Available online at www.sciencedirect.com

ScienceDirect

C. R. Acad. Sci. Paris, Ser. I 347 (2009) 735–738

COMPTES RENDUS MATHEMATIQUE

Mathematical Analysis/Harmonic Analysis

Reproducing kernels for harmonic Besov spaces on the ball ☆

Seçil Gergün [a], H. Turgay Kaptanoğlu [b], A. Ersin Üreyen [c]

[a] *Department of Mathematics and Computer Science, Çankaya University, Ankara 06530, Turkey*

[b] *Department of Mathematics, Bilkent University, Ankara 06800, Turkey*

[c] *Department of Mathematics, Anadolu University, Eskişehir 26470, Turkey*

Received 27 February 2009; accepted 24 March 2009

Available online 9 May 2009

Presented by Jean-Pierre Kahane

Abstract

Besov spaces of harmonic functions on the unit ball of $\mathbb{R}^n$ are defined by requiring sufficiently high-order derivatives of functions lie in harmonic Bergman spaces. We compute the reproducing kernels of those Besov spaces that are Hilbert spaces. The kernels turn out to be weighted infinite sums of zonal harmonics and natural radial fractional derivatives of the Poisson kernel. ***To cite this article: S. Gergün et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Noyaux reproduisants pour les espaces harmoniques de Besov sur la boule. Les espaces de Besov de fonctions harmoniques sur la boule unité de $\mathbb{R}^n$ sont défini en exigeant que suffisamment des dérivés de haut ordre de fonctions appartiennent aux espaces de Bergman harmoniques. Nous calculons les noyaux reproduisants de ces espaces de Besov qui sont des espaces de Hilbert. Les noyaux se révèlent être, de façon tout naturel, des sommes infinies pondérées des harmoniques zonalles et des dérivés fractionnels radiaux du noyau de Poisson. ***Pour citer cet article : S. Gergün et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction and preliminaries

Let $\mathbb{B}$ and $\mathbb{S}$ be the unit ball and the unit sphere in $\mathbb{R}^n$ with respect to the usual inner product $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ and the norm $|x| = \sqrt{x \cdot x}$. Always $n \geqslant 2$, and we often write $x = r\xi$, $y = \rho\eta$ with $r = |x|$, $\rho = |y|$ and $\xi, \eta \in \mathbb{S}$. We let ν and σ be the volume and surface measures on $\mathbb{B}$ and $\mathbb{S}$ normalized as $\nu(\mathbb{B}) = 1$ and $\sigma(\mathbb{S}) = 1$. We always take $q \in \mathbb{R}$, and define on $\mathbb{B}$ also the measures $d\nu_q(x) = N_q (1 - |x|^2)^q \, d\nu(x)$. These measures are finite only for $q > -1$ and then we choose N_q so as to have $\nu_q(\mathbb{B}) = 1$. For $q \leqslant -1$, we set $N_q = 1$. We denote the Lebesgue classes with respect to ν_q by L^p_q, and we always consider $1 \leqslant p < \infty$.

We let $h(\mathbb{B})$ denote the space of complex-valued harmonic functions on $\mathbb{B}$, those annihilated by the usual Laplacian Δ, with the topology of uniform convergence on compact subsets of $\mathbb{B}$. The Besov spaces under consideration in

☆ This research is supported by TÜBİTAK under Research Project Grant 108T329.

E-mail addresses: gergun@cankaya.edu.tr (S. Gergün), kaptan@fen.bilkent.edu.tr (H.T. Kaptanoğlu), aureyen@anadolu.edu.tr (A.E. Üreyen).
URL: http://www.fen.bilkent.edu.tr/~kaptan/ (H.T. Kaptanoğlu).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.04.016

this Note form a two-parameter Sobolev-type family within $h(\mathbb{B})$. They extend Bergman spaces of harmonic functions. They have been studied early in [4–8] from a different perspective.

