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Polyhedral analysis for the two-item uncapacitated lot-sizing problem with one-way substitution

Hande Yaman

Bilkent University, Department of Industrial Engineering, Bilkent 06800 Ankara, Turkey

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ABSTRACT

We consider a production planning problem for two items where the high quality item can substitute the demand for the low quality item. Given the number of periods, the demands, the production, inventory holding, setup and substitution costs, the problem is to find a minimum cost production and substitution plan. This problem generalizes the well-known uncapacitated lot-sizing problem. We study the projection of the feasible set onto the space of production and setup variables and derive a family of facet defining inequalities for the associated convex hull. We prove that these inequalities together with the trivial facet defining inequalities describe the convex hull of the projection if the number of periods is two. We present the results of a computational study and discuss the quality of the bounds given by the linear programming relaxation of the model strengthened with these facet defining inequalities for larger number of periods.

1. Introduction

In this paper we consider the two-item uncapacitated lot-sizing problem with one-way substitution, that is the high quality item can substitute the demand for the low quality item. Given the planning horizon and the demands for the two items in each period, the aim of the problem is to propose a production and substitution plan which minimizes the production, inventory, setup and substitution costs over the planning horizon and meets the demand for the two items on time.

Most of the literature on product substitution is concerned with problems in a stochastic setting (see, e.g., [7,10,17,20,28,30,31,36,37]). The literature in the deterministic setting is quite limited. Balakrishnan and Geunes [5] model the production planning problem with substitutable components and an arbitrary substitution structure. They derive properties of optimal solutions and propose a dynamic programming algorithm. The proposed algorithm finds a shortest path in a graph with $O(n^m)$ nodes and $O(n^{m+1})$ arcs where n is the number of periods and m is the number of components. Hence the method's worst case running time is exponential in the general case. When applied to the two-item problem, the algorithm runs in polynomial time and hence our problem is polynomially solvable. Geunes [18] models the same problem as an *Uncapacitated Facility Location (UFL) Problem*, solves using the dual-ascent method of Erlenkotter [15] and presents computational results where the performance of the heuristic approach is tested in comparison to the exact shortest path approach.

Hsu, Li and Xiao [21] consider two versions of the production planning problem with one-way substitution. In the first version called *Substitution with Conversion (SWC)*, a lower-index product requires a physical transformation to be able to substitute the demand of a higher-index product. Once the product undergoes this transformation it may be stocked as a higher-index product if it is not used immediately. In the second version of the problem called *Substitution without Conversion*

E-mail address: hyaman@bilkent.edu.tr.

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(SWO), there is no need for a physical transformation and the lower-index product can substitute the demand for a higher-index product immediately but cannot be stocked as a higher-index product. The authors prove that the *SWO* is a special case of *SWC* and is strongly NP-hard. They propose dynamic programming algorithms as well as a heuristic.

Li, Chen and Cai [25,26] consider product substitution together with remanufacturing. In [25], the authors propose a dynamic programming approach by extending the one of [5] to handle remanufacturing. They also propose a heuristic algorithm and present computational results. In [26], a genetic algorithm is proposed for a capacitated version of the problem with batch processing.

We have not encountered any study on the polyhedral analysis of the production planning problem with substitution. In this paper we focus on the simplest case of the problem, that is, we consider only two items and one-way substitution. We call this problem the *Two-item Uncapacitated Lot-sizing Problem with One-way Substitution* and abbreviate by *2ULS*.

The problem we consider generalizes the well-known *Uncapacitated Lot-sizing Problem (ULS)*. If the demand for the low quality item is zero for all periods in the planning horizon, then *2ULS* reduces to *ULS*. There is a huge literature on the polyhedral properties of *ULS* and many of its variants (some examples are [2,4,3,6,12,13,19,23,24,27,29,32,34,38,39]). Even though *ULS* is a polynomially solvable problem, the knowledge on strong valid inequalities for the convex hull of its feasible solutions is useful in solving more complicated production planning problems for which *ULS* is a relaxation (see, e.g., [8,9,33,42]). Our motivation to study strong valid inequalities for *2ULS* is similar. This simple problem may arise as a substructure in more complicated problems and the results derived in this paper can be used in devising solution algorithms for these problems.

Before concluding this section, we review the results on *ULS* and then summarize our results in this paper for *2ULS* with emphasis on similarities and differences with the results for *ULS*.

There has been a lot of research on *ULS* and its variants since the seminal paper of Wagner and Whitin [41]. In [41], a dynamic programming algorithm that runs in $O(n^2)$ time with n being the number of periods is given. Later, more efficient implementations that run in $O(nlogn)$ time are proposed [1,16,40]. Barany et al. [6] give a description of the convex hull of feasible solutions using the so-called (l, S)-inequalities. Krarup and Bilde [22] give an extended formulation as a *UFL* and show that the linear programming (LP) relaxation of this formulation always has an optimal solution with integer setup variables. A shortest path extended formulation is given in Eppen and Martin [14]. For details, we refer the reader to Pochet and Wolsey [35].

In this paper, we first give a model of the *2ULS* that we refer to as *2ULS model with substitution variables*. Then we study the projection of the feasible set of the *2ULS model with substitution variables* onto the space of production and setup variables. This projection gives a valid formulation for the case where the substitution costs are zero and the inventory holding costs are equal for the two items. We refer to this formulation as the *2ULS model without substitution variables* and to the convex hull of its feasible solutions as the *2ULS polytope*.

We also develop a *UFL* model and project it onto the space of production and setup variables. We characterize the nondominated projection inequalities; these inequalities are generalizations of the (l, S)-inequalities. We provide necessary and sufficient conditions for these inequalities to be facet defining for the *2ULS polytope*. As in the case of *ULS*, the LP relaxation of the *2ULS model without substitution variables* strengthened with these (l, S)-like-inequalities gives the same lower bound as the LP relaxation of the *UFL* model. Unlike in the case of *ULS*, if the planning horizon is longer than two periods, these LP relaxations may not have optimal solutions with integral setup variables. In other words, the projection of the *UFL* model onto the space of production and setup variables is not necessarily the same as the *2ULS polytope* for three or more periods.

The rest of the paper is organized as follows. In Section 2, we present the two models, the *2ULS model with substitution variables* and the *2ULS model without substitution variables* and introduce a family of valid inequalities which generalize the (l, S)-inequalities. In Section 3, we investigate the dimension and trivial facets of the *2ULS polytope*, derive some properties of its facet defining inequalities, present a *UFL* model, project it onto the space of production and setup variables, and derive a class of facet defining inequalities. At the end of this section, we show that these inequalities together with trivial facet defining inequalities describe the *2ULS polytope* if the number of periods is two. We report the results of a computational experiment to see the quality of the lower bounds obtained by solving the LP relaxation of the *UFL* model in Section 4. We conclude with future research directions in Section 5.

2. The models and the (l_1, l_2, S^1, S^2)-inequalities

Let n be a positive integer and $T = \{1, \dots, n\}$. Let $\bar{p}_t^i$, h_t^i, q_t^i, and d_t^i denote the unit production cost, the unit inventory holding cost, the setup cost and the demand for item $i = 1, 2$ in period $t \in T$, respectively. The cost of substituting a unit demand of item 2 with item 1 in period $t \in T$ is denoted by $\bar{c}_t$. We assume that the costs are nonnegative and the starting and ending inventories are zero for both items. For $t_1, t_2 \in T$ and $i = 1, 2$, we define $D_{t_1 t_2}^i = \sum_{t=t_1}^{t_2} d_t^i$ if $t_1 \le t_2$ and $D_{t_1 t_2}^i = 0$ if $t_1 > t_2$.

To model the *2ULS*, we define the following decision variables. Let x_t^i be the amount of production of item $i = 1, 2$ in period $t \in T$, s_t^i be the amount of item $i = 1, 2$ in inventory at the end of period $t \in T \cup \{0\}$, and y_t^i be 1 if production of item $i = 1, 2$ takes place in period $t \in T$ and 0 otherwise. For $t \in T$, let a_t^{12} and a_t^{22} be the amounts of items 1 and 2 used to satisfy the demand of item 2 in period t, respectively.

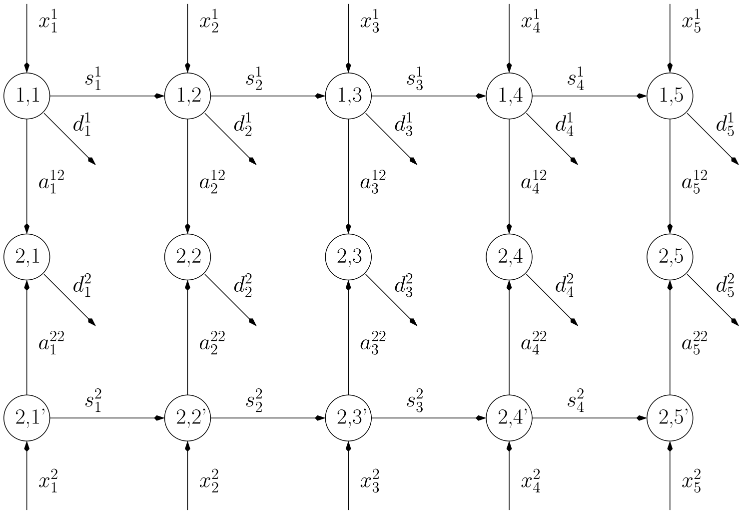

Fig. 1. 5 period example as a fixed charge network model.

Now the *2ULS model with substitution variables* is as follows:

$$z = \min \sum_{t=1}^{n} \left(\bar{p}_t^1 x_t^1 + \bar{p}_t^2 x_t^2 + h_t^1 s_t^1 + h_t^2 s_t^2 + q_t^1 y_t^1 + q_t^2 y_t^2 + \bar{c}_t a_t^{12} \right) \quad (1)$$

$$\text{s.t.} \quad x_t^1 + s_{t-1}^1 = a_t^{12} + d_t^1 + s_t^1 \quad \forall t \in T \quad (2)$$

$$x_t^2 + s_{t-1}^2 = a_t^{22} + s_t^2 \quad \forall t \in T \quad (3)$$

$$a_t^{12} + a_t^{22} = d_t^2 \quad \forall t \in T \quad (4)$$

$$x_t^1 \le (D_{tn}^1 + D_{tn}^2) y_t^1 \quad \forall t \in T \quad (5)$$

$$x_t^2 \le D_{tn}^2 y_t^2 \quad \forall t \in T \quad (6)$$

$$s_0^1 = s_0^2 = s_n^1 = s_n^2 = 0 \quad (7)$$

$$s_t^1, s_t^2, a_t^{22} \ge 0 \quad \forall t \in T \quad (8)$$

$$a_t^{12} \ge 0 \quad \forall t \in T \quad (9)$$

$$x_t^1, x_t^2 \ge 0 \quad \forall t \in T \quad (10)$$

$$y_t^1, y_t^2 \in \{0, 1\} \quad \forall t \in T. \quad (11)$$

Constraints (2) and (3) are balance equations for items 1 and 2 respectively. Constraints (2) imply that the amount of item 1 on hand at period t is used to satisfy the demands of items 1 and 2 and the remaining amount will be the inventory at the end of this period. Constraints (3) are similar, but item 2 can only be used to satisfy its own demand. Constraints (4) ensure that the demand of item 2 is satisfied on time using items 1 and 2. Due to constraints (5) and (6), if there is no setup for a given item in a given period, then this item cannot be produced in that period. Constraints (7) fix the values of initial and ending inventories to zero. Constraints (8)–(11) are nonnegativity and binary requirements. The objective function (1) is the total production, inventory holding, setup and substitution costs for the two items over the planning horizon.

The multi-item problem with an arbitrary substitution structure is modeled in [5]. The above model is a simplified version of their model for 2 items and one-way substitution. Balakrishnan and Geunes [5] view their problem as a generalized network flow problem with concave costs defined on a specific directed network and derive some properties of optimal solutions. In our case, as we have only one-way substitution, the corresponding network also simplifies. An example for 5 periods is depicted in Fig. 1. All production arcs originate at a given dummy node.

In this representation, for given setup vectors for the two items, the problem reduces to a minimum cost network flow problem. Thus any extreme point solution satisfies $s_{t-1}^i x_t^i = 0$ for all $t \in T$ and $i = 1, 2$ and $a_t^{12} a_t^{22} = 0$ for all $t \in T$. These properties are given for the general case by Balakrishnan and Geunes [5] under the names *Zero Inventory Production (ZIP) Property* and *Homogeneous Product Lots (HPL) Property*. As a result of these two properties, the authors conclude that the problem has an optimal solution where the demand of each item is satisfied from the most recent setup of the same item or an item that can substitute it. This is called the *Most Recent Usage (MRU) Property*.

Next, we present the *2ULS model without substitution variables*. Let $K = -\sum_{t=1}^{n}(h_t^1 D_{1t}^1 + h_t^2 D_{1t}^2)$, $p_t^1 = \overline{p}_t^1 + \sum_{l=t}^{n} h_l^1$, $p_t^2 = \overline{p}_t^2 + \sum_{l=t}^{n} h_l^2$ and $c_t = \sum_{l=t}^{n}(h_l^2 - h_l^1) + \overline{c}_t$ for $t \in T$.

Theorem 1. *If $c_t = 0$ for all $t \in T$, then the 2ULS can be formulated as:*

$$z = K + \min \sum_{t=1}^{n} \left(p_t^1 x_t^1 + p_t^2 x_t^2 + q_t^1 y_t^1 + q_t^2 y_t^2\right) \tag{12}$$

s.t. (5), (6), (10), (11)

$$\sum_{t=1}^{n} x_t^1 + \sum_{t=1}^{n} x_t^2 = D_{1n}^1 + D_{1n}^2 \tag{13}$$

$$\sum_{l=1}^{t} x_l^1 \geq D_{1t}^1 \quad \forall t \in T \tag{14}$$

$$\sum_{l=1}^{t_1} x_l^1 + \sum_{l=1}^{t_2} x_l^2 \geq D_{1t_1}^1 + D_{1t_2}^2 \quad \forall t_1, t_2 \in T : t_1 \geq t_2. \tag{15}$$

Proof. See Appendix A where we prove that the projection of the feasible set of the *2ULS model with substitution variables* onto the space of production and setup variables is given by (5), (6), (10), (11) and (13)–(15). □

To see the necessity of constraints (15) for $t_1 > t_2$, consider a three period problem and assume that $d_t^i = 1$ for $i = 1, 2$ and $t = 1, 2, 3$. The solution (x^1, x^2, y^1, y^2) where $x_1^1 = x_3^1 = x_2^2 = 2$, $x_2^1 = x_1^2 = x_3^2 = 0$, $y_1^1 = y_3^1 = y_2^2 = 1$ and $y_2^1 = y_1^2 = y_3^2 = 0$ satisfies constraints (5), (6), (10), (11), (13), (14), and constraints (15) for $t_1 = t_2$. But this solution is infeasible as there is no set up for item 1 in period 2 and no set up for item 2 in period 1, we need to have $x_1^1 \geq 3$. Now constraint (15) for $t_1 = 2$ and $t_2 = 1$ which reads $x_1^1 + x_2^1 + x_1^2 \geq 3$ eliminates this solution.

In this paper, we introduce a family of valid inequalities which generalize the (l, S)-inequalities for the *ULS* polytope. Let l_1 and l_2 in T be such that $l_1 \geq l_2$, $l_2 < n$, $A^1 = \{1, \ldots, l_1\}$, $A^2 = \{1, \ldots, l_2\}$, $S^1 \subseteq A^1$, $S^2 \subseteq A^2$ with $1 \in S^1$. The (l_1, l_2, S^1, S^2)-inequality

$$\sum_{t \in A^1 \setminus S^1} (D_{tl_1}^1 + D_{tl_2}^2) y_t^1 + \sum_{t \in A^2 \setminus S^2} D_{tl_2}^2 y_t^2 + \sum_{t \in S^1} x_t^1 + \sum_{t \in S^2} x_t^2 \geq D_{1l_1}^1 + D_{1l_2}^2. \tag{16}$$

is a valid inequality for the feasible sets of both the *2ULS model with substitution variables* and the *2ULS model without substitution variables*.

