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Capital dependent population growth induces cycles

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ABSTRACT

Cobb–Douglas type production functions and time-delay are not sufficient for the economy to behave cyclic. However, capital dependent population dynamics can enforce Hopf bifurcation.

1. Introduction

Kaleckian investment lag is historically important since Kalecki laid a mathematical foundation of the economic cycles as early as mid 30s. The main mathematical apparatus (namely the Hayes' Theorem) which analyzes the characteristic roots of quasi-polynomials emerged at fifties. Hayes gives a complete stability characterization for the first order linear delay differential equations. However, as Zak [20] points out, the first thorough analysis of a general class delay differential equations is by Bellman and Cooke [3] with later fundamental work by Hale [7].

Kalecki [8] introduces production lags, a time delay between the investment decisions and delivery of the capital goods, to show the generation of endogenous cycles. Kalecki employ a linear delay differential equation of the deviation of investment which is denoted by J. The investment equation is $\dot{J}(t) = AJ(t) - BJ(t-\theta)$. Model of Kalecki [8] exhibits endogenous cycles by employing simple time lags in a linear DDE.[1]

Periodic solutions to dynamic systems are also analyzed extensively in control theory. One way to detect limit cycles is Hopf bifurcation. Hopf bifurcation discards tedious calculations and provides a powerful and easy tool to detect limit cycles. Hopf cycles appear when a fixed point loses or gains stability due to a change in a parameter and meanwhile a cycle either emerges from or collapses into the fixed point [1]. Under the circumstances the system can either have a stable fixed point sorrounded by an unstable cycle (called a *subcritical* Hopf bifurcation); or a stable cycle loses its stability and a stable cycle appears (called a *supercritical* Hopf bifurcation) as the parameter(s) approaches to a critical value [1]. Both cases can be economically significantly meaningful. Supercritical case which implies a stable cycle can be considered as a stylized business cycle or growth cycles and the subcritical case can correspond to the corridor stability [9]. The Hopf bifurcation dominates the literature when the problem reduces to detect cycles in dynamic models. The analysis further boils down to finding a pair of pure imaginary roots, since the non-zero speed condition is not actually necessary for having a Hopf bifurcation[2] (see [6, p. 418]; [16, p. 578]). Zak [20], [17,18] and [19] applied the improvements of Hopf theorem to the Solow–Kalecki type of growth models.

According to the model presented by Zak [20], the capital becomes productive after a time period, say τ. That is, the productive capital at time t is $k(t-\tau)$. Moreover, capital also depreciates through production. Therefore, the evolution of capital is governed by the following DDE:

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[1] The lag structure find itself a place in two dimensional non-optimizing business cycle models à *la* Kaldor–Kalecki. The dynamics of Kaldor–Kalecki type models is extensively studied on a series of papers by [10–13] and Krawiec et al. [14]. Kaldor–Kalecki models have two mechanisms which would lead to cyclic behaviour, one being the nonlinearity of the investment function and the other being the time delay in investment [12].

[2] To be more specific, let us quote [6, p.418]: "[The non-zero speed condition] is expressed by saying that the pair of complex conjugate eigenvalues crosses the imaginary axis with non-zero speed. This is also a generic requirement, though it is not absolutely necessary: the existence part of the Theorem remains valid even in the degenerate case when this derivative is zero [etc.]".

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$$\dot{k}(t) = f(k(t-\tau)) - \delta k(t-\tau). \quad (1)$$

However, Brandt-Pollman et al. [4] classify the lag structure given in Eq. (1) as a *delivery lag*[3] rather than a *time-to-build lag*.[4] Yet, we will employ *time-to-build lag* structure, which is of the form

$$\dot{k}(t) = f(k(t-\tau)) - \delta k(t). \quad (2)$$

We show in the paper that the capital evolution with the lag structure in Eq. (2) will not yield Hopf cycles if the production function is of Cobb–Douglas type.

The population growth in Zak [20] is assumed to be zero. However, the results will mostly remain if constant population growth is used. Cigno [5] introduced a capital dependent (variable) population growth. The said population growth equation tries to link the growth of population with per capita consumption and degree of industrialization, where the relation is positive for the former, but negative for the latter. That is, the dynamics of the population reflect the positive effect of higher per capita consumption and the negative effect of higher degree of industrialization. Denoting the per capita consumption with $(1-s)Q/L$, the dynamics of the population in the paper is governed by $n(t) = \{(1-s)(Q/L)\}^{\upsilon_1}(K/L)^{\upsilon_2}$, where υ_1, $\upsilon_2 > 0$. Cigno [5] found out the stability characterization of endogenous population growth in an exhaustible resource framework. Cigno [5] concludes that, for certain parameter settings the steady state is stable.

