_1+f_3)-2f_1f_3]^2}{4f_1f_3(f_1-f_2)(f_2-f_3)}.$$

The corresponding solution is

$$u(x,t) = \frac{2f_1f_3\text{cn}[\beta(x-ct)]}{(f_1+f_3)\text{cn}[\beta(x-ct)]+(f_1-f_3)}, \tag{4.50}$$

where

$$c = -\frac{f_1+f_2+f_3}{4} - \frac{f_1f_2f_3}{4(f_2f_3+f_1f_2-f_1f_3)}.$$

For $k=0$, we have the relation $f_2(f_1+f_3)=2f_1f_3$ and the solution becomes

$$u(x,t) = \frac{2f_1f_3\cos\left[\frac{f_1-f_3}{f_1+f_3}\sqrt{-f_1f_3}(x-ct)\right]}{(f_1+f_3)\cos\left[\frac{f_1-f_3}{f_1+f_3}\sqrt{-f_1f_3}(x-ct)\right]+(f_1-f_3)}, \quad -f_1f_3>0, \tag{4.51}$$

where $c = -\dfrac{f_1^2 + 6f_1f_3 + f_3^2}{4(f_1 + f_3)}$. The case for $k = 1$ gives the condition $f_2^2(f_1 - f_3)^2 = 0$ to be satisfied. Hence we have two possibilities: $f_2 = 0$ or $f_1 = f_3$. If $f_2 = 0$ then $\beta = \pm\sqrt{-f_1f_3}$, $-f_1f_3 > 0$ and the corresponding solution is

$$u(x,t) = \frac{2f_1f_3\text{sech}[\sqrt{-f_1f_3}(x + \frac{f_1+f_3}{4}t)]}{(f_1+f_3)\text{sech}[\sqrt{-f_1f_3}(x + \frac{f_1+f_3}{4}t)] + (f_1 - f_3)}, \quad -f_1f_3 > 0. \tag{4.52}$$

If $f_1 = f_3$ then $a = \dfrac{1}{f_1}$ and $b = 0$ so we have a constant solution $u(x,t) = f_1$.

2.g dn tn solution

Let $y = \text{dn}(\beta\xi)\,\text{tn}(\beta\xi)$ with $\xi = x - ct$ where the function y satisfies the first order differential equation (4.7). Hence when we compare the coefficients of (4.28) and (4.7), we get

$$\beta^2 = \nu_4 \qquad k^2 = \frac{2\nu_4 - \nu_2}{4\nu_4} \qquad b^2 = \frac{\nu_0}{\nu_4}.$$

Let us take $f_4 = \dfrac{f_1f_2f_3}{f_2f_3 + f_1f_2 - f_1f_3}$. The third equality above gives four choices for b:

$$\pm\frac{\sqrt{-f_2(f_3 - f_1)(2f_1f_3 - f_2(f_1+f_3))}}{f_1f_2f_3} \qquad \pm\frac{\sqrt{f_2(f_3 - f_1)(2f_1f_3 - f_2(f_1+f_3))}}{f_1f_2f_3}. \tag{4.53}$$

To have real solutions, the parameters must be real. Hence from the expressions for b we have either $-f_2(f_3 - f_1)(2f_1f_3 - f_2(f_1+f_3)) \geq 0$ or $f_2(f_3 - f_1)(2f_1f_3 - f_2(f_1+f_3)) \geq 0$. If the first one is true then

$$\beta^2 = \frac{f_2(f_3 - f_1)(2f_1f_3 - f_2(f_1+f_3))}{4(f_2f_3 + f_1f_2 - f_1f_3)} \qquad k^2 = \frac{-f_3^2(f_1 - f_2)^2}{f_2(f_3 - f_1)(2f_1f_3 - f_2(f_1+f_3))}. \tag{4.54}$$

If the second one is true then

$$\beta^2 = \frac{f_2(f_1 - f_3)(2f_1f_3 - f_2(f_1+f_3))}{4(f_2f_3 + f_1f_2 - f_1f_3)} \qquad k^2 = \frac{f_1^2(f_2 - f_3)^2}{f_2(f_1 - f_3)(-2f_1f_3 + f_2(f_1+f_3))}. \tag{4.55}$$

From the equality for k^2 in (4.54) we get

$$\frac{k^2 - 1}{k^2} = \frac{f_1^2(f_3 - f_2)^2}{f_3^2(f_2 - f_1)^2} \geq 0.$$

This gives that $k^2 \geq 1$. We know that for the parameter k^2 of Jacobi elliptic functions we have $0 \leq k^2 \leq 1$. Additionally, at the limiting points $k = 0$ and $k = 1$ it yields that $F(f)$ has two double zeros that is the case which does not give real solution as we stated in section 2.2. We also have the similar result for (4.55). Hence we do not have real solutions for all $k^2 \in [0,1]$.

