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PID controller design for fractional-order systems with time delays

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ABSTRACT

Classical proper PID controllers are designed for linear time invariant plants whose transfer functions are rational functions of s^α, where $0 < \alpha < 1$, and s is the Laplace transform variable. Effect of input–output time delay on the range of allowable controller parameters is investigated. The allowable PID controller parameters are determined from a small gain type of argument used earlier for finite dimensional plants.

1. Introduction

Fractional order system models have been widely studied over the past two decades (see e.g., [1–7] and their references), where stability analysis and controller design problems are studied. Another line of research in this context is the design of fractional order controllers, including fractional order PID controllers, for fractional order as well as rational (finite dimensional) systems [8–14].

Fractional order systems appear in various engineering applications, see, e.g., [15–20]. It is interesting to see that they might appear in two ways. First, through theoretical modeling of physical phenomena and second from frequency domain experiments when traditional integer order models do not fit the data (for instance when Bode diagrams do not show slopes of integer multiples of 20dB/decade [21]). Many fields are concerned. In electricity, models of polarization emittance of metal electrodes [22] as well as capacitor models (based on purely empirical Curie's law of 1889) [23] are of fractional type. In material sciences, fractional order derivatives are used to model visco-elastic materials [24], non-laminated ferromagnetic components [25] or magnetic core coils [21]. Other physical phenomena such as heat conduction [26] or flexible structures [27] give rise to transfer functions with fractional powers of s (typically square root of s).

The topic of the present work is the design of *classical* proper PID controllers for fractional order systems.

Many different PID controller design techniques are available for rational (finite dimensional) systems with time delays; e.g. [28–31]. In this paper, we extend the approach of [28] to fractional order systems with time delays.

The class of plants considered and the feedback control problem studied are defined in Section 2. The proposed PID controller design method is described in Section 3. A numerical example is given in Section 4, and concluding remarks are made in Section 5.

2. Problem definition

Consider the standard single input–single output feedback system shown in Fig. 1, where C is the controller to be designed for the plant P.

We assume that the plant is linear and time invariant. Its dynamical behavior is represented by the transfer function

$$P(s) = e^{-hs}\,\frac{G(s^\alpha)}{s^\alpha - p} \tag{1}$$

where s is the Laplace transform variable, $h > 0$ is the total input–output time delay, $\alpha \in (0, 1)$ is the fractional order, $p \geq 0$ ($p^{1/\alpha}$ being the location of the unstable pole of the plant), and $G(w)$ is a rational stable transfer function in the variable $w = s^\alpha$ with $G(p) \neq 0$ and $G(0) \neq 0$. Such a plant was considered with $h = 0$ in [25] when modeling non-laminated electromagnetic suspensions.

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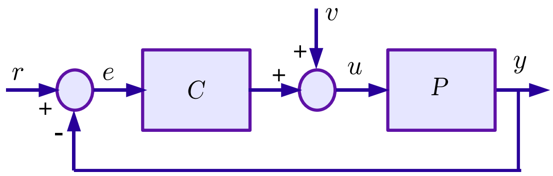

Fig. 1. Standard feedback system.

It is clear that we need $G(0) \neq 0$ for stabilizability of (1) by a controller which includes an integrator. We assume that α is a rational number, i.e., we are restricting ourselves to the class of *fractional systems of commensurate order*, [6]. There is a simple stability test for this type of systems, which can be seen below.

Given all the parameters of plant (1), our goal is to design a classical Proportional + Integral + Differential (PID) controller in the form

$$C(s) = K_p + \frac{K_i}{s} + K_d \frac{s}{\tau_d s + 1} \tag{2}$$

where K_p, K_i, K_d are free parameters and τ_d is an arbitrarily small positive number making the controller proper.

