Discretization of hyperbolic type Darboux integrable equations preserving integrability

Ismagil Habibullin[1]

Ufa Institute of Mathematics, Russian Academy of Science,

Chernyshevskii Str., 112, Ufa, 450077, Russia

Natalya Zheltukhina[2]

Department of Mathematics, Faculty of Science,

Bilkent University, 06800, Ankara, Turkey

Alfia Sakieva[3]

Ufa Institute of Mathematics, Russian Academy of Science,

Chernyshevskii Str., 112, Ufa, 450077, Russia

Abstract

A method of integrable discretization of the Liouville type nonlinear partial differential equations is suggested based on integrals. New examples of discrete Liouville type models are presented.

1 Introduction

The problem of integrable discretization of the integrable PDE is very complicated and not enough studied. The same is true for evaluating the continuum limit for discrete models [1]. In the present paper we undertake an attempt to clarify the connection between Liouville type partial differential equations and their discrete analogues. One unexpected observation is that there are pairs of equations, one continuous and the other one semi-discrete, having a common integral. Inspired by these examples, we introduced a method of discretization of PDE having a nontrivial integral. Similar ideas are used in [2] where a method of construction of difference scheme for ordinary differential equations preserving the classical Lie group is suggested. Let us begin with the necessary definitions.

[1]e-mail: habibullinismagil@gmail.com

[2]e-mail: natalya@fen.bilkent.edu.tr

[3]e-mail: alfiya85.85@mail.ru

We consider discrete equations of the form

$$v(n+1,m+1) = f(v(n,m), v(n+1,m), v(n,m+1)) \qquad (1)$$

and semi-discrete chains

$$t(n+1,x) = f(x, t(n,x), t(n+1,x), t_x(n,x))\,. \qquad (2)$$

Equations (1) and (2) are discrete and semi-discrete analogues of hyperbolic equations

$$u_{xy} = f(x,y,u,u_x,u_y)\,. \qquad (3)$$

Functions $v = v(n,m)$, $t = t(n,x)$ and $u = u(x,y)$ depend on discrete variables n and m and continuous variables x and y. Through the paper we use the following notations:

$$v_{i,j} = v(n+i, m+j); \qquad v_i = v_{i,0}; \qquad \bar{v}_j = v_{0,j}; \qquad t_i = t(n+i,x)\,.$$

For equation (3) function $W(x,y,u,u_y,u_{yy},\ldots,\partial^k u/\partial y^k)$ is called an x-integral of order k if $D_x W = 0$ and $W_{\partial^k u/\partial y^k} \neq 0$, and function $\bar{W}(x,y,u,u_x,u_{xx},\ldots,\partial^m u/\partial x^m)$ is called a y-integral of order m if $D_y\bar{W} = 0$ and $\bar{W}_{\partial^m u/\partial x^m} \neq 0$. Here, D_x and D_y denote the total derivatives with respect to x and y. Equation (3) is called Darboux integrable if it possesses nontrivial x- and y-integrals.

For equation (2) function $F(x,n,t_m,t_{m+1},t_{m+2},\ldots,t_{m'})$ is called an x-integral of order $m'-m+1$ if $D_x F = 0$ and $F_{t_m} \neq 0$, $F_{t'_m} \neq 0$ and function $I(x,n,t,t_x,t_{xx},\ldots,\frac{d^k t}{dt^k})$ is called an n-integral of order k, if $DI = I$ and $I_{\frac{d^k t}{dt^k}} \neq 0$. Here, D is the forward shift operator in n, i.e. $Dh(n,x) = h(n+1,x)$. Equation (2) is called Darboux integrable if it possesses nontrivial x- and n- integrals.

For equation (1) function $I(n,m,\bar{v}_k,\bar{v}_{k+1},\ldots,\bar{v}_{k'})$ is called an n-integral of order $k'-k+1$ if $DI = I$ and $I_{\bar{v}_k} \neq 0$, $I_{\bar{v}_{k'}} \neq 0$, and function $\bar{I}(n,m,v_r,v_{r+1},v_{r+2},\ldots,v_{r'})$ is called an m-integral of order $r'-r+1$, if $\bar{D}\bar{I} = \bar{I}$ and $\bar{I}_{v_r} \neq 0$, $\bar{I}_{v_{r'}} \neq 0$. Here, D and $\bar{D}$ are the forward shift operators in n and m respectively. Equation (1) is called Darboux integrable, if it possesses nontrivial n- and m-integrals (see also [3]).

Continuous equations (3) are very-well studied. In particular, the question of describing all Darboux integrable equations (3) is completely solved (see [4] - [7]). All equations (3) possessing x- and y-integrals of order 2 are described by the following theorem.

Theorem 1.1 *(see [7]) Any equation (3), for which there exist second order x- and y-integrals, under the change of variables $x \to X(x)$, $y \to Y(y)$, $u \to U(x,y,u)$, can be reduced to one of the kind:*

(1) $u_{xy} = e^u$, $\bar{W} = u_{xx} - 0.5u_x^2$, $W = u_{yy} - 0.5u_y^2$;

(2) $u_{xy} = e^y u_y$, $\bar{W} = u_x - e^u$, $W = \frac{u_{yy}}{u_y} - u_y$;

(3) $u_{xy} = e^u\sqrt{u_y^2 - 4}$, $\bar{W} = u_{xx} - 0.5u_x^2 - 0.5e^{2u}$, $W = \frac{u_{yy} - u_y^2 + 4}{\sqrt{u_y^2 - 4}}$;

(4) $u_{xy} = u_x u_y \left(\frac{1}{u-x} - \frac{1}{u-y}\right)$, $\bar{W} = \frac{u_{xx}}{u_x} - \frac{2u_x}{u-x} + \frac{1}{u-x}$, $W = \frac{u_{yy}}{u_y} - \frac{2u_y}{u-y} + \frac{1}{u-y}$;

(5) $u_{xy} = \psi(u)\beta(u_x)\bar{\beta}(u_y)$, $(ln\psi)'' = \psi^2$, $\beta\beta' = -u_x$, $\bar{\beta}\bar{\beta}' = -u_y$,
$\bar{W} = \frac{u_{xx}}{\beta(u_x)} - \psi(u)\beta(u_x)$, $W = \frac{u_{yy}}{\bar{\beta}(u_y)} - \psi(u)\bar{\beta}(u_y)$;

(6) $u_{xy} = \frac{\beta(u_x)\bar{\beta}(u_y)}{u}$, $\beta\beta' + c\beta = -u_x$, $\bar{\beta}\bar{\beta}' + c\bar{\beta} = -u_y$,
$\bar{W} = \frac{u_{xx}}{\beta} - \frac{\beta}{u}$, $W = \frac{u_{yy}}{\bar{\beta}} - \frac{\bar{\beta}}{u}$;

(7) $u_{xy} = -2\frac{\sqrt{u_x u_y}}{x+y}$, $\bar{W} = \frac{u_{xx}}{\sqrt{u_x}} + 2\frac{\sqrt{u_x}}{x+y}$, $W = \frac{u_{yy}}{\sqrt{u_y}} + 2\frac{\sqrt{u_y}}{x+y}$;

(8) $u_{xy} = \frac{1}{(x+y)\beta(u_x)\bar{\beta}(u_y)}$, $\beta' = \beta^3 + \beta^2$, $\bar{\beta}' = \bar{\beta}^3 + \bar{\beta}^2$,
$\bar{W} = u_{xx}\beta(u_x) - \frac{1}{(x+y)\beta(u_x)}$, $W = u_{yy}\bar{\beta}(u_y) - \frac{1}{(x+y)\bar{\beta}(u_y)}$.