4.3 Discussion About the Special Solutions

When $F(f)$ has one double f_2 and two simple zeros f_1 and f_3 we have the following system of equations:

$$\begin{aligned} -4c &= f_1 + 2f_2 + f_3 \\ 4(d_1 - c^2) &= -[f_1 f_3 + 2f_2(f_1 + f_3) + f_2^2] \\ 8d_2 &= 2f_1 f_2 f_3 + f_2^2(f_1 + f_3) \\ 8d_3 &= -f_1 f_2^2 f_3. \end{aligned} \tag{4.56}$$

The exact solutions in terms of the Jacobi elliptic functions take the following forms.

(i) In Case 1.a and Case 1.b, we have $d_2 = cd_1$. Using this in (4.56) we obtain that either $2f_2 = f_1 + f_3$ or $f_1 = f_3$. The second one is not allowed due to the discussion in the section 3.2. By using (4.56), the first one leads to $k^2 = 1$. In this case the solution is given in (4.22) and (4.25) which are compatible with the limiting solutions discussed in section 5, part a.

(ii) In Case 2.b and Case 2.c, we have $d_2 = -4d_3 a$. From the first equation of (4.56) we have $a = \frac{1}{f_2}$. Then this implies $d_2 = -\frac{4d_3}{f_2}$. This constraint gives $2f_1 f_3 = f_2(f_1 + f_3)$ which yields $k^2 = 1$. In this case the solutions are given in (4.39) and (4.43) which are compatible with the limiting solutions discussed in section 5, part b.

(iii) If $f_2 = 0$ then $d_3 = 0$ hence $d_2 = 0$ which leads to $k^2 = 1$. In this case the solution is (4.52) given in Case 2.f which are compatible with the limiting solutions discussed in section 5, part c.

4.4 General Solutions of (1.3) in Terms of Elliptic Functions

Here we shall deal with the most general form of solutions

$$u(x,t) = f(\xi) = \frac{a_1 + b_1 y(\beta\xi)}{a_2 + b_2 y(\beta\xi)}, \quad \xi = x - ct. \tag{4.57}$$

When we insert this form into the equation $F(f)$ we get

$$(y')^2 = \frac{1}{\beta^2(b_1 a_2 - b_2 a_1)^2}(\Omega_4 y^4 + \Omega_3 y^3 + \Omega_2 y^2 + \Omega_1 y + \Omega_0), \; b_1 a_2 - b_2 a_1 \neq 0, \tag{4.58}$$

where

$$\begin{aligned}
\Omega_4 &= -b_1^2(b_1+2cb_2)^2+4b_2^2(b_1^2d_1+2b_1b_2d_2+2b_2^2d_3)=b_2^4F(b), \quad b=b_1/b_2 && (4.59)\\
\Omega_3 &= 4(2d_2a_1b_2^3-cb_1^3a_2+8d_3a_2b_2^3-3ca_1b_1^2b_2+2d_1a_1b_1b_2^2+2d_1b_1^2a_2b_2 \\
&\quad -2c^2a_1b_1b_2^2-2c^2b_1^2a_2b_2+6d_2b_1a_2b_2^2-a_1b_1^3) && (4.60)\\
\Omega_2 &= 4d_1a_1^2b_2^2+4d_1b_1^2a_2^2-4c^2a_1^2b_2^2-4c^2b_1^2a_2^2+48d_3a_2^2b_2^2-12ca_1^2b_1b_2-12ca_1b_1^2a_2 \\
&\quad +24d_2a_1a_2b_2^2+24d_2b_1a_2^2b_2-6a_1^2b_1^2+16d_1a_1b_1a_2b_2-16c^2a_1b_1a_2b_2 && (4.61)\\
\Omega_1 &= 4(2d_2b_1a_2^3-ca_1^3b_2+8d_3a_2^3b_2-3ca_1^2b_1a_2+2d_1a_1^2a_2b_2+2d_1a_1b_1a_2^2 \\
&\quad -2c^2a_1^2a_2b_2-2c^2a_1b_1a_2^2+6d_2a_1a_2^2b_2-a_1^3b_1) && (4.62)\\
\Omega_0 &= -a_1^4+8d_3a_2^4-4ca_1^3a_2+4d_1a_1^2a_2^2-4c^2a_1^2a_2^2+8d_2a_1a_2^3=a_2^4F(a), \quad a=a_1/a_2. && (4.63)
\end{aligned}$$

We have four arbitrary constants c, d_1, d_2, d_3 in the differential equation (1.5). In (4.58) we have effectively four independent parameters. By choosing these constants properly we get several solutions in terms of elliptic functions. We can analyze these solutions in two groups:

i) If $F(f)$ has zeros then we can make the coefficients of y^4 to vanish by taking $F(b)=0$. This means that $b=b_1/b_2$ is a zero of $F(f)$. In addition to that choosing the constant $\Omega_0=0$ yields that $F(a)=0$ where $a=a_1/a_2$. This also means that $a=a_1/a_2$ is another zero of $F(f)$. Note that $a\neq b$ since $b_1a_2-b_2a_1\neq 0$. Then the equation (4.58) takes the form where the square of elliptic functions and their inverses given in (4.8)-(4.14) satisfy. By making substitution $a=a_1/a_2$ and $b=b_1/b_2$, the equation (4.58) becomes

