ON THE ARTAL–CARMONA–COGOLLUDO CONSTRUCTION

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Abstract. We derive explicit defining equations for a number of irreducible maximizing plane sextics with double singular points only. For most real curves, we also compute the fundamental group of the complement; all groups found are abelian, which suffices to complete the computation of the groups of all non-maximizing irreducible sextics. As a by-product, examples of Zariski pairs in the strongest possible sense are constructed.

1. Introduction

During the last dozen of years, the geometry and topology of singular complex plane projective curves of degree six (*plane sextics* in the sequel) has been a subject of substantial interest. Due to the fast development, there seems to be no good contemporary survey; I can only suggest [9] for a few selected topics and a number of references. Apart from the more subtle geometric properties that some special classes of sextics may possess, the principal questions seem to be

- the equisingular deformation classification of sextics,
- the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus D)$ of the complement,
- the defining equations.

The last one seems more of a practical interest: the defining equations may serve as a tool for attacking other problems. However, the equations may also shed light on the arithmetical properties of the so called *maximizing* (*i.e.*, those with the maximal total Milnor number $\mu = 19$) sextics, as such curves are rigid (have discrete moduli spaces) and are defined over algebraic number fields, see [17].

At present, the work is mostly close to its completion, at least for *irreducible* sextics. (Reducible sextics are too large in number on the one hand and seem less interesting on the other.) This paper bridges some of the remaining gaps.

In fact, the development of the subject is so fast that new results appear and become available before old ones are published. Thus, the new papers [1] and [16] substantially complement and complete the results of the present work. For the reader's convenience, these new findings are either cited next to or incorporated into the corresponding statements.

All sextics with at least one triple or more complicated singular point, including non-simple ones, are completely covered in [9] (the combinatorial approach used there seems more effective than the defining equations); for this reason, such curves are almost ignored in this paper. Thus, modulo a few quite reasonable conjectures, which have mostly been proved, it remains to study a few maximizing sextics with double singular points only.

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TABLE 1. Sextics considered in the paper

#	Singularities	(r, c)	π_1	References, remarks
4.	$\mathbf{A}_{16} \oplus \mathbf{A}_3$	$(2, 0)$	$\mathbb{Z}_6$	(5.13), see also [3]
6.	$\mathbf{A}_{15} \oplus \mathbf{A}_4$	$(0, 1)^*$	$\mathbb{Z}_6$	(5.14), see also [3]
7.	$\mathbf{A}_{14} \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(0, 3)$		(5.16)
10.	$\mathbf{A}_{13} \oplus \mathbf{A}_6$	$(0, 2)$		(5.19)
11.	$\mathbf{A}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2, 0)$	$\mathbb{Z}_6$	(5.18)
12.	$\mathbf{A}_{12} \oplus \mathbf{A}_7$	$(0, 1)$		(5.7)
13.	$\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	$(1, 1)$	$\mathbb{Z}_6$	(5.20), see Remark 1.3
14.	$\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$	$(1, 0)$	$\mathbb{Z}_6$	(5.21)
16.	$\mathbf{A}_{11} \oplus 2\mathbf{A}_4$	$(2, 0)$	$\mathbb{Z}_6$	(5.23), see Remark 1.4
18.	$\mathbf{A}_{10} \oplus \mathbf{A}_9$	$(2, 0)^*$	$\mathbb{Z}_6$	(5.10)
19.	$\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$	$(1, 1)$	$\mathbb{Z}_6$	(5.11), see Remark 1.3
20.	$\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$	$(2, 0)$	$\mathbb{Z}_6$	(5.8)
21.	$\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_3$	$(0, 1)$		(5.24)
23.	$\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$	$(2, 0)$	$\mathbb{Z}_6$	(5.28)
24.	$\mathbf{A}_{10} \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$	$(1, 1)$		(5.29), (5.30)
25.	$\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(1, 0)$	$\mathbb{Z}_6$	(5.25)
27.	$\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$	$(1, 1)^*$	$\mathbb{Z}_6$	(5.34), see Remark 1.3
30.	$\mathbf{A}_8 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4$	$(0, 1)$		(5.32)
31.	$\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1, 1)$	$\mathbb{Z}_6$	(5.33), see Remark 1.3
34.	$\mathbf{A}_7 \oplus 2\mathbf{A}_6$	$(0, 1)$		(5.39)
35.	$\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(2, 0)$	$\mathbb{Z}_6$	(5.38)

* These sets of singularities are realized by reducible sextics as well

The classification of maximizing sextics is known: it can be obtained from [6, 25] (a reduction to an arithmetical problem), [25] (a list of the sets of singularities), and [20] (the deformation classification). The resulting list of *irreducible* maximizing sextics with double singular points only is found in [10]: altogether, there are 39 sets of singularities realized by 42 real and 20 pairs of complex conjugate curves.

Some of these sets of singularities have been studied and their defining equations and fundamental groups are known, see [10] for references. Here, we obtain the equations for 21 set of singularities, leaving only six sets unsettled, see (1.7). Then, we try to derive a few topological and arithmetical consequences.

1.1. **Principal results.** Strange as it seems, the main result of the paper does not appear in the paper. We compute explicit defining equations for most irreducible maximizing plane sextics with double singular points only. Unfortunately, many equations are too complicated, and it seems neither possible nor meaningful to reproduce them in a journal article. In both human and machine readable form they can be downloaded from my web page [8]; here, in §5, we outline the details of the computation and provide information that is just sufficient to recover the equations using the rather complicated formulas of §4. Note that [8] incorporates as well the results of [16], thus containing the defining equations of *all* irreducible maximizing sextics with double singular points only.

We deal with *maximizing irreducible non-special sextics* (see the definition prior to Theorem 2.2). The sets of singularities for which equations are obtained are listed in Table 1; for consistency, we retain the numbering introduced in [10]. Also listed are the number of curves realizing each set of singularities (in the form (r, c), where r is the number of real curves and c is the number of pairs of complex conjugate ones), the fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \smallsetminus D)$, when known, and references to the equations, other sources, and remarks.

The equations obtained are used to make a few observations stated in the four theorems below; they concern the minimal fields of definition (Theorem 1.1), the fundamental group of the complement (Theorem 1.2), and a few examples of the so-called arithmetic Zariski pairs (Theorems 1.8 and 1.9). It seems feasible that, with appropriate modifications, these statements would extend to all irreducible maximizing sextics.

Theorem 1.1. *Let $n := r + 2c$ be the total number of irreducible sextics realizing a maximizing set of singularities* $\mathbf{S}$ *with double singular points only. Then, the n curves are defined over an algebraic number field $\Bbbk$, $[\Bbbk : \mathbb{Q}] = n$; they differ by the n embeddings $\Bbbk \hookrightarrow \mathbb{C}$. If $n > 2$, the Galois closure of $\Bbbk$ has Galois group $\mathbb{D}_{2n}$. This field $\Bbbk$ is* minimal *in the sense that it is contained in the coefficient field of any defining polynomial.*

This theorem is proved in §6.1, and the minimal fields of definition are described in §5 together with the equations, see references in Table 1. (Unless stated otherwise, $\Bbbk$ is the minimal field containing the parameters listed in §5.) This proof works as well for the few sextics with triple singular points mentioned below, see (1.6), and, probably, for most other maximizing sextics. In particular, it works for the equation newly found in [16] (see [8] for details), and this fact is incorporated into the statement. Another proof, using the concept of *dessins d'enfants*, is discussed in §6.2; it leads to somewhat disappointing consequences, see Remark 6.3.

Theorem 1.2. *With two exceptions, the fundamental group of a* real *maximizing non-special sextic with double singular points only is $\mathbb{Z}_6$. The exceptions are the real curve realizing $\mathbf{A}_{10} \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$, line 24 and one of the two curves realizing $\mathbf{A}_{11} \oplus 2\mathbf{A}_4$, line 16, see Remark 1.4 below.*

In the two exceptional cases, the fundamental group is unknown (at least, to the author). I expect that these groups are also abelian, as well as those of the non-real curves in Table 1. The proof of this theorem is partially based upon [16]; it is explained in §6.3, and all technical details are found in [8].

Remark 1.3. The sets of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$, line 13, $\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$, line 19, $\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$, line 27, and $\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 31 are realized by three Galois conjugate curves each. In each case, only one of the three curves is real, and *only for this real curve* the fundamental group $\pi_1 = \mathbb{Z}_6$ has been computed.

Remark 1.4. The set of singularities $\mathbf{A}_{11} \oplus 2\mathbf{A}_4$, line 16 is realized by two Galois conjugate curves. Both curves are real, but the group $\pi_1 = \mathbb{Z}_6$ is computed for one of them only; for the other curve, the presentation obtained is incomplete and I cannot assert that the group is finite.

The following corollary of Theorem 1.2 relies on the deformation classification of irreducible sextics, which is now completed, see [1]; the proof will appear elsewhere.

Corollary 1.5 (see [1]). *Let $D \subset \mathbb{P}^2$ be a non-maximizing non-special irreducible simple plane sextic. Then, unless the set of singularities of D is*

$$2\mathbf{D}_7 \oplus 2\mathbf{A}_2, \quad \mathbf{D}_7 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2, \quad 2\mathbf{D}_4 \oplus 4\mathbf{A}_2, \quad \text{or} \quad 2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2,$$

one has $\pi_1(\mathbb{P}^2 \smallsetminus D) = \mathbb{Z}_6$. ▷

In the four exceptional cases, the groups are known to be non-abelian: they are $SL(2, \mathbb{F}_5) \odot \mathbb{Z}_{12}$ for the last curve and $SL(2, \mathbb{F}_3) \times \mathbb{Z}_2$ for the three others. Here, the notation $SL(2, \mathbb{F}_5) \odot \mathbb{Z}_{12}$ stands for the *central product*, *i.e.*, the direct product $SL(2, \mathbb{F}_5) \times \mathbb{Z}_{12}$ with the center $\mathbb{Z}_2 \subset SL(2, \mathbb{F}_5)$ identified with $\mathbb{Z}_2 \subset \mathbb{Z}_{12}$.

Although special care has been taken to avoid sextics with triple singular points, see Remark 4.12, some of them do appear in the computation. These are the curves realizing the following eight sets of singularities:

(1.6)
$\mathbf{E}_6 \oplus \mathbf{A}_{13}$, see (5.17), $\mathbf{D}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$, see (5.35),
$\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_3$, see (5.26), $\mathbf{D}_5 \oplus \mathbf{A}_{14}$, see (5.15),
$\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_6$, see (5.37), $\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_4$, see (5.27),
$\mathbf{D}_9 \oplus \mathbf{A}_{10}$, see (5.9), $\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_6$, see (5.36).

Their equations are also described in §5, and the conclusion of Theorem 1.1 extends to these curves literally, together with the proof. The fundamental groups of all these curves are abelian, see [9].

In this paper, we confine ourselves to the sextics that can be obtained from a pencil of cubics with at most four basepoints (see §1.2 for the explanation). The remaining six sets of singularities are

(1.7)
15. $\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, 36. $\mathbf{A}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$,
22. $\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, 38. $2\mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$,
26. $\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$, 39. $\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$.

For these curves, the models used in §5 are not applicable: pencils with more than four basepoints need to be considered and the computation seems to become much more involved. In [8], the defining equations of these six curves are derived from the parametric equations found in [16].

We conclude with a few examples of the so-called *Zariski pairs*, see [2]. Roughly, two plane curves $D_1, D_2 \subset \mathbb{P}^2$ constitute a Zariski pair if they are combinatorially equivalent but topologically distinct, in the sense that may vary from problem to problem. In Theorem 1.8, we use the strongest combinatorial equivalence relation (the curves are Galois conjugate; in the terminology of [21], the Zariski pairs are *arithmetic*) and the weakest topological one (the complements $\mathbb{P}^2 \smallsetminus D_i$, $i = 1, 2$ are not properly homotopy equivalent). In Theorem 1.9, the topological relation is slightly stronger. The first example of Galois conjugate but not homeomorphic algebraic varieties is due to Serre [19]. Still, very few other examples are known; a brief survey of the subject, including arithmetic Zariski pairs on plane curves, is contained in [21]. For Zariski pairs in general, see [4].

Theorem 1.8. *Let* $\mathbf{S}$ *be one of the following twelve sets of singularities*:

(1) $\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_4$, $\mathbf{A}_{18} \oplus \mathbf{A}_1$, $\mathbf{A}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, $\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$, *line 13*, $\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, $\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$, *line 19*, $\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, $\mathbf{A}_{10} \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$, *line 24*, $\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, *line 31*;
(2) $\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_3$, $\mathbf{A}_{16} \oplus \mathbf{A}_3$, *line 4*, $\mathbf{A}_{10} \oplus \mathbf{A}_9$, *line 18*, $\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$, *line 20*, $\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$, $\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$, *line 35*.

In case 1, with $(r, c) = (1, 1)$*, let* D_1*,* D_2 *be a real and a non-real sextic realizing* $\mathbf{S}$*; in case 2, with* $(r, c) = (2, 0)$*, let* D_1*,* D_2 *be the two real sextics. Then* (D_1, D_2) *is a Zariski pair in the following strongest sense: the two curves are Galois conjugate, but the complements* $\mathbb{P}^2 \smallsetminus D_i$*,* $i = 1, 2$*, are not properly homotopy equivalent.*

For the four sets of singularities $\mathbf{A}_{18} \oplus \mathbf{A}_1$, $\mathbf{A}_{16} \oplus \mathbf{A}_3$, line 4, $\mathbf{A}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$, and $\mathbf{A}_{10} \oplus \mathbf{A}_9$, line 18, the fact that the spaces $\mathbb{P}^2 \smallsetminus D_i$, $i = 1, 2$, are not homeomorphic was originally established in [21], and for Theorem 1.8 we use essentially the same topological invariant.

Theorem 1.9. *Let* $\mathbf{S}$ *be either* $\mathbf{A}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$*, line 11, or* $\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$*, line 23, or* $\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$*. Then the pair* (D_1, D_2) *of two distinct real sextics realizing* $\mathbf{S}$ *is a Zariski pair in the following sense: the two curves are Galois conjugate, but the topological pairs* $(\mathbb{P}^2, D_i)$*,* $i = 1, 2$*, are not homotopy equivalent. In fact, a slightly stronger statement holds: the pairs*

$$(\mathbb{P}^2 \smallsetminus \operatorname{Sing} D_i, D_i \smallsetminus \operatorname{Sing} D_i), \quad i = 1, 2 \tag{1.10}$$

are not properly homotopy equivalent, where $\operatorname{Sing} D_i$ *stands for the set of all singular points of* D_i*.*

These theorems are proved in §6.4 and §6.6. For the sets of singularities as in Theorem 1.8(2) and Theorem 1.9, we can also state that the Zariski pairs are π_1-*equivalent*, *i.e.*, the fundamental groups $\pi_1(\mathbb{P}^2 \smallsetminus D_i)$, $i = 1, 2$, are isomorphic (as they are both $\mathbb{Z}_6$). In fact, the spaces $\mathbb{P}^2 \smallsetminus D_i$ are homotopy equivalent, see Proposition 6.4, but they are not homeomorphic! Probably, this conclusion (the homotopy equivalence of the complements) also holds for Theorem 1.8(1).