The *Pochhammer symbol* is defined by $(a)_b := \Gamma(a+b)/\Gamma(a)$ when a and $a+b$ are off the pole set of the gamma function Γ. Stirling formula gives

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b} \quad \text{and} \quad \frac{(a)_c}{(b)_c} \sim c^{a-b} \quad (\operatorname{Re} c \to \infty), \tag{1}$$

where $x \sim y$ means that $|x/y|$ is bounded above and below by two positive constants.

The rest of the material in this section is classical and can be found in [1, Chapter 5] or [11, Chapter IV]. For $m = 0, 1, 2, \ldots$, let $\mathcal{H}_m$ denote the space of all harmonic homogeneous polynomials of degree m. By homogeneity, a $u \in \mathcal{H}_m$ is determined by its restriction to $\mathbb{S}$, and we freely identify u with its restriction. The restrictions are called *spherical harmonics*. The space $L^2(\sigma)$ is the orthogonal direct sum of all the $\mathcal{H}_m$.

Lemma 1.1. *If $u \in h(\mathbb{B})$, then u has a unique homogeneous expansion $u = \sum_{m=0}^{\infty} u_m$ with $u_m \in \mathcal{H}_m$, the series converging absolutely on $\mathbb{B}$, and uniformly on its compact subsets.*

The spaces $\mathcal{H}_m$ are finite-dimensional; we set $\delta_m = \dim \mathcal{H}_m$. Then the evaluation functionals at points $\eta \in \mathbb{S}$ are bounded on $\mathcal{H}_m$, and so $\mathcal{H}_m$ is a reproducing kernel Hilbert space. Its reproducing kernel $Z_m(\xi, \eta)$ is called the *zonal harmonic* of degree m; thus Z_m is a positive definite function. Zonal harmonics can be extended to positive definite functions on $\overline{\mathbb{B}} \times \overline{\mathbb{B}}$ as $Z_m(x, y) = r^m \rho^m Z_m(\xi, \eta)$ by homogeneity. Zonal harmonics are real-valued and symmetric in their variables, that is, $Z_m(x, y) = Z_m(y, x)$ for $x, y \in \overline{\mathbb{B}}$. Consequently, Z_m is harmonic in either of its variables since it lies in $\mathcal{H}_m$. There are explicit formulas for $Z_m(x, y)$ in the two books mentioned above, but we do not need them in this Note.

Our main results are given in Section 3. Detailed proofs and further results will be presented elsewhere.

2. Harmonic Bergman spaces and kernels

For $q > -1$, the Bergman space b^p_q is that closed subspace of L^p_q whose members are also in $h(\mathbb{B})$. The Bergman spaces with $p = 2$ are reproducing kernel Hilbert spaces, and the reproducing kernel of b^2_q is

$$R_q(x, y) = \sum_{m=0}^{\infty} \frac{(n/2+1+q)_m}{(n/2)_m} Z_m(x, y) =: \sum_{m=0}^{\infty} \gamma_m(q) Z_m(x, y) \quad (q > -1,\ x, y \in \mathbb{B}), \tag{2}$$

which also defines $\gamma_m(q)$; see [9, Proposition 3]; also [6, p. 25], [8], and [1, pp. 156–157] for $q = 0$; and [13, p. 357] for $n = 2$; and also [2, (3.1)] for integer q. The kernels R_q converge absolutely on $\mathbb{B} \times \mathbb{B}$, and uniformly if one variable lives in a compact subset of $\mathbb{B}$. Therefore the R_q are symmetric in their variables and harmonic as a function of each.

The computation yielding R_q is valid only for $q > -1$, but R_{-1} also perfectly makes sense and is nothing but the homogeneous extension of the Poisson kernel P to $\mathbb{B} \times \mathbb{B}$ since $\gamma_m(-1) = 1$ for all m. In fact, the coefficients $\gamma_m(q)$ make sense down to $q > -(n/2+1)$, and for all such q, they satisfy

$$\gamma_m(q) \sim m^{1+q} \quad (m \to \infty) \tag{3}$$

by (1). With smaller $\gamma_m(q)$, the infinite sums in (2) for $q \leqslant -1$ have at least the same convergence properties on $\mathbb{B} \times \mathbb{B}$ as those for $q > -1$. Since $\gamma_m(q) > 0$ for all m and $q > -(n/2+1)$, and the Z_m are positive definite kernels, by convergence we conclude that R_q given as in (2) is a positive definite function, and thus is a reproducing kernel and generates a reproducing kernel Hilbert space on $\mathbb{B}$ for all $q > -(n/2+1)$. All the references cited in the previous paragraph restrict themselves to $q > -1$ or so. The point of view of getting from kernels to spaces apparently has never been utilized before.