We present an example of the (l_1, l_2, S^1, S^2)-inequalities.

Example 1. Consider the following instance of *2ULS*. Let $n = 3$, $d_t^1 = d_t^2 = 1$, $p_t^1 = 2$, $p_t^2 = 1.5$, $h_t^1 = h_t^2 = 0.5$, and $q_t^1 = q_t^2 = 2$ for $t = 1, 2, 3$. When we solve the LP relaxation of the *2ULS model with substitution variables* together with the constraint $y_1^1 = 1$, we obtain the following optimal solution: $x_1^1 = 2$, $x_2^1 = 1$, $x_3^1 = 2$, $x_2^2 = 1$, $a_1^{12} = 1$, $a_3^{12} = 1$, $a_2^{22} = 1$, $y_1^1 = 1$, $y_2^1 = 0.25$, $y_3^1 = 1$, $y_2^2 = 0.5$ and other variables are zero.

Now if in a feasible solution, $y_2^1 = y_2^2 = 0$, then we need $x_1^1 + x_1^2 \geq 4$ to be able to satisfy the demand for the first two periods. Hence this inequality is valid when $y_2^1 = y_2^2 = 0$. Next, we lift this inequality with variables y_2^2 and y_2^1. Suppose that $y_2^2 = 1$. As the production of item 2 in period 2 cannot be used to satisfy the demand of item 2 in period 1 and as it is not possible to substitute item 2 for item 1, we still need $x_1^1 + x_1^2 \geq 3$. So inequality $x_1^1 + x_1^2 + y_2^2 \geq 4$ is a valid inequality when $y_2^1 = 0$. Finally, if $y_2^1 = 1$, we still need $x_1^1 + x_1^2 + y_2^2 \geq 2$. Hence, inequality $x_1^1 + x_1^2 + 2y_2^1 + y_2^2 \geq 4$ is a valid inequality. This is the (l_1, l_2, S^1, S^2)-inequality for $l_1 = l_2 = 2$, $A^1 = \{1, 2\}$, $A^2 = \{1, 2\}$, $S^1 = \{1\}$, and $S^2 = \{1\}$ and is violated by the optimal solution of the LP relaxation.

3. Polyhedral analysis

Let X be the set of vectors $(x^1, x^2, y^1, y^2) \in \mathbb{R}_+^{2n} \times \{0, 1\}^{2n}$ that satisfy constraints (5), (6) and (13)–(15) and $P = conv(X)$ be the *2ULS polytope*.

In what follows, we assume that $d_t^1 > 0$ and $d_t^2 > 0$ for all $t \in T$. This implies that $y_1^1 = 1$ for all $(x^1, x^2, y^1, y^2) \in X$.

Let e_t be the unit vector of size n with the tth entry equal to 1 and other entries equal to 0.

3.1. Dimension and trivial facets

All solutions (x^1, x^2, y^1, y^2) in X satisfy $\sum_{t \in T} x_t^1 + \sum_{t \in T} x_t^2 = D_{1n}^1 + D_{1n}^2$ and $y_1^1 = 1$. In the theorem below, we show that these are the only equalities satisfied by all solutions in X.

Theorem 2. $\dim(P) = 4n - 2$.

Proof. Suppose that all solutions (x^1, x^2, y^1, y^2) in P satisfy $\alpha^1 x^1 + \alpha^2 x^2 + \beta^1 y^1 + \beta^2 y^2 = \gamma$. Let $x^1 = (D^1_{1n} + D^2_{1n})e_1$, $x^2 = 0, y^1 = \sum_{t \in T} e_t, y^2 = \sum_{t \in T} e_t$. The solution (x^1, x^2, y^1, y^2) is in P. For $t \in T$, both solutions $(x^1 + \epsilon e_t - \epsilon e_1, x^2, y^1, y^2)$ and $(x^1 - \epsilon e_1, x^2 + \epsilon e_t, y^1, y^2)$ are in P for small enough $\epsilon > 0$. Hence $\alpha^1_t = \alpha^2_t = \alpha^1_1$ for all $t \in T$. For $t \in T \setminus \{1\}$, $(x^1, x^2, y^1 - e_t, y^2)$ is in P. So $\beta^1_t = 0$ for all $t \in T \setminus \{1\}$. For $t \in T$, $(x^1, x^2, y^1, y^2 - e_t)$ is in P. Thus $\beta^2_t = 0$ for all $t \in T$. So all solutions (x^1, x^2, y^1, y^2) in P satisfy $\sum_{j \in T} \alpha^1_1 x^1_j + \sum_{j \in T} \alpha^1_1 x^2_j + \beta^1_1 y^1_1 = \gamma$. As (x^1, x^2, y^1, y^2) is in P, we have $\gamma = \beta^1_1 + \alpha^1_1(D^1_{1n} + D^2_{1n})$. Then $\alpha^1 x^1 + \alpha^2 x^2 + \beta^1 y^1 + \beta^2 y^2 = \gamma$ is the sum of α^1_1 times $\sum_{t \in T} x^1_t + \sum_{t \in T} x^2_t = D^1_{1n} + D^2_{1n}$ and β^1_1 times $y^1_1 = 1$. Hence the rank of the equality matrix of P is equal to 2. As $P \subseteq \mathbb{R}^{4n}$, it follows that $\dim(P) = 4n - 2$. □

Next, we prove that some of the constraints of the *2ULS model without substitution variables* define facets of P.

Theorem 3. *The following inequalities define facets of P:*

$$x^1_t \ge 0 \quad \text{for } t \in T \setminus \{1\} \tag{17}$$

$$x^2_t \ge 0 \quad \text{for } t \in T \tag{18}$$

$$y^1_t \le 1 \quad \text{for } t \in T \setminus \{1\} \tag{19}$$

$$y^2_t \le 1 \quad \text{for } t \in T \tag{20}$$

$$x^1_t \le (D^1_{tn} + D^2_{tn}) y^1_t \quad \text{for } t \in T \setminus \{1\} \tag{21}$$

$$x^2_t \le D^2_{tn} y^2_t \quad \text{for } t \in T. \tag{22}$$

Proof. Let $t \in T \setminus \{1\}$ and define $F = \{(x^1, x^2, y^1, y^2) \in P : x^1_t = 0\}$. Suppose that all solutions in F also satisfy $\alpha^1 x^1 + \alpha^2 x^2 + \beta^1 y^1 + \beta^2 y^2 = \gamma$. The solution (x^1, x^2, y^1, y^2) where $x^1 = (D^1_{1n} + D^2_{1n})e_1$, $x^2 = 0$, $y^1 = \sum_{j \in T} e_j$, $y^2 = \sum_{j \in T} e_j$ is in F. For $j \in T \setminus \{t\}$, both solutions $(x^1 + \epsilon e_j - \epsilon e_1, x^2, y^1, y^2)$ and $(x^1 - \epsilon e_1, x^2 + \epsilon e_j, y^1, y^2)$ are in F for small enough $\epsilon > 0$. Also the solution $(x^1 - \epsilon e_1, x^2 + \epsilon e_t, y^1, y^2)$ is in F. Hence $\alpha^1_j = \alpha^1_1$ for all $j \in T \setminus \{t\}$ and $\alpha^2_j = \alpha^1_1$ for all $j \in T$. For $j \in T \setminus \{1\}$, as $(x^1, x^2, y^1 - e_j, y^2)$ is in F, we have $\beta^1_j = 0$. Similarly, for $j \in T$, as $(x^1, x^2, y^1, y^2 - e_j)$ is in F, $\beta^2_j = 0$. As (x^1, x^2, y^1, y^2) is in F, $\gamma = \alpha^1_1(D^1_{1n} + D^2_{1n}) + \beta^1_1$. Hence $\alpha^1 x^1 + \alpha^2 x^2 + \beta^1 y^1 + \beta^2 y^2 = \gamma$ is $\sum_{j \in T} \alpha^1_1 x^1_j + \sum_{j \in T} \alpha^1_1 x^2_j + (\alpha^1_t - \alpha^1_1)x^1_t + \beta^1_1 y^1_1 = \alpha^1_1(D^1_{1n} + D^2_{1n}) + \beta^1_1$. This is a weighted sum of $\sum_{j \in T} x^1_j + \sum_{j \in T} x^2_j = D^1_{1n} + D^2_{1n}$, $y^1_1 = 1$ and $x^1_t = 0$. The proof for inequalities (18)–(20) can be done in a similar way.

Let $t \in T \setminus \{1\}$, define $F = \{(x^1, x^2, y^1, y^2) \in P : x^1_t = (D^1_{tn} + D^2_{tn}) y^1_t\}$ and suppose that all solutions in F also satisfy $\alpha^1 x^1 + \alpha^2 x^2 + \beta^1 y^1 + \beta^2 y^2 = \gamma$. Consider the solution (x^1, x^2, y^1, y^2) where $x^1 = (D^1_{1n} + D^2_{1n})e_1$, $x^2 = 0$, $y^1 = \sum_{j \in T \setminus \{t\}} e_j$, $y^2 = \sum_{j \in T} e_j$. This solution is in F. Applying the same ideas as above, we can show that $\alpha^1_j = \alpha^1_1$ and $\beta^1_j = 0$ for all $j \in T \setminus \{1, t\}$, $\alpha^2_j = \alpha^1_1$ and $\beta^2_j = 0$ for all $j \in T$. Moreover as (x^1, x^2, y^1, y^2) is in F, we also have $\gamma = \alpha^1_1(D^1_{1n} + D^2_{1n}) + \beta^1_1$.

Now consider the solution $(x^1 - (D^1_{tn} + D^2_{tn})e_1 + (D^1_{tn} + D^2_{tn})e_t, x^2, y^1 + e_t, y^2)$. This solution is also in F and hence $(\alpha^1_t - \alpha^1_1)(D^1_{tn} + D^2_{tn}) + \beta^1_t = 0$. So the equation $\alpha^1 x^1 + \alpha^2 x^2 + \beta^1 y^1 + \beta^2 y^2 = \gamma$ is $\sum_{j \in T} \alpha^1_1 x^1_j + \sum_{j \in T} \alpha^1_1 x^2_j + (\alpha^1_t - \alpha^1_1)x^1_t + \beta^1_1 y^1_1 - (\alpha^1_t - \alpha^1_1)(D^1_{tn} + D^2_{tn}) y^1_t = \alpha^1_1(D^1_{1n} + D^2_{1n}) + \beta^1_1$ which is the sum of α^1_1 times $\sum_{j \in T} x^1_j + \sum_{j \in T} x^2_j = D^1_{1n} + D^2_{1n}$, β^1_1 times $y^1_1 = 1$ and $(\alpha^1_t - \alpha^1_1)$ times $x^1_t - (D^1_{tn} + D^2_{tn}) y^1_t = 0$. The proof for inequalities (22) can be done in a similar way. □

3.2. Properties of nontrivial facet defining inequalities of P

Now let inequality $\sum_{j \in T} \alpha^1_j x^1_j + \sum_{j \in T} \alpha^2_j x^2_j + \sum_{j \in T} \beta^1_j y^1_j + \sum_{j \in T} \beta^2_j y^2_j \ge \gamma$ be a facet defining inequality for P. Assume that the facet defined by this inequality is different from those defined by inequalities (17)–(22). Without loss of generality, we can assume that $\beta^1_1 = 0$, $\alpha^1_j \ge 0$ and $\alpha^2_j \ge 0$ for all $j \in T$, and $\prod_{j \in T} \alpha^1_j \alpha^2_j = 0$ (add $-\beta^1_1$ times $y^1_1 = 1$ and $-a$ times $\sum_{j \in T} x^1_j + \sum_{j \in T} x^2_j = D^1_{1n} + D^2_{1n}$ where $a = \min\{\min_{j \in T}\{\alpha^1_j\}, \min_{j \in T}\{\alpha^2_j\}\}$).

Next theorem presents some properties of the coefficients of variables in facet defining inequalities that are different from the trivial facet defining inequalities (17)–(22).

Theorem 4. *If inequality $\sum_{j \in T} \alpha^1_j x^1_j + \sum_{j \in T} \alpha^2_j x^2_j + \sum_{j \in T} \beta^1_j y^1_j + \sum_{j \in T} \beta^2_j y^2_j \ge \gamma$ is facet defining for P, the facet defined by this inequality is different from those defined by inequalities (17)–(22), $\beta^1_1 = 0$, $\alpha^1_j \ge 0$ and $\alpha^2_j \ge 0$ for all $j \in T$ and $\prod_{j \in T} \alpha^1_j \alpha^2_j = 0$, then*

i. *$\beta^1_j \ge 0$ and $\beta^2_j \ge 0$ for all $j \in T$,*
ii. *$\gamma > 0$ and $\alpha^1_1 > 0$,*
iii. *for $j \in T$, if $\beta^1_j = 0$ then $\alpha^1_j \ge \alpha^1_t$ for all $t \in T$ with $t > j$ and $\alpha^1_j \ge \alpha^2_t$ for all $t \in T$ with $t \ge j$,*

iv. *for $j \in T$, if $\beta_j^2 = 0$ then $\alpha_j^2 \geq \alpha_t^2$ for all $t \in T$ with $t > j$,*
v. *for $j \in T \setminus \{1\}$ and $t \in T$ such that $t < j$ and $\beta_t^1 = 0$, $(\alpha_t^1 - \alpha_j^1)(D_{jn}^1 + D_{jn}^2) \geq \beta_j^1$,*
vi. *for $j \in T \setminus \{1\}$, if there exists $t \in T$ such that $t < j$, $\beta_t^1 = 0$ and $\alpha_j^1 = \alpha_t^1$, then $\beta_j^1 = 0$,*
vii. *for $j \in T$ and $t \in T$ such that $t \leq j$ and $\beta_t^1 = 0$, $(\alpha_t^1 - \alpha_j^2)D_{jn}^2 \geq \beta_j^2$,*
viii. *for $j \in T$ and $t \in T$ such that $t < j$ and $\beta_t^2 = 0$, $(\alpha_t^2 - \alpha_j^2)D_{jn}^2 \geq \beta_j^2$,*
ix. *for $j \in T$, if there exists $t \in T$ such that $t \leq j$, $\beta_t^1 = 0$ and $\alpha_j^2 = \alpha_t^1$, or if there exists $t \in T$ such that $t < j$, $\beta_t^2 = 0$ and $\alpha_j^2 = \alpha_t^2$, then $\beta_j^2 = 0$,*
x. *for $j \in T \setminus \{1\}$, if $\beta_j^1 = 0$, then there exists $t \in T$ with $t < j$ such that $\alpha_j^1 \geq \alpha_t^1$,*
xi. *for $j \in T$, if $\beta_j^2 = 0$, then there exists $t \in T$ with $t < j$ such that $\alpha_j^2 \geq \min\{\alpha_t^1, \alpha_t^2\}$ or $\alpha_j^2 \geq \alpha_j^1$,*
xii. *$\sum_{j \in T \setminus \{1\}} \beta_j^1 + \sum_{j \in T} \beta_j^2 > 0$,*
xiii. *if $\alpha_n^1 = 0$ and $\beta_n^1 = 0$, then let t_1 be the smallest index in T with $\alpha_{t_1}^1 = 0$ and $\beta_{t_1}^1 = 0$ and t_2 be the smallest index in T with $\alpha_{t_2}^2 = 0$ and $\beta_{t_2}^2 = 0$. Then $t_2 \leq t_1$, $\alpha_j^1 = \beta_j^1 = 0$ for all $j \in T$ with $j \geq t_1$ and $\alpha_j^2 = \beta_j^2 = 0$ for all $j \in T$ with $j \geq t_2$,*
xiv. *if $\alpha_n^1 + \beta_n^1 > 0$, $\alpha_n^2 = 0$, and $\beta_n^2 = 0$, let t_2 be the smallest index in T with $\alpha_{t_2}^2 = 0$ and $\beta_{t_2}^2 = 0$. Then $\alpha_j^2 = \beta_j^2 = 0$ for all $j \in T$ with $j \geq t_2$,*
xv. *$\beta_n^2 = 0$.*