1.2. **The idea.** The main tool used in the paper is the Artal–Carmona–Cogolludo construction (*ACC-construction* in the sequel) developed in [3]. This construction is outlined in §3.1. We confine ourselves to generic (in the sense of Definition 2.3) irreducible non-special sextics with double singular points only, see Convention 3.3; using the theory of $K3$-surfaces, we show that the ramification locus of the ACC-model of such a curve is irreducible and with $\mathbf{A}$ type singularities only, see §3.3. Furthermore, we describe the singular fibers of the corresponding Jacobian elliptic surface Y, see §3.2, and prove that, geometrically, the blow-down map $Y \to \mathbb{P}^2$ is as shown in Figure 1 on page 16. This fact eliminates the need in the tedious case-by-case analysis of the possible configurations of divisors (*cf.* [3]), and the construction of the ACC-models of all maximizing sextics becomes a relatively easy task.

The other key ingredient is Moody's paper [14]. With a little effort (and `Maple`) its results can be extended to explicit formulas for the rational two-to-one map $\mathbb{P}^2 \dashrightarrow \Sigma_2$ related to the ACC-model, see §4.2; they are used to pass from the defining equations of the ACC-models in Σ_2 to those of the original sextics. (In fact, we incorporate Moody's formulas from the very beginning and describe the ACC-models in terms of pencils of cubics; it may be due to this fact that, in the six missing cases, the equations on the parameters are too complicated to be solved or even to be written down.)

The fundamental groups are computed as suggested in [10], representing sextics as tetragonal curves. Alternatively, one could try to use the ACC-models, which are *trigonal* curves and look simpler; unfortunately, one would have to keep track of too many (three to four) extra sections, which makes this approach about as difficult as the direct computation, especially when the curve is not real. Certainly, given equations, one can also use the modern technology and compute the monodromy by brute force; however, at this stage I prefer to refrain from a computer aided solution to a problem that is not discrete in its nature.

The other theorems are proved in §6 by constructing appropriate invariants.

1.3. **Contents of the paper.** In §2, after a brief introduction to the basic concepts related to plane sextics, we use the theory of $K3$-surfaces to describe the rational curves splitting in the double covering ramified at a generic non-special irreducible sextic. These results are used in §3, where we introduce the ACC-model and show that the models of non-special irreducible sextics are particularly simple. In §4, we recall and extend the results of [14] concerning the Bertini involution $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and explain how these results apply to the ACC-construction. In §5, the details of deriving the defining equations of maximizing sextics are outlined and their minimal fields of definition are described. Finally, in §6 we give formal proofs of Theorems 1.1, 1.2, 1.8, and 1.9 and make a few concluding remarks. As a digression, we discuss the homotopy type of the complement of an irreducible plane curve with abelian fundamental group, see §6.5.

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2. The covering $K3$-surface

The principal goal of this section is Theorem 2.5, which describes the rational curves in the $K3$-surface ramified at a generic (see Definition 2.3 below) non-special irreducible sextic.

2.1. **Terminology and notation.** A plane sextic $D \subset \mathbb{P}^2$ is called *simple* if all its singular points are simple, *i.e.*, those of type $\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$, see [12]. Given a sextic D, we denote by $P_i \in \mathbb{P}^2$ its singular points.

We will also use the classical concept of infinitely near points: given a point P in a surface S, all points Q in the exceptional divisor in the blow-up $S(P)$ of S at P are said to be *infinitely near* to P (notation $Q \to P$). A curve $D \subset S$ *passes* through a point $Q \to P$ (notation $Q \in D$) if D passes through P and the strict transform of D in $S(P)$ passes through Q. Similarly, a point $Q \to P$ is *singular* for D if it is singular for the strict transform of D in $S(P)$. This construction can be iterated and one can consider sequences $\ldots \to Q' \to Q \to P$ of infinitely near points. Starting from level 0 for the points of the original plane $\mathbb{P}^2$, we define the *level* of an infinitely near point *via* $\operatorname{level}(Q) = \operatorname{level}(P) + 1$ whenever $Q \to P$.

For a sextic $D \subset \mathbb{P}^2$ with $\mathbf{A}$-type singular points only, denote by $DP(D)$ the set of all *double points* of D, including infinitely near. This set is a union of disjoint maximal (with respect to inclusion) chains, each singular point P_i of type $\mathbf{A}_p$ giving rise to a chain $Q_r \to \ldots \to Q_1 = P_i$ of length $r = [\frac{1}{2}(p+1)]$. A subset $\mathfrak{Q} \subset DP(D)$ is called *complete* if, whenever $Q \in \mathfrak{Q}$ and $Q \to Q'$, also $Q' \in \mathfrak{Q}$.

2.2. **The homological type.** Given a simple sextic D, denote by $X := X_D$ the minimal resolution of singularities of the double covering of the plane $\mathbb{P}^2$ ramified at D. It is a $K3$-surface, see, *e.g.*, [17]. With a certain abuse of the language, X is referred to as the *covering $K3$-surface*.

Denote by $\mathbf{L} := H_2(X) \cong 2\mathbf{E}_8 \oplus 3\mathbf{U}$ the intersection index lattice of X (where $\mathbf{U} = \mathbb{Z}\mathbf{u}_1 + \mathbb{Z}\mathbf{u}_2$, $\mathbf{u}_1^2 = \mathbf{u}_2^2 = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1$ is the *hyperbolic plane*). Let $h \in \mathbf{L}$, $h^2 = 2$, be the class of a hyperplane section (pull-back of a line in $\mathbb{P}^2$), and let $\mathbf{P}_i \subset \mathbf{L}$ be the lattice spanned by the classes of the exceptional divisors over a singular point P_i of D. It is a negative definite even root lattice of the same name $\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$ as the type of P_i, and $\operatorname{rk}\mathbf{P}_i = \mu(P_i)$ is the Milnor number. This lattice has a *canonical basis*, constituted by the classes of the exceptional divisors. The basis vectors are the walls of a single Weyl chamber; they can be identified with the vertices of the Dynkin graph of $\mathbf{P}_i$.

We will consider the sublattice $\mathbf{S} := \bigoplus_i \mathbf{P}_i$, referred to as the *set of singularities* of D, the obviously orthogonal sum $\mathbf{S} \oplus \mathbb{Z}h$, and its *primitive hull*

$$(\mathbf{S} \oplus \mathbb{Z}h)\tilde{} := \big((\mathbf{S} \oplus \mathbb{Z}h) \otimes \mathbb{Q}\big) \cap \mathbf{L}.$$

The sequence of lattice extensions

$$\mathbf{S} \subset \mathbf{S} \oplus \mathbb{Z}h \subset \mathbf{L} \tag{2.1}$$

is called the *homological type* of D. Clearly, $\mathbf{S}$ is an even negative definite lattice and $\operatorname{rk}\mathbf{S} = \mu(D)$ is the total Milnor number of D. Since $\sigma_-\mathbf{L} = 19$, one has $\mu(D) \leqslant 19$, see [17]. A simple sextic D with $\mu(D) = 19$ is called *maximizing*. Note that both the inequality and the term apply to simple sextics only.

An irreducible sextic $D \subset \mathbb{P}^2$ is called *special*, or $\mathbb{D}_{2n}$-*special*, see [5], if its fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus D)$ admits a dihedral quotient $\mathbb{D}_{2n}$, $n \geqslant 3$. If this is the case, one has $n = 3$, 5, or 7, and $\mathbb{D}_6$-special sextics are those of *torus type*, see [5]. By definition, the fundamental groups of special sextics are not abelian.

Theorem 2.2 (see [5])**.** *A simple sextic D is irreducible and non-special if and only if $\mathbf{S} \oplus \mathbb{Z}h \subset \mathbf{L}$ is a primitive sublattice,* i.e., $\mathbf{S} \oplus \mathbb{Z}h = (\mathbf{S} \oplus \mathbb{Z}h)\tilde{}$. ▷

Since both $\mathbf{S}$ and $\mathbb{Z}h$ are generated by algebraic curves (and since the Néron–Severi lattice $NS(X)$ is primitive in $\mathbf{L}$), one has $(\mathbf{S} \oplus \mathbb{Z}h)\tilde{} \subset NS(X)$.

Definition 2.3. A simple sextic $D \subset \mathbb{P}^2$ is called *generic* if $(\mathbf{S} \oplus \mathbb{Z}h)\tilde{} = NS(X)$.

In each equisingular stratum of the space of simple sextics, generic ones form a dense Zariski open subset. A maximizing sextic is always generic.

Let $\tau\colon X \to X$ be the deck translation of the ramified covering $X \to \mathbb{P}^2$. This automorphism induces an involutive autoisometry $\tau_*\colon \mathbf{L} \to \mathbf{L}$.

Lemma 2.4. *The induced autoisometry $\tau_*\colon \mathbf{L} \to \mathbf{L}$ acts as follows: $\tau_*(h) = h$; the restriction of τ_* to $\mathbf{P}_i$ is induced by the symmetry s_i of the Dynkin graph of $\mathbf{P}_i$ (in the canonical basis), where*

- *s_i is the only nontrivial symmetry if $\mathbf{P}_i = \mathbf{A}_{p\geqslant 2}$, $\mathbf{D}_{\mathrm{odd}}$, or $\mathbf{E}_6$, and*
- *s_i is the identity otherwise;*

the restriction of τ_* to $(\mathbf{S} \oplus \mathbb{Z}h)^\perp$ is $-\mathrm{id}$.

Proof. The action of τ_* on $\mathbf{S} \oplus \mathbb{Z}h$ is given by a simple computation using the minimal resolution of the singularities of D in $\mathbb{P}^2$. For the last statement, since $X/\tau = \mathbb{P}^2$ is rational, τ is anti-symplectic, *i.e.*, $\tau_*(\omega) = -\omega$ for the class ω of a holomorphic form on X. Since also τ_* is defined over $\mathbb{Z}$, the (-1)-eigenspace of τ_* contains the minimal *rational* subspace $V \subset \mathbf{L} \otimes \mathbb{Q}$ such that $\omega \in V \otimes \mathbb{C}$. On the other hand, τ_* is invariant under equisingular deformations and, deforming D to a generic sextic, one has $V = (\mathbf{S} \oplus \mathbb{Z}h)^\perp \otimes \mathbb{Q}$. (Recall that $NS(X) = \omega^\perp \cap \mathbf{L}$.) □

2.3. **Rational curves in X.** The goal of this section is the following theorem, which is proved at the end of the section.

Theorem 2.5. *Let $D \subset \mathbb{P}^2$ be a generic irreducible non-special sextic with $\mathbf{A}$-type singularities only, and let X be the covering $K3$-surface. Let, further, $R \subset X$ be a nonsingular rational curve whose projection $\bar{R} \subset \mathbb{P}^2$ is a curve of degree at most 3. Then the projection $R \to \bar{R}$ is two-to-one (in other words, R is the pull-back of the strict transform of $\bar{R}$), and $\bar{R}$ is one of the following:*

1. *a line through a complete pair $\mathfrak{Q}_2 \subset DP(D)$;*
2. *a conic through a complete quintuple $\mathfrak{Q}_5 \subset DP(D)$;*
3. *a cubic through a complete septuple $\mathfrak{Q}_7 \subset DP(D)$ with a double point at a distinguished point $P \in \mathfrak{Q}_7$ of level zero.*

Conversely, given a complete set $\mathfrak{Q}_2$, $\mathfrak{Q}_5$ or pair $P \in \mathfrak{Q}_7$ as above, there is a unique, respectively, line, conic, or cubic $\bar{R}$ as in items 1–3. This curve $\bar{R}$ is irreducible, and the pull-back of its strict transform is a nonsingular rational curve in X.

Corollary 2.6. *Under the hypotheses of Theorem 2.5, the configuration $DP(D)$ is* almost del Pezzo, *in the sense that*

1. *there is no line passing through three points;*
2. *there is no conic passing through six points;*
3. *there is no cubic passing through eight points and singular at one of them.*

More generally, there is no line or conic whose local intersection index with D at each intersection point is even. ◁

Remark 2.7. If D is a special sextic, there *are* conics (not necessarily irreducible) passing through some complete sextuples $\mathfrak{Q}_6 \subset DP(D)$, see [5]. Thus, the existence of such conics is yet another characterization of *generic* irreducible special sextics. See [22] for more details.

We precede the proof of Theorem 2.5 with a few observations.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ be the canonical basis for a summand of $\mathbf{S}$ of type $\mathbf{A}_k$. For an element $a \in \mathbf{A}_k$, $a = \sum a_i \mathbf{e}_i$, let $a_0 = a_{k+1} = 0$ and denote $d_i = a_i - a_{i-1}$, $i = 1, \ldots, k+1$. Recall that $\mathbf{A}_k$ is the orthogonal complement $(\mathbf{v}_1 + \ldots + \mathbf{v}_{k+1})^\perp$ in the lattice $\mathbf{B}_{k+1} := \bigoplus_{i=1}^{k+1} \mathbb{Z}\mathbf{v}_i$, $\mathbf{v}_i^2 = -1$, so that $\mathbf{e}_i = \mathbf{v}_i - \mathbf{v}_{i+1}$, $i = 1, \ldots, k$. In this notation, one has $a = \sum d_i \mathbf{v}_i$. Hence, $a^2 = -\sum d_i^2$ and $a \cdot a' = -\sum d_i d_i'$ for another element $a' = \sum d_i' \mathbf{v}_i$. The following statement is straightforward.