The feedback system formed by the controller C and the plant P is stable if $(1 + PC)^{-1}$, $C(1 + PC)^{-1}$ and $P(1 + PC)^{-1}$ are stable transfer functions. These transfer functions are indeed fractional delay systems of retarded type and it has been proven [32] that H_∞-stability of these systems is equivalent to their BIBO-stability, a necessary and sufficient condition being that the system has no poles in the right half-plane (including no pole of fractional order at $s = 0$) and a numerical algorithm to test this property is available in [33]. In the case of fractional systems of commensurate order, checking stability can be done as follows (see e.g. [3,6]). Let $w = s^\alpha$ and assume that $T(w)$ is a rational function with poles $w_1, \ldots, w_n$. Enumerate the poles so that $w_1, \ldots, w_{2n_c}$ are complex conjugate, with $w_{n_c+k} = \bar{w}_k$ and $w_k = |w_k| e^{j\theta_k}$ where $\theta \in (0, \pi)$ for $k = 1, \ldots, n_c$, and $w_{2n_c+1}, \ldots, w_n$ are real. Then, the system $T(s^\alpha)$ is stable if and only if

$$\alpha \frac{\pi}{2} < \theta_k \quad \text{for } k = 1, \ldots, n_c, \text{ and}$$
$$w_k < 0 \quad \text{for } k = 2n_c + 1, \ldots, n.$$

We say that C is a stabilizing controller for the plant P if the feedback system formed by this pair is stable.

3. PID controller design

In this section, we design classical PID controllers in form (2) for plant (1). As in [28], the design will be done in two steps: first, PD controllers will be investigated, and then the integral action will be added.

3.1. PD controller design

A typical PD controller can be written in the form

$$C_{pd}(s) = K_p \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right). \tag{3}$$

We can express the non-delayed part of the plant as the ratio of two stable factors:

$$P(s) = e^{-hs} Y(s)^{-1} X(s) \quad \text{with } Y(s) := \frac{s^\alpha - p}{s^\alpha + x}$$
$$X(s) := \frac{G(s^\alpha)}{s^\alpha + x} \tag{4}$$

where $x > 0$ is the free parameter. While it is an arbitrary positive number at this stage, x plays an important role in the controller design.

With the notation introduced in (4), the feedback system stability is equivalent to stability of U^{-1}, where

$$U(s) := Y(s) + e^{-hs} X(s) C_{pd}(s). \tag{5}$$

Inserting C_{pd}, X and Y into (5), we have

$$U(s) = 1 - \frac{(p + x)}{s^\alpha + x} + e^{-hs} \frac{G(s^\alpha)}{s^\alpha + x} K_p \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right).$$

By choosing

$$K_p = (p + x) G(0)^{-1} \tag{6}$$

we obtain

$$\begin{aligned} U(s) &= 1 - \frac{(p + x)}{s^\alpha + x} \\ &\quad \times \left(1 - e^{-hs} G(s^\alpha) G(0)^{-1} \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right)\right) \\ &= 1 - \frac{(p + x)\, s^\alpha}{s^\alpha + x} \\ &\quad \times \left(\frac{1 - e^{-hs} G(s^\alpha) G(0)^{-1}}{s^\alpha} - \frac{\widetilde{K}_d e^{-hs} G(s^\alpha)\, s^{1-\alpha}}{G(0)\, (\tau_d s + 1)}\right). \end{aligned} \tag{7}$$

Since $\|\frac{s^\alpha}{s^\alpha + x}\|_\infty = 1$ for all $x > 0$, by the small gain theorem, U^{-1} is stable if

$$\left\| \left(\frac{1 - e^{-hs} G(s^\alpha) G(0)^{-1}}{s^\alpha}\right) - \widetilde{K}_d\, e^{-hs} G(s^\alpha) G(0)^{-1} \frac{s^{1-\alpha}}{\tau_d s + 1} \right\|_\infty < \frac{1}{(p + x)}.$$

The following results are immediate consequences of the above discussion.

Lemma 1. *For plant (1) there exists a stabilizing proportional controller, $C(s) = K_p$, if*

$$p < \left\| \frac{1 - e^{-hs} G(s^\alpha) G(0)^{-1}}{s^\alpha} \right\|_\infty^{-1} =: \psi_o. \tag{8}$$

When (8) holds, all proportional controllers in the form (6) are stabilizing, where x satisfies $0 < x < (\psi_o - p)$. □

Lemma 2. *Suppose there exist $\widetilde{K}_d \in \mathbb{R}$ and $\tau_d > 0$, such that*

$$p < \left\| \left(\frac{1 - e^{-hs} G(s^\alpha) G(0)^{-1}}{s^\alpha}\right) - \widetilde{K}_d\, e^{-hs} G(s^\alpha) G(0)^{-1} \times \frac{s^{1-\alpha}}{\tau_d s + 1} \right\|_\infty^{-1} =: \psi_d. \tag{9}$$