Proof. Let $F = \{(x^1, x^2, y^1, y^2) \in P : \sum_{j \in T} \alpha_j^1 x_j^1 + \sum_{j \in T} \alpha_j^2 x_j^2 + \sum_{j \in T} \beta_j^1 y_j^1 + \sum_{j \in T} \beta_j^2 y_j^2 = \gamma\}$. We prove the above items one by one.

i. For $j \in T \setminus \{1\}$, since F is different from the facet defined by $y_j^1 \leq 1$, there exists a solution (x^1, x^2, y^1, y^2) in F such that $y_j^1 < 1$. The solution $(x^1, x^2, y^1 + \epsilon e_j, y^2)$ is in P for small enough $\epsilon > 0$. Hence, $\beta_j^1 \geq 0$. Similarly, we can show that $\beta_j^2 \geq 0$ for all $j \in T$.
ii. In addition, as $\alpha_j^1 \geq 0$ and $\alpha_j^2 \geq 0$ for all $j \in T$, x^1, x^2, y^1 and y^2 are nonnegative and F is different from the facets defined by inequalities (17)–(22), we have $\gamma > 0$.
Assume $\alpha_1^1 = 0$. Then as $((D_{1n}^1 + D_{1n}^2)e_1, 0, e_1, 0)$ is in P, we have $0 \geq \gamma$. This contradicts $\gamma > 0$. So $\alpha_1^1 > 0$.
iii. Let $j \in T$ with $\beta_j^1 = 0$. For $t \in T$ with $t > j$, there exists a solution (x^1, x^2, y^1, y^2) in F such that $x_t^1 > 0$ since F is different from the facet defined by $x_t^1 \geq 0$. The solution $(x^1 + \epsilon(e_j - e_t), x^2, y^1 - (y_j^1 - 1)e_j, y^2)$ is in P for small enough $\epsilon > 0$. Hence $\alpha_j^1 \geq \alpha_t^1$. Similarly, we can prove that $\alpha_j^1 \geq \alpha_t^2$ for all $j \in T$ with $\beta_j^1 = 0$ and $t \in T$ with $t \geq j$.
iv. Similar to the proof of item iii.
v. Let $j \in T \setminus \{1\}$. There exists a solution (x^1, x^2, y^1, y^2) in F such that $x_j^1 < (D_{jn}^1 + D_{jn}^2)y_j^1$ since F is different from the facet defined by $x_j^1 \leq (D_{jn}^1 + D_{jn}^2)y_j^1$. Let t be in T such that $t < j$. The solution $(x^1 - x_j^1(e_j - e_t), x^2, y^1 + (1 - y_t^1)e_t - y_j^1 e_j, y^2)$ is in P. If $\beta_t^1 = 0$, then $(\alpha_t^1 - \alpha_j^1)x_j^1 \geq \beta_j^1 y_j^1$. As $\alpha_t^1 \geq \alpha_j^1$ and $x_j^1 < (D_{jn}^1 + D_{jn}^2)y_j^1$, we have $(\alpha_t^1 - \alpha_j^1)(D_{jn}^1 + D_{jn}^2) \geq \beta_j^1$.
vi. Let $j \in T \setminus \{1\}$. Suppose there exists $t \in T$ such that $t < j$, $\beta_t^1 = 0$ and $\alpha_j^1 = \alpha_t^1$. Then by (i) and (v), we have $\beta_j^1 = 0$.
vii. Let $j \in T$. There exists a solution $(x^1, x^2, y^1, y^2) \in F$ such that $x_j^2 < D_{jn}^2 y_j^2$ since F is different from the facet defined by $x_j^2 \leq D_{jn}^2 y_j^2$. Let t be in T is such that $t \leq j$ and $\beta_t^1 = 0$. As the solution $(x^1 + x_j^2 e_t, x^2 - x_j^2 e_j, y^1 + (1 - y_t^1)e_t, y^2 - y_j^2 e_j)$ is in P, we have $(\alpha_t^1 - \alpha_j^2)D_{jn}^2 > \beta_j^2$.
viii. Similar to the proof of item (vii).
ix. Let $j \in T$. If there exists $t \in T$ such that $t \leq j$, $\beta_t^1 = 0$ and $\alpha_j^2 = \alpha_t^1$, then by (i) and (vii), we have $\beta_j^2 = 0$. Similarly, if there exists $t \in T$ such that $t < j$, $\beta_t^2 = 0$ and $\alpha_j^2 = \alpha_t^2$, then by (i) and (viii) we have $\beta_j^2 = 0$.
x. For $j \in T \setminus \{1\}$ with $\beta_j^1 = 0$, there exists a solution $(x^1, x^2, y^1, y^2) \in F$ such that $y_j^1 = 0$. Then $\sum_{i=1}^{j-1} x_i^1 > D_{1j-1}^1$ and $\sum_{i=1}^{j-1} x_i^1 + \sum_{i=1}^{j-1} x_i^2 > D_{1j-1}^1 + D_{1j-1}^2$. Let $t \in T$ be the largest index with $t < j$ and $x_t^1 > 0$. Then $(x^1 + \epsilon(e_j - e_t), x^2, y^1 + e_j, y^2)$ is in P for small enough $\epsilon > 0$. Hence $\alpha_j^1 \geq \alpha_t^1$.
xi. For $j \in T$ with $\beta_j^2 = 0$, there exists a solution $(x^1, x^2, y^1, y^2) \in F$ such that $y_j^2 = 0$. Then $\sum_{i=1}^{j} x_i^1 + \sum_{i=1}^{j-1} x_i^2 > D_{1j}^1 + D_{1j-1}^2$. Let $t \in T$ be the largest index with $t \leq j$ and $x_t^1 > 0$ or $x_t^2 > 0$. Then either $(x^1 - \epsilon e_t, x^2 + \epsilon e_j, y^1, y^2 + e_j)$ or $(x^1, x^2 + \epsilon(e_j - e_t), y^1, y^2 + e_j)$ is in P for small enough $\epsilon > 0$. Hence $\alpha_j^2 \geq \min\{\alpha_t^1, \alpha_t^2\}$. If $t = j$, then $\alpha_j^2 \geq \alpha_j^1$.
xii. If $\beta_j^1 = \beta_j^2 = 0$ for all $j \in T$, then by item (iii), we have $\alpha_1^1 \geq \alpha_2^1 \geq \ldots \geq \alpha_n^1$. Item (x) for $j = 2$ implies that $\alpha_2^1 \geq \alpha_1^1$. Hence $\alpha_2^1 = \alpha_1^1$. Applying (x) for $j = 3$ yields $\alpha_3^1 = \alpha_2^1 = \alpha_1^1$. Repeating this argument, we obtain $\alpha_j^1 = a$ for all $j \in T$ and for some $a \in \mathbb{R}$.
Applying item (xi) for $j = 1$, we get $\alpha_1^2 \geq \alpha_1^1$. Together with $\alpha_1^2 \leq \alpha_1^1$ from item (iii), this gives $\alpha_1^2 = \alpha_1^1$. Now by item (iv), we have $\alpha_1^2 \geq \alpha_2^2 \geq \ldots \geq \alpha_n^2$. Item (xi) for $j = 2$ implies $\alpha_2^2 \geq \min\{\alpha_1^1, \alpha_1^2\}$ or $\alpha_2^2 \geq \alpha_2^1$. Both cases give $\alpha_2^2 \geq \alpha_1^1$. As $\alpha_2^2 \leq \alpha_2^1$ from item (iii) and $\alpha_2^1 = \alpha_1^1$, we obtain $\alpha_2^2 = \alpha_1^1$. Repeating this argument iteratively, we can show that $\alpha_j^2 = a$ for all $j \in T$.
Hence we conclude that if $\beta_j^1 = \beta_j^2 = 0$ for all $j \in T$, then $\alpha_j^1 = \alpha_j^2 = a$ for all $j \in T$ and for some $a \in \mathbb{R}$. But as $\alpha_1^1 > 0$ and $\prod_{j \in T} \alpha_j^1 \alpha_j^2 = 0$ this is not possible. So $\sum_{j \in T \setminus \{1\}} \beta_j^1 + \sum_{j \in T} \beta_j^2 > 0$.

xiii. Suppose that $\alpha_n^1 = 0$ and $\beta_n^1 = 0$. Let t_1 be the smallest index in T with $\alpha_{t_1}^1 = 0$ and $\beta_{t_1}^1 = 0$. Then (iii) implies that for $j \in T$ with $j > t_1$, $\alpha_j^1 \leq \alpha_{t_1}^1 = 0$. As $\alpha_j^1 \geq 0$, we have $\alpha_j^1 = 0$. Now as $\alpha_j^1 = \alpha_{t_1}^1$, by (vi), we have $\beta_j^1 = \beta_{t_1}^1 = 0$. Similarly, for $j \in T$ with $j \geq t_1$, (iii) implies that $\alpha_j^2 \leq \alpha_{t_1}^1 = 0$. Again as $\alpha_j^2 \geq 0$, we have $\alpha_j^2 = 0$. Then by (ix), we have $\beta_j^2 = \beta_{t_1}^1 = 0$.
Let t_2 be the smallest index in T with $\alpha_{t_2}^2 = 0$ and $\beta_{t_2}^2 = 0$. Then as $\alpha_j^2 = \beta_j^2 = 0$ for all $j \in T$ with $j \geq t_1$, we have $t_2 \leq t_1$. Let $j \in T$ with $j > t_2$. By (iv), we have $\alpha_j^2 = 0$. Then by (ix), we have $\beta_j^2 = 0$.

xiv. Similar to the proof of item (xiii).

xv. There exists a solution in $(x^1, x^2, y^1, y^2) \in F \cap X$ such that $x_n^2 < d_n^2 y_n^2$ since F is different from the facet defined by $x_n^2 \leq d_n^2 y_n^2$. Now as $(x^1, x^2, y^1, y^2) \in X$, we have $y_n^2 = 1$. If $x_n^2 = 0$, then as $(x^1, x^2, y^1, y^2 - e_n)$ is in P, we have that $\beta_n^2 \leq 0$. As $\beta_n^2 \geq 0$ by (i), we have $\beta_n^2 = 0$. Now suppose that $x_n^2 > 0$. Then there exists $t \in T$ such that both $(x^1 - \epsilon e_t, x^2 + \epsilon e_n, y^1, y^2)$ and $(x^1 + \epsilon e_t, x^2 - \epsilon e_n, y^1, y^2)$ are in P or there exists $t \in T \setminus \{n\}$ such that both $(x^1, x^2 + \epsilon(e_n - e_t), y^1, y^2)$ and $(x^1, x^2 - \epsilon(e_n - e_t), y^1, y^2)$ are in P. We will give the proof for the first case. So let $t \in T$ be such that both $(x^1 - \epsilon e_t, x^2 + \epsilon e_n, y^1, y^2)$ and $(x^1 + \epsilon e_t, x^2 - \epsilon e_n, y^1, y^2)$ are in P. Then $\alpha_n^2 = \alpha_t^1$. Now consider the solution $(x^1 + x_n^2 e_t, x^2 - x_n^2 e_n, y^1, y^2 - e_n)$. This solution is also in P since (x^1, x^2, y^1, y^2) is in X and so $y_t^1 = 1$ and $x_t^1 + x_n^2 \leq D_{tn}^1 + D_{tn}^2$. As $\alpha_n^2 = \alpha_t^1$, $\beta_n^2 \leq 0$. Together with (i), this implies $\beta_n^2 = 0$. The proof for the second case is similar. □

By item (xii) of Theorem 4, we know that $\sum_{j \in T \setminus \{1\}} \beta_j^1 + \sum_{j \in T} \beta_j^2 > 0$ in any facet defining inequality that satisfies the assumptions of the theorem. This leads to the following corollary.

Corollary 1. *If inequality $\sum_{j \in T} \alpha_j^1 x_j^1 + \sum_{j \in T} \alpha_j^2 x_j^2 \geq \alpha_0$ defines a face of P different from those defined by $x_t^1 \geq 0$ for some $t \in T \setminus \{1\}$ and $x_t^2 \geq 0$ for some $t \in T$, then the inequality is not facet defining for P.*

Item (ii) of Theorem 4 states that $\alpha_1^1 > 0$ in any facet defining inequality that satisfies the assumptions of the theorem. Since $\prod_{j \in T} \alpha_j^1 \alpha_j^2 = 0$ is among the assumptions of the theorem, we have the following corollary.

Corollary 2. *If inequality $\sum_{j \in T} \beta_j^1 y_j^1 + \sum_{j \in T} \beta_j^2 y_j^2 \geq \beta_0$ defines a face of P different from those defined by $y_t^1 \leq 1$ for some $t \in T \setminus \{1\}$ or $y_t^2 \leq 1$ for some $t \in T$, then the inequality is not facet defining for P.*

These two corollaries imply that nontrivial facet defining inequalities of P involve both the production and setup variables. In the following section, we derive such a family of facet defining inequalities.

3.3. The UFL formulation and its projection onto the space of the production and setup variables

In this section, we derive the (l_1, l_2, S^1, S^2)-inequalities using the projection of the *UFL* formulation onto the space of production and setup variables.

Geunes [18] gives a *UFL* formulation for the multi-item problem with an arbitrary substitution structure. The proposed model in [18] is the so-called aggregate or weak model whereas here we focus on the strong *UFL* model.

For u and t in T such that $u \leq t$, define v_{ut}^1 and v_{ut}^2 to be the amount of production of items 1 and 2 in period u to satisfy their own demands in period t, respectively. Define also v_{ut}^{12} to be the amount of production of item 1 in period u to satisfy the demand of item 2 in period t.

The *UFL* formulation for the *2ULS* with $c_t = 0$, $p_t^1 \geq 0$ and $p_t^2 \geq 0$ for all $t \in T$ is as follows.

$$z = K + \min \sum_{t=1}^{n} \left(p_t^1 x_t^1 + p_t^2 x_t^2 + q_t^1 y_t^1 + q_t^2 y_t^2\right) \tag{23}$$

s.t. (5), (6), (10), (11), (13)

$$\sum_{u=1}^{t} v_{ut}^1 = d_t^1 \quad \forall t \in T \tag{24}$$

$$\sum_{u=1}^{t} v_{ut}^{12} + \sum_{u=1}^{t} v_{ut}^2 = d_t^2 \quad \forall t \in T \tag{25}$$

$$v_{ut}^1 \leq d_t^1 y_u^1 \quad \forall u, t \in T : u \leq t \tag{26}$$

$$v_{ut}^{12} \leq d_t^2 y_u^1 \quad \forall u, t \in T : u \leq t \tag{27}$$

$$v_{ut}^2 \leq d_t^2 y_u^2 \quad \forall u, t \in T : u \leq t \tag{28}$$

$$\sum_{t=u}^{n} (v_{ut}^1 + v_{ut}^{12}) \leq x_u^1 \quad \forall u \in T \tag{29}$$

$$\sum_{t=u}^{n} v_{ut}^2 \leq x_u^2 \quad \forall u \in T \tag{30}$$

$$v_{ut}^1, v_{ut}^2, v_{ut}^{12} \geq 0 \quad \forall u, t \in T : u \leq t. \tag{31}$$

Constraints (24) and (25) ensure that the demands of items 1 and 2 are satisfied on time. Constraints (26)–(28) imply that if a setup for an item does not take place in a given period, then the demand of later periods cannot be satisfied from production of that item in this period. Constraints (29) and (30) compute the amounts of production of items 1 and 2 in terms of v_{ut}^1, v_{ut}^2 and v_{ut}^{12} variables. Here the use of inequality constraints rather than equations does not change the optimal value since $p_t^1 \geq 0$ and $p_t^2 \geq 0$ for all $t \in T$. Finally, constraints (31) are nonnegativity constraints.

