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Characterizing killing vector fields of standard static space-times

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ABSTRACT

We provide a global characterization of the Killing vector fields of a standard static space-time by a system of partial differential equations. By studying this system, we determine all the Killing vector fields in the same framework when the Riemannian part is compact. Furthermore, we deal with the characterization of Killing vector fields with zero curl on a standard static space-time.

1. Introduction

The main concern of the current paper is to study the existence and characterization of Killing vector fields (KVF for short) of a standard static space-time (SSS-T for short).[1] Our approach partially follows that of Sánchez for Robertson–Walker space-times in [2], which is centrally supported by the structure of KVFs on warped products of pseudo-Riemannian manifolds, already obtained in the pioneering article of Bishop and O'Neill [3].

A *standard static space-time* (also called *globally static*, see [4]) is a Lorentzian warped product where the warping function is defined on a Riemannian manifold (called the *natural space* or *Riemannian part*) and acting on the negative definite metric on an open interval of real numbers (see Definition 3.2). This structure can be considered as a generalization of the Einstein static universe. In [5], it was shown that any static space-time[2] is *locally* isometric to a standard static one. There are many interesting and recent studies about several questions in SSS-Ts, see for instance [10–12,1,13–18] and references therein.

The existence of KVFs on pseudo-Riemannian manifolds was considered by many researchers (physicists [19] and mathematicians) from several points of view and by using different techniques. One of the first articles by Sánchez (i.e., [20]) is devoted to provide a review about these questions in the framework of Lorentzian geometry. In [2], Sánchez studied the structure of KVFs on a generalized Robertson–Walker space-time. He obtained necessary and sufficient conditions for a

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1 We would like to inform the reader that some of the results provided in this article were previously announced in the survey [1].

2 An n-dimensional space-time (M, g) is called *static* if there exists a nowhere vanishing time-like KVF X on M such that the distribution of $(n-1)$-plane orthogonal to X is integrable (see [6, Section 3.7] and also the general relativity texts [7–9]).

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vector field to be Killing on generalized Robertson–Walker space-times and gave a characterization of them as well as an explicit list for the globally hyperbolic case. In the recent survey [21] about general relativity, there appears a rich variety of questions where KVFs, stationary vector fields and black hole solutions play central roles.

Our first main result is about the characterization of KVFs on a SSS-T by a set of conditions similar to the conditions obtained by Carot and da Costa in [22] for the analogous *local* problem. Unfortunately, in their article (see [22, Section 4.2]) there are couple of computational mistakes that compromise the validity of their procedure but not their conclusions (see the Appendix). Here we apply an intrinsic notation (as in [2]) to obtain and provide global characterization conditions of KVFs on a SSS-T, obtaining as a side-product the correct relations corresponding to the procedure of Carot and da Costa.

In our second main result, we establish the central role of a particular over-determined system of partial differential equations involving the Hessian in the characterization of KVFs on SSS-Ts and studying these systems we completely characterize the KVFs of a SSS-T with compact Riemannian part. As an interesting application, we deal with the characterization of KVFs with *zero curl* (called here *non-rotating*) on a SSS-T.

The article is organized in the following way: in Section 2 we establish the main results. In Section 3 we give some useful preliminaries along the article. In Section 4 we prove the central results announced in Section 2 and other supplementary statements. In Section 5 we give some applications of the main results.

2. Description of main results

Throughout the article "*I will be an open real interval of the form* $I = (t_1, t_2)$ *where* $-\infty \leq t_1 < t_2 \leq \infty$". and "$(F, g_F)$ *will be a connected Riemannian manifold without boundary with* $\dim F = s$". We will denote the set of all strictly positive C^∞ functions defined on F by $C^\infty_{>0}(F)$.

Let $\mathbb{V}$ be an $\mathbb{R}$-vector space. For any subset S of $\mathbb{V}$, we use $\langle S\rangle$ to denote the $\mathbb{R}$-subspace of $\mathbb{V}$ generated by S. Briefly, if $\mathrm{x} \in \mathbb{V}$ we will write $\langle \mathrm{x}\rangle$ instead of $\langle\{\mathrm{x}\}\rangle$. Also, we will write $\dot{\mathbb{R}} = \mathbb{R} \setminus \{0\}$.

Suppose that $\mathscr{M}$ is a module over a ring $\mathbb{A}$ and $\mathscr{W} \subseteq \mathscr{M}$. If $\mathrm{v} \in \mathscr{M}$, then we will use the following notation $\mathrm{v} + \mathscr{W} = \{\mathrm{v} + W : W \in \mathscr{W}\}$.

Let $\mathscr{K}$ be the real Lie algebra of KVFs on (F, g_F). Given $\varphi, \psi \in C^\infty(F)$ we denote

$$\mathscr{K}^{\psi}_{\varphi} = \{K \in \mathscr{K} : K(\varphi) = \psi\}$$

and

$$\mathscr{K}^{\langle\psi\rangle}_{\varphi} = \{K \in \mathscr{K} : K(\varphi) \in \langle\psi\rangle\},$$

where the $\langle\psi\rangle$ is considered as an $\mathbb{R}$-subspace of $C^\infty(F)$. Notice that $\mathscr{K}^{\psi}_{\varphi}$ is not a real vector space unless ψ is identically zero.

In Section 4, we study KVFs of SSS-Ts. Firstly, we show necessary and sufficient conditions for a vector field of the form $h\partial_t + V$ to be a conformal Killing (see Proposition 4.2).

Then adapting the techniques of Sánchez in [2] to SSS-Ts, we give our first main result, namely.

Theorem 2.1. *Let* $f \in C^\infty_{>0}(F)$ *and* $I_f \times F := (I \times F, g := -f^2 \mathrm{d}t^2 \oplus g_F)$ *the* f*-associated SSS-T. Then, given an arbitrary* $t_0 \in I$*, the set of KVFs on* $I_f \times F$ *is*

$$\psi h\partial_t + \int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\, f^2 \mathrm{grad}_F \psi + \widehat{K} + \mathscr{K}^{0}_{\ln f}, \tag{2.1}$$

where $h \in C^\infty(I)$ *verifies*

$$-h'' = \nu h, \quad \nu \in \mathbb{R}; \tag{2.2}$$

$\psi \in C^\infty(F)$ *verifies*

$$f^2 \mathrm{grad}_F \psi \in \mathscr{K}^{\nu\psi}_{\ln f} \neq \varnothing \tag{2.3}$$

and

$$\widehat{K} \in \mathscr{K}^{-h'(t_0)\psi}_{\ln f} \neq \varnothing, \tag{2.4}$$

where $\varnothing$ *is the empty set.*

If $\nu \neq 0$*, then* $-\frac{h'(t_0)}{\nu} f^2 \mathrm{grad}_F \psi$ *may be taken as* $\widehat{K}$ *and* (2.1) *takes the form*

$$\psi h\partial_t - \frac{h'}{\nu} f^2 \mathrm{grad}_F \psi + \mathscr{K}^{0}_{\ln f}. \tag{2.5}$$

We remark here the central role of the problem (2.3) in Theorem 2.1. Our approach essentially reduces (2.3) to the study of a parametric overdetermined system of partial differential equations (involving the Hessian) on the Riemannian part (F, g_F).

By studying (2.3) (see also (4.27)) and applying the well known results about the solutions (ν, u) of a weighted elliptic problem

$$-\Delta_{g_F} u = \nu w u \quad \text{on } (F, g_F),$$

where $\Delta_{g_F}(\cdot) := g_F^{ij}\nabla_i^{g_F}\nabla_j^{g_F}(\cdot)$ is the Laplace–Beltrami operator, $w \in C^\infty_{>0}(F)$ and F is compact, we obtain our second main result, namely.

Theorem 2.2. *Let $f \in C^\infty_{>0}(F)$ and $I_f \times F$ the f-associated SSS-T. If (F, g_F) is compact then, the set of all KVFs on $I_f \times F$ is given by*

$$\{a\partial_t + \tilde{K} \mid a \in \mathbb{R}, \tilde{K} \text{ is a KVF on } (F, g_F) \text{ and } \tilde{K}(f) = 0\}.$$

Furthermore, the decomposition given above is unique.

In Remark 4.17, we show the relation between the above results and those of Sharipov [23] about KVFs of a closed homogeneous and isotropic universe.

In Section 5, we apply Theorems 2.1 and 2.2 to deal with the existence of non-rotating KVFs on a SSS-T. Applying Theorem 2.1, we obtain a set of conditions that characterize the parallel KVFs on a SSS-T of the type $\mathbb{R}_f \times F$ in Theorem 5.4. As a consequence, we give a classification of non-rotating KVFs on SSS-Ts where the natural part is either complete with nonnegative Ricci curvature (see Theorem 5.6, Corollary 5.7 and Proposition 5.8) or compact and simply connected (see Proposition 5.10).

3. Preliminaries

On an arbitrary differentiable manifold N, $C^\infty_{>0}(N)$ denotes the set of all strictly positive C^∞ functions defined on N, $TN = \bigcup_{p\in N} T_pN$ denotes the tangent bundle of N and $\mathfrak{X}(N)$ denotes the $C^\infty(N)$-module of smooth vector fields on N.[3]

We also recall the canonical (usually called "musical") isomorphisms $TF \overset{\flat}{\underset{\sharp}{\rightleftarrows}} T^*F$ between the tangent bundle TF and the cotangent bundle T^*F, induced by the metric g_F. More explicitly, for $\mathrm{u} \in TF$ and $\eta \in T^*F$, we write

$$g_F(\cdot, \eta^\sharp) = \eta(\cdot),$$

and

$$\mathrm{u}^\flat(\cdot) = g_F(\mathrm{u}, \cdot).$$

Sharp ($\sharp$) and flat ($\flat$) correspond to the classical raising and lowering indices, respectively. For instance, $\mathrm{grad}\psi = \sharp d\psi$ (or $(d\psi)^\sharp$) and $g_F(\mathrm{grad}\psi, \cdot) = d\psi(\cdot)$ (or $(\mathrm{grad}\psi)^\flat = d\psi$), for any smooth function ψ on F (for details see for instance [24–27] among many others).

In order to provide a complete picture to the reader, we recall the general definitions of singly warped products and SSS-Ts below.

Definition 3.1. Let (B, g_B) and (N, g_N) be pseudo-Riemannian manifolds and $b \in C^\infty_{>0}(B)$. Then the (singly) *warped product* $B \times_b N$ is the product manifold $B \times N$ furnished with the metric tensor

$$g = \pi^*(g_B) \oplus (b \circ \pi)^2 \sigma^*(g_N),$$

where $\pi : B \times N \to B$ and $\sigma : B \times N \to N$ are the usual projection maps and * denotes the pull-back operator on tensors.

Definition 3.2. Let $f \in C^\infty_{>0}(F)$. The $n(= 1 + s)$-dimensional product manifold $I \times F$ furnished with the metric tensor $g = -f^2 \mathrm{d}t^2 \oplus g_F$ is called a *standard static space-time* [15] (also usually called *globally static*, see [4]) and is denoted by $I_f \times F$. From now on, we will frequently refer to this as the f-associated SSS-T.

