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Inventory and coordination issues with two substitutable products

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ABSTRACT

This study considers a two level supply chain in a newsboy setting with two substitutable products. Demands for the two products are assumed independent as long as both are available. If, however, a product stocks out, some of its demand is transferred to the available one with a known probability which ultimately creates a dependence on the amount of purchased items. The retailer is allowed to return some or all of the unsold products to the manufacturer with some credit. The expected chain profit, the retailer's and the manufacturer's profit expressions are derived under general conditions. Special cases are inspected to investigate the conditions under which channel coordination is achieved. It is demonstrated that channel coordination can not be achieved if unlimited returns are allowed with full credit, a result that agrees with the findings of Pasternak [B.A. Pasternack, Optimal pricing and return policies for perishable commodities, Market. Sci. 4 (1985) 166–176] for the single item case. For the cases of unlimited returns with partial credit, the conditions for coordination are derived for one way full substitutions. For exponential demand explicit expressions for the channel and retailer's expected profit functions are provided.

1. Introduction

Supply chain management and contracts between levels of a supply chain have gained considerable attention in the last decade.

In supply chains, uncertainties arising from factors such as market demand, process yield, product quality, competition and promotions introduce risks to both the manufacturers and the retailers. In order to increase the performance of the system by sharing the risks involved, contracts that include specifications regarding the quality, quantity, return rates and wholesale prices are undertaken between the manufacturer and the retailer with the purpose that such agreements would be beneficial to both parties. Most commonly studied examples of contracts are sales rebate, quantity flexibility, wholesale price, buyback and revenue sharing contracts, each of which provides the retailer with different incentives to make them order more than they would with only a wholesale price scheme. Quantity flexibility contracts provide some refund to the retailer when demand is lower than the order quantity, whereas the sales rebate contracts offer the retailer some incentive when demand is greater than a threshold, so that the retailer pays less for the units sold beyond this threshold. In revenue-sharing contracts, the manufacturer gets some credit per unit sold to the retailer in addition to a percentage of the retailers revenue. In buyback contracts, all or some of the unsold products are returned to the manufacturer for some credit.

Coordination among the retailer and the manufacturer is an important issue in designing contracts. In channel coordination, the objective is to bring the decentralized expected profit closer to the centralized expected profit and if they are equal, *channel coordination* is achieved.

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In this study, we consider a two level supply chain with a retailer and a manufacturer, for two substitutable products where the retailer is allowed to return some unsold products to the manufacturer. It is assumed that substitution takes place only at stockout situations. General expressions are derived for the expected total profit of the supply chain, the expected profit of the retailer and the expected profit of the manufacturer. Some special cases, regarding the substitution probabilities and return proportions are considered to obtain the necessary conditions for channel coordination. It is found that contracts that allow for unlimited returns with full credit can not coordinate the system, whereas unlimited returns with partial credits allow for coordination under one way or two way full substitution. These findings agree with the early work of [1] with single product, in that unlimited returns with full credit does not coordinate the system. Furthermore if one way substitution is in effect, the demand distribution for the stock-out product has an impact on the coordinating parameters. It is also observed that substitution dynamics have significant effect on the conditions under which coordination is achieved. The main contribution of this study is twofold: derivation of the expected profit functions of both parties in a two level chain with two substitutable products under a buyback contract; and the identification of the cases where the coordination is achieved.

A vast literature has accumulated about contracts and coordination in the last years, where an excellent review can be found in [2]. We briefly review below some work related to our study.

One of the earliest studies about channel coordination and buyback contracts is provided by Pasternack [1] for a newsboy setting where the retailer is allowed to return some or all of the unsold items to the manufacturer with some credit. Pasternack [1] found that neither a policy that allows for unlimited returns at full credit, nor the one that allows for no returns can achieve channel coordination, whereas coordination is achieved by a buyback contract with full returns at partial credit. An important finding was that the channel coordinating parameters were independent of the demand distribution, which facilitates the task of the manufacturer to design a contract. In another study, [3] consider a manufacturer that uses a buyback contract to manipulate the competition between the retailers. Buyback contracts intensify the degree of competition between the retailers. More intense retail competition means lower retailer prices and greater sales which results in larger profits for the manufacturer. Emmons and Gilbert [4] study buyback contracts where the retailer commits to both a stocking quantity and a selling price and Donohue [5] studies buyback contracts in a model with multiple production modes that allows for forecast updating. In another related work, manufacturer's pricing and return policies are studied by Lau and Lau [6]. In traditional studies, the retailer can order any quantity from the manufacturer at any time. However, this is undesirable from the manufacturer‘s point of view mostly due to the bullwhip effect which increases demand variance. To avoid the increases in demand variability, minimum purchase agreements are suggested as studied by Anupindi and Akella [7]. The advantages and limitations of revenue sharing contracts, where the retailer pays the manufacturer a percentage of the revenue he generates in addition to the wholesale price, is studied by Cachon and Lariviere [8].

Although contracts and coordination issues for supply chains have been investigated extensively as briefed above, there has been very limited work considering contracts with multiple products. To the best of our knowledge the only work with multiple products (no substitution) is by Anupindi and Bassok [9], who consider contracts for multiple products when the supplier offers business volume discounts. They argue that the optimal policy structure is complex and provide approximations based on the optimal policy of a similar contract with a single product.

Regarding the inventory control of multiple products with substitution, one of the early works is by Ignall and Veinott [10] who studied the conditions under which myopic solution is optimal in the long run. McGillivray and Silver [11] investigated the effects of the substitutability on stock control rules and inventory costs. Their model assumed that if an item is out of stock there is a fixed probability of the customer to substitute another available item. They considered the case of total substitutability (probability of substitution equaling one) and compared this with the case of no substitutability to obtain limits on the potential benefits achievable from substitution. Their results indicate that full substitution results in a decrease in the total optimal order quantity and substitution is less effective if the stock levels and substitution probabilities are low. Parlar and Goyal [12] studied a two product single period inventory model in which substitution occurs with a known probability. They showed that the total profit function is concave for a wide variety of problem parameters and developed necessary conditions for an optimal solution. In another study, Parlar [13] used a game theoretic approach to model two independent decision makers whose products can be substituted if one becomes out of stock. He showed that there exist a Nash equilibrium solution. See also Pasternack and Drezner [14] and Drezner et al. [15] for models with two substitutable products and Gurnani and Drezner [16] for a deterministic nested substitution problem with multiple substitutable products, and Ernst and Kouvelis [17] for a problem with three substitutable products where the objective function is shown to be jointly convex. Bassok et al. [18] consider a multiproduct single period inventory problem with downward substitution and show that the benefits of considering substitution in ordering decisions are higher with high demand variability, low substitution costs and low price to cost ratios. Smith and Agrawal [19] developed a probabilistic demand model capturing the effects of substitution, where inventory optimization includes both the selection of the set of items to stock and their stock levels under resource constraints. See also Rajaram and Tang [20] who studied the impact of product substitution on order quantities and profits. Using a consumer choice model based on utility maximization, Mahajan and van Ryzin [21] analyze a single period model with dynamic partial substitution. They show that the expected profit is in general not even quasi concave. Netessine and Rudi [22] consider both centralized and competitive inventory models under substitution with deterministic proportions and Netessine et al. [23] consider a multi-product environment with multivariate demand, allowing one level substitution and elaborate on the impact of correlation. Kraiselburd et al. [24] compare the vendor managed and retailer managed inventory systems in the substitutable products setting with stochastic demand. Yadavalli et al. [25] consider a model with two substitutable products, Poisson demands and joint ordering and study the stationary behavior of the inventory system. In a recent work, Karakul and Chan [26] consider the joint optimization of the pricing and procurement decisions for two products when one of the products can be substituted by the other product. Due to the complexity of the objective function, they provide sufficient conditions under which the objective function is unimodal.