Lemma 2.8. *An element $a = \sum d_i \mathbf{v}_i \in \mathbf{B}_{k+1}$ is in $\mathbf{A}_k$ if and only if $\sum d_i = 0$. Furthermore, $a \cdot \mathbf{e}_i \geqslant 0$ for all $i = 1, \ldots, k$ if and only if $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_{k+1}$.* ◁

Corollary 2.9. *The elements $a \in \mathbf{A}_k$ such that $a^2 \geqslant -10$ and $a \cdot \mathbf{e}_i \geqslant 0$ for all $i = 1, \ldots, k$ are as follows:*

- $\mathbf{a}^q := -\mathbf{v}_1 - \ldots - \mathbf{v}_q + \mathbf{v}_{k+2-q} + \ldots + \mathbf{v}_{k+1}$ $(1 \leqslant q \leqslant 5)$: $(\mathbf{a}^q)^2 = -2q$;
- $\mathbf{b}^q := \mathbf{a}^1 + \mathbf{a}^q$ $(1 \leqslant q \leqslant 2)$: $(\mathbf{b}^q)^2 = -6 - 2q$;
- $\mathbf{c}^+ := -2\mathbf{v}_1 + \mathbf{v}_k + \mathbf{v}_{k+1}$ *or* $\mathbf{c}^- := -\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_{k+1}$: $(\mathbf{c}^\pm)^2 = -6$. ◁

Corollary 2.10. *The* symmetric (*with respect to the only nontrivial symmetry of the Dynkin graph*) *elements* $a \in \mathbf{A}_k$ *such that* $a^2 \geqslant -20$ *and* $a \cdot \mathbf{e}_i \geqslant 0$ *for all* $i = 1, \ldots, k$ *are* $\mathbf{a}^q$ $(1 \leqslant q \leqslant 10)$, $\mathbf{b}^q$ $(1 \leqslant q \leqslant 7)$, $\mathbf{a}^2 + \mathbf{a}^q$ $(2 \leqslant q \leqslant 4)$, *and* $2\mathbf{a}^1 + \mathbf{a}^q$ $(1 \leqslant q \leqslant 2)$, *see Corollary 2.9 for the notation.* ◁

Proof of Theorem 2.5. Recall a description of rational curves on a $K3$-surface X, see [18] or [11, Theorem 6.9.1]. Let $\mathcal{C} := \{x \in NS(X) \otimes \mathbb{R} \,|\, x^2 > 0\}$ be the positive cone and $\mathbb{P}(\mathcal{C}) := \mathcal{C}/\mathbb{R}^*$ its projectivization. Consider the group G of motions of the hyperbolic space $\mathbb{P}(\mathcal{C})$ generated by the reflections against hyperplanes orthogonal to vectors $v \in NS(X)$ of square (-2) and let $\Pi \subset \mathbb{P}(\mathcal{C})$ be the fundamental polyhedron of G containing the class of a Kähler form $\rho \in NS(X) \otimes \mathbb{R}$. Denote by $\Delta_+(X)$ the set of vectors $v \in NS(X)$ such that $v^2 = -2$, $v \cdot \rho > 0$, and v is orthogonal to a face of Π. Then $\Delta_+(X) \subset \mathbf{L}$ *is precisely the set of homology classes realized by nonsingular rational curves on* X. Each class $v \in \Delta_+(X)$ is realized by a unique such curve.

The set $\Delta_+ := \Delta_+(X)$ can be found step by step, using Vinberg's algorithm [24] and taking for ρ a small perturbation of h. At Step 0, one adds to Δ_+ the classes of all exceptional divisors, *i.e.*, the canonical basis for $\mathbf{S}$. Then, at each step s, $s > 0$, one adds to Δ_+ all vectors $v \in NS(X)$ such that $v^2 = -2$, $v \cdot h = s$ (there are finitely many such vectors), and $v \cdot u \geqslant 0$ for any $u \in \Delta_+$ with $u \cdot h < s$, *i.e.*, *v has non-negative intersection with any vector added to Δ_+ at the previous steps.* *A priori*, $\Delta_+(X)$ may be infinite and this process does not need to terminate; for further details and the termination condition, see [24].

Under the hypotheses of the theorem, $NS(X) = \mathbf{S} \oplus \mathbb{Z}h$ and all vectors v used in Vinberg's algorithm are of the form $v = a + rh$, $r > 0$, $a \in \mathbf{S}$, $a^2 = -2r^2 - 2$. In particular, all odd steps are vacuous and the condition $v \cdot u \geqslant 0$ for all $u \in \Delta_+$, $u \cdot h = 0$ is equivalent to the requirement that each component $a_i \in \mathbf{P}_i$ of a should be as in Lemma 2.8. With one exception (the curve D itself, see below), the rational curve R represented by $v = a + rh$ projects to a curve $\bar{R} \subset \mathbb{P}^2$ of degree $2r$ or r, depending on whether v is or is not τ_*-invariant; the projection is two-to-one in the former case and one-to-one in the latter. (Thus, the projection is two-to-one whenever $\deg \bar{R}$ is odd, and only the case of conics needs special attention.)

Summarizing, we need to consider Steps 2, 4, and 6 of Vinberg's algorithm, in the last case confining ourselves to τ_*-invariant vectors only.

Introduce the notation $\mathbf{a}_i^q \in \mathbf{P}_i$ *etc.* for the elements $\mathbf{a}^q$ *etc.* as in Corollary 2.9 in the lattice $\mathbf{P}_i$. Throughout the rest of the proof, we assume the convention that distinct subscripts represent distinct indices; for example, an expression $\mathbf{a}_i^1 + \mathbf{a}_j^1$ implies implicitly that $i \neq j$.

Added at Step 2 are all elements of the form $\mathbf{a}_i^2 + h$ (whenever $\mu(P_i) \geqslant 3$) and $\mathbf{a}_i^1 + \mathbf{a}_j^1 + h$. These elements are in a one-to-one correspondence with complete pairs $\mathfrak{Q}_2 \subset DP(D)$, and the corresponding rational curves are the pull-backs of the lines as in Theorem 2.5(1). In particular, we conclude that there are no conics in $\mathbb{P}^2$ whose pull-backs split into pairs of rational curves in X.

Added at Step 4 are all elements of the form $\sum \mathbf{a}_{i_\alpha}^{q_\alpha} + 2h$, $\sum q_\alpha = 5$. These elements are in a one-to-one correspondence with complete quintuples $\mathfrak{Q}_5 \subset DP(D)$, and the rational curves are the pull-backs of the conics as in Theorem 2.5(2).

Note that elements containing $\mathbf{b}^q$ or $\mathbf{c}^{\pm}$, see Corollary 2.9, cannot be added at Step 4. Indeed, any such element would be one of the following:

- $a = \mathbf{b}_i^2 + 2h$; then $a \cdot (\mathbf{a}_i^2 + h) = -2 < 0$;

- $a = \mathbf{b}_i^1 + \mathbf{a}_j^1 + 2h$; then $a \cdot (\mathbf{a}_i^1 + \mathbf{a}_j^1 + h) = -2 < 0$;
- $a = \mathbf{c}_i^{\pm} + \mathbf{a}_j^2 + 2h$; then $a \cdot (\mathbf{a}_i^1 + \mathbf{a}_j^1 + h) = -1 < 0$;
- $a = \mathbf{c}_i^{\pm} + \mathbf{a}_j^1 + \mathbf{a}_k^1 + 2h$; then $a \cdot (\mathbf{a}_i^1 + \mathbf{a}_j^1 + h) = -1 < 0$.

Finally, we characterize the τ_*-*invariant* elements added at Step 6.

An element of the form $\sum \mathbf{a}_{i_\alpha}^{q_\alpha} + 3h$, $\sum q_\alpha = 10$, exists (and then is unique) if and only if D has ten double points. In this case, genus$(D) = 0$ and the above element represents D itself.

Elements of the form $\mathbf{b}_i^q + \sum \mathbf{a}_{i_\alpha}^{q_\alpha} + 3h$, $q + \sum q_\alpha = 7$, *are* added at Step 6. (It is immediate that any such element has non-negative intersection with all those added at Steps 2 and 4.) Such elements are parametrized by pairs $P_i \in \mathfrak{Q}_7$, where $\mathfrak{Q}_7 \subset DP(D)$ is a complete septuple and P_i is a distinguished point of level zero. The corresponding rational curve is the pull-back of a cubic as in Theorem 2.5(3). No other τ_*-invariant element can be added at this step. Indeed, by Corollary 2.10, such an element would be one of the following:

- $a = (\mathbf{a}_i^2 + \mathbf{a}_i^q) + \ldots + 3h$ $(2 \leqslant q \leqslant 4)$; then $a \cdot (\mathbf{a}_i^2 + h) = -2 < 0$;
- $a = 3\mathbf{a}_i^1 + \mathbf{a}_j^1 + 3h$; then $a \cdot (\mathbf{a}_i^1 + \mathbf{a}_j^1 + h) = -2 < 0$;
- $a = (2\mathbf{a}_i^1 + \mathbf{a}_i^2) + 3h$; then $a \cdot (\mathbf{a}_i^2 + h) = -2 < 0$.

This observation completes the proof of Theorem 2.5. □

3. The ACC-construction

In this section, we recall the principal results of [3] concerning the properties of the ACC-construction and describe the singular fibers and the ramification locus of the ACC-model of a generic non-special irreducible sextic.

3.1. **The construction** (see [3]). Consider a sextic $D \subset \mathbb{P}^2$ with **A**-type singular points only and fix a complete octuple $\mathfrak{Q}_8 = \{Q_1, \ldots, Q_8\} \subset DP(D)$. (Thus, we assume that D has at least eight double points. If D is irreducible, this assumption is equivalent to the requirement that genus$(D) \leqslant 2$.)

Let $\mathcal{P} := \mathcal{P}(\mathfrak{Q}_8)$ be the closure of the set of cubics passing through all points of $\mathfrak{Q}_8$. As shown in [3], $\mathcal{P}$ is a pencil and a generic member of $\mathcal{P}$ is a nonsingular cubic; hence, $\mathcal{P}$ has nine basepoints: the points $Q_1, \ldots, Q_8$ of $\mathfrak{Q}_8$ and another *implicit* point Q_0, which *a priori* may be infinitely close to some of $Q_i \in \mathfrak{Q}_8$. Let $\mathfrak{Q}_8^* := \mathfrak{Q}_8 \cup \{Q_0\}$. It follows that the result $Y := \mathbb{P}^2(\mathfrak{Q}_8^*)$ of the blow-up of the nine points $Q_0, \ldots, Q_8$ is a relatively minimal rational Jacobian elliptic surface, the distinguished section being the exceptional divisor $\tilde{Q}_0$ over Q_0. With $\mathfrak{Q}_8$ (and hence Q_0 and Y) understood, we use the notation $\tilde{A}$ for the strict transform in Y of a curve $A \subset \mathbb{P}^2$. Let also $\tilde{Q}_i \subset Y$ be the strict transform of the exceptional divisor obtained by blowing up the basepoint Q_i, $i = 0, \ldots, 8$, and let $\tilde{\mathcal{P}}$ be the resulting elliptic pencil on Y.

The fiberwise multiplication by (-1) is an involutive automorphism $\beta\colon Y \to Y$, and the quotient Y/β blows down to the Hirzebruch surface Σ_2, *i.e.*, geometrically ruled rational surface with an exceptional section E of self-intersection (-2) (the image of $\tilde{Q}_0$). Conversely, Y is recovered as the minimal resolution of singularities of the double covering of Σ_2 ramified at the exceptional section E and a certain *proper trigonal curve* $\bar{K}$ (*i.e.*, a reduced curve disjoint from E and intersecting each fiber of the ruling at three points). This representation of the Jacobian elliptic surface Y is often referred to as its *Weierstraß model*.

Theorem 3.1 (see [3])**.** *One has $\tilde{D} \cdot \tilde{Q}_0 = 0$ and $\tilde{D} \cdot \tilde{F} = 2$, where $\tilde{F}$ is a generic fiber of $\tilde{\mathcal{P}}$. Furthermore, one has $\beta(\tilde{D}) = \tilde{D}$; hence, the image $\bar{D} \subset \Sigma_2$ of $\tilde{D}$ is a section of Σ_2 disjoint from E.* ▷

Definition 3.2. Given a sextic $D \subset \mathbb{P}^2$ and a complete octuple $\mathfrak{Q}_8 \subset DP(D)$, the pair $(Y, \tilde{D})$ equipped with the projections $\mathbb{P}^2 \leftarrow Y \to \Sigma_2$ is called the *ACC-model* of D (defined by $\mathfrak{Q}_8$). Here, Y is a relatively minimal rational Jacobian elliptic surface and $\tilde{D} \subset Y$ is the bisection that projects onto D.

3.2. **The singular fibers.** A complete octuple $\mathfrak{Q}_8 \subset DP(D)$, see §3.1, is a union of maximal chains, one chain (possibly empty) over each singular point P_i of D. Given D, this octuple is determined by assigning the *height* $h_i := \operatorname{ht} P_i$ (the length of the corresponding chain) to each singular point P_i. One has $0 \leqslant h_i \leqslant \frac{1}{2}(p+1)$ if P_i is of type $\mathbf{A}_p$ and $\sum h_i = 8$.

Convention 3.3. Till the rest of this section, we fix a sextic $D \subset \mathbb{P}^2$ satisfying the following conditions:

- $D \subset \mathbb{P}^2$ is a generic (see Definition 2.3) irreducible non-special sextic,
- D has **A**-type singular points only and $\operatorname{genus}(D) \leqslant 2$.

Fix, further, a collection of heights $\{h_i\}$ of the singular points of D satisfying the conditions above. Hence, we have also fixed a complete octuple $\mathfrak{Q}_8 \subset DP(D)$ and a pencil $\mathcal{P} := \mathcal{P}(\mathfrak{Q}_8)$ of cubics as in §3.1, *i.e.*, an ACC-model of D.

For a singular point $P_i \in \mathfrak{Q}_8$, we denote by $P_i^\top$ the topmost (*i.e.*, that of maximal level) element of $\mathfrak{Q}_8$ that is infinitely near to P_i. If, in addition, $\operatorname{ht} P_i \geqslant 2$, then $F_i \in \mathcal{P}$ is the member of the pencil singular at P_i: such a cubic obviously exists and, since a generic member of $\mathcal{P}$ is nonsingular, it is unique.

The following three statements are proved at the end of the section.

Theorem 3.4. *Each* reducible *singular fiber of $\tilde{\mathcal{P}}$ contains a (unique) cubics $\tilde{F}_i$ corresponding to a singular point P_i of D of height $h_i \geqslant 2$.*

Addendum 3.5. *Let P_i be a singular point of D and $h := \operatorname{ht} P_i \geqslant 2$. Then F_i is an irreducible nodal (possibly cuspidal if $h \leqslant 3$) cubic. The corresponding fiber of $\tilde{\mathcal{P}}$ is $\tilde{F}_i + \sum \tilde{Q}_j$, the summation running over all points $Q_j \in \mathfrak{Q}_8$ such that $Q_j \to P_i$ and $Q_j \neq P_i^\top$. This fiber is of type $\tilde{\mathbf{A}}_{h-1}$, possibly degenerating to $\tilde{\mathbf{A}}_h^*$ if $h \leqslant 3$.*

Addendum 3.6. *Let P_i be a singular point of D and $h := \operatorname{ht} P_i \geqslant 1$. Then $\tilde{P}_i^\top$ is a section of $\tilde{\mathcal{P}}$ disjoint from $\tilde{Q}_0$. As a consequence, the implicit basepoint Q_0 of $\mathcal{P}$ is a point of level zero.*

Proof of Theorem 3.4 and Addendum 3.5. Let $\tilde{F} = m_1\tilde{E}_1 + \ldots + m_r\tilde{E}_r$, $r \geqslant 2$, be a *reducible* singular fiber. Each component $\tilde{E}_k$ of $\tilde{F}$ is a nonsingular rational curve and one has $\tilde{E}_k \cdot \tilde{D} \leqslant 2$, see Theorem 3.1. Hence, $\tilde{E}_k$ lifts to a nonsingular rational curve or a pair of such curves in the covering $K3$-surface X.