Then, the controller $C_{pd}(s) = K_p(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1})$ is a stabilizing controller for plant (1) with $K_p = (p + x) G(0)^{-1}$ for all x satisfying $0 < x < (\psi_d - p)$. □

From the PD controller design method proposed in Lemma 2, we see that the allowable values of the proportional gain are in the range

$$K_p^{\min} := p G(0)^{-1} < K_p < \psi_d G(0)^{-1} =: K_p^{\max}.$$

Therefore, we would like to maximize ψ_d in order to maximize the allowable range for K_p. This problem is equivalent to finding the optimal $\widetilde{K}_d \in \mathbb{R}$ so that

$$\psi_d^{-1} = \left\| \left(\frac{1 - \mathrm{e}^{-hs} G(s^\alpha) G(0)^{-1}}{s^\alpha} \right) - \widetilde{K}_d\, \mathrm{e}^{-hs} G(s^\alpha) G(0)^{-1} \times \frac{s^{1-\alpha}}{\tau_d s + 1} \right\|_\infty \tag{10}$$

is minimized for a given fixed $\tau_d > 0$. A similar problem has been studied in [30] for the case $\alpha = 1$, i.e., for rational systems with time delays. In general, minimization of ψ_d^{-1} is a two-dimensional search: for each fixed $\widetilde{K}_d \in \mathbb{R}$, compute the infinity norm by a frequency sweep. In [30], it is shown that, for a large class of rational systems with time delays, this computation can be reduced to a one dimensional search. Currently, we do not know if a similar result can be obtained for the class of plants studied here; we leave this problem open for a future study.

Once ψ_d is maximized, we would like to choose K_p so that the gain margin is maximized, i.e.,

$$\min \left\{ \frac{K_p}{K_p^{\min}}, \frac{K_p^{\max}}{K_p} \right\}$$

is maximized, [34]. Clearly, the optimal choice is $K_p^{\mathrm{opt}} = \sqrt{K_p^{\min} K_p^{\max}}$, i.e.

$$K_p^{\mathrm{opt}} = \sqrt{p\, \psi_d}\, G(0)^{-1}. \tag{11}$$

3.2. Adding integral action to the PD controller

Assume that condition (9) of Lemma 2 is satisfied and hence a stabilizing PD controller C_{pd} can be found for plant (1). We now try to find

$$C_i(s) = \frac{K_i}{s} \tag{12}$$

so that $C_{pid}(s) = C_{pd}(s) + C_i(s)$ is a stabilizing controller for the plant. This is a two step design process and it works as follows; see e.g. [28,35]. Define

$$H(s) := P(s)(1 + P(s)C_{pd}(s))^{-1} \tag{13}$$

and note that $H(0) = G(0)/x$ which is non-zero by the assumption that $G(0) \neq 0$ and by design $x > 0$. If C_i defined by (12) is a stabilizing controller for the "new plant" H (13), then C_{pid} is a stabilizing controller for the original plant P. Now let

$$K_i := \gamma H(0)^{-1}, \quad \text{with } \gamma > 0 \tag{14}$$

then

$$(1 + C_i(s)H(s))^{-1} = \frac{s}{s+\gamma} \left(1 + \frac{\gamma\, s^\alpha}{s+\gamma} \left(\frac{H(s)H(0)^{-1} - 1}{s^\alpha} \right) \right)^{-1}. \tag{15}$$

Let us define

$$R_\alpha(\gamma) := \left\| \frac{\gamma\, s^\alpha}{s+\gamma} \right\|_\infty. \tag{16}$$

Then by the small gain theorem $C_i(s) = \gamma H(0)^{-1}/s$ is a stabilizing controller for $H(s)$ if

$$0 < R_\alpha(\gamma) < \left\| \frac{H(s)H(0)^{-1} - 1}{s^\alpha} \right\|_\infty^{-1}. \tag{17}$$