$$(y')^2=\frac{b_2^2F(b)}{\beta^2a_2^2(b-a)^2}y^4+\frac{4b_2}{\beta^2a_2(b-a)^2}\omega_3y^3+\frac{2}{\beta^2(b-a)^2}\omega_2y^2+\frac{4a_2}{\beta^2b_2(b-a)^2}\omega_1y+\frac{a_2^2F(a)}{\beta^2b_2^2(b-a)^2} \tag{4.64}$$

where

$$\begin{aligned}
\omega_3 &= 2ad_3-cb^3+8d_3-3acb^2+2d_1ab+2d_1b^2-2c^2ab-2c^2b^2+6d_2b-ab^3\\
\omega_2 &= 2d_1a^2+2d_1b^2-2c^2a^2-2c^2b^2+24d_3-6ca^2b-6cab^2+12d_2a+12d_2b\\
&\quad -3a^2b^2+8d_1ab-8c^2ab\\
\omega_1 &= 2d_2b-ca^3+8d_3-3bca^2+2d_1ab+2d_1a^2-2c^2ab-2c^2a^2+6d_2a-ba^3.
\end{aligned}$$

If a and b are the zeros of $F(f)$, then $F(a)=F(b)=0$ and we do not have the terms with y^4 and the constant term in (4.64). For instance, let $a=f_1$ and $b=f_2$, then we can write $F(f)=-(f-f_1)(f-f_2)(f-f_3)(f-f_4)$ such that f_1,f_2,f_3,f_4 are zeros of $F(f)$. Let us write ω_1, ω_2 and ω_3 in terms of the zeros of the function $F(f)$ by the help of (2.8).

$$\begin{aligned}
\omega_1 &= \frac{(f_1-f_2)^2}{4}(f_1-f_4)(f_1-f_3) && (4.65)\\
\omega_2 &= \frac{1}{2}(f_1-f_2)^2\{(f_2-f_3)(f_1-f_4)+(f_1-f_3)(f_2-f_4)\} && (4.66)\\
\omega_3 &= \frac{(f_1-f_2)^2}{4}(f_2-f_3)(f_2-f_4). && (4.67)
\end{aligned}$$

Now we give all solutions of (1.3) of the form (4.57). Let $y = \mathrm{sn}^2(\beta\xi)$ with $\xi = x - ct$ where the function y satisfies the first order differential equation (4.8). Hence when we compare the coefficients of (4.64) and (4.8), we get

$$\beta^4 = \frac{\omega_1\omega_3}{k^2(b-a)^4} \qquad \frac{b_2}{a_2} = \frac{-2(1+k^2)\omega_1}{\omega_2} \qquad k^2 = -1 + \frac{\omega_2^2}{8\omega_1\omega_3} \pm \frac{\omega_2}{8\omega_1\omega_3}\sqrt{\omega_2^2 - 16\omega_1\omega_3}.$$

Here without loosing any generality we take $f_1 \le f_2 \le f_3 \le f_4$.

(1) Let $a = f_1$**,** $b = f_2$. For this choice we have

$$\beta = \frac{1}{2}\sqrt{(f_1 - f_3)(f_2 - f_4)} \qquad \frac{b_2}{a_2} = \frac{f_4 - f_1}{f_2 - f_4} \qquad k^2 = \frac{(f_2 - f_3)(f_1 - f_4)}{(f_1 - f_3)(f_2 - f_4)},$$

and hence the solution with the initial condition $f(0) = f_1$ is

$$u(x,t) = f(\xi) = \frac{f_1(f_2 - f_4) + f_2(f_4 - f_1)\mathrm{sn}^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)}\xi)}{(f_2 - f_4) + (f_4 - f_1)\mathrm{sn}^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)}\xi)}. \tag{4.68}$$

Since a and b are any zeros of $F(f)$ we have other choices of these parameters.

(2) Let $a = f_2$**,** $b = f_1$. For this choice we have

$$\beta = \frac{1}{2}\sqrt{(f_1 - f_3)(f_2 - f_4)} \qquad \frac{b_2}{a_2} = \frac{f_3 - f_2}{f_1 - f_3} \qquad k^2 = \frac{(f_2 - f_4)(f_1 - f_3)}{(f_1 - f_4)(f_2 - f_3)},$$

and hence the solution with the initial condition $f(0) = f_2$ is

$$u(x,t) = f(\xi) = \frac{f_2(f_1 - f_3) + f_1(f_3 - f_2)\mathrm{sn}^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)}\xi)}{(f_1 - f_3) + (f_3 - f_2)\mathrm{sn}^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)}\xi)}. \tag{4.69}$$

(3) Let $a = f_3$**,** $b = f_4$. For this choice we have

$$\beta = \frac{1}{2}\sqrt{(f_1 - f_3)(f_2 - f_4)} \qquad \frac{b_2}{a_2} = \frac{f_2 - f_3}{f_4 - f_2} \qquad k^2 = \frac{(f_4 - f_1)(f_3 - f_2)}{(f_3 - f_1)(f_4 - f_2)},$$

and hence the solution with the initial condition $f(0) = f_3$ is

$$u(x,t) = f(\xi) = \frac{f_3(f_4 - f_2) + f_4(f_2 - f_3)\mathrm{sn}^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)}\xi)}{(f_4 - f_2) + (f_2 - f_3)\mathrm{sn}^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)}\xi)}. \tag{4.70}$$