3. Harmonic Besov kernels and spaces

Our purpose now is to extend the kernels R_q even further to all $q \in \mathbb{R}$. The main idea is to replace the coefficients $\gamma_m(q)$ of Z_m in R_q by new $\gamma_m(q)$ that preserve the growth rate of (3) for $q \leqslant -(n/2+1)$ as well.

Definition 3.1. For $m = 0, 1, 2, \ldots$, we set

$$\gamma_m(q) = \begin{cases} \frac{(n/2+1+q)_m}{(n/2)_m}, & \text{if } q > -(n/2+1); \\ \frac{(m!)^2}{(1-n/2-q)_m\,(n/2)_m}, & \text{if } q \leqslant -(n/2+1); \end{cases}$$

and define

$$R_q(x, y) = \sum_{m=0}^{\infty} \gamma_m(q)\, Z_m(x, y). \tag{4}$$

It seems the kernels (4) for $q \leqslant -(n/2+1)$ are completely new here. Note that $\gamma_0(q) = 1$ for all q and (3) is satisfied for all real q by (1) as promised. Moreover, $\gamma_m(q) \neq 0$ for all $m = 0, 1, 2, \ldots$ and all $q \in \mathbb{R}$.

Proposition 3.2. *The series* (4) *converges absolutely for any $x, y \in \mathbb{B}$. If one of x or y lies in a compact subset of $\mathbb{B}$ and the other in $\overline{\mathbb{B}}$, then the series converges uniformly.*

Definition 3.3. Let $u = \sum_{m=0}^{\infty} u_m \in h(\mathbb{B})$ be given as in Lemma 1.1. We define operators D_s^t by

$$D_s^t u := \sum_{m=0}^{\infty} d_m(s, t)\, u_m := \sum_{m=0}^{\infty} \frac{\gamma_m(s+t)}{\gamma_m(s)}\, u_m.$$

For any s, $D_s^0 = I$, the identity. If λ is a multi-index, then $D_s^t x^\lambda = d_{|\lambda|}(s, t)\, x^\lambda$. So in every case $D_s^t(\mathcal{H}_m) = \mathcal{H}_m$. By Definition 3.1 and (1), we have $d_m(s, t) \sim m^t$ as $m \to \infty$ for all $s \in \mathbb{R}$. Particularly, $D_{-n/2}^1 = \mathcal{R} + I$, where $\mathcal{R}$ is the radial derivative given by $\mathcal{R}u(x) = \nabla u(x) \cdot x = \sum_{m=0}^{\infty} m^1\, u_m(x)$ in which ∇ is the usual gradient. Summing up, each D_s^t is a radial differential operator of fractional order t.

Moreover, $d_m(s, t) \neq 0$ for all choices of m, s, t. Then every D_s^t is invertible with two-sided inverse $(D_s^t)^{-1} = D_{s+t}^{-t}$, which follows from the additive property $D_{s+t}^u D_s^t = D_s^{t+u}$. The operators D_s^t are constructed so that $D_s^t R_s(x, y) = R_{s+t}(x, y)$ in all cases, where differentiation is performed only on one of the variables x, y; and by symmetry it does not matter which. In particular, $R_q(x, y) = D_{-1}^{1+q} P(x, y)$ extending [2, (3.1)], [1, 8.12], and a formula in [9, p. 29].

