Mathematical Social Sciences 61 (2011) 157–165

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

# Hedonic coalition formation games: A new stability notion

Mehmet Karakaya *

*Department of Economics, Bilkent University, 06800, Bilkent, Ankara, Turkey*

ARTICLE INFO

*Article history:*
Received 22 December 2009
Received in revised form
21 March 2011
Accepted 23 March 2011
Available online 5 April 2011

ABSTRACT

This paper studies hedonic coalition formation games where each player's preferences rely only upon the members of her coalition. A new stability notion under free exit-free entry membership rights, referred to as strong Nash stability, is introduced which is stronger than both core and Nash stabilities studied earlier in the literature. Strong Nash stability has an analogue in non-cooperative games and it is the strongest stability notion appropriate to the context of hedonic coalition formation games. The weak top-choice property is introduced and shown to be sufficient for the existence of a strongly Nash stable partition. It is also shown that descending separable preferences guarantee the existence of a strongly Nash stable partition. Strong Nash stability under different membership rights is also studied.



## 1. Introduction

Individuals act by forming coalitions under certain economic and political circumstances such as the provision of public goods in local communities or forming clubs and organizations. One way to describe such an environment is to model it as a (pure) hedonic coalition formation game.

A hedonic coalition formation game consists of a finite nonempty set of players and a list of players' preferences where every player's preferences depend only on the members of her coalition.[1] An outcome of such a game is a partition of the player set (coalition structure) -that is, a collection of coalitions whose union is equal to the set of players, and which are pairwise disjoint. Marriage problems and roommate problems (Gale and Shapley (1962), Roth and Sotomayor (1990)) can be seen as special cases of hedonic coalition formation games, where each agent only considers who will be his/her mate. In fact, hedonic games are reduced forms of general coalition formation games where, for each coalition, how its total payoff is to be divided among its members is fixed in advance and made known to all agents.[2]

Given a hedonic coalition formation game, the main concern is the existence of partitions that are stable in some sense. The stability concepts that have been mostly studied so far are core stability and Nash stability of coalition structures.[3] A partition is core stable if there is no coalition each of whose members strictly prefers it to the coalition to which she belongs under the given partition. A partition is said to be Nash stable if there is no player who benefits from leaving her present coalition to join another coalition of the partition which might be the "empty coalition" in this context. Note that a Nash stable partition need not be core stable, and a core stable partition need not be Nash stable.

One needs to focus attention on two key points when considering or comparing stability concepts, namely: (i) who can deviate from the given partition (e.g., a coalition of players as in core stability, a singleton as in Nash stability), and (ii) what the deviators are entitled to do (e.g., form a new, self standing coalition as in core stability, join an already existing coalition —irrespective of how the incumbent members are effected- as in Nash stability). For hedonic coalition formation games, the second point can be examined by introducing membership rights. Sertel (1992) introduced four possible membership rights in an abstract setting. Given a hedonic game and a partition, the membership rights employed specify the set of agents whose approval is needed for each particular deviation of a subset of players.

Under *free exit-free entry* (FX-FE) membership rights, every agent is entitled to make any movements among the coalitions of a given partition without taking any permission of members of the coalitions that she leaves or joins. An example in the context of the roommate problem would be that whenever an agent finds a place in a room, she has the right to move into that room. So, two agents in different rooms may benefit by exchanging their rooms without asking anyone else. Another example is that a citizen of a country

* Tel.: +90 3122902370; fax: +90 3122665140.
*E-mail addresses:* kmehmet@bilkent.edu.tr, karakayamehmet@gmail.com.

1 The dependence of a player's utility on the identity of members of her coalition is referred to as the "hedonic aspect" in Drèze and Greenberg (1980), and the formal model of (pure) hedonic coalition formation games was introduced by Banerjee et al. (2001) and Bogomolnaia and Jackson (2002).

2 The reader is referred to Bogomolnaia and Jackson (2002) for a detailed motivation of (pure) hedonic games.

3 See the taxonomy introduced in Sung and Dimitrov (2007) for all stability concepts which were studied in the literature.

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doi:10.1016/j.mathsocsci.2011.03.004

which is a member of the EU can move to another country in the EU without the permission of either country.

Under *free exit-approved entry* (FX-AE) membership rights, an agent can leave her current coalition without the permissions of her current partners, but she can join another coalition only if all members of that coalition welcome her, that is her joining does not hurt any member of the coalition she joins. A typical example is provided by club membership, where a member of a club can leave her current club without taking into account whether her leaving hurts some members of that club. However, she needs the approval of the members of a club that she wants to join. Another example is that of a researcher, who is a member of a research team and can leave the team without the permissions of other team members, while her joining another team is usually subject to the approval of that team's present members.

Under *approved exit-free entry* (AX-FE), every agent is endowed with rights, under which she can leave her current coalition only if that coalition's members approve her leaving, while her joining requires no one else's permission. An example would be that of an army recruiting volunteers. Every healthy citizen in a certain age interval may enter the army if he volunteers to do so, but is not allowed to freely exit once he is in.

Under *approved exit-approved entry* (AX-AE) membership rights each player needs to get the unanimous permission of the coalition that she leaves or joins. A typical example is that of a criminal organization. An agent who is a member of a criminal organization cannot leave it without permission as she may have information about some secrets of the organization. Similarly, one cannot join a criminal organization without permission by a similar token.

Note that under the definition of Nash stability, a player can deviate by leaving her current coalition to join another coalition of the partition without any permission of the players of the coalitions that she leaves or joins, although she might thereby be hurting some of these. In other words, Nash stability is defined under FX-FE membership rights.[4]

The aim of this paper is to study coalitional extension of Nash stability under FX-FE membership rights, referred to as strong Nash stability, which has not been studied yet. Note that strong Nash stability is not defined in Sung and Dimitrov (2007) but they identified some weaker versions of strong Nash stability.

Two approaches will be employed while defining a strongly Nash stable partition. The first approach is posed in terms of an induced non-cooperative game. A hedonic coalition formation game induces a non-cooperative game in which each player chooses a "label"; players who choose the same label are placed in a common coalition. Strong Nash (respectively, Nash) stability in this induced game then corresponds to strong Nash (respectively, Nash) of the corresponding partition in the coalitional form of the game. The second approach is posed in terms of movements and reachability. A partition is said to be strongly Nash stable if there is no subset of players who reach a new partition via certain admissible movements such that these players strictly prefer the new partition to the initial one.

Banerjee et al. (2001) introduced the *top-coalition* and the *weak top-coalition* properties and proved that each property suffices for a hedonic game to have a core stable partition. Bogomolnaia and Jackson (2002) introduced two conditions, called *ordinal balancedness* and *weak consecutiveness*. They showed that if a hedonic game is ordinally balanced or weakly consecutive, then there exists a core stable partition.[5]

Bogomolnaia and Jackson (2002) showed that a hedonic game which is *additively separable* and satisfies *symmetry* has a Nash stable partition. However, Banerjee et al. (2001) provided an example of a hedonic game which is additively separable and satisfies symmetry, but has no core stable partition. Burani and Zwicker (2003) considered *descending separable preferences* posed in the form of several ordinal axioms, and showed that it is sufficient for the simultaneous existence of Nash and core stable partition.

The weak top-choice property is introduced by borrowing the definition of weak top-coalition from Banerjee et al. (2001), and shown that it guarantees the existence of a strongly Nash stable partition (Proposition 1). It is also shown that descending separable preferences suffice for a hedonic game to have a strongly Nash stable partition (Proposition 2).

How the concept of strong Nash stability changes under different membership rights is also examined. It is shown that under FX-AE membership rights, a partition is FX-AE strictly strongly Nash stable if and only if it is strictly core stable (Proposition 3), showing that core stability entails an FX-AE rights structure. Sung and Dimitrov (2007) defined *contractual strict core* stability and showed that for any hedonic game such a partition always exists. It is proved that under AX-AE membership rights, a partition is AX-AE strictly strongly Nash stable if and only if it is contractual strictly core stable (Proposition 4).

The paper is organized as follows: Section 2 presents the basic notions. Section 3 introduces the weak top-choice property and provides an existence result. Descending separable preferences are studied in Section 4 and it is shown that there always exists a strongly Nash stable partition if players have descending separable preferences. In Section 5, strong Nash stability under different membership rights is studied.

