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Modified Korteweg–de Vries surfaces

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In this work, we consider 2-surfaces in $\mathbb{R}^3$ arising from the modified Korteweg–de Vries (mKdV) equation. We give a method for constructing the position vector of the mKdV surface explicitly for a given solution of the mKdV equation. mKdV surfaces contain Willmore-like and Weingarten surfaces. We show that some mKdV surfaces can be obtained from a variational principle where the Lagrange function is a polynomial of the Gaussian and mean curvatures. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2409523]

I. INTRODUCTION

In this work we study the 2-surfaces in $\mathbb{R}^3$ arising from the deformations of the modified Korteweg–de Vries (mKdV) equation and its Lax pair. Deformation technique was developed by several authors. Here we mainly follow Refs. 1–12.

Let $u(x,t)$ satisfy the mKdV equation

$$u_t = u_{3x} + \frac{3}{2}u^2 u_x. \tag{1}$$

Substituting the traveling wave ansatz $u_t - \alpha\, u_x = 0$ in Eq. (1), where α is an arbitrary real constant, we get

$$u_{2x} = \alpha u - \frac{u^3}{2}. \tag{2}$$

Here and in what follows, subscripts x, t, and λ denote the derivatives of the objects with respect to x, t, and λ, respectively. The subscript nx stands for n times x derivative, where n is a positive integer, e.g., u_{2x} indicates the second derivative of u with respect to x. We use Einstein's summation convention on repeated indices over their range. Equation (2) can be obtained from a Lax pair U and V, where

$$U = \frac{i}{2}\begin{pmatrix} \lambda & -u \\ -u & -\lambda \end{pmatrix}, \tag{3}$$

$$V = -\frac{i}{2}\begin{pmatrix} \frac{1}{2}u^2 - (\alpha + \alpha\lambda + \lambda^2) & (\alpha+\lambda)u - iu_x \\ (\alpha+\lambda)u + iu_x & -\frac{1}{2}u^2 + (\alpha + \alpha\lambda + \lambda^2) \end{pmatrix}, \tag{4}$$

and λ is the spectral parameter. The Lax equations are given as

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$$\Phi_x = U\Phi, \quad \Phi_t = V\Phi, \tag{5}$$

where the integrability of these equations are guaranteed by the mKdV equation or the zero curvature condition

$$U_t - V_x + [U,V] = 0. \tag{6}$$

A connection of the mKdV equation to surfaces in $\mathbb{R}^3$ can be achieved by defining $su(2)$ valued 2×2 matrices A and B satisfying

$$A_t - B_x + [A,V] + [U,B] = 0. \tag{7}$$

Let F be an $su(2)$ valued position vector of the surface S corresponding to the mKdV equation such that

$$y_j = F_j(x,t;\lambda), \quad j = 1,2,3, \quad F = i\sum_{k=1}^{3} F_k\sigma_k, \tag{8}$$

where σ_k's are the Pauli sigma matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

The connection formula (connecting integrable systems to 2-surfaces in $\mathbb{R}^3$) is given by

$$F_x = \Phi^{-1}A\Phi, \quad F_t = \Phi^{-1}B\Phi. \tag{10}$$

Then at each point on S, there exists a frame $\{F_x, F_t, \Phi^{-1}C\Phi\}$ forming a basis of $\mathbb{R}^3$, where $C = [A,B]/\|[A,B]\|$ and $[A,B]$ denotes the usual commutator $[A,B]=AB-BA$. The inner product $\langle , \rangle$ of $su(2)$ valued vectors X and Y are given by $\langle X,Y\rangle = -\frac{1}{2}\mathrm{tr}(XY)$. Hence $\|X\| = \sqrt{|\langle X,X\rangle|}$. The first and second fundamental forms of S are