On the contrary, the problem of describing all equations (1) or (2) possessing both integrals (so-called Darboux integrable equations) is very far from being solved (the problem of classification is solved only for a very special kind of semi-discrete equations [8]), it would be beneficial for further classification to obtain new Darboux-integrable equations (1) and semi-discrete chains (2). It was observed that many chains (2) and their continuum limit equations (3) possess the same n- and y-integrals:

$semi-discrete\ chain$	$n-integral\, I$	$continuous$ $analogue$	$y-integral\, \bar{W}$
$t_{1x} = t_x + 0.5t_1^2 - 0.5t^2$	$t_x - 0.5t^2$	$u_{xy} = uu_y$	$u_x - 0.5u^2$
$t_{1x} = t_x + Ce^{0.5(t+t_1)}, C = Const$	$t_{xx} - 0.5t_x^2$	$u_{xy} = e^u$	$u_{xx} - 0.5u_x^2$
$t_{1x} = t_x + \sqrt{e^{2t} + Ce^{t+t_1} + e^{2t_1}}$	$t_{xx} - 0.5t_x^2 - 0.5e^{2t}$	$u_{xy} = e^u\sqrt{1+u_y^2}$	$u_{xx} - 0.5u_x^2 - 0.5e^{2u}$

The main aim of the present paper is the discretization of equations (3) preserving the structure of y-integrals of order 2: we take y-integral for each of eight classes of Theorem 1.1 and find the semi-discrete chain (2) possessing the given n-integral (y-integral). The next Theorem presents a list of semi-discrete models of Darboux integrable equations (3) from Theorem 1.1 with integrals of order 2.

Theorem 1.2 *Below is the list of equations (2) possessing the given n-integral I :*

given $n-integral$	*the corresponding chain*	
$I = t_{xx} - 0.5t_x^2$	$t_{1x} = t_x + Ce^{0.5(t+t_1)}$, $C = Const$	(1^*)
$I = t_x - e^t$	$t_{1x} = t_x - e^t + e^{t_1}$	(2^*a)
$I = \frac{t_{xx}}{t_x} - t_x$	$t_{1x} = K(t, t_1)t_x$, *where* $K_tK^{-1} + K_{t_1} = K - 1$	(2^*b)
$I = t_{xx} - 0.5t_x^2 - 0.5e^{2t}$	$t_{1x} = t_x + \sqrt{e^{2t} + Re^{t+t_1} + e^{2t_1}}$, $R = Const$	(3^*a)
$I = \frac{t_{xx}-t_x^2+4}{\sqrt{t_x^2-4}}$	$t_{1x} = (1 + Re^{t+t_1})t_x + \sqrt{R^2e^{2(t+t_1)} + 2Re^{t+t_1}}\sqrt{t_x^2 - 4}$	(3^*b)
$I = \frac{t_{xx}}{t_x} - \frac{2t_x}{t-x} + \frac{1}{t-x}$	$t_{1x} = \frac{(t_1+L)(t_1-x)}{(t+L)(t-x)}t_x$, $L = Const$	(4^*)
$I = \frac{t_{xx}}{\beta(t_x)} - \psi(t)\beta(t_x)$ $(ln\psi)'' = \psi^2, \beta\beta' = -t_x$	$\beta(t_x) = it_x$ *and* $t_{1x} = K(t, t_1)t_x$, *where* $K_t + KK_{t_1} + K^2\psi(t_1) - K\psi(t) = 0$	(5^*)
$I = \frac{t_{xx}}{\beta(t_x)} - \frac{\beta(t_x)}{t}$, $\beta\beta' + c\beta = -t_x$	$\beta(t_x) = Rt_x$, *and* $t_{1x} = K(t, t_1)t_x$, *where* $\frac{K_t}{K} + K_{t_1} = \frac{R^2(tK-t_1)}{tt_1}$	(6^*)
$I = \frac{t_{xx}}{\sqrt{t_x}} + \frac{2\sqrt{t_x}}{x+R}$	$t_{1x} = \left(\sqrt{t_x} + \frac{C}{x+R}\right)^2$, $R = Const$, $C = Const$	(7^*)
$I = \beta(t_x)t_{xx} - \frac{1}{(x+R)\beta(t_x)}$, $\beta' = \beta^3 + \beta^2$	$\beta(t_x) = -1$ *and* $t_{1x} = t_x + \frac{t_1-t+C}{x+R}$	(8^*)

It is remarkable that each equation in Theorem 1.2 also admits a nontrivial x-integral. It means that discretization preserving the structure of y-integrals sends Darboux integrable equations (3) into Darboux integrable chains (2).

Note that equation (1^*) was found in [9]. Equation (3^*a) for $R = 2$ was found in [3], equations (2^*a) and (3^*a) are found in [8]. To our knowledge, the other equations from Theorem 1.2 are new.

The next theorem lists x-integrals for chains from Theorem 1.2.

Theorem 1.3 *(I) The equations* (2^*b), (5^*) *and* (6^*) *from Theorem 1.2 having the form* $t_{1x} = K(t, t_1)t_x$ *admit x-integral* $F(t, t_1)$, *where function* F *is a solution of* $F_t + K(t, t_1)F_{t_1} = 0$ *with a given function* $K(t, t_1)$.
(II) x-integrals of equations (8^*), (1^*), (3^*a), (3^*b), (4^*), (7^*) *and* (2^*a) *are* $F = (t_1-t+C)/(x+y)$, $F = e^{(t_1-t)/2} + e^{(t_1-t_2)/2}$, $F = arcsinh(ae^{t_1-t_2} + b) + arcsinh(ae^{t_1-t} + b)$ *with* $a = 2(4 - R^2)^{-1/2}$, $b = R(4 - R^2)^{-1/2}$, $F = \sqrt{Re^{2t_1} + 2e^{t_1-t}} + \sqrt{Re^{2t_1} + 2e^{t_1-t_2}}$, $F = (t_1 - t)(t_2 + L)(t_2 - t)^{-1}(t_1 + L)^{-1}$, $F = (2t_1 - t - t_2)/(2C^2) - 1/(x + R)$ *and* $F = (e^t - e^{t_2})(e^{t_1} - e^{t_3})(e^t - e^{t_3})^{-1}(e^{t_1} - e^{t_2})^{-1}$ *correspondingly.*

One can also apply the discretization method preserving the structure of integrals for semi-discrete chains (2): take x-integral for a semi-discrete chain and find discrete equation (1) with the given m-integral (x-integral).