(4) Let $a = f_4, b = f_3$. For this choice we have

$$\beta = \frac{1}{2}\sqrt{(f_1 - f_3)(f_2 - f_4)} \qquad \frac{b_2}{a_2} = \frac{f_1 - f_4}{f_3 - f_1} \qquad k^2 = \frac{(f_3 - f_2)(f_4 - f_1)}{(f_3 - f_1)(f_4 - f_2)},$$

and hence the solution with the initial condition $f(0) = f_4$ is

$$u(x,t) = f(\xi) = \frac{f_4(f_3 - f_1) + f_3(f_1 - f_4)\mathrm{sn}^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)}\xi)}{(f_3 - f_1) + (f_1 - f_4)\mathrm{sn}^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)}\xi)}. \tag{4.71}$$

Similarly, we can also find other type of solutions including square of Jacobi elliptic functions and inverses of them. But they are equivalent because of the relations $\text{cn}^2 v + \text{sn}^2 v = 1$ and $\text{tn}^2 v = \dfrac{\text{sn}^2 v}{1 - \text{sn}^2 v}$. Hence the solutions given in **(1)**, **(2)**, **(3)** and **(4)** are all the most general solutions of (1.3) depending upon the initial conditions.

ii) Another choice is taking a_1, b_1, a_2, b_2 so that $\Omega_3 = \Omega_1 = 0$. Then the equation (4.58) takes the form where elliptic functions and their inverses given in (4.1)-(4.6) satisfy. Note that if we take $b_2 = 0$ to make $\Omega_1 = \Omega_3 = 0$ we have $c = -a_1/a_2$, $d_2 = cd_1$ and the solution becomes $f(\xi) = \gamma + \alpha y$ which we have already studied in section 4.2, Case 1. If $a_2 = 0$ we can use inverse of Jacobi elliptic functions for y and then the case turns to $b_2 = 0$ case. When we take $b_1 = 0$, to make $\Omega_1 = \Omega_3 = 0$ we have $d_2 = -4d_3 a_2/a_1$ and $d_1 = 8d_3(a_2/a_1)^2 + c^2 + c/2a_2$ and the solution becomes $f(\xi) = \dfrac{1}{\alpha + \gamma y}$ that is the case we have already studied in section 4.2, Case 2.

In the next section we mention about the system (1.2) when $\ell = 3$ and $\ell = 4$.

5 $\ell = 3$ and $\ell = 4$ Cases

1) The degenerate coupled KdV equation for $\ell = 3$ is

$$
\begin{aligned}
u_t &= \frac{3}{2} u u_x + v_x \\
v_t &= v u_x + \frac{1}{2} u v_x + \omega_x \\
\omega_t &= -\frac{1}{4} u_{xxx} + \omega u_x + \frac{1}{2} u \omega_x.
\end{aligned} \tag{5.1}
$$

Here we will show that unlike the case $\ell = 2$, we have real traveling wave solution with asymptotically vanishing boundary condition in $\ell = 3$ case. Let $u(x,t) = f(\xi)$, $v(x,t) = g(\xi)$, and $\omega(x,t) = h(\xi)$, where $\xi = x - ct$. From the first equation of (5.1) we have

$$-cf' = \frac{3}{2} f f' + g',$$

which gives

$$g(\xi) = -cf - \frac{3}{4} f^2 + d_1,$$

where d_1 is an integration constant. Using $g(\xi)$ in the second equation of (5.1) yields

$$h' = 3cff' + \frac{3}{2} f^2 f' + (c^2 - d_1) f'.$$

Integrating above equation once we get

$$h(\xi) = \frac{3}{2} c f^2 + \frac{1}{2} f^3 + (c^2 - d_1) f + d_2,$$

where d_2 is an integration constant. Using $h(\xi)$ in the third equation of (5.1) yields

$$\frac{1}{4}f''' = \Big(\frac{9c^2}{2} - \frac{3d_1}{2}\Big)ff' + \frac{9c}{2}f^2f' + \frac{5}{4}f^3f' + (c^3 - cd_1 + d_2)f'.$$

Integrating above equation once we obtain

$$\frac{1}{4}f'' = \Big(\frac{9c^2}{4} - \frac{3d_1}{4}\Big)f^2 + \frac{3c}{2}f^3 + \frac{5}{16}f^4 + (c^3 - cd_1 + d_2)f + d_3.$$

By using f' as an integrating factor, we integrate once more. Finally, we get

$$(f')^2 = \frac{f^5}{2} + 3cf^4 + (6c^2 - 2d_1)f^3 + 4(c^3 - cd_1 + d_2)f^2 + 8d_3f + 8d_4,$$

where c, d_1, d_2, d_3, d_4 are constants. If we apply the boundary conditions $f, f', f'', f''', g, g', h, h' \to 0$ as $\xi \to \pm\infty$ we get $d_1 = d_2 = d_3 = d_4 = 0$. Hence we have

$$\begin{aligned} (f')^2 &= \frac{f^5}{2} + 3cf^4 + 6c^2f^3 + 4c^3f^2 \\ &= \frac{f^2}{2}(f + 2c)^3. \end{aligned} \tag{5.2}$$