Assume that $\tilde{E}_k$ is not one of the exceptional divisors $\tilde{Q}_j$, *i.e.*, $\tilde{E}_k$ projects to a curve $E_k \subset \mathbb{P}^2$. Then $\deg E_k \leqslant 3$ and, due to Theorem 2.5, the pull-back of $\tilde{E}_k$ in X is irreducible. Hence, $\tilde{E}_k \cdot \tilde{D} = 2$ and, since also $\tilde{F} \cdot \tilde{D} = 2$, we conclude that $\tilde{E}_k$ is the *only* component of $\tilde{F}$ that does not contract to a point in $\mathbb{P}^2$. Thus, the image of $\tilde{F}$ in $\mathbb{P}^2$ is either an irreducible rational cubic or a triple line. In the former case, the singular point of the cubic should be resolved in Y; hence, this singular point is at one of those of D. The latter case is easily ruled out as, due to Corollary 2.6, the line passes through two double points of D only.

According to Theorem 2.5(3), the cubic E_k passes through seven double points $Q_1, \ldots, Q_7$ of D and is singular at one of these points, say, $P_i := Q_1$. It is easy to see that E_k belongs to $\mathcal{P}$ if and only if all seven points are in $\mathfrak{Q}_8$ and the eighths element of $\mathfrak{Q}_8$ is infinitely near to P_i. Hence, the height of P_i is at least 2 and the corresponding singular fiber of $\mathcal{P}$ is as stated in Addendum 3.6. □

Proof of Addendum 3.6. We have $K_Y \sim \tilde{F}$, where $\tilde{F}$ is a fiber of $\mathcal{P}$. Furthermore, $(\tilde{P}_i^\top)^2 = -2$ if $Q_0 \to P_i^\top$ or $(\tilde{P}_i^\top)^2 = -1$ otherwise. By the adjunction formula, in the former case $\tilde{P}_i^\top \cdot \tilde{F} = 0$, *i.e.*, $\tilde{P}_i^\top$ is a component of a (necessarily reducible) singular fiber of $\tilde{\mathcal{P}}$. This possibility is ruled out by Theorem 3.4 and Addendum 3.5. In the latter case, $\tilde{P}_i^\top \cdot \tilde{F} = 1$, *i.e.*, $\tilde{P}_i^\top$ is a section. Since Q_0 is not infinitely near to $P_i^\top$, this section is disjoint from $\tilde{Q}_0$. □

3.3. **The ramification locus $\bar{K} \subset \Sigma_2$.** Fix a sextic $D \subset \mathbb{P}^2$ and the other data as in Convention 3.3 and let $\bar{K} \subset \Sigma_2$ be the trigonal part of the ramification locus of its ACC-model.

Corollary 3.7 (of Theorem 3.4 and Addendum 3.5)**.** *The curve $\bar{K} \subset \Sigma_2$ has a type $\mathbf{A}_{h_i-1}$ singular point $\bar{P}_i$ for each singular point P_i of D of height $h_i \geqslant 2$; this curve has no other singular points.* ◁

Proposition 3.8. *The curve $\bar{K} \subset \Sigma_2$ is irreducible.*

Proof. The curve $\bar{K}$ is reducible if and only if the Mordell–Weil group $MW(Y)$ has 2-torsion, see, *e.g.*, [9, Corollary 6.13 and Proposition 6.2]. One has

$$MW(Y) = H_2(Y)/\mathbf{S}', \qquad H_2(Y) = \mathbb{Z}[\tilde{L}] \oplus \bigoplus \mathbb{Z}[\tilde{Q}_i],$$

where $\mathbf{S}' \subset H_2(Y)$ is the sublattice generated by the classes of the section and the components of all fibers (see [23]), $L \subset \mathbb{P}^2$ is a generic line, and $Q_i \in \mathfrak{Q}_8^*$. In view of Theorem 3.4 and Addendum 3.5, we can decompose $\mathbf{S}' = \mathbf{S}'' + \mathbb{Z}[\tilde{F}]$, where $\mathbf{S}''$ is generated by $[\tilde{Q}_0]$ and the classes $[\tilde{Q}_i]$ of *all but the topmost* elements $Q_i \in \mathfrak{Q}_8$ and F is a generic fiber. One has $[\tilde{F}] = 3[\tilde{L}] - \sum(l_i + 1)[\tilde{Q}_i]$, where $Q_i \in \mathfrak{Q}_8^*$ and $l_i := \operatorname{level}(Q_i)$. Modulo $\mathbf{S}''$, the summation can be restricted to the topmost elements $Q_i = P_j^\top$ only. Then $l_i + 1 = h_j$ and, since g.c.d.$\{h_j\} \mid 8$ is prime to 3, the quotient $H_2(Y)/\mathbf{S}'$ is torsion free. □

Corollary 3.9. *Each singular point P_i of D of height $h_i \geqslant 1$ gives rise to a section $L_i := \bar{P}_i^\top$ of Σ_2* (viz. *the image of $\tilde{P}_i^\top$*) *disjoint from E. This section is triple tangent to $\bar{K}$ (if $h_i = 1$) or double tangent to $\bar{K}$ and passing through $\bar{P}_i$ (if $h_i \geqslant 2$); it does not pass through any other singular point of $\bar{K}$.*

In Corollary 3.9, we do not exclude the possibility that two or three points of tangency of $\bar{P}_i^\top$ and $\bar{K}$ may collide. Furthermore, these points of tangency may also collide with $\bar{P}_i$ (if $h_i \geqslant 2$).

Proof. In view of Proposition 3.8, $\bar{P}_i^\top$ is not a component of $\bar{K}$, and the structure of the singular fibers given by Addendum 3.5 implies that $\bar{P}_i^\top$ passes through $\bar{P}_i$ (if $h_i \geqslant 2$). Indeed, in the notation of Addendum 3.5, the cubic F_i does *not* pass through $P_i^\top$, *i.e.*, $\tilde{F}_i \cdot \tilde{P}_i^\top = 0$. On the other hand, it is $\tilde{F}_i$ that is the only component of the singular fiber that does not contract in Σ_2, see Addendum 3.6. Similarly,

$\bar{P}_i^\top$ does not pass through any other singular point $\bar{P}_j$, $P_j \neq P_i$, $h_j \geqslant 2$, as the corresponding cubic F_j *does* pass through $P_i^\top$. □

4. The Bertini involution

The ACC-construction can also be described as follows. Let Y' be the plane blown up at the eight points $Q_1, \ldots, Q_8$. It is a (nodal, in general) del Pezzo surface of degree 1. The anti-bicanonical linear system defines a map $\varphi\colon Y' \to \Sigma_2' \subset \mathbb{P}^3$, where Σ_2' is a quadratic cone. This map is of degree 2; it is ramified over the vertex (the image of Q_0) and a curve $\bar{K}' \subset \Sigma_2'$ cut off by a cubic surface disjoint from the vertex. The image $\bar{D}' \subset \Sigma_2'$ of D is a plane section. The deck translation of φ is called the *Bertini involution* (defined by the above pencil of cubics and its distinguished basepoint Q_0); it has one isolated fixed point, which is the implicit basepoint Q_0. The objects appearing in the original construction, see §3.1, are obtained from those just described by blowing this implicit point Q_0 (or its image in Σ_2', whichever is appropriate) up.

4.1. **Explicit equations** (see [14])**.** In the exposition below, we try to keep the notation of [14]. Consider the pencil defined by two plane cubics $\{w(x) = 0\}$ and $\{w'(x) = 0\}$, where $x = (x_1 : x_2 : x_3)$, and assume that $P_0(0 : 0 : 1)$, $P_1(0 : 1 : 0)$, and $P_2(1 : 0 : 0)$ are among its basepoints, whence

$$w(x) = x_3^2(a_1x_1 + a_2x_2) + x_3(b_1x_1^2 + b_2x_1x_2 + b_3x_2^2) + (c_1x_1^2x_2 + c_2x_1x_2^2)$$

and similar for w'. The cubic passing through a point $y = (y_1 : y_2 : y_3)$ is given by

$$\mathcal{W}_3(x) := w(x)w'(y) - w'(x)w(y) = 0.$$

Clearly,

$$\mathcal{W}_3(x) = x_3^2(A_1x_1 + A_2x_2) + x_3(B_1x_1^2 + B_2x_1x_2 + B_3x_2^2) + (C_1x_1^2x_2 + C_2x_1x_2^2),$$

where $A_i(y) := a_iw'(y) - a_i'w(y)$, $B_i(y) := b_iw'(y) - b_i'w(y)$, and $C_i(y) := c_iw'(y) - c_i'w(y)$. Let $\kappa := a_1b_1' - a_1'b_1$ and consider the polynomials

$$\begin{aligned}
C_5\,(y) &:= A_2[B_1 + \kappa y_1y_3^2]_{y_2} + [A_1 - \kappa y_1^2y_3]_{y_2}[A_2y_3 + B_3y_2]_{y_1} + \kappa B_3y_1y_3,\\
\phi_6\,(y) &:= A_1C_2 + y_3C_5,\\
\psi_6\,(y) &:= A_2C_1 + y_3C_5,\\
r_1'\,(y) &:= B_1A_2^2 - B_2A_1A_2 + B_3A_1^2.
\end{aligned}$$

Here, following [14], we use the notation $[e]_u$ to indicate that e has a common factor u and this factor has been removed. In these notations, the Bertini involution is the birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $y \mapsto z = (z_1 : z_2 : z_3)$, where

$$z_1 = \phi_6[A_2^2\phi_6 + B_3r_1']_{y_1}, \quad z_2 = \psi_6[A_1^2\psi_6 + B_1r_1']_{y_2}, \quad z_3 = \psi_6\phi_6C_5.$$

Apart from the basepoint P_0, the fixed point locus is the order nine curve $K \subset \mathbb{P}^2$ given by the equation

$$\mathcal{K}\,(y) := \psi_6[A_1y_3 + B_1y_1]_{y_2} - \phi_6[A_2y_3 + B_3y_2]_{y_1} = 0.$$

The sextics $\{\phi_6 = 0\}$ and $\{\psi_6 = 0\}$ play a special rôle: they are the loci contracted by the Bertini involution to the basepoints P_1 and P_2, respectively.

4.2. **The map** $\mathbb{P}^2 \dashrightarrow \Sigma_2$ (see [7]). The anti-bicanonical linear system $|-2K_{Y'}|$ is generated by the strict transforms of the sextics $\{\phi_6 = 0\}$, $\{w^2 = 0\}$, $\{ww' = 0\}$, and $\{w'^2 = 0\}$. Hence, in appropriate coordinates $(\bar{z}_0 : \bar{z}_1 : \bar{z}_2 : \bar{z}_3)$ in $\mathbb{P}^3$, the anti-bicanonical map $Y' \to \mathbb{P}^3$, regarded as a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$, is given by

$$\bar{z}_0 = \phi_6(y), \quad \bar{z}_1 = w^2(y), \quad \bar{z}_2 = w(y)w'(y), \quad \bar{z}_3 = w'^2(y).$$

Its image is the cone $\bar{z}_1\bar{z}_3 = \bar{z}_2^2$, which is further rationally mapped to the Hirzebruch surface Σ_2 *via* $\bar{z} \mapsto (\bar{x}, \bar{y})$, $\bar{x} = \bar{z}_1/\bar{z}_2$, $\bar{y} = \bar{z}_0/\bar{z}_2$. Here, $(\bar{x}, \bar{y})$ are *affine* coordinates in Σ_2 such that the exceptional section E is the line $\{\bar{y} = \infty\}$. Summarizing, the composed rational map $\mathbb{P}^2 \dashrightarrow \Sigma_2$ defined by a pair of cubics as in §4.1 is $y \mapsto (\bar{x}, \bar{y})$,

$$\bar{x} = w(y)/w'(y), \quad \bar{y} = \phi_6(y)/w'^2(y).$$

Under this map, the pull-back of a proper section

$$\{\bar{y} = \bar{\mathcal{D}}_2(\bar{x})\}, \quad \bar{\mathcal{D}}_2(\bar{x}) := d_0 + d_1\bar{x} + d_2\bar{x}^2 \tag{4.1}$$

of Σ_2 is the plane sextic D given by the equation

$$\phi_6(x) = d_0 w'^2(x) + d_1 w'(x)w(x) + d_2 w^2(x). \tag{4.2}$$

In particular, $\{\phi_6 = 0\}$ is the pull-back of the section $\{\bar{y} = 0\}$.

The following conventions simplify the few further identities used in the sequel.

Notation 4.3. Given a degree n monomial e in the coefficients $a_1, \ldots, c_2$ of w and an integer $0 \leqslant m \leqslant n$, denote by $\{e\}_m$ the sum of $\binom{n}{m}$ monomials, each obtained from e by replacing m of its n factors with their primed versions. For example,

$$\{a_1c_2\}_1 = a_1c_2' + a_1'c_2, \quad \{b_2^2\}_1 = 2b_2b_2', \quad \{a_1b_1c_1\}_2 = a_1b_1'c_1' + a_1'b_1c_1' + a_1'b_1'c_1.$$

This definition extends to homogeneous polynomials by linearity.

Convention 4.4. Without further notice, we use same small letters to denote the coefficients of a homogeneous bivariate polynomial: $\mathcal{P}_n(t_1, t_2) = \sum_{i=0}^n p_i t_1^i t_2^{n-i}$ for a polynomial $\mathcal{P}_n$ of degree n. With the common abuse of notation, we freely treat homogeneous bivariate polynomials as univariate ones: $\mathcal{P}_n(\bar{x}) := \mathcal{P}_n(\bar{x}, 1)$. This convention corresponds to the passage from homogeneous coordinates $(t_1 : t_2)$ to the affine coordinate $\bar{x} := t_1/t_2$ in the projective line $\mathbb{P}^1$.