In spite of the absence of the complete classification for Darboux-integrable semi-discrete chains (2) there is a large variety of such chains in literature (see, for instance, [3], [8] and [10]). The procedure of obtaining fully discrete equations for a given integral is a difficult task and requires further investigation. As a rule it is reduced to a very complicated functional equation. We illustrate the application of the discretization method on chains (1^*), (4^*) and (7^*) from Theorem 1.2. The discrete analogues of the chains are presented in the next Remark.

Remark 1.4 *Below is the list of equations (1) possessing the given m-integral $\bar{I}$:*

given m − integral	*the corresponding equation*	
$\bar{I} = e^{(v_1-v)/2} + e^{(v_1-v_2)/2}$	$e^{v_{1,1}+v} = \frac{1}{C+e^{-(v_1+\bar{v}_1)}}$	(1^{**})
$\bar{I} = (v_1 - v)(v_2 + L)(v_2 - v)^{-1}(v_1 + L)^{-1}$	$v_{1,1} = \frac{L(v_1+\bar{v}_1-v)+v_1\bar{v}_1}{L+v}$	(4^{**})
$\bar{I} = 2v_1 - v - v_2$	$v_{1,1} = v_1 + h(\bar{v}_1 - v)$, $z = h(2z - h(z))$	(7^{**})

The equations (1^{**}), (4^{**}) and (7^{**}) have respectively the following n-integrals $I = e^{(\bar{v}_1-v)/2} + e^{(\bar{v}_1-\bar{v}_2)/2}$, $I = (\bar{v}_1 - \bar{v})(\bar{v}_2 + L)(\bar{v}_2 - t)^{-1}(\bar{v}_1 + L)^{-1}$ and $I = \bar{v}_1 - v - h^{-1}(\bar{v}_1 - v)$ with h^{-1} being the inverse function of function h that satisfies the functional equation $z = h(2z - h(z))$.

Equation (1^{**}) from Remark 1.4 appeared in [11], equations (4^{**}) and (7^{**}) seem to be new, unfortunately we failed to answer the question whether equation $z = h(2z-h(z))$ has any solution different from linear one $h(z) = z + C$.

The article is organized as follows. Theorem 1.2 is proved in Section 2. The proof of Theorem 1.3 is omitted. Chains (1^*), (2^*a) and (3^*a) are of the form $t_{1x} = t_x + d(t, t_1)$, and their x-integrals can be seen in [8]. One can find x-integrals for chains (3^*b), (4^*), (7^*) and (8^*) by direct calculations. In Section 3 the discretization of chains (1^*), (4^*) and (7^*) from Remark 1.4 are presented and for each obtained discrete equation the second integral is found. In Section 4 the Conclusion is drawn.

2 Proof of Theorem 1.2

Case (1^*): Consider all chains (2) with n-integral of the form $I = t_{xx} - \frac{1}{2}{t_x}^2$. Equality $DI = I$ implies

$$f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx} - \frac{1}{2} f^2 = t_{xx} - \frac{1}{2}{t_x}^2. \tag{4}$$

By comparing the coefficients before t_{xx} in (4) we have $f_{t_x} = 1$. Therefore,

$$f(x, t, t_1, t_x) = t_x + d(x, t, t_1)\,. \tag{5}$$

We substitute (5) into (4) and get $d_x + d_t t_x + d_{t_1} t_x + d_{t_1} d - \frac{1}{2}{t_x}^2 - dt_x - \frac{1}{2}d^2 = -\frac{1}{2}{t_x}^2$, or equivalently, $d_t + d_{t_1} - d = 0$ and $d_x + d_{t_1} d - \frac{1}{2}d^2 = 0$. We solve the last two equations simultaneously and find that $d = e^{t_1} K(x, t_1 - t)$, where $K = Ce^{-\frac{1}{2}(t_1 - t)}$ and C is an arbitrary constant. Therefore, chain (2) with n-integral $I = t_{xx} - \frac{1}{2}{t_x}^2$ becomes $t_{1x} = t_x + Ce^{(t_1+t)/2}$.

Case (2^*a): Consider all chains (2) with n-integral $I = t_x - e^t$. Equality $DI = I$ implies $f - e^{t_1} = t_x - e^t$, which gives the equation $t_{1x} = f = t_x - e^t + e^{t_1}$.

Case (2^*b): Consider all chains (2) with n-integral $I = \frac{t_{xx}}{t_x} - t_x$. Equality $DI = I$ implies

$$\frac{f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx}}{f} - f = \frac{t_{xx}}{t_x} - t_x. \tag{6}$$

By comparing the coefficients before t_{xx} in (6) we have $f_{t_x}/f = 1/t_x$, that is $f = K(x, t, t_1) t_x$. Substitute $f = K(x, t, t_1) t_x$ into (6) and have $\frac{K_x}{K} + \frac{K_t}{K} t_x + K_{t_1} t_x - K t_x = -t_x$, or equivalently (by comparing the coefficients before t_x and ${t_x}^0$), we get $\frac{K_t}{K} + K_{t_1} = K - 1$ and $K_x = 0$. Therefore, equations $t_{1x} = K(t, t_1) t_x$, where K satisfies $\frac{K_t}{K} + K_{t_1} = K - 1$ are the only chains (2) that admit n-integral I of the form $I = \frac{t_{xx}}{t_x} - t_x$.