Organization of the paper is as follows: In Section 2, the general model is introduced and the expected profit expressions are provided. In Section 3, special cases are considered and necessary conditions to achieve channel coordination are obtained. Finally, in Section 4 concluding remarks are made and future research directions are stated.

2. Model and analysis

We consider a single period newsboy type inventory problem with two substitutable perishable products in a two level supply chain, consisting of a retailer and a manufacturer. Among the several contract types that are introduced in the previous section, we focus on the return contracts, where the retailer is allowed to return some or all of the unsold products to the manufacturer with partial or full credit. Our set-up is similar to that of Pasternack [1], except that we generalize his study for two substitutable products.

We first derive the expressions for the total expected channel profit, manufacturers and the retailer's expected profits under general model parameters. We then investigate the special cases under which channel coordination is achieved.

For product i $(i = 1, 2)$, the following notation is used: the manufacturing cost per item is c_i, the wholesale price paid by the retailer to the manufacturer is d_i and p_i is the selling price of the retailer. We denote the the order quantity of the retailer and the production quantity of the manufacturer by Q_i and the percentage of Q_i that the retailer can return to the manufacturer is R_i. The credit paid by the manufacturer to the retailer for a returned item is denoted by s_i. The random demands for products 1 and 2 are denoted by X and Y, respectively with density (or probability mass) functions $f(x)$, $g(y)$ and distribution functions $F(x)$, $G(y)$, respectively. A customer will accept a unit of Product 2 when Product 1 is out of stock with probability a and the probability of substituting Product 1 when Product 2 is out of stock is b. There is no cost for substitution and the salvage value is zero. For consistency, we assume $c_i \leqslant d_i \leqslant p_i$. It is assumed that the demand for the two products are independent in order to get more explicit structural results. Although the derivation of the objective function would be straightforward, the analysis would be highly complicated for correlated demand, as discussed by Netessine et al. [23]. On the other hand, substitution dynamics eventually create a dependency between the effective demands of the two products. As to the realization of demand and substitution, we assume that the demand for both products occur at the beginning of the period and the original demand to each product is satisfied first. If there is excess inventory from one product and there is excess demand in the other, some or all of the excess demand is satisfied from the other product according to the probabilistic substitution behavior.

In the next section we derive the expressions for the expected total supply chain profit, the retailer's and the manufacturer's expected profits.

2.1. Total supply chain expected profit

Using the notation and the assumptions discussed above, our aim is to derive the expression for the total expected profit of the supply chain, which will be denoted by $EP_T(Q_1, Q_2)$, where Q_1, Q_2. Total expected profit is obtained assuming that the

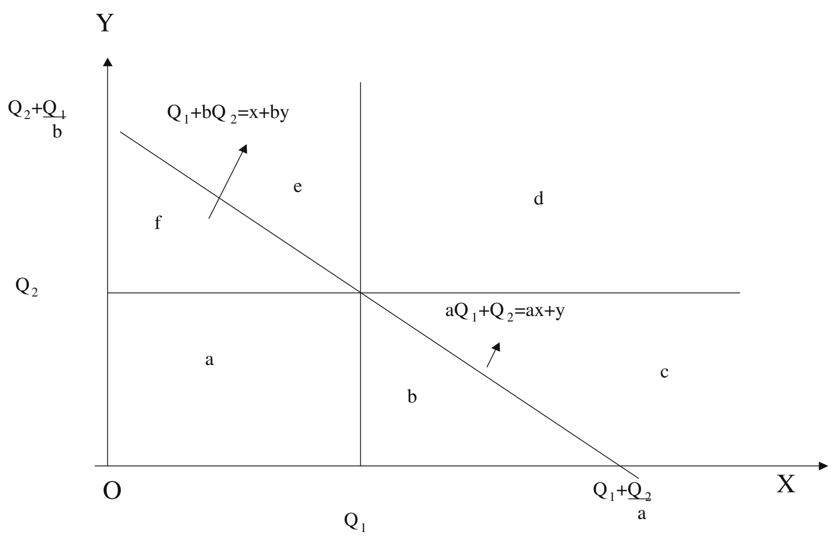

Fig. 1. Six regions giving rise to the total expected profit function.

producer sells directly to the customer and the derivation is based on the six regions $a-f$ depicted in Fig. 1. Let π_i denote the profit over region $i, i = a, \dots, f$, and $X = x, Y = y$ be the realized demands for products 1 and 2, respectively. Suppose the initial stocks for the products are Q_1, Q_2. In region a, demands for both of the products are less than their stock levels. In region b, demand for product 1 exceeds its inventory level and the excess demand can be fully satisfied by product 2. In region c, demand for product 1 exceeds its inventory level but the excess demand can only be partially satisfied by product 2. In region d, demands for both products are greater than their inventory levels. In region e, demand for product 2 exceeds its inventory level and the excess demand can only be partially satisfied by product 1. In region f, demand for product 2 exceeds its inventory level and can be fully satisfied by product 1. A simplified expression for the total supply chain expected profit expression is obtained by integrating the corresponding profit expressions over their respective regions and adding the cost of production $-c_1Q_1 - c_2Q_2$ (see also Parlar and Goyal, Eq. (11), p. 5). All the expressions in this section are given in terms of integrals, which should be replaced by summations for discrete demands.

Proposition 2.1. *Total expected profit of the supply chain is given by:*

$$EP_T(Q_1, Q_2) = -p_1 \int_0^{Q_1} F(x) G\left(Q_2 + \frac{(Q_1 - x)}{b}\right) dx + (p_2 - c_2) Q_2 - p_2 \int_0^{Q_2} G(x) F\left(Q_1 + \frac{(Q_2 - x)}{a}\right) dx + (p_1 - c_1) Q_1. \quad (1)$$

Parlar and Goyal [12] shows that $EP_T(Q_1, Q_2)$ is jointly concave in (Q_1, Q_2) provided that $bp_1 \leqslant p_2 \leqslant p_1/a$.

2.2. Retailer's expected profit

Next we consider the expected profit of the retailer who orders from the manufacturer according to the buyback agreement described above. The retailer orders Q_1 and Q_2 items from the two products at the beginning of the period at a cost of $d_1Q_1 + d_2Q_2$. Possible realizations of demand and substitutions together with returnable quantities are described in the eleven regions $a-k$ as illustrated in Fig. 2. As before, let $X = x$ and $Y = y$ be the realized demands for the two products.