By using trigonometric substitution $f = -2c\sin^2\theta$ and making the cancelations, above equality becomes

$$\frac{d\theta}{c^{3/2}\sin\theta\cos^2\theta} = \mp d\xi \quad \Rightarrow \frac{\sin\theta d\theta}{c^{3/2}\sin^2\theta\cos^2\theta} = \mp d\xi.$$

Making the substitution $u = \cos\theta$ gives

$$\frac{du}{c^{3/2}(u^2 - 1)u^2} = \mp d\xi,$$

which is solved as

$$\frac{1}{c^{3/2}}\Big\{\frac{1}{u} + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|\Big\} = \mp(\xi - \xi_0), \tag{5.3}$$

where ξ_0 is an integration constant. Note that $u = \cos\theta = \pm\Big(1 + \dfrac{f}{2c}\Big)^{1/2}$. When the solution $f = 0$, u is either 1 or -1. Insert the expression for u into the above equation so we get the relation defining the solution f,

$$\frac{1}{c^{3/2}}\left\{\pm\Big(1 + \frac{f}{2c}\Big)^{-1/2} + \ln\left|\frac{\pm\Big(1 + \frac{f}{2c}\Big)^{1/2} - 1}{\pm\Big(1 + \frac{f}{2c}\Big)^{1/2} + 1}\right|\right\} = \mp(\xi - \xi_0). \tag{5.4}$$

Hence we have asymptotically vanishing real traveling solution for $\ell = 3$. We expect that this is true for all odd ℓ.

2) Now let us analyze $\ell = 4$ case. The degenerate coupled KdV equation for $\ell = 4$ is

$$\begin{aligned} u_t &= \frac{3}{2}uu_x + v_x \\ v_t &= vu_x + \frac{1}{2}uv_x + \omega_x \\ \omega_t &= \omega u_x + \frac{1}{2}u\omega_x + \rho_x \\ \rho_t &= -\frac{1}{4}u_{xxx} + \rho u_x + \frac{1}{2}u\rho_x. \end{aligned} \tag{5.5}$$

Proposition 5.1. *There is no real asymptotically vanishing traveling wave solution of the equation (5.5) in the form $u(x,t) = f(\xi)$, $v(x,t) = g(\xi)$, $\omega(x,t) = h(\xi)$ and $\rho(x,t) = r(\xi)$, where $\xi = x - ct$.*

Proof. Let $u(x,t) = f(\xi)$, $v(x,t) = g(\xi)$, $\omega(x,t) = h(\xi)$ and $\rho(x,t) = r(\xi)$, where $\xi = x - ct$. From the first equation of (5.5) we have

$$-cf' = \frac{3}{2}ff' + g',$$

which gives

$$g(\xi) = -cf - \frac{3}{4}f^2 + d_1,$$

where d_1 is an integration constant. Using $g(\xi)$ in the second equation of (5.5) yields

$$h' = 3cff' + \frac{3}{2}f^2f' + (c^2 - d_1)f'.$$

Integrating above equation once we have

$$h(\xi) = \frac{3}{2}cf^2 + \frac{1}{2}f^3 + (c^2 - d_1)f + d_2,$$

where d_2 is an integration constant. Using $h(\xi)$ in the third equation of (5.5) yields

$$r' = -\frac{5}{4}f^3f' - \frac{9}{2}cf^2f' - \Big(\frac{9}{2}c^2 + \frac{3}{2}d_1\Big)ff' + (-c^3 + cd_1 - d_2)f'.$$

Integrating this equation once gives

$$r(\xi) = -\frac{5}{16}f^4 - \frac{3}{2}cf^3 + \Big(-\frac{9}{4}c^2 + \frac{3}{4}d_1\Big)f^2 + (-c^3 + cd_1 - d_2)f + d_3,$$

where d_3 is an integration constant. Using $r(\xi)$ in the fourth equation of (5.5) gives

$$\frac{1}{4}f''' = -\frac{15}{16}f^4f' - 5cf^3f' + \Big(\frac{3}{2}d_1 - 9c^2\Big)f^2f' + \Big(3cd_1 - 6c^3 - \frac{3}{2}d_2\Big)ff' + (c^2d_1 + d_3 - c^4 - cd_2)f'.$$