Now we project the feasible set of the LP relaxation of the above *UFL* formulation onto the space of production and setup variables.

Theorem 5. *The projection of the feasible set of the LP relaxation of the UFL formulation onto the space of production and setup variables is given by inequalities (5), (6), (10), (13), $y_t^1 \leq 1$ and $y_t^2 \leq 1$ for all $t \in T$, and the (l_1, l_2, S^1, S^2)-inequalities (16) for all l_1 and l_2 in T such that $l_1 \geq l_2$, $l_2 < n$, $A^1 = \{1, \dots, l_1\}$, $A^2 = \{1, \dots, l_2\}$, $S^1 \subseteq A^1$, $S^2 \subseteq A^2$ with $1 \in S^1$.*

Proof. For given values of x_t^1, x_t^2, y_t^1, and y_t^2 for $t \in T$ that satisfy (5), (6), (10), (13), and $y_t^1 \leq 1$ and $y_t^2 \leq 1$ for all $t \in T$, there exists an assignment of values v_{ut}^1, v_{ut}^{12} and v_{ut}^2 for all $u, t \in T$ such that $u \leq t$ satisfying (24)–(31) if and only if

$$\sum_{t=1}^{n}\sum_{u=1}^{t}(d_t^1 y_u^1 \beta_{ut}^1 + d_t^2 y_u^1 \beta_{ut}^{12} + d_t^2 y_u^2 \beta_{ut}^2) + \sum_{t=1}^{n}(x_t^1\sigma_t^1 + x_t^2\sigma_t^2) \geq \sum_{t=1}^{n}(d_t^1\alpha_t^1 + d_t^2\alpha_t^2)$$

for all $(\alpha, \beta, \sigma) \geq 0$ such that

$$\beta_{ut}^1 \geq \alpha_t^1 - \sigma_u^1 \quad \forall u, t \in T : u \leq t \tag{32}$$

$$\beta_{ut}^{12} \geq \alpha_t^2 - \sigma_u^1 \quad \forall u, t \in T : u \leq t \tag{33}$$

$$\beta_{ut}^2 \geq \alpha_t^2 - \sigma_u^2 \quad \forall u, t \in T : u \leq t. \tag{34}$$

Note here that we limited our attention to nonnegative α, since $\sigma \geq 0$, $\beta \geq 0$ and $d_t^i \geq 0$ for all $i = 1, 2$ and $t \in T$.

Let $C = \{(\alpha, \beta, \sigma) \geq 0 : (32)\text{–}(34)\}$. Define $B = \{(u, t) : u, t \in T, u \leq t\}$. For a given $(\alpha, \beta, \sigma) \in C$, define $A^1 = \{t \in T : \alpha_t^1 > 0\}$, $A^2 = \{t \in T : \alpha_t^2 > 0\}$, $S^1 = \{t \in T : \sigma_t^1 > 0\}$, $S^2 = \{t \in T : \sigma_t^2 > 0\}$, $B^1 = \{(u, t) \in B : \beta_{ut}^1 > 0\}$, $B^2 = \{(u, t) \in B : \beta_{ut}^2 > 0\}$, and $B^{12} = \{(u, t) \in B : \beta_{ut}^{12} > 0\}$. Define also t_a^1, t_a^2, t_s^1, and t_s^2 to be the largest indices in A^1, A^2, S^1, and S^2, respectively.

Next, we investigate the extreme rays of the projection cone C. If (α, β, σ) is an extreme ray of C and $|B^1 \cup B^{12} \cup B^2 \cup A^1 \cup A^2 \cup S^1 \cup S^2| = 1$, then $A^1 \cup A^2 = \emptyset$. These extreme rays give the redundant inequalities $y_t^1 \geq 0$ and $y_t^2 \geq 0$ for $t \in T$ and $x_1^1 \geq 0$ and the facet defining inequalities $x_t^1 \geq 0$ for $t \in T \setminus \{1\}$ and $x_t^2 \geq 0$ for $t \in T$. Now, we study the remaining extreme rays in the following lemma.

Lemma 1. *If (α, β, σ) is an extreme ray of C, $|B^1 \cup B^{12} \cup B^2 \cup A^1 \cup A^2 \cup S^1 \cup S^2| \geq 2$, then $\alpha_t^1 = \rho$ for $t \in A^1$, $\alpha_t^2 = \rho$ for $t \in A^2$, $\sigma_t^1 = \rho$ for $t \in S^1$, $\sigma_t^2 = \rho$ for $t \in S^2$, $\beta_{ut}^1 = \rho$ for $(u, t) \in B^1$, $\beta_{ut}^2 = \rho$ for $(u, t) \in B^2$, and $\beta_{ut}^{12} = \rho$ for $(u, t) \in B^{12}$ for some $\rho > 0$. Moreover $\beta_{ut}^1 = (\alpha_t^1 - \sigma_u^1)^+$, $\beta_{ut}^{12} = (\alpha_t^2 - \sigma_u^1)^+$, and $\beta_{ut}^2 = (\alpha_t^2 - \sigma_u^2)^+$ for all $(u, t) \in B$, $t_s^1 \leq \max\{t_a^1, t_a^2\}$ and $t_s^2 \leq t_a^2$.*

Proof. Let $(\alpha, \beta, \sigma) \in C$ be such that $|B^1 \cup B^{12} \cup B^2 \cup A^1 \cup A^2 \cup S^1 \cup S^2| \geq 2$. Then $(\alpha, \beta, \sigma) = 1/2(\overline{\alpha}, \overline{\beta}, \overline{\sigma}) + 1/2(\underline{\alpha}, \underline{\beta}, \underline{\sigma})$ where $\overline{\alpha}_t^1 = \alpha_t^1 - \epsilon$ and $\underline{\alpha}_t^1 = \alpha_t^1 + \epsilon$ for $t \in A^1$, $\overline{\alpha}_t^1 = \underline{\alpha}_t^1 = \alpha_t^1$ for $t \in T \setminus A^1$, $\overline{\alpha}_t^2 = \alpha_t^2 - \epsilon$ and $\underline{\alpha}_t^2 = \alpha_t^2 + \epsilon$ for $t \in A^2$, $\overline{\alpha}_t^2 = \underline{\alpha}_t^2 = \alpha_t^2$ for $t \in T \setminus A^2$, $\overline{\sigma}_t^1 = \sigma_t^1 - \epsilon$ and $\underline{\sigma}_t^1 = \sigma_t^1 + \epsilon$ for $t \in S^1$, $\overline{\sigma}_t^1 = \underline{\sigma}_t^1 = \sigma_t^1$ for $t \in T \setminus S^1$, $\overline{\sigma}_t^2 = \sigma_t^2 - \epsilon$ and $\underline{\sigma}_t^2 = \sigma_t^2 + \epsilon$ for $t \in S^2$, $\overline{\sigma}_t^2 = \underline{\sigma}_t^2 = \sigma_t^2$ for $t \in T \setminus S^2$, $\overline{\beta}_{ut}^1 = \beta_{ut}^1 - \epsilon$ and $\underline{\beta}_{ut}^1 = \beta_{ut}^1 + \epsilon$ if $(u, t) \in B^1$ and $\sigma_u^1 = 0$, $\overline{\beta}_{ut}^1 = \underline{\beta}_{ut}^1 = \beta_{ut}^1$ if $(u, t) \in B^1$ and $\sigma_u^1 > 0$ or $(u, t) \in B \setminus B^1$, $\overline{\beta}_{ut}^2 = \beta_{ut}^2 - \epsilon$ and $\underline{\beta}_{ut}^2 = \beta_{ut}^2 + \epsilon$ if $(u, t) \in B^2$ and $\sigma_u^2 = 0$, $\overline{\beta}_{ut}^2 = \underline{\beta}_{ut}^2 = \beta_{ut}^2$ if $(u, t) \in B^2$ and $\sigma_u^2 > 0$ or $(u, t) \in B \setminus B^2$, $\overline{\beta}_{ut}^{12} = \beta_{ut}^{12} - \epsilon$ and $\underline{\beta}_{ut}^{12} = \beta_{ut}^{12} + \epsilon$ if $(u, t) \in B^{12}$ and $\sigma_u^1 = 0$, $\overline{\beta}_{ut}^{12} = \underline{\beta}_{ut}^{12} = \beta_{ut}^{12}$ if $(u, t) \in B^{12}$ and $\sigma_u^1 > 0$ or $(u, t) \in B \setminus B^{12}$. Besides, both $(\overline{\alpha}, \overline{\beta}, \overline{\sigma})$ and $(\underline{\alpha}, \underline{\beta}, \underline{\sigma})$ are in C for some small enough $\epsilon > 0$. Hence if (α, β, σ) is an extreme ray of C, then all its positive entries should be equal. The second part is easy to prove. $\square$

These extreme rays give the following inequalities:

$$\sum_{t \in T \setminus S^1}\left(\sum_{j \in A^1 : j \geq t} d_j^1 + \sum_{j \in A^2 : j \geq t} d_j^2\right) y_t^1 + \sum_{t \in T \setminus S^2}\sum_{j \in A^2 : j \geq t} d_j^2 y_t^2 + \sum_{t \in S^1} x_t^1 + \sum_{t \in S^2} x_t^2 \geq \sum_{t \in A^1} d_t^1 + \sum_{t \in A^2} d_t^2. \tag{35}$$

Some of the inequalities (35) are redundant as they are implied by the constraints in the formulation and some are dominated by other inequalities (35). The following lemmas derive conditions under which inequalities (35) are dominated.

Lemma 2. *If* $1 \notin S^1$, *then inequality* (35) *is redundant.*

Proof. If $1 \notin S^1$, then y_1^1 appears with coefficient $\sum_{j\in A^1:j\geq 1} d_j^1 + \sum_{j\in A^2:j\geq 1} d_j^2$ which is equal to $\sum_{j\in A^1} d_j^1 + \sum_{j\in A^2} d_j^2$. As y_1^1 is always 1, its coefficient is equal to the right-hand side of the inequality and all other variables have nonnegative coefficients, the inequality is redundant. □

Lemma 3. *If* $A^1 \subset T$ *and* $\min_{i\in T\setminus A^1} i < t_a^1$, *then inequality* (35) *is dominated by other inequalities* (35).

Proof. Suppose that $A^1 \subset T$. Let $l = \min_{i\in T\setminus A^1} i$. If $l < t_a^1$, then consider inequalities (35) for $A^1 \cup \{l\}$ and $A^1 \setminus \{t_a^1\}$ and for the same choices of S^1, A^2 and S^2. The inequality (35) for $A^1 \cup \{l\}$ is

$$\sum_{t\in T\setminus S^1}\left(\sum_{j\in A^1\cup\{l\}:j\geq t} d_j^1 + \sum_{j\in A^2:j\geq t} d_j^2\right) y_t^1 + \sum_{t\in T\setminus S^2}\sum_{j\in A^2:j\geq t} d_j^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2 \geq \sum_{t\in A^1\cup\{l\}} d_t^1 + \sum_{t\in A^2} d_t^2$$

which is the same as

$$\sum_{t\in T\setminus S^1}\left(\sum_{j\in A^1:j\geq t} d_j^1 + \sum_{j\in A^2:j\geq t} d_j^2\right) y_t^1 + \sum_{t\in T\setminus S^2}\sum_{j\in A^2:j\geq t} d_j^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2$$
$$\geq \sum_{t\in A^1} d_t^1 + d_l^1\left(1 - \sum_{t\in T\setminus S^1:l\geq t} y_t^1\right) + \sum_{t\in A^2} d_t^2. \tag{36}$$

The inequality (35) for $A^1 \setminus \{t_a^1\}$ is

$$\sum_{t\in T\setminus S^1}\left(\sum_{j\in A^1\setminus\{t_a^1\}:j\geq t} d_j^1 + \sum_{j\in A^2:j\geq t} d_j^2\right) y_t^1 + \sum_{t\in T\setminus S^2}\sum_{j\in A^2:j\geq t} d_j^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2 \geq \sum_{t\in A^1\setminus\{t_a^1\}} d_t^1 + \sum_{t\in A^2} d_t^2$$

and is the same as

$$\sum_{t\in T\setminus S^1}\left(\sum_{j\in A^1:j\geq t} d_j^1 + \sum_{j\in A^2:j\geq t} d_j^2\right) y_t^1 + \sum_{t\in T\setminus S^2}\sum_{j\in A^2:j\geq t} d_j^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2$$
$$\geq \sum_{t\in A^1} d_t^1 - d_{t_a^1}^1\left(1 - \sum_{t\in T\setminus S^1:t_a^1\geq t} y_t^1\right) + \sum_{t\in A^2} d_t^2. \tag{37}$$

Adding (36) and $\frac{d_l^1}{d_{t_a^1}^1}$ times (37) and dividing by $1 + \frac{d_l^1}{d_{t_a^1}^1}$ yield:

$$\sum_{t\in T\setminus S^1}\left(\sum_{j\in A^1:j\geq t} d_j^1 + \sum_{j\in A^2:j\geq t} d_j^2\right) y_t^1 + \sum_{t\in T\setminus S^2}\sum_{j\in A^2:j\geq t} d_j^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2$$
$$\geq \sum_{t\in A^1} d_t^1 + \sum_{t\in A^2} d_t^2 + \frac{d_{t_a^1}^1}{d_l^1 + d_{t_a^1}^1} d_l^1\left(1 - \sum_{t\in T\setminus S^1:l\geq t} y_t^1\right) - \frac{d_{t_a^1}^1}{d_l^1 + d_{t_a^1}^1} d_l^1\left(1 - \sum_{t\in T\setminus S^1:t_a^1\geq t} y_t^1\right)$$
$$= \sum_{t\in A^1} d_t^1 + \sum_{t\in A^2} d_t^2 + \frac{d_{t_a^1}^1 d_l^1}{d_l^1 + d_{t_a^1}^1}\left(-\sum_{t\in T\setminus S^1:l\geq t} y_t^1 + \sum_{t\in T\setminus S^1:t_a^1\geq t} y_t^1\right).$$

Now as $t_a^1 > l$, we have $\sum_{t\in T\setminus S^1:t_a^1\geq t} y_t^1 \geq \sum_{t\in T\setminus S^1:l\geq t} y_t^1$. Hence inequality (35) for A^1 is dominated. □

Lemma 4. *If* $A^2 \subset T$ *and* $\min_{i\in T\setminus A^2} i < t_a^2$, *then inequality* (35) *is dominated by other inequalities* (35).