Let $\overline{R_i} = 1 - R_i$ be the proportion of the order quantity for which return is not allowed for product i. In region a, where $x \leqslant \overline{R_1}Q_1$ and $y \leqslant \overline{R_2}Q_2$, R_iQ_i of the unsold items are returned to the manufacturer according to the permitted return percentages. In region b, where $x \leqslant \overline{R_1}Q_1$ and $\overline{R_2}Q_2 \leqslant y \leqslant Q_2$, only R_1Q_1 of the unsold items of product 1 is returned to the manufacturer but all the unsold ones from product 2 are returned since the leftovers are below the allowed return quantity. In region c, where $y \geqslant Q_2, Q_1 - (x + b(y - Q_2)) > R_1Q_1$, demand for product 2 exceeds the available inventory, the excess demand is fully satisfied by product 1 and R_1Q_1 units of product 1 is returned to the manufacturer. Similarly, in region d, $y \geqslant Q_2, Q_1 - (x + b(y - Q_2)) < R_1Q_1, x + b(y - Q_2) < Q_1$, demand for product 2 exceeds its inventory level, the excess demand is fully satisfied by product 1 and all unsold units of product 1 is returned to the manufacturer. In region f, where $\overline{R_1}Q_1 \leqslant x \leqslant Q_1$ and $\overline{R_2}Q_2 \leqslant y \leqslant Q_2$, all the unsold items of product 1 and 2 are returned to the manufacturer. In region i, since $x \geqslant Q_1$ and $y \geqslant Q_2$, demands for both products exceed their inventory levels therefore no substitution and returns take place. Finally in region j, where $y \geqslant Q_2, x + b(y - Q_2) < Q_1$ and $x < Q_1$, demand for product 2 exceeds its inventory level, the excess demand is only partially satisfied by product 1. Retailer's expected profit expression, $EP_R(Q_1, Q_2)$, is obtained by integrating the profit expressions over their respective regions and adding the cost $-d_1Q_1 - d_2Q_2$. The result is given below, the proof of which is given in Appendix.

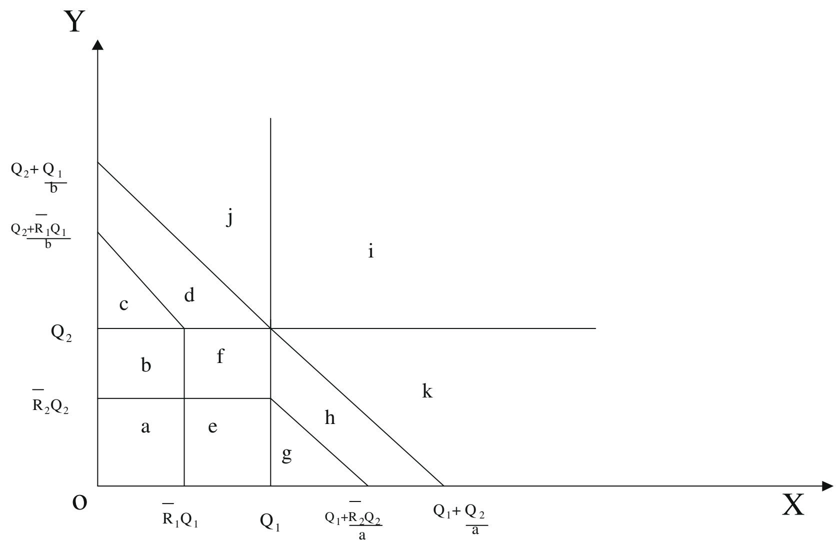

Fig. 2. Eleven regions giving rise to the retailer's expected profit function

Proposition 2.2. *Under the buyback contract, the retailer's expected profit is given by:*

$$\begin{aligned} EP_R(Q_1,Q_2) = & -p_1\int_0^{Q_1}F(x)G\left(Q_2+\frac{(Q_1-x)}{b}\right)dx+(p_2-d_2)Q_2+(p_1-d_1)Q_1-p_2\int_0^{Q_2}G(x)F\left(Q_1+\frac{(Q_2-x)}{a}\right)dx \\ & +F(Q_1)s_2\int_{\overline{R_2}Q_2}^{Q_2}G(y)dy+G(Q_2)s_1\int_{\overline{R_1}Q_1}^{Q_1}F(x)dx+\int_{Q_2}^{\infty}\int_{\overline{R_1}Q_1-b(y-Q_2)}^{Q_1-b(y-Q_2)}[Q_1-x-b(y-Q_2)]s_1dF(x)dG(y) \\ & +R_1Q_1s_1\int_{Q_2}^{Q_2+\frac{(\overline{R_1}Q_1)}{b}}F(\overline{R_1}.Q_1-b(y-Q_2))dG(y)+\int_{Q_1}^{\infty}\int_{\overline{R_2}Q_2-a(x-Q_1)}^{Q_2-a(x-Q_1)}[Q_2-y-a(x-Q_1)]s_2dG(y)dF(x) \\ & +R_2Q_2s_2\int_{Q_1}^{Q_1+\frac{(\overline{R_2}Q_2)}{a}}G(\overline{R_2}Q_2-a(x-Q_1))dF(x). \end{aligned} \tag{2}$$

The general expression above unfortunately does not allow to derive further insights due to its complexity. Hence we elaborate below some special cases.

2.3. Special cases with one-way full substitution

Now we elaborate some special cases with full substitution and/or full return. Note first that when two way full substitution is allowed, the customers would buy the other product with certainty in stock-out cases, and no substitution cost is incurred. Hence it would be optimal to carry inventory of only the product with higher profit margin, which reduces the problem to a single product case. Therefore, it is of interest to consider only the cases with one-way full substitution. Below we introduce three cases with one-way full substitution accompanied with (a) no returns, (b) full returns and (c) full return with one product and no return with the other.

Corollary 2.1

(a) *No returns with one-way full substitution* $(a=1,b=0,R_1=R_2=0)$

$$EP_R(Q_1,Q_2) = -p_1\int_0^{Q_1}F(x)dx+(p_1-d_1)Q_1-p_2\int_0^{Q_2}F(Q_1+Q_2-y)G(y)dy+(p_2-d_2)Q_2. \tag{3}$$

(b) *Full returns with one-way full substitution* $(a=1,b=0,R_1=R_2=1)$

$$EP_R(Q_1,Q_2) = -(p_1-s_1)\int_0^{Q_1}F(x)dx+(p_1-d_1)Q_1-(p_2-s_2)\int_0^{Q_2}F(Q_1+Q_2-y)G(y)]dy+(p_2-d_2)Q_2. \tag{4}$$

(c) *One-way full return with one-way full substitution* $(a=1,b=0,R_1=1,R_2=0)$

$$EP_R(Q_1,Q_2) = -(p_1-s_1)\int_0^{Q_1}F(x)dx+(p_1-d_1)Q_1-p_2\int_0^{Q_2}F(Q_1+Q_2-y)G(y)dy+(p_2-d_2)Q_2. \tag{5}$$

Proposition 2.3. *For the special cases given in Corollary 2.1, $EP_R(Q_1,Q_2)$ is jointly concave in (Q_1,Q_2), provided that the following conditions hold in each case:*

$$(a): p_2 \leqslant p_1 \quad (b): (p_2-s_2) \leqslant (p_1-s_1) \quad (c): p_2 \leqslant (p_1-s_1)$$

Proof. Directly follows from the concavity result of [12] after observing the similarity of the structures of the profit functions in (a)–(c) to that of (1) with modified parameters and rewriting the conditions for concavity accordingly with the modified parameters.

Note that the conditions for the proposition above can be interpreted as the consistency conditions and in all cases product 2 is substituted for product one. In part (a) returns are not allowed, hence p_2, the price of the substituted product is required to be less or equal p_1, price of the first choice product. In part (b) full returns are allowed and in this case it is required that, the loss due to returns to manufacturer for the substituted product (p_2-s_2) is less or equal the loss from the return of the first choice product, p_1-s_1. Finally in (c), the price of the product which is always purchased in place of the other in case of stock-outs is needed to be less than the loss incurred when the first product is returned to the manufacturer instead of sold to a customer.