Integrating the above equation once we get

$$\frac{1}{4}f'' = -\frac{3}{16}f^5 - \frac{5c}{4}f^4 + \Big(\frac{d_1}{2} - 3c^2\Big)f^3 + \Big(\frac{3}{2}cd_1 - 3c^3 - \frac{3}{4}d_2\Big)f^2 + (c^2d_1 + d_3 - c^4 - cd_2)f + d_4,$$

where d_4 is an integration constant. By using f' as an integrating factor, we integrate once more. Finally, we get

$$(f')^2 = -\frac{1}{4}f^6 - 2cf^5 + (d_1 - 6c^2)f^4 + (4cd_1 - 8c^3 - 2d_2)f^3 + 4(c^2d_1 + d_3 - c^4 - cd_2)f^2 + 8d_4 f + 8d_5,$$

where d_5 is an integration constant. If we apply the boundary conditions f, f', f'', f''', g, g', h, h', r, $r' \to 0$ as $\xi \to \infty$, we get $d_1 = d_2 = d_3 = d_4 = d_5 = 0$. Hence the above equation becomes

$$\begin{aligned}(f')^2 &= -\frac{1}{4}f^6 - 2cf^5 - 6c^2f^4 - 8c^3f^3 - 4c^4f^2 \\ &= -\frac{f^2}{4}(f+2c)^4.\end{aligned}$$

Obviously, there is no real traveling wave solution of the case $\ell = 4$ with asymptotically vanishing boundary conditions.

Conjecture: For all even ℓ, since we have the following equality

$$(f')^2 = -\frac{f^2}{2^{2l-4}}(f+2c)^l, \quad u(x,t) = f(\xi) \quad \xi = x - ct, \tag{5.6}$$

the degenerate coupled KdV equation (1.2) does not have real traveling wave solution with asymptotically vanishing boundary conditions.

6 Graphs of the Exact Solutions

Here we give the graphs of exact solutions to see the behavior of the solutions.

Case $1.a$ and Case $1.b$ for $k = 1$:
According to the conditions on parameters, they are chosen as

$$\alpha = \beta = 1 \qquad c = 2 \qquad d_1 = -7/4 \qquad d_2 = -7/2 \qquad d_3 = -3/2.$$

Hence the solution becomes

$$u(x,t) = \text{sech}(\xi) - 2, \quad \xi = x - 2t, \tag{6.1}$$

and the graph of this function is

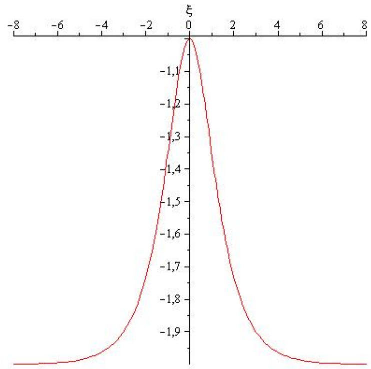


Figure 1: Graph of Case $1.a$-$1.b$ $(k = 1)$

Note that by the choice of the parameters of this case the equation (1.5) becomes

$$F(f) = -(f+3)(f+1)(f+2)^2.$$

The numerical values of the zeros of $F(f)$ are such that the graph corresponds to the exact solitary wave solution given in section 3.3, part (a).

Case $1.a$ for different values of k:
Here to see the behavior of the solution by the change of the value of k we give the following graph:

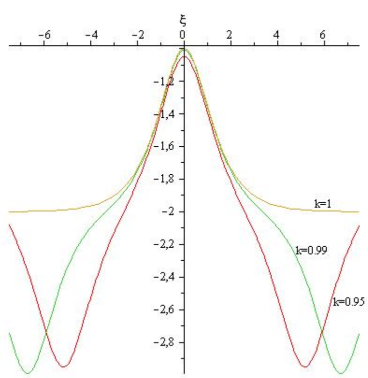


Figure 2: Graph of Case $1.a$ (different values of k)

Case $1.b$ for $k = 0.5$:
The parameters are chosen as

$$\alpha = \beta = 1 \qquad c = 2 \qquad d_1 = -\frac{25}{16} \qquad d_2 = -\frac{25}{8} \qquad d_3 = -\frac{39}{32}.$$

The solution is

$$u(x,t) = \mathrm{dn}(\xi) - 2, \quad \xi = x - 2t, \tag{6.2}$$

and the graph of this function is

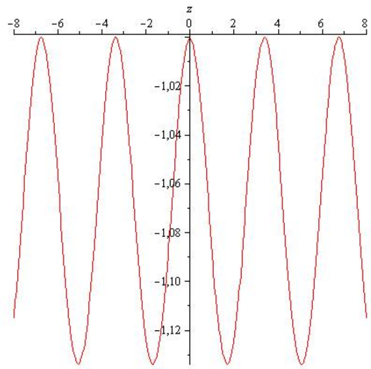


Figure 3: Graph of Case $1.b$ $(k = 0.5)$

Note that by the choice of the parameters of this case the equation (1.5) becomes

$$F(f) = -(f+3)(f+1)\Big(f - (-2 + \frac{1}{2}\sqrt{3})\Big)\Big(f - (-2 - \frac{1}{2}\sqrt{3})\Big).$$

Since $F(f)$ has four different simple zeros, we expect periodic solution as in the graph.