## 2. Basic notions

Let $N = \{1, 2, \ldots, n\}$ be a nonempty finite set of players. A nonempty subset $H$ of $N$ is called a coalition. Let $i \in N$ be a player, and $\Sigma_i = \{H \subseteq N \mid i \in H\}$ denote the set of coalitions each of which contains player $i$. Each player $i$ has a reflexive, complete and transitive preference relation $\succeq_i$ over $\Sigma_i$. So, a player's preferences depend only on the members of her coalition. The strict and indifference preference relations associated with $\succeq_i$ will be denoted by $\succ_i$ and $\sim_i$, respectively. Let $\succeq = (\succeq_1, \ldots, \succeq_n)$ denote a preference profile for the set of players.

**Definition 1.** A pair $G = (N, \succeq)$ denote a ***hedonic coalition formation game***, or simply a ***hedonic game***.

Given a hedonic game, it is required that the set of coalitions which might form to be a partition of $N$.

**Definition 2.** ***A partition (coalition structure)*** of a finite set of players $N = \{1, \ldots, n\}$ is a set $\pi = \{H_1, H_2, \ldots, H_K\}$ ($K \leq n$ is a positive integer) such that

(i) for any $k \in \{1, \ldots, K\}$, $H_k \neq \emptyset$,
(ii) $\bigcup_{k=1}^{K} H_k = N$, and
(iii) for any $k, l \in \{1, \ldots, K\}$ with $k \neq l$, $H_k \cap H_l = \emptyset$.

4 Other stability concepts that consider individual deviations under different membership rights have already been studied in the literature. That is, *individual stability* is defined under FX-AE membership rights (see Bogomolnaia and Jackson (2002)), *contractual Nash stability* is defined under AX-FE membership rights (see Sung and Dimitrov (2007)), and *contractual individual stability* is defined under AX-AE membership rights (see Bogomolnaia and Jackson (2002) and Ballester (2004)).

5 For other studies concerning the existence of core stable partitions, the reader is referred to Alcalde and Romero-Medina (2006), Alcalde and Revilla (2004), Dimitrov et al. (2006) and Pápai (2004), among others.

Let $\Pi(N)$ denote the set of all partitions of $N$. Given any $\pi \in \Pi(N)$ and any $i \in N$, let $\pi(i) \in \pi$ denote the unique coalition which contains the player $i$. Since we are working with hedonic games, for any player $i \in N$, the preference relation $\succeq_i$ over $\Sigma_i$ can be extended over the set of all partitions $\Pi(N)$ in a usual way as follows: For any $\pi, \acute{\pi} \in \Pi(N)$, $[\pi \succeq_i \acute{\pi}]$ if and only if $[\pi(i) \succeq_i \acute{\pi}(i)]$.

**Definition 3.** Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***individually rational for player*** $i$ if $\pi(i) \succeq_i \{i\}$ and is ***individually rational*** if it is individually rational for every player $i \in N$.

A partition is individually rational if each player prefers the coalition that she is a member of to being single, i.e., each agent $i$ prefers $\pi(i)$ to $\{i\}$.

**Definition 4.** Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***core stable*** if there does not exist a coalition $T \subseteq N$ such that for all $i \in T$, $T \succ_i \pi(i)$. If such a coalition $T$ exists, then it is said that $T$ ***blocks*** $\pi$.[6]

**Definition 5.** Let $G = (N, \succeq)$ be a hedonic game and $\pi \in \Pi(N)$ a partition. We say that a player $i \in N$ ***Nash blocks*** $\pi$ if there exists a coalition $H \in (\pi \cup \{\emptyset\})$ such that $H \cup \{i\} \succ_i \pi(i)$. A partition is ***Nash stable*** if there does not exist a player who Nash blocks it.

Two approaches will be employed while defining the strongly Nash stable partition. In the first one, the non-cooperative game induced by a hedonic game is used.

Every hedonic game induces a non-cooperative game as defined below.[7]

Let $G = (N, \succeq)$ be a hedonic game with $|N| = n$ players. Consider the following induced non-cooperative game $\Gamma^G = (N, (S_i)_{i\in N}, (R_i)_{i\in N})$ which is defined as follows:

- The *set of players* in $\Gamma^G$ is the player set $N$ of $G$.
- Let $\mathcal{L} = \{L_1, \ldots, L_m\}$ be a finite set of labels such that $m = n+1$. Take $\mathcal{L}$ to be the *set of strategies* available to each player, so $S_i = \mathcal{L}$ for each $i \in N$. Let $S = \prod_{i\in N} S_i$ denote the strategy space. A strategy profile $s = (s_1, \ldots, s_n) \in S$ induces a partition $\pi_s$ of $N$ as follows: two players $i, j$ of $N$ are in the same piece of $\pi_s$ if and only if $s_i = s_j$ ($i$ and $j$ choose the same strategy according to $s$).
- *Preferences* for $\Gamma^G$ is defined as follows: a player $i$ prefers the strategy profile $s$ to the strategy profile $\acute{s}$, $sR_i\acute{s}$, if and only if $\pi_s(i) \succeq_i \pi_{\acute{s}}(i)$, i.e., player $i$ prefers the coalition of those who choose the same strategy as she does according to $s$, to the coalition of those who choose the same strategy as she does according to $\acute{s}$.

Now, the main stability concept of this paper will be defined by using the induced non-cooperative game approach.

**Definition 6.** Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***strongly Nash stable*** if it is induced by a strategy profile which is a strong Nash equilibrium of the induced non-cooperative game $\Gamma^G$.

Thus, the Nash equilibria of $\Gamma^G$ correspond to the Nash stable partitions of $G$, and the strong Nash equilibria of $\Gamma^G$ correspond to the strongly Nash stable partitions of $G$. Hence, strong Nash stability has an analogue in non-cooperative games, and it is the strongest natural stability notion appropriate to the context of hedonic games.

If the strategy profile $s$ which induces the partition $\pi_s$ is not a strong Nash equilibrium of $\Gamma^G$, then there is a subset of players $H \subseteq N$ which deviates from $s$ (according to $s$) and this deviation is beneficial to all agents in $H$. In such a case, it is said that $H$ ***strongly Nash blocks*** the partition $\pi_s$.

The second approach is posed in terms of movements and reachability which is derived from the first one.

Let $\pi_s$ be a partition which is induced by the strategy profile $s$, and $H \subseteq N$ be a deviating subset of players. The deviation of these players from $s$ can be explained as movements among the coalitions of the partition $\pi_s$, where the allowable movements of these players are as follows[8]:

(i) All players in $H \notin \pi_s$ choose a label which is not chosen by any player under $s$.[9] Let $\acute{s}$ denote the strategy profile that is obtained by this deviation. Now, $H \in \pi_{\acute{s}}$. This deviation means in terms of movements that all players in $H$ leave their current coalitions and form the coalition $H \in \pi_{\acute{s}}$ (which is the movement used in the definition of blocking in the core stability).

(ii) All players in $H$[10] choose the label which is chosen by members of a coalition $T \in \pi_s$. Let $\widetilde{s}$ denote the strategy profile that is obtained by this deviation. Now, $(H \cup T) \in \pi_{\widetilde{s}}$. This deviation means all players in $H$ leave their current coalitions and join another coalition $T$ of $\pi_s$, so for each $i \in H$, $\pi_{\widetilde{s}}(i) = T \cup H$.

(iii) Players in $H \notin \pi_s$ partition among themselves as $\{H_1, \ldots, H_t\}$, and for any $k \in \{1, \ldots, t\}$, agents in $H_k$ choose the label which is chosen under $s$ by an agent $j \in H_{k+1}$, where it is taken $t + 1 = 1$. Let $\widehat{s}$ denote the strategy profile that is obtained by this deviation. Now, for any $i \in H_k$, $\pi_{\widehat{s}}(i) = (\pi_s(j) \setminus H) \cup H_k$. This deviation means individual players in $H$ (or subsets of $H$) exchange their current coalitions in the partition $\pi_s$. For instance, let $H = \{i, j\} \notin \pi_s$ and player $i$ leaves $\pi_s(i)$ and joins $\pi_s(j) \setminus \{j\}$, and player $j$ leaves $\pi_s(j)$ and joins $\pi_s(i) \setminus \{i\}$. So, $\pi_{\widehat{s}}(i) = (\pi_s(j) \setminus \{j\}) \cup \{i\}$ and $\pi_{\widehat{s}}(j) = (\pi_s(i) \setminus \{i\}) \cup \{j\}$. Note that more complicated movements are possible when the size of $H$ increases.[11]

Given a partition $\pi$ and a subset of players $H \subseteq N$, by any movements of $H$ among the coalitions of the partition $\pi$, players of $H$ obtain a new partition $\acute{\pi}$, and it is said that $\acute{\pi}$ is reachable from the partition $\pi$ via $H$.