$$\begin{aligned} (\mathrm{d}s_{\mathrm{I}})^2 &\equiv g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \langle A,A\rangle \mathrm{d}x^2 + 2\langle A,B\rangle \mathrm{d}x\mathrm{d}t + \langle B,B\rangle \mathrm{d}t^2, \\ (\mathrm{d}s_{\mathrm{II}})^2 &\equiv h_{ij}\mathrm{d}x^i\mathrm{d}x^j = \langle A_x + [A,U],C\rangle \mathrm{d}x^2 + 2\langle A_t + [A,V],C\rangle \mathrm{d}x\mathrm{d}t + \langle B_t + [B,V],C\rangle \mathrm{d}t^2, \end{aligned} \tag{11}$$

where $i,j=1,2$, $x^1=x$, and $x^2=t$. Here g_{ij} and h_{ij} are coefficients of the first and second fundamental forms, respectively. The Gauss and the mean curvatures of S are, respectively, given by $K=\det(g^{-1}h)$ and $H=\frac{1}{2}\mathrm{tr}(g^{-1}h)$, where g and h denote the matrices (g_{ij}) and (h_{ij}), and g^{-1} stands for the inverse of the matrix g.

In order to calculate the fundamental forms in Eq. (11) and the curvatures K and H, one needs the deformation matrices A and B. Given U and V, finding A and B from Eq. (7) is a difficult task in general. However, there are some deformations which provide A and B directly. They are given as follows:

- Spectral parameter λ invariance of the equation:

$$A = \mu\frac{\partial U}{\partial\lambda}, \quad B = \mu\frac{\partial V}{\partial\lambda}, \quad F = \mu\Phi^{-1}\frac{\partial\Phi}{\partial\lambda}, \tag{12}$$

  where μ is an arbitrary function of λ.
- Symmetries of the (integrable) differential equations:

$$A = \delta U, \quad B = \delta V, \quad F = \Phi^{-1}\delta\Phi, \tag{13}$$

  where δ represents the classical Lie symmetries and (if integrable) the generalized symmetries of the nonlinear partial differential equations (PDEs).
- Gauge symmetries of the Lax equation:

$$A = M_x + [M,U], \quad B = M_t + [M,V], \quad F = \Phi^{-1} M \Phi, \tag{14}$$

where M is any traceless 2×2 matrix.

There are some surfaces which may be obtained from a variational principle. For this purpose, we consider a functional $\mathcal{F}$ which is defined by

$$\mathcal{F} \equiv \int_S \mathcal{E}(H,K)\mathrm{d}A + p\int_V \mathrm{d}V, \tag{15}$$

where $\mathcal{E}$ is some function of the curvatures H and K, p is a constant, and V is the volume enclosed by the surface S. For open surfaces, we let $p=0$. The first variation of the functional $\mathcal{F}$ gives the following Euler-Lagrange equation for the Lagrange function $\mathcal{E}$:[13–16]

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla\cdot\bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0, \tag{16}$$

where ∇^2 and $\nabla\cdot\bar{\nabla}$ are defined as

$$\nabla^2 = \frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial x^i}\left(\sqrt{\tilde{g}}g^{ij}\frac{\partial}{\partial x^j}\right), \quad \nabla\cdot\bar{\nabla} = \frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial x^i}\left(\sqrt{\tilde{g}}Kh^{ij}\frac{\partial}{\partial x^j}\right), \tag{17}$$

and $\tilde{g}=\det(g_{ij})$, where g^{ij} and h^{ij} are the inverse components of the first and second fundamental forms, $x^1=x$, $x^2=t$. The following are examples of surfaces derived from a variational principle:

(i) Minimal surfaces: $\mathcal{E}=1$, $p=0$;
(ii) constant mean curvature surfaces: $\mathcal{E}=1$;
(iii) linear Weingarten surfaces: $\mathcal{E}=aH+b$, where a and b are some constants;
(iv) Willmore surfaces: $\mathcal{E}=H^2$;[17,18] and
(v) surfaces solving the shape equation of lipid membrane: $\mathcal{E}=(H-c)^2$, where c is a constant.[13–16,19–21]