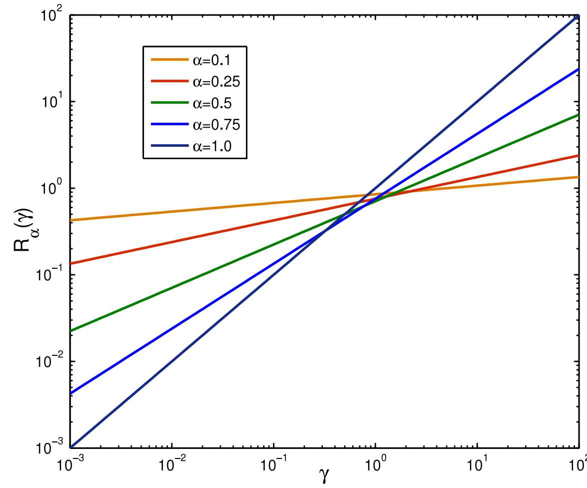

Fig. 2. $R_\alpha(\gamma)$ versus γ.

Note that for $\alpha = 1$ we have $R_1(\gamma) = \gamma$, and for the rational case the function $(H(s)H(0)^{-1} - 1)/s$ is stable. However, when $H(s)$ is a fractional transfer function, $(H(s)H(0)^{-1} - 1)/s$ might be unstable due to problems of boundedness at zero. Therefore, writing

$$(1 + C_i(s)H(s))^{-1} = \frac{s}{s+\gamma} \left(1 + \frac{\gamma\, s}{s+\gamma} \left(\frac{H(s)H(0)^{-1} - 1}{s} \right) \right)^{-1}, \tag{18}$$

rather than (15), and then applying the small gain theorem, as was done in [28], does not work in the case of fractional systems. So, we have to compute $R_\alpha(\gamma)$ as a function of γ for the specific α value appearing in the plant transfer function. It is a simple exercise to show that

$$R_\alpha(\gamma) = \alpha^{\alpha/2}\, (1-\alpha)^{(1-\alpha)/2}\, \gamma^\alpha. \tag{19}$$

The graphs of $R_\alpha(\gamma)$ versus γ for different values of α are shown in Fig. 2. Another observation we can make from (15) is that if $\|H(s)H(0)^{-1} - 1\|_\infty < 1$ then all $C_i(s) = \gamma H(0)^{-1}/s$ stabilize H, for any $\gamma > 0$.

The above discussion is summarized with the following results.

Lemma 3. *Assume that condition (8) of Lemma 1 is satisfied and the proportional controller $K_p = (p+x)G(0)^{-1}$ is designed to stabilize the plant $P(s) = \mathrm{e}^{-hs} G(s^\alpha)(s^\alpha - p)^{-1}$. Then the PI controller*

$$C_{pi}(s) = K_p + \frac{\gamma H(0)^{-1}}{s} = \left((p+x) + \frac{\gamma\, x}{s} \right) G(0)^{-1} \tag{20}$$

is a stabilizing controller for the plant P for all γ satisfying

$$0 < R_\alpha(\gamma) < \left\| \frac{H(s)H(0)^{-1} - 1}{s^\alpha} \right\|_\infty^{-1} \tag{21}$$

where $H(s) = P(s)(1 + K_p P(s))^{-1}$.

Lemma 4. *Assume that condition (9) is satisfied for some $\widetilde{K}_d \in \mathbb{R}$ and $\tau_d > 0$. Let C_{pd} be a stabilizing controller for the plant, $P(s) = \mathrm{e}^{-hs} G(s^\alpha)(s^\alpha - p)^{-1}$, as designed in Lemma 2. Then the PID controller*

$$C_{pid}(s) = C_{pd}(s) + \frac{\gamma H(0)^{-1}}{s} = \left((p+x) \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1} \right) + \frac{\gamma\, x}{s} \right) G(0)^{-1} \tag{22}$$

is a stabilizing controller for P for all γ satisfying

$$0 < R_\alpha(\gamma) < \left\| \frac{H(s)H(0)^{-1} - 1}{s^\alpha} \right\|_\infty^{-1} \tag{23}$$

where $H(s) = P(s)(1 + C_{pd}(s)P(s))^{-1}$. □

The above PI and PID controller design methods lead to an interesting question: what are the optimal choices of $x > 0$ such that the ranges of allowable γ, i.e. the right hand sides of (21) and (23), are the largest possible? For example, in the PI design, for each fixed x in the range $0 < x < (\psi_o - p)$, one can compute the upper bound in (21) numerically. Therefore, the largest allowable γ range and the corresponding optimal x can be found from a one dimensional numerical search. Clearly, it is not possible to find an analytical solution for this problem.