Proof. Suppose that $A^2 \subset T$, $l = \min_{i\in T\setminus A^2} i$ and $l < t_a^2$. The inequalities (35) for $A^2 \cup \{l\}$ and $A^2 \setminus \{t_a^1\}$ are

$$\sum_{t\in T\setminus S^1}\left(\sum_{j\in A^1:j\ge t} d_j^1 + \sum_{j\in A^2:j\ge t} d_j^2\right) y_t^1 + \sum_{t\in T\setminus S^2}\sum_{j\in A^2:j\ge t} d_j^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2$$
$$\ge \sum_{t\in A^1} d_t^1 + \sum_{t\in A^2} d_t^2 + d_l^2\left(1 - \sum_{t\in T\setminus S^2:l\ge t} y_t^2 - \sum_{t\in T\setminus S^1:l\ge t} y_t^1\right) \quad (38)$$

and

$$\sum_{t\in T\setminus S^1}\left(\sum_{j\in A^1:j\ge t} d_j^1 + \sum_{j\in A^2:j\ge t} d_j^2\right) y_t^1 + \sum_{t\in T\setminus S^2}\sum_{j\in A^2:j\ge t} d_j^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2$$
$$\ge \sum_{t\in A^1} d_t^1 + \sum_{t\in A^2} d_t^2 - d_{t_a^2}^2\left(1 - \sum_{t\in T\setminus S^2:t_a^2\ge t} y_t^2 - \sum_{t\in T\setminus S^1:t_a^2\ge t} y_t^1\right), \quad (39)$$

respectively. Adding (38) and $\frac{d_l^2}{d_{t_a^2}^2}$ times (39) and dividing by $1 + \frac{d_l^2}{d_{t_a^2}^2}$ yield:

$$\sum_{t\in T\setminus S^1}\left(\sum_{j\in A^1:j\ge t} d_j^1 + \sum_{j\in A^2:j\ge t} d_j^2\right) y_t^1 + \sum_{t\in T\setminus S^2}\sum_{j\in A^2:j\ge t} d_j^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2$$
$$\ge \sum_{t\in A^1} d_t^1 + \sum_{t\in A^2} d_t^2 + \frac{d_{t_a^2}^2 d_l^2}{d_l^2 + d_{t_a^2}^2}\left(-\sum_{t\in T\setminus S^2:l\ge t} y_t^2 - \sum_{t\in T\setminus S^1:l\ge t} y_t^1 + \sum_{t\in T\setminus S^2:t_a^2\ge t} y_t^2 + \sum_{t\in T\setminus S^1:t_a^2\ge t} y_t^1\right).$$

As $\sum_{t\in T\setminus S^2:t_a^2\ge t} y_t^2 + \sum_{t\in T\setminus S^1:t_a^2\ge t} y_t^1 \ge \sum_{t\in T\setminus S^2:l\ge t} y_t^2 + \sum_{t\in T\setminus S^1:l\ge t} y_t^1$, inequality (35) for A^2 is dominated. □

Hence we are interested in inequalities (35) for choices of A^1, A^2, S^1, and S^2 such that for l_1 and l_2 in $T \cup \{0\}$, A^1 and A^2 are the sets of the first l_1 and l_2 elements of T, $S^1 \subseteq A^1 \cup A^2$ with $1 \in S^1$, and $S^2 \subseteq A^2$. Inequality (35) becomes

$$\sum_{t\in T\setminus S^1}(D_{tl_1}^1 + D_{tl_2}^2) y_t^1 + \sum_{t\in T\setminus S^2} D_{tl_2}^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2 \ge D_{1l_1}^1 + D_{1l_2}^2. \quad (40)$$

Lemma 5. *If $l_1 < l_2$, then inequality (40) is dominated by other inequalities (40).*

Proof. Let $A^1 = A^2 = \{1, \ldots, l_2\}$. Inequality (40) is

$$\sum_{t\in T\setminus S^1}(D_{tl_1}^1 + D_{tl_2}^2) y_t^1 + \sum_{t\in T\setminus S^2} D_{tl_2}^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2 \ge D_{1l_1}^1 + D_{1l_2}^2 + \sum_{l_1<j\le l_2} d_j^1\left(1 - \sum_{t\in T\setminus S^1} y_t^1\right).$$

Let $A^1 = A^2 = \{1, \ldots, l_1\}$. Inequality (40) is

$$\sum_{t\in T\setminus S^1}(D_{tl_1}^1 + D_{tl_2}^2) y_t^1 + \sum_{t\in T\setminus S^2} D_{tl_2}^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2 \ge D_{1l_1}^1 + D_{1l_2}^2 - \sum_{l_1<j\le l_2} d_j^1\left(1 - \sum_{t\in T\setminus S^1} y_t^1 - \sum_{t\in T\setminus S^2} y_t^2\right).$$

Summing up these two inequalities and dividing by 2 yields:

$$\sum_{t\in T\setminus S^1}(D_{tl_1}^1 + D_{tl_2}^2) y_t^1 + \sum_{t\in T\setminus S^2} D_{tl_2}^2 y_t^2 + \sum_{t\in S^1} x_t^1 + \sum_{t\in S^2} x_t^2 \ge D_{1l_1}^1 + D_{1l_2}^2 + 1/2\sum_{l_1<j\le l_2} d_j^1 \sum_{t\in T\setminus S^2} y_t^2.$$

This inequality dominates inequality (40) for $A^1 = \{1, \ldots, l_1\}$ and $A^2 = \{1, \ldots, l_2\}$ with $l_1 < l_2$. □

Lemma 6. *If $l_2 = 0$ and $1 \in S^1$, then inequality (40) is dominated by other inequalities (40).*

Proof. If $l_2 = 0$, inequality (40) simplifies to

$$\sum_{t\in T\setminus S^1} D^1_{tl_1} y^1_t + \sum_{t\in S^1} x^1_t \geq D^1_{1l_1}. \tag{41}$$

Now consider inequality (40) for $l_2 = 1$ and $S^2 = \emptyset$.

$$\sum_{t\in T\setminus S^1} D^1_{tl_1} y^1_t + \sum_{t\in S^1} x^1_t \geq D^1_{1l_1} + d^2_1(1 - y^2_1). \tag{42}$$

As $y^2_1 \leq 1$, inequality (42) dominates (41). □

Lemma 7. *If $l_1 = l_2 = n$, then inequality (40) is redundant.*

Proof. Let $l_1 = l_2 = n$. Inequality (40) reads

$$\sum_{t\in T\setminus S^1} (D^1_{tn} + D^2_{tn}) y^1_t + \sum_{t\in T\setminus S^2} D^2_{tn} y^2_t + \sum_{t\in S^1} x^1_t + \sum_{t\in S^2} x^2_t \geq D^1_{1n} + D^2_{1n}.$$

This inequality is the sum of $\sum_{t\in T} x^1_t + \sum_{t\in T} x^2_t = D^1_{1n} + D^2_{1n}$, $(D^1_{tn} + D^2_{tn})y^1_t - x^1_t \geq 0$ for $t \in T \setminus S^1$ and $D^2_{tn} y^2_t - x^2_t \geq 0$ for $t \in T \setminus S^2$. □

Now we can conclude that for given values of x^1_t, x^2_t, y^1_t, and y^2_t for $t \in T$ that satisfy (5), (6), (10), (13), and $y^1_t \leq 1$ and $y^2_t \leq 1$ for all $t \in T$, there exists an assignment of values v^1_{ut}, v^{12}_{ut} and v^2_{ut} for all $u, t \in T$ such that $u \leq t$ satisfying (24)–(31) if and only if inequalities (40) are satisfied for all l_1 and l_2 in T such that $l_1 \geq l_2, l_2 < n, A^1 = \{1, \ldots, l_1\}, A^2 = \{1, \ldots, l_2\}$, $S^1 \subseteq A^1, S^2 \subseteq A^2$ with $1 \in S^1$. Note here that under these conditions, as $D^1_{tl_1} + D^2_{tl_2} = 0$ for all $t \in T \setminus A_1$ and $D^2_{tl_2} = 0$ for all $t \in T \setminus A_2$, inequality (40) is the same as the (l_1, l_2, S^1, S^2)-inequality (16). □ (end of the proof of Theorem 5).

The next theorem gives necessary and sufficient conditions for the (l_1, l_2, S^1, S^2)-inequalities to be facet defining.

Theorem 6. *Let l_1 and l_2 be in T with $l_1 \geq l_2$ and $l_2 < n$. Let $A^1 = \{1, \ldots, l_1\}$, $A^2 = \{1, \ldots, l_2\}$, $S^1 \subseteq A^1$, and $S^2 \subseteq A^2$ with $1 \in S^1$. The (l_1, l_2, S^1, S^2)-inequality (16) is valid for P. The inequality is facet defining for P if and only if at least one of the following conditions is satisfied:*

i. *$A^1 \setminus S^1 \neq \emptyset$ and $l_2 + 1 \geq \min_{t\in A^1\setminus S^1} t$,*
ii. *$A^1 \setminus S^1 \neq \emptyset$ and $A^2 \setminus S^2 \neq \emptyset$,*
iii. *$A^1 = S^1 = T$ and $A^2 \setminus S^2 \neq \emptyset$.*

Proof. Let $F = \{(x^1, x^2, y^1, y^2) \in P : \sum_{t\in A^1\setminus S^1} (D^1_{tl_1} + D^2_{tl_2}) y^1_t + \sum_{t\in A^2\setminus S^2} D^2_{tl_2} y^2_t + \sum_{t\in S^1} x^1_t + \sum_{t\in S^2} x^2_t = D^1_{1l_1} + D^2_{1l_2}\}$. Suppose that all solutions in F also satisfy $\sum_{t\in T} \alpha^1_t x^1_t + \sum_{t\in T} \alpha^2_t x^2_t + \sum_{t\in T} \beta^1_t y^1_t + \sum_{t\in T} \beta^2_t y^2_t = \gamma$. Let $y^1 = \sum_{j\in(T\setminus A^1)\cup S^1} e_j$ and $y^2 = \sum_{j\in(T\setminus A^2)\cup S^2} e_j$.

Suppose that condition (i) is satisfied, i.e., $A^1\setminus S^1 \neq \emptyset$ and $j_1 = \min_{t\in A^1\setminus S^1} t \leq l_2+1$. The solution $(x^1, x^2, e_{j_1}+y^1, y^2)$ where $x^1 = (D^1_{1j_1-1} + D^2_{1j_1-1})e_1 + (D^1_{j_1n} + D^2_{j_1n})e_{j_1}$ and $x^2 = 0$ is in F. For $t \in (T\setminus A^1)\cup(S^1\setminus\{1\})$, the solution $(x^1, x^2, e_{j_1}+y^1-e_t, y^2)$ is also in F and hence $\beta^1_t = 0$. Similarly, for $t \in (T \setminus A^2) \cup S^2$, as the solution $(x^1, x^2, e_{j_1} + y^1, y^2 - e_t)$ is also in F, we have $\beta^2_t = 0$.

For $t \in T\setminus A^1$, the solution $(x^1 - \epsilon e_{j_1} + \epsilon e_t, x^2, e_{j_1} + y^1, y^2)$ is in F for small enough $\epsilon > 0$. Hence $\alpha^1_t = \alpha^1_{j_1}$. For $t \in T\setminus A^2$, as the solution $(x^1 - \epsilon e_{j_1}, x^2 + \epsilon e_t, e_{j_1} + y^1, y^2)$ is in F for small enough $\epsilon > 0$, we have $\alpha^2_t = \alpha^1_{j_1}$.

The solution (x^1, x^2, y^1, y^2) where $x^1 = (D^1_{1l_1} + D^2_{1l_2})e_1 + D^1_{l_1+1,n} e_{l_1+1}$ and $x^2 = D^2_{l_2+1,n} e_{l_2+1}$ is in F. For $t \in S^1 \setminus \{1\}$, the solution $(x^1 - \epsilon e_1 + \epsilon e_t, x^2, y^1, y^2)$ is also in F for small enough $\epsilon > 0$. Hence $\alpha^1_t = \alpha^1_1$. For $t \in S^2$, the solution $(x^1 - \epsilon e_1, x^2 + \epsilon e_t, y^1, y^2)$ is in F for small enough $\epsilon > 0$. This shows that $\alpha^2_t = \alpha^1_1$.

Let $t \in A^1 \setminus S^1$ with $t > j_1$. Consider the solutions $(x^1, x^2, e_t + y^1, y^2)$ where $x^1 = (D^1_{1t-1} + \min\{D^2_{1t-1}, D^2_{1l_2}\})e_1 + (D^1_{tl_1} + D^2_{tl_2})e_t + D^1_{l_1+1,n} e_{l_1+1}$ and $x^2 = D^2_{l_2+1,n} e_{l_2+1}$ and $(x^1 + \epsilon e_t, x^2 - \epsilon e_{l_2+1}, e_t + y^1, y^2)$. As both solutions are in F, we have $\alpha^1_t = \alpha^2_{l_2+1} = \alpha^1_{j_1}$.

For $t \in A^2 \setminus S^2$, consider the solutions $(x^1, x^2, y^1, e_t + y^2)$ where $x^1 = (D^1_{1l_1} + D^2_{1t-1})e_1 + D^1_{l_1+1,n} e_{l_1+1}$ and $x^2 = D^2_{tl_2} e_t + D^2_{l_2+1,n} e_{l_2+1}$ and $(x^1, x^2 - \epsilon e_{l_2+1} + \epsilon e_t, y^1, e_t + y^2)$. Both of these solutions are in F. Hence $\alpha^2_t = \alpha^2_{l_2+1} = \alpha^1_{j_1}$.

Now subtracting $\alpha^1_{j_1}$ times $\sum_{t=1}^n x^1_t + \sum_{t=1}^n x^2_t = D^1_{1n} + D^2_{1n}$ and β^1_1 times $y^1_1 = 1$ from $\sum_{t\in T} \alpha^1_t x^1_t + \sum_{t\in T} \alpha^2_t x^2_t + \sum_{t\in T} \beta^1_t y^1_t + \sum_{t\in T} \beta^2_t y^2_t = \gamma$ yields

$$\sum_{t\in S^1} (\alpha^1_1 - \alpha^1_{j_1}) x^1_t + \sum_{t\in S^2} (\alpha^1_1 - \alpha^1_{j_1}) x^2_t + \sum_{t\in A^1\setminus S^1} \beta^1_t y^1_t + \sum_{t\in A^2\setminus S^2} \beta^2_t y^2_t = \gamma - \alpha^1_{j_1}(D^1_{1n} + D^2_{1n}) - \beta^1_1.$$

Let $x^1 = (D^1_{1l_1} + D^2_{1l_2})e^1_1 + D^1_{l_1+1,n}e_{l_1+1}$ and $x^2 = D^2_{l_2+1,n}e_{l_2+1}$ and consider the solution (x^1, x^2, y^1, y^2). As this solution is in F, $\gamma - \alpha^1_{j_1}(D^1_{1n} + D^2_{1n}) - \beta^1_1 = (\alpha^1_1 - \alpha^1_{j_1})(D^1_{1l_1} + D^2_{1l_2})$.

For $t \in A^1\setminus S^1$, the solution $(x^1-(D^1_{tl_1}+D^2_{tl_2})e_1+(D^1_{tl_1}+D^2_{tl_2})e_t, x^2, y^1+e_t, y^2)$ is also in F. Hence $\beta^1_t = (D^1_{tl_1}+D^2_{tl_2})(\alpha^1_1-\alpha^1_{j_1})$. For $t \in A^2 \setminus S^2$, the solution $(x^1 - D^2_{tl_2}e_1, x^2 + D^2_{tl_2}e_t, y^1, y^2 + e_t)$ is also in F. So $\beta^2_t = D^2_{tl_2}(\alpha^1_1 - \alpha^1_{j_1})$.

Hence $\sum_{t\in T} \alpha^1_t x^1_t + \sum_{t\in T} \alpha^2_t x^2_t + \sum_{t\in T} \beta^1_t y^1_t + \sum_{t\in T} \beta^2_t y^2_t = \gamma$ is a weighted sum of $\sum_{t\in A^1\setminus S^1}(D^1_{tl_1} + D^2_{tl_2})y^1_t + \sum_{t\in A^2\setminus S^2} D^2_{tl_2}y^2_t + \sum_{t\in S^1} x^1_t + \sum_{t\in S^2} x^2_t = D^1_{1l_1} + D^2_{1l_2}$, $\sum_{t=1}^n x^1_t + \sum_{t=1}^n x^2_t = D^1_{1n} + D^2_{1n}$ and $y^1_1 = 1$.