Case (3^*a): Consider all chains (2) with n-integral $I = t_{xx} - \frac{1}{2}{t_x}^2 - \frac{1}{2}e^{2t}$. Equality $DI = I$ implies

$$f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx} - \frac{1}{2} f^2 - \frac{1}{2} e^{2t_1} = t_{xx} - \frac{1}{2}{t_x}^2 - \frac{1}{2} e^{2t}. \tag{7}$$

By comparing the coefficients before t_{xx} in (7) we have $f_{t_x} = 1$, that is $f(x, t, t_1, t_x) = t_x + d(x, t, t_1)$. Substitute $f(x, t, t_1, t_x) = t_x + d(x, t, t_1)$ into (7) and have

$$d_x + d_t t_x + d_{t_1}(t_x + d) - \frac{1}{2}(t_x + d)^2 - \frac{1}{2} e^{2t_1} = -\frac{1}{2}{t_x}^2 - \frac{1}{2} e^{2t}. \tag{8}$$

Compare the coefficients before t_x and ${t_x}^0$ in (8) and get

$$d_t + d_{t_1} - d = 0, \qquad d_x + d_{t_1} d - \frac{1}{2} d^2 - \frac{1}{2} e^{2t_1} = -\frac{1}{2} e^{2t}. \tag{9}$$

The first equation in (9) has a solution $d = e^{t_1}K(x, t_1 - t)$. Substitution of this expression into the second equation of (9) gives $e^{-t_1}K_x + K_{t_1-t}K + \frac{1}{2}K^2 - \frac{1}{2} + \frac{1}{2}e^{-2(t_1-t)} = 0$. Since K depends on $U = t_1 - t$ and x, then $K_x = 0$ and the last equation becomes $2K'K + K^2 = 1 - e^{-2U}$, and hence, $d = e^{t_1}K = \sqrt{e^{2t_1} + e^{2t} + Re^{t+t_1}}$, where R is and arbitrary constant. Therefore, chain (2) with n-integral $I = t_{xx} - \frac{1}{2}{t_x}^2 - \frac{1}{2}e^{2t}$ becomes $t_{1x} = t_x + \sqrt{e^{2t_1} + e^{2t} + Re^{t+t_1}}, R = const$.

Case (3^*b): Consider all chains (2) with n-integral $I = \frac{t_{xx} - {t_x}^2 + 4}{\sqrt{{t_x}^2 - 4}}$. Equality $DI = I$ implies

$$\frac{f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx} - f^2 + 4}{\sqrt{f^2 - 4}} = \frac{t_{xx} - {t_x}^2 + 4}{\sqrt{{t_x}^2 - 4}}. \tag{10}$$

By comparing the coefficients before t_{xx} in (10) we get $\frac{f_{t_x}}{\sqrt{f^2-4}} = \frac{1}{\sqrt{{t_x}^2-4}}$, that is $arccosh\frac{f}{2} = arccosh\frac{t_x}{2} + K(x, t, t_1)$. Thus,

$$f(x, t, t_1, t_x) = At_x + B\sqrt{{t_x}^2 - 4}, \tag{11}$$

where $A(x, t, t_1) = coshK$, $B(x, t, t_1) = sinhK$, $A^2 - B^2 = 1$. Note that $f = 2cosh((arccosh\frac{t_x}{2}) + K)$, i.e. $\sqrt{f^2 - 4} = 2sinh((arccosh\frac{t_x}{2}) + K) = 2(\sqrt{\frac{{t_x}^2}{4} - 1}coshK + \frac{t_x}{2}sinhK)$, or $\sqrt{f^2 - 4} = Bt_x + A\sqrt{{t_x}^2 - 4}$. Substitute (11) into (10) and have

$$t_xA_x + B_x\sqrt{{t_x}^2 - 4} + {t_x}^2A_t + t_xB_t\sqrt{{t_x}^2 - 4} + (t_xA_{t_1} + B_{t_1}\sqrt{{t_x}^2 - 4})(At_x + B\sqrt{{t_x}^2 - 4})$$
$$-(At_x + B\sqrt{{t_x}^2 - 4})^2 + 4 = -(Bt_x + A\sqrt{{t_x}^2 - 4})\sqrt{{t_x}^2 - 4},$$

that can be written shortly as

$$({t_x}^2 - 4)(\alpha_1 + \alpha_2 t_x)^2 = (\alpha_3 + \alpha_4 t_x + \alpha_5 {t_x}^2)^2, \tag{12}$$

where $\alpha_1 = B_x, \alpha_2 = B_t + A_{t_1}B + B_{t_1}A - 2AB + B, \alpha_3 = -4B_{t_1}B + 4B^2 + 4 - 4A, \alpha_4 = A_x, \alpha_5 = A_t + A_{t_1}A + B_{t_1}B - A^2 - B^2 + A$. We compare the coefficients before ${t_x}^4$, ${t_x}^3$, ${t_x}^2$, t_x, t_x^0 in (12) and have ${\alpha_2}^2 = {\alpha_5}^2$, $2\alpha_1\alpha_2 = 2\alpha_4\alpha_5$, ${\alpha_1}^2 - 4{\alpha_2}^2 = {\alpha_4}^2 + 2\alpha_3\alpha_5$, $-8\alpha_1\alpha_2 = 2\alpha_3\alpha_4$, $-4{\alpha_1}^2 = {\alpha_3}^2$, that implies $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$, which is possible only if $A = 1 + Re^{t+t_1}$ and $B = \sqrt{R^2e^{2(t+t_1)} + 2Re^{(t+t_1)}}$, where $R = const$. Therefore, by (11), the chain (2) with n-integral $I = \frac{t_{xx} - {t_x}^2 + 4}{\sqrt{{t_x}^2 - 4}}$ becomes $t_{1x} = (1 + Re^{t+t_1})t_x + \sqrt{R^2e^{2(t+t_1)} + 2Re^{(t+t_1)}}\sqrt{{t_x}^2 - 4}$.

Case (4^*): Consider chains (2) with n-integral $I = \frac{t_{xx}}{t_x} - \frac{2t_x}{t-x} + \frac{1}{t-x}$. Equality $DI = I$ implies

$$\frac{f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx}}{f} - \frac{2f}{t_1 - x} + \frac{1}{t_1 - x} = \frac{t_{xx}}{t_x} - \frac{2t_x}{t - x} + \frac{1}{t - x}. \tag{13}$$

We compare the coefficients before t_{xx} and have $f_{t_x}/f = 1/t_x$, that is $f = t_xK(x, t, t_1)$. Substitute $f = t_xK$ into (13) and have

$$\frac{K_xt_x + K_t{t_x}^2 + K_{t_1}K{t_x}^2}{Kt_x} - \frac{2Kt_x}{t_1 - x} + \frac{1}{t_1 - x} = -\frac{2t_x}{t - x} + \frac{1}{t - x}. \tag{14}$$

By comparing the coefficients before t_x and ${t_x}^0$ in (14) we get

$$\frac{K_t}{K} + K_{t_1} = \frac{2K}{t_1 - x} - \frac{2}{t - x}, \qquad \frac{K_x}{K} = -\frac{1}{t_1 - x} + \frac{1}{t - x}. \tag{15}$$