Case $2.a$ **for** $k = 0$**:** The parameters are

$$a = -2 \qquad b = \sqrt{3} \qquad c = 1 \qquad d_1 = \frac{3}{4} \qquad d_2 = d_3 = 0 \qquad \beta = 1.$$

Hence the solution becomes

$$u(x,t) = \frac{1}{-2 - \sqrt{3}\sin(\xi)}, \quad \xi = x - t, \tag{6.3}$$

and the graph of this function is

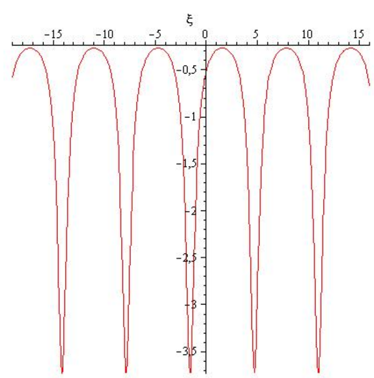


Figure 4: Graph of Case $2.a$ $(k = 0)$

Note that by the choice of the parameters of this case the equation (1.5) becomes

$$F(f) = -f^2(f - (-2+\sqrt{3}))(f - (-2-\sqrt{3})).$$

Here the function $F(f)$ has one double zero $f_2 = 0$ and two simple zeros $f_1 = -2-\sqrt{3}$ and $f_3 = -2+\sqrt{3}$ so $f_1 < f_3 < f_2$. As it is stated in section 2.2, part (4) we have periodic solution which can also be seen in the above graph.

Case $2.b$ **for** $k = 0$**:** The parameters are chosen as

$$a = 2 \qquad b = -\sqrt{3} \qquad c = -1 \qquad d_1 = \frac{3}{4} \qquad d_2 = d_3 = 0 \qquad \beta = 1.$$

Hence the solution becomes

$$u(x,t) = \frac{1}{2-\sqrt{3}\cos(\xi)}, \quad \xi = x + t, \tag{6.4}$$

and the graph of this function is

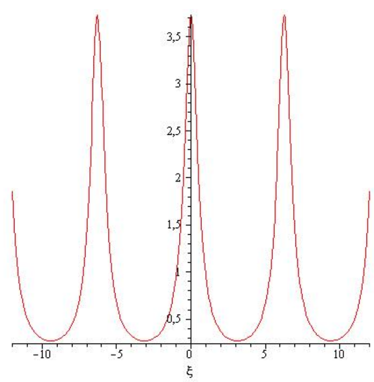


Figure 5: Graph of Case $2.b$ $(k=0)$

Note that by the choice of the parameters of this case the equation (1.5) becomes

$$F(f) = -f^2(f - (2+\sqrt{3}))(f - (2-\sqrt{3})).$$

The function $F(f)$ has one double zero $f_2 = 0$ and two simple zeros $f_1 = 2-\sqrt{3}$ and $f_3 = 2+\sqrt{3}$ so $f_2 < f_1 < f_3$. As it is given in section 2.2, part (4), the solution is periodic, which can be easily seen in the graph.

Case $2.b$ **and Case** $2.c$ **for** $k = 1$**:** The parameters are chosen as

$$a = 1 \qquad b = -\sqrt{\frac{7}{8}} \qquad c = -\frac{9}{2} \qquad d_1 = 10 \qquad d_2 = 4 \qquad d_3 = -1 \qquad \beta = \sqrt{7}.$$

Hence the solution becomes

$$u(x,t) = \frac{1}{1-\sqrt{\frac{7}{8}}\operatorname{sech}(\sqrt{7}\xi)}, \quad \xi = x + \frac{9}{2}t, \tag{6.5}$$

and the graph of this function is

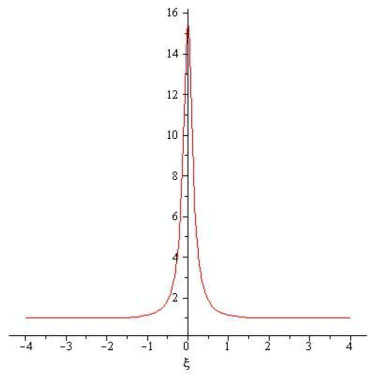


Figure 6: Graph of Case 2.b-2.c $(k = 1)$

Note that by the choice of the parameters of this case the equation (1.5) becomes

$$F(f) = -(f - (8 + 2\sqrt{14}))(f - (8 - 2\sqrt{14}))(f - 1)^2.$$

The numerical values of the zeros of $F(f)$ are such that the graph corresponds to the exact solitary wave solution given in section 3.3, part (a).

Case 2.e **for** $k = 0$**:** The parameters are chosen as

$$a = 1 \qquad b = 2 \qquad c = -\frac{1}{3} \qquad d_1 = \frac{5}{18} \qquad d_2 = -\frac{1}{6} \qquad d_3 = \frac{1}{24} \qquad \beta = \frac{2\sqrt{3}}{3}.$$