On a warped product of the form $B \times_f N$, we will denote the set of lifts to the product by the corresponding projection of the vector fields in $\mathfrak{X}(B)$ (respectively, $\mathfrak{X}(N)$) by $\mathfrak{L}(B)$ (respectively, $\mathfrak{L}(N)$) (see [5]). We will use the same symbol for a tensor field and its lift.

Two of the most famous examples of SSS-Ts are the *Minkowski space-time* and the *Einstein static universe* [6,28,29] which is $\mathbb{R} \times \mathbb{S}^3$ equipped with the metric

$$g = -\mathrm{d}t^2 + (\mathrm{d}r^2 + \sin^2 r \mathrm{d}\theta^2 + \sin^2 r \sin^2\theta \mathrm{d}\phi^2),$$

where $\mathbb{S}^3$ is the usual 3-dimensional Euclidean sphere and the warping function $f \equiv 1$ (see Remark 4.17). Another well-known example is the universal covering space of anti-de Sitter space-time, a SSS-T of the form $\mathbb{R}_f \times \mathbb{H}^3$ where $\mathbb{H}^3$ is the 3-dimensional hyperbolic space with constant negative sectional curvature and the warping function $f : \mathbb{H}^3 \to (0, \infty)$ defined as $f(r, \theta, \phi) = \cosh r$ [6,29]. Finally, we can also mention the *exterior Schwarzschild space-time* [6,29], a SSS-T of the form $\mathbb{R}_f \times (2m, \infty) \times \mathbb{S}^2$, where $\mathbb{S}^2$ is the 2-dimensional Euclidean sphere, the warping function $f : (2m, \infty) \times \mathbb{S}^2 \to (0, \infty)$

[3] As long as it is possible, our computations will be intrinsic and coordinate free. It is remarkable that we do not use special coordinates for particular dimensions such as three or four, which can obscure the computations.

is given by $f(r,\theta,\phi)=\sqrt{1-2m/r}$, $r>2m$ and the line element on $(2m,\infty)\times\mathbb{S}^2$ is

$$\mathrm{d}s^2=\left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2).$$

Now, we will recall the definition of Killing and conformal-Killing vector fields (CKVF for short) on an arbitrary pseudo-Riemannian manifold. More explicitly, let (M,g) be a pseudo-Riemannian manifold of dimension n and $X\in\mathfrak{X}(M)$ be a vector field on M. Then

- X is said to be Killing if $\mathrm{L}_Xg=0$,
- X is said to be conformal-Killing if there exists a smooth function $\sigma\colon M\to\mathbb{R}$ such that $\mathrm{L}_Xg=2\sigma g$,

where L_X denotes the Lie derivative with respect to X. Moreover, for any Y and Z in $\mathfrak{X}(M)$, we have the following identity (see [5, p. 250 and p. 61])

$$\mathrm{L}_Xg(Y,Z)=g(\nabla_YX,Z)+g(Y,\nabla_ZX). \tag{3.1}$$

Remark 3.3. On $(I,g_I=\pm\mathrm{d}t^2)$ any vector field is conformal Killing. Indeed, if X is a vector field on (I,g_I), then X can be expressed as $X=h\partial_t$ for some smooth function $h\in C^\infty(I)$. Hence, $\mathrm{L}_Xg_I=2\sigma g_I$ with $\sigma=h'$.

In the next remark we enumerate a set of properties of the families of KVFs introduced in the first paragraph of Section 2. We do not apply some of them in the rest of the article, but several of them clarify some paragraphs in [22, pp. 476–478] (see also Appendix here).

Remark 3.4. Let $\phi,\psi\in C^\infty(F)$ be.

(1) $0\in\mathscr{K}_\varphi^0=\mathscr{K}_\varphi^{\langle 0\rangle}\subseteq\mathscr{K}_\varphi^{\langle\psi\rangle}$.
(2) For all $k\in\dot{\mathbb{R}}$ is $\frac{1}{k}\mathscr{K}_\varphi^{k\psi}=\mathscr{K}_\varphi^{\psi}$.
(3) $\{K\in\mathscr{K}:K(\varphi)\in\dot{\mathbb{R}}\,\psi\}=\dot{\mathbb{R}}\,\mathscr{K}_\varphi^{\psi}$.
(4) $\mathbb{R}\mathscr{K}_\varphi^{\psi}\subseteq\mathscr{K}_\varphi^{\langle\psi\rangle}$. Furthermore, $(\mathscr{K}_\varphi^0\setminus 0)\cap\mathbb{R}\mathscr{K}_\varphi^{\psi}$ is empty if $\psi\not\equiv 0$.
(5) By definition $\mathscr{K}_\varphi^{\langle\psi\rangle}$ is an $\mathbb{R}$-subspace of $\mathscr{K}$. But in general it is *not* an $\mathbb{R}$-sub-Lie algebra of $\mathscr{K}$.
(6) If $\psi\in\langle\varphi\rangle$, i.e., $\psi=k\varphi$ with $k\in\mathbb{R}$, then $\mathscr{K}_\varphi^{\langle k\varphi\rangle}$ is an $\mathbb{R}$-sub-Lie algebra of $\mathscr{K}$.
(7) If $\psi=\psi_0$ is a *non zero constant* in $\mathbb{R}$, then $\mathscr{K}_\varphi^{\psi_0}\subsetneq\mathscr{K}_\varphi^{\langle\psi_0\rangle}$.
(8) $\mathscr{K}_\varphi^{\langle 1\rangle}$ is an $\mathbb{R}$-sub-Lie algebra of $\mathscr{K}$.
(9) $\mathscr{K}_\varphi^{0}=\mathscr{K}_\varphi^{\langle 0\rangle}$ and hence, it is an $\mathbb{R}$-sub-Lie algebra of $\mathscr{K}$.
(10) By linear algebra arguments, it is clear that for a fixed $\widehat{K}\in\mathscr{K}_\varphi^{\psi}$ we have,
$$\mathscr{K}_\varphi^{\psi}=\widehat{K}+\mathscr{K}_\varphi^{0}.$$
(11) Given two elements in $\mathscr{K}_\varphi^{\langle\psi\rangle}$, there exists a linear combination of them in $\mathscr{K}_\varphi^{0}$. Thus as above, for a fixed $\bar{K}\in\mathscr{K}_\varphi^{\langle\psi\rangle}\setminus\mathscr{K}_\varphi^{0}$ there results
$$\mathscr{K}_\varphi^{\langle\psi\rangle}=\bar{K}+\mathscr{K}_\varphi^{0}.$$
(12) As $\mathbb{R}$-vector spaces
$$0\le\dim\mathscr{K}_\varphi^{0}\le\dim\mathscr{K}_\varphi^{\langle\psi\rangle}\le\dim\mathscr{K}_\varphi^{0}+1\le\dim\mathscr{K}+1\le\frac{s(s+1)}{2}+1.$$

Remark 3.5. Let $\varphi\in C^\infty(F)$ be. The *Hessian of* φ is the symmetric $(0,2)$-tensor defined by

$$\mathrm{H}_F^{\varphi}(X,Y)=g_F(\nabla_X^F\mathrm{grad}_F\varphi,Y)=\nabla^F\nabla^F\varphi(X,Y) \tag{3.2}$$

for any $X,Y\in\mathfrak{X}(F)$, where ∇^F is the Levi–Civita connection and grad_F is the g_F-gradient operator. The g_F-trace of H_F^{φ} is the Laplace–Beltrami operator denoted by $\Delta_F\varphi$. Notice that Δ_F is elliptic when (F,g_F) is Riemannian (see [5, pp. 85–87]).

Applying the properties that characterize the Levi–Civita connection, it is easy to prove that the following conditions are equivalent:

(1) $\mathrm{grad}_F\phi$ is a KVF on (F,g_F);
(2) $H_F^{\phi}=0$;
(3) $\mathrm{grad}_F\phi$ is parallel.

Furthermore, if these are verified, then $\Delta_F\varphi=0$ (i.e., φ is harmonic) and $g_F(\mathrm{grad}_F\phi,\mathrm{grad}_F\phi)$ is a nonnegative constant (thus the norm $|\mathrm{grad}_F\phi|_F:=\sqrt{g_F(\mathrm{grad}_F\phi,\mathrm{grad}_F\phi)}$ results constant too). In particular, this implies that: if a KVF is a gradient, then it is identically zero when (F,g_F) is compact (see [30, p. 43]).

4. Killing vector fields on SSS-Ts

We will begin by stating a simple result which will be useful in our study (see [2,31] and p. 126 of [32]).

Proposition 4.1. *Let $f \in C^\infty_{>0}(F)$ and $I_f \times F$ the f-associated SSS-T (i.e., with metric tensor $g = f^2 g_I \oplus g_F$, where $g_I = -\mathrm{d}t^2$). Suppose that $X, Y, Z \in \mathfrak{X}(I)$ and $V, W, U \in \mathfrak{X}(F)$. Then*

$$\mathrm{L}_{X+V} g(Y+W, Z+U) = f^2 \mathrm{L}^I_X g_I(Y,Z) + 2fV(f) g_I(Y,Z) + \mathrm{L}^F_V g_F(W,U),$$

where L^I (respectively, L^F) is the Lie derivative on (I, g_I) (respectively, (F, g_F)).

On the other hand, if $h: I \to \mathbb{R}$ is smooth and $Y, Z \in \mathfrak{X}(I)$, then

$$\mathrm{L}_{h\partial_t} g_I(Y,Z) = Y(h) g_I(Z, \partial_t) + Z(h) g_I(Y, \partial_t). \tag{4.1}$$

By combining the previous statements we can prove the following.

Proposition 4.2. *Let $f \in C^\infty_{>0}(F)$ and $I_f \times F$ the f-associated SSS-T. Suppose that $h: I \to \mathbb{R}$ is smooth and $V \in \mathfrak{X}(F)$. Then $h\partial_t + V$ is a CKVF on $I_f \times F$ with $\sigma \in C^\infty(I \times F)$ if and only if the following properties are satisfied:*

(1) *V is conformal-Killing on (F, g_F) with associated $\sigma \in C^\infty(F)$,*
(2) *h is affine, i.e., there exist real numbers μ and ν such that $h(t) = \mu t + \nu$ for any $t \in I$,*
(3) *$V(f) = (\sigma - \mu) f$.*

Proof. (1) follows from Proposition 4.1 by taking $Y = Z = 0$ and a separation of variables argument. On the other hand, from Proposition 4.1 with $W = U = 0$, (4.1) and Remark 3.3, we have $(\sigma - h')f = V(f)$. Hence, again by separation of variables, h' is constant and then (2) is obtained. Thus, (3) is clear.