The surfaces obtained from the solutions of the equation

$$\nabla^2 H + aH^3 + bHK = 0 \tag{18}$$

are called *Willmore-like* surfaces, where ∇^2 is the Laplace-Beltrami operator defined on the surface and a,b are arbitrary constants. Unless $a=2$ and $b=-2$, these surfaces do not arise from a variational problem. The case $a=-b=2$ corresponds to the Willmore surfaces. For compact 2-surfaces, the constant p may be different than zero, but for noncompact surfaces we assume it to be zero. For the latter, we require asymptotic conditions, where K goes to a constant and H goes to zero. This requires that the mKdV equation have solutions decaying rapidly to zero as $|x|\to\pm\infty$. Soliton solutions of the mKdV equation satisfy this requirement. In this work, using solitonic solutions of the mKdV equation, we find the corresponding 2-surfaces and then solve the Euler-Lagrange equation [Eq. (16)] for polynomial Lagrange functions of H and K, i.e.,

$$\mathcal{E} = a_N H^N + \cdots + b_{10}KH + b_{11}KH^2 + \cdots + e_1 K + \ldots . \tag{19}$$

For each N, we find the constants a_l, b_{nk}, and e_m in terms of others and the parameters of the surface.

From a solution of the mKdV equation, we first find the fundamental forms in Eq. (11) and the curvatures K and H of the corresponding 2-surface S. To find the position vector $\mathbf{y}(x,t)$ of S, we use Eq. (10). To solve this equation, we need the matrix Φ satisfying the Lax equation [Eq. (5)] for a given function $u(x,t)$. Hence, in general, our method for constructing the position vector $\mathbf{y}$ of integrable surfaces consists of the following steps:

(i) Find a solution $u(x,t)$ of the mKdV equation.

(ii) Find a solution of the Lax equation [Eq. (5)] for a given $u(x,t)$.
(iii) Find the corresponding deformation matrices A, B, and find F from Eq. (10).

In this work more specifically, starting with *one soliton* solution of the mKdV equation and following the steps above, we solve the Lax equations and find the corresponding SU(2) valued function $\Phi(x,t)$. Then using the spectral deformations and combination of the gauge and spectral deformations, we find the parametric representations (position vectors) of the mKdV surfaces and plot some of them for some special values of constants. We show that there are some Weingarten and Willmore-like mKdV surfaces obtained from spectral deformations. Surfaces arising from a combination of the gauge and spectral deformations do not contain Willmore-like surfaces. We study also the mKdV surfaces corresponding to the symmetry deformations. We determine all geometric quantities in terms of the function $u(x,t)$ and the symmetry $\phi(x,t)$. For the simplest symmetry $\phi=u_x$, the surface turns out to be the surface of the sphere with radius $|(\alpha\mu)/(2\lambda)|$, where λ is the spectral parameter and α and μ are constants.

II. mKdV SURFACES FROM SPECTRAL DEFORMATIONS

In this section, we find surfaces arising from the spectral deformation of Lax pair for the mKdV equation. We start with the following proposition.

Proposition 1: Let u satisfy (which describes a traveling mKdV wave) Eq. (2). The corresponding su(2) valued Lax pair U and V of the mKdV equation are given by Eqs. (3) and (4), respectively. Then, su(2) valued matrices A and B are

$$A=\frac{i}{2}\begin{pmatrix}\mu & 0\\ 0 & -\mu\end{pmatrix}, \tag{20}$$

$$B=-\frac{i}{2}\begin{pmatrix}-(\alpha\mu+2\mu\lambda) & \mu u\\ \mu u & \alpha\mu+2\mu\lambda\end{pmatrix}, \tag{21}$$

where $A=\mu U_\lambda$, $B=\mu V_\lambda$, μ is a constant, and λ is the spectral parameter. The surface S, generated by U, V, A and B, has the following first and second fundamental forms $(j,k=1,2)$:

$$(\mathrm{d}s_{\mathrm{I}})^2=g_{jk}\mathrm{d}x^j\mathrm{d}x^k=\frac{\mu^2}{4}([\mathrm{d}x+(\alpha+2\lambda)\mathrm{d}t]^2+u^2\mathrm{d}t^2), \tag{22}$$

$$(\mathrm{d}s_{\mathrm{II}})^2=h_{jk}\mathrm{d}x^j\mathrm{d}x^k=\frac{\mu u}{2}(\mathrm{d}x+(\alpha+\lambda)\mathrm{d}t)^2+\frac{\mu u}{4}(u^2-2\alpha)\mathrm{d}t^2, \tag{23}$$

with the corresponding Gaussian and mean curvatures

$$K=\frac{2}{\mu^2}(u^2-2\alpha),\quad H=\frac{1}{2\mu u}(3u^2+2(\lambda^2-\alpha)), \tag{24}$$

where $x^1=x$, $x^2=t$.

By using U, V, A, and B and the method given in Sec. I, Proposition 1 provides the first and second fundamental forms, and the Gaussian and mean curvatures of the surface corresponding to spectral deformation. The following proposition gives a class of surfaces which are Willmore-like.

Proposition 2: Let $u_x^2=\alpha u^2-u^4/4$. Then the surface S, defined in Proposition 1, is a Willmore-like surface, i.e., the Gaussian and mean curvatures satisfy Eq. (18), where

$$a=\frac{4}{9},\quad b=1,\quad \alpha=\lambda^2, \tag{25}$$

and λ is an arbitrary constant.

It is important to search for mKdV surfaces arising from a variational principle.[13–16] For this purpose, we do not need a parametrization of the surface. The fundamental forms and the Gauss and mean curvatures are enough to look for such mKdV surfaces. The following proposition gives a class of mKdV surfaces that solves the Euler-Lagrange equation [Eq. (16)].

Proposition 3: Let $u_x^2 = \alpha u^2 - u^4/4$. *Then there are mKdV surfaces defined in Proposition 1 satisfying the generalized shape equation [Eq. (16)] when* $\mathcal{E}$ *is a polynomial function of H and K.*

Here are several examples:

Example 1: Let deg $(\mathcal{E}) = N$, then

(i) for $N=3$:

$$\mathcal{E} = a_1 H^3 + a_2 H^2 + a_3 H + a_4 + a_5 K + a_6 KH,$$

$$\alpha = \lambda^2, \quad a_1 = -\frac{p\mu^4}{72\lambda^4}, \quad a_2 = a_3 = a_4 = 0, \quad a_6 = \frac{p\mu^4}{32\lambda^4},$$

where $\lambda \neq 0$, and μ, p, and a_5 are arbitrary constants;

(ii) for $N=4$:

$$\mathcal{E} = a_1 H^4 + a_2 H^3 + a_3 H^2 + a_4 H + a_5 + a_6 K + a_7 KH + a_8 K^2 + a_9 KH^2,$$

$$\alpha = \lambda^2, \quad a_2 = -\frac{p\mu^4}{72\lambda^4}, \quad a_3 = -\frac{8\lambda^2}{15\mu^2}(27a_1 - 8a_8), \quad a_4 = 0,$$

$$a_5 = \frac{\lambda^4}{5\mu^4}(81a_1 + 16a_8), \quad a_7 = \frac{p\mu^4}{32\lambda^4}, \quad a_9 = -\frac{1}{120}(189a_1 + 64a_8),$$

where $\lambda \neq 0$, $\mu \neq 0$, and p, a_1, a_6, and a_8 are arbitrary constants;