On the other hand, we can find a suboptimal analytical solution as follows. Recall that $H(0) = G(0)/x$ and

$$H(s) = e^{-hs}G(s^\alpha)((s^\alpha - p) + x - x + (p+x)G(0)^{-1}e^{-hs}G(s^\alpha))^{-1}.$$

Then we have

$$\frac{H(s)H(0)^{-1} - 1}{s^\alpha} = \frac{\frac{e^{-hs}G(s^\alpha)G(0)^{-1}x}{s^\alpha + x}\left(1 - \left(\frac{(p+x)s^\alpha}{s^\alpha+x}\right)\frac{1 - e^{-hs}G(s^\alpha)G(0)^{-1}}{s^\alpha}\right)^{-1} - 1}{s^\alpha} = \frac{\frac{p}{(s^\alpha+x)}\left(\frac{1 - e^{-hs}G(s^\alpha)G(0)^{-1}}{s^\alpha}\right) - \frac{1}{s^\alpha+x}}{1 - \frac{(p+x)s^\alpha}{s^\alpha+x}\left(\frac{1 - e^{-hs}G(s^\alpha)G(0)^{-1}}{s^\alpha}\right)}.$$

Recall that

$$\psi_o = \left\| \frac{1 - e^{-hs}G(s^\alpha)G(0)^{-1}}{s^\alpha} \right\|_\infty^{-1}.$$

So, from the above

$$\left\| \frac{H(s)H(0)^{-1} - 1}{s^\alpha} \right\|_\infty \le \frac{p\,\psi_o^{-1} + 1}{x}\left(1 - (p+x)\psi_o^{-1}\right)^{-1}.$$

Thus we have the following lower bound for the upper bound in (21),

$$\widetilde{\gamma} := x\,\frac{\psi_o - (p+x)}{\psi_o + p} \le \left\| \frac{H(s)H(0)^{-1} - 1}{s^\alpha} \right\|_\infty^{-1}. \tag{24}$$

Now we can maximize $\widetilde{\gamma}$ by an appropriate choice of x. It is a simple exercise to show that the optimal choice of x maximizing $\widetilde{\gamma}$ is

$$x_{opt} = \frac{\psi_o - p}{2} \tag{25}$$

and the corresponding maximal $\widetilde{\gamma}$ is $\frac{x_{opt}^2}{\psi_o + p}$. This means that, by (19), γ should be in the range

$$0 < \gamma < \frac{c_\alpha\, x_{opt}^{2/\alpha}}{(\psi_o + p)^{1/\alpha}} =: \gamma_{max}$$

$$\text{where } c_\alpha := \left(\sqrt{\alpha}\,(1-\alpha)^{(1-\alpha)/2\alpha}\right)^{-1}. \tag{26}$$

For example $c_{0.5} = 2$. We propose to choose

$$\gamma_{opt} := \frac{\gamma_{max}}{2} \tag{27}$$

as the (sub)optimal γ value to be used in the PI controller. Inserting (25) into the PI controller expression (20), we obtain

$$C_{pi}(s) = \left(1 + \frac{\gamma_{opt}}{s}\right) x_{opt}\, G(0)^{-1} \tag{28}$$

as the suboptimal PI controller, where x_{opt} is given by (25) and γ_{opt} is determined from (26) to (27).

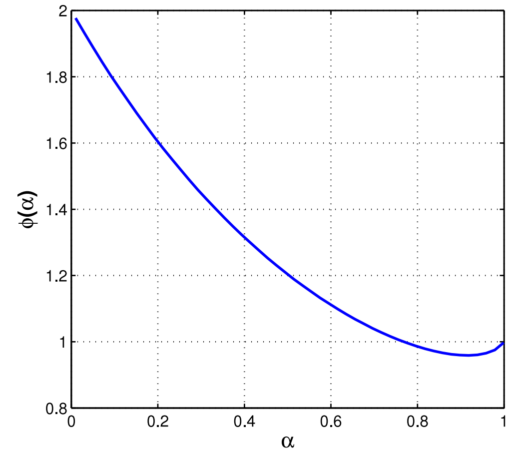

Fig. 3. $\phi(\alpha) = h^{-\alpha}\psi_o^{-1}$ versus α.