The case $a=1,b=0,R_1=0,R_2=1$, which implies that the second product is always substituted for the first one and all the left overs of the second product are allowed to be fully returned turns out to be more complicated. We first present below the resulting expression for the expected retailer profit and then present some results to aid in the analysis of specific cases. □

Corollary 2.2. *Let $a = 1, b = 0, R_1 = 0, R_2 = 1$, corresponding to one-way full substitution and full return of Product 2. For this case,*

$$\begin{aligned} EP_R(Q_1,Q_2) = &-p_1\int_0^{Q_1}F(x)dx + (p_1-d_1)Q_1 + (p_2-d_2)Q_2 - (p_2-s_2)\int_0^{Q_2}F(Q_1+Q_2-x)G(x)dx \\ &- s_2\int_0^{Q_2}[F(Q_1+Q_2)-F(Q_1+Q_2-x)]xdG(x). \end{aligned} \tag{6}$$

Proposition 2.4. *For the special case of Corollary 2.2, let*

$$\eta(Q_1,Q_2) = -(p_2-s_2)\int_0^{Q_2}f(Q_1+Q_2-x)G(x)dx - s_2\int_0^{Q_2}[f(Q_1+Q_2)-f(Q_1+Q_2-x)]xdG(x). \tag{7}$$

Also let $\eta_1(Q_1,Q_2) \equiv \partial\eta_1(Q_1,Q_2)/\partial Q_1$, $\eta_2 Q_1,Q_2) \equiv \partial\eta_1 Q_1,Q_2)/\partial Q_2$, $\eta_{12} Q_1,Q_2) \equiv \partial^2\eta_1 Q_1,Q_2)/\partial Q_1\partial Q_2$ and

$$C(Q_1,Q_2) = -(p_2-s_2)g(Q_2)F(Q_1) - s_2\{Q_2 f(Q_1)g(Q_2) + [F(Q_1+Q_2)-F(Q_1)][Q_2 g'(Q_2)+g(Q_2)]\}$$

Then,

(a) the first order conditions are given as

$$0 = (p_1-d_1) - p_1F(Q_1) + \eta(Q_1,Q_2),$$
$$0 = (p_2-d_2) - (p_2-s_2)G(Q_2)F(Q_1) - s_2[F(Q_1+Q_2)-F(Q_1)]Q_2g(Q_2) + \eta(Q_1,Q_2).$$

(b) Let $H(Q_1,Q_2) \equiv \{h_{ij}\}, i,j = 1,2$ be the Hessian matrix corresponding to $EP_R(Q_1,Q_2)$. Then we have

$$h_{11} = -p_1\eta_1(Q_1,Q_2),$$
$$h_{12} = \eta_2(Q_1,Q_2),$$
$$h_{22} = \eta_2(Q_1,Q_2) + C(Q_1,Q_2).$$

Note that if $h_{11} < 0$, $h_{22} < 0$ and the determinant $h_{11}h_{22} - h_{12}^2 < 0$, then $EP_R(Q_1,Q_2)$ is jointly concave. This must be checked for any special application for the unique maximum to exist. The above analysis illustrate the difficulty of obtaining general results even for some special cases. Nevertheless, to get some further insights, we consider another special case regarding the demand distributions that allow for explicit expressions for the expected profit functions.

2.3.1. Exponential demand

In this section, we elaborate the case where the demand for both products have exponential distribution with parameters λ and μ for products 1 and 2, respectively, with $F(x) = 1 - e^{-\lambda x}$ and $G(y) = 1 - e^{-\mu y}$. In order to evaluate the expressions for the total expected profit and the expected profit of the retailer in special cases, define:

$$U(\alpha,\beta,Q,\tau) = (\alpha-\beta)Q + \frac{\beta}{\tau}(1-e^{-\tau Q}),$$
$$W(\alpha,\beta,Q_1,Q_2,\tau_1,\tau_2) = (\alpha-\beta)Q_1 + \beta\left[\frac{1}{\tau_2}e^{-\tau_2Q_2}(1-e^{-\tau_2Q_1}) + \frac{1}{\tau_1}\left(1-e^{-\tau_1Q_1}\right) - \frac{e^{-\tau_2Q_2}}{\tau_1-\tau_2}\left(e^{-\tau_2Q_1}-e^{-\tau_1Q_1}\right)\right].$$

Then, after some algebra, it can be shown that the total expected profit reduces to the following for one way full substitution:

$$EP_T(Q_1,Q_2) = U(p_1-c_1,p_1,Q_1,\lambda) + W(p_2-c_2,p_2,Q_2,Q_1,\mu,\lambda).$$

Similarly, the retailer's expected profit is given as below for the special cases presented in Corollary 2.1:

$$EP_R(Q_1,Q_2) = \begin{cases} U(p_1-d_1,p_1,Q_1,\lambda) + W(p_2-d_2,p_2,Q_2,Q_1,\mu,\lambda) & \text{for (a)} \\ U(p_1-d_1,p_1-s_1,Q_1,\lambda) + W(p_2-d_2,p_2-s_2,Q_2,Q_1,\mu,\lambda) & \text{for (b)} \\ U(p_1-d_1,p_1-s_1,Q_1,\lambda) + W(p_2-d_2,p_2,Q_2,Q_1,\mu,\lambda) & \text{for (c).} \end{cases}$$

Example 1. Suppose the demand for the products are independent exponential, with $\lambda = 0.02$ and $\mu = 0.05$. As will become clear in the next section, if the manufacturer allows no returns or allows full returns with full credit, the two parts of the supply chain do not coordinate. Therefore in this example we consider one way full substitution ($a = 1, b = 0$), full return ($R_1 = R_2 = 1$) case with partial credit. For Product 1, the system parameters are set to $c_1 = 2.00$, $d_1 = 4.2$, $p_1 = 7.0$ and $s_1 = 3.0$; and for Product 2 $c_2 = 3.00$, $d_2 = 5.2$, $p_2 = 7.0$ and $s_2 = 3.3$.

Fig. 3a depicts the total channel profit and b the expected profit of the retailer when partial credit for the returned items are offered as above. We find that if substitution was not allowed, the news vendor values for Product 1 and 2 would be $Q_1 = 62$ and $Q_2 = 16$. When coordination issues are not considered, if the manufacturer directly sells to the market, his

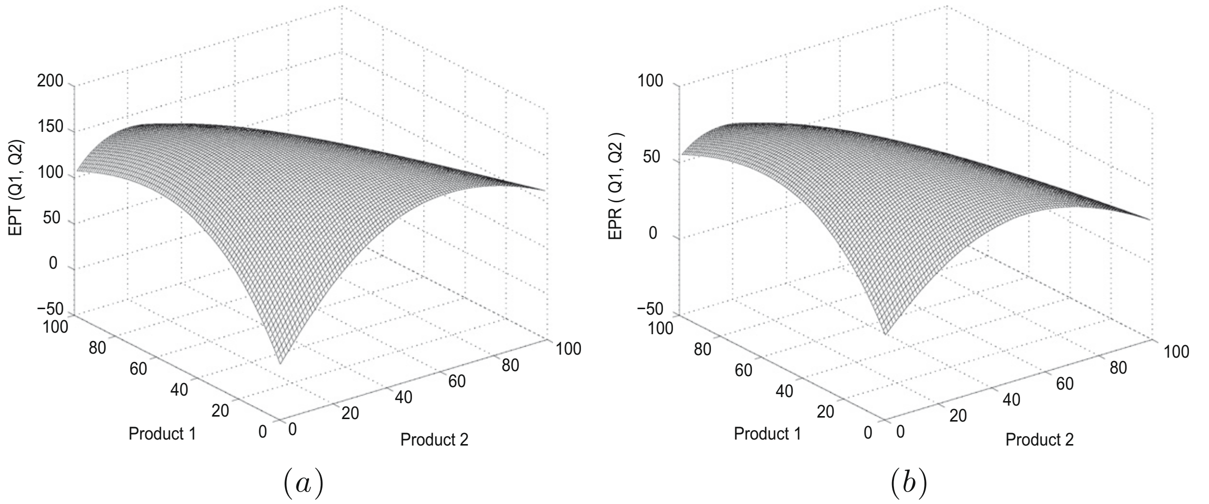

Fig. 3. The expected total profit and the expected profit of the retailer. For Product 1, the parameters c_1, d_1, p_1, s_1 are respectively, 2.0, 4.20, 7.0 and 3.0. For Product 2, c_2, d_2, p_2, s_2 are 3.0, 5.20, 7.0 and 3.30.