By computations similar to the previous ones, the converse turns out to be a consequence of the decomposition of any vector field on $I_f \times F$, i.e., as a sum of its horizontal and vertical parts. □

Corollary 4.3. *Let $f \in C^\infty_{>0}(F)$ and $I_f \times F$ the f-associated SSS-T. Suppose that $h: I \to \mathbb{R}$ is smooth and $V \in \mathfrak{X}(F)$. Then $h\partial_t + V$ is a KVF on $I_f \times F$ if and only if the following properties are satisfied:*

(1) *V is Killing on (F, g_F),*
(2) *h is affine, i.e., there exist real numbers μ and ν such that $h(t) = \mu t + \nu$ for any $t \in I$,*
(3) *$V(f) = -\mu f$.*

Proof. It is sufficient to apply Proposition 4.2 with $\sigma \equiv 0$. □

In what follows, we will make use of some arguments given in [2] (see also [22]) about the structure of Killing and CKVFs in warped products. In [2] by applying them, Sánchez obtains full characterizations of the Killing and CKVFs in a generalized Robertson–Walker space-time. In order to be more explanatory, we begin by adapting his procedure to our scenario.

Let (B, g_B) be a semi-Riemannian manifold with dimension r and $f \in C^\infty_{>0}(F)$. Consider the warped product $B_f \times F := (B \times F, g := f^2 g_B + g_F)$. Given a vector field Z on $B \times F$, we will write $Z = Z_B + Z_F$ with $Z_B = (\pi_{B*}(Z), 0)$ and $Z_F = (0, \pi_{F*}(Z))$, the projections onto the natural foliations ($B_q = B \times \{q\}$, $q \in F$ and $F_p = \{p\} \times F$, $p \in B$). Any covariant or contravariant tensor field ω on one of the factors (B or F) induces naturally a tensor field on $B \times F$ (i.e., the lift), which either will be denoted by the same symbol ω, or else (when necessary) will be distinguished by putting a bar on it, i.e., $\overline{\omega}$.

Proposition 4.4 (*See Proposition 3.6 in [2]*). *If K is a KVF on $B_f \times F$, then K_B is a CKVF on B_q for any $q \in F$ and K_F is a KVF on F_p for any $p \in B$.*

Suppose that $\{C_{\overline{a}} \in \mathfrak{X}(B) \mid \overline{a} = 1, \dots, \overline{r}\}$ is a basis for the set of all CKVFs on B and $\{K_{\overline{b}} \in \mathfrak{X}(F) \mid \overline{b} = 1, \dots, \overline{s}\}$ is a basis for the set of all KVFs on F.

By Proposition 4.4 (see [2, Section 3.3] and also [3, Sections 7 and 8]), KVFs on a warped product $B_f \times F$ can be given as

$$K = \underbrace{\psi^{\overline{a}} C_{\overline{a}}}_{K_B} + \underbrace{\phi^{\overline{b}} K_{\overline{b}}}_{K_F}, \tag{4.2}$$

where $\phi^{\overline{b}} \in C^\infty(B)$ and $\psi^{\overline{a}} \in C^\infty(F)$. Moreover, we consider $\hat{K}_{\overline{b}} := g_F(K_{\overline{b}}, \cdot)$ and $\hat{C}_{\overline{a}} := g_B(C_{\overline{a}}, \cdot)$. Notice that $\hat{(\cdot)}$ denotes the musical isomorphism $\flat$ with respect to the corresponding metric.

Then Proposition 3.8 of [2] implies that a vector field K of the form (4.2) is Killing on $B_f \times F$ if and only if the following equations are satisfied:

$$\begin{cases} \psi^{\overline{a}} \sigma_{\overline{a}} + K_F(\ln f) = 0 \\ \mathrm{d}\phi^{\overline{b}} \otimes \hat{K}_{\overline{b}} + \hat{C}_{\overline{a}} \otimes f^2 \mathrm{d}\psi^{\overline{a}} = 0, \end{cases} \tag{4.3}$$

where $C_{\overline{a}}$ is a CKVF on B with $\sigma_{\overline{a}} \in C^\infty(B)$, i.e., $\mathrm{L}^B_{C_{\overline{a}}} g_B = 2\sigma_{\overline{a}} g_B$.

Let us assume that (F, g_F) admits at least one *nonzero* KVF. Thus, there exists a basis $\{K_{\overline{b}} \in \mathfrak{X}(F) \mid \overline{b} = 1, \dots, \overline{s}\}$ for the set of KVFs on F.

Recalling Remark 3.3, we observe that the dimension of the set of CKVFs on $(I, -\mathrm{d}t^2)$ is infinite so that one cannot apply directly the above procedure due to Sánchez before observing that the form of the CKVFs on $(I, -\mathrm{d}t^2)$ is explicit (i.e., any vector field on $(I, -\mathrm{d}t^2)$ is conformal Killing). Indeed, it is easy to prove that all the computations are valid by considering the form of any CKVF on $(I, -\mathrm{d}t^2)$, namely $h\partial_t$ where $h \in C^\infty(I)$, instead of the finite basis of CKVFs in the Sánchez approach.

If we apply the latter technique adapted to the SSS-T $I_f \times F$ with the metric given by $g = f^2 g_I \oplus g_F$ where $g_I = -\mathrm{d}t^2$, then a $K \in \mathfrak{X}(I_f \times F)$ is Killing if and only if K can be written in the form

$$K = \psi h \partial_t + \phi^{\overline{b}} K_{\overline{b}}, \tag{4.4}$$

where h and $\phi^{\overline{b}} \in C^\infty(I)$ for any $\overline{b} \in \{1, \ldots, \overline{m}\}$ and $\psi \in C^\infty(F)$ satisfy the following version of system (4.3)

$$\begin{cases} h'\psi + \phi^{\overline{b}} K_{\overline{b}}(\ln f) = 0 \\ \mathrm{d}\phi^{\overline{b}} \otimes g_F(K_{\overline{b}}, \cdot) + g_I(h\partial_t, \cdot) \otimes f^2 \mathrm{d}\psi = 0. \end{cases} \tag{4.5}$$

Thus, in order to study KVFs on SSS-Ts we will concentrate our attention to the existence of solutions for the system (4.5).

Since $\mathrm{d}\phi^{\overline{b}} = (\phi^{\overline{b}})'\mathrm{d}t$ with $\phi^{\overline{b}} \in C^\infty(I)$ and $g_I(h\partial_t, \cdot) = -h\mathrm{d}t$, (4.5) is equivalent to

$$h'\psi + \phi^{\overline{b}} K_{\overline{b}}(\ln f) = 0 \tag{4.6a}$$

$$(\phi^{\overline{b}})'\mathrm{d}t \otimes g_F(K_{\overline{b}}, \cdot) = h\mathrm{d}t \otimes f^2 \mathrm{d}\psi, \tag{4.6b}$$

and by raising indices in (4.6b), (4.6) is also equivalent to

$$h'\psi + \phi^{\overline{b}} K_{\overline{b}}(\ln f) = 0 \tag{4.7a}$$

$$(\phi^{\overline{b}})'\partial_t \otimes K_{\overline{b}} = h\partial_t \otimes f^2 \mathrm{grad}_F \psi. \tag{4.7b}$$

First of all, we will apply a separation of variables procedure to (4.7b). Recall that $\{K_{\overline{b}}\}_{1\le \overline{b} \le \overline{m}}$ is a basis of the KVFs in (F, g_F). Thus by simple computations, each $(\phi^{\overline{b}})'$ verifies

$$\begin{aligned} (\phi^{\overline{b}})'(t) &= [h(t) - h(t_0)]\gamma^{\overline{b}} + (\phi^{\overline{b}})'(t_0), \\ &= \gamma^{\overline{b}} h(t) + \delta^{\overline{b}}, \end{aligned} \tag{4.8}$$

where $\gamma^{\overline{b}}$ and $\delta^{\overline{b}} (= -h(t_0)\gamma^{\overline{b}} + (\phi^{\overline{b}})'(t_0)$, for some fixed $t_0 \in I$ that is independent of $\overline{b})$ are real constants.

The solutions of the first order ordinary differential equation in (4.8) are given by

$$\phi^{\overline{b}}(t) = \gamma^{\overline{b}} \int_{t_0}^t h(s)\,\mathrm{d}s + \delta^{\overline{b}} t + \eta^{\overline{b}}, \tag{4.9}$$

where $\eta^{\overline{b}}$ is a constant for each $\overline{b}$.

By introducing (4.8) in (4.7b), the latter takes the following equivalent form:

$$h\partial_t \otimes [\gamma^{\overline{b}} K_{\overline{b}} - f^2 \mathrm{grad}_F \psi] = \partial_t \otimes [-\delta^{\overline{b}} K_{\overline{b}}]. \tag{4.10}$$

Thus, by recalling again the fact that $\{K_{\overline{b}}\}_{1\le \overline{b} \le \overline{m}}$ is a basis of the KVFs in (F, g_F), there results two different cases, namely.

h nonconstant: First of all, note that by applying the separation of variables method in (4.10), the <u>non</u>-constancy of h implies that

$$\begin{cases} \gamma^{\overline{b}} K_{\overline{b}} - f^2 \mathrm{grad}_F \psi = 0 \\ \delta^{\overline{b}} = 0 \quad \forall \overline{b}. \end{cases} \tag{4.11}$$

Thus, by (4.9),

$$\phi^{\overline{b}}(t) = \gamma^{\overline{b}} \int_{t_0}^t h(s)\,\mathrm{d}s + \eta^{\overline{b}}. \tag{4.12}$$

On the other hand, by differentiating (4.7a) with respect to t and then by considering (4.11), we obtain

$$h''\psi + h(f^2 \mathrm{grad}_F \psi)(\ln f) = 0.$$

Besides, by considering (4.11), (4.12) and again (4.7a) there results

$$\left[hh' - h'' \int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\right] \psi + h\eta^{\overline{b}} K_{\overline{b}}(\ln f) = 0.$$

Thus, we proved that (4.7) is sufficient to

$$f^2\mathrm{grad}_F\psi \in \mathscr{K}; \tag{4.13a}$$

$$h''\psi + h(f^2\mathrm{grad}_F\psi)(\ln f) = 0; \tag{4.13b}$$

$$\begin{cases} \forall \bar{b} : \phi^{\bar{b}}(t) = \tau^{\bar{b}} \displaystyle\int_{t_0}^{t} h(s)\,\mathrm{d}s + \omega^{\bar{b}} \quad \text{where} \\ \tau^{\bar{b}}, \omega^{\bar{b}} \in \mathbb{R} : f^2\mathrm{grad}_F\psi = \tau^{\bar{b}}K_{\bar{b}} \quad \text{and} \\ \left[hh' - h'' \displaystyle\int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\right]\psi + h\omega^{\bar{b}}K_{\bar{b}}(\ln f) = 0; \end{cases} \tag{4.13c}$$

on I.[4]

By (4.13b), it is not difficult to show that if $-\frac{h''}{h}$ is nonconstant, then $\psi \equiv 0$[5]and the latter infers

$$K = \phi^{\bar{b}}K_{\bar{b}} \quad \text{with } \phi^{\bar{b}}(t) = \omega^{\bar{b}} \quad \text{and} \quad \omega^{\bar{b}} \in \mathbb{R} : \omega^{\bar{b}}K_{\bar{b}} \in \mathscr{K}^0_{\ln f}. \tag{4.14}$$