4. Examples

Example 1. We will first consider the plant

$$P(s) = \frac{e^{-hs}}{s^\alpha - p}, \quad \text{with } h > 0,\ p \ge 0 \tag{29}$$

and design PID controllers using the method developed in Section 3.

For P and PI controller design we need to compute the quantity

$$\psi_o = \left\| \frac{1 - e^{-hs}G(s^\alpha)G(0)^{-1}}{s^\alpha} \right\|_\infty^{-1}.$$

When $\alpha = 1$, we have $\psi_o = h^{-1}$. In the case $0 < \alpha < 1$, we compute ψ_o from

$$\psi_o^{-1} = \sup_{\omega\in\mathbb{R}} \frac{|1 - e^{-jh\omega}|}{|(j\omega)^\alpha|} = h^\alpha\sqrt{2}\sup_{\tilde{\omega}\in\mathbb{R}} \frac{\sqrt{1 - \cos(\tilde{\omega})}}{\tilde{\omega}^\alpha}.$$

Therefore,

$$h^{-\alpha}\psi_o^{-1} = \sqrt{2}\sup_{\tilde{\omega}\in\mathbb{R}} \frac{\sqrt{1 - \cos(\tilde{\omega})}}{\tilde{\omega}^\alpha} =: \phi(\alpha). \tag{30}$$

Fig. 3 shows how $\phi(\alpha)$ varies with α. As expected, for $\alpha = 1$ we have $\phi = 1$. But it is interesting to observe that behavior of ϕ is not monotonic, and there is a minimum value near $\alpha = 0.9$.

According to Lemma 1, there is a stabilizing controller for plant (29) if $p < \psi_o$, i.e., if

$$p\,h^\alpha < \frac{1}{\phi(\alpha)}$$

where $\phi(\alpha)$ is as shown in Fig. 3. In particular, for $\alpha = 0.5$, we have $\phi = 1.2$ and we can find a stabilizing proportional controller using Lemma 1 if

$$h < \frac{1}{1.2^2\,p^2} = \frac{0.6944}{p^2}.$$

Recall that the sufficient conditions of Section 3 are obtained using the small gain arguments, so there is some conservatism. We can also use the results of [36] and find that there exists a stabilizing proportional controller for all $h < h_{max}$ as follows.

The stability for $h = 0$ is guaranteed with $K_p > p$. When h increases, the position of the infinite number of new poles poses no restriction, since for a delay system of retarded type (the closed-loop $[P, K_p]$ is indeed a fractional delay system of retarded type) they appear in the left-half plane. The exact value of the delay for

Table 1

c	0.10	0.17	0.28	0.46	0.77	1.29	2.15	3.59	5.99	1.00
$\widetilde{h}_{\max}$	4.14	3.13	2.35	1.75	1.29	0.94	0.68	0.48	0.34	0.23
$h_{\max}$ [37]	4.58	3.56	2.78	2.17	1.68	1.35	1.05	0.83	0.64	0.46
K_d	0.46	0.59	0.78	1.01	1.31	1.78	2.32	3.18	4.44	6.50

which some poles cross the imaginary axis are related to the non-negative real roots ω_R of the quasi-polynomial

$$W(\omega) = \omega - p\sqrt{2\omega} + p^2 - K_p^2$$

which leads to $\omega_R = K_p^2 + p\sqrt{2K_p^2 - p^2}$.

The maximum delay h is given by

$$h = \frac{1}{\omega_R} \arcsin\left(\frac{\sqrt{2\omega_R}}{2K_p}\right) \tag{31}$$

and, maximizing (31) with respect to $K_p > p$ results in $K_p \to p$, and hence $h \to h_{\max}$ with

$$h_{\max} = \frac{\pi}{4p^2} \approx \frac{0.7854}{p^2}.$$

The value of $h_{\max}$ is exact, in the sense that if $h \geq h_{\max}$ then there does not exist a stabilizing proportional controller. Thus the level of conservatism in our approach is less than 12% (to be exact $(0.7854 - 0.6944)/0.7854 = 0.1159$).