Now suppose that condition (ii) is satisfied but (i) is not, i.e., $A^1 \setminus S^1 \neq \emptyset$, $j_1 = \min_{t\in A^1\setminus S^1} t > l_2 + 1$, and $A^2 \setminus S^2 \neq \emptyset$. Let $j_2 = \min_{t\in A^2\setminus S^2} t$. The solution $(x^1, x^2, e_{j_1}+y^1, e_{j_2}+y^2)$ where $x^1 = (D^1_{1j_1-1}+D^2_{1j_2-1})e_1+(D^1_{j_1n}+D^2_{j_1n})e_{j_1}$ and $x^2 = D^2_{j_2j_1-1}e_{j_2}$ is in F. For $t \in (T \setminus A^1) \cup (S^1 \setminus \{1\})$, the solution $(x^1, x^2, e_{j_1} + y^1 - e_t, e_{j_2} + y^2)$ is also in F and hence $\beta^1_t = 0$. Similarly, for $t \in (T \setminus A^2) \cup S^2$, as the solution $(x^1, x^2, e_{j_1} + y^1, e_{j_2} + y^2 - e_t)$ is also in F, we have $\beta^2_t = 0$. For $t \in T \setminus A^1$, the solution $(x^1 - \epsilon e_{j_1} + \epsilon e_t, x^2, e_{j_1} + y^1, e_{j_2} + y^2)$ is in F for small enough $\epsilon > 0$. Hence $\alpha^1_t = \alpha^1_{j_1}$. For $t \in T \setminus A^2$, as the solution $(x^1 - \epsilon e_{j_1}, x^2 + \epsilon e_t, e_{j_1} + y^1, e_{j_2} + y^2)$ is in F for small enough $\epsilon > 0$, we have $\alpha^2_t = \alpha^1_{j_1}$. The remaining part of the proof is the same as above as none of the solutions used depend on j_1.

Next suppose that condition (iii) is satisfied, i.e., $A^1 = S^1 = T$ and $A^2 \setminus S^2 \neq \emptyset$. Let $j_2 = \min_{t\in A^2\setminus S^2} t$. Consider the solution $(x^1, x^2, y^1, e_{j_2} + y^2)$ where $x^1 = (D^1_{1n} + D^2_{1j_2-1})e_1$ and $x^2 = D^2_{j_2n}e_{j_2}$. This solution is in F. For $t \in T \setminus \{1\}$, the solution $(x^1, x^2, y^1 - e_t, e_{j_2} + y^2)$ is also in F and hence $\beta^1_t = 0$. Similarly, for $t \in (T \setminus A^2) \cup S^2$, as the solution $(x^1, x^2, y^1, e_{j_2} + y^2 - e_t)$ is also in F, we have $\beta^2_t = 0$. For $t \in T \setminus A^2$, the solution $(x^1, x^2 - \epsilon e_{j_2} + \epsilon e_t, y^1, e_{j_2} + y^2)$ is in F for small enough $\epsilon > 0$, hence $\alpha^2_t = \alpha^2_{j_2}$.

The solution (x^1, x^2, y^1, y^2) where $x^1 = (D^1_{1n} + D^2_{1l_2})e_1$ and $x^2 = D^2_{l_2+1,n}e_{l_2+1}$ is in F. For $t \in T \setminus \{1\}$, the solution $(x^1 - \epsilon e_1+\epsilon e_t, x^2, y^1, y^2)$ is also in F for small enough $\epsilon > 0$. Hence $\alpha^1_t = \alpha^1_1$. For $t \in S^2$, as the solution $(x^1 - \epsilon e_1, x^2 + \epsilon e_t, y^1, y^2)$ is in F for small enough $\epsilon > 0$, we obtain $\alpha^2_t = \alpha^1_1$.

For $t \in A^2 \setminus S^2$, consider the solutions $(x^1, x^2, y^1, e_t + y^2)$ where $x^1 = (D^1_{1n} + D^2_{1t-1})e_1$ and $x^2 = D^2_{tl_2}e_t + D^2_{l_2+1,n}e_{l_2+1}$ and $(x^1, x^2 - \epsilon e_{l_2+1} + \epsilon e_t, y^1, e_t + y^2)$. Both of these solutions are in F showing that $\alpha^2_t = \alpha^2_{l_2+1} = \alpha^2_{j_2}$.

If we subtract $\alpha^2_{j_2}$ times $\sum_{t=1}^n x^1_t+\sum_{t=1}^n x^2_t = D^1_{1n}+D^2_{1n}$ and β^1_1 times $y^1_1 = 1$ from $\sum_{t\in T} \alpha^1_t x^1_t+\sum_{t\in T} \alpha^2_t x^2_t+\sum_{t\in T} \beta^1_t y^1_t + \sum_{t\in T} \beta^2_t y^2_t = \gamma$, we obtain

$$\sum_{t\in T}(\alpha^1_1 - \alpha^2_{j_2})x^1_t + \sum_{t\in S^2}(\alpha^1_1 - \alpha^2_{j_2})x^2_t + \sum_{t\in A^2\setminus S^2} \beta^2_t y^2_t = \gamma - \alpha^2_{j_2}(D^1_{1n} + D^2_{1n}) - \beta^1_1.$$

Let $x^1 = (D^1_{1n} + D^2_{1l_2})e^1_1$ and $x^2 = D^2_{l_2+1,n}e_{l_2+1}$ and consider the solution (x^1, x^2, y^1, y^2). As this solution is in F, $\gamma - \alpha^2_{j_2}(D^1_{1n} + D^2_{1n}) - \beta^1_1 = (\alpha^1_1 - \alpha^2_{j_2})(D^1_{1n} + D^2_{1l_2})$. For $t \in A^2 \setminus S^2$, the solution $(x^1 - D^2_{tl_2}e_1, x^2 + D^2_{tl_2}e_t, y^1, y^2 + e_t)$ is also in F. So $\beta^2_t = D^2_{tl_2}(\alpha^1_1 - \alpha^2_{j_2})$. Hence $\sum_{t\in T} \alpha^1_t x^1_t + \sum_{t\in T} \alpha^2_t x^2_t + \sum_{t\in T} \beta^1_t y^1_t + \sum_{t\in T} \beta^2_t y^2_t = \gamma$ is a weighted sum of $\sum_{t\in A^2\setminus S^2} D^2_{tl_2}y^2_t + \sum_{t\in T} x^1_t + \sum_{t\in S^2} x^2_t = D^1_{1n} + D^2_{1l_2}$, $\sum_{t=1}^n x^1_t + \sum_{t=1}^n x^2_t = D^1_{1n} + D^2_{1n}$ and $y^1_1 = 1$.

If $A^1 \setminus S^1 \neq \emptyset$, $\min_{t\in A^1\setminus S^1} t > l_2+1$ and $A^2 \setminus S^2 = \emptyset$, then observe that the coefficient for $t \in A^1 \setminus S^1$ is $D^1_{tl_1}$ since $t > l_2+1$. Consider the inequality (16) for the same choices of A^1, S^1, and S^2, but for $A^2 \cup \{l_2 + 1\}$. The inequality is

$$\sum_{t\in A^1\setminus S^1} D^1_{tl_1}y^1_t + d^2_{l_2+1}y^2_{l_2+1} + \sum_{t\in S^1} x^1_t + \sum_{t\in S^2} x^2_t \geq D^1_{1l_1} + D^2_{1l_2+1}$$

which is the same as

$$\sum_{t\in A^1\setminus S^1} D^1_{tl_1}y^1_t + \sum_{t\in S^1} x^1_t + \sum_{t\in S^2} x^2_t \geq D^1_{1l_1} + D^2_{1l_2} + d^2_{l_2+1}(1 - y^2_{l_2+1})$$

and hence dominates the previous (l_1, l_2, S^1, S^2)-inequality as $y^2_{l_2+1} \leq 1$.

If $A^1 = S^1 \neq T$, then consider the inequality (16) for the same choices of S^1, A^2, and S^2, but for $A^1\cup\{l_1+1\}$. The inequality is

$$\sum_{t\in A^1\setminus S^1} (D^1_{tl_1} + D^2_{tl_2})y^1_t + d^1_{l_1+1}y^1_{l_1+1} + \sum_{t\in A^2\setminus S^2} D^2_{tl_2}y^2_t + \sum_{t\in S^1} x^1_t + \sum_{t\in S^2} x^2_t \geq D^1_{1l_1+1} + D^2_{1l_2}.$$

Since $y^1_{l_1+1} \leq 1$, this inequality dominates the previous (l_1, l_2, S^1, S^2)-inequality.

If $A^1 = S^1 = T$ and $A^2 \setminus S^2 = \emptyset$, then by Corollary 1, the (l_1, l_2, S^1, S^2)-inequality cannot be facet defining. □

To conclude this section, we investigate the complexity of the separation problem associated with (l_1, l_2, S^1, S^2)-inequalities. For fixed l_1 and l_2 with $l_1 \geq l_2$ and $l_2 < n$, let $A^1 = \{1, \ldots, l_1\}$, $A^2 = \{1, \ldots, l_2\}$, $S^1 = \{t \in A^1 : x^1_t < (D^1_{tl_1} + D^2_{tl_2})y^1_t\}$ and $S^2 = \{t \in A^2 : x^2_t < D^2_{tl_2}y^2_t\}$. If the corresponding (l_1, l_2, S^1, S^2)-inequality is not violated, there is no violated (l_1, l_2, S^1, S^2)-inequality for this choice of l_1 and l_2. Hence (l_1, l_2, S^1, S^2)-inequalities can be separated in polynomial time.

The (l_1, l_2, S^1, S^2)-inequalities are not sufficient in general to describe P. In Appendix B, we give the fractional extreme points of the LP relaxation of the *2ULS* formulation strengthened with the (l_1, l_2, S^1, S^2)-inequalities for the instance with $n = 3$ and $d_t^i = 1$ for $i = 1, 2$ and $t = 1, 2, 3$ (obtained using PORTA [11]).

3.4. Description of the convex hull for two periods

Here we prove that the (l_1, l_2, S^1, S^2)-inequalities together with the trivial facet defining inequalities describe P for $n = 2$. Notice that there are only three facet defining (l_1, l_2, S^1, S^2)-inequalities for two periods.

Theorem 7. *For $n = 2$, P is described by:*

$$x_1^1 + x_2^1 + x_1^2 + x_2^2 = D_{12}^1 + D_{12}^2 \tag{43}$$
$$y_1^1 = 1 \tag{44}$$
$$x_2^1 \geq 0 \tag{45}$$
$$x_1^2 \geq 0 \tag{46}$$
$$x_2^2 \geq 0 \tag{47}$$
$$y_2^1 \leq 1 \tag{48}$$
$$y_1^2 \leq 1 \tag{49}$$
$$y_2^2 \leq 1 \tag{50}$$
$$x_2^1 \leq (d_2^1 + d_2^2)y_2^1 \tag{51}$$
$$x_1^2 \leq (d_1^2 + d_2^2)y_1^2 \tag{52}$$
$$x_2^2 \leq d_2^2 y_2^2 \tag{53}$$
$$x_1^1 + d_2^1 y_2^1 + d_1^2 y_1^2 \geq d_1^1 + d_2^1 + d_1^2 \tag{54}$$
$$x_1^1 + d_2^1 y_2^1 + x_1^2 \geq d_1^1 + d_2^1 + d_1^2 \tag{55}$$
$$x_1^1 + x_2^1 + d_1^2 y_1^2 \geq d_1^1 + d_2^1 + d_1^2. \tag{56}$$

Proof. Let $\overline{P}$ be the set of solutions that satisfy (43)–(45) and (56). It is easy to see that $\dim(\overline{P}) = \dim(P) = 6$. We can assume, without loss of generality, that any inequality $\alpha_1^1 x_1^1 + \alpha_2^1 x_2^1 + \alpha_1^2 x_1^2 + \alpha_2^2 x_2^2 + \beta_1^1 y_1^1 + \beta_2^1 y_2^1 + \beta_1^2 y_1^2 + \beta_2^2 y_2^2 \geq \gamma$ which defines a facet of P different from those defined by the above inequalities, satisfies $\beta_1^1 = 0$, $\alpha_1^1, \alpha_2^1, \alpha_1^2, \alpha_2^2 \geq 0$ and $\alpha_1^1\alpha_2^1\alpha_1^2\alpha_2^2 = 0$. Moreover, by Theorem 4, we know that $\alpha_1^1 > 0$, $\alpha_1^1 \geq \alpha_2^1$, $\alpha_1^1 \geq \alpha_1^2$, $\alpha_1^1 \geq \alpha_2^2$, $\gamma > 0$, $\beta_2^2 = 0$, $\beta_2^1, \beta_1^2 \geq 0$, and $\beta_2^1 + \beta_1^2 > 0$.

Now suppose that inequality $\alpha_1^1 x_1^1 + \alpha_2^1 x_2^1 + \alpha_1^2 x_1^2 + \alpha_2^2 x_2^2 + \beta_2^1 y_2^1 + \beta_1^2 y_1^2 \geq \gamma$ defines a facet of P different from those defined by the above inequalities. Let $F = \{(x^1, x^2, y^1, y^2) \in P : \alpha_1^1 x_1^1 + \alpha_2^1 x_2^1 + \alpha_1^2 x_1^2 + \alpha_2^2 x_2^2 + \beta_2^1 y_2^1 + \beta_1^2 y_1^2 = \gamma\}$.

Let (x^1, x^2, y^1, y^2) be a solution in F such that $y_2^1 = 0$. Such a solution exists since F is different from the facet defined by inequality $y_2^1 \leq 1$. Then $x_1^1 \geq d_1^1 + d_2^1$ and the solution $(x^1 - d_2^1(e_1 - e_2), x^2, y^1 + e_2, y^2)$ is in P. Hence $\beta_2^1 \geq (\alpha_1^1 - \alpha_2^1)d_2^1$. Now if $\alpha_2^1 = 0$, this implies that $\beta_2^1 \geq \alpha_1^1 d_2^1$. Similarly, we can show that if $\alpha_1^2 = 0$ then $\beta_1^2 \geq \alpha_1^1 d_1^2$.

There exists a solution (x^1, x^2, y^1, y^2) in F such that $x_2^1 < (d_2^1 + d_2^2)y_2^1$ since F is different from the facet defined by inequality (51). Then $(x^1 - \epsilon(e_1 - e_2), x^2, y^1, y^2)$ or $(x^1 + \epsilon e_2, x^2 - \epsilon e_1, y^1, y^2)$ or $(x^1 + \epsilon e_2, x^2 - \epsilon e_2, y^1, y^2)$ is in P for small enough $\epsilon > 0$. Hence $\alpha_2^1 \geq \alpha_1^1$ or $\alpha_2^1 \geq \alpha_1^2$ or $\alpha_2^1 \geq \alpha_2^2$. Similarly, we can show that $\alpha_1^2 \geq \alpha_1^1$ or $\alpha_1^2 \geq \alpha_2^1$ or $\alpha_1^2 \geq \alpha_2^2$ and $\alpha_2^2 \geq \alpha_1^1$ or $\alpha_2^2 \geq \alpha_2^1$ or $\alpha_2^2 \geq \alpha_1^2$.

If any two of α_2^1, α_1^2 and α_2^2 are equal to α_1^1, then the remaining one is zero. This contradicts with the above conditions since $\alpha_1^1 > 0$. So at most one of α_2^1, α_1^2, and α_2^2 can be equal to α_1^1 and the remaining ones should be strictly less than α_1^1. Among the values that are less than α_1^1, at least two should be equal to zero. Hence we have the following cases:

1. $\alpha_1^1 = \alpha_1^2 > \alpha_2^1 = \alpha_2^2 = 0$

   As $\alpha_1^1 = \alpha_1^2$ and $\beta_1^1 = 0$, we have $\beta_1^2 = 0$ by item (vi) of Theorem 4. And as $\alpha_2^1 = 0$, we have $\beta_2^1 \geq \alpha_1^1 d_2^1$.

   Note that the solution $((d_1^1 + d_2^1 + d_1^2)e_1, d_2^2 e_2, e_1, e_2)$ is in P. Hence $\gamma \leq \alpha_1^1(d_1^1 + d_2^1 + d_1^2)$. Then the inequality is dominated by (55) and cannot be facet defining.