optimal production quantities would be 49 and 30, respectively with an optimum profit of 170.1. On the other hand with s_1 and s_2 as given above, the retailer's optimum order quantities would be 53 and 22, respectively, resulting in an expected profit of 84.2. We observe that the manufacturer and retailer can not coordinate the channel with the above choices of d_i's and s_i's. In the next section we will see how these parameters should be modified to achieve channel coordination.

Example 2. To illustrate how the expected profit expressions react to the changes in system parameters, we consider another example. Now set $c_1 = 2.0$, $d_1 = 4.0$, $p_1 = 7.0$ and $s_1 = 3.0$; and for Product 2 $c_1 = 3.00$, $d_1 = 4.5$, $p_1 = 7.0$ and $s_1 = 4$. We again have the optimum channel quantities as 49 and 30 resulting in a profit of 170.1 (since the profit margin has not changed). However the retailer's optimal order quantities has changed to 32 and 84, respectively yielding a profit of 135. The general shape of the profit functions are as given in the previous figure.

2.4. Manufacturer's expected profit

We simply obtain the manufacturer's expected profit by noting that $EP_T(Q_1, Q_2) = EP_R(Q_1, Q_2) + EP_M(Q_1, Q_2)$. However, for the purpose of completeness, we provide below the resulting expression.

Proposition 2.4. *The manufacturer's expected profit is given by:*

$$\begin{aligned} EP_M(Q_1, Q_2) = {} & (d_1 - c_1)Q_1 + (d_2 - c_2)Q_2 - F(Q_1)s_2 \int_{\overline{R_2}Q_2}^{Q_2} G(y)d(y) - G(Q_2)s_1 \int_{\overline{R_1}Q_1}^{Q_1} F(x)d(x) \\ & - \int_{Q_2}^{\infty} \int_{\overline{R_1}Q_1 - b(y-Q_2)}^{Q_1 - b(y-Q_2)} [Q_1 - x - b(y - Q_2)]s_1 dF(x)dG(y) \\ & - R_1 Q_1 s_1 \int_{Q_2}^{Q_2 + \frac{(\overline{R_1}Q_1)}{b}} F(\overline{R_1}Q_1 - b(y - Q_2))dG(y) \\ & - \int_{Q_1}^{\infty} \int_{\overline{R_2}Q_2 - a(x-Q_1)}^{Q_2 - a(x-Q_1)} [Q_2 - y - a(x - Q_1)]s_2 dG(y)dF(x) \\ & - R_2 Q_2 s_2 \int_{Q_1}^{Q_1 + \frac{(\overline{R_2}Q_2)}{a}} G(\overline{R_2}Q_2 - a(x - Q_1))dF(x). \end{aligned} \tag{8}$$

3. Channel coordination

We next consider several special cases regarding the substitution probabilities and return percentages, and investigate the conditions under which channel coordination is achieved. Concavity of the total profit function $EP_T(Q_1, Q_2)$ is proved by Parlar and Goyal [12] under general conditions, from which the concavity of the $EP_R(Q_1, Q_2)$ follows as discussed in the previous section. Hence, there exist unique inventory levels for both products that maximize the expected channel profit as well as the expected profit of the retailer. These quantities can be obtained from the first order conditions. In some special cases, these first order conditions for the retailer are satisfied only when the order quantities are infinite, in which case we say that the system is sub-optimal. Similarly, when infeasible conditions are required for the channel coordination, such as zero profit of the manufacturer or the retailer, we refer to that as system sub-optimality. Below we provide the main results concerning the channel coordinations, the proofs of which are given in the Appendix.

Case-1: *Full returns with partial credit, no substitution*

Suppose the retailer is allowed to return all unsold products to the manufacturer and there is no substitution between the two products. This case is similar to two independent products and the results of [1] are valid for each one. Namely, a policy that allows unlimited returns for full credit or that allows no returns is system suboptimal. However, a policy which allows for unlimited returns at partial credit will be system optimal for appropriately chosen values of model parameters, as stated below. Similarly, as discussed before the two way full substitution also reduces to a single product and the following result is valid with the product that offers a higher profit margin for the manufacturer.

Proposition 3.1. *Let $a = 0, b = 0, R_1 = R_2 = 1$. Then channel coordination is achieved if the following conditions are satisfied:*

$$\frac{p_1 - c_1}{p_1} = \frac{p_1 - d_1}{p_1 - s_1}, \tag{9}$$

$$\frac{p_2 - c_2}{p_2} = \frac{p_2 - d_2}{p_2 - s_2}. \tag{10}$$

The above conditions indicate that for channel coordination with two independent products is achieved if the ratio of the channel profit per unit to the selling price is the same as the ratio of the retailer's profit per unit to the difference between the selling price and the return credit, which requires that the return credit should not exceed the wholesale price. We see from the above conditions that the coordinating parameters are independent of the demand distribution.

Case-2: *Full returns with partial credit, one-way full substitution*

Consider the case where the retailer is allowed to return all unsold products to the manufacturer with partial credit and only product 1 is substituted with product 2 with probability one, if stock-out occurs. The condition under which coordination is achieved is given below. If one-way substitution is effective for the other product, the indices will simply be interchanged and F will be replaced by G.

Proposition 3.2. *Let $a = 1, b = 0, R_1 = R_2 = 1$. Then, channel coordination is achieved if*

$$F(Q_1) = \frac{c_2(p_2 - s_2) + p_2(s_2 - d_2 + d_1 - c_1) + s_2(p_1 + c_1)}{s_1 p_2 - s_2 p_1}, \tag{11}$$

provided that the r.h.s of (11) lies in (0,1).

Note that, unlike the previous case, the asymmetry in the substitution behavior resulted in a condition that depends on the demand distribution. In particular, this condition requires that the service level for product 1 satisfies the condition given in the r.h.s. of (11).

Case-3: *One-way full substitution with no returns*. Suppose again we have one-way full substitution but returns are not allowed. Such an agreement fails to coordinate the channel.