We show that constant population growth is not sufficient to obtain cyclical behaviour in certain type of capital accumulations, given that the production is Cobb–Douglas. However, a capital-dependent population growth rule leads to Hopf cycles.

This paper is organized as follows. In Section 2, we show that Cobb–Douglas production function and constant population growth model does not contain Hopf cycles. We have introduced the theorem from Louisell [15], which gives an easier method to detect pure imaginary roots. In Section 3, we extend the model so that the population growth is now capital dependent. Employing similar techniques, we have found out that the latter model gives Hopf cycles. Section 3 is the conclusion.

2. Constant population growth

Finding pure imaginary roots has been widely discussed in the literature. The following theorem from Louisell [15] constitutes a shortcut to detect the pure imaginary roots of certain type of difference-differential systems.

Let $A_0, A_1 \in \mathbb{R}^{n\times n}, \tau > 0$. Consider the following difference-differential equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \quad (3)$$

which has a characteristic function of

$$T(\lambda) = \lambda I - A_0 - A_1 e^{-\tau\lambda}. \quad (4)$$

[3] *Delivery lag* is such that investment for new capital goods is made at time t but the new capital goods need some time τ to be delivered and, thus, to be productive [4].

[4] *Time-to-build lag* is such that capital goods need some time τ over which they require investments in order to be produced [4].

Theorem 1 (*Louisell* [15]). *Let $A_0, A_1 \in \mathbb{R}^{n\times n}, \tau > 0$ and let*

$$J = \begin{pmatrix} A_0 \otimes I & A_1 \otimes I \\ -I \otimes A_1 & -I \otimes A_0 \end{pmatrix}, \quad (5)$$

where $\otimes$ denotes the Kronecker product.[5] *Then, all imaginary axis eigenvalues of the delay Eq. (3) are the eigenvalues of J.*

Assume that we are faced with an economy endowed with Cobb–Douglas production function and capital lag[6] which is given as follows:

$$\dot{k}(t) = sk^{\alpha}(t-\tau) - (n(t)+\delta)k(t), \quad (6)$$

where $\alpha \in (0,1)$ is the constant capital's share in production, $\tau > 0$ is the constant capital lag, $\delta > 0$ is the constant depreciation of capital and $s > 0$ is the constant rate of savings. Denote $n(t) = \frac{\dot{L}(t)}{L(t)}$. Under the standard growth model with time lag, where the rate of population growth is assumed to be constant, i.e. $n(t) = n$ for all $t > 0$, we will show that this Solow–Kalecki growth model does not induce any Hopf cycles.

The steady state level of capital is

$$k_{ss} = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}} \quad (7)$$

and the linearization of the dynamic system around its steady state will yield

$$\dot{z}(t) = \left(\alpha s k_{ss}^{\alpha-1}\right) z(t-\tau) - (n+\delta)z(t), \quad (8)$$

with the change of variable $z(t) = k(t) - k_{ss}$. The matrix which should be used to employ the result of the theorem from Louisell (2001) is as follows:[7]

$$J = \begin{pmatrix} A_0 & A_1 \\ -A_1 & -A_0 \end{pmatrix},$$

where $A_0 = -(n+\delta)$ and $A_1 = \alpha s k_{ss}^{\alpha-1} = \alpha(n+\delta)$. In this case, we have $\lambda_{1,2} = \pm(n+\delta)\sqrt{1-\alpha^2} \in \mathbb{R}$ as eigenvalues. Since, this matrix does not possess any pure imaginary eigenvalues, the linearized system which is characterized by Eq. (8) has no pure imaginary eigenvalues therefore, Kaleckian growth models with Cobb–Douglas type of production functions and capital lag do not admit any Hopf bifurcation, thus persistent cycles.[8]

[5] Let $W \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{p\times q}$. Then $W \otimes Y \in \mathbb{R}^{pm\times qn}$ is as follows

$$W \otimes Y = \begin{pmatrix} w_{11}Y & \dots & w_{1n}Y \\ \cdot & \cdot & \cdot \\ w_{m1}Y & \dots & w_{mn}Y \end{pmatrix}.$$

[6] Capital lagged Cobb–Douglas type production function is assumed to be

$$Y(t) = K^{\alpha}(t-\tau)L^{1-\alpha}(t).$$

[7] Note that $A \otimes I = A$ if $I \in \mathbb{R}^{1x1}$ for any $A \in \mathbb{R}^{1x1}$:

[8] This does not mean that the solutions exhibit no oscillations at all. Note that the characteristic equation which is associated with the capital accumulation equation in (6) is as follows

$$h(\lambda) := \lambda - \left(s\alpha k_{ss}^{\alpha-1}\right)e^{-\tau\lambda} - (n+\delta).$$

This is a quasi-polynomial of order one which has infinite number of complex roots. Thus, assuming stabilizing initial conditions and parameter combinations, the resulting system will exhibit dampened oscillations.