2. $\alpha_1^1 = \alpha_2^1 > \alpha_1^2 = \alpha_2^2 = 0$
   Similar to the previous case.
3. $\alpha_1^1 = \alpha_2^2 > \alpha_2^1 = \alpha_1^2 = 0$
   Suppose that there exists a solution (x^1, x^2, y^1, y^2) in F such that $y_2^1 = y_1^2 = 0$. Then $x_1^1 \geq d_1^1 + d_2^1 + d_1^2$ and the solution $(x^1 - (d_2^1 + d_2^2 - x_2^2)e_1 + (d_2^1 + d_2^2)e_2, x^2 - x_2^2 e_2, y^1 + e_2, y^2)$ is in P. This implies that $\beta_2^1 \geq \alpha_1^1(d_2^1 + d_2^2)$. This contradicts item (v) of Theorem 4. So all solutions (x^1, x^2, y^1, y^2) in F satisfy $y_2^1 + y_1^2 \geq 1$.
   Now suppose all solutions (x^1, x^2, y^1, y^2) in F satisfy $y_2^1 + y_1^2 = 1$. Then $F = \{(x^1, x^2, y^1, y^2) \in P : y_2^1 + y_1^2 = 1\}$. This contradicts the assumptions that $\alpha_1^1 > 0$ and $\alpha_1^1\alpha_2^1\alpha_1^2\alpha_2^2 = 0$. Hence there exists a solution (x^1, x^2, y^1, y^2) in F such that $y_2^1 = y_1^2 = 1$. If $x_1^1 > d_1^1$, then at least one of the solutions $(x^1 - \epsilon(e_1 - e_2), x^2, y^1, y^2)$ and $(x^1 - \epsilon e_1, x^2 + \epsilon e_1, y^1, y^2)$ are in P for small enough $\epsilon > 0$. This implies that $\alpha_1^1 \leq 0$. Similarly, we can show that if $x_2^2 > 0$, then $\alpha_2^2 \leq 0$. Hence all solutions (x^1, x^2, y^1, y^2) in F such that $y_2^1 = y_1^2 = 1$ satisfy $x_1^1 = d_1^1$ and $x_2^2 = 0$. Now consider the solution $(x^1 + d_2^1 e_1 - x_2^1 e_2, x^2 + (x_2^1 - d_2^1)e_1, y^1 - e_2, y^2)$. This solution is in P and so $\alpha_1^1 d_2^1 \geq \beta_2^1$. We also have $\beta_2^1 \geq \alpha_1^1 d_2^1$. So $\beta_2^1 = \alpha_1^1 d_2^1$. Similarly, we can show that $\beta_1^2 = \alpha_1^1 d_1^2$.
   Now consider the solution $(d_1^1 e_1 + (d_2^1 + d_2^2)e_2, d_1^2 e_1, e_1 + e_2, e_1)$. This solution is in P. Hence $\gamma \leq \alpha_1^1(d_1^1 + d_2^1 + d_1^2)$. So the inequality is dominated by (54).
4. $\alpha_1^1 > \alpha_2^2 > \alpha_2^1 = \alpha_1^2 = 0$
   Suppose there exists a solution (x^1, x^2, y^1, y^2) in F such that $x_2^2 > 0$ and $y_2^1 = 1$ or $y_1^2 = 1$. Then either $(x^1, x^2 - \epsilon(e_2 - e_1), y^1, y^2)$ or $(x^1 + \epsilon e_2, x^2 - \epsilon e_2, y^1, y^2)$ is in P for small enough $\epsilon > 0$ and so $\alpha_2^2 \leq 0$. So all solutions (x^1, x^2, y^1, y^2) in F such that $x_2^2 > 0$ satisfy $y_2^1 = 0$ and $y_1^2 = 0$. Then $x_1^1 \geq d_1^1 + d_2^1 + d_1^2$. If $x_2^2 < d_2^2$, then the solution $(x^1 - \epsilon e_1, x^2 + \epsilon e_2, y^1, y^2 + (1 - y_2^2)e_2)$ is in P for small enough $\epsilon > 0$ and so $\alpha_2^2 \geq \alpha_1^1$. This implies that all solutions (x^1, x^2, y^1, y^2) in F such that $x_2^2 > 0$ satisfy $x_1^1 = d_1^1 + d_2^1 + d_1^2$, $y_2^1 = y_1^2 = 0$ and $x_2^2 = d_2^2$ and $\gamma = \alpha_1^1(d_1^1 + d_2^1 + d_1^2) + \alpha_2^2 d_2^2$.
   Let (x^1, x^2, y^1, y^2) be in F such that $x_2^2 = 0$. Then we have $y_2^1 + y_1^2 \geq 1$ since $\alpha_1^1(d_1^1 + d_2^1 + d_1^2 + d_2^2) > \gamma$. Now suppose that $y_2^1 = 1$ and $y_1^2 = 0$. Then if $x_1^1 > d_1^1 + d_1^2$, the solution $(x^1 - \epsilon(e_1 - e_2), x^2, y^1, y^2)$ is in P for small enough $\epsilon > 0$ and so $\alpha_1^1 \leq 0$. So $x_1^1 = d_1^1 + d_1^2$. Similarly, if $y_2^1 = 0$ and $y_1^2 = 1$, then $x_1^1 = d_1^1 + d_2^1$. If both y_2^1 and y_1^2 are 1, then if $x_1^1 > d_1^1$, either $(x^1 - \epsilon(e_1 - e_2), x^2, y^1, y^2)$ or $(x^1 - \epsilon e_1, x^2 + \epsilon e_1, y^1, y^2)$ is in P for small enough $\epsilon > 0$ and $\alpha_1^1 \leq 0$. Hence $x_1^1 = d_1^1$.
   Now we can conclude that all points (x^1, x^2, y^1, y^2) in F satisfy $x_1^1 = d_1^1 + d_2^1(1 - y_2^1) + d_1^2(1 - y_1^2)$ and the inequality cannot be facet defining.
5. $\alpha_1^1 > \alpha_2^1 > \alpha_1^2 = \alpha_2^2 = 0$
   Let (x^1, x^2, y^1, y^2) be a solution in F with $y_1^2 = 0$ and $x_2^2 < d_2^2$. Then either $(x^1 - \epsilon e_1, x^2 + \epsilon e_2, y^1, y^2 + (1 - y_2^2)e_2)$ or $(x^1 - \epsilon e_2, x^2 + \epsilon e_2, y^1, y^2 + (1 - y_2^2)e_2)$ is in P for small enough $\epsilon > 0$ implying either $\alpha_1^1 \leq 0$ or $\alpha_2^1 \leq 0$. So all solutions (x^1, x^2, y^1, y^2) in F with $y_1^2 = 0$ satisfy $x_2^2 = d_2^2$ and hence $x_1^1 + x_2^1 = d_1^1 + d_2^1 + d_1^2$.
   Now suppose that (x^1, x^2, y^1, y^2) is a solution in F with $y_1^2 = 1$. Now if $x_1^2 + x_2^2 < d_1^2 + d_2^2$, then at least one of the solutions $(x^1 - \epsilon e_1, x^2 + \epsilon e_2, y^1, y^2 + (1 - y_2^2)e_2)$, $(x^1 - \epsilon e_2, x^2 + \epsilon e_2, y^1, y^2 + (1 - y_2^2)e_2)$, $(x^1 - \epsilon e_1, x^2 + \epsilon e_1, y^1, y^2)$ and $(x^1 - \epsilon e_2, x^2 + \epsilon e_1, y^1, y^2)$ is in P for small enough $\epsilon > 0$. This implies that $\alpha_1^1 \leq 0$ or $\alpha_2^1 \leq 0$. Hence all solutions (x^1, x^2, y^1, y^2) in F with $y_1^2 = 1$ satisfy $x_1^2 + x_2^2 = d_1^2 + d_2^2$ and $x_1^1 + x_2^1 = d_1^1 + d_2^1$.
   Thus all solutions (x^1, x^2, y^1, y^2) in F satisfy $x_1^1 + x_2^1 = d_1^1 + d_2^1 + d_1^2(1 - y_1^2)$.
6. $\alpha_1^1 > \alpha_1^2 > \alpha_2^1 = \alpha_2^2 = 0$
   Similar to the previous case.
7. $\alpha_1^1 > \alpha_2^2 = \alpha_2^1 = \alpha_1^2 = 0$
   Then $\beta_2^1 \geq \alpha_1^1 d_2^1$ and $\beta_1^2 \geq \alpha_1^1 d_1^2$. As the solution $((d_1^1 + d_2^1 + d_1^2)e_1, d_2^2 e_2, e_1, e_2)$ is in P, we have that $\gamma \leq \alpha_1^1(d_1^1 + d_2^1 + d_1^2)$. So the inequality is dominated by (54). □

4. Computational study

In this section, we perform an experiment to evaluate the quality of the lower bound obtained by solving the LP relaxation of the formulation strengthened with the (l_1, l_2, S^1, S^2)-inequalities. To this end, we solve the *2ULS model without substitution variables* and the *UFL* formulation and their LP relaxations for randomly generated instances and report the % gaps, i.e., $\frac{opt-lp}{opt} * 100$ where *opt* is the optimal value of *2ULS* and *lp* is the optimal value of the LP relaxation of the corresponding formulation.

For a given planning horizon, we generate instances with varying cost and demand structures using three parameters, η, χ, and δ. The parameter η is used to compute inventory holding costs as a fraction of production costs and takes values in $\{0.05, 0.1, 0.2\}$. We set the setup costs equal to the parameter χ and assign the values 5000 and 20 000 to χ. Finally, the parameter δ controls the variability of demand over periods. We let the demand values differ between 100 and $\delta + 100$ where $\delta \in \{300, 600, 900\}$.

For given values of η, χ, and δ, we generate an instance as follows. We let $\bar{p}_t^1 = 50 + \lceil \rho_t^1 * 10 \rceil$ and $\bar{p}_t^2 = 40 + \lceil \rho_t^2 * 10 \rceil$ where ρ_t^1 and ρ_t^2 are uniformly distributed in the interval [0, 1] and $\bar{c}_t = 0$ for $t \in T$. The inventory holding costs are computed using the production costs as $h_t^1 = h_t^2 = \lceil \eta(\bar{p}_t^1 + \bar{p}_t^2) \rceil$ for $t \in T$. We take the setup costs $q_t^1 = q_t^2 = \chi$ for $t \in T$.

Table 1
Average and maximum % gaps and the number of problems with zero gap for 10-period problems.

		$\chi = 5000$						$\chi = 20\,000$					
		Model1			Model2			Model1			Model2		
δ	η	Ave	Min	Max	Ave	Max	No	Ave	Min	Max	Ave	Max	No
300	0.05	5.60	4.33	7.01	0.01	0.06	15	6.60	4.91	8.35	0.02	0.23	18
	0.1	5.78	5.01	6.62	0.01	0.10	18	6.62	3.51	8.80	0.02	0.25	18
	0.2	5.42	4.28	6.46	0.01	0.08	15	6.45	3.74	8.70	0.03	0.59	19
600	0.05	4.41	3.42	5.41	0.00	0.06	19	6.55	4.30	8.24	0.00	0.00	20
	0.1	4.66	3.50	5.73	0.00	0.00	20	6.91	4.07	8.55	0.03	0.34	17
	0.2	4.57	2.84	5.89	0.00	0.05	18	6.98	4.53	8.61	0.05	0.24	14
900	0.05	3.54	2.66	4.20	0.00	0.00	20	6.18	4.10	7.17	0.02	0.11	15
	0.1	3.68	2.59	4.79	0.00	0.00	20	6.94	5.89	7.80	0.03	0.30	16
	0.2	3.75	3.02	4.83	0.00	0.00	20	6.67	5.56	8.29	0.01	0.16	17

Table 2
Average and maximum % gaps and the number of problems with zero gap for 20-period problems.

		$\chi = 5000$						$\chi = 20\,000$					
		Model1			Model2			Model1			Model2		
δ	η	Ave	Min	Max	Ave	Max	No	Ave	Min	Max	Ave	Max	No
300	0.05	5.24	4.11	6.29	0.00	0.02	14	8.28	7.31	9.33	0.01	0.12	17
	0.1	5.13	3.93	6.30	0.01	0.06	13	8.20	7.15	9.44	0.01	0.10	18
	0.2	4.99	3.81	5.93	0.01	0.03	14	8.01	7.06	9.36	0.02	0.19	16
600	0.05	3.99	2.72	5.01	0.00	0.00	20	7.46	5.57	8.92	0.01	0.09	15
	0.1	3.96	2.70	5.14	0.00	0.00	20	7.44	5.79	8.69	0.01	0.07	13
	0.2	3.87	2.80	4.83	0.00	0.00	20	7.32	6.07	8.58	0.02	0.09	10
900	0.05	3.26	2.16	4.54	0.00	0.00	20	6.55	4.94	8.45	0.02	0.13	10
	0.1	3.28	2.10	4.43	0.00	0.00	20	6.52	5.09	7.80	0.01	0.06	10
	0.2	3.13	2.09	4.14	0.00	0.00	20	6.39	4.70	7.63	0.02	0.09	10

Table 3
Average and maximum % gaps and the number of problems with zero gap for 50-period problems.

		$\chi = 5000$						$\chi = 20\,000$					
		Model1			Model2			Model1			Model2		
δ	η	Ave	Min	Max	Ave	Max	No	Ave	Min	Max	Ave	Max	No
300	0.05	3.26	2.16	4.25	0.00	0.02	11	6.13	4.51	7.61	0.01	0.11	9
	0.1	3.16	2.10	4.21	0.00	0.01	10	6.06	4.45	7.60	0.02	0.08	10
	0.2	3.17	2.17	4.30	0.00	0.02	8	5.98	4.46	7.44	0.01	0.07	12
600	0.05	2.40	1.51	3.27	0.00	0.01	16	5.02	3.53	6.70	0.01	0.05	5
	0.1	2.37	1.48	3.36	0.00	0.00	18	4.99	3.50	6.37	0.01	0.04	2
	0.2	2.32	1.45	3.18	0.00	0.00	16	4.93	3.40	6.56	0.02	0.04	2
900	0.05	1.86	1.15	2.53	0.00	0.00	19	4.29	2.88	5.61	0.01	0.04	4
	0.1	1.90	1.18	2.64	0.00	0.00	19	4.27	3.02	5.63	0.01	0.03	2
	0.2	1.91	1.17	2.73	0.00	0.00	19	4.32	2.90	5.80	0.01	0.03	3

Finally, we generate the demands as $d_t^1 = 100 + \lceil \sigma_t^1 \delta \rceil$ and $d_t^2 = 100 + \lceil \sigma_t^2 \delta \rceil$ where σ_t^1 and σ_t^2 are uniformly distributed in the interval [0, 1] for $t \in T$.

We generated random instances for planning horizons of 10, 20, 50, and 100 periods. For a given number of periods, and fixed values of η, χ, and δ, we solved our problem for 20 instances. In total, we solved 1440 instances. The results are reported in Tables 1–4.

In these tables, Model1 refers to the original model (12) subject to (5), (6), (10), (11), (13), (14), (15) with the additional constraint $y_1^1 = 1$ and Model2 refers to the extended *UFL* model (23)–(31) with $y_1^1 = 1$. For each setting of parameters and each model, we report the average (in column ave), minimum (in column min), and maximum (in column max) percentage gaps and the number of instances for which the gap turned out be zero (in column no) over the 20 instances. For Model1, there was no instance for which the gap was zero, so we omit this column. With Model2, we omit the min gap column since they were equal to zero in most of the cases.

We observe that the original formulation results in large duality gaps whereas the optimal value of the LP relaxation of the *UFL* formulation is very close to the optimal value of *2ULS*. The largest percentage gap is less than 0.6% and the overall average percentage gap is 0.01%. The gap is zero for 900 instances over 1440 with this formulation.

Table 4
Average and maximum % gaps and the number of problems with zero gap for 100-period problems.