Since the strict transform of the sextic $\{\psi_6 = 0\}$ is also an anti-bicanonical curve, there must be a relation $\psi_6 = \phi_6 + \mathcal{S}_2(w, w')$ for a certain homogeneous polynomial $\mathcal{S}_2$ of degree two. Such a relation indeed exists: one has

$$s_0 = a_2c_1 - a_1c_2, \qquad s_i = (-1)^i\{s_0\}_i \quad \text{for } i = 1, 2. \tag{4.5}$$

Hence, $\{\psi_6 = 0\}$ is the pull-back of the section $\{\bar{y} = -\mathcal{S}_2(\bar{x})\}$ of Σ_2.

Assume that the pencil has another basepoint $P_3 \notin (P_0P_1)$ of level zero and normalize its coordinates *via* $u = (1 : u_2 : u_3)$. This point gives rise to another sextic $\{\psi_6^u = 0\}$, the one contracted to P_3 by the Bertini involution. Changing the coordinate triangle to $(P_0P_1P_3)$ and then changing it back to $(P_0P_1P_2)$, one can easily see that $\psi_6^u = \phi_6 + \mathcal{S}_2^u(w, w')$, where

$$\begin{gathered} s_0^u = s_0 + \,(a_2c_2u_2 + \,(a_2b_2 - a_1b_3)\,u_3) \,+ a_2b_3u_2u_3 + a_2^2u_3^2, \\ s_i^u = (-1)^i\{s_0^u\}_i \quad \text{for } i = 1, 2. \end{gathered} \tag{4.6}$$

As above, $\{\psi_6^u = 0\}$ is the pull-back of the section $\{\bar{y} = -\mathcal{S}_2^u(\bar{x})\}$ of Σ_2.

Finally, since $K \subset \mathbb{P}^2$ is the pull-back of the ramification locus $\bar{K} \subset \Sigma_2$ (other than the exceptional section $E \subset \Sigma_2$) and the latter is a proper trigonal curve, there must be a relation

$$\mathcal{K}^2 = -4\phi_6^3 + \phi_6^2\mathcal{P}_2(w, w') + \phi_6\mathcal{Q}_4(w, w') + \mathcal{R}_3^2(w, w'),$$

where $\mathcal{P}_2$, $\mathcal{Q}_4$, and $\mathcal{R}_3$ are homogeneous polynomials of the degrees indicated. (The coefficient (-4) is obtained by comparing the leading terms.) The coefficients of these polynomials are

$$\begin{aligned} r_0 &= -a_1b_2c_2 + a_1b_3c_1 + a_2b_1c_2, \\ q_0 &= 4\,(a_1c_2 - b_1b_3)\,s_0 + 2b_2r_0, \\ p_0 &= b_2^2 - 4a_2c_1 - 4b_1b_3 + 8a_1c_2, \end{aligned} \tag{4.7}$$
$$p_i = (-1)^i\{p_0\}_i, \quad q_i = (-1)^i\{q_0\}_i, \quad r_i = (-1)^i\{r_0\}_i \quad \text{for } i > 0.$$

It follows that the defining equation of the ramification locus $\bar{K} \subset \Sigma_2$ is

$$\bar{\mathcal{K}}(\bar{x}, \bar{y}) := -4\bar{y}^3 + \bar{y}^2\mathcal{P}_2(\bar{x}) + \bar{y}\mathcal{Q}_4(\bar{x}) + \mathcal{R}_3^2(\bar{x}) = 0. \tag{4.8}$$

Remark 4.9. Strictly speaking, most statements in §4.1 and §4.2 hold only if the pencil is sufficiently generic. Most important is the requirement that the pencil should have no basepoints infinitely near to P_0. Otherwise, many expressions above acquire common factors; after the cancellation, the Bertini involution degenerates to the so-called *Geiser involution* and, instead of a map $\mathbb{P}^2 \dashrightarrow \Sigma_2$, we obtain a generically two-to-one map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ ramified at a quartic curve (the anti-canonical map of a nodal del Pezzo surface of degree 2). See [14, 7] for details. Thus, in agreement with the ACC-construction, we always assume that $P_0 = Q_0$ is a simple basepoint of the pencil; the other basepoints may be multiple.

4.3. **An implementation of the ACC-construction.** Together with Moody's formulas, the ACC-construction gives us a relatively simple way to obtain defining equations of sextics with large Milnor number.

Consider the pencil $\mathcal{P}$ generated by a pair of cubics and, as in §4.1, assume that it has at least three level zero basepoints P_0 (necessarily simple), P_1, P_2 at the coordinate vertices and, possibly, some other basepoints P_i, $i \geqslant 3$. The pencil gives rise to a two-to-one rational map $\mathbb{P}^2 \dashrightarrow \Sigma_2$, see §4.2. We make use of the following curves in Σ_2:

- the ramification locus $\bar{K} = \{\bar{\mathcal{K}}(\bar{x}, \bar{y}) = 0\}$, see (4.8);
- the section $L_1 := \bar{P}_1^\top = \{\bar{y} = 0\}$, the image of $\{\phi_6 = 0\}$, see §4.2.
- the section $L_2 := \bar{P}_2^\top = \{\bar{y} = -\mathcal{S}_2(\bar{x})\}$, the image of $\{\psi_6 = 0\}$, see (4.5);
- the sections $L_i := \bar{P}_i^\top = \{\bar{y} = -\mathcal{S}_2^u(\bar{x})\}$, $i \geqslant 3$, if present, see (4.6);
- the section $\bar{D} = \{\bar{y} = \bar{\mathcal{D}}_2(\bar{x})\}$, the image of sextic D to be constructed.

Here, $\bar{\mathcal{D}}_2$ is a degree 2 polynomial as in (4.1); once found, it produces a sextic $D \subset \mathbb{P}^2$ given by the defining equation (4.2).

Remark 4.10. The pull-back of L_1 in the elliptic surface Y splits into two sections interchanged by the deck translation. One of them projects to $\{\phi_6 = 0\}$, which is contracted to P_1 by the Bertini involution, see §4.1. Hence, the other is $\tilde{P}_1^\top$ and L_1 is indeed $\bar{P}_1^\top$. The same argument applies to the other sections L_i.

Let h_i be the multiplicity of the basepoint P_i, $i \geqslant 1$, *i.e.*, the local intersection index of the two cubics at this point. Then any sextic D constructed as above is guaranteed to have a singular point adjacent to $\mathbf{A}_{2h_i-1}$ at P_i, and the construction is indeed the ACC-model of D with $\operatorname{ht} P_i = h_i$. The further degeneration of D depends on the position of $\bar{D}$ with respect to $\bar{K}$ and L_i, $i \geqslant 1$. These degenerations are discussed in details in §5.1. Geometrically, the singularities of D can be understood by running the constraction backwards, *i.e.*, considering the bisection $\tilde{D} \subset Y$ and blowing it down to $\mathbb{P}^2$. Under the assumptions of Convention 3.3, the blow-down map $Y \to \mathbb{P}^2$ is described by Addendum 3.5 and Corollary 3.9. More precisely, for each $i \geqslant 1$, we choose for $\tilde{P}_i^\top$ one of the two components of the strict transform of L_i in Y, so that all components chosen are pairwise disjoint. (The existence of such a choice is guaranteed by the construction; there are two coherent choices interchanged by the deck translation.) If $h_i \geqslant 2$, the section L_i passes through a singular point $\bar{P}_i$ of $\bar{K}$, which is in a certain singular fiber $\bar{F}_i$. The corresponding reducible singular fibers of Y has several components: the strict transform $\tilde{F}_i$ of $\bar{F}_i$ and a number of other components $\tilde{Q}_j$. In these notations, the map $Y \to \mathbb{P}^2$ is the blow-down of all chosen sections $\tilde{P}_i^\top$, followed by the consecutive blow-down of the components $\tilde{Q}_j \neq \tilde{F}_i$ of the reducible singular fibers, starting from the one intersecting $\tilde{P}_i^\top$. The components $\tilde{F}_i$ left uncontracted project to the members of the original pencil of cubics singular at P_i, see §3.2.

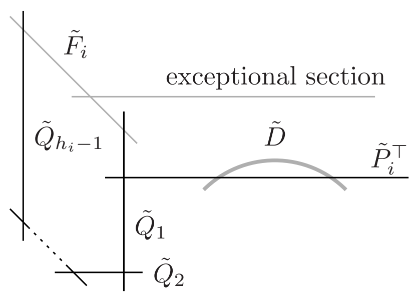

FIGURE 1. The divisors in Y blown down to $P_i \in \mathbb{P}^2$

The divisors $\tilde{P}_i^\top, \tilde{Q}_j \subset Y$ blown down to a single singular point $P_i \in \mathbb{P}^2$ are shown in black in Figure 1, where the components $\tilde{Q}_j$ are numbered in the order that they are contracted.

Remark 4.11. Since we are trying to find maximizing sextics, which are rare, we need to consider families of pencils depending on parameters. It is easy to show that the moduli space of the pencils that have basepoints of multiplicities $h_0 = 1$, $h_1, \ldots, h_r$, $\sum h_i = 9$, has dimension r. This agrees with the dimension of the equisingular stratum of the moduli space of trigonal curves in Σ_2: assuming $\tilde{\mathbf{A}}$ type singular fibers only, this dimension equals $8 - \mu(\bar{K})$.

Remark 4.12. Since we are interested in generic non-special irreducible sextics with double singular points only, see Convention 3.3, some values of the parameters are forbidden, and we use this fact to simplify the equations. Mostly, the following restrictions are used:

1. The cubics $\{w = 0\}$ and $\{w' = 0\}$ are irreducible, see Theorem 3.4 and Addendum 3.5. (In fact, *all* members of the pencil must be irreducible.)
2. No three basepoints of level zero are collinear, see Corollary 2.6(1).

(3) The basepoint multiplicities are as stated, *i.e.*, basepoints do not collide.
(4) The section $\bar{D}$ is distinct from each L_i (as $\{\phi_6 = 0\}$ has a triple point at P_1, and similar for $\{\psi_6 = 0\}$ *etc.*, see [14]).
(5) More generally, $\bar{D}$ does not pass through a singular point of $\bar{K}$, as otherwise D would also have a triple singular point, *cf.* Figure 1.
(6) The section $\bar{D}$ is not tangent to the ramification locus $\bar{K}$ within one of its reducible singular fibers $\bar{F}_i$, see Corollary 2.6(3).

A number of other restrictions are ignored: we merely check the sextics obtained and select those with the desired set of singularities.

5. The computation

In this section, we outline some details of the computation. Most polynomials obtained are too bulky to be reproduced here; they can be downloaded from my web page [8], in both human and machine readable form. (I will extend this manuscript should there be any new development.) Here, we provide information that is just enough to recover the defining polynomials using the formulas in §4.

We use the notation and setup introduced in §4, especially in §4.3.

5.1. **Common equations.** We always assume that the multiplicities h_1, h_2 of the basepoints P_1, P_2 are at least two and choose for $\{w' = 0\}$ and $\{w = 0\}$ the unique members of the pencil that are singular at P_1 and P_2, respectively. Such pairs of cubics (with the necessary number of parameters) are easily constructed by an appropriate triangular Cremona transformation from appropriate pairs of conics.

Under the assumptions, the ramification locus $\bar{K} \subset \Sigma_2$ has singular points $\mathbf{A}_{h_1-1}$ and $\mathbf{A}_{h_2-1}$ over $\bar{x} = \infty$ and $\bar{x} = 0$, respectively, and all sextics obtained have singularities at least $\mathbf{A}_{2h_i-1}$ at P_i, $i = 1, 2$. In the final equations, we change to affine coordinates (x, y) in $\mathbb{P}^2$ so that $P_1(0, \infty)$, $P_2(0, 0)$, and the tangents to D at these points are the lines $\{x = \infty\}$ and $\{y = 0\}$, respectively. This final change of coordinates is indicated below for each pair of cubics.

The zero section L_1 intersects $\bar{K}$ at three double points, $\bar{x} = \mu, \nu, \infty$, and in all cases considered (some of) the parameters present in the equations are expressed rationally in terms of μ, ν. For most equations, we use this re-parameterization.

The further degeneration of D can be described using Figure 1 and the fact that each point of a p-fold, $p \geqslant 2$, intersection of $\bar{D}$ and $\bar{K}$ smooth for $\bar{K}$ gives rise to a type $\mathbf{A}_{p-1}$ singular point of the strict transform $\tilde{D} \subset Y$.

If the section $\bar{D}$ is tangent to L_1,

$$\bar{\mathcal{D}}_2(\bar{x}) = a(\bar{x} - \lambda)^2, \qquad a \in \mathbb{C}^*, \quad \lambda \in \mathbb{C} \smallsetminus \{\mu, \nu\},$$

the $\mathbf{A}_{2h_1-1}$ type singular point P_1 of D degenerates to $\mathbf{A}_{2h_1}$. If $\lambda = \mu$, this point degenerates further to $\mathbf{A}_{2h_1+1}$. In this case, substituting $\bar{y} = a(\bar{x} - \mu)^2$ to the equation $\bar{\mathcal{K}}(\bar{x}, \bar{y}) = 0$ of the ramification locus, we obtain

$$(\bar{x} - \mu)^2 \mathcal{M}_4(\bar{x} - \mu) = 0, \quad \mathcal{M}_4(u) := m_0 + m_1 u + m_2 u^2 + m_3 u^3 + m_4 u^4, \tag{5.1}$$

and the point P_1 degenerates to $\mathbf{A}_{2h_1+1+k}$, $k \geqslant 0$, if

$$m_0 = \ldots = m_{k-1} = 0. \tag{5.2}$$

The first equation $m_0 = 0$ is linear in a; hence, a can be expressed rationally in terms of the other parameters and substituted to the other equations. Geometrically, this equation corresponds to the inflection tangency of $\bar{D}$ and $\bar{K}$.

The degenerations of the other singular point P_2 can be described similarly, by analyzing the intersection of $\bar{D}$ and L_2. Thus, the degeneration $\mathbf{A}_{2h_2-1} \to \mathbf{A}_{2h_2}$ is given by the equation

$$\operatorname{discriminant}(\bar{\mathcal{D}}_2 + \mathcal{S}_2) = 0. \tag{5.3}$$

Other singular points of D are due to the extra tangency of $\bar{D}$ and $\bar{K}$. Assuming that $\bar{\mathcal{D}}_2(\bar{x}) = a(\bar{x}-\mu)^2$, the sextic has an extra singular point if

$$\operatorname{discriminant}(\mathcal{M}_{4-k}) = 0, \quad \mathcal{M}_{4-k} := \mathcal{M}_4/u^k. \tag{5.4}$$

Note, though, that this discriminant may also vanish due to the further degeneration $\mathbf{A}_{2h_2-1} \to \mathbf{A}_{2h_2+1}$ of P_2 or another 'fixed' singular point, if present. The sextic has an extra cusp (typically) if, in addition to (5.4), one has

$$\operatorname{resultant}(\mathcal{M}_{4-k}, \mathcal{M}''_{4-k}) = \operatorname{resultant}(\mathcal{M}'_{4-k}, \mathcal{M}''_{4-k}) = 0. \tag{5.5}$$

Remark 5.6. When simplifying equations and their intermediate resultants, we routinely disregard all factors that would result in forbidden (see Remark 4.12) or otherwise 'unlikely' values of the parameters involved. It is important to notice that, since the classification of sextics is known, we do not need to be too careful not to lose a solution: it suffices to find the right number of distinct curves realizing a given set of singularities. The latter fact is given by Corollary 6.1 below.