**Definition 7.** Let $G = (N, \succeq)$ be a hedonic game and $\pi \in \Pi(N)$ be a partition. Another partition $\acute{\pi} \in (\Pi(N) \setminus \{\pi\})$ is said to be ***reachable from*** $\pi$ ***by movements of a subset of players*** $H \subseteq N$, denoted by $\pi \xrightarrow{H} \acute{\pi}$, if, for all $i, j \in (N \setminus H)$ with $i \neq j$, $\pi(i) = \pi(j) \Leftrightarrow \acute{\pi}(i) = \acute{\pi}(j)$.

6 A partition $\pi \in \Pi(N)$ is ***strictly core stable*** if there does not exist a coalition $T \subseteq N$ such that for all $i \in T$, $T \succeq_i \pi(i)$, and for some $i \in T$, $T \succ_i \pi(i)$. If such a coalition $T$ exists, then it is said that $T$ ***weakly blocks*** $\pi$.

7 I am grateful to the Associate Editor for suggesting this approach.

8 Movements of $H$ are coordinated and simultaneous.

9 Such a label always exists, since $m = n + 1$.

10 It is possible in here that $H \in \pi_s$.

11 Movements of $H$ among the coalitions of the partition $\pi_s$ can also be explained as follows: Each player in $H$ leaves the coalition that she belongs under partition $\pi_s$. Let $\pi_s^{-H} = \{T \setminus H \mid T \in \pi_s \text{ and } T \setminus H \neq \emptyset\}$ denote the set of coalitions after each player in $H$ leaves her current coalition. Now, individual players or subsets of $H$ can join any coalition (or an empty set) of $(\pi_s^{-H} \cup \{\emptyset\})$. This approach is similar to the one given by Conley and Konishi (2002). In their approach, a set of agents is only allowed to form coalitions among themselves, i.e., individual players or subsets of $H$ are only permitted to join the empty set. However, in our approach individual players or subsets of $H$ are allowed to join not only the empty set but also any coalition of $\pi_s^{-H}$.

Reachability by movements of a subset of agents simply says that agents who are not deviators are passive, and a non-deviator remains with all former mates who are not deviators. Notice that a subset of players $\widehat{H} \supseteq H$ can do all movements that $H$ can. Note that for any $\pi \in \Pi(N)$ and $\acute{\pi} \in (\Pi(N) \setminus \{\pi\})$, $\pi \xrightarrow{N} \acute{\pi}$, i.e., given any partition $\pi$ all other partitions can be reached by movements of the grand coalition $N$.

Now, the strong Nash stability of a partition can also be defined in terms of movements and reachability.

**Definition 8.** Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***strongly Nash stable*** if there does not exist a pair $(\acute{\pi}, H)$ (where $\acute{\pi} \in (\Pi(N) \setminus \{\pi\})$ and $\emptyset \neq H \subseteq N$) such that

(i) $\pi \xrightarrow{H} \acute{\pi}$ ($\acute{\pi}$ is reachable from $\pi$ by movements of $H$), and
(ii) for all $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$.

If such a pair $(\acute{\pi}, H)$ exists, then it is said that $H$ ***strongly Nash blocks*** $\pi$ (by inducing $\acute{\pi}$).

Note that the two definitions of strongly Nash stable partitions are equivalent (Definitions 6 and 8).

It is clear that a strongly Nash stable partition is both core and Nash stable. However, a hedonic game which has a partition that is both core and Nash stable may not have a strongly Nash stable partition.

**Example 1.** Let $G = (N, \succeq)$, where $N = \{1, 2, 3, 4\}$ and the preferences of players are as follows:

$$\{1, 4\} \succ_1 \{1, 2\} \succ_1 \{1, 3, 4\} \succ_1 \{1, 3\} \succ_1 \{1\} \succ_1 \dots,$$ [12]

$$\{2, 4\} \succ_2 \{1, 2\} \succ_2 \{2, 3, 4\} \succ_2 \{2\} \succ_2 \dots,$$

$$\{1, 3\} \succ_3 \{3, 4\} \succ_3 \{1, 2, 3\} \succ_3 \{3\} \succ_3 \dots,$$

$$\{3, 4\} \succ_4 \{1, 2, 4\} \succ_4 \{2, 4\} \succ_4 \{4\} \succ_4 \dots.$$

The partitions $\widetilde{\pi} = \{\{1, 2\}, \{3, 4\}\}$ and $\widehat{\pi} = \{\{1, 3\}, \{2, 4\}\}$ are the only partitions which are both core stable and Nash stable, and there is no partition $\pi \in (\Pi(N) \setminus \{\widetilde{\pi}, \widehat{\pi}\})$ which is either core stable or Nash stable. However, neither $\widetilde{\pi}$ nor $\widehat{\pi}$ is strongly Nash stable.

Let $\widetilde{s}$ denote the strategy profile in $\Gamma^G$ which induces the partition $\widetilde{\pi}$. So, players 1 and 2 choose the same label under $\widetilde{s}$, say $\acute{L}$, and players 3 and 4 choose the same label under $\widetilde{s}$, say $\bar{L}$. Thus $\widetilde{s} = (\acute{L}, \acute{L}, \bar{L}, \bar{L})$. The strategy profile $\widetilde{s}$ is not a strong Nash equilibrium of $\Gamma^G$, since players 2 and 3 deviate from $\widetilde{s}$ as follows:[13]Player 2 chooses label $\bar{L}$ and player 3 chooses label $\acute{L}$. Let $\widehat{s} = (\acute{L}, \bar{L}, \acute{L}, \bar{L})$ denote the strategy profile that is obtained by the deviation of players 2 and 3. Now, the strategy profile $\widehat{s}$ induces the partition $\widehat{\pi} = \{\{1, 3\}, \{2, 4\}\}$.[14] This deviation is beneficial to both players 2 and 3, since $\widehat{\pi}(2) \succ_2 \widetilde{\pi}(2)$ and $\widehat{\pi}(3) \succ_3 \widetilde{\pi}(3)$. Therefore, $\widetilde{\pi}$ is not strongly Nash stable.

Now consider the partition $\widehat{\pi} = \{\{1, 3\}, \{2, 4\}\}$. $\widehat{\pi}$ is not strongly Nash stable, since players 1 and 4 strongly Nash block the partition $\widehat{\pi}$ by exchanging their current coalitions, i.e., $\widehat{\pi} \xrightarrow{\{1,4\}} \widetilde{\pi}$, and $\widetilde{\pi}(1) \succ_1 \widehat{\pi}(1)$ and $\widetilde{\pi}(4) \succ_4 \widehat{\pi}(4)$.

Hence the partitions $\widetilde{\pi}$ and $\widehat{\pi}$ are not strongly Nash stable, whereas they are both core and Nash stable. Therefore there is no strongly Nash stable partition for this game.

12 Note that only individually rational coalitions are listed in a player's preference list, since remaining coalitions for the player can be listed in any way.

13 Note that players 2 and 3 dislike each other, that is $\{2\} \succ_2 \{2, 3\}$ and $\{3\} \succ_3 \{2, 3\}$.

14 This deviation means in terms of movements that players 2 and 3 exchange the coalitions that they are in under $\widetilde{\pi}$, and the partition $\widehat{\pi}$ is reached by this movement.

Iehlé (2007) introduced *pivotal balancedness* and showed that it is both necessary and sufficient for the existence of a core stable partition. As strong Nash stability implies core stability, and the hedonic game in Example 1 has a core stable partition but lacks any strongly Nash stable partitions, it follows that pivotal balancedness is a necessary but not sufficient condition for strong Nash stability.

## 3. The weak top-choice property

Banerjee et al. (2001) introduced two top-coalition properties and showed that each property is sufficient for a hedonic game to have a core stable partition.

Given a nonempty set of players $\widetilde{N} \subseteq N$, a nonempty subset $H \subseteq \widetilde{N}$ is a ***top-coalition*** of $\widetilde{N}$ if for any $i \in H$ and any $T \subseteq \widetilde{N}$ with $i \in T$, we have $H \succeq_i T$. A game $G = (N, \succeq)$ satisfies the ***top-coalition property*** if for any nonempty set of players $\widetilde{N} \subseteq N$, there exists a top-coalition of $\widetilde{N}$.