We solve two equations of (15) simultaneously and have $K = \frac{t_1+L}{t+L}\frac{t_1-x}{t-x}$, where L is an arbitrary constant. Therefore, any chain (2) with n-integral $I = \frac{t_{xx}}{t_x} - \frac{2t_x}{t-x} + \frac{1}{t-x}$ becomes $t_{1x} = \frac{t_1+L}{t+L}\frac{t_1-x}{t-x}t_x$.
Case (5^*) : Consider all chains (2) with n-integral $I = \frac{t_{xx}}{\beta} - \psi\beta$, where $\beta = \beta(t_x), \psi = \psi(t), \beta\beta' = -t_x$. We have, $2\beta\beta' = -2t_x$, i.e. $\beta^2 = -{t_x}^2 + M^2$, or $\beta = \sqrt{M^2 - {t_x}^2}$, where M is an arbitrary constant. Equality $DI = I$ implies

$$\frac{f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx}}{\beta(f)} - \psi(t_1)\beta(f) = \frac{t_{xx}}{\beta(t_x)} - \psi(t)\beta(t_x). \tag{16}$$

We compare the coefficients before t_{xx} and have $f_{t_x}/\beta(f) = 1/\beta(t_x)$ which implies that
either (5^*a): $M = 0$, $\beta(t_x) = it_x$ and $t_{1x} = K(x,t,t_1)t_x$,
or (5^*b): $M \neq 0$ and then $arcsin\frac{f}{M} = arcsin\frac{t_x}{M} + L(x,t,t_1)$, that is,

$$f = t_x A(x,t,t_1) + \sqrt{M^2 - {t_x}^2}B(x,t,t_1), A^2 + B^2 = 1. \tag{17}$$

In case (5^*a) we substitute $t_{1x} = f = K(x,t,t_1)t_x$ into (16), use that $\beta(t_x) = it_x$, and obtain

$$K_x = 0, \qquad \frac{K_t}{K} + K_{t_1} + \psi(t_1)K = \psi(t). \tag{18}$$

Therefore, the chains (2) with n-integral $I = \frac{t_{xx}}{it_x} - i\psi(t)t_x$ are equations $t_{1x} = K(t,t_1)t_x$, where function K satisfies (18).
Let us consider case (5^*b). Note that

$$M^2 - f^2 = M^2 - A^2{t_x}^2 - 2ABt_x\sqrt{M^2 - {t_x}^2} - B^2M^2 + B^2{t_x}^2 = (Bt_x - A\sqrt{M^2 - {t_x}^2})^2$$

and $\beta(f) = \pm(Bt_x - A\sqrt{M^2 - {t_x}^2})$, $\beta(t_x) = \sqrt{M^2 - {t_x}^2}$. Substitute (17) into (16) and get

$$\frac{A_x t_x + B_x\sqrt{M^2 - {t_x}^2} + A_t{t_x}^2 + B_t t_x\sqrt{M^2 - {t_x}^2} + (A_{t_1}t_x + B_{t_1}\sqrt{M^2 - {t_x}^2})(At_x + B\sqrt{M^2 - {t_x}^2})}{\pm(Bt_x - A\sqrt{M^2 - {t_x}^2})}$$

$$= \pm(Bt_x - A\sqrt{M^2 - {t_x}^2})\psi(t_1) - \sqrt{M^2 - {t_x}^2}\psi(t),$$

or the same,

$$(M^2 - {t_x}^2)(\alpha_1 + \alpha_2 t_x)^2 = (\alpha_3 + \alpha_4 t_x + \alpha_5{t_x}^2)^2, \tag{19}$$

where $\alpha_1 = B_x$, $\alpha_4 = A_x$, $\alpha_2 = B_t + A_{t_1}B + AB_{t_1} + 2AB\psi(t_1) + B\psi(t)$, $\alpha_3 = BB_{t_1}M^2 - A^2M^2\psi(t_1) - A\psi(t)M^2$, $\alpha_5 = A_t + A_{t_1}A - B_{t_1}B - B^2\psi(t_1) + A^2\psi(t_1) + A\psi(t)$. We compare the coefficients before t_x^k, $k=0,1,2,3,4$, in (19) and find that $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5=0$,which is possible only if $\psi=R$ is a constant function, that contradicts to the equation $(ln\psi)''=\psi^2$. Therefore, case (5^*b) is not realized.

Case (6^*): Consider chains (2) with n-integrals $I=\frac{t_{xx}}{\beta(t_x)}-\frac{\beta(t_x)}{t}$, where $\beta=\beta(t_x)$ and $\beta\beta'+c\beta=-t_x$. The equality $DI=I$ implies

$$\frac{f_{t_x}}{\beta(f)}=\frac{1}{\beta(t_x)} \tag{20}$$

and

$$\frac{f_x+f_tt_x+f_{t_1}f}{\beta(f)}-\frac{\beta(f)}{t_1}=-\frac{\beta(t_x)}{t}. \tag{21}$$

Differentiation of (20) with respect to x, t, t_1 gives

$$f_{xt_x}=\frac{\beta'(f)}{\beta(t_x)}f_x, \qquad f_{t_xt}=\frac{\beta'(f)}{\beta(t_x)}f_t, \qquad f_{t_xt_1}=\frac{\beta'(f)}{\beta(t_x)}f_{t_1}. \tag{22}$$

First we differentiate (21) with respect to t_x, use (22), and get

$$\frac{1}{\beta(f)}f_t+\frac{1}{\beta(t_x)}f_{t_1}=-\frac{(c\beta(f)+f)}{t_1\beta(t_x)}+\frac{c}{t}+\frac{t_x}{t\beta(t_x)}. \tag{23}$$

Next we differentiate (23) with respect to t_x, use (22), and arrive to the equality

$$\left\{\frac{t_x}{\beta(t_x)}-\frac{f}{\beta(f)}\right\}f_{t_1}=-\frac{(c\beta(f)+f)t_x}{t_1\beta(t_x)}-\frac{\beta(f)}{t_1}+\frac{\beta(t_x)}{t}+\frac{ct_x}{t}+\frac{t_x^2}{t\beta(t_x)}.$$

There are two possibilities:

either (6^*a), when

$$A:=\frac{t_x}{\beta(t_x)}-\frac{f}{\beta(f)}=0, \tag{24}$$

or (6^*b), when

$$f_{t_1}=\frac{\beta(f)\beta(t_x)}{t_x\beta(f)-f\beta(t_x)}\left\{\frac{-(c\beta(f)+f)t_x}{t_1\beta(t_x)}-\frac{\beta(f)}{t_1}+\frac{\beta(t_x)}{t}+\frac{ct_x}{t}+\frac{t_x^2}{t\beta(t_x)}\right\}. \tag{25}$$