On the other hand, if $-\frac{h''}{h} = \nu$ is constant[6], (4.13b) implies

$$\begin{cases} -\dfrac{h''}{h} = \nu \\ (f^2\mathrm{grad}_F\psi)(\ln f) = \nu\psi. \end{cases} \tag{4.15}$$

Furthermore, by (4.13c) (see footnote 4)

$$\omega^{\bar{b}}K_{\bar{b}} \in \mathscr{K}^{-h'(t_0)\psi}_{\ln f}. \tag{4.16}$$

Hence, by (4.13) and (4.14) the problem (4.6) is sufficient for:

$$\begin{cases} (a) \quad \begin{cases} \psi \equiv 0; \\ \phi^{\bar{b}}(t) \equiv \omega^{\bar{b}} \text{ on } I \quad \text{where } \omega^{\bar{b}} \in \mathbb{R} : \omega^{\bar{b}}K_{\bar{b}} \in \mathscr{K}^0_{\ln f}; \end{cases} \\ \text{or} \\ (b) \quad \exists \nu \in \mathbb{R} : \\ \qquad \begin{cases} -\dfrac{h''}{h} = \nu; \\ f^2\mathrm{grad}_F\psi \in \mathscr{K}^{\nu\psi}_{\ln f}; \\ \begin{cases} \forall \bar{b} : \phi^{\bar{b}}(t) = \tau^{\bar{b}} \displaystyle\int_{t_0}^{t} h(s)\,\mathrm{d}s + \omega^{\bar{b}} \quad \text{where } \tau^{\bar{b}}, \omega^{\bar{b}} \in \mathbb{R} : \\ f^2\mathrm{grad}_F\psi = \tau^{\bar{b}}K_{\bar{b}} \quad \text{and} \quad \omega^{\bar{b}}K_{\bar{b}} \in \mathscr{K}^{-h'(t_0)\psi}_{\ln f}. \end{cases} \end{cases} \end{cases}$$

Notice that the case (a) is a subcase of (b), for instance taking $\nu = 0$. This allows us to say that (4.6) is sufficient for:

$\exists \nu \in \mathbb{R}$ such that

$$-h'' = \nu h; \tag{4.17a}$$

$$f^2\mathrm{grad}_F\psi \in \mathscr{K}^{\nu\psi}_{\ln f}; \tag{4.17b}$$

$$\begin{cases} \forall \bar{b} : \phi^{\bar{b}}(t) = \tau^{\bar{b}} \displaystyle\int_{t_0}^{t} h(s)\,\mathrm{d}s + \omega^{\bar{b}} \quad \text{where } \tau^{\bar{b}}, \omega^{\bar{b}} \in \mathbb{R} : \\ f^2\mathrm{grad}_F\psi = \tau^{\bar{b}}K_{\bar{b}} \quad \text{and} \quad \omega^{\bar{b}}K_{\bar{b}} \in \mathscr{K}^{-h'(t_0)\psi}_{\ln f}. \end{cases} \tag{4.17c}$$

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4 Clearly, $h\int_{t_0}^{(\cdot)} h''(s)\,\mathrm{d}s - h''\int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s = hh' - hh'(t_0) - h''\int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s$. So, if $-\frac{h''}{h} = \nu$ with ν constant, then the left hand side is 0; as a consequence $hh' - h''\int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s = hh'(t_0)$.

5 Suppose that t_1 and t_2 are such that $-\frac{h''}{h}(t_1) \neq -\frac{h''}{h}(t_2)$. Since $-\frac{h''}{h}(t_1)\psi = (f^2\mathrm{grad}_F\psi)\ln f$ and $-\frac{h''}{h}(t_2)\psi = (f^2\mathrm{grad}_F\psi)\ln f$, $\underbrace{\left(-\frac{h''}{h}(t_1) + \frac{h''}{h}(t_2)\right)}_{\neq 0}\psi = 0$. So $\psi \equiv 0$.

6 Recall the Courant theorem about the number of nodal points of the eigenfunctions of a Sturn–Liouville problem with Dirichlet boundary conditions (see [33, p. 454], [34, p. 174]). Roughly speaking this says that the number of nodal sets of the n-th eigenfunction of such a problem is n. Since the latter particularly implies that no node of an eigenfunction is an accumulation point of nodes of the same eigenfunction, it allows us to consider the ratio $\frac{h''}{h}$ defined on the whole interval I.

$h \equiv h_0$ *constant*: By (4.9), (4.7) takes the form

$$[(t-t_0)h_0\gamma^{\bar b} + t\delta^{\bar b} + \eta^{\bar b}]K_{\bar b}(\ln f) = 0 \tag{4.18a}$$

$$\left(\gamma^{\bar b}h_0 + \delta^{\bar b}\right)K_{\bar b} = h_0 f^2 \mathrm{grad}_F\psi. \tag{4.18b}$$

We consider two subcases

$h_0 = 0$: Since $\{K_{\bar b}\}_{1\le\bar b\le\bar m}$ is a basis, (4.18b) implies $\delta^{\bar b} = 0$ for all $\bar b$. So $K = \eta^{\bar b}K_{\bar b}$. Thus, it is clear that (4.17) is verified choosing $\nu = 0$, $\tau^{\bar b} = 0$ and $\omega^{\bar b} = \eta^{\bar b}$ for all $\bar b$. Notice that "$\tau^{\bar b} = 0$ for all $\bar b$" is equivalent to $\psi \equiv 0$.

$h_0 \neq 0$: In this case (4.18b) implies that $f^2\mathrm{grad}_F\psi$ is Killing on (F, g_F) and gives the coefficients of $f^2\mathrm{grad}_F\psi$ with respect to the basis $\{K_{\bar b}\}_{1\le\bar b\le\bar m}$. On the other hand, differentiating (4.18a) with respect to t and then considering (4.18b), we obtain

$$0 = (\gamma^{\bar b}h_0 + \delta^{\bar b})K_{\bar b}(\ln f) = h_0(f^2\mathrm{grad}_F\psi)(\ln f).$$

Furthermore, the latter and (4.18a) imply that

$$(\eta^{\bar b} - h_0t_0\gamma^{\bar b})K_{\bar b}(\ln f) = 0.$$

Thus, we proved that (4.17) is verified choosing $\nu = 0$, $\tau^{\bar b} = \frac{1}{h_0}(\gamma^{\bar b}h_0 + \delta^{\bar b})$ and $\omega^{\bar b} = \eta^{\bar b} - h_0t_0\gamma^{\bar b}$ for all $\bar b$.

Conversely, it is easy to prove that if for a set of sufficiently regular functions h, ψ and $\{\phi_{\bar b}\}_{1\le\bar b\le\bar m}$, where h and $\phi^{\bar b} \in C^\infty(I)$ for any $\bar b \in \{1,\ldots,\bar m\}$ and $\psi \in C^\infty(F)$, there exists $\nu \in \mathbb{R}$ such that (4.17) is verified, then the vector field $\psi h\partial_t + \phi^{\bar b}K_{\bar b}$ on the SSS-T $I_f \times F$ is Killing. Indeed, this set satisfies (4.7).

Hence, in the precedent discussion we proved the following result.

Theorem 4.5. *Let $f \in C^\infty_{>0}(F)$, $\{K_{\bar b}\}_{1\le\bar b\le\bar m}$ a basis of KVFs on (F, g_F) and $I_f \times F$ the f-associated SSS-T. Then, any KVF on $I_f \times F$ admits the structure*

$$K = \psi h\partial_t + \phi^{\bar b}K_{\bar b}, \tag{4.19}$$

where h and $\phi^{\bar b} \in C^\infty(I)$ for any $\bar b \in \{1,\ldots,\bar m\}$ and $\psi \in C^\infty(F)$.

Furthermore, assume that K is a vector field on $I_f \times F$ with the structure as in (4.19). Hence, for an arbitrary fixed $t_0 \in I$, K is Killing on $I_f \times F$ if and only if there exists a real number $\nu \in \mathbb{R}$ such that

$$-h'' = \nu h; \tag{4.20a}$$

$$f^2\mathrm{grad}_F\psi \in \mathscr{K}^{\nu\psi}_{\ln f}; \tag{4.20b}$$

$$\begin{cases} \forall \bar b : \phi^{\bar b}(t) = \tau^{\bar b}\displaystyle\int_{t_0}^{t} h(s)\,\mathrm{d}s + \omega^{\bar b} \quad \text{where } \tau^{\bar b}, \omega^{\bar b} \in \mathbb{R}: \\ f^2\mathrm{grad}_F\psi = \tau^{\bar b}K_{\bar b} \quad \text{and} \quad \omega^{\bar b}K_{\bar b} \in \mathscr{K}^{-h'(t_0)\psi}_{\ln f}. \end{cases} \tag{4.20c}$$

For clarity we also state the following lemma, which covers the case where the Riemannian part of the SSS-T admits no non identically zero KVF.

Lemma 4.6. *Let $f \in C^\infty_{>0}(F)$ and $I_f \times F$ the f-associated SSS-T. If the only KVF on (F, g_F) is the zero vector field, then all the KVFs on $I_f \times F$ are given by $h_0\partial_t$ where h_0 is a constant.*

Proof. Indeed, by Proposition 4.4 if K is a KVF on $I_f \times F$, then $K = \psi h\partial_t$ where $\psi \in C^\infty(F)$ and $h \in C^\infty(I)$. Then, by similar arguments to those applied to system (4.7), a vector field of the latter form is Killing if and only if the following equations are verified

$$h'\psi = 0 \tag{4.21a}$$

$$h\partial_t \otimes f^2\mathrm{grad}_F\psi = 0. \tag{4.21b}$$

As an immediate consequence, either "h and ψ are constants" or "$\psi \equiv 0$". $\square$

Proof of Theorem 2.1. It is sufficient to apply Theorem 4.5, Remark 3.4 (10) and Lemma 4.6. In order to obtain (2.5) for the case $\nu \neq 0$, notice that (2.1) can be written as

$$\psi h\partial_t + \underbrace{\left(\int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s - \frac{h'(t_0)}{\nu}\right)}_{=-\frac{h'}{\nu}} f^2\mathrm{grad}_F\psi + \underbrace{\widehat{K} + \frac{h'(t_0)}{\nu}f^2\mathrm{grad}_F\psi}_{\in\mathscr{K}^0_{\ln f}} + \mathscr{K}^0_{\ln f}. \quad \square$$

Remark 4.7. If the Riemannian part (F, g_F) admits a non identically zero KVF, then the family of KVFs obtained in Corollary 4.3 corresponds to the case of $\psi \equiv 1$ in Theorem 4.5. Thus, (4.20) implies that $\nu = 0$, and $\tau^{\bar{b}} = 0$ for any $\bar{b}$, and also $h(t) = at + b$ is affine, and $\phi^{\bar{b}} = \omega^{\bar{b}}$ is constant such that $\phi^{\bar{b}} K_{\bar{b}}(\ln f) = -a$. The latter conditions agree with those in Corollary 4.3.