Hence the solution becomes

$$u(x,t) = \frac{\sin(\frac{2\sqrt{3}}{3}\xi)}{\sin(\frac{2\sqrt{3}}{3}\xi) + 2}, \quad \xi = x + \frac{1}{3}t, \tag{6.6}$$

and the graph of this function is

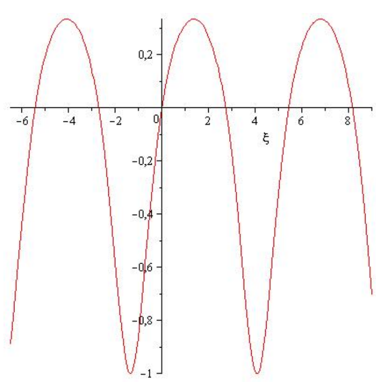


Figure 7: Graph of Case 2.e $(k = 0)$

Note that by the choice of the parameters of this case the equation (1.5) becomes

$$F(f) = -(f-1)^2(f+1)\left(f - \frac{1}{3}\right).$$

Here the function $F(f)$ has one double zero $f_2 = 1$ and two simple zeros $f_1 = -1$ and $f_3 = \frac{1}{3}$ so $f_1 < f_3 < f_2$. As it is noted in section 2.2, part (4) we have periodic solution which can be seen in the graph.

Case $2.f$ **for** $k = 0$**:** The parameters are chosen as

$$a = 1 \qquad b = 2 \qquad c = -\frac{1}{3} \qquad d_1 = \frac{5}{18} \qquad d_2 = -\frac{1}{6} \qquad d_3 = \frac{1}{24} \qquad \beta = \frac{2\sqrt{3}}{3}.$$

Hence the solution becomes

$$u(x,t) = \frac{\cos(\frac{2\sqrt{3}}{3}\xi)}{\cos(\frac{2\sqrt{3}}{3}\xi) + 2}, \quad \xi = x + \frac{1}{3}t, \tag{6.7}$$

and the graph of this function is

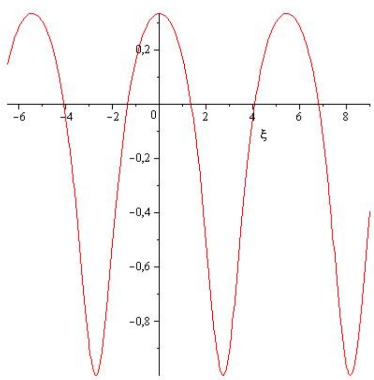


Figure 8: Graph of Case $2.f$ $(k = 0)$

Note that by the choice of the parameters of this case the equation (1.5) becomes

$$F(f) = -(f-1)^2(f+1)\left(f - \frac{1}{3}\right).$$

The zeros of the function $F(f)$ are same as in the previous case. So the graph fits to the fact given in section 2.2, part (4).

Case $2.f$ **for** $k = 1$**:** The parameters are chosen as

$$a = 1 \qquad b = 2 \qquad c = \frac{1}{6} \qquad d_1 = \frac{1}{9} \qquad d_2 = d_3 = 0 \qquad \beta = \frac{\sqrt{3}}{3}.$$

Hence the solution becomes

$$u(x,t) = \frac{\operatorname{sech}(\frac{\sqrt{3}}{3}\xi)}{\operatorname{sech}(\frac{\sqrt{3}}{3}\xi) + 2}, \quad \xi = x - \frac{1}{6}t, \tag{6.8}$$

and the graph of this function is

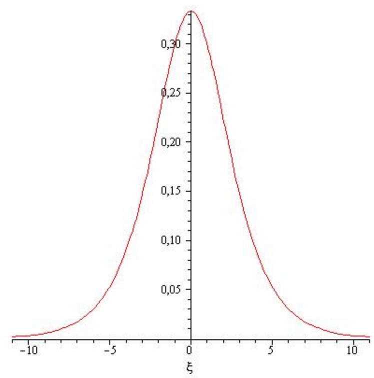


Figure 9: Graph of Case $2.f$ $(k = 1)$

Note that by the choice of the parameters of this case the equation (1.5) becomes

$$F(f) = -f^4 - \frac{2}{3}f^3 + \frac{1}{3}f^2 = -\Big(f - \frac{1}{3}\Big)(f + 1)f^2.$$

The numerical values of the zeros of $F(f)$ are such that the graph corresponds to the exact solitary wave solution given in section 3.3, part (a).

7 Conclusion

We have studied symmetry reduced (traveling waves) equations of the Kaup-Boussinesq (KB) type of coupled degenerate KdV equations for $\ell = 2$. The reduced equation turns out to be such that the square of the derivative of the dependent variable is equal to a fourth degree polynomial of the dependent variable. There are four arbitrary constants in the polynomial function. We have investigated all possible cases and gave all solitary wave solutions which rapidly decay to some constants of the ($\ell = 2$) KB equations. There are periodic solutions of this set of coupled KdV equations in terms of the Jacobi elliptic functions. We first introduced special solutions of this type where the zeros of $F(f)$ satisfy certain constraints. If we remove these constraints among the zeros we obtained the most general solution in terms of the elliptic functions of KB system under the assumed symmetry. There are four different such solutions which differ by the initial values at the origin. For illustration we have given the graphs of some interesting solutions. We have also initiated the work on the cases for $\ell = 3$ and $\ell = 4$. We have given some results concerning these cases. A detailed study of the traveling wave solutions of the cases $\ell = 3$ and $\ell = 4$ will be communicated later.

8 Acknowledgment

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