The suboptimal PI controller (28) for $P(s) = \frac{e^{-hs}}{\sqrt{s}-p}$ can be computed from

$$\psi_o = \frac{1}{1.2\sqrt{h}}, \qquad x_{\text{opt}} = \frac{\psi_o - p}{2}$$

$$\gamma_{\text{opt}} = \frac{1}{4}\left(\frac{\psi_o - p}{\psi_o + p}\right)^2 \left(\frac{\psi_o - p}{2}\right)^2.$$

In particular, when $p = 0$, we have

$$C_{pi}(s) = \frac{1}{2.4\sqrt{h}}\left(1 + \frac{1/16}{1.2^2\ h\, s}\right).$$

For the optimal PD controller proposed in Section 3, we need to find the optimal $\widetilde{K}_d \in \mathbb{R}$, say $\widetilde{K}_d^{\text{opt}}$, so that ψ_d^{-1}, (10), is minimized for a small fixed value of $\tau_d > 0$.

Considering $h = 1$, we calculated the optimal PD control which results in the parameters $\tau_d = 4.2$ and $\widetilde{K}_d^{\text{opt}} = -1.7346$, and hence $\psi_d^{-1} = 0.9873$. Then the optimal PD controller is given by

$$G(0)^{-1}\sqrt{p\,\psi_d}\left(1 + \widetilde{K}_d^{\text{opt}} \frac{s}{\tau_d s + 1}\right)$$

where stability is assured for all systems with $p < \psi_d = 1.0165$. Notice that with just the proportional controller, we could only guarantee stability for systems with $p < 0.8333$, which indicates an increase of about 22%.

Example 2. Now consider the following plant modeling a non-laminated magnetic suspension system as studied in [25]:

$$P_2(s) = e^{-hs}\frac{G(s^\alpha)}{s^\alpha - p} = e^{-hs}\frac{1}{(s^\alpha)^5 + (s^\alpha)^4 - c} \quad \alpha = 0.5 \tag{32}$$

where c is a positive real constant and in the ideal case $h = 0$. This system has exactly one real positive pole and four poles in the left-half plane; see [25]. Hence, the techniques presented in Section 3 are applicable. We investigate the largest allowable time delay h (which may exist due to communication constraints between the controller and the plant) for which the PD controller design technique proposed in this paper gives a stable feedback system.

Table 1 shows the results for 10 values of c logarithmically spaced between 0.1 and 10. For each one of those points, a PD controller that maximizes the allowable value of delay was calculated using the results of Section 3. The maximal allowable delay for which our technique finds an admissible PD controller is denoted by $\widetilde{h}_{\max}$. The optimal PD controller determined using the techniques of Section 3, has the proportional gain $K_p = c$, and K_d is shown in Table 1 for various values of c, for the delay $\widetilde{h}_{\max}$. For the PD controllers designed, the exact value of maximal allowable delay, denoted by $h_{\max}$, can be calculated using the numerical techniques presented in [37]. We see that the degree of conservatism (i.e. the gap between $\widetilde{h}_{\max}$ and $h_{\max}$) is low.

5. Conclusions

In this paper, we developed a method to design classical PID controllers (with proper derivative action) for a class of fractional order plants with time delays. The main idea behind this approach was to use the small gain type of arguments used in [28]. The fractional order plant is factored into a stable part and an unstable part, where the unstable part is in the form $(s^\alpha - p)^{-1}$ with $p > 0$. There is no restriction on the stable part $G(s^\alpha)$ except that $G(0) \neq 0$ and $G(p) \neq 0$. It may be possible to extend this method to fractional order plants with a higher degree unstable part, but in that situation there are some technical difficulties even for the case of rational plants; see [28] and its references.

The (sub)optimal PD and PI controller design method proposed here also works for rational plants with time delays and single pole in $\mathbb{R}_+$; see [30]. However, in the case of fractional systems, there is a major difference for the minimization of ψ_d^{-1}, (10): when $\alpha \neq 1$ we cannot let $\tau_d = 0$, because, otherwise $s^{1-\alpha}$ term multiplying $\widetilde{K}_d$ will make the norm equal to infinity unless $\widetilde{K}_d = 0$. Therefore, the selection of a small positive τ_d plays an important role in this case, and hence, search for the optimal $\widetilde{K}_d$ and τ_d pair is more difficult compared to the problem studied in [30]. On the other hand, having a positive τ_d makes the "proper PD controller" a stable first order controller. So, in this sense optimization of $\widetilde{K}_d$ and τ_d solves the optimal first order stable controller design problem.

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