In other words, if ν is nonzero, then the family of KVFs in Theorem 4.5 are different form those in Corollary 4.3, they correspond to the so called nontrivial KVFs in [2].

Remark 4.8 (*Uniqueness of the Decomposition*)**.** Under the assumptions of Theorem 4.5, further suppose that K is a KVF on $I_f \times F$. We know that K admits a decomposition given by (4.19). If K admits a different decomposition of the same type, more explicitly, $K = \psi_1 h_1 \partial_t + \phi_1^{\bar{b}} K_{\bar{b}}$, it is easy to prove that $h\psi = h_1\psi_1$ and $\phi^{\bar{b}} = \phi_1^{\bar{b}}$ for each $\bar{b}$, i.e., such decomposition is essentially unique. More specifically, $h_1 = \lambda h$, $\psi_1 = \frac{1}{\lambda}\psi$ and $\phi^{\bar{b}} = \phi_1^{\bar{b}}$ for each $\bar{b}$, where $\lambda \neq 0$ is a real constant.

Remark 4.9. Let $f \in C^\infty_{>0}(F)$ be smooth. For any $\nu \in \mathbb{R}$, we consider the problem

$$f^2 \mathrm{grad}_F \psi \in \mathscr{K}^{\nu\psi}_{\ln f} \quad \text{with } \psi \in C^\infty(F) \tag{4.22}$$

and define

$$\mathcal{K}^\nu_f = \{\psi \in C^\infty(F) : \psi \text{ verifies (4.22)}\}$$

and

$$\mathfrak{K}^\nu_f = \{K \in \mathfrak{X}(F) : \exists \psi \in \mathcal{K}^\nu_f \text{ such that } f^2 \mathrm{grad}_F \psi = K\}.$$

It is easy to show that $\mathcal{K}^\nu_f$ (respectively, $\mathfrak{K}^\nu_f$) is an $\mathbb{R}$-subspace of $C^\infty(F)$ (respectively, $\mathscr{K}$). In particular, if $\psi \in \mathcal{K}^\nu_f$ then

$$(f^2 \mathrm{grad}_F \lambda\, \psi)(\ln f) = \lambda\nu\, \psi, \quad \forall \lambda \in \mathbb{R}. \tag{4.23}$$

Consequently, if $\{\tau_{\bar{b}}\}_{1\le \bar{b}\le \bar{m}}$ is the set of coefficients of a KVF of the form $f^2 \mathrm{grad}_F \psi$ with respect to the basis $\{K_{\bar{b}}\}_{1\le \bar{b}\le \bar{m}}$ and $\lambda \in \mathbb{R}$, then

$$-\lambda\nu\, \psi + \omega^{\bar{b}} K_{\bar{b}}(\ln f) = 0, \tag{4.24}$$

where $\omega^{\bar{b}} = \lambda \tau^{\bar{b}}$, for any $\bar{b}$.

Notice that, this is particularly useful in order to simplify the condition (4.20c) when $\nu \neq 0$, taking $\lambda\nu = -h'(t_0)$.

We observe also that it is easy to prove that the Lie bracket of two elements in $\mathfrak{K}^\nu_f$ belongs to $\mathscr{K}^0_{\ln f}$.

Now we deal with the existence of nontrivial solutions for the problem (4.22), which is relevant for Theorem 4.5 and as a consequence for Theorems 2.1 and 2.2.

Lemma 4.10. *Let $f \in C^\infty_{>0}(F)$ and $\psi \in C^\infty(F)$. Then the vector field $f^2 \mathrm{grad}_F \psi$ is Killing on (F, g_F) if and only if*

$$\mathrm{H}^\psi_F + \frac{1}{f}[\mathrm{d}f \otimes d\psi + d\psi \otimes \mathrm{d}f] = 0, \tag{4.25}$$

where H^ψ_F is the g_F-Hessian of the function ψ.

Proof. We begin by recalling two results. By (3.1), for all $\varphi \in C^\infty(F)$

$$\mathrm{L}^F_{\mathrm{grad}_F \varphi} g_F = 2\mathrm{H}^\varphi_F. \tag{4.26}$$

Moreover, for any $Z \in \mathfrak{X}(F)$,

$$\mathrm{L}^F_{\varphi Z} g_F = \varphi \mathrm{L}^F_Z g_F + \mathrm{d}\varphi \otimes Z^\flat + Z^\flat \otimes \mathrm{d}\varphi.$$

So the latter formulas with $\varphi = f^2$ and $Z = \mathrm{grad}_F \psi$ imply

$$\mathrm{L}^F_{f^2 \mathrm{grad}_F \psi} g_F = f^2 \mathrm{L}^F_{\mathrm{grad}_F \psi} g_F + \mathrm{d}f^2 \otimes (\mathrm{grad}_F \psi)^\flat + (\mathrm{grad}_F \psi)^\flat \otimes \mathrm{d}f^2.$$

But $(\mathrm{grad}_F \psi)^\flat = d\psi$, so

$$\mathrm{L}^F_{f^2 \mathrm{grad}_F \psi} g_F = 2f^2 \left[\mathrm{H}^\psi_F + \frac{1}{f}[\mathrm{d}f \otimes d\psi + d\psi \otimes \mathrm{d}f]\right].$$

Then $f^2 \mathrm{grad}_F \psi$ is a KVF on (F, g_F) if and only if (4.25) is verified. $\square$

Thus, by Lemma 4.10 and the identity $fg_F(\mathrm{grad}_F\psi, \mathrm{grad}_F f) = (f\,\mathrm{grad}_F\psi)(f)$, (4.22) is equivalent to

$$\psi \in C^\infty(F); \tag{4.27a}$$

$$\mathrm{H}_F^\psi + \frac{1}{f}[\mathrm{d}f \otimes d\psi + d\psi \otimes \mathrm{d}f] = 0; \tag{4.27b}$$

$$fg_F(\mathrm{grad}_F\psi, \mathrm{grad}_F f) = \nu\psi \quad \text{where } \nu \text{ is a constant.} \tag{4.27c}$$

Remark 4.11. By Lemma 4.10, if the dimension of the Lie algebra of KVFs of (F, g_F) is zero, then the system (4.27) has only the trivial solution given by a constant ψ (this constant is not identically 0 only if $\nu = 0$). This happens, for instance when (F, g_F) is a compact Riemannian manifold of negative-definite Ricci curvature without boundary, indeed it is sufficient to apply the vanishing theorem due to Bochner (see for instance [35], [30, p. 44], [25, Theorem 1.84] or [36, Proposition 6.6 of Chapter III]).

Lemma 4.12. *Let $f \in C^\infty_{>0}(F)$. If (ν, ψ) satisfies (4.27), then ν is an eigenvalue and ψ is an associated ν-eigenfunction of the elliptic problem:*

$$-\Delta_{g_F}\psi = \nu\frac{2}{f^2}\psi \quad \textit{on } (F, g_F). \tag{4.28}$$

Proof. It is enough to apply the general identity

$$\mathrm{trace}_{g_F}[\mathrm{d}f \otimes \mathrm{d}\psi + \mathrm{d}\psi \otimes \mathrm{d}f] = 2g_F(\mathrm{grad}_F\psi, \mathrm{grad}_F f) \tag{4.29}$$

to the g_F-trace of (4.27b) and then consider (4.27c). □

Remark 4.13. i: Notice that similar arguments to those applied in Lemma 4.12 allow us to prove that the system (4.27) is *equivalent* to

$$\psi \in C^\infty(F); \tag{4.30a}$$

$$\mathrm{H}_F^\psi + \frac{1}{f}[\mathrm{d}f \otimes d\psi + d\psi \otimes \mathrm{d}f] = 0; \tag{4.30b}$$

$$-\Delta_{g_F}\psi = \nu\frac{2}{f^2}\psi \quad \text{where } \nu \text{ is a constant.} \tag{4.30c}$$

ii: Assuming (4.30) (or equivalently (4.27)), if $p \in F$ is a critical point of f or ψ, then $\nu = 0$ or $\psi(p) = 0$.

Remark 4.14 (*See* Theorem 5.4 *for an Application*)**.** Suppose that $f \in C^\infty_{>0}(F)$ and take $\psi = \frac{C}{f}$ with $C \neq 0$ constant. Then it is easy to prove that

$$\mathrm{H}_F^\psi + \frac{1}{f}[\mathrm{d}f \otimes d\psi + d\psi \otimes \mathrm{d}f] = -C\frac{1}{f^2}\mathrm{H}_F^f \tag{4.31}$$

and

$$fg_F(\mathrm{grad}_F\psi, \mathrm{grad}_F f) = -\frac{C}{f}\underbrace{g_F(\mathrm{grad}_F f, \mathrm{grad}_F f)}_{\geq 0}. \tag{4.32}$$

Thus, ψ verifies (4.27) iff $\mathrm{H}_F^f = 0$ and $\nu = -g_F(\mathrm{grad}_F f, \mathrm{grad}_F f)$.

Note that $\mathrm{H}_F^f = 0$ implies $g_F(\mathrm{grad}_F f, \mathrm{grad}_F f)$ is constant and nonnegative (see Remark 3.5). Besides $\nu = -g_F(\mathrm{grad}_F f, \mathrm{grad}_F f)$ infers ν is non-positive.

Besides, since $f^2\mathrm{grad}_F\psi = -C\mathrm{grad}_F f$ and $C \neq 0$, $f^2\mathrm{grad}_F\psi$ is a KVF iff $\mathrm{grad}_F f$ is a KVF.

Example 4.15. Let $f \in C^\infty_{>0}(F)$ such that

$$H_F^f = 0 \quad \text{and} \quad \nu := -g_F(\mathrm{grad}_F f, \mathrm{grad}_F f) < 0 \tag{4.33}$$

and let $\psi = \frac{C}{f}$ with $C \neq 0$ constant. So, $f^2\mathrm{grad}_F\psi = -C\mathrm{grad}_F f \in \mathscr{K}^{\nu\psi}_{\ln f}$ (cfr. (4.20b)). Then, by some computations and applying Theorem 4.5, Lemma 4.10 and Remarks 4.9 and 4.14 (see also the proof of Theorem 2.1) we obtain that if h is a solution of $-h'' = \nu h$ on an interval I, then

$$C\left(\frac{h}{f}\partial_t + \frac{h'}{\nu}\mathrm{grad}_F f\right) \tag{4.34}$$

is a KVF on the SSS-T $I_f \times F$.

Proposition 4.16. *Let (F, g_F) be compact and $f \in C^{\infty}_{>0}(F)$. Then (ν, ψ) satisfies (4.27) if and only if $\nu = 0$ and ψ is constant.*

Proof. It is clear that $(0, \psi)$ with ψ constant verifies (4.27). So, we will concentrate our attention in the converse direction.

First of all, notice that by (4.27c), if $p \in F$ is a critical point of ψ, then $\nu\psi(p) = 0$. Then, since (F, g_F) is compact, there exists a point $p_0 \in F$ such that $\psi(p_0) = \inf_F \psi$ and consequently, $\nu\psi(p_0) = 0$.