Typically, solutions to the equations appear in groups of Galois conjugate ones: all unknowns are expressed as rational polynomials in a certain algebraic number. These groups are referred to as *solution clusters*.

5.2. **The ramification locus $2\mathbf{A}_3$.** We start with a pair of cubics

$$w := \big((\beta-\alpha)\,x_1 + (\beta-\alpha+\alpha\beta)\,x_2\big)\,x_3^2 + \big((\beta-2\alpha)\,x_2x_1 - \alpha x_2^2\big)\,x_3 - \alpha x_1x_2^2,$$
$$w' := \big((\alpha-\beta+\alpha\beta)\,x_1 + (\alpha-\beta)\,x_2\big)\,x_3^2 - \big(\beta x_1^2 - (\alpha-2\beta)\,x_2x_1\big)\,x_3 - \beta x_1^2x_2,$$

where $\alpha, \beta \in \mathbb{C}$, $\alpha, \beta \neq 0$, and $\alpha \neq \beta$. The change of coordinates for the final equations is

$$x_1 = 1-x, \qquad x_2 = y-x, \qquad x_3 = x.$$

The re-parameterization in terms of μ, ν, see §5.1, is as follows

$$\alpha = -\frac{(\mu+1)\,(\nu+1)}{(\mu+2)\,(\nu+2)}, \qquad \beta = -\frac{(\mu+1)\,(\nu+1)}{\mu\nu+\mu+\nu+2},$$

and the equation $m_0 = 0$ (point P_1 adjacent to $\mathbf{A}_{10}$, see (5.2)) yields

$$a = a_1 := \frac{(\mu+\nu+2)\,(\mu-\nu)^2\,(\nu+1)^2}{4\,(\mu+1)\,(\mu+2)^2\,(\nu+2)^4\,(\mu\nu+\nu+\mu+2)}.$$

All sextics below are obtained from sections $\bar{D}$ of the form $\{\bar{y} = a_1(\bar{x}-\mu)^2\}$.

For the set of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_7$, line 12, the two additional equations are $m_1 = m_2 = 0$, see (5.2). They have two solutions

$$\mu = \frac{1}{13}(-4 \pm 6i), \quad \nu = 2\mu - 2, \tag{5.7}$$

producing two complex conjugate sextics.

For the set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$, line 20, the additional equations are (5.4) and (5.5); their solutions are

$$\mu = 2\epsilon, \quad \nu = -10 - 12\epsilon, \qquad \epsilon = \frac{1}{11}(-6 \pm \sqrt{3}). \tag{5.8}$$

Finally, assume $k = 1$ in (5.2) and consider equations (5.3) and (5.4). They have three solution clusters. One of them,

$$\mu = -5, \qquad \nu = -\frac{19}{7}, \tag{5.9}$$

results in a sextic with the set of singularities $\mathbf{D}_9 \oplus \mathbf{A}_{10}$. The two others are

$$\mu = \epsilon, \quad \nu = \frac{1}{5}(-7 + \epsilon), \qquad \epsilon = \frac{1}{2}(-11 \pm 3\sqrt{5}) \tag{5.10}$$

for the set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_9$, line 18 and

$$\mu = \frac{\epsilon}{3}, \quad \nu = \frac{1}{3}(\epsilon^2 + 15\epsilon + 13), \qquad \epsilon^3 + 17\epsilon^2 + 51\epsilon + 43 = 0. \tag{5.11}$$

for the set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$, line 19. In the former case, it is immediate that the line $y = 0$ is not a component of the curve; hence, the curves are irreducible.

5.3. **The ramification locus $\mathbf{A}_5 \oplus \mathbf{A}_1$.** We start with a pair of cubics

$$\begin{aligned} w &:= \bigl(\alpha x_1 - \beta x_2\bigr)\, x_3^2 - \bigl(x_2^2 + (\beta+1)\, x_1 x_2\bigr)\, x_3 - x_1 x_2^2, \\ w' &:= \bigl((\beta + 2\alpha)\, x_1 + \alpha x_2\bigr)\, x_3^2 \\ &\qquad + \bigl((2\alpha+1)\, x_1 x_2 + (2\alpha+\beta+1)\, x_1^2\bigr)\, x_3 + (\alpha+1)\, x_1^2 x_2, \end{aligned}$$

where $\alpha, \beta \in \mathbb{C}$, $\alpha \ne 0, -1$, and $\alpha + \beta \ne 0$. The final change of variables is

$$x_1 = 1 - x, \qquad x_2 = y - \frac{(2\alpha+\beta+1)\, x}{\alpha+1}, \qquad x_3 = x.$$

The re-parameterization in terms of μ, ν, see §5.1, is as follows

$$\alpha = 2 - \mu - \nu, \qquad \beta = \mu\nu - 1, \tag{5.12}$$

and the equation $m_0 = 0$, see (5.2), result in

$$a = a_1 := -\frac{(\mu+\nu-2)\,(\mu-\nu)^2}{4\,(\mu-1)^3\,(\nu-2)}.$$

The first three sextics are obtained from sections $\bar{D}$ of the form $\{\bar{y} = a_1(\bar{x} - \mu)^2\}$.

For the set of singularities $\mathbf{A}_{16} \oplus \mathbf{A}_3$, line 4, the two additional equations are $m_1 = m_2 = 0$, see (5.2). They have two solutions

$$\mu = \epsilon, \quad \nu = \frac{7}{2} - \epsilon, \qquad \epsilon = \frac{1}{32}(59 \pm 3\sqrt{17}). \tag{5.13}$$

These curves were first studied in [3].

For the set of singularities $\mathbf{A}_{15} \oplus \mathbf{A}_4$, line 6, the equations are $m_1 = 0$, see (5.2), and (5.3). They have two solutions

$$\mu = 1 \pm 3i, \quad \nu = \frac{1}{5}(4 - 2\mu). \tag{5.14}$$

These curves and their fundamental groups were studied in [3]. It is easily seen that the conic maximally tangent to a curve at its type $\mathbf{A}_{15}$ point is not a component. Hence, the curves are irreducible.

Consider additional equations (5.3) and (5.4). They have two solution clusters. The first one,

$$\mu = 4 - 3\nu, \quad \nu = \frac{1}{6}(5 \pm i\sqrt{15}), \tag{5.15}$$

results in a sextic with the set of singularities $\mathbf{D}_5 \oplus \mathbf{A}_{14}$. The other one

$$\mu = \epsilon, \quad \nu = \frac{1}{15}(9\epsilon^5 - 9\epsilon^4 - 72\epsilon^3 + 69\epsilon^2 + 172\epsilon - 166),$$
$$9\epsilon^6 - 27\epsilon^5 - 45\epsilon^4 + 195\epsilon^3 - 20\epsilon^2 - 372\epsilon + 276 = 0 \tag{5.16}$$

gives us $\mathbf{A}_{14} \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 7. It is easily seen that $\mathbb{Q}(\epsilon)$ is a purely imaginary extension with Galois group $\mathbb{D}_{12}$.

Next two sextics are obtained from a section $\{\bar{y} = a_2(\bar{x} - \mu)^2\}$, where

$$a_2 := \frac{(\mu + \nu - 2)\left(\mu^2\nu^2 - 6\mu\nu + 4\mu + 4\nu - 3\right)}{4\left(\mu - 1\right)^3}$$

is found from equation (5.3), which is linear in a. Adding equations (5.4) and (5.5), we obtain four solution clusters. One of them corresponds to a non-maximizing sextic, and another one,

$$\mu = 5 - 3\nu, \quad \nu = \frac{1}{6}(7 \pm i\sqrt{3}), \tag{5.17}$$

results in a sextic with the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_{13}$. The two others are

$$\mu = \epsilon, \quad \nu = \frac{1}{7}(25 - 9\epsilon), \qquad \epsilon = \frac{1}{2}(7 \pm \sqrt{21}) \tag{5.18}$$

for the set of singularities $\mathbf{A}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$, line 11 and

$$\mu = \epsilon, \quad \nu = \frac{1}{8}(9\epsilon^3 - 45\epsilon^2 + 73\epsilon - 26),$$
$$9\epsilon^4 - 63\epsilon^3 + 175\epsilon^2 - 224\epsilon + 112 = 0 \tag{5.19}$$

for the set of singularities $\mathbf{A}_{13} \oplus \mathbf{A}_6$, line 10. In the latter case, $\mathbb{Q}(\epsilon)$ is a purely imaginary extension with Galois group $\mathbb{D}_8$.

For the last set of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$, line 13, substitution (5.12) is not used. A section $\bar{D} = \{\bar{y} = a(\bar{x} - \lambda)^2\}$ is tangent to L_2 if and only if

$$a = a_2' := -\frac{\alpha\left(4\alpha + 4\beta - \beta^2\right)}{4\left(\lambda^2\alpha + \lambda^2 + \lambda\beta - 2\lambda + 1\right)},$$

see (5.3), the point of tangency being over

$$\bar{x} = \lambda_2 := -\frac{\lambda\beta - 2\lambda + 2}{2\lambda\alpha + \beta + 2\lambda - 2}.$$

Substitute $\bar{y} = a_2'(\bar{x} - \lambda)^2$ to $\bar{\mathcal{K}}(\bar{x}, \bar{y}) = 0$ and expand the result as $\mathcal{N}_6(\bar{x} - \lambda_2)$, $\mathcal{N}_6(u) := \sum_{i=0}^{6} n_i u^i$. For the $\mathbf{A}_6$ type point P_2, we have $n_0 = 0$ (which also implies $n_1 = 0$) and $n_2 = 0$, and an extra $\mathbf{A}_1$ type point results from the third equation

$$\operatorname{discriminant}(\mathcal{N}_6/u^3) = 0.$$

This is a lengthy computation, and we use a certain 'cheating', *cf.* Remark 5.6: since the curves are expected to be defined over a cubic algebraic number field, in all univariant resultants computed on the way we ignore all irreducible factors of degree other than 3. At the end, we arrive at the following solutions:

$$\alpha = 4\epsilon, \quad \beta = -\frac{1}{27}(252\epsilon^2 + 468\epsilon + 76), \quad \lambda = -\frac{1}{27}(882\epsilon^2 + 1638\epsilon + 329),$$
$$441\epsilon^3 + 315\epsilon^2 + 79\epsilon + 7 = 0. \tag{5.20}$$

Collecting all data, the sextic is the pull-back of the section

$$5103\bar{y} = (441\epsilon^2 + 204\epsilon + 28)(27\bar{x} + 882\epsilon^2 + 1638\epsilon + 329)^2.$$

5.4. **The ramification locus $\mathbf{A}_3 \oplus 2\mathbf{A}_1$.** We start with a pair of cubics

$$\begin{aligned} w &:= \bigl((\beta-\alpha)\,x_1 + \bigl(\alpha\beta - 2\alpha + 1\bigr)\,x_2\bigr)\,x_3^2 \\ &\quad + \bigl((\alpha\beta\rho - 2\alpha\rho + 1)\,x_1x_2 + (\alpha - \alpha\rho)\,x_2^2\bigr)\,x_3 + \bigl(\alpha\rho - \alpha\rho^2\bigr)\,x_1x_2^2, \\ w' &:= \bigl((\alpha\beta - 2\beta + 1)\,x_1 + (\alpha - \beta)\,x_2\bigr)\,x_3^2 \\ &\quad + \bigl((\alpha\beta\rho - 2\beta\rho + 1)\,x_1^2 + (-2\beta\rho + \alpha + \alpha\rho)\,x_1x_2\bigr)\,x_3 + \bigl(\alpha\rho - \beta\rho^2\bigr)\,x_1^2x_2, \end{aligned}$$

where $\alpha, \beta, \rho \in \mathbb{C}$, $\alpha \neq 0, 1$, $\beta \neq 0, 1$, $\rho \neq 0, 1$, $\alpha \neq \beta$, and $\beta\rho \neq 1$. The final change of variables is

$$x_1 = 1 - \frac{x}{\rho}, \qquad x_2 = y + \frac{(\alpha\beta\rho - 2\beta\rho + 1)\,x}{\rho\,(\beta\rho - \alpha)}, \qquad x_3 = x.$$

This pencil of cubics has another 2-fold basepoint $P_3(u)$,

$$u = (1 : u_2 : u_3) := \left(1 : -\frac{\beta\rho - 1}{\rho - 1} : -\frac{\beta\rho - 1}{\beta - 1}\right),$$

resulting in an $\mathbf{A}_1$ type point of $\bar{K}$ over $\bar{x} = (\rho - 1)/(\beta\rho - 1)$. The corresponding section is $L_3 := \bar{P}_3^{\top} = \{\bar{y} + \mathcal{S}_2^u(\bar{x}) = 0\}$, see (4.6).

The re-parameterization in terms of μ, ν, see §5.1, is as follows

$$\beta = \frac{\mu\nu\alpha + 2\mu\alpha + 2\nu\alpha - \mu - \nu + 3\alpha - 2}{\mu\nu + \mu\alpha + \nu\alpha + 2\alpha - 1}, \qquad \rho = \frac{\mu\nu + \mu\alpha + \nu\alpha + 2\alpha - 1}{(\mu + \alpha)\,(\nu + \alpha)},$$

and the equation $m_0 = 0$, see (5.2), results in

$$a = a_1 := \frac{\alpha\,(\alpha - 1)^4\,(\mu + \nu + 2)\,(\mu - \nu)^2}{4\,(\mu + 1)^2\,(\nu + 2)\,(\mu + \alpha)^3\,(\nu + \alpha)^2}.$$

The first six sextics, with the sets of singularities adjacent to $\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$, are obtained from sections $\bar{D} = \{\bar{y} = a_1(\bar{x} - \mu)^2\}$, with β, ρ as above and

$$\alpha = \frac{(\mu\nu - 1)^2\,(\nu + 2)}{4\mu\nu^2 + 10\mu\nu + \mu^2 + 5\nu^2 + 8\mu + 12\nu + 8}$$

found from (5.3). We need two more equations for the two parameters μ, ν left.