3. Capital dependent population growth

Note that any variation in the population growth rate within some certain limits does not change the above result. Suppose that the population growth is not constant but exogenously time dependent. Moreover, suppose that the $n(t)$ is convergent for some n_{ss}, that is $n(t) \to n_{ss}$ as time goes to infinity. Then neither the steady state values, nor the linearized system dynamics which is given by (8) is effected. Thus, time-varying population growth is not sufficient for cyclic behaviour,[9] since the only mechanism that would give this kind of behaviour is a Hopf cycle.

On the other hand, the behaviour can drastically change if we use wealth-induced population dynamics, even if we stick to the Cobb–Douglas production function. Cigno [5] proposes the following population growth

$$n(t) = (1-s)^{v_1} k(t)^{\alpha v_1 - v_2},$$

where v_1 and v_2 are positive constants. For the ease of calculations, assume zero depreciation, i.e. $\delta = 0$. Substituting this into the capital accumulation equation, we obtain

$$\dot{k}(t) = sk^{\alpha}(t-\tau) - (1-s)^{v_1} k(t)^{1+\alpha v_1 - v_2}. \tag{9}$$

Steady state equation will adjust accordingly:

$$k_{ss} = \left(\frac{s}{(1-s)^{v_1}}\right)^{\frac{1}{1-\alpha(1-v_1)-v_2}}, \tag{10}$$

whence the linearized system around the steady state will be governed by

$$\dot{z}(t) = \left(\alpha s k_{ss}^{\alpha-1}\right) z(t-\tau) - (1-s)^{v_1}(1+\alpha v_1 - v_2) k_{ss}^{\alpha v_1 - v_2} z(t), \tag{11}$$

with the change of variable $z(t) = k(t) - k_{ss}$.

Proposition 2. *The growth model with endogenous population but without positive delay admits monotonic solutions.*

Proof. The above equations of capital accumulation (9), steady state capital (10) and linearized dynamics (11) are preserved with $\tau = 0$. The eigenvalue associated with this system is

$$\lambda = s k_{ss}^{\alpha-1}(\alpha - (1+\alpha v_1 - v_2)) \lesseqgtr 0,$$

when $(\alpha - (1+\alpha v_1 - v_2)) \lesseqgtr 0$. Thus, the solutions will be monotonically converging to the steady state (diverging to infinity) if $\alpha < 1 + \alpha v_1 - v_2$ ($\alpha > 1 + \alpha v_1 - v_2$). □

This propositions implies that the endogenous population growth à *la* Cigno [5] alone is not sufficient to create oscillatory behaviour, not even temporary ones.

To characterize limit cycle behaviour of the model with time delay, we have to calculate the corresponding matrix J in accordance with Louisell [15] which is cast as follows

$$J = \begin{pmatrix} A_0 & A_1 \\ -A_1 & -A_0 \end{pmatrix},$$

where $A_0 = -(1-s)^{v_1}(1+\alpha v_1 - v_2)k_{ss}^{\alpha v_1 - v_2}$ and $A_1 = \left(\alpha s k_{ss}^{\alpha-1}\right)$. The two eigenvalues of J are

$$\lambda_{1,2} = \pm\sqrt{A_0^2 - A_1^2}. \tag{12}$$

Proposition 3. *If $-\alpha < 1 + \alpha v_1 - v_2 < \alpha$, then the system undergoes a Hopf bifurcation.*

Proof. The eigenvalues are pure imaginary given that $A_0^2 - A_1^2 < 0$. This is the case if and only if $| 1 + \alpha v_1 - v_2 | < | \alpha | = \alpha$. Next, we have to check the transversality condition. Note that the characteristic equation associated with the law of capital accumulation (11) is

$$\lambda = A_1 e^{-\tau\lambda} + A_0.$$

Differentiating both sides with respect to τ, we have

$$\frac{d\lambda}{d\tau} = A_1 e^{-\tau\lambda}\left(\lambda + \tau\frac{d\lambda}{d\tau}\right) = (\lambda - A_0)\left(\lambda + \tau\frac{d\lambda}{d\tau}\right).$$