On the other hand, by applying Lemma 4.12, one can conclude that ν is an eigenvalue and ψ is an associated ν-eigenfunction of the elliptic problem (4.28). Besides, since (F, g_F) is compact, it is well known that the eigenvalues of (4.28) form a sequence in $\mathbb{R}_{\geq 0}$ and the only eigenfunctions without changing sign are the constants corresponding to the eigenvalue 0.

Thus, if $\psi(p_0) \geq 0$, then $\nu = 0$ and ψ results a nonnegative constant. Alternatively, if $\psi(p_0) < 0$, then $\nu\psi(p_0) = 0$, so $\nu = 0$. As a consequence of that, ψ is a negative constant. $\square$

Proof of Theorem 2.2. If (F, g_F) has only the zero KVF, the result is an easy consequence of Lemma 4.6.

Let us consider now the case there exists a basis $\{K_{\overline{b}}\}_{1\leq\overline{b}\leq\overline{m}}$ for the space of KVFs on (F, g_F). Theorem 4.5 and Proposition 4.16 imply that a vector field K on the SSS-T $I_f \times F$ is Killing if and only if it admits the structure

$$K = \psi h\partial_t + \phi^{\overline{b}} K_{\overline{b}}, \tag{4.35}$$

where

(1) $h(t) = at + b$ with constants a and b;
(2) ψ is constant;
(3) $\phi^{\overline{b}}$ are constants satisfying $a\psi + \phi^{\overline{b}} K_{\overline{b}}(\ln f) = 0$.

Since (F, g_F) is compact, then $\inf_F \ln f$ is reached at a point $p_0 \in F$. Set $\tilde{K} = \phi^{\overline{b}} K_{\overline{b}}$. Thus $\tilde{K}(\ln f)|_{p_0} = 0$ and by (3) $a = 0$ or $\psi = 0$. Hence we proved that any KVF on $I_f \times F$ is given by a KVF on (F, g_F) plus eventually a real multiple of ∂_t. Note that $\tilde{K}(\ln f) = \frac{1}{f}\tilde{K}(f)$, so by (3) we have $\tilde{K}(f) = 0$.

The uniqueness of the decomposition is easily obtained by evaluating the KVF at the function $\sigma(t, x) = t$. $\square$

Remark 4.17 (*KVFs in the Einstein Static Universe*)**.** In [23], the author studied KVFs of a closed homogeneous and isotropic universe (for related questions in quantum field theory and cosmology see [37,28,38,39]). Theorem 6.1 of [23] corresponds to our Theorem 2.2 for the spherical universe $\mathbb{R} \times \mathbb{S}^3$ with the pseudo-metric $-(R^2\mathrm{d}t^2 - R^2 h_0)$, where the sphere $\mathbb{S}^3$ is endowed with the usual metric h_0 induced by the canonical Euclidean metric of $\mathbb{R}^4$ and R is a real constant (i.e., a stable universe).

As we have already mentioned in Remark 4.11, any KVF of a compact Riemannian manifold of negative-definite Ricci curvature is equal to zero. Thus, one can easily state the following result.

Corollary 4.18. *Let $I_f \times F$ be a SSS-T. If (F, g_F) is compact with negative-definite Ricci curvature, then any KVF on $I_f \times F$ is given by $a\partial_t$ where $a \in \mathbb{R}$.*

In [17, Theorem 5], it is shown that the decomposition of a space-time as a standard static one is essentially unique when the fiber F is compact. We observe that Corollary 4.18 enables us to establish a stronger conclusion (i.e., nonexistence of nontrivial (it means independent of ∂_t) strictly stationary[7] fields) under a stronger assumption involving the definiteness of the Ricci curvature.

We would like to make some comments about the case where the Riemannian part of the SSS-T is noncompact. While Theorem 4.5 does not require the compactness of the Riemannian manifold (F, g_F), this condition is central for a complete characterization similar to the one provided in Theorem 2.2. The key question in our approach is the full characterization of the solutions of (4.30) (or the equivalent problems (4.22) and (4.27)), which is obtained by means of the theory of weighted elliptic eigenvalue problems on compact Riemannian manifolds when (F, g_F) is *compact*. In the *noncompact* case, the latter question is more difficult. Through the application of Liouville type arguments about the nonexistence of one side bounded subharmonic functions on complete and noncompact Riemannian manifolds, it is possible to obtain partial nonexistence results of nontrivial solutions for (4.30), but the global question is still open. However, there are particular situations, like the following well known example where the application of Theorem 2.1 is sufficient for a complete classification.

Example 4.19 (*KVFs in the Minkowski Space–time*)**.** Let the Riemannian part $(F, g_F) = (\mathbb{R}^s, g_0)$ where g_0 is the canonical metric and $f \equiv 1$ be. Thus, it is easy to show that the solutions of (4.30) are (ν, ψ) where $\nu = 0$ or $\psi \equiv 0$. Furthermore if $\nu = 0$, then $\psi(x) = c^i x_i + d$ where $\forall i : 1 \leq i \leq s$, $c^i \in \mathbb{R}$ and $d \in \mathbb{R}$. Recall that for $\nu = 0$, $h(t) = at + b$ where $a, b \in \mathbb{R}$.

On the other hand, the condition (2.4) implies $h'(t_0)(c^i x_i + d) \equiv 0$.

7 Stationary means Killing and time-like (see [17]).

Hence, all the KVFs of the Minkowski space-time are

$$(c^i x_i + d)(at + b)\partial_t + \int_{t_0}^{t} (as + b)\,\mathrm{d}s\, c^i \partial_i + \mathscr{K},$$

where $a, b, c_i, d \in \mathbb{R}$ satisfy $a(c^i x_i + d) \equiv 0$. Precisely, these are

$$(c^i x_i + d)\partial_t + (t - t_0)\, c^i \partial_i + \mathscr{K}$$

or equivalently (taking $t_0 = 0$)

$$c^i \underbrace{(x_i \partial_t + t\partial_i)}_{\text{Lorentz boosts}} + d\partial_t + \mathscr{K},$$

where $c_i, d \in \mathbb{R}$. Thus the dimension of the Lie algebra of the KVFs of the Minkowski space-time is $s + 1 + s(s + 1)/2 = (s + 1)(s + 2)/2$.

5. Non-rotating killing vector fields

In this section we will apply Theorems 2.1 and 2.2 to the analysis of non-rotating KVFs on SSS-Ts also called *static regular predictable space-times* in [29, p. 325] (also see the recent article [17] for a related question).

We first recall the definition of the curl operator on semi-Riemannian manifolds of arbitrary finite dimension, namely: if V is a vector field on a semi-Riemannian manifold (N, g_N), then curl$V$ is the antisymmetric 2-covariant tensor field defined by

$$\operatorname{curl} V(X, Y) := g_N(\nabla_X V, Y) - g_N(\nabla_Y V, X), \tag{5.1}$$

where $X, Y \in \mathfrak{X}(N)$ (see for instance [5,40] and for other close approach [41]). Thus, it is easy to prove for all $\phi \in C^\infty(N)$

$$\operatorname{curl}(\phi V) = g_N(V, T_\phi) + \phi \operatorname{curl} V, \tag{5.2}$$

where T_ϕ is the so called torsion of ϕ.[8]

We will consider the following definitions (see [40,42]): A vector field V on a semi-Riemannian manifold (N, g_N) is said to be

non-rotating: [9] if $\operatorname{curl} V(X, Y) = 0$ for all $X, Y \in \mathfrak{X}(N)$.
orthogonally irrotational: [10] if $\operatorname{curl} V(X, Y) = 0$ for any $X, Y \in \mathfrak{X}(N)$ orthogonal to V. This condition is equivalent to "V has an integrable orthogonal distribution".

It is clear that if a vector field is non-rotating, then it is orthogonally irrotational. The converse is not true (see below Example 5.3). Moreover, (5.2) implies that if V is orthogonally irrotational, then so is ϕV for any $\phi \in C^\infty_{>0}(N)$. Indeed, since ϕ does not vanish, X is orthogonal to ϕV if and only if it is orthogonal to V. However, if V is non-rotating and $\phi \in C^\infty_{>0}(N)$, ϕV is not necessarily non-rotating (see (5.2)).

Remark 5.1. Let V a KVF on a semi-Riemannian manifold (N, g_N). Then, V is non-rotating iff it is parallel. Indeed, for any $X, Y \in \mathfrak{X}(N)$

$$0 = \mathrm{L}_V g_N(X, Y) = \operatorname{curl} V(X, Y) + 2g_N(\nabla_Y V, X). \tag{5.3}$$

Thus,

(1) $\operatorname{curl} V(X, Y) = 0$ for any $X, Y \in \mathfrak{X}(N)$;
(2) for any $Y \in \mathfrak{X}(N)$ "$g_N(\nabla_Y V, X) = 0$ for any $X \in \mathfrak{X}(N)$";
(3) $\nabla_Y V = 0$ for any $Y \in \mathfrak{X}(N)$;
(4) V is parallel (see [5, p. 63]);

are equivalent.

Remark 5.2. Let $\mathbb{R}_f \times F$ be a SSS-T. Recall that any $V \in \mathfrak{X}(\mathbb{R} \times F)$ admits a decomposition as $V_\mathbb{R} + V_F$ (see above Proposition 4.4).

[8] For any $\phi \in C^\infty(N)$, the torsion of the function ϕ, T_ϕ, is the antisymmetric 2-covariant tensor field defined by $T_\phi(X, Y) := X(\phi)Y - Y(\phi)X$ for all $X, Y \in \mathfrak{X}(N)$ (taking attention to the sign in the definition, see for instance [41, p. 139]).

[9] In [40,42] this condition is called irrotational.

[10] In [5,17] this condition is called irrotational.

Let also

$$\mathrm{Tan} : T_{(q,p)}(\mathbb{R} \times F) \longrightarrow T_{(q,p)}(\mathbb{R} \times \{p\}) \tag{5.4}$$

the projection onto the subspace $T_{(q,p)}(\mathbb{R} \times \{p\})$ and

$$\mathrm{Nor} : T_{(q,p)}(\mathbb{R} \times F) \longrightarrow (T_{(q,p)}(\mathbb{R} \times \{p\}))^{\perp} = T_{(q,p)}(\{q\} \times F) \tag{5.5}$$

the orthogonal projection onto the subspace $T_{(q,p)}(\{q\} \times F)$.

Then applying [5, Proposition 35] we obtain that for any $X, Y \in \mathfrak{L}(F)$ and any $V, W \in \mathfrak{L}(\mathbb{R})$ hold

(1) $\nabla_X Y = \nabla^F_X Y$,
(2) $\nabla_X V = \nabla_V X = \frac{X(f)}{f} V$,
(3) $\nabla_V W = \underbrace{\nabla^{\mathbb{R}^1_1}_V W}_{\mathrm{Tan}\nabla_V W} + \underbrace{f dt^2(V, W)\mathrm{grad}_F f}_{\mathrm{Nor}\nabla_V W}$,

where $\mathbb{R}^1_1 = (\mathbb{R}, -\mathrm{d}t^2)$ and ∇ (respectively, $\nabla^{\mathbb{R}^1_1}$ and ∇^F) is the Levi–Civita connection of $\mathbb{R}_f \times F$ (respectively, $\mathbb{R}^1_1$ and (F, g_F)).