Let us first consider case (6^*a). It follows from (24) and (20) that $f_{t_x}/f=1/t_x$, that is $f=K(x,t,t_1)t_x$. We substitute $f=K(x,t,t_1)t_x$ into (21), use $\beta(f)/t_1=(\beta(t_x)f)/(t_xt_1)=\frac{\beta(t_x)K}{t_1}$, and obtain

$$K_x+t_x\left\{\frac{K_t}{K}+K_{t_1}\right\}=\frac{\beta^2(t_x)}{t_x}\left\{\frac{K}{t_1}-\frac{1}{t}\right\},$$

that is, $K_x=0$, $\beta(t_x)=\sqrt{R^2t_x^2+Ct_x}$, $R=Const$, $B=Const$, and

$$\frac{K_t}{K}+K_{t_1}=R^2\left\{\frac{K}{t_1}-\frac{1}{t}\right\}. \tag{26}$$

Substitution of $\beta(t_x)=\sqrt{R^2t_x^2+Ct_x}$ into (24) shows that $\beta(t_x)=Rt_x$. Therefore, in case (6^*a), the n-integral is $I=\frac{t_{xx}}{Rt_x}-\frac{Rt_x}{t}$ and the corresponding chain (2) is of the form $t_{1x}=K(t,t_1)t_x$, where K satisfies (26).

Let us now study case (6^*b). It follows from (25) and (23) that

$$f_t=\frac{f\beta(t_x)\beta(f)}{\beta(f)t_x-f\beta(t_x)}\left\{\frac{c\beta(f)+f}{t_1\beta(t_x)}-\frac{c}{t}-\frac{t_x}{t\beta(t_x)}+\frac{\beta^2(f)}{t_1f\beta(t_x)}-\frac{\beta(f)}{tf}\right\}. \tag{27}$$

First we differentiate (25) with respect to t and find f_{t_1t}, use the expression for f_t from (27) and $\beta'(f)=-(f+c\beta(f))/\beta(f)$ to express f_{t_1t} in terms of $\beta(f)$, $\beta(t_x)$, f, t, t_1, t_x. Then we differentiate (27) with respect to t_1 and find f_{tt_1}, use the expression for f_{t_1} from (25) and $\beta'(f)=-(f+c\beta(f))/\beta(f)$ to express f_{tt_1} in terms of $\beta(f)$, $\beta(t_x)$, f, t, t_1, t_x.

Direct calculations show that

$$f_{tt_1}-f_{t_1t}=\frac{2\beta(f)ct_x(\beta^2(f)+cf\beta(f)+f^2)(-tf+t_1(c\beta(t_x)+t_x))}{tt_1^2(\beta(t_x)f-\beta(f)t_x)^2}.$$

Equality $f_{tt_1}=f_{t_1t}$ yields (i) $\beta^2(f)+cf\beta(f)+f^2=0$, i.e. $\beta(f)=Af$, $\beta(t_x)=At_x$, where $A=\frac{-c\pm\sqrt{c^2-4}}{2}$, or (ii) $f=t_1t^{-1}(c\beta(t_x)+t_x)$.

Let us consider case (i). It follows from (20) that $f=K(x,t,t_1)t_x$. The same considerations as in part (6^*a) show that the chain (2) in this case is $t_{1x}=K(t,t_1)t_x$, where function $K(t,t_1)$ satisfies (26).

Let us consider case (ii). It follows from (20) that $\beta(f)=t_1t^{-1}((1-c^2)\beta(t_x)-ct_x)$. We substitute this expression for $\beta(f)$ into (21) and get $c^2(2-c^2)\beta^2(t_x)+2c(1-c^2)t_x\beta(t_x)-c^2t_x^2=0$, that implies that (I) $c=0$, (II) $c^2=2$, (III) $\beta(t_x)=\frac{c}{2-c^2}t_x$, or (IV) $\beta(t_x)=-\frac{1}{c}t_x$. Cases (II) and (IV) are not realized, each of them is incompatible with $\beta\beta'+c\beta=-t_x$. Case (III) is realized only for $c=2$ (with $\beta(t_x)=-t_x$) and $c=-2$ (with $\beta(t_x)=t_x$). Therefore, using $f=t_1t^{-1}(c\beta(t_x)+t_x)$ and the fact that $c=0$ (with $\beta(t_x)=\pm it_x$) or $c=\pm2(\beta(t_x)=-\pm t_x)$ we arrive to a chain (2) of the form $t_{1x}=\pm\frac{t_1}{t}t_x$. Note that chains $t_{1x}=\pm t_1t^{-1}t_x$ with $\beta(t_x)=\pm t_x$ or $\beta(t_x)=\pm it_x$ is of the form $t_{1x}=K(t,t_1)t_x$, where K satisfies (26) with $R^2=1$ (for $t_{1x}=-t_1t^{-1}t_x$) or $R^2=-1$ (for $t_{1x}=t_1t^{-1}t_x$).

Case (7^*): Consider chains (2) with n-integral $I=\frac{t_{xx}}{\sqrt{t_x}}+2\frac{\sqrt{t_x}}{x+y}$, $y=Const$. Equality $DI=I$ implies

$$\frac{f_x+f_tt_x+f_{t_1}f+f_{t_x}t_{xx}}{\sqrt{f}}+2\frac{\sqrt{f}}{x+y}=\frac{t_{xx}}{\sqrt{t_x}}+2\frac{\sqrt{t_x}}{x+y}. \tag{28}$$

By comparing the coefficients before t_{xx} we have $f_{t_x}/\sqrt{f}=1/\sqrt{t_x}$, or

$$f=(\sqrt{t_x}+K(x,t,t_1))^2. \tag{29}$$

Substitute (29) into (28) and get $K_x + K_t t_x + K_{t_1} t_x + 2K_{t_1}\sqrt{t_x}K + K_{t_1}K^2 + \frac{K}{x+y} = 0$. We compare the coefficients before $\sqrt{t_x}$, t_x, ${t_x}^0$ and have $2K_{t_1}K = 0$, i.e. $K = L(x,t)$; $K_t + K_{t_1} = 0$, i.e. $K = L(x)$; and $K_x + K_{t_1}K^2 + \frac{K}{x+y} = 0$, i.e. $K = \frac{C}{x+y}$, $C = const$. Therefore, chain (2) with n-integral $I = \frac{t_{xx}}{\sqrt{t_x}} + 2\frac{\sqrt{t_x}}{x+y}$ becomes $t_{1x} = (\sqrt{t_x} + \frac{C}{x+y})^2$, where C and y are arbitrary constants.