Thus,

$$\frac{d\lambda}{d\tau} = \frac{-\lambda(\lambda - A_0)}{1 + \tau(\lambda - A_0)}.$$

Finally,

$$\mathrm{Re}\left.\frac{d\lambda}{d\tau}\right|_{\lambda = i\omega, \tau = \tau_{bi}} = \frac{\omega^2}{(1-\tau A_0)^2 + (\tau\omega)^2} > 0,$$

where ω is the eigenvalue in (12) and τ_{bi} is the bifurcating delay which we do not need to find explicitly thanks to [15]. □

We know from *D*-subdivision method that the Hopf boundary is obtained in either the first or second quadrant of the coefficient space.[10] The sign of the coefficient of $z(t)$, which is $-(1-s)^{v_1}(1+\alpha v_1 - v_2)$, determines on which quadrant the coefficients lie. If $(1 + \alpha v_1 - v_2) > 0$, the coefficients are on the second quadrant and otherwise they are on the first. We should also note that the saddle-path stability is sacrificed for a limit cycle. That is, endogenous population growth eliminates the unstable manifold, however we obtained a limit cycle.

The Hopf cycles exist when the parameters are in a relationship within some limits. To see this, we utilize the following restatement of proposition 3:

Proposition 4. *If $v_1 < 1$, then the system undergoes a Hopf bifurcation if $\frac{1-v_2}{1-v_1} < \alpha$ and $\frac{1-v_2}{1+v_1} > -\alpha$.*

Proposition 5. *If $v_1 > 1$, then the system undergoes a Hopf bifurcation if $\frac{1-v_2}{1-v_1} > \alpha$ and $\frac{1-v_2}{1+v_1} > -\alpha$.*

[9] Time-varying population growth case is exploited for the insights it presents. Other than that, the author is fully informed that this kind of population growth functions are not employed in the literature.

[10] The coefficients can lie on the first or second quadrant of the parameter space (a,b), since $b > 0$ and these quadrants are those on where the Hopf boundary (the boundary where the system loses it stability) lies (see [2]). The parameters (a,b) are the coefficients of the characteristic equation $h(z) = z + a + be^{-z\tau} = 0$.

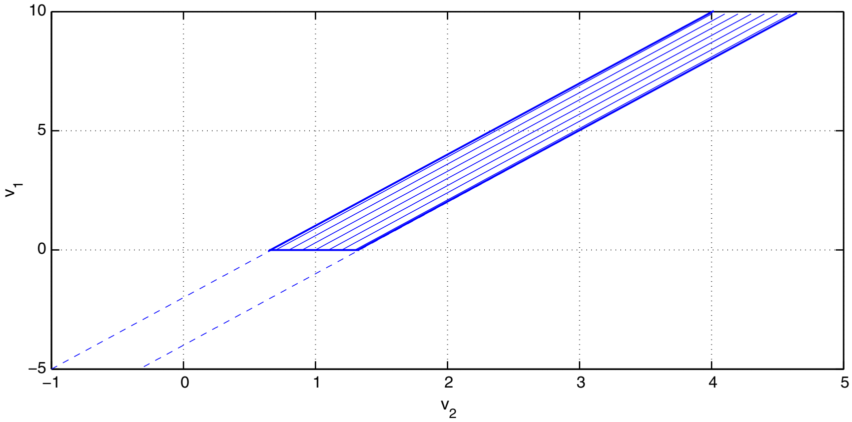

Fig. 1. v_1 and v_2 combinations which allows for Hopf bifurcation when $\alpha = \frac{1}{3}$ (the horizontal axis is v_2 and the vertical axis is v_1).

Table 1
Behaviour of the solutions in different setups where population growth is constant/endogenous and time delay structure exists/does not exist.

Behaviour of the solutions		
	$\tau = 0$	$\tau > 0$
Constant pop. growth	Monotonic	Dampened oscillations
Endogenous pop. growth	Monotonic	Dampened oscillations ($\tau \neq \tau_{bi}$) Persistent oscillations ($\tau = \tau_{bi}$)

Both propositions keep the parameters v_1 and v_2 close enough to ensure nonexplosive dynamics[11] where cyclic behaviour is not possible. In the both propositions, the relative ratio of distance to one should not exceed α given a lower bound to v_2 for a given v_1. Whereas, the other inequality is an upper bound to v_2. To be more illustrative, we can substitute for a common value for the constant of the share of capital in production, α, is $\alpha = \frac{1}{3}$ and further analyze the parameter combinations that allows for Hopf cycles.