In particular

(1) $\nabla_X Y = \nabla^F_X Y$,
(2) $\nabla_X \partial_t = \nabla_{\partial_t} X = \frac{X(f)}{f} \partial_t = X(\ln f)\, \partial_t$,
(3) $\nabla_{\partial_t} \partial_t = f \mathrm{grad}_F f$.

Example 5.3 (*Elementary but Important*)**.** Clearly, ∂_t is a stationary KVF on any SSS-T $(\mathbb{R} \times F, g := -f^2 dt^2 + g_F)$. Furthermore, "$\partial_t$ is non-rotating iff f is a positive constant". Indeed, the identity

$$-f^2 \mathrm{grad} t = \partial_t \tag{5.6}$$

and (5.2) imply that

$$\mathrm{curl}\, \partial_t = \frac{1}{f^2} g(\partial_t, T_{f^2}). \tag{5.7}$$

So, if f is constant, ∂_t is non-rotating (because $T_{f^2} \equiv 0$).

On the other hand, $T_{f^2}(\partial_t, Z) = -Z(f^2)\partial_t \ \forall Z \in \mathfrak{X}(\mathbb{R} \times F)$, so

$$(\mathrm{curl}\, \partial_t)(\partial_t, Z) = Z(f^2) \quad \forall Z \in \mathfrak{X}(\mathbb{R} \times F). \tag{5.8}$$

Thus, if ∂_t is non-rotating, then f is constant.

However, ∂_t is always orthogonally irrotational. Indeed, it is sufficient to observe that by (5.7), there results

$$(\mathrm{curl}\, \partial_t)(X, Y) = 0 \quad \forall X, Y \in \mathfrak{X}(\mathbb{R} \times F)\ g\text{-orthogonal to } \partial_t. \tag{5.9}$$

Theorem 5.4. *Let κ be a KVF on a SSS-T of the form $(\mathbb{R} \times F, g := -f^2 dt^2 + g_F)$. If κ is parallel (or non-rotating) then κ belongs to the set (2.1) and satisfies one of the following conditions:*

(1) *$h \equiv 0$ or $\psi \equiv 0$; in these cases κ is the lift of a $\kappa_1 \in \mathscr{K}^0_{\ln f}$ parallel (or non-rotating) on (F, g_F);*
(2) *$\psi = \frac{C}{f}$ where $C \neq 0$ is a constant, $H^f_F = 0$ (in particular f is harmonic of constant sign, $\sqrt{g_F(\mathrm{grad} f, \mathrm{grad} f)}$ is constant and* grad*f is Killing on (F, g_F)) and the part in $\widehat{K} + \mathscr{K}^0_{\ln f}$ is parallel (or non-rotating) on (F, g_F). Furthermore $\nu = -g_F(\mathrm{grad}_F f, \mathrm{grad}_F f)$ and hence non-positive.*

Proof. By Theorem 2.1 κ takes the form (2.1), i.e., there exists $\check{K} \in \mathscr{K}^0_{\ln f}$ such that

$$\kappa = \psi h \partial_t + \int_{t_0}^{(\cdot)} h(s)\, \mathrm{d}s\, f^2 \mathrm{grad}_F \psi + \widehat{K} + \check{K}, \tag{5.10}$$

where h, ψ and $\widehat{K}$ are like in Theorem 2.1. Then, by Remark 5.2, we obtain for any $X \in \mathfrak{L}(F)$ the following expressions:

i:

$$\begin{aligned} \nabla_{\partial_t}[\psi h \partial_t] &= \psi h \nabla_{\partial_t} \partial_t + \partial_t(\psi h)\partial_t \\ &= \psi h f \mathrm{grad} f + \psi h' \partial_t \\ &= \psi h f^2 \mathrm{grad}(\ln f) + \psi h' \partial_t; \end{aligned} \tag{5.11}$$

ii:

$$\begin{aligned}\nabla_X[\psi h\partial_t] &= \psi h\nabla_X\partial_t + X(\psi h)\partial_t \\ &= \psi hX(\ln f)\partial_t + hX(\psi)\partial_t;\end{aligned} \tag{5.12}$$

iii:

$$\begin{aligned}\nabla_{\partial_t}\left[\int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\, f^2\mathrm{grad}_F\psi\right] &= \int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\,\nabla_{\partial_t}[f^2\mathrm{grad}_F\psi] + hf^2\mathrm{grad}_F\psi \\ &= \int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\,\underbrace{[f^2\mathrm{grad}_F\psi](\ln f)}_{=\nu\psi}\partial_t + hf^2\mathrm{grad}_F\psi \\ &= \int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\,\nu\psi\,\partial_t + hf^2\mathrm{grad}_F\psi;\end{aligned} \tag{5.13}$$

iv:

$$\begin{aligned}\nabla_X\left[\int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\, f^2\mathrm{grad}_F\psi\right] &= \int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\,\nabla_X[f^2\mathrm{grad}_F\psi] + \underbrace{X\left[\int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\right]}_{=0} f^2\mathrm{grad}_F\psi \\ &= \int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\,\nabla_X^F[f^2\mathrm{grad}_F\psi];\end{aligned} \tag{5.14}$$

v:

$$\nabla_{\partial_t}\left[\widehat{K} + \check{K}\right] = \widehat{K}(\ln f)\partial_t; \tag{5.15}$$

vi:

$$\nabla_X\left[\widehat{K} + \check{K}\right] = \nabla_X^F\left[\widehat{K} + \check{K}\right]. \tag{5.16}$$

Thus and since κ is parallel, we obtain:

$$\begin{aligned}0 = \nabla_{\partial_t}\kappa &= \psi hf^2\mathrm{grad}_F(\ln f) + \psi\underbrace{\left[h' + \nu\int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\right]}_{=h'(t_0)}\partial_t + hf^2\mathrm{grad}_F\psi + \underbrace{\widehat{K}(\ln f)}_{-h'(t_0)\psi}\partial_t \\ &= f^2h[\psi\,\mathrm{grad}_F(\ln f) + \mathrm{grad}_F\psi] \\ &= fh[\psi\,\mathrm{grad}_F f + f\,\mathrm{grad}_F\psi] \\ &= fh\,\mathrm{grad}_F[f\psi]\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}0 = \nabla_X\kappa &= h[\psi X(\ln f) + X(\psi)]\partial_t + \int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\,\nabla_X^F[f^2\mathrm{grad}_F\psi] + \nabla_X^F\left[\widehat{K} + \check{K}\right] \\ &= \frac{h}{f}X(f\psi)\partial_t + \int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s\,\nabla_X^F[f^2\mathrm{grad}_F\psi] + \nabla_X^F\left[\widehat{K} + \check{K}\right].\end{aligned} \tag{5.18}$$

Since $f > 0$, (5.17) infers $h \equiv 0$ or "$f\psi = C$ constant with $\mathrm{sign}\psi = \mathrm{sign}C$".
Suppose that $h \not\equiv 0$. So $f\psi = C$ constant,

$$f^2\mathrm{grad}_F\psi = -\mathrm{grad}_F(Cf) \tag{5.19}$$

and considering (5.18) there results

$$\nabla_X^F\mathrm{grad}(Cf)\int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s = \nabla_X^F\left[\widehat{K} + \check{K}\right]. \tag{5.20}$$

So, if there exists $X \in \mathfrak{X}(F)$ such that $\nabla_X^F\mathrm{grad}(Cf) \neq 0$, then by a separation of variables argument $\int_{t_0}^{(\cdot)} h(s)\,\mathrm{d}s$ is constant (the right hand side of (5.20) is independent of t). But this implies that $h \equiv 0$, the contrary of our assumption. So, $\nabla_X^F\mathrm{grad}(Cf) = 0$ for all $X \in \mathfrak{X}(F)$, or equivalently $H_F^{(Cf)} = 0$ (see Remark 3.5). In particular Cf is harmonic on (F, g_F) with constant sign.

Hence, if $h \equiv 0$, then κ is the lift of a $\kappa_1 \in \mathscr{K}^0_{\ln f}$ parallel on (F, g_F) (by (5.10) and (2.4)). Otherwise, if $C = 0$ then $\psi \equiv 0$ and again by (5.10) and (2.4), κ is the lift of a $\kappa_1 \in \mathscr{K}^0_{\ln f}$ parallel on (F, g_F); if $C \neq 0$, then (5.19) holds and the left hand side of (5.20) is 0 for all $X \in \mathfrak{X}(F)$, so κ belongs to (2.1) with $\psi = \frac{C}{f}$, $H^f_F = 0$ (particularly f results harmonic of positive sign), $\sqrt{g_F(\mathrm{grad} f, \mathrm{grad} f)}$ constant and grad$f$ is Killing on (F, g_F) (see Remark 3.5) and the part in $\widehat{K} + \mathscr{K}^0_{\ln f}$ parallel on (F, g_F). The properties of ν in the latter case are consequence of Remark 4.14.

Note that the case non-rotating is equivalent to the case parallel by Remark 5.1. □

Remark 5.5. Taking into account the role played by the positive solutions of the equation

$$H^f_F = 0 \quad \text{on } (F, g_F), \tag{5.21}$$

we observe that Kanai [43, Theorem B] and Tashiro [44] proved, applying the de Rham's decomposition theorem [45], that

> A complete Riemannian manifold (F, g_F) of dimension $s \geq 2$ has a nontrivial (i.e., nonconstant) solution of (5.21) iff (F, g_F) is a Riemannian product $(\bar{F} \times \mathbb{R}, g_{\bar{F}} + g_0)$ of a complete Riemannian manifold $(\bar{F}, g_{\bar{F}})$ and the real line $(\mathbb{R}, g_0)$, where g_0 is the canonical metric of $\mathbb{R}$.

On the other hand, clearly the solutions of (5.21) are harmonic, but we are interested in positive solutions (note that these will be warping functions in our study). Thus, it is interesting to recall the so called Liouville type theorems on complete Riemannian manifolds about the existence of harmonic functions of constant sign. Among others we mention the following pioneering Yau [46, Corollary 1] result:

> On a complete Riemannian manifold with nonnegative Ricci curvature, every positive harmonic function on the whole manifold is constant.

Furthermore, we know that the solutions of (5.21) has gradient of constant norm. About questions of existence of these type of functions we reference the reader to the Sakai article [47], where it is proved (among other results) that

> On a complete Riemannian manifold with nonnegative Ricci curvature, any smooth function with gradient of constant norm is an affine function.

Now, taking into account the latter remark, we will characterize the non-rotating KVFs on a SSS-T under an hypothesis about the Ricci curvature of the Riemannian part when this is complete.