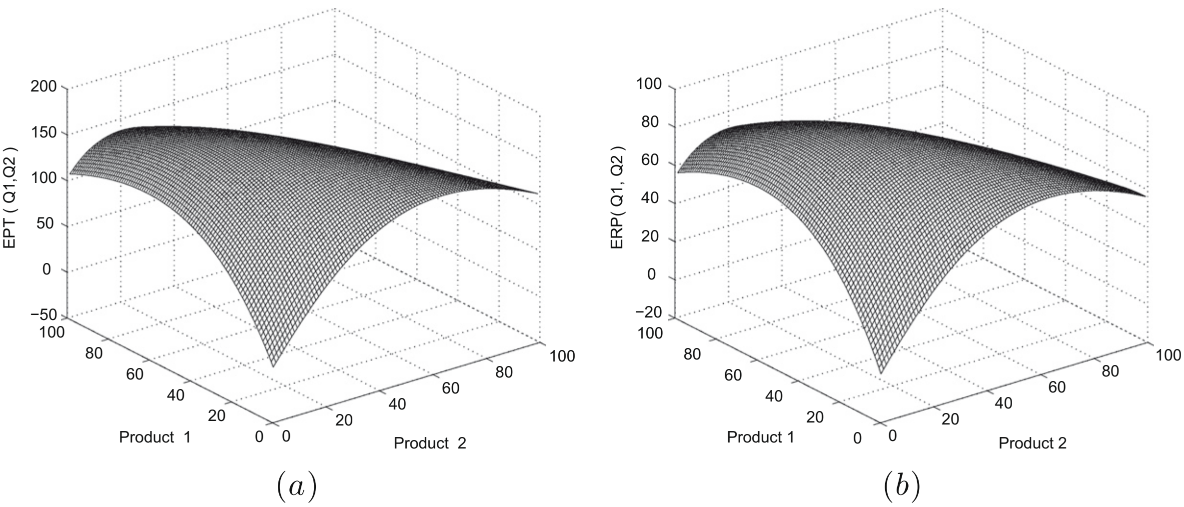

Fig. 4. The expected total profit and the expected profit of the retailer. For Product 1, the parameters c_1, d_1, p_1, s_1 are respectively, 2.0, 4.22, 7.0 and 3.05. For Product c_2, d_2, p_2, s_2 are 3.0, 5.0, 7.0 and 3.5.

Table 1
Case-2: Profit share for one-way full substitution with partial credit, full returns.

s_1	s_2	EP_R	$\%EP_R$	EP_R	$\%EP_R$	EP_R	$\%EP_R$
		$d_1 = 4.5$, $d_2 = 3$,		$d_1 = 6$, $d_2 = 4.5$		$d_1 = 7$, $d_2 = 5.5$	
1,5	0.18	129.30	73,08	64,802	36,63	21,80	12,32
2	0.77	133.53	75,47	69,03	39,01	26,03	14,71
2,5	1,36	137.76	77,86	73,2568	41,40	30,26	17,10
3	1,96	141.98	80,25	77,48	43,79	34,48	19,49
3,5	2,55	146.21	82,64	81,71	46,18	38,71	21,88
4	3,15	150.44	85,03	85,94	48,57	42,94	24,27

Proposition 3.3. *Let $a = 1, b = 0$, $R_1 = 0$ and $R_2 = 0$. Then, channel coordination requires $c_1 = d_1$ and $c_2 = d_2$. Hence, the system is suboptimal, unless the manufacturer makes zero profit.*

Case-4: *One-way full substitution with full returns and full credit*

This is a special case of Case-2 with $s_1 = d_1, s_2 = d_2$. The manufacturer pays the wholesale price back for all the unsold items. This unbalanced system in favor of the retailer is suboptimal.

Proposition 3.4. *Suppose $a = 1, b = 0, R_1 = 1$, $R_2 = 1$, $s_1 = d_1$ and $s_2 = d_2$. Then the system is suboptimal.*

Example 3. Recall that in Examples 1 and 2 we have observed that the coordinatin can not be achieved with the given wholesale prices and return credits. Coordinating parameters are obtained as follows. With the same production costs and selling prices, i.e. $c_1 = 2.0, p_1 = 7, c_2 = 3.0, p_2 = 7$, the values of the wholesale price and the return credits that coordinate the channel are obtained as $d_1 = 4.22,\ s_1 = 3.05$ and $d_2 = 5.0,\ s_2 = 3.50$, respectively. These parameters yield the same optimal retailer order quantities 49 and 30 with the optimal profit of 90.8 for the retailer. Since the total expected channel profit is 170.1, we observe that the retailer gets most of the profit. However, other coordinating parameters would result in different shares of the profit among the manufacturer and the retailer, as will be discussed in the next example. The resulting profit functions with these particular coordinating parameters are given in Fig. 4.

Next we consider an example that illustrates how the total profit is splitted between the manufacturer and the retailer under different values wholesale prices and return credits that coordinate the system.

Example 4. For this example we consider identical negative binomial demand distributions for the two products, the parameters of which are set to $r_i = 5, p_i = 0.25$ for i=1,2. The resulting means and variances are $E(Y) = E(X) = r/p$ and $V(X) = V(Y) = r(1-p)/p^2$. We consider Case 2 where coordination is achieved under full returns with partial credit. We set the costs of producing one unit of Product 1 and 2 as $c_1 = 3, c_2 = 2$, respectively and the corresponding selling prices as $p_1 = 9$ and $p_2 = 7$. The optimal production quantitities Q_1^*, Q_2^* of the manufacturer that maximizes, $EP_T(Q_1, Q_2)$ are found and then the transfer payments and buyback credits that achieve channel coordination are investigated using the results of Proposition 3.2. The optimal production quantities for Case 2 are found as $Q_1^* = 14$ and $Q_2^* = 31$ with the corresponding total expected chain profit, $EP_T(Q_1, Q_2)$ of 177.808. In this case recall that Product 2 is substituted for Product 1 with probability one. Hence as expected, the optimal production quantity of the Product 2 is larger despite the fact that the unit profit of Product 1 is larger.

It is of interest to see the impact of the coordinating wholesale prices and return credits on the profit share among the parts of the channel. To illustrate this, the expected profit of the retailer and the percentage of his share, denoted by $\%EP_R$, are obtained for different choices of wholesale prices d_1, d_2 and return credits s_1 and s_2, and the results are displayed in Table 1. As expected the retailer's share increase with the return credit and decrease with the wholesale price, and wholesale prices have more impact on how the profit is shared among the parts of the channel.

4. Conclusion

In this study, a simple supply chain structure with a single retailer and a manufacturer is considered for two substitutable products. The retailer is allowed to return some products to the manufacturer according to the contract between the retailer and the manufacturer. We provide the expressions for the total expected channel profit, manufacturers expected profit and the retailers expected profit under general model parameters. Special cases regarding the substitution probabilities, return credits and return percentages are investigated for channel coordination. In a similar study of [1] with a single product, it was found that channel coordination was not achieved with full returns and full credits. This is consistent with our result. We have found that channel coordination is not achieved for no returns cases. We have also provided expected profit expressions for the special case of exponential demand and elaborated on the model with several examples where the demand has exponential or negative binomial distributions, respectively.

It would be interesting to extend the results of this study to correlated multi products and multi-period settings. Other contract types with multiple products are also worthwhile to consider.