Case (8*): Consider chains (2) with n-integral $I = \beta(t_x)t_{xx} - \frac{1}{(x+y)\beta(t_x)}$, where y is an arbitrary constant and $\beta'(t_x) = \beta^3(t_x) + \beta^2(t_x)$. The equality $DI = I$ gives

$$\beta(f)\{f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx}\} - \frac{1}{(x+y)\beta(f)} = \beta(t_x)t_{xx} - \frac{1}{(x+y)\beta(t_x)},$$

that implies

$$\beta(f) f_{t_x} = \beta(t_x) \tag{30}$$

and

$$\beta(f)\{f_x + f_t t_x + f_{t_1} f\} = \frac{1}{(x+y)\beta(f)} - \frac{1}{(x+y)\beta(t_x)}. \tag{31}$$

Differentiate (30) with respect to x, t, t_1 and get

$$f_{xt_x} = -(\beta(f)+1)\beta(t_x)f_x, \quad f_{tt_x} = -(\beta(f)+1)\beta(t_x)f_t, \quad f_{t_1t_x} = -(\beta(f)+1)\beta(t_x)f_{t_1}. \tag{32}$$

Now differentiate (31) with respect to t_x, we have

$$\beta(f)f_t + \beta(t_x)f_{t_1} = \frac{1}{x+y} - \frac{\beta(t_x)}{(x+y)\beta(f)}. \tag{33}$$

Differentiate (33) with respect to t_x and get $f_{t_1} = -\frac{1}{(x+y)\beta(f)}$. The last equation together with (33), (30) and (31) gives

$$f_{t_1} = -\frac{1}{(x+y)\beta(f)}, \quad f_t = \frac{1}{(x+y)\beta(f)}, \quad f_{t_x} = \frac{\beta(t_x)}{\beta(f)} \tag{34}$$

and

$$f_x = \frac{1}{x+y}\left\{\frac{1}{\beta^2(f)} - \frac{1}{\beta(f)\beta(t_x)} - \frac{t_x}{\beta(f)} + \frac{f}{\beta(f)}\right\}. \tag{35}$$

Since, by (34) and (35), $f_{t_1x} - f_{xt_1} = \frac{1}{\beta^2(f)(x_y)^2}(\beta(f)+1)$, then $\beta(f) = -1$, and, therefore, by (34), we have $f_{t_1} = (x+y)^{-1}$, $f_t = -(x+y)^{-1}$, $f_{t_x} = 1$. Hence, $f(x,t,t_1,t_x) = t_x + \frac{t_1 - t}{x+y} + C(x)$. We substitute this expression for f into (35) and obtain $C(x) = C(x+y)^{-1}$, where C is an arbitrary constant. Therefore, with the n-integral $I = \frac{t_{xx}}{\sqrt{t_x}} + 2\frac{\sqrt{t_x}}{x+y}$ the chain (2) becomes $t_{1x} = t_x + \frac{t_1 - t}{x+y} + C(x+y)^{-1}$, where y is arbitrary constant.

3 Proof of Remark 1.4

Case 1**: Consider all equations (1) with m-integral $\bar{I} = e^{v_1-v} + e^{v_1-v_2}$. Denote by $e^{-v_j} = w_j$, $j = 0, 1, 2$, and $e^{-\bar{v}_1} = \bar{w}_1$. In new variables $\bar{I} = \frac{v+v_2}{v_1}$ is an m-integral of equation $w_{1,1} = g(w, w_1\bar{w}_1)$. $\bar{D}\bar{I} = \bar{I}$ implies

$$\frac{w_2 + w}{w_1} = \frac{g_1 + \bar{w}_1}{g}. \tag{36}$$

We differentiate both sides of (36) with respect to w_2 and apply the shift operator D^{-1}, we have

$$\frac{1}{w_1} = \frac{g_{1w_2}}{g} \quad \Rightarrow \quad D^{-1}\left(\frac{1}{w_1}\right) = D^{-1}\left(\frac{g_{1w_2}}{g}\right) \quad \Rightarrow \quad g_{w_1} = \frac{\bar{w}_1}{w}.$$

Therefore,

$$g = \frac{\bar{w}_1 w_1}{w} + c(w, \bar{w}_1), \qquad g_1 = \frac{g w_2}{w_1} + c(w_1, g). \tag{37}$$

We substitute (37) into (36) and get

$$g\frac{w}{w_1} = c(w_1, g) + \bar{w}_1. \tag{38}$$

Substitution of (37) into (38) implies that $c(w, \bar{w}_1)w = c(w_1, g)w_1$, or the same, $c(w, \bar{w}_1)w = D(c(w, \bar{w}_1)w)$. Suppose that equation $w_{1,1} = g(w, w_1\bar{w}_1)$ does not admit an m-integral of the first order, then $c(w, \bar{w}_1)w = D(c(w, \bar{w}_1)w) = C = const$. Thus, $c(w, \bar{w}_1) = C/w$. Finally, $g(w, w_1, \bar{w}_1) = \frac{\bar{w}_1 w_1}{w} + Cw^{-1}$. Therefore, the equations (1) with m-integral $\bar{I} = e^{v_1-v} + e^{v_1-v_2}$ becomes $e^{v_{1,1}+v} = (C + e^{-(v_1+\bar{v}_1)})^{-1}$, where C is an arbitrary constant. Note that this equation is symmetric with respect to variables v_1 and $\bar{v}_1$. Therefore, n-integral for the equation can be obtained by simply changing in m-integral variables v_j into variables $\bar{v}_j$, $j = 1, 2$.

Case 4**: Consider equations (1) with m-integral $\bar{I} = \frac{(v_1-v)(v_2+L)}{(v_2-v)(v_1+L)}$. Equation $v_{1,1} = f(v, v_1, \bar{v}_1)$ can be rewritten as $v_{-1,1} = r(v, v_{-1}, \bar{v}_1)$. Equality $\bar{D}\bar{I} = \bar{I}$ implies

$$\frac{f - \bar{v}_1)(f_1 + L)}{(f_1 - \bar{v}_1)(f + L)} = \frac{(v_1 - v)(v_2 + L)}{(v_2 - v)(v_1 + L)}. \tag{39}$$

Take the logarithmic derivative of (39) with respect to v_2 and then apply the shift operator D^{-1}, we get

$$\frac{f_{1v_2}}{f_1 + L} - \frac{f_{1v_2}}{f_1 - \bar{v}_1} = \frac{1}{v_2 + L} - \frac{1}{v_2 - v} \quad \Rightarrow \quad \frac{f_{v_1}(r + L)}{(f + L)(f - r)} = \frac{v_{-1} + L}{(v_1 + L)(v_1 - v_{-1})}. \tag{40}$$

We conclude from the second equation of (40) that

$$\frac{f + L}{f - r} = \frac{v_1 + L}{v_1 - v_{-1}} K(v, \bar{v}_1). \tag{41}$$

Take the logarithmic derivative of (41) with respect to v_{-1} and get $f - r = r_{v_{-1}}(v_1 - v_{-1})$. Differentiation of the last equality with respect to v_1 yields $f_{v_1} = r_{v_{-1}}$. We differentiate (40) with respect to v_{-1} and use the fact that $f_{v_1} = r_{v_{-1}}$, we obtain $f_{v_1} = \pm\frac{f-r}{v_1-v_{-1}}$.