Given a nonempty set of players $\widetilde{N} \subseteq N$, a nonempty subset $H \subseteq \widetilde{N}$ is a ***weak top-coalition*** of $\widetilde{N}$ if $H$ has an ordered partition $\{H^1, \dots, H^l\}$ such that

(i) for any $i \in H^1$ and any $T \subseteq \widetilde{N}$ with $i \in T$, we have $H \succeq_i T$, and
(ii) for any $k > 1$, any $i \in H^k$ and any $T \subseteq \widetilde{N}$ with $i \in T$, we have $T \succ_i H \Rightarrow T \cap (\bigcup_{m<k} H^m) \neq \emptyset$.

A game $G = (N, \succeq)$ satisfies the ***weak top-coalition property*** if for any nonempty set of players $\widetilde{N} \subseteq N$, there exists a weak top-coalition of $\widetilde{N}$.

For any nonempty set of players $H \subseteq N$, let $W(H)$ denote the weak top-coalitions of $H$. Thus, $W(N)$ denote the weak top-coalitions of the grand coalition $N$.

**Definition 9.** A hedonic game $G = (N, \succeq)$ satisfies the ***weak top-choice property*** if $W(N)$ partitions $N$.

**Proposition 1.** *If a hedonic game satisfies the weak top-choice property, then it has a strongly Nash stable partition.*

**Proof.** Let $G = (N, \succeq)$ be a hedonic game which satisfies the weak top-choice property. Let $W(N) = \{H_1, \dots, H_K\}$ with corresponding partitions $\{H_1^1, \dots, H_1^{l(1)}\}, \dots, \{H_K^1, \dots, H_K^{l(K)}\}$. Clearly, $W(N)$ is a partition for $N$ since the game satisfies the weak top-choice property. Let $W(N) = \pi^\star$. It will be shown that $\pi^\star$ is strongly Nash stable. Suppose that $\pi^\star$ is not strongly Nash stable. Then, there exists a nonempty subset of players $H \subseteq N$ which strongly Nash blocks the partition $\pi^\star$.

Note that $H \cap (\bigcup_{j=1}^K H_j^1) = \emptyset$, since for any $j \in \{1, \dots, K\}$, for any $i \in H_j^1$ and any $T \in \Sigma_i$, $H_j \succeq_i T$. Now it will be shown that $H \cap (\bigcup_{j=1}^K H_j^2) = \emptyset$. For any $j \in \{1, \dots, K\}$, any agent $i \in H_j^2$ needs the cooperation of at least one agent in $H_j^1$ in order to form a better coalition than $H_j$. That is, for any $i \in H_j^2$ and any $T \in \Sigma_i$, $T \succ_i H_j$ implies $T \cap H_j^1 \neq \emptyset$. However, it is known that $H \cap H_j^1 = \emptyset$ for all $j \in \{1, \dots, K\}$, so $H \cap (\bigcup_{j=1}^K H_j^2) = \emptyset$.

Continuing with the similar arguments it is shown that $H \cap (\bigcup_{j=1}^K H_j^k) = \emptyset$ for all $k \in \{1, \dots, \bar{l}\}$, where $\bar{l} = max \{l(1), \dots, l(K)\}$. However, this implies that there does not exist a nonempty subset of players $H \subseteq N$ which strongly Nash blocks the partition $\pi^\star$, a contradiction. Hence $\pi^\star$ is strongly Nash stable.

We have constructed examples showing that the weak top-choice property and the weak top-coalition property are independent of each other.[15] If a game satisfies the weak top-choice

15 These examples are provided as a supplementary material to the Associate Editor and two referees, but are not included in the paper. A reader who wants to see these examples may contact the Author.

property and players have strict preferences, then the game may have more than one strongly Nash stable partition, we provided such an example.

A stronger version of the weak top-choice property can be defined as follows (by using the definition of top-coalition): A hedonic game $G = (N, \succeq)$ satisfies the ***top-choice property*** if the top-coalitions of the grand coalition $N$ form a partition of $N$. Now, if a hedonic game satisfies the top-choice property then it has a strongly Nash stable partition, where the proof is left to the reader. Moreover, if every player's best coalition is unique then there exists a unique strongly Nash stable partition which consists of the top-coalitions of $N$. We have constructed examples showing that the top-choice property and the top-coalition property (respectively, the weak top-coalition property) are independent of each other. It is clear that if a hedonic game satisfies the top-choice property then it also satisfies the weak top-choice property. However, a hedonic game satisfying the weak top-choice property may fail to satisfy the top-choice property.

An application of the weak top-choice property is Benassy (1982)'s *uniform reallocation rule*.[16] Banerjee et al. (2001) showed that a hedonic game which is induced by the uniform reallocation rule satisfies the weak top-coalition property, by proving that any subset $\widetilde{N} \subseteq N$ is a weak top-coalition of itself. Hence, the weak top-choice property is satisfied, and the partition $\{N\}$ is strongly Nash stable. Note that a hedonic game which is induced by the uniform reallocation rule may violate the top-choice property.[17]

## 4. Descending separable preferences

In a well established paper, Burani and Zwicker (2003) study hedonic games when players have *descending separable preferences*, and show that such a hedonic game always has a partition, which is called the *top segment partition*, that is both core and Nash stable. Burani and Zwicker (2003) will be followed to define descending separable preferences and the top segment partition.[18]

Let $p : N \to N$ be a permutation of the set of players and assume that $p$ yields a strict reference ranking of players

$$p_1 > p_2 > \cdots > p_n. \tag{1}$$

The following conditions are defined for an individual player's preferences.

**Condition 1** (*Common Ranking of Individuals, CRI*)**.** For any three distinct players $p_i, p_j$ and $p_k$, if $p_j > p_k$ then $\{p_i, p_j\} \succeq_{p_i} \{p_i, p_k\}$.

**Condition 2** (*Descending Desire, DD*)**.** For any pair $p_i, p_j$ of distinct players with $p_i > p_j$ and for any coalition $C$ containing neither player $p_i$ nor $p_j$, if $\{p_j\} \cup C \succeq_{p_j} \{p_j\}$ then $\{p_i\} \cup C \succeq_{p_i} \{p_i\}$ and if $\{p_j\} \cup C \succ_{p_j} \{p_j\}$ then $\{p_i\} \cup C \succ_{p_i} \{p_i\}$.

**Condition 3** (*Separable Preferences, SP*)**.** A profile of players' preferences is *separable* if, for every $i, j \in N$ and every coalition $C$ such that $C \in \Sigma_i$ and $j \notin C$, $\{i, j\} \succeq_i \{i\} \Leftrightarrow C \cup \{j\} \succeq_i C$ and $\{i, j\} \succ_i \{i\} \Leftrightarrow C \cup \{j\} \succ_i C$.

Condition SP implies the property of iterated separable preferences.

**Definition 10** (*Iterated Separable Preferences*)**.** For any player $p_i$ and for any two disjoint coalitions $C$ and $D$ with $C \ni p_i$, if $\{p_i, d\} \succeq_{p_i} \{p_i\}$ for every $d \in D$ then $C \cup D \succeq_{p_i} C$, and if $\{p_i, d\} \succ_{p_i} \{p_i\}$ for every $d \in D$ then $C \cup D \succ_{p_i} C$.

**Condition 4** (*Group Separable Preferences, GSP*)**.** For any player $p_i$ and for any two disjoint coalitions $C$ and $D$ with $C \ni p_i$, if $\{p_i\} \cup D \succeq_{p_i} \{p_i\}$ then $C \cup D \succeq_{p_i} C$ and if $\{p_i\} \cup D \succ_{p_i} \{p_i\}$ then $C \cup D \succ_{p_i} C$.

**Condition 5** (*Responsive Preferences, RESP*)**.** For any triple of players $p_i, p_j, p_k$ and for any coalition $C$ such that $p_j, p_k \notin C$ and $p_i \in C$, $\{p_i, p_j\} \succeq_{p_i} \{p_i, p_k\}$ if and only if $\{p_j\} \cup C \succeq_{p_i} \{p_k\} \cup C$ and $\{p_i, p_j\} \succ_{p_i} \{p_i, p_k\}$ if and only if $\{p_j\} \cup C \succ_{p_i} \{p_k\} \cup C$.

**Condition 6** (*Replaceable Preferences, REP*)**.** For any pair $p_i, p_j$ of distinct players with $p_i > p_j$ and for any coalition $C$ containing neither player $p_i$ nor $p_j$, if $\{p_i, p_j\} \cup C \succeq_{p_j} \{p_j\}$ then $\{p_i, p_j\} \cup C \succeq_{p_i} \{p_i\}$ and if $\{p_i, p_j\} \cup C \succ_{p_j} \{p_j\}$ then $\{p_i, p_j\} \cup C \succ_{p_i} \{p_i\}$.