Lemma 3.4. *Every D_s^t maps $h(\mathbb{B})$ into itself continuously. Thus $D_s^t u$ is harmonic on $\mathbb{B}$ if u is.*

Definition 3.5. Consider the linear transformation I_s^t defined for $u \in h(\mathbb{B})$ by $I_s^t u(x) = (1 - |x|^2)^t\, D_s^t u(x)$. For $q \in \mathbb{R}$ and $1 \leqslant p < \infty$, we define the *harmonic Besov space* b_q^p to consist of all $u \in h(\mathbb{B})$ for which $I_s^t u$ belongs to L_q^p for some s, t satisfying

$$q + pt > -1. \tag{5}$$

Condition (5) assures that all b_q^p contain the polynomials and therefore are nontrivial. Thus $\mathcal{H}_m \subset b_q^p$ for all possible values of the parameters. For any s, t satisfying (5), by the invertibility of D_s^t and that $1 - |x|^2 \neq 0$ for $x \in \mathbb{B}$, the map I_s^t imbeds b_q^p into L_q^p. Then $\|u\|_{b_q^p} := \|I_s^t f\|_{L_q^p}$ defines a norm on b_q^p for each such s, t, and only $0 \in h(\mathbb{B})$ has norm 0. Similarly, each pair s, t satisfying (5) with $p = 2$ gives rise to an inner product on b_q^2 by $[u, v]_{b_q^2} = [I_s^t u, I_s^t v]_{L_q^2}$. Explicitly,

$$\|u\|_{b_q^p}^p = N_q \int_{\mathbb{B}} \left|D_s^t u(x)\right|^p \left(1 - |x|^2\right)^{q+pt} \mathrm{d}\nu(x), \qquad [u, v]_{b_q^2} = N_q \int_{\mathbb{B}} D_s^t u(x)\, \overline{D_s^t v(x)} \left(1 - |x|^2\right)^{q+2t} \mathrm{d}\nu(x).$$

Proposition 3.6. *Different values of s, t satisfying* (5) *give equivalent norms $\|\cdot\|_{b_q^p}$ on b_q^p.*

Definition 3.5 assigns the space b_q^p to the point (p, q) in the half plane $\{\operatorname{Re} p \geqslant 1\}$. When $q > -1$, we can take $t = 0$, and thus the spaces b_q^p are the well-known harmonic weighted Bergman spaces. Our main interest lies in the region $q \leqslant -1$, but our results cover and generalize what is known for $q > -1$ as well.

Harmonic Besov spaces are studied in [4–8] in the generality of Definition 3.5, but recently only smaller subfamilies are considered such as with $q = -n$ or $t = 1$ or both; see [3,10,12,14].

Theorem 3.7. *Each b_q^2 for $q \in \mathbb{R}$ is a reproducing kernel Hilbert space, and its reproducing kernel is R_q.*

For another description of b_q^2, let's start with the homogeneous expansion of $u \in h(\mathbb{B})$ given in Lemma 1.1. If $\{Y_{m1}, \dots, Y_{m\delta_m}\}$ is an orthonormal basis for $\mathcal{H}_m \subset L^2(\sigma)$, then each u_m restricted to $\mathbb{S}$ has itself an expansion in terms of the $\{Y_{mk}\}$, and thus

$$u(x) = \sum_{m=0}^{\infty} r^m\, u_m(\xi) = \sum_{m=0}^{\infty} r^m \sum_{k=1}^{\delta_m} c_{mk}\, Y_{mk}(\xi) = \sum_{m=0}^{\infty}\sum_{k=1}^{\delta_m} c_{mk}\, Y_{mk}(x) \quad (x \in \mathbb{B}), \tag{6}$$

$$c_{mk} = \int_{\mathbb{S}} u_m(\xi)\overline{Y_{mk}(\xi)}\,\mathrm{d}\sigma(\xi) = \frac{1}{r^m}\int_{\mathbb{S}} u_m(r\xi)\,\overline{Y_{mk}(\xi)}\,\mathrm{d}\sigma(\xi) = \frac{1}{r^m}\int_{\mathbb{S}} u(r\xi)\overline{Y_{mk}(\xi)}\,\mathrm{d}\sigma(\xi) \quad (0 < r < 1)$$

by orthogonality. We also see that this computation of c_{mk} is independent of $r \in (0,1)$.