First assume that $f_{v_1} = -\frac{f-r}{v_1-v_{-1}}$. We have, $f - r = D(v, v_{-1}, \bar{v}_1)(v_1 - v_{-1})^{-1}$. It follows from $r_{v_{-1}} = -\frac{f-r}{v_1-v_{-1}}$ that $f-r = C(v, v_1, \bar{v}_1)(v_1-v_{-1})^{-1}$, and, therefore, $f-r = C(v, \bar{v}_1)(v_1-v_{-1})^{-1}$. We substitute this expression for $f-r$ into (41) and see that $f+L = C(v, \bar{v}_1)K(v, \bar{v}_1)(v_1+L)(v_1-v_{-1})^{-2}$ which is impossible since f does not depend on v_{-1}.

Now consider the case when $f_{v_1} = \frac{f-r}{v_1-v_{-1}}$. We have, $f - r = (v_1 - v_{-1})D(v, v_{-1}, \bar{v}_1)$. Also, $r_{v_{-1}} = \frac{f-r}{v_1-v_{-1}}$ implies that $f - r = (v_1 - v_{-1})C(v, v_1, \bar{v}_1)$. One can see that $D(v, v_{-1}, \bar{v}_1) = C(v, v_1, \bar{v}_1) =: C(v, \bar{v}_1)$. Therefore, $f - r = C(v, \bar{v}_1)(v_1 - v_{-1})$. It follows from (41) that

$$f = A(v, \bar{v}_1)v_1 + A(v, \bar{v}_1)L - L, \tag{42}$$

where $A = CK$. Note that $A = A(v, \bar{v}_1)$ and $A_1 = A(v_1, f(v, v_1, \bar{v}_1))$. Substitute (42) into (39), get

$$(Av_1 + AL - L - \bar{v}_1)(A_1v_2 + A_1L)(v_2 - v)(v_1 + L)$$
$$= (A_1v_2 + A_1L - L - \bar{v}_1)(Av_1 + AL)(v_1 - v)(v_2 + L),$$

and compare the coefficients before v_2^2, we have

$$A_1(Av_1 + AL - L - \bar{v}_1)(v_1 + L) = A_1(Av_1 + A)(v_1 - v). \tag{43}$$

It follows from (43) that $A_1 = 0$ or, by comparing the coefficients before v_1, one gets $A = \frac{L+\bar{v}_1}{L+v}$. Therefore, by (42), we have the equation $v_{1,1} = f = \frac{L(\bar{v}_1+v_1-v)+v_1\bar{v}_1}{L+v}$. Note that the equation is symmetric with respect to variables v_1 and $\bar{v}_1$. This observation allows one to write down an n-integral I by a given m-integral $\bar{I}$ by changing in $\bar{I}$ variables v_j into variables $\bar{v}_j$, $j = 1, 2$.

Case 7**: Consider all equations (1) with m-integral $F = 2v - v_1 - v_{-1} = D^{-1}\bar{I}$, where $\bar{I} = 2v_1 - v - v_2$. Equation $v_{1,1} = f(v, v_1, \bar{v}_1)$ can be rewritten as $v_{-1,1} = r(v, v_{-1}, \bar{v}_1)$. Equality $\bar{D}F = F$ implies

$$2\bar{v}_1 - f - r = 2v - v_1 - v_{-1}. \tag{44}$$

We apply $\frac{\partial}{\partial v_1}$ and $\frac{\partial}{\partial v_{-1}}$ to (44) and find that $f_{v_1} = 1$ and $r_{v_{-1}} = 1$. Therefore, $f = v_1 + h(v, \bar{v}_1)$ and $r = v_{-1} + q(v, \bar{v}_1)$. Substitute these expressions for f and r into (44) and get

$$q = 2\bar{v}_1 - 2v - h. \tag{45}$$

Equation $v_{1,1} = f = v_1 + h(v, v_1)$ can be rewritten as

$$\bar{v}_1 = v + h(v_{-1}, v_{-1,1}) = v + h(v_{-1}, v_{-1} + q(v, \bar{v}_1)). \tag{46}$$

First differentiate (46) with respect to v_{-1} and then apply the shift operator D^{-1}, we get $D^{-1}h_v + D^{-1}h_{\bar{v}_1} = 0$, that is $h = h(\bar{v}_1 - v)$. Equations (44) - (46) give $\bar{v}_1 - v = h(2\bar{v}_1 - 2v - h)$, or by taking $\epsilon = \bar{v}_1 - v$ one gets $\epsilon = h(2\epsilon - h(\epsilon))$. Therefore, the equation with m-integral $\bar{I} = 2v_1 - v - v_2$ becomes $v_{1,1} = v_1 + h(\bar{v}_1 - v)$, where h solves a functional equation $\epsilon = h(2\epsilon - h(\epsilon))$. This equation $v_{1,1} = v_1 + h(\bar{v}_1 - v)$ admits also an n-integral. Since the equation is of the form $Dz = h(z)$ with $z = \bar{v}_1 - v_1$ then we have $D(z - h^{-1}(z)) = z - h^{-1}(z)$. Actually, $D(z - h^{-1}(z)) = D(z) - z = h(z) - z = z - h^{-1}(z) = z - h^{-1}(z)$. Here we use the identity $h(z) - z = z - h^{-1}(z)$ which is equivalent to the functional equation $z = h(2z - h(z))$.

4 Conclusions

The problem of discretization of Liouville type equations is discussed. Besides purely theoretical interest as a bridge between two parallel realizations of the integrability theory, this subject has an important practical significance. There are two-dimensional Toda field equations corresponding to each semisimple or of Kac-Moody type Lie algebra (see [12], [13]). The question is open whether there exist integrable discrete versions of these. Different particular cases are studied in [14], [15], [16]. In the article a step is done towards the solution of the problem. An effective method of discretization is suggested based on integrals. It is known that the Bäcklund transform is a kind of discretization (see [3], [17]). We would like to stress that our method of discretization essentially differs from that one. Even though for some exceptional cases the semi-discrete equation obtained realizes the Bäcklund transformation of the original equation for the other examples it is not the case.

Acknowledgments

This work is partially supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) grant #209 T 062, Russian Foundation for Basic Research (RFBR) (grants # 11-01-00732-a, # 11-01-97005-r-povoljie-a, # 10-01-91222-CT-a and # 10-01-00088-a), and MK-8247.2010.1.

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