Condition REP implies descending mutual preferences.

**Definition 11** (*Descending Mutual Preferences*)**.** For any pair $p_i, p_j$ of distinct players with $p_i > p_j$, if $\{p_i, p_j\} \succeq_{p_j} \{p_j\}$ then $\{p_i, p_j\} \succeq_{p_i} \{p_i\}$ and if $\{p_i, p_j\} \succ_{p_j} \{p_j\}$ then $\{p_i, p_j\} \succ_{p_i} \{p_i\}$.

**Definition 12.** A profile of agents' preferences is ***descending separable*** if there exists a reference ordering (1) under which Conditions 1 (CRI), 2 (DD), 3 (SP), 4 (GSP), 5 (RESP), and 6 (REP) all hold.

Let $G = (N, \succeq)$ be a hedonic game where players have descending separable preferences. A partition $\pi^\star = \{T^\star, \{p_{l+1}\}, \ldots, \{p_n\}\}$ is called a *top-segment partition* which is obtained in terms of the reference ordering (1) as follows: First, the *top-segment coalition* $T^\star$ is formed. Player $p_1$, the first agent in the ordering, belongs to the top-segment coalition. If the next agent, player $p_2$, strictly prefers being alone to joining $p_1$, then $T^\star$ is completed and $T^\star = \{p_1\}$. If, however, $\{p_1, p_2\} \succeq_{p_2} \{p_2\}$, then player $p_2$ is added to $T^\star$. Continue to add players from left to right until a player, denoted as $p_{l+1}$, is reached who strictly prefers staying alone to joining the growing coalition (or until everyone joins, if such an agent $p_{l+1}$ is never reached). The top-segment coalition is denoted by $T^\star = \{p_1, \ldots, p_l\}$. Second, let players from $p_{l+1}$ to $p_n$ each form a one member coalition.

Following results are taken from Burani and Zwicker (2003) which will be helpful while proving that a hedonic game with descending separable preferences always has a strongly Nash stable partition.

**Lemma 1** (*Burani and Zwicker (2003), Lemma 1, page 37*)**.** *Every individually rational coalition contains at most $l$ members.*

It is shown in Burani and Zwicker (2003) that there exists a coalition $\emptyset \neq T^{\star\star} = \{p_1, \ldots, p_f\}$ contained in $T^\star$ such that $\{p_i, p_l\} \succeq_{p_i} \{p_i\}$ holds for each agent $p_i \in T^{\star\star}$, where such an agent with the highest index is denoted by $p_f$.

**Lemma 2** (*Burani and Zwicker (2003), Lemma 3, page 38*)**.** *For each of the players in $T^{\star\star} = \{p_1, \ldots, p_f\} \subset T^\star$, coalition $T^\star$ is top-ranked among individually rational coalitions (or tied for top). Therefore, no deviating coalition can contain any of the players in $T^{\star\star}$.*

We will also need the following lemma.

**Lemma 3.** *For each player $p_k \in \{p_{l+1}, \ldots, p_n\}$, $\{p_k\} \succ_{p_k} \{p_j, p_k\}$ holds for any $p_j \in \{p_{f+1}, \ldots, p_l\} = T^\star \setminus T^{\star\star}$.*

[16] See Banerjee et al. (2001) for details of the hedonic game derived from the uniform reallocation rule.

[17] See example 3 (page 152) of Banerjee et al. (2001) for such an example.

[18] The reader is referred to Burani and Zwicker (2003) for more details of descending separable preferences and the construction of the top segment partition.

**Proof.** First, it is shown that the lemma holds for agent $p_{l+1}$. Consider agent $p_{f+1}$. Since $p_{f+1} \notin T^{\star\star}$, $\{p_{f+1}\} \succ_{p_{f+1}} \{p_{f+1}, p_l\}$. Then, condition CRI and transitivity of preferences imply, $\{p_{f+1}\} \succ_{p_{f+1}} \{p_{f+1}, p_{l+1}\}$. This fact, together with descending mutual preferences, yields that $\{p_{l+1}\} \succ_{p_{l+1}} \{p_{f+1}, p_{l+1}\}$. Now, by condition CRI, $\{p_{l+1}\} \succ_{p_{l+1}} \{p_j, p_{l+1}\}$ holds for any $p_j \in \{p_{f+1}, \dots, p_l\}$. It is also needed to show independently that $\{p_{l+1}\} \succ_{p_{l+1}} \{p_l, p_{l+1}\}$ holds, in case $T^{\star\star} = \{p_1, \dots, p_{l-1}\}$. Suppose not. Condition CRI then implies that $\{p_j, p_{l+1}\} \succeq_{p_{l+1}} \{p_{l+1}\}$ for all $p_j \in T^\star$. Now, iterated separable preferences imply that $(T^\star \cup \{p_{l+1}\}) \succeq_{p_{l+1}} \{p_{l+1}\}$ which is in contradiction with $p_{l+1} \notin T^\star$. So, $\{p_{l+1}\} \succ_{p_{l+1}} \{p_l, p_{l+1}\}$ also holds. Hence, $\{p_{l+1}\} \succ_{p_{l+1}} \{p_j, p_{l+1}\}$ for any $p_j \in \{p_{f+1}, \dots, p_l\}$.

Second, by condition DD, it holds for any $p_k < p_{l+1}$ that $\{p_k\} \succ_{p_k} \{p_j, p_k\}$ for every $p_j \in \{p_{f+1}, \dots, p_l\}$, completing the proof. □

Our main result with descending separable preferences is now stated and proved.

**Proposition 2.** *Let $G = (N, \succeq)$ be a hedonic game. If players have descending separable preferences, then there always exists a strongly Nash stable partition.*

**Proof.** Let $G = (N, \succeq)$ be a hedonic game where players have descending separable preferences. Let $\pi^\star$ be a top-segment partition. It is known by Burani and Zwicker (2003) that $\pi^\star$ is both core and Nash stable. It will be shown that $\pi^\star$ is strongly Nash stable. Suppose that $\pi^\star$ is not strongly Nash stable. Then, there exists a pair $(\pi, H)$ where $\pi \in (\Pi(N) \setminus \{\pi^\star\})$ and $\emptyset \neq H \subseteq N$ such that $\pi^\star \xrightarrow{H} \pi$ and for all $i \in H, \pi(i) \succ_i \pi^\star(i)$. Note that $|H| > 1$ since $\pi^\star$ is Nash stable.

Since $\pi^\star$ is both core and Nash stable, and it is supposed that $H$ strongly Nash blocks the partition $\pi^\star$, another remaining four possible cases will be checked.

*Case* 1. $H \subseteq \{p_{l+1}, \dots, p_n\}$ and $H$ strongly Nash blocks the top-segment partition $\pi^\star$ by joining $T^\star$.[19]

Since $H$ strongly Nash blocks the partition $\pi^\star$ by joining $T^\star$, $(T^\star \cup H) \succ_{p_j} \{p_j\}$ for all $p_j \in H$. For any $p_i \in T^\star$ and any $p_j \in H$, $p_i > p_j$. So, by condition REP, it holds for each $p_i \in T^\star$ that $(T^\star \cup H) \succ_{p_i} \{p_i\}$. Hence, $(T^\star \cup H)$ would be an individually rational coalition which contradicts with Lemma 1, since $|(T^\star \cup H)| > l$. So, there is no subset $H$ of $\{p_{l+1}, \dots, p_n\}$ which strongly Nash blocks the top-segment partition $\pi^\star$ by joining $T^\star$.

*Case* 2. $H \subsetneqq \{p_{l+1}, \dots, p_n\}$, $p_i \in [N \setminus (T^\star \cup H)]$, and $H$ strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$.[20]

Since $H$ strongly Nash blocks the partition $\pi^\star$ by joining $\{p_i\}$, $(H \cup \{p_i\}) \succ_{p_j} \{p_j\}$ for all $p_j \in H$. Note that since $\pi^\star$ is Nash stable, it is true for every $p_j \in H$ that $\{p_j\} \succeq_{p_j} \{p_j, p_k\}$ for all $p_k \in [(H \setminus \{p_j\}) \cup \{p_i\}]$. Then, iterated separable preferences imply that $\{p_j\} \succeq_{p_j} (H \cup \{p_i\})$ for every $p_j \in H$. This is in contradiction with the fact that $H$ strongly Nash blocks the partition $\pi^\star$ by joining $\{p_i\}$. Hence, there does not exist a proper subset $H$ of $\{p_{l+1}, \dots, p_n\}$ which strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$, where $p_i \in [N \setminus (T^\star \cup H)]$.