(iii) for $N=5$:

$$\mathcal{E} = a_1 H^5 + a_2 H^4 + a_3 H^3 + a_4 H^2 + a_5 H + a_6 + a_7 K + a_8 KH + a_9 K^2 + a_{10} KH^2 + a_{11} K^2 H + a_{12} KH^3,$$

$$\alpha = \lambda^2, \quad a_3 = -\frac{1}{504\mu^2\lambda^2}(\lambda^6[4212a_1 + 256a_{11}] + 7p\mu^6),$$

$$a_4 = -\frac{8\lambda^2}{15\mu^2}(27a_2 - 8a_9), \quad a_5 = \frac{6\lambda^4}{7\mu^4}(135a_1 - 88a_{11}),$$

$$a_6 = \frac{\lambda^4}{5\mu^4}(81a_2 + 16a_9), \quad a_8 = \frac{1}{32\mu^2\lambda^2}(\lambda^6[-324a_1 + 512a_{11}] + p\mu^6),$$

$$a_{10} = -\frac{1}{120}(189a_2 + 64a_9), \quad a_{12} = -\frac{1}{756}(1053a_1 + 512a_{11}),$$

where $\lambda \neq 0$, $\mu \neq 0$, and p, a_1, a_2, a_7, a_9, and a_{11} are arbitrary constants;

(iv) for $N=6$:

$$\mathcal{E} = a_1 H^6 + a_2 H^5 + a_3 H^4 + a_4 H^3 + a_5 H^2 + a_6 H + a_7 + a_8 K + a_9 KH + a_{10} K^2 + a_{11} KH^2 + a_{12} K^2 H + a_{13} KH^3 + a_{14} K^3 + a_{15} K^2 H^2 + a_{16} KH^4,$$

$$\alpha=\lambda^2,$$

$$a_4=-\frac{1}{504\mu^2\lambda^2}(\lambda^6[4212a_2+256a_{12}]+7p\mu^6),$$

$$a_5=-\frac{\lambda^4}{900\mu^4}(-359\,397a_1+191\,488a_{14}-203\,472a_{16})-\frac{8\lambda^2}{15\mu^2}(27a_3-8a_{10}),$$

$$a_6=\frac{6\lambda^4}{7\mu^4}(135a_2-88a_{12}),$$

$$a_7=\frac{\lambda^6}{25\mu^6}(29\,889a_1-9856a_{14}+11\,664a_{16})+\frac{\lambda^4}{5\mu^4}(81a_3+16a_{10}),$$

$$a_9=\frac{1}{32\mu^2\lambda^4}(\lambda^6[-324a_2+512a_{12}]+p\mu^6),$$

$$a_{11}=-\frac{\lambda^2}{1800\mu^2}(59\,778a_1-13\,312a_{14}+23\,328a_{16})-\frac{1}{120}(189a_3+64a_{10}),$$

$$a_{13}=-\frac{1}{756}(1053a_2+512a_{12}),$$

$$a_{15}=-\frac{1}{2880}(5103a_1+2048a_{14}+3888a_{16}),$$

where $\lambda\neq 0$, $\mu\neq 0$, and p, a_1, a_2, a_3, a_8, a_{10}, a_{12}, a_{14}, and a_{16} are arbitrary constants.

For general N, from the above examples, the polynomial function $\mathcal{E}$ takes the form

- for odd N:

$$\mathcal{E}=\sum_{l=0}^{M}a_lH^{2l+1}+\sum_{n=1}^{M}\left(\sum_{k=0}^{(M-n)}b_{kn}H^{2k+1}\right)K^n+eK,$$

where $N=2M+1$, $M=1,2,3\ldots$;
- for even N:

$$\mathcal{E}=\sum_{l=0}^{M}a_lH^{2l}+\sum_{n=1}^{(M-1)}\left(\sum_{k=0}^{(M-1-n)}b_{kn}H^{2k+2}\right)K^n+\sum_{m=1}^{M}e_mK^m,$$

where $N=2M$, $M=2,3,4,\ldots$. In both cases a_l, b_{kn}, and e_m are constants.