Theorem 5.6. *Let κ be a KVF on a SSS-T of the form $(\mathbb{R} \times F, g := -f^2 dt^2 + g_F)$ where (F, g_F) is a complete Riemannian manifold with nonnegative Ricci curvature. If κ is non-rotating then one of the following conditions hold*

(1) *κ is the lift of a $\kappa_1 \in \mathscr{K}^0_{\ln f}$ non-rotating on (F, g_F);*
(2) *f is a positive constant and κ belongs to*

$$\psi a \partial_t + \mathscr{K}, \tag{5.22}$$

where $\psi = \frac{C}{f}$ with $C \neq 0$ a constant and the element in $\mathscr{K}$ is non-rotating on (F, g_F).

Conversely, if κ is a vector field on $(\mathbb{R} \times F, g := -f^2 dt^2 + g_F)$ verifying (1) or (2), then κ is a non-rotating KVF on $(\mathbb{R} \times F, g := -f^2 dt^2 + g_F)$.

Proof. The condition (1) is clear by the analogous condition in Theorem 5.4. If (2) is the condition verified in Theorem 5.4, by the Yau result cited in Remark 5.5 f is a positive constant and $\psi = \frac{C}{f} \neq 0$, where C is a constant. Since any vector field is zero on any constant, (2.3) implies $\nu = 0$ and (2.4) implies $0 = \widehat{K}(\ln f) = -h'(t_0)\psi$, so by (2.2) h is a constant. Furthermore, again because f is constant $\mathscr{K} = \mathscr{K}^0_{\ln f}$ and the element in $\mathscr{K}$ is non-rotating by Theorem 5.4.

The converse is an immediate consequence of Remark 5.2, Corollary 4.3 and Example 5.3. □

Corollary 5.7. *Let κ be a KVF on a SSS-T $(\mathbb{R} \times F, g := -f^2 dt^2 + g_F)$ where (F, g_F) is a complete Riemannian manifold with nonnegative Ricci curvature and f nonconstant. If κ is non-rotating then it is the lift of a $\kappa_1 \in \mathscr{K}^0_{\ln f}$ non-rotating on (F, g_F).*

Proposition 5.8. *Let $(\mathbb{R} \times F, g := -f^2 dt^2 + g_F)$ be a SSS-T where (F, g_F) is a complete Riemannian manifold and f is nonconstant. If κ is a non-rotating KVF on $\mathbb{R}_f \times F$ then it is the lift of a $\kappa_1 \in \mathscr{K}^0_{\ln f}$ non-rotating on (F, g_F) or (F, g_F) has negative Ricci curvature somewhere and is a Riemannian product $(\bar{F} \times \mathbb{R}, g_{\bar{F}} + g_0)$ of a complete Riemannian manifold $(\bar{F}, g_{\bar{F}})$ and the real line $(\mathbb{R}, g_0)$, where g_0 is the canonical metric of $\mathbb{R}$.*

Proof. It is a consequence of Theorem 5.4 and the de Rham–Tashiro–Kanai result mentioned in Remark 5.5. □

At this point we deal with the case where the stronger hypothesis of compactness of the Riemannian part holds.

Lemma 5.9. *Let (F, g_F) be compact and simply connected. The unique KVF on (F, g_F) with zero curl is the zero field.*

Proof. Let $K \in \mathfrak{X}(F)$ a KVF such that $\operatorname{curl} K = 0$. Since (F, g_F) is simply connected, K is a gradient (see [48]), i.e., there exists $\psi \in C^\infty(F)$ such that $\operatorname{grad}_F \psi = K$. Then, by Remarks 3.5 and 5.1 we infer that ψ is harmonic, but (F, g_F) is compact, so ψ is constant and as a consequence K is zero. □

Proposition 5.10. *Let (F, g_F) be compact and simply connected. If $f \in C^\infty_{>0}(F)$ is nonconstant, then there is no nontrivial (non-identically zero) non-rotating KVF on the SSS-T $(\mathbb{R} \times F, g := -f^2 \mathrm{d}t^2 + g_F)$.*

Proof. Let κ be a non-rotating KVF on $(\mathbb{R} \times F, g := -f^2 \mathrm{d}t^2 + g_F)$. Item (2) in Theorem 5.4 is not verified because f would be harmonic and nonconstant on a compact manifold. So (1) is verified, in particular κ is the lift of a non-rotating $\kappa_1 \in \mathscr{K}^0_{\ln f}$ on (F, g_F). Hence, applying Lemma 5.9, κ_1 and as a consequence its lift κ is zero. □

We will finish with the following additional result.

Proposition 5.11. *Let κ be a KVF on a SSS-T of the form $(\mathbb{R} \times F, g := -f^2 dt^2 + g_F)$ such that $(\operatorname{curl} \kappa)(X, Y) = 0$ for all $X, Y \in \mathfrak{X}(\mathbb{R} \times F)$ g-orthogonal to ∂_t. If (F, g_F) is compact and simply connected, then $\kappa = a\partial_t$ where a is a real constant. In particular, κ becomes time-like if $a \neq 0$.*

Proof. Notice that letting κ be a KVF on $(\mathbb{R} \times F, g)$ where (F, g_F) is compact, Theorem 2.2 implies that $\kappa = a\partial_t + K$ where a is a constant and K is a KVF on (F, g_F) with $K(f) = 0$. Thus, by hypothesis, linearity of the curl and (5.9), we obtain that

$$\operatorname{curl} K(X, Y) = 0 \quad \text{for all } X, Y \in \mathfrak{X}(\mathbb{R} \times F)\ g\text{-orthogonal to } \partial_t. \tag{5.23}$$

Since the lifts of elements in $\mathfrak{X}(F)$ are g-orthogonal to ∂_t that verifies (5.23). Then, applying Remark 5.2 and the definition of the curl , there results $\operatorname{curl}_F K = 0$ (curl_F denotes the curl on the Riemannian manifold (F, g_F)), i.e., K is non-rotating on (F, g_F). Hence applying Lemma 5.9 $K \equiv 0$. Thus we have established that $\kappa = a\partial_t$, where a is constant. □

Remark 5.12. Notice that in Proposition 5.11 the involved vector fields necessarily turn out to be causal (i.e., non-spacelike). The conclusion would be invalid if we eliminate the simply connectedness. For example, consider the vector field $a\partial_t + \partial_\theta$ where $a < 1$ on $(\mathbb{R} \times \mathbb{S}^1, g := -\mathrm{d}t^2 + \mathrm{d}\theta^2)$. This is a non-rotating KVF and yet not time-like due to $a < 1$, indeed $g(a\partial_t + \partial_\theta, a\partial_t + \partial_\theta) = -a^2 + 1 > 0$.

6. Conclusions

It is very well known that a space-time possesses a symmetry if it admits nontrivial KVFs. Thus existence and characterization problems of KVFs are extremely important in the geometry of space-times (see [21]). In Theorem 2.1 we give a description of the set of KVFs of a SSS-T, where the role of an over-determined system of partial differential equations on the Riemannian part is central, namely (4.22) (or equivalently (4.27) and (4.30)). Our analysis corrects the computational mistakes in [22] mentioned in our Section 1.

As a consequence of Theorem 2.1 and the well known results about the eigenvalues and eigenfunctions of a positively weighted elliptic problem on a compact Riemannian manifold without boundary, we also provide a characterization of the KVFs on a SSS-T with compact Riemannian part in Theorem 2.2. Note that by combining this theorem with the vanishing results of Bochner (see Remark 4.11), we obtain that in a SSS-T with compact Riemannian part of negative Ricci curvature without boundary, the only KVFs are of the form $c\partial_t$ and yet time-like where $c \in \mathbb{R}$ is constant. The study of analogous results to Theorem 2.2 but with *noncompact* Riemannian part is an open question.

However, Theorem 2.1 allow us to obtain a characterization of parallel KVFs on SSS-Ts of the form $\mathbb{R}_f \times F$ in Theorem 5.4. Then, combining the latter with the Liouville type results of Yau about the existence of harmonic functions bounded from below on complete Riemannian manifolds with nonnegative Ricci curvature and with the results of de Rham–Tashiro–Kanai about the existence of concircular scalar fields, we partially classify the KVFs on SSS-Ts where the natural part is complete (see Theorem 5.6 and its consequences in Section 5). The classification is complete if the Riemannian part is either of nonnegative Ricci curvature or compact and simply connected.

We would like to observe that other relevant problem is the full classification of the conformal KVFs of a SSS-T. There are partial recent results in this direction (see for instance [49,18] and the references therein).

In principle, one can apply the technique developed here to characterize KVFs or CKVFs of some other space-time models such as stationary space-times[11] and multiply generalized Robertson–Walker space-times.[12] However, the expressions for the Lie derivative and the KVF or CKVF equations of such space-time models result more complex ones than those of SSS-Ts. All these equations bring to the study of more sophisticated linear and nonlinear partial differential equations on manifolds. We will deal with the study of these questions in future works.

11 A space-time is called *stationary* if it admits a time-like Killing vector field [15].

12 See [50] for a definition and properties of *multiply generalized Robertson–Walker space-times*.

Acknowledgments

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Appendix

Notice that fixed $\nu \in \mathbb{R} \setminus \{0\}$, (4.20) is equivalent to

$$-h'' = \nu h; \tag{A.1a}$$

$$f^2 \mathrm{grad}_F \psi \in \mathscr{K}_{\ln f}^{\nu\psi}; \tag{A.1b}$$

$$\begin{cases} \forall \overline{b} : \phi^{\overline{b}}(t) = \tau^{\overline{b}} \displaystyle\int_{t_0}^{t} h(s)\,\mathrm{d}s + \omega^{\overline{b}} \quad \text{where } \tau^{\overline{b}}, \omega^{\overline{b}} \in \mathbb{R} : \\ f^2 \mathrm{grad}_F \psi = \tau^{\overline{b}} K_{\overline{b}} \quad \text{and} \\ \left(\omega^{\overline{b}} + \dfrac{h'(t_0)}{\nu} \tau^{\overline{b}} \right) K_{\overline{b}} \in \mathscr{K}_{\ln f}^{0}. \end{cases} \tag{A.1c}$$

Now we correct a couple of computational mistakes in [22] that result in wrong conclusions in that article. In any case the local approach of the authors is generically correct. First of all we observe that SSS-Ts of dimension 4 with $I = \mathbb{R}$ correspond to "*warped space-times of class* A_2" with "$\epsilon = -1$" in their notation. Carot and da Costa deal with the study of KVFs of A_2 warped product space-times applying local techniques in [22, Section 4.2].

The first mistake is in [22, Eq. (70) p. 475] namely, the second term on the left hand side must be multiplied by ϵ. It is particularly relevant for SSS-Ts, indeed in this case $\epsilon = -1$.

The second mistake is connected to the case $\nu \neq 0$ in (4.20) (or equivalently (A.1)): in this situation it is not possible to assume a priori that $h'(t_0) = 0$, which contradicts with [22, left Eq. (78a) p. 476]. The right hand side of the latter would be $c\lambda$ where c is a real constant arising in the integration constant term of [22, Eq. (72) p. 476]. This mistake propagates along the authors' analysis of their case $k = 1, \epsilon = 1$.

In any case, it is possible to justify expressions like (79) or (84) in [22], applying (A.1) or (2.5).

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