Appendix

The following results on integrals are needed in some of the derivations below:

$$\int_{Q_2}^{\infty}\int_0^{Q_1}[Q_1-x]dF(x)dG(y)=\int_0^{Q_1}F(x)dx-G(Q_2)\int_0^{Q_1}F(x)dx$$
$$\int_{Q_1}^{\infty}\int_{Q_1-x}^{Q_1+Q_2-x}[Q_2-y+Q_1-x]dG(y)dF(x)=\int_{Q_1}^{\infty}\int_{Q_1-x}^{Q_1+Q_2-x}G(y)dydF(x). \tag{12}$$

Calculation of $EP_T(Q_1,Q_2)$

Referring to Fig. 1, profit expressions in each region can be written as

(a)	$\pi_a=p_1x+p_2y$	$x\leqslant Q_1, y\leqslant Q_2$
(b)	$\pi_b=p_2y+p_2ax+Q_1(p_1-p_2a)$	$x\geqslant Q_1, y\leqslant Q_2, a(x-Q_1)<Q_2-y$
(c)	$\pi_c=p_1Q_1+p_2Q_2$	$x\geqslant Q_1, y\leqslant Q_2, a(x-Q_1)>Q_2-y$
(d)	$\pi_d=p_1Q_1+p_2Q_2$	$x\geqslant Q_1, y\geqslant Q_2$
(e)	$\pi_e=p_1Q_1+p_2Q_2$	$x\leqslant Q_1, y\geqslant Q_2, Q_1-x<b(y-Q_2)$
(f)	$\pi_f=p_1x+p_1by+Q_2(p_2-p_1b)$	$x\leqslant Q_1, y\geqslant Q_2, Q_1-x>b(y-Q_2)$

Similar terms in different profit expressions are collected and their contribution to the overall expected profit are given as follows:

Term p_1x in region $(a\cup f)$:

$$p_1\int_0^{Q_1}x\int_0^{Q_2+\frac{(Q_1-x)}{b}}dG(y)dF(x)=p_1\int_0^{Q_1}xG(Q_2+\frac{(Q_1-x)}{b})dF(x).$$

Applying integration by parts we write the above term as

$$p_1\left[Q_1G(Q_2)F(Q_1)-\int_0^{Q_1}G\left(Q_2+\frac{(Q_1-x)}{b}\right)F(x)d_x+(Q_1+bQ_2)\int_{Q_2}^{Q_2+\frac{Q_1}{b}}F(Q_1+b(Q_2-u))dG(u)\right.$$
$$\left.-b\int_{Q_2}^{Q_2+\frac{Q_1}{b}}F(Q_1+b(Q_2-u))udG(u)\right]. \tag{13}$$

Term p_2y in region $(a\cup b)$

$$p_2\left[Q_2G(Q_2)F(Q_1)-\int_0^{Q_2}F\left(Q_1+\frac{(Q_2-x)}{a}\right)G(x)dx+(Q_2+aQ_1)\int_{Q_1}^{Q_1+\frac{Q_2}{a}}G(Q_2+a(Q_1-u))dF(u)\right.$$
$$\left.-a\int_{Q_1}^{Q_1+\frac{Q_2}{a}}G(Q_2+a(Q_1-u))udF(u)\right]. \tag{14}$$

Term p_1by in region (f)

$$p_1b\int_{Q_2}^{Q_2+\frac{Q_1}{b}}y\int_0^{Q_1+b(Q_2-y)}dF(x)dG(y)=p_1b\int_{Q_2}^{Q_2+\frac{Q_1}{b}}yF(Q_1+b(Q_2-y))dG(y). \tag{15}$$

Term p_2ax in region (b)

$$p_2a\int_{Q_1}^{Q_1+\frac{Q_2}{a}}xG(Q_2+a(Q_1-x))dF(x). \tag{16}$$

Term $Q_2(p_2-p_1b)$ in region (f)

$$Q_2(p_2-p_1b)\left[-F(Q_1)G(Q_2)+\int_0^{Q_1}G\left(Q_2+\frac{(Q_1-x)}{b}\right)dF(x)\right]. \tag{17}$$

Term $Q_1(p_1-p_2a)$ in region (b)

$$Q_1(p_1-p_2a)\left[-F(Q_1)G(Q_2)+\int_0^{Q_2}F\left(Q_1+\frac{(Q_2-y)}{a}\right)dG(y)\right]. \tag{18}$$

Term $p_1Q_1 + p_2Q_2$ appears in regions $(d), (e)$ and (c) and the corresponding contributions in these regions are:

$$(d)\ (p_1Q_1 + p_2Q_2)\overline{F}(Q_1)\overline{G}(Q_2), \tag{19}$$

$$(e)\ (p_1Q_1 + p_2Q_2)\int_{Q_1}^{\infty}\int_{Q_2+\frac{(Q_1-x)}{b}}^{\infty} dG(y)dF(x) = (p_1Q_1 + p_2Q_2)\int_0^{Q_1}\overline{G}\left(Q_2 + \frac{(Q_1-x)}{b}\right)dF(x)$$
$$= (p_1Q_1 + p_2Q_2)(F(Q_1) - \int_0^{Q_1} G\left(Q_2 + \frac{(Q_1-x)}{b}\right)dF(x)), \tag{20}$$

$$(f)\ (p_1Q_1 + p_2Q_2)\left(G(Q_2) - \int_0^{Q_2} F\left(Q_1 + \frac{(Q_2-y)}{a}\right)dG(y)\right). \tag{21}$$

The sum of (19)–(21) results in the following for the contribution of $p_1Q_1 + p_2Q_2$

$$(p_1Q_1 + p_2Q_2)\left[1 + G(Q_2)F(Q_1) - \int_0^{Q_2} F\left(Q_1 + \frac{(Q_2-y)}{a}\right)dG(y) - \int_0^{Q_1} G\left(Q_2 + \frac{(Q_1-x)}{b}\right)dF(x).\right. \tag{22}$$

Finally $EP_T(Q_1, Q_2)$ is obtained by the sum of (13)–(18), (22) and $-c_1Q_1 - c_2Q_2$.

Calculation of the $EP_R(Q_1, Q_2)$

The derivation of the expression for the retailer's profit is done similarly by considering different regions as given in Fig. 2. The profit expressions in tese regions are written as

$(a)\quad \pi_a = p_1x + p_2y + R_1Q_1s_1 + R_2Q_2s_2$
$(b)\quad \pi_b = p_1x + p_2y + R_1Q_1s_1 + (Q_2 - y)s_2$
$(c)\quad \pi_c = p_1x + p_1(b(y - Q_2)) + R_1Q_1s_1 + p_2Q_2$
$(d)\quad \pi_d = p_2Q_2 + p_1(x + b(y - Q_2)) + (Q_1 - x - b(y - Q_2))s_1$
$(e)\quad \pi_e = p_1x + p_2y + R_2Q_2s_2 + (Q_1 - x)s_1$
$(f)\quad \pi_f = p_1x + p_2y + (Q_1 - x)s_1 + (Q_2 - y)s_2$
$(g)\quad \pi_g = p_2y + p_2(a(x - Q_1)) + R_2Q_2s_2 + p_1Q_1$
$(h)\quad \pi_h = p_1Q_1 + p_2(y + a(x - Q_1)) + (Q_2 - y - a(x - Q_1))s_2$
$(i)\quad \pi_i = p_1Q_1 + p_2Q_2$
$(j)\quad \pi_j = p_1Q_1 + p_2Q_2$
$(k)\quad \pi_k = p_1Q_1 + p_2Q_2$

As before, similar terms in the above expressions are collected to calculate the contribution to the expected profit as follows:

Term p_1x in region $a \cup b \cup e \cup f$

$$p_1\int_0^{Q_1} x\int_0^{Q_2} dG(y)dF(x) = p_1G(Q_2)\left[Q_1F(Q_1) - \int_0^{Q_1} F(x)dx\right] \tag{23}$$

Term p_2y in region $(a \cup b \cup e \cup f)$

$$p_2F(Q_1)\left[Q_2G(Q_2) - \int_0^{Q_2} G(y)dy\right] \tag{24}$$

Term $R_1Q_1s_1$ in region $(a \cup b)$

$$\int_0^{Q_2} x\int_0^{\overline{R_1}.Q_1} R_1Q_1s_1dF(x)dG(y) = R_1Q_1s_1F(\overline{R_1}Q_1)G(Q_2) \tag{25}$$