		$\chi = 5000$						$\chi = 20\,000$					
		Model1			Model2			Model1			Model2		
δ	η	Ave	Min	Max	Ave	Max	No	Ave	Min	Max	Ave	Max	No
300	0.05	1.97	1.23	2.68	0.00	0.01	4	3.99	2.71	5.23	0.01	0.05	1
	0.1	1.95	1.22	2.75	0.00	0.01	2	3.92	2.72	5.36	0.01	0.03	4
	0.2	1.88	1.21	2.64	0.00	0.01	5	3.81	2.65	5.17	0.00	0.02	7
600	0.05	1.42	0.85	1.98	0.00	0.00	14	3.13	2.06	4.20	0.01	0.03	0
	0.1	1.43	0.82	2.03	0.00	0.00	10	3.14	2.03	4.31	0.01	0.03	1
	0.2	1.37	0.85	1.96	0.00	0.00	12	3.03	2.07	4.13	0.01	0.02	0
900	0.05	1.17	0.67	1.65	0.00	0.00	17	2.75	1.77	3.71	0.01	0.02	1
	0.1	1.12	0.66	1.61	0.00	0.00	18	2.64	1.70	3.63	0.01	0.02	0
	0.2	1.10	0.66	1.64	0.00	0.00	18	2.60	1.68	3.80	0.00	0.01	2

Table 5
Summary of results.

n	gap1	gap2	perc2	χ	gap1	gap2	perc2	δ	gap1	gap2	perc2	η	gap1	gap2	perc2
10	5.63	0.01	88.61	5 000	3.17	0.00	78.06	300	5.07	0.01	57.92	0.05	4.38	0.01	63.33
20	5.72	0.01	77.78	20 000	5.59	0.01	46.94	600	4.32	0.01	62.92	0.1	4.42	0.01	62.29
50	3.80	0.01	51.39					900	3.74	0.01	66.67	0.2	4.33	0.01	61.88
100	2.36	0.00	32.22												

We summarize the results for different values of parameters in Table 5. For a given value of a parameter, we report the average percentage gaps with Model1 (in column gap1) and Model2 (in column gap2) and percentage of the number of instances for which the gap is zero for Model2 (in column perc2). Here we can see that as the number of periods increases, the average % gaps decrease for both models, but the number of problems with zero gap with Model2 also decreases. The problems with $\chi = 20\,000$ have larger gaps compared to problems with $\chi = 5000$ with both models. Also the number of instances for which the duality gap is zero with Model2 is smaller with $\chi = 20\,000$. As δ increases, the average gap remains the same with Model2 but decreases with Model1. However, the number of instances with zero gap with Model2 increases as δ increases. Finally, as the parameter η increases, the number of instances with zero gap decreases but we cannot observe a significant effect on the average gaps with both models.

To summarize, these results suggest that even though the LP relaxation of the formulation of the *2ULS model without substitution variables* strengthened with the (l_1, l_2, S^1, S^2)-inequalities is not necessarily equal to the polytope associated with *2ULS*, its optimal value yields a strong lower bound and improves significantly over the lower bound obtained by solving the LP relaxation of the original formulation.

Finally, we note that the instances with 100 periods are solved to optimality using the *UFL* formulation in less than 2 seconds of cpu time using GAMS 22.6 with CPLEX 11.0.0 on a Xeon 2.83 GHz quadcore processor with 8 GB of Ram running under 64 bit Ubuntu Linux.

5. Concluding remarks

In this paper, we presented polyhedral results for the *2ULS*. We investigated the dimension, trivial facets and properties of the nontrivial facet defining inequalities of the convex hull of the projection of the feasible set of *2ULS* onto the space of production and setup variables. Using the projection of the *UFL* formulation on the same space, we derived a family of facet defining inequalities. These inequalities together with the trivial facet defining inequalities describe the *2ULS polytope* if the number of periods is two. The computational results showed that for larger number of periods, the % gaps between the optimal value of the problem and the optimal value of the LP relaxation of the *UFL* formulation are quite small.

There are several further questions that are interesting to investigate. It is possible to come up with an extended formulation – for instance, using the dynamic programming recursion of [5] – whose projection onto the space of production and setup variables is the same as the convex hull. If the extreme rays of the projection cone can be characterized, then we may obtain the remaining facet defining inequalities that complement the (l_1, l_2, S^1, S^2)-inequalities in the description of the convex hull.

Also we can generalize the problem investigated in this paper in several ways. For instance, we may consider the problem with substitution costs that are not necessarily zero. Then a natural formulation of the problem involves also the substitution variables. The (l_1, l_2, S^1, S^2)-inequalities remain valid, but we do not know whether they define facets or not. Another interesting area is to investigate the problem for larger number of items and generalize the (l_1, l_2, S^1, S^2)-inequalities and other facet defining inequalities.

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Appendix A. Projection onto the space of production and setup variables

We first obtain a formulation for *2ULS* using only production, substitution and setup variables. Then we also project out the substitution variables. Substituting $s_t^1 = \sum_{l=1}^t (x_l^1 - a_l^{12}) - D_{1t}^1$, $s_t^2 = \sum_{l=1}^t (x_l^2 + a_l^{12}) - D_{1t}^2$, and $a_t^{22} = d_t^2 - a_t^{12}$ yields:

$$z = K + \min \sum_{t=1}^{n} \left(p_t^1 x_t^1 + p_t^2 x_t^2 + c_t a_t^{12} + q_t^1 y_t^1 + q_t^2 y_t^2\right)$$

s.t. (5), (6), (9)–(11), (13)

$$\sum_{l=1}^{t} \left(x_l^1 - a_l^{12}\right) \geq D_{1t}^1 \quad \forall t \in T \tag{57}$$

$$\sum_{l=1}^{t} \left(x_l^2 + a_l^{12}\right) \geq D_{1t}^2 \quad \forall t \in T \tag{58}$$

$$a_t^{12} \leq d_t^2 \quad \forall t \in T. \tag{59}$$

Notice that (13) implies that $s_n^1 + s_n^2 = 0$. But as both s_n^1 and s_n^2 are nonnegative due to (57) and (58), both quantities are zero.

By Farkas' Lemma, for given values of x_t^i for $i = 1, 2$ and $t \in T$, there exists an assignment of values $a_t^{12} \geq 0$ for $t \in T$ that satisfies (57)–(59) if and only if

$$\sum_{t=1}^{n} \left(\sum_{l=1}^{t} x_l^1 - D_{1t}^1\right) \alpha_t + \sum_{t=1}^{n} \left(\sum_{l=1}^{t} x_l^2 - D_{1t}^2\right) \beta_t + \sum_{t=1}^{n} d_t^2 \gamma_t \geq 0 \tag{60}$$

for all (α, β, γ) such that

$$\sum_{l=t}^{n} \alpha_l - \sum_{l=t}^{n} \beta_l + \gamma_t \geq 0 \quad \forall t \in T$$

$$\alpha_t, \beta_t, \gamma_t \geq 0 \quad \forall t \in T.$$

Let $C = \{(\alpha, \beta, \gamma) \in \mathbb{R}_+^{3n} : \sum_{l=t}^n \alpha_l - \sum_{l=t}^n \beta_l + \gamma_t \geq 0 \;\; \forall t \in T\}$. We characterize the extreme rays of the cone C to be able to obtain a formulation in production and setup variables.

Lemma 8. *For a given $(\alpha, \beta, \gamma) \in C$, define $A = \{t \in T : \alpha_t > 0\}$, $B = \{t \in T : \beta_t > 0\}$, and $\Gamma = \{t \in T : \gamma_t > 0\}$. Suppose (α, β, γ) is an extreme ray of C. Then one of the following is true.*

1. *$A = \emptyset$, $B = \emptyset$ and $|\Gamma| = 1$.*
2. *$A = \emptyset$, $B = \{t_B\}$ for some $t_B \in T$, $\Gamma = \{1, \ldots, t_B\}$ and $\gamma_t = \beta_{t_B}$ for all $t \in \Gamma$.*
3. *$B = \emptyset$, $\Gamma = \emptyset$ and $|A| = 1$.*
4. *$\Gamma = \emptyset$, $A = \{t_A\}$, $B = \{t_B\}$ for some t_A and t_B in T with $t_A \geq t_B$ and $\alpha_{t_A} = \beta_{t_B}$.*
5. *$A = \{t_A\}$, $B = \{t_B\}$ for some t_A and t_B in T with $t_B > t_A$, $\Gamma = \{t_A + 1, \ldots, t_B\}$ and $\alpha_{t_A} = \beta_{t_B} = \gamma_t$ for all $t \in \{t_A + 1, \ldots, t_B\}$.*

Proof. Suppose (α, β, γ) is an extreme ray of C. Let $\epsilon > 0$ be very small. If $A = \emptyset$ and $\Gamma = \emptyset$, then feasibility of (α, β, γ) implies that $B = \emptyset$. Hence we consider the cases where $A \neq \emptyset$ or $\Gamma \neq \emptyset$. If $A = \emptyset$ and $\Gamma \neq \emptyset$, then there are two cases. If $B = \emptyset$, then $|\Gamma| = 1$. If $B \neq \emptyset$ then let t_B be the smallest index in B. As $(\alpha, \beta, \gamma) \in C$ we have $\{1, \ldots, t_B\} \subseteq \Gamma$. Then $(\alpha, \beta, \gamma) = 1/2(\alpha, \beta^1, \gamma^1) + 1/2(\alpha, \beta^2, \gamma^2)$ where $\beta_{t_B}^1 = \beta_{t_B} - \epsilon$, $\beta_{t_B}^2 = \beta_{t_B} + \epsilon$, $\gamma_t^1 = \gamma_t - \epsilon$, $\gamma_t^2 = \gamma_t + \epsilon$ for $t \in \{1, \ldots, t_B\}$, $\beta_t^1 = \beta_t^2 = \beta_t$ for all $t \neq t_B$ and $\gamma_t^1 = \gamma_t^2 = \gamma_t$ for all $t \in \{t_B + 1, \ldots, n\}$. Hence $B = \{t_B\}$, $\Gamma = \{1, \ldots, t_B\}$ and $\gamma_t = \beta_{t_B}$ for all $t \in \Gamma$. If $A \neq \emptyset$ and $\Gamma = \emptyset$, then we again consider two cases. If $B = \emptyset$, then $|A| = 1$. If $B \neq \emptyset$, then let t_A and t_B be the largest indices in A and B, respectively. Feasibility implies that $t_A \geq t_B$. Then $(\alpha, \beta, \gamma) = 1/2(\alpha^1, \beta^1, \gamma) + 1/2(\alpha^2, \beta^2, \gamma)$ where $\alpha_{t_A}^1 = \alpha_{t_A} - \epsilon$, $\alpha_{t_A}^2 = \alpha_{t_A} + \epsilon$, $\beta_{t_B}^1 = \beta_{t_B} - \epsilon$, $\beta_{t_B}^2 = \beta_{t_B} + \epsilon$, $\alpha_t^1 = \alpha_t^2 = \alpha_t$ for all $t \neq t_A$, and $\beta_t^1 = \beta_t^2 = \beta_t$ for all $t \neq t_B$. Hence, $A = \{t_A\}$, $B = \{t_B\}$, and $\alpha_{t_A} = \beta_{t_B}$. If $A \neq \emptyset$, $\Gamma \neq \emptyset$, and $B = \emptyset$, then $(\alpha, \beta, \gamma) = 1/2(\alpha, \beta, \gamma^1) + 1/2(\alpha, \beta, \gamma^2)$ where $\gamma_{t^*}^1 = \gamma_{t^*} - \epsilon$ and $\gamma_{t^*}^2 = \gamma_{t^*} + \epsilon$ for some $t^* \in \Gamma$ and $\gamma_t^1 = \gamma_t^2 = \gamma_t$ for all $t \neq t^*$. This contradicts (α, β, γ) being an extreme ray. Finally, if $A \neq \emptyset$, $\Gamma \neq \emptyset$, and $B \neq \emptyset$, then $A \cap B = \emptyset$. Let t_A and t_B be the largest indices in A and B, respectively. If $t_A \geq t_B$, then $(\alpha, \beta, \gamma) = 1/2(\alpha^1, \beta^1, \gamma) + 1/2(\alpha^2, \beta^2, \gamma)$ where $\alpha_{t_A}^1 = \alpha_{t_A} - \epsilon$, $\alpha_{t_A}^2 = \alpha_{t_A} + \epsilon$, $\beta_{t_B}^1 = \beta_{t_B} - \epsilon$, $\beta_{t_B}^2 = \beta_{t_B} + \epsilon$, $\alpha_t^1 = \alpha_t^2 = \alpha_t$ for all $t \neq t_A$, and $\beta_t^1 = \beta_t^2 = \beta_t$ for all $t \neq t_B$. Hence $t_B > t_A$. Then $(\alpha, \beta, \gamma) = 1/2(\alpha^1, \beta^1, \gamma^1) + 1/2(\alpha^2, \beta^2, \gamma^2)$ where $\alpha_{t_A}^1 = \alpha_{t_A} - \epsilon$, $\alpha_{t_A}^2 = \alpha_{t_A} + \epsilon$, $\beta_{t_B}^1 = \beta_{t_B} - \epsilon$, $\beta_{t_B}^2 = \beta_{t_B} + \epsilon$, $\alpha_t^1 = \alpha_t^2 = \alpha_t$ for all $t \neq t_A$, $\beta_t^1 = \beta_t^2 = \beta_t$ for all $t \neq t_B$, $\gamma_t^1 = \gamma_t - \epsilon$, $\gamma_t^2 = \gamma_t + \epsilon$ for $t \in \{t_A + 1, \ldots, t_B\}$ and $\gamma_t^1 = \gamma_t^2 = \gamma_t$ for all $t \notin \{t_A + 1, \ldots, t_B\}$. This implies that $A = \{t_A\}$, $B = \{t_B\}$ and $\Gamma = \{t_A + 1, \ldots, t_B\}$ and $\alpha_{t_A} = \beta_{t_B} = \gamma_t$ for all $t \in \{t_A + 1, \ldots, t_B\}$. □

Now using the extreme rays of the projection cone, we obtain the projection of the feasible set of the *2ULS* onto the space of production and setup variables.

Theorem 8. *The projection of the feasible set of the 2ULS onto the space of production and setup variables is given by* (5), (6), (10), (11) *and* (13)–(15)*.*

Proof. The first and second types of rays of C give the inequalities $d_t^2 \geq 0$ and $\sum_{l=1}^{t} x_l^2 \geq 0$ for $t \in T$, respectively. Both families of inequalities are redundant. The rays of type three lead to inequalities $\sum_{l=1}^{t} x_l^1 \geq D_{1t}^1$ for $t \in T$. Type four rays give inequalities $\sum_{l=1}^{t_1} x_l^1 + \sum_{l=1}^{t_2} x_l^2 \geq D_{1t_1}^1 + D_{1t_2}^2$ for t_1 and t_2 in T such that $t_1 \geq t_2$. The rays of the last type give the redundant inequalities $\sum_{l=1}^{t_1} x_l^1 + \sum_{l=1}^{t_2} x_l^2 \geq D_{1t_1}^1 + D_{1t_1}^2$ for t_1 and t_2 in T such that $t_1 < t_2$. □

Appendix B. The fractional extreme points of the LP relaxation of the original model together with (l_1, l_2, S^1, S^2)-inequalities for $n = 3$, $d_t^1 = d_t^2 = 1$ for $t = 1, 2, 3$

x_1^1	x_2^1	x_3^1	x_1^2	x_2^2	x_3^2	y_1^1	y_2^1	y_3^1	y_1^2	y_2^2	y_3^2
2	3/2	1	3/2	0	0	1	1/2	1/2	1/2	0	0
5/2	3/2	1	0	1	0	1	1/2	1/2	0	1/2	0
2	3/2	1	3/2	0	0	1	1/2	1/2	1/2	0	1
2	3/2	1	3/2	0	0	1	1/2	1/2	1/2	1	0
5/2	3/2	1	0	1	0	1	1/2	1/2	0	1/2	1
5/2	3/2	1	0	1	0	1	1/2	1/2	1	1/2	0
2	3/2	1	3/2	0	0	1	1/2	1/2	1/2	1	1
3/2	3/2	1	1	1	0	1	1/2	1/2	1	1/2	0
5/2	3/2	1	0	1	0	1	1/2	1/2	1	1/2	1
3/2	3/2	1	1	1	0	1	1/2	1/2	1	1/2	1

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