Theorem 3.8. *The Hilbert space b_q^2 coincides with the space β_q of functions $u \in h(\mathbb{B})$ with expansions of the form* (6) *for which*

$$\sum_{m=0}^{\infty}\sum_{k=1}^{\delta_m} \frac{|c_{mk}|^2}{\gamma_m(q)} < \infty \tag{7}$$

equipped with the inner product $\langle u, v\rangle_q = \sum_{m=0}^{\infty}\sum_{k=1}^{\delta_m} \frac{1}{\gamma_m(q)} c_{mk}\overline{c'_{mk}}$ and the associated norm $|||u|||_q = \langle u, u\rangle^{1/2}$, where primes denote the coefficients of $v \in h(\mathbb{B})$.

Corollary 3.9. *The norms $|||\cdot|||_q$ and $\|\cdot\|_{b_q^2}$ are equivalent on b_q^2.*

Example 3.10. Harmonic Besov spaces b_q^2 with different q are different. Let $q_1 < q_2$. By Definition 3.5, it is clear that $b_{q_1}^2 \subset b_{q_2}^p$. Next, define $u(x) = \sum_{m=1}^{\infty} m^{(q_1+q_2)/4}\, Y_{m1}(x)$. Then $u \in b_{q_2}^2 \setminus b_{q_1}^2$, because by (7),

$$|||u|||_{q_1} \sim \sum_{m=1}^{\infty} \frac{1}{m^{1-(q_2-q_1)/2}} = \infty \quad \text{while} \quad |||u|||_{q_2} \sim \sum_{m=1}^{\infty} \frac{1}{m^{1+(q_2-q_1)/2}} < \infty.$$

References

[1] S. Axler, P. Bourdon, W. Ramey, Harmonic Function Theory, Springer, New York, 1992.
[2] R.R. Coifman, R. Rochberg, Representation theorems for holomorphic and harmonic functions in L^p, Astérisque 77 (1980) 12–66.
[3] M. Jevtić, M. Pavlović, Harmonic Besov spaces on the unit ball of $\mathbb{R}^n$, Rocky Mountain J. Math. 31 (2001) 1305–1316.
[4] E. Ligocka, The Sobolev spaces of harmonic functions, Studia Math. 84 (1986) 79–87.
[5] E. Ligocka, Estimates in Sobolev norms $\|\cdot\|_p^s$ for harmonic and holomorphic functions and interpolation between Sobolev and Hölder spaces of harmonic functions, Studia Math. 86 (1987) 255–271.
[6] E. Ligocka, On the reproducing kernel for harmonic functions and the space of Bloch harmonic functions on the unit ball in $\mathbb{R}^n$, Studia Math. 87 (1987) 23–32.
[7] E. Ligocka, On the space of Bloch harmonic functions and interpolation of spaces of harmonic and holomorphic functions, Studia Math. 87 (1987) 223–238.
[8] E. Ligocka, Corrigendum to the paper "On the reproducing kernel for harmonic functions and the space of Bloch harmonic functions on the unit ball in $\mathbb{R}^n$", Studia Math. 101 (1992) 319.
[9] J. Miao, Reproducing kernels for harmonic Bergman spaces of the unit ball, Monatsh. Math. 125 (1998) 25–35.
[10] G. Ren, U. Kähler, Weighted Lipschitz continuity and harmonic Bloch and Besov spaces in the unit real ball, Proc. Edinb. Math. Soc. 48 (2005) 743–755.
[11] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ., Princeton, 1971.
[12] S. Stević, On harmonic function spaces, J. Math. Soc. Japan 57 (2005) 781–802.
[13] Z. Wu, Operators on harmonic Bergman spaces, Integral Equations Operator Theory 24 (1996) 352–371.
[14] R. Yoneda, A characterization of the harmonic Bloch space and the harmonic Besov spaces by an oscillation, Proc. Edinb. Math. Soc. 45 (2002) 229–239.