*Case* 3. $H \subseteq T^\star$, $p_i \in \{p_{l+1}, \dots, p_n\}$, and $H$ strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$.[21]

Since $H \subseteq T^\star$ strongly Nash blocks the partition $\pi^\star$ by joining $\{p_i\}$, $(H \cup \{p_i\}) \succ_{p_j} T^\star$ for all $p_j \in H$. This fact, together with Lemma 2, implies that $H \cap T^{\star\star} = \emptyset$. Let $p_h \in H$ be a player such that $p_h > p_j$ for all $p_j \in (H \setminus \{p_h\})$. Note that $p_h \neq p_l$, because $|H| > 1$. Since $p_h \notin T^{\star\star}$, agent $p_h$ has preferences such that $\{p_h\} \succ_{p_h} \{p_h, p_l\}$. Condition CRI yields that $\{p_h, p_l\} \succeq_{p_h} \{p_h, p_i\}$ because $p_l > p_i$, and transitivity of preferences implies, $\{p_h\} \succ_{p_h} \{p_h, p_i\}$. Then, descending mutuality implies, $\{p_j\} \succ_{p_j} \{p_j, p_i\}$ holds for each $p_j \in H$. This result combined with condition SP implies that $H \succ_{p_j} (H \cup \{p_i\})$ for every $p_j \in H$. Now, transitivity of preferences yields for each $p_j \in H$ that $H \succ_{p_j} T^\star$. However, this is in contradiction with $\pi^\star$ being core stable, i.e., $H$ would block the partition $\pi^\star$. Hence, there is no subset $H$ of $T^\star$ which strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$, where $p_i \in \{p_{l+1}, \dots, p_n\}$.

*Case* 4. $H = H_1 \cup H_2$, where $H_1 \subseteq T^\star$ and $H_2 \subsetneqq \{p_{l+1}, \dots, p_n\}$, $p_i \in N \setminus (T^\star \cup H_2)$, and $H$ strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$.[22]

So, $(H \cup \{p_i\}) \succ_{p_j} T^\star$ for all $p_j \in H_1$, and $(H \cup \{p_i\}) \succ_{p_k} \{p_k\}$ for all $p_k \in H_2$. Since $\pi^\star$ is Nash stable, it holds for each $p_k \in H_2$ that, $\{p_k\} \succeq_{p_k} \{p_k, p_h\}$ for any $p_h \in [(H_2 \setminus \{p_k\}) \cup \{p_i\}]$. Now, Lemma 2 implies that $H_1 \cap T^{\star\star} = \emptyset$, i.e., $H_1 \subseteq \{p_{f+1}, \dots, p_l\}$. This fact, together with Lemma 3, implies that, for each $p_k \in H_2$, $\{p_k\} \succ_{p_k} \{p_k, p_j\}$ for any $p_j \in H_1$. Hence, for each $p_k \in H_2$ it holds that $\{p_k\} \succeq_{p_k} \{p_k, p_x\}$ for all $p_x \in [(H \setminus \{p_k\}) \cup \{p_i\}]$. Then, iterated separable preferences imply that $\{p_k\} \succeq_{p_k} (H \cup \{p_i\})$ for all $p_k \in H_2$, which is the desired contradiction. Hence, there does not exist $H = H_1 \cup H_2$, where $H_1 \subseteq T^\star$ and $H_2 \subsetneqq \{p_{l+1}, \dots, p_n\}$, which strongly Nash blocks the top-segment partition $\pi^\star$ by joining $\{p_i\}$, where $p_i \in N \setminus (T^\star \cup H_2)$.

Since the four cases cover all possibilities, it is concluded that there does not exist a subset of players $\emptyset \neq H \subseteq N$ which strongly Nash blocks the top-segment partition $\pi^\star$. Hence $\pi^\star$ is strongly Nash stable. □

Based on Proposition 2, one can argue that Burani and Zwicker (2003) were studying the wrong solution concept; they really should have been applying their methods to strong Nash stability. We have constructed examples showing that preferences are descending separable and the weak top-choice properties are independent of each other.

Burani and Zwicker (2003) also studied hedonic games on additively separable and symmetric domain of preferences where players' preferences are *purely cardinal.*

A hedonic game $G = (N, \succeq)$ is ***additively separable*** if for any $i \in N$, there exists a function $v_i : N \to \mathbb{R}$ such that for any $H, T \in \Sigma_i, H \succeq_i T \Leftrightarrow \sum_{j \in H} v_i(j) \geq \sum_{j \in T} v_i(j)$, where $v_i(j) = 0$ for $i = j$. An additively separable hedonic game satisfies ***symmetry*** if for any $i, j \in N$, $v_i(j) = v_j(i)$.

**Definition 13.** A profile of additively separable and symmetric preferences is ***purely cardinal*** if there exists an assignment of individual weights $w(i)$ to the players for which the following vector $v$ represents the profile: for all $i, j \in N$,

$$v(i,j) = \begin{cases} w(i) + w(j) & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

For any player $i$, her individual weight $w(i)$ represents the fixed individual contribution that she brings to any member of the coalition that she belongs. Purely cardinal preferences are descending separable, where the reference ranking (1) of agents is the permutation that ranks them in non-increasing order of their weights. Hence, a hedonic game with purely cardinal preferences

19 So, $\pi = \{T^\star \cup H\} = \{\{N\}\}$ if $H = \{p_{l+1}, \dots, p_n\}$, and $\pi = \{T^\star \cup H, \{\{p_j\} \mid p_j \in N \setminus (T^\star \cup H)\}\}$ if $H \subsetneqq \{p_{l+1}, \dots, p_n\}$.

20 So, $\pi = \{T^\star, H \cup \{p_i\}, \{\{p_j\} \mid p_j \in [N \setminus (T^\star \cup H \cup \{p_i\})]\}\}$ if $H \neq N \setminus (T^\star \cup \{p_i\})$, and $\pi = \{T^\star, H \cup \{p_i\}\}$ if $H = N \setminus (T^\star \cup \{p_i\})$.

21 Now, $\pi = \{T^\star \setminus H, H \cup \{p_i\}, \{\{p_j\} \mid p_j \in N \setminus (T^\star \cup \{p_i\})\}\}$ if $H \subsetneqq T^\star$, and $\pi = \{H \cup \{p_i\}, \{\{p_j\} \mid p_j \in N \setminus (T^\star \cup \{p_i\})\}\}$ if $H = T^\star$.

22 So, $\pi = \{T^\star \setminus H_1, H \cup \{p_i\}, \{\{p_j\} \mid p_j \in [N \setminus (T^\star \cup H_2 \cup \{p_i\})]\}\}$ if $H_1 \subsetneqq T^\star$, and $\pi = \{H \cup \{p_i\}, \{\{p_j\} \mid p_j \in [N \setminus (T^\star \cup H_2 \cup \{p_i\})]\}\}$ if $H_1 = T^\star$. Note that $H_1 \neq \emptyset$ by case 2 and $H_2 \neq \emptyset$ by case 3.

always has a strongly Nash stable partition. However, we have provided an example showing that purely cardinal preferences is not a necessary condition for a game to have a strongly Nash stable partition.

We have constructed examples showing that preferences being purely cardinal and the weak top-choice property are independent of each other. Note that players' preferences need not be purely cardinal for a separable[23] and *anonymous* game.

A hedonic game $G = (N, \succeq)$ satisfies ***anonymity*** if for any $i \in N$, for any $H, T \in \Sigma_i$ with $|H| = |T|$, $H \sim_i T$.

The proof of the following lemma is left to the reader.

**Lemma 4.** *If a hedonic game is anonymous, additively separable and symmetric, then players' preferences are purely cardinal (hence has a strongly Nash stable partition).*

Also note that if a hedonic game is anonymous, additively separable and symmetric, then it satisfies the top-choice property, where the proof is left to the reader. However, we have given an example showing that a hedonic game which satisfies the top-choice property may not be additively separable and symmetric.