For the set of singularities $\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$, line 14, the two extra equations are $m_1 = m_2 = 0$, see (5.2). Their only solution is

$$\mu = -\frac{33}{13}, \quad \nu = -\frac{29}{39}. \tag{5.21}$$

For the set of singularities $\mathbf{A}_{11} \oplus 2\mathbf{A}_4$, line 16, we have $m_1 = 0$, see (5.2), and

$$\operatorname{discriminant}(\bar{\mathcal{D}}_2 + \mathcal{S}_2^u) = 0, \tag{5.22}$$

cf. (5.3). These equations have four solutions

$$\begin{gathered} \mu = -\frac{1}{11}(10\epsilon^3 + 70\epsilon^2 + 141\epsilon + 109), \quad \nu = \epsilon, \\ 50\epsilon^4 + 300\epsilon^3 + 685\epsilon^2 + 720\epsilon + 302 = 0. \end{gathered} \tag{5.23}$$

Analysing the discriminant, one can easily guess that the splitting field of the above minimal polynomial for ϵ is $\mathbb{Q}(\sqrt{2}, i\sqrt{15})$. An extra change of variables, making the two $\mathbf{A}_4$ type points complex conjugate, takes the four curves into two real ones, defined and Galois conjugate over $\mathbb{Q}(\sqrt{2})$.

The pair of equations (5.4) and (5.5) has six solution clusters. One is

$$\mu = -\frac{1}{3}, \quad \nu = \frac{1}{7}(-14 \pm i\sqrt{7}) \tag{5.24}$$

for the set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_3$, line 21, and another one is

$$\mu = \frac{17}{11}, \quad \nu = \frac{41}{11} \tag{5.25}$$

for $\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$, line 25. Two solution clusters produce non-maximizing sextics, and the two others are

$$\mu = \frac{1}{6}(-13 \pm \sqrt{33}), \quad \nu = 3\mu + 2 \tag{5.26}$$

for the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_3$ and

$$\mu = \epsilon, \quad \nu = \frac{1}{5}(-\epsilon^2 + 4\epsilon + 20), \qquad \epsilon^3 + \epsilon^2 - 10\epsilon + 10 = 0 \tag{5.27}$$

for the set of singularities $\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_4$.

The pair of equations (5.4) and (5.22) has, among others, solutions

$$\mu = \frac{1}{3}(-13 + 7\nu), \quad \nu = \frac{1}{7}(-11 \pm 3\sqrt{15}), \tag{5.28}$$

resulting in the set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$, line 23, and

$$\begin{gathered} \mu = -\frac{53025}{51748}\nu^5 - \frac{14425}{3044}\nu^4 - \frac{417315}{51748}\nu^3 - \frac{440835}{51748}\nu^2 - \frac{96864}{12937}\nu - \frac{59839}{12937}, \\ 875\nu^6 + 5375\nu^5 + 13375\nu^4 + 18025\nu^3 + 14770\nu^2 + 7180\nu + 1592 = 0, \end{gathered} \tag{5.29}$$

resulting in $\mathbf{A}_{10} \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$, line 24. The minimal polynomial for ν is reducible over $\mathbb{Q}(\omega)$, $\omega := i\sqrt{55}$. Furthermore, one can easily see that

$$\nu = \frac{1}{770}(21\omega\epsilon^2 + 87\omega\epsilon + 385\epsilon + 35\omega), \qquad 7\epsilon^3 + 43\epsilon^2 + 77\epsilon + 49 = 0, \tag{5.30}$$

and another change of variables in $\mathbb{P}^2$, making the two $\mathbf{A}_4$ points complex conjugate, converts the six sextics into three ones, defined and Galois conjugate over $\mathbb{Q}(\epsilon)$.

The three other solutions to (5.4) and (5.22) provide an alternative representation of $\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$ and two alternative representations for $\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_4$.

5.5. **The ramification locus $\mathbf{A}_3 \oplus 2\mathbf{A}_1$ (continued).** For the remaining five sets of singularities, we start with the same pair of cubics as in §5.4 and observe that ρ can be expressed rationally in terms of the $\bar{x}$-coordinate λ of one of the points of tangency of $\bar{K}$ and L_3:

$$\rho = \frac{(1 + \lambda\beta)\,(\lambda\beta\alpha + \lambda\alpha - 2\lambda\beta + 2\alpha - \beta - 1)}{\beta\,(\lambda + 1)\,(\lambda\beta\alpha + \lambda\alpha - 2\lambda\beta - \beta\alpha + 3\alpha - 2)}.$$

In all five cases, P_3 is adjacent to $\mathbf{A}_6$; hence, $\bar{\mathcal{D}}_2(\bar{x}) = -\mathcal{S}_2^u(\bar{x}) + a(\bar{x} - \lambda)^2$. Then, substituting $\bar{y} = \bar{\mathcal{D}}_2(\bar{x})$ to $\bar{\mathcal{K}}(\bar{x}, \bar{y}) = 0$, we obtain

$$(\bar{x} - \lambda)^2 \mathcal{M}_4(\bar{x} - \lambda) = 0$$

for a certain polynomial $\mathcal{M}_4$ of degree four, *cf.* (5.1), and the coefficient a is found from the equation $m_0 = 0$. (The expression is too bulky to be reproduced here.) In all five cases, we also have equation (5.3) making P_2 adjacent to $\mathbf{A}_4$.

For the set of singularities $\mathbf{A}_8 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4$, line 30, we have additional equations

$$\operatorname{discriminant}(\bar{\mathcal{D}}_2) = 0 \tag{5.31}$$

(P_1 is adjacent to $\mathbf{A}_8$) and $m_1 = 0$ (P_2 is adjacent to $\mathbf{A}_7$). The solutions are

$$\alpha = \frac{1}{15}(27 - 14\epsilon), \quad \beta = -\frac{1}{45}(64 + 23\epsilon), \quad \lambda = \frac{1}{37}(15 + 90\epsilon), \qquad \epsilon = \pm i. \tag{5.32}$$

In the other four cases, we use a 'cheating' as above: since the curves are expected to be defined over algebraic number fields of degree two or three, we precompute univariate resultants and ignore their factors of degree greater than four. (In the case $\mathbf{A}_7 \oplus 2\mathbf{A}_6$, line 34, the presence of the two $\mathbf{A}_6$ points treated 'asymmetrically' may and does increase the field of definition.)

Equations (5.31) and (5.4), $k = 1$, have four solution clusters. One of them is

$$\begin{gathered} \alpha = \frac{1}{2576595}\left(-820\epsilon^2 + 559955\epsilon + 3862092\right), \quad \beta = \frac{\epsilon}{9}, \\ \lambda = \frac{15}{1098463796}\left(-44995\epsilon^2 + 31556708\epsilon - 151837233\right), \\ 5\epsilon^3 - 3495\epsilon^2 + 8047\epsilon - 10925 = 0 \end{gathered} \tag{5.33}$$

for the set of singularities $\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$, line 31, and another is

$$\begin{gathered} \alpha = \frac{1}{226590}\left(-20121\epsilon^2 + 1110632\epsilon + 22549\right), \quad \beta = 4\epsilon, \\ \lambda = \frac{1}{5395}\left(61959\epsilon^2 - 3470518\epsilon + 41949\right), \\ 57\epsilon^3 - 3196\epsilon^2 + 221\epsilon - 7 = 0 \end{gathered} \tag{5.34}$$

for the set of singularities $\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$, line 27. The two others are

$$\alpha = -\frac{13}{7}, \quad \beta = 91, \quad \lambda = -\frac{1}{13} \tag{5.35}$$

for the set of singularities $\mathbf{D}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$ and

$$\alpha = \frac{1}{21}(7 \pm 2i\sqrt{7}), \quad \beta = \frac{1}{66}(49\alpha - 7), \quad \lambda = -\frac{3}{4}(21\alpha + 11) \tag{5.36}$$

for the set of singularities $\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_6$.

Finally, consider (5.3) and the equations

$$3m_2m_4 = m_3^2, \qquad 3m_1m_3 = m_2^2, \qquad 9m_1m_4 = m_2m_3.$$

(This is a simplified version of (5.4) and (5.5), stating that a cubic polynomial is a perfect cube.) They have three solution clusters:

$$\alpha = \frac{2}{7}(2 \pm i\sqrt{3}), \quad \beta = \frac{1}{28}(19\alpha - 6), \quad \lambda = \frac{1}{4}(7\alpha - 22), \tag{5.37}$$

resulting in the set of singularities $\mathbf{E}_6 \oplus \mathbf{A}_7 \oplus \mathbf{A}_6$,

$$\alpha = \frac{2}{13}(40\beta + 9), \quad \beta = \frac{1}{50}(57 \pm 13\sqrt{21}), \quad \lambda = \frac{1}{39}(25\beta - 22), \tag{5.38}$$

resulting in the set of singularities $\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$, line 35, and

$$\begin{gathered} \alpha = \frac{2}{27}(7\beta^3 + 42\beta^2 + 66\beta + 20), \quad \lambda = \frac{1}{54}(49\beta^3 + 203\beta^2 + 203\beta - 104), \\ 49\beta^4 + 245\beta^3 + 357\beta^2 + 56\beta + 22 = 0. \end{gathered} \tag{5.39}$$

resulting in $\mathbf{A}_7 \oplus 2\mathbf{A}_6$, line 34. In this latter case, the minimal polynomial for β becomes reducible over $\mathbb{Q}(i\sqrt{7})$, and an extra change of variables converts the four curves found into two complex conjugate curves defined over $\mathbb{Q}(i\sqrt{7})$.

6. Proofs

In this concluding section, we outline the proofs of the principal theorems stated in the introduction. For Theorem 1.1, we suggest two slightly different proofs. The one in §6.1 would work, with appropriate modifications and computation, for any maximizing sextic, with or without triple singular points. The other one, see §6.2, is more limited, but it reveals additional information (rather negative) about the dessin of the cubic resolvent of a sextic, see Remark 6.3.

6.1. **Proof of Theorem 1.1.** The fields $\Bbbk$ are described together with the equations of the curves, and the computation of their Galois groups is straightforward. For the minimality, we construct a projective invariant $J \in \mathbb{C}$ of (some) maximizing sextics, depending rationally on the coefficients of their defining polynomials.

We use the following obvious observation: given a polynomial $d \in \mathbb{C}[x]$, the product of all linear factors of d of the same given multiplicity is defined over the same field as d. Each curve in question has a distinguished singular point P_1 of type $\mathbf{A}_m$, $m \geqslant 3$: for example, we can choose the point of the maximal Milnor number. Let $\mathcal{D} \in \Bbbk[x, y]$ be a defining polynomial of the curve in some coordinate system. Applying the above observation to the discriminant of $\mathcal{D}$ with respect to y (and then to the restriction to the corresponding fiber $x = \text{const}$), we conclude that the coordinates of P_1 are in $\Bbbk$. Hence, up to the action of $PGL(3, \Bbbk)$, we can assume that P_1 is $(\infty, 0)$ and that the tangent to D at P_1 is the line $x = \infty$.

Let $d \in \Bbbk[x]$ be the discriminant of the new defining polynomial with respect to y. It has six distinct roots. (This is a common count for any maximizing sextic with $\mathbf{A}$-type singular points only, provided that all its singular fibers are maximally generic.) All curves considered have two to four singular points, each but P_1 contributing a multiple root to d. Hence, d has three to five simple roots; let $\tilde{d}(x) = \sum d_i x^i$ be the product of the corresponding linear factors. As explained above, $\tilde{d} \in \Bbbk[x]$, and one can take for J any rational function of its coefficients d_i invariant under the action of the group of affine linear transformations $x \mapsto ax + b$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$. For our purposes, the following invariants are sufficient:

- if $\deg \tilde{d} = 3$, then $J = J_3$ is the j-invariant of ∞ and the three roots of $\tilde{d}$;
- if $\deg \tilde{d} = 4$, then $J = J_4$ is the j-invariant of the four roots of $\tilde{d}$;
- if $\deg \tilde{d} = 5$, then $J = J_5 := \bigl(5d_3d_5 - 2d_4^2\bigr)^{10} / \bigl(5^{10} d_5^{12} \operatorname{discriminant}(\tilde{d})\bigr)$.

Explicitly, in the former two cases one has

$$J_3 = -\frac{4\left(3d_1d_3 - d_2^2\right)^3}{27d_3^2\Delta}, \qquad J_4 = \frac{4\left(d_2^2 + 12d_0d_4 - 3d_1d_3\right)^3}{27\Delta},$$

where $\Delta := \operatorname{discriminant}(\tilde{d})$. By construction, one has $J \in \Bbbk$ and J does not depend on the choice of coordinates or particular defining equation.

Now, on the case by case basis, one can check that, for each curve considered in §5, the invariant J is well defined, *i.e.*, the singular fibers are maximally generic. In most cases, the field $\Bbbk$ obtained in the computation equals $\mathbb{Q}(J)$, and this fact concludes the proof. The exceptional cases are $\mathbf{A}_{11} \oplus 2\mathbf{A}_4$, line 16, $\mathbf{A}_{10} \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$, line 24, and $\mathbf{A}_7 \oplus 2\mathbf{A}_6$, line 34. Each of these curves has a pair of isomorphic singular points, which are treated asymmetrically by the construction, and the field obtained is twice as large as predicted. In each case, an extra change of variables reduces the field of definition to $\mathbb{Q}(J)$. □

Corollary 6.1 (of the proof)**.** *All sextics obtained in §5 are pairwise distinct. As a consequence, all sextics listed in Table 1 are present in §5.*

Proof. Within each set of singularities, the curves differ by the value of J, regarded as a complex number. (Since the curves are Galois conjugate, the values $J \in \Bbbk$ in the abstract field of definition are equal.) □

6.2. **An alternative proof *via dessins d'enfants*.** In this section, we discuss another projective invariant $j_0 := j_0(D)$ with the same property as above: $\mathbb{Q}(j_0) = \Bbbk$ is the minimal field of definition of D. (This property is easily verified by a direct case-by-case computation.)

As above, pick a distinguished singular point P_1 of type $\mathbf{A}_m$, $m \geqslant 3$, for example, the one of the maximal Milnor number. Consider the plane $\mathbb{P}^2(P_1)$ blown up at P_1: it is a Hirzebruch surface Σ_1, and the strict transform of D is a tetragonal curve intersecting the exceptional section at a single point, which is a singular point of type $\mathbf{A}_{m-2}$. Blowing this point up and blowing down the corresponding fiber, we convert $\mathbb{P}^2(P_1)$ to a Hirzebruch surface Σ_2; the strict transform of D is a *proper* (*i.e.*, disjoint from the exceptional section) tetragonal curve $\tilde{D} \subset \Sigma_2$. This curve has a *cubic resolvent* $C \subset \Sigma_4$: if $\tilde{D}$ is given by a *reduced* equation

$$\mathcal{D}(x, y) := y^4 + p(x)y^2 + q(x)y + r(x) = 0,$$

then C is the proper trigonal curve given by

$$\mathcal{C}(x, y) := y^3 - 2p(x)y^2 + b_1(x)y + q(x)^2 = 0, \qquad b_1 := p^2 - 4r.$$

One can see that C is equipped with a distinguished section $L := \{y = 0\}$ that splits into two components in the covering elliptic surface. Furthermore, $\tilde{D}$ is recovered from the pair (C, L) uniquely up to the transformation $(y, x) \mapsto (-y, x)$.