Term $R_2Q_2s_2$ in region $(a \cup e)$

$$R_2Q_2s_2G(\overline{R_2}Q_2)F(Q_1) \tag{26}$$

Term $(Q_2 - y)s_2$ in region $(b \cup f)$

$$-R_2Q_2s_2F(Q_1)G(\overline{R_2}Q_2) + s_2F(Q_1)\int_{\overline{R_2}Q_2}^{Q_2} G(y)dy \tag{27}$$

Term $(Q_1 - x)s_1$ in region $(b \cup f)$

$$-R_1Q_1s_1G(Q_2)F(\overline{R_1}Q_1) + s_1G(Q_2)\int_{\overline{R_1}Q_1}^{Q_1} F(x)dx \tag{28}$$

Term $p_1Q_1 + p_2Q_2$ in region (i)

$$\int_{Q_1}^{\infty}\int_{Q_2}^{\infty}(p_1Q_1 + p_2Q_2)dG(y)dF(x) = (p_1Q_1 + p_2Q_2)\overline{G}(Q_2)\overline{F}(Q_1) \tag{29}$$

p_2Q_2 in $(c \cup d \cup j)$

$$\int_{Q_2}^{\infty}\int_0^{Q_1} p_2Q_2dF(x)dG(y) = p_2Q_2\overline{G}(Q_2)F(Q_1) \tag{30}$$

Term p_1Q_1 in (j)

$$p_1Q_1\overline{G}(Q_2)F(Q_1) - p_1Q_1\int_{Q_2}^{\infty} F(Q_1 - (b(y-Q_2)))dG(y) \tag{31}$$

Term $p_1(x + b(y - Q_2))$ in region $(c \cup d)$

$$\begin{aligned}\int_{Q_2}^{Q_2+\frac{Q_1}{b}}\int_0^{Q_1-(b(y-Q_2))} p_1(x+b(y-Q_2))dF(x)dG(y) &= p_1G(Q_2)\int_0^{Q_1}F(x)dx \\ -p_1\int_0^{Q_1}F(x)G\left(Q_2+\frac{(Q_1-x)}{b}\right)dx &+ p_1Q_1\int_{Q_2}^{Q_2+\frac{Q_1}{b}}F(Q_1-(b(y-Q_2)))dG(y)\end{aligned} \tag{32}$$

Term $R_1Q_1s_1$ in region (c)

$$\int_{Q_2}^{Q_2+\frac{\overline{R_1}Q_1}{b}}\int_0^{\overline{R_1}Q_1-b(y-Q_2)} R_1Q_1s_1dF(x)dG(y) = R_1Q_1s_1\int_{Q_2}^{Q_2+\frac{\overline{R_1}Q_1}{b}}F(\overline{R_1}Q_1-b(y-Q_2))dG(y) \tag{33}$$

Finally term $[Q_1 - x - b(y - Q_2)]s_1$ in region (d) contributes

$$\int_{Q_2}^{\infty}\int_{\overline{R_1}.Q_1-b(y-Q_2)}^{Q_1-b(y-Q_2)} [Q_1 - x - b(y-Q_2)]s_1dF(x)dG(y) \tag{34}$$

The expression for $EP_R(Q_1, Q_2)$ is then obtained after some algebra, by summing the terms in (23)–(34).

Proof of Proposition 3.2. For this special case $EP_T(Q_1, Q_2)$ reduces to

$$EP_T(Q_1,Q_2) = -p_1\int_0^{Q_1}F(x)dx + (p_2-c_2)Q_2 - p_2\int_0^{Q_2}G(y)F(Q_1+Q_2-y)dy + (p_1-c_1)Q_1$$

Using Leibniz's rule and setting the first partial derivatives to zero we have;

$$0 = p_1 - c_1 - p_1F(Q_1) - p_2\int_0^{Q_2}G(y)f(Q_1+Q_2-y)dy \tag{35}$$

$$0 = p_2 - c_2 - p_2G(Q_2)F(Q_1) - p_2\int_0^{Q_2}G(y)f(Q_1+Q_2-y)dy \tag{36}$$

From which we obtain;

$$1 - F(Q_1)G(Q_2) = \frac{c_2 + p_1 - c_1 - F(Q_1)p_1}{p_2}. \tag{37}$$

The partial derivatives of (3) set to zero result in:

$$0 = (s_1 - p_1)F(Q_1) + (p_1 - d_1) + (s_2 - p_2)\int_0^{Q_2}f(Q_1+Q_2-y)G(y)d(y) \tag{38}$$

$$0 = (s_2 - p_2)F(Q_1)G(Q_2) + (p_2 - d_2) + (s_2 - p_2)\int_0^{Q_2}f(Q_1+Q_2-y)G(y)d(y) \tag{39}$$

Solving (38) and (39), we get

$$[(s_1 - p_1)F(Q_1) + (p_1 - d_1) - (p_2 - d_2)]/(s_2 - p_2) = F(Q_1)G(Q_2) \tag{40}$$

Combining (40) and (41) we get the result.

Proof of Proposition 3.3. In this case, $EP_T(Q_1, Q_2)$, is given by:

$$EP_T(Q_1,Q_2) = -p_1\int_0^{Q_1}F(x)dx + (p_2-c_2)Q_2 - p_2\int_0^{Q_2}G(y)F(Q_1+Q_2-y)dy + (p_1-c_1)Q_1$$

From this expression we obtain the first order conditions as

$$0 = p_1 - c_1 - p_1 F(Q_1) - p_2 \int_0^{Q_2} G(y) f(Q_1 + Q_2 - y) dy \tag{41}$$

$$0 = p_2 - c_2 - p_2 G(Q_2) F(Q_1) - p_2 \int_0^{Q_2} G(y) f(Q_1 + Q_2 - y) dy \tag{42}$$

From (4) we get the first order conditions as

$$0 = p_1 - d_1 - p_1 F(Q_1) - p_2 \int_0^{Q_2} f(Q_1 + Q_2 - y) G(y) dy \tag{43}$$

$$0 = p_2 - d_2 - p_2 F(Q_1) G(Q_2) - p_2 \int_0^{Q_2} f(Q_1 + Q_2 - y) G(y) dy \tag{44}$$

Eqs. (41), (43), (42) and (44) imply that $c_1 = d_1, c_2 = d_2$ which is not feasible.

Proof of Proposition 3.4. This is a special case of case 2. Consider the expression given by (39) for the first order conditions of the retailer's profit. Letting $s_2 = d_2$, we get

$$0 = (p_2 - s_2)\left[1 - F(Q_1)G(Q_2)) - \int_0^{Q_2} f(Q_1 + Q_2 - y) G(y) dy\right] \tag{45}$$

Noting that

$$\int_0^{Q_2} f(Q_1 + Q_2 - y) G(y) dy = \int_{Q_1}^{Q_1+Q_2} G(Q_1 + Q_2 - u) dF(u)$$

(45) is written as

$$1 - F(Q_1)G(Q_2) = \int_{Q_1}^{Q_1+Q_2} G(Q_1 + Q_2 - u) dF(u) \leqslant \int_{Q_1}^{\infty} dF(u) = 1 - F(Q_1)$$

which is impossible unless $Q_1 = Q_2 = \infty$. Hence, the system is suboptimal.

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