The strong Nash stability for hedonic games is not the unique stability notion which has not been studied earlier. In fact, two other stability notions for hedonic games can be defined.[24]

**Definition 14.** Let $G = (N, \succeq)$ be a hedonic game and $\pi \in \Pi(N)$ a partition. We say that a subset of players $T \subseteq N$ ***coalitionally Nash blocks*** $\pi$ if there exists a coalition $H \in (\pi \cup \{\emptyset\})$ such that for each player $i \in T$, $(H \cup T) \succ_i \pi(i)$. A partition is ***coalitionally Nash stable*** if there does not exist a subset of players which coalitionally Nash blocks it.

**Definition 15.** Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***core-exchange stable*** if it is core stable and there does not exist a subset of players $T \subseteq N$ such that individual players in $T$ or subsets of $T$ (strongly Nash) block $\pi$ by exchanging their current coalitions under $\pi$.

It is clear that these two concepts are independent of each other, and each of these concepts is weaker than strong Nash stability. Moreover, a partition is both coalitionally Nash stable and core-exchange stable if and only if it is strongly Nash stable.

**Open question.** We have constructed an example showing that neither the weak top-choice property nor the preferences being descending separable is necessary for a hedonic game to have a strongly Nash stable partition. Hence, it is an open question to find a condition which is both necessary and sufficient for the existence of a strongly Nash stable partition.

## 5. Strong Nash stability under different membership rights

Different societies may have different membership rights, and a designer employs a certain rights structure to achieve some aims. This section studies how the concept of strong Nash stability changes under different membership rights. We will see that strong Nash stability under different membership rights fits with the earlier concepts.

FX-FE strong Nash stability is what has been called strong Nash stability in previous sections. Now, its strict version is defined.

**Definition 16.** Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***free exit-free entry strictly strongly Nash stable*** (FX-FE strictly strongly Nash stable) if there does not exist a pair $(\acute{\pi}, H)$ (where $\acute{\pi} \in (\Pi(N) \setminus \{\pi\})$ and $\emptyset \neq H \subseteq N$) such that

(i) $\pi \xrightarrow{H} \acute{\pi}$ ($\acute{\pi}$ is reachable from $\pi$ by movements of $H$),
(ii) for all $i \in H$, $\acute{\pi}(i) \succeq_i \pi(i)$, and for some $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$.

**Definition 17.** Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***free exit-approved entry strongly Nash stable*** (FX-AE strongly Nash stable) if there does not exist a pair $(\acute{\pi}, H)$ such that

(i) $\pi \xrightarrow{H} \acute{\pi}$,
(ii) for all $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$, and
(iii) for all $i \in H$, for all $k \in (\acute{\pi}(i) \setminus \{i\})$, $\acute{\pi}(k) \succeq_k \pi(k)$.

**Definition 18.** Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***approved exit-approved entry strongly Nash stable*** (AX-AE strongly Nash stable) if there does not exist a pair $(\acute{\pi}, H)$ such that

(i) $\pi \xrightarrow{H} \acute{\pi}$,
(ii) for all $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$, and
(iii) for all $k \in (N \setminus H)$, $\acute{\pi}(k) \succeq_k \pi(k)$.

**Definition 19.** Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is ***approved exit-free entry strongly Nash stable*** (AX-FE strongly Nash stable) if there does not exist a pair $(\acute{\pi}, H)$ such that

(i) $\pi \xrightarrow{H} \acute{\pi}$,
(ii) for all $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$, and
(iii) for all $i \in H$, for all $j \in (\pi(i) \setminus \{i\})$, $\acute{\pi}(j) \succeq_j \pi(j)$.

Strict versions of concepts given in Definitions 17–19 are defined by replacing item (ii) with [for all $i \in H$, $\acute{\pi}(i) \succeq_i \pi(i)$, and for some $i \in H$, $\acute{\pi}(i) \succ_i \pi(i)$].

**Lemma 5.** *Let $G = (N, \succeq)$ be a hedonic game. If a partition $\pi \in \Pi(N)$ is FX-AE strongly Nash stable, then it is core stable.*

**Proof.** Let $G = (N, \succeq)$ be a hedonic game and $\pi \in \Pi(N)$ be an FX-AE strongly Nash stable partition. Suppose that $\pi$ is not core stable. Then, there is a coalition $T \subseteq N$ such that for all $i \in T$, $T \succ_i \pi(i)$. Let $\acute{\pi} = \{T, \{\{H \setminus T\} \mid H \in \pi \text{ and } H \setminus T \neq \emptyset\}\}$ denote the partition that is obtained from coalition $T$'s blocking of $\pi$. Now, it is shown that the pair $(\acute{\pi}, T)$ satisfies the three conditions of FX-AE strong Nash stability. First, it is clear that $\acute{\pi}$ is reachable from $\pi$ by $T$, i.e., $\pi \xrightarrow{T} \acute{\pi}$. Second, since it is supposed that $T$ blocks $\pi$, i.e., for any $i \in T$, $T = \acute{\pi}(i) \succ_i \pi(i)$. Third, for any $i \in T$, $\acute{\pi}(i) \setminus \{i\} = T \setminus \{i\}$. So, for all $i \in T$, for all $k \in (\acute{\pi}(i) \setminus \{i\})$, we have $\acute{\pi}(k) \succ_k \pi(k)$. Hence, the pair $(\acute{\pi}, T)$ satisfies the three conditions of FX-AE strong Nash stability, in contradiction with $\pi$ being FX-AE strongly Nash stable, i.e., coalition $T$ would block the partition $\pi$ under FX-AE membership rights. Hence, $\pi$ is core stable. □

Note that this lemma implies that if a partition is FX-AE strictly strongly Nash stable, then it is strictly core stable. Now, it is shown that the converse of this lemma is true under the assumption that players have strict preferences.

**Lemma 6.** *Let $G = (N, \succ)$ be a hedonic game where players have strict preferences. If a partition $\pi \in \Pi(N)$ is core stable, then it is FX-AE strongly Nash stable.*

**Proof.** Let $G = (N, \succ)$ be a hedonic game where players have strict preferences. Let $\pi \in \Pi(N)$ be a core stable partition. Suppose that $\pi$ is not FX-AE strongly Nash stable. Then, there exists a pair $(\acute{\pi}, H)$ such that players in $H$ strongly Nash block the partition $\pi$ by inducing $\acute{\pi}$ under FX-AE membership rights.

Since $\pi$ is core stable, $H$ cannot block $\pi$. So, $H$ strongly Nash blocks the partition $\pi$ by either players in $H$ (or subsets of $H$) exchange their current coalitions that they belong under $\pi$ or

23 A hedonic game is *separable* if players' preferences satisfy Condition 3 (SP).

24 These definitions and the following open question are motivated by questions posed by two anonymous referees.

all players in $H$ leave their current coalitions and join another coalition of the partition $\pi$. In either case, there exists a coalition $T \in \acute{\pi}$ such that $T \cap H \neq \emptyset$. Since the membership rights is FX-AE and players have strict preferences, we have $\acute{\pi}(j) \succ_j \pi(j)$ for all players $j \in (T \setminus H)$. This result, together with the fact that $H$ strongly Nash blocks the partition $\pi$, implies that $\acute{\pi}(i) \succ_i \pi(i)$ for all $i \in T$. However, this is in contradiction with $\pi$ being core stable, i.e., coalition $T$ would block the partition $\pi$. Hence, $\pi$ is FX-AE strongly Nash stable. □

This lemma is not true without the assumption of strict preferences.

**Example 2.** Let $G = (N, \succeq)$, where $N = \{1, 2\}$ and players' preferences are as follows: $\{1, 2\} \sim_1 \{1\}$, and $\{1, 2\} \succ_2 \{2\}$.

The partition $\pi = \{\{1\}, \{2\}\}$ is core stable. However $\pi$ is not FX-AE strongly Nash stable, since player 2 strongly Nash blocks the partition $\pi$ by joining $\{1\}$ under FX-AE membership rights, i.e., $\pi \xrightarrow{\{2\}} \acute{\pi} = \{\{1, 2\}\}$, and $\acute{\pi}(2) \succ_2 \pi(2)$ and $\acute{\pi}(1) \sim_1 \pi(1)$.

Lemma 6 implies, if a partition is strictly core stable, then it is FX-AE strictly strongly Nash stable. The following proposition is an implication of Lemmas 5 and 6.

**Proposition 3.** *Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is FX-AE strictly strongly Nash stable if and only if it is strictly core stable.*

Note that if players have strict preferences then a partition is FX-AE strictly strongly Nash stable if and only if it is FX-AE strongly Nash stable, and a partition is strictly core stable if and only if it is core stable. Proposition 3 shows that core stability entails an FX-AE rights structure.

Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is *contractual core stable* (defined in Sung and Dimitrov (2007)) if there does not exist a coalition $T \subseteq N$ such that

(i) for all $i \in T$, $T \succ_i \pi(i)$ and
(ii) for all $j \in (N \setminus T)$, $\pi(j) \setminus T \succeq_j \pi(j)$.

Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is *contractual strictly core stable* (defined in Sung and Dimitrov (2007)) if there does not exist a coalition $T \subseteq N$ such that

(i) for all $i \in T$, $T \succeq_i \pi(i)$,
(ii) for some $i \in T$, $T \succ_i \pi(i)$, and
(iii) for all $j \in (N \setminus T)$, $\pi(j) \setminus T \succeq_j \pi(j)$.

**Lemma 7.** *Let $G = (N, \succeq)$ be a hedonic game. If a partition $\pi \in \Pi(N)$ is AX-AE strongly Nash stable, then it is contractual core stable.*

**Proof.** Let $G = (N, \succeq)$ be a hedonic game and $\pi \in \Pi(N)$ be an AX-AE strongly Nash stable partition. Suppose that $\pi$ is not contractual core stable. Then, there is a coalition $T \subseteq N$ such that for all $i \in T$, $T \succ_i \pi(i)$ and for all $j \in (N \setminus T)$, $\pi(j) \setminus T \succeq_j \pi(j)$.

Let $\acute{\pi} = \{T, \{\{\pi(j) \setminus T\} \mid j \in (N \setminus T) \text{ and } \pi(j) \setminus T \neq \emptyset\}\}$ denote the partition that is obtained from coalition $T$'s blocking of $\pi$. Now, it is shown that the pair $(\acute{\pi}, T)$ satisfies the three conditions of AX-AE strong Nash stability. The first two conditions are trivially satisfied, i.e., $\pi \xrightarrow{T} \acute{\pi}$, and for all $i \in T$, $T = \acute{\pi}(i) \succ_i \pi(i)$.

Let $H = \{j \in N \mid j \notin T \text{ and } j \in \pi(i) \text{ for some } i \in T\}$ and $\bar{H} = \{\bar{j} \in N \mid \bar{j} \notin \pi(i) \text{ for any } i \in T\}$. Note that $H$, $\bar{H}$ and $T$ are pairwise disjoint, and $N = H \cup \bar{H} \cup T$. Since it is supposed that $T$ blocks $\pi$ and this blocking does not hurt any player, for any $j \in H$ we have $\acute{\pi}(j) \succeq_j \pi(j)$. Note that $\bar{H} = \{\bar{j} \in N \mid \acute{\pi}(\bar{j}) = \pi(\bar{j})\}$, so for any $\bar{j} \in \bar{H}$ we have $\acute{\pi}(\bar{j}) \sim_{\bar{j}} \pi(\bar{j})$. Hence, for any $k \in (N \setminus T)$ we have $\acute{\pi}(k) \succeq_k \pi(k)$, i.e., the third condition of AX-AE strong Nash stability is also satisfied by the pair $(\acute{\pi}, T)$. Hence, the pair $(\acute{\pi}, T)$ satisfies the three conditions of AX-AE strong Nash stability, this contradicts with $\pi$ being AX-AE strongly Nash stable. That is, $T$ would strongly Nash block the partition $\pi$ under AX-AE membership rights. Hence, $\pi$ is contractual core stable. □

By this lemma, it can be said that if a partition is AX-AE strictly strongly Nash stable, then it is contractual strictly core stable. Now, it is shown that the converse of Lemma 7 is true under the assumption that players have strict preferences.

**Lemma 8.** *Let $G = (N, \succ)$ be a hedonic game where players have strict preferences. If a partition $\pi \in \Pi(N)$ is contractual core stable, then it is AX-AE strongly Nash stable.*

**Proof.** Let $G = (N, \succ)$ be a hedonic game where players have strict preferences. Let $\pi \in \Pi(N)$ be a contractual core stable partition. Suppose that $\pi$ is not AX-AE strongly Nash stable. Then, there exists a pair $(\acute{\pi}, H)$ such that $H$ strongly Nash blocks the partition $\pi$ by inducing $\acute{\pi}$ under AX-AE membership rights.

Let $T = \{i \in N \mid \acute{\pi}(i) \neq \pi(i)\}$ denote the set of agents whose coalitions changed from $\pi$ to $\acute{\pi}$. Note that $T \neq \emptyset$. Now, for any $i \in T$ we have $\acute{\pi}(i) \succ_i \pi(i)$, since players have strict preferences and it is supposed that $\pi$ is not AX-AE strongly Nash stable.

However, each player in $T$ leaves her current coalition under $\pi$, and forms the coalitions $T_1, \ldots, T_K$ which are pairwise disjoint and their union is equal to $T$ such that for any $k \in \{1, \ldots, K\}$, $T_k \in \acute{\pi}$. Now, for any $k \in \{1, \ldots, K\}$, we have, for all $i \in T_k$, $T_k \succ_i \pi(i)$ and for all $j \in (N \setminus T_k)$, $\pi(j) \setminus T_k \succeq_j \pi(j)$. This is in contradiction with $\pi$ being contractual core stable, i.e., for any $k \in \{1, \ldots, K\}$, a coalition $T_k$ would block the partition $\pi$ without hurting other players. Hence, $\pi$ is AX-AE strongly Nash stable. □

Lemma 8 may fail to be true if the assumption that players have strict preferences is relaxed.[25] By Lemma 8, it can be said that, if a partition is contractual strictly core stable, then it is AX-AE strictly strongly Nash stable. The next proposition follows from Lemmas 7 and 8.

**Proposition 4.** *Let $G = (N, \succeq)$ be a hedonic game. A partition $\pi \in \Pi(N)$ is AX-AE strictly strongly Nash stable if and only if it is contractual strictly core stable.*

Sung and Dimitrov (2007) showed that for any hedonic game a contractual strictly core stable partition always exists. This result together with Proposition 4 implies that an AX-AE strictly strongly Nash stable partition always exists for any hedonic game.

Note that, if a partition is AX-FE strongly Nash stable, then it is AX-AE strongly Nash stable. This fact and Lemma 7 imply, if a partition $\pi \in \Pi(N)$ is AX-FE strongly Nash stable then it is contractual core stable. However, the converse is not true.

**Example 3.** Let $G = (N, \succeq)$, where $N = \{1, 2\}$ and players' preferences are as follows: $\{1\} \succ_1 \{1, 2\}$, and $\{1, 2\} \succ_2 \{2\}$.

The partition $\pi = \{\{1\}, \{2\}\}$ is contractual core stable. However $\pi$ is not AX-FE strongly Nash stable, since player 2 strongly Nash blocks the partition $\pi$ by joining $\{1\}$ under AX-FE membership rights, i.e., $\pi \xrightarrow{\{2\}} \acute{\pi} = \{\{1, 2\}\}$, and $\acute{\pi}(2) \succ_2 \pi(2)$ and $\pi(2) \setminus \{2\} = \emptyset$, i.e., there is no player that player 2 needs to get a permission to leave from the coalition $\pi(2)$.[26]

## Acknowledgements

I would like to thank my professors Tarık Kara, Semih Koray and M. Remzi Sanver for elaborate discussions and helpful comments.

25 Consider Example 2. The partition $\pi = \{\{1\}, \{2\}\}$ is contractual core stable. However, it is not AX-AE strongly Nash stable, since $\pi \xrightarrow{\{2\}} \acute{\pi} = \{\{1, 2\}\}$, and $\acute{\pi}(2) \succ_2 \pi(2)$ and $\acute{\pi}(1) \sim_1 \pi(1)$.

26 The partition $\acute{\pi} = \{\{1, 2\}\}$ is AX-FE strongly Nash stable since player 2 does not permit player 1 to leave from $\{1, 2\}$. However, $\acute{\pi}$ is not individually rational, since $\{1\} \succ_1 \acute{\pi}(1)$. So, there is no individually rational and AX-FE strongly Nash stable partition for this game.

I thank Göksel Aşan, Gabrielle Demange, Kevin Hasker, Farhad Husseinov, Özgür Kıbrıs, Wolfgang Leininger, William Thomson, and participants of the Economic Theory seminars at Bilkent University for their valuable comments and conversations. I am also indebted to an anonymous Associate Editor and two anonymous referees for their insightful comments and suggestions which significantly improved the results and the presentation of the paper. Of course, I am the sole person responsible for all possible errors.

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