Proposition 6. *Let* $\alpha = \frac{1}{3}$. *If* $-4 + 3v_2 < v_1 < -2 + 3v_2$, *then the system undergoes a Hopf bifurcation.*

Proof. Plug $\alpha = \frac{1}{3}$. The rest is straightforward. □

This relation between parameters v_1 and v_2 is visualized in Fig. 1.

The shaded region gives the v_1 and v_2's which induces Hopf cycles when $\alpha = \frac{1}{3}$, whereas the bold lines gives the boundaries of this region.

4. Conclusion

In this paper, we have analyzed the effects of varying population growth in a Solow–Kalecki type of growth model. We show that Cobb–Douglas type production functions and time-delay are not sufficient for the economy to have persistent cycles, yet it exhibits dampened oscillations. This is contrary to the common belief that delay is sufficient to obtain cyclic dynamics.

We extend the model so that population growth is endogenized. Then we show that capital dependent population dynamics supports Hopf bifurcation and thus limit cycles. However, it should be noted that without the delay structure, the economy may not, exhibit cycles. We summarize the results in Table 1.

Thus, the interaction between the delay structure and endogenous population causes limit cycles, whereas the delay or the endogenized population is not sufficient for limit cycle solutions. The mechanism that leads to cycles is an *adjustment failure* between the level of capital and level of population, where the failure is a result of delay structure. In the constant population case, failure is corrected after some period (dampened oscillations), yet in the endogenized population case, for a specific set of parameters (bifurcating parameters), failure cannot be corrected and persistent oscillations are possible.

[11] The positivity constraint of the parameters v_1 and v_2 maintains the economic intuition as in Cigno [5], that the population growth rate is positively related to per capita consumption and inversely related to the degree of industrialization. We do not see these explicitly since we are employing per capita variables. Yet, Cigno [p. 285] [5] also finds a similar result and underlines that these parameters should be close to each other to obtain stable growth.

References

[1] Asea PK, Zak PJ. Time-to-build and cycles. J Econ Dyn Control 1999;23:1155–75.
[2] Bambi Mauro. Endogenous growth and time-to-build: the AK case. J Econ Dyn Control 2008;32:1015–40.
[3] Bellman R, Cooke KL. Differential-difference equations. London: Academic Press; 1963.
[4] Brandt-Pollman U, Winkler R, Sager S, Moslener U, Schlöder PS. Numerical solutions of optimal control problems with constant control delays. Comput Econ 2008;31:181–206.
[5] Cigno Alessandro. Growth with exhaustible resources and endogenous population. Rev Econ Stud 1981;48(2):281–7.
[6] Farkas Miklós. Periodic motions. New York: Springer-Verlag; 1994.
[7] Hale J. Theory of functional differential equations. New York: Springer; 1977.
[8] Kalecki Michal. A macrodynamic theory of the business cycle. Econometrica 1935;3:327–44.
[9] Kind Christoph. Remarks on the economic interpretation of Hopf bifurcations. Econ Lett 1999;62:147–54.
[10] Krawiec A, Szydłowski M. The Kaldor–Kalecki business cycle model. Ann Oper Res 1999;89:89–100.
[11] Krawiec A, Szydłowski M. On nonlinear mechanics of business cycle model. Regular Chaotic Dyn 2000;6:101–17.

[12] Krawiec A, Szydłowski M. The Kaldor–Kalecki model of business cycle as a two-dimensional dynamical system. J Nonlinear Math Phys 2001;8:266–71.
[13] Krawiec A, Szydłowski M. The stability problem in the Kaldor–Kalecki business cycle model. Chaos Soliton Fract 2005;25:299–305.
[14] Krawiec A, Szydłowski M, Toboła J. Nonlinear oscillations in business cycle model with time lags. Chaos Soliton Fract 2001;12:505–51.
[15] Louisell James. Stability exponent and eigenvalue Abscissas by way of the imaginary axis eigenvalues. In: Niculescu S, Gu K, editors. Time-delay systems. Lecture notes in computational science and engineering, vol. 38. Springer-Verlag; 2004. p. 193–206.
[16] Manfredi P, Fanti L. Cycles in dynamic economic modelling. Econ Modell 2004;21:573–94.
[17] Szydłowski Marek. Time-to-build in dynamics of economic models I: Kalecki's model. Chaos Soliton Fract 2002;14:697–703.
[18] Szydłowski Marek. Time-to-build in dynamics of economic models II: Models of economic growth. Chaos Soliton Fract 2003;14:355–64.
[19] Szydłowski M, Krawiec A. A note on the Kaleckian lags in the Solow model. Rev Polit Econ 2004;16:501–6.
[20] Zak P. Kaleckian lags in general equilibrium. Rev Polit Econ 1999;11:321–30.