Associated to C is its *functional j-invariant*

$$j(x) = \frac{4(p^2 + 12r)^3}{27\,\mathrm{discriminant}(\mathcal{C}, y)};$$

it is a rational map $\mathbb{P}^1 \to \mathbb{C} \cup \{\infty\}$. The graph $j^{-1}(\mathbb{R} \cup \{\infty\}) \subset \mathbb{P}^1$, decorated as shown in Figure 2, is called the *dessin* of C. This construction appears in a number

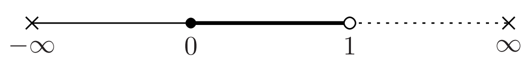

FIGURE 2. The decoration of the dessin

of places; a detailed exposition and further references can be found in [9]. Typically, the •- and ∘-vertices of the dessin correspond to *nonsingular* fibers of C and have valency six and four, respectively, whereas each ×-vertex of valency $2p$ corresponds to a singular fiber of Kodaira's type I_p. The dessin may also have *monochrome* vertices, *viz.* the critical points of j with *real* critical values other that 0, 1, or ∞.

It is easily seen that the total Milnor number of C is $\mu(C) = \mu(D) - 2$ (assuming $\mathbf{A}$ type singularities only). Thus, if D is maximizing, C is at most one unit short of being a so-called *maximal* trigonal curve, see [9]. It follows that j has at most one critical point with critical value $j_0 \neq 0, 1, \infty$; by definition, this critical value j_0, if defined, is the invariant being constructed. □

Remark 6.2. The computation shows that the invariant $j_0(D)$ is well defined (and has the property $\mathbb{Q}(j_0) = \Bbbk$) for all maximizing sextics with known equations except $(\mathbf{A}_{17} \oplus \mathbf{A}_2)$ (torus type) and $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$ ($\mathbb{D}_{10}$-special). In the two offending cases, the $\tilde{\mathbf{A}}_2$ type singular fiber of C degenerates to $\tilde{\mathbf{A}}_2^*$ and C is maximal.

Remark 6.3. The fact that $\mathbb{Q}(j_0) = \Bbbk$ proves also a certain negative result. As explained above, typically, the trigonal curve $C \subset \Sigma_4$ associated to a maximizing sextic D is almost maximal but not maximal. On the other hand, this curve is equipped with a distinguished section L splitting in the covering elliptic $K3$-surface, so that the latter has maximal Picard rank, and this fact makes the pair (C, L) rigid and defined over an algebraic number field. One might expect that the existence of such a section would manifest itself in the combinatorial properties of the dessin of C, *e.g.*, in the presence of a monochrome vertex, so that maximizing sextics with $\mathbf{A}$ type singular points only could also be studied in purely combinatorial terms. However, this is not so: since $\mathbb{Q}(j_0) = \Bbbk$ is the minimal field of definition, the only critical value j_0 that could result in a monochrome vertex is non-real whenever the chosen embedding $\Bbbk \hookrightarrow \mathbb{C}$ is non-real.

6.3. **Proof of Theorem 1.2.** All groups are computed as explained in [10], using real curves and applying the Zariski–van Kampen theorem to real singular fibers only. In the three exceptional cases, the presentations obtained are incomplete and the computation is inconclusive. Further details are found in [8].

Two cases need special attention. One is the set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$, line 20. This set is realized by two real sextics D_1, D_2, one having one pair of complex conjugate singular fibers, the other having two. For the first curve, we have $\pi_1(\mathbb{P}^2 \smallsetminus D_1) = \mathbb{Z}_6$. For the other one, the presentation is incomplete and the computation only gives us an epimorphism $G \twoheadrightarrow \pi_1(\mathbb{P}^2 \smallsetminus D_2)$, where G fits into a short exact sequence

$$1 \to SL(2, \mathbb{F}_5) \to G \to \mathbb{Z}_6 \to 1.$$

(In other words, $[G, G] = SL(2, \mathbb{F}_5)$, as found by `GAP`.) Hence, the group $\pi_1(\mathbb{P}^2 \smallsetminus D_2)$ is finite. On the other hand, since the two curves are Galois conjugate, the profinite completions of their fundamental groups are isomorphic, and we conclude that $\pi_1(\mathbb{P}^2 \smallsetminus D_2) = \mathbb{Z}_6$.

Alternatively, D_2 can be projected from its $\mathbf{A}_7$ type point. This projection has only one pair of complex conjugate singular fibers, and we obtain a complete presentation confirming that $\pi_1(\mathbb{P}^2 \smallsetminus D_2) = \mathbb{Z}_6$.

The other special case is the set of singularities $\mathbf{A}_{10} \oplus \mathbf{A}_9$, line 18 realized by two real curves. For one of them, the presentation obtained using the projection from the $\mathbf{A}_{10}$ type point (as the equations suggest) is incomplete. However, projecting from the $\mathbf{A}_9$ type point, we conclude that both groups are abelian. ☐

6.4. **Proof of Theorem 1.8.** We need to show that, within each pair (D_1, D_2), the spaces $\mathbb{P}^2 \smallsetminus D_i$, $i = 1, 2$, are not properly homotopy equivalent. (The sextics realizing each of the sets of singularities $\mathbf{A}_{18} \oplus \mathbf{A}_1$ or $\mathbf{A}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ are also Galois conjugate; this fact is proved in [3].)

Let $D \subset \mathbb{P}^2$ be an irreducible sextic and consider the complement $\mathbb{P}^2 \smallsetminus D$. Since $H_1(\mathbb{P}^2 \smallsetminus D) = \mathbb{Z}_6$, there is a unique double covering $X^\circ := X_D^\circ \to \mathbb{P}^2 \smallsetminus D$. It is an oriented 4-manifold; hence, the group $H_2(X^\circ)$ is naturally equipped with the intersection index form $H_2(X^\circ) \otimes H_2(X^\circ) \to \mathbb{Z}$. The quotient $\mathbf{T}_D := H_2(X^\circ)/\ker$ (where $\ker H_2 = H_2^\perp$ stands for the kernel of the form) is a nondegenerate integral lattice. Obviously, up to sign (a choice of orientation, *i.q.* a generator for the group $H^4_c(X) = \mathbb{Z}$), this lattice is preserved by proper homotopy equivalences of the complement $\mathbb{P}^2 \smallsetminus D$. In the case of maximizing sextics, the sign is determined by the requirement that $\mathbf{T}_D$ should be positive definite.

Let $X \to \mathbb{P}^2$ be the covering $K3$-surface of D, see §2.2, and let $D' \subset X$ be the preimage of D. Then $X^\circ = X \smallsetminus D'$ and, by Poincaré–Lefschetz duality, $H_2(X^\circ) = H^2(X, D')$. From the exact sequence

$$H^1(X) = 0 \to H^1(D') \to H^2(X, D') \to H^2(X) \to H^2(D') \to \ldots$$

of pair (X, D') we conclude that the invariant $\mathbf{T}_D$ is isomorphic to the orthogonal complement of $H_2(D')$ in $H_2(X)$. Since D' contains all exceptional divisors and the divisorial pull-back of D in X is equivalent to $6h$, the primitive hulls $H_2(D')\tilde{}$ and $(\mathbf{S} \oplus \mathbb{Z}h)\tilde{}$, see §2.2, coincide; hence, $\mathbf{T}_D = (\mathbf{S} \oplus \mathbb{Z}h)^\perp$. The latter orthogonal complement is often called the *transcendental lattice* of D; it plays an important rôle in the classification of plane sextics.

Now, consulting Shimada's tables [20], one can see that, in each pair as in the statement, the two curves do differ by their transcendental lattices. □

6.5. **The homotopy type of the complement.** In §6.4, as well as in §6.6 below, we have to speak about *proper* homotopy equivalence only. The reason is the fact that the definition of the intersection index form uses Poincaré duality which, in the case of non-compact manifolds, involves cohomology with compact supports. In general, this form does *not* need to be a homotopy invariant. As an example, we show that the complements of most maximizing sextics are homotopy equivalent.

Following [13], denote by P_m the *pseudo-projective plane* of degree m: this space is obtained by adjoining a 2-cell e^2 to the circle S^1 *via* a degree m map $\partial e^2 \to S^1$.

Proposition 6.4. *Let $D \subset \mathbb{P}^2$ be a maximizing sextic with $\pi_1(\mathbb{P}^2 \smallsetminus D) = \mathbb{Z}_6$. Then there is a homotopy equivalence $\mathbb{P}^2 \smallsetminus D \sim P_6 \vee S^2$.*

Proof. The complement $\mathbb{P}^2 \smallsetminus D$ is a Stein manifold; hence, it has homotopy type of a *CW*-complex of dimension 2. Since the group $\pi_1(\mathbb{P}^2 \smallsetminus D) = \mathbb{Z}_6$ is finite cyclic, from [13] one has $\mathbb{P}^2 \smallsetminus D \sim P_6 \vee S^2 \vee \ldots \vee S^2$, where the number of copies of S^2 equals the Betti number $b_2(\mathbb{P}^2 \smallsetminus D)$. Using Poincaré–Lefschetz duality, exact sequence of pair $(\mathbb{P}^2, D)$, the fact that D is irreducible, and the additivity of the topological Euler characteristic χ, we obtain

$$b_2(\mathbb{P}^2 \smallsetminus D) = b_1(D) = 2 - \chi(D) = 2g - \mu(D) = 1,$$

where $g = 10$ is the genus of a nonsingular sextic. □

The proof of the following generalization is literally the same.

Proposition 6.5. *Let $D \subset \mathbb{P}^2$ be a plane curve of degree m with $\pi_1(\mathbb{P}^2 \smallsetminus D) = \mathbb{Z}_m$. Then there is a homotopy equivalence $\mathbb{P}^2 \smallsetminus D \sim P_m \vee S^2 \vee \ldots \vee S^2$, where the number of copies of the 2-sphere S^2 equals $(m-1)(m-2) - \mu(C)$.* ◁

Proposition 6.5 explains why π_1-equivalent Zariski pairs on irreducible curves are so difficult to construct: if the fundamental group is abelian, the complements are not distinguished by the conventional homotopy invariants. In this respect, maximizing plane sextics are indeed very special, as their transcendental lattices $\mathbf{T}$ are positive definite and thus provide additional invariants. (The isomorphism class of an indefinite lattice is usually determined by its signature and discriminant form, and these invariants can be computed in terms of the combinatorial type.) Another important class of curves with a similar property are the so-called *maximal trigonal curves* in Hirzebruch surfaces, see [9].

Remark 6.6. In a forthcoming paper, we will show that, with as few as about a dozen of exceptions, the fundamental group of a non-special irreducible simple sextic D is $\mathbb{Z}_6$. Hence, in most cases the homotopy type of the complement $\mathbb{P}^2 \smallsetminus D$ is completely determined by $\mu(D)$.

6.6. **Proof of Theorem 1.9.** Given a *non-special* irreducible sextic D, in addition to X°, see §6.4, consider the double covering $\bar{X}^\circ \to \mathbb{P}^2 \smallsetminus \operatorname{Sing} D$ ramified at $D \smallsetminus \operatorname{Sing} D$. Let $E \subset X$ be the union of the exceptional divisors in the covering $K3$-surface, see §2.2. Then we have Poincaré–Lefschetz duality $H_2(\bar{X}^\circ) = H^2(X, E)$ and exact sequence

$$H^1(E) = 0 \to H^2(X, E) \to H^2(X) \to H^2(E) = \mathbf{S}.$$

It follows that $H_2(\bar{X}^\circ) = \mathbf{S}^\perp$ is a non-degenerate lattice and the inclusion $X^\circ \hookrightarrow \bar{X}^\circ$ induces a primitive embedding $\mathbf{T}_D \hookrightarrow \mathbf{S}^\perp$, *cf.* §6.4. Thus, the primitive lattice extension $\mathbf{S}^\perp \supset \mathbf{T}_D$ (considered up to sign) is a proper homotopy equivalence invariant of pairs (1.10). The orthogonal complement of $\mathbf{T}_D$ in $\mathbf{S}^\perp$ is $\mathbb{Z}h$, see §2.2.

Now, using Nikulin's theory of discriminant forms, see [15], and the assumption that $\mathbf{S} \oplus \mathbb{Z}h \subset \mathbf{L}$ is a primitive sublattice, see Theorem 2.2, it is easy to show that the isomorphism classes of primitive lattice extensions as above are in a one-to-one correspondence with the isomorphism classes of pairs $(\mathbf{T}_D, v \bmod 2\mathbf{T}_D)$, where $v \in \mathbf{T}_D$ is a vector such that $v \cdot \mathbf{T}_D \in 2\mathbb{Z}$ and $v^2 = 6 \bmod 8$. (The lattice $\mathbf{S}^\perp$ is the index 2 extension $(\mathbf{T}_D \oplus \mathbb{Z}h) + \mathbb{Z}v'$, where $v' := \frac{1}{2}(h + v) \in (\mathbf{T}_D \oplus \mathbb{Z}h) \otimes \mathbb{Q}$; the arithmetical details are left to the reader.)

Using Shimada's tables [20] again, we see that $\mathbf{T}_D = \mathbb{Z}a \oplus \mathbb{Z}b$, where

- $a^2 = 6$, $b^2 = 70$ for $\mathbf{A}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$, line 11,
- $a^2 = 22$, $b^2 = 30$ for $\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$, line 23, and
- $a^2 = 30$, $b^2 = 70$ for $\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$.

In each case, there are two $O(\mathbf{T}_D)$-orbits of classes $(v \bmod 2\mathbf{T}_D)$ as above, *viz.* those of $(a \bmod 2\mathbf{T}_D)$ and $(b \bmod 2\mathbf{T}_D)$. These orbits distinguish the two sextics realizing $\mathbf{S}$. □

Remark 6.7. According to [6], two simple sextics $D_1, D_2 \subset \mathbb{P}^2$ have isomorphic oriented homological types if and only if the pairs $(\mathbb{P}^2, D_i)$, $i = 1, 2$, are related by an orientation preserving diffeomorphism subject to a certain regularity condition at the singular points. Diffeomorphisms of 4-manifolds are a delicate subject, and the regularity at the singular points is a subtle technical condition (roughly, it is required that the structure of the exceptional divisors should be preserved). The reader may observe that what is essentially done in §6.4 and §6.6 is merely an attempt to extract *topological* invariants from the homological types.

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