A. The parametrized form of the three parameter family of mKdV surfaces

In the previous section, we found possible mKdV surfaces satisfying certain equations. In this section, we find the position vector

$$\mathbf{y} = (y_1(x,t), y_2(x,t), y_3(x,t)) \tag{26}$$

of the mKdV surfaces for a given solution of the mKdV equation and the corresponding Lax pair. To determine $\mathbf{y}$, we use the following steps:

(i) Find a solution u of the mKdV equation with a given symmetry: Here we consider Eq. (2), which is obtained from the mKdV equation by using the traveling wave solutions $u_t = \alpha u_x$, where $\alpha = -1/c$, $c \neq 0$ are arbitrary constants.

(ii) Find the matrix Φ of the Lax equation [Eq. (5)] for given U and V: In our case, the corresponding $su(2)$ valued U and V of the mKdV equation are given by Eqs. (3) and (4). Consider the 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}. \tag{27}$$

By using this and Eq. (3) for U, we can write $\Phi_x = U\Phi$ in matrix form as

$$\begin{pmatrix} (\Phi_{11})_x & (\Phi_{12})_x \\ (\Phi_{21})_x & (\Phi_{22})_x \end{pmatrix} = \begin{pmatrix} \frac{1}{2}i\lambda\Phi_{11} - \frac{1}{2}iu\Phi_{21} & \frac{1}{2}i\lambda\Phi_{12} - \frac{1}{2}iu\Phi_{22} \\ -\frac{1}{2}i\lambda\Phi_{21} - \frac{1}{2}iu\Phi_{11} & -\frac{1}{2}i\lambda\Phi_{22} - \frac{1}{2}iu\Phi_{12} \end{pmatrix}. \tag{28}$$

Combining $(\Phi_{11})_x = \frac{1}{2}i\lambda\Phi_{11} - \frac{1}{2}iu\Phi_{21}$ and $(\Phi_{21})_x = -\frac{1}{2}i\lambda\Phi_{21} - \frac{1}{2}iu\Phi_{11}$, we get

$$(\Phi_{21})_{xx} - \frac{u_x}{u}(\Phi_{21})_x + \left[\frac{1}{4u}(u(\lambda^2 + u^2) - 2i\lambda u_x)\right]\Phi_{21} = 0. \tag{29}$$

Similarly, a second order equation can be written for Φ_{22} by using the first order equations of Φ_{12} and Φ_{22}. By solving the second order equation [Eq. (29)] of Φ_{21} and the equation for Φ_{22}, we determine the explicit x dependence of Φ_{21}, Φ_{22} and also Φ_{11}, Φ_{12}. The components of $\Phi_t = V\Phi$ read

$$(\Phi_{11})_t = -\frac{i}{2}\left[\frac{u^2}{2} - \alpha - \alpha\lambda - \lambda^2\right]\Phi_{11} - \frac{i}{2}[(\alpha+\lambda)u - iu_x]\Phi_{21}, \tag{30}$$

$$(\Phi_{21})_t = \frac{i}{2}\left[\frac{u^2}{2} - \alpha - \alpha\lambda - \lambda^2\right]\Phi_{21} - \frac{i}{2}[(\alpha+\lambda)u + iu_x]\Phi_{11}, \tag{31}$$

and

$$(\Phi_{12})_t = -\frac{i}{2}\left[\frac{u^2}{2} - \alpha - \alpha\lambda - \lambda^2\right]\Phi_{12} - \frac{i}{2}[(\alpha+\lambda)u - iu_x]\Phi_{22}, \tag{32}$$

$$(\Phi_{22})_t = \frac{i}{2}\left[\frac{u^2}{2} - \alpha - \alpha\lambda - \lambda^2\right]\Phi_{22} - \frac{i}{2}[(\alpha+\lambda)u + iu_x]\Phi_{12}. \tag{33}$$

By solving these equations, we determine the explicit t dependence of Φ_{11}, Φ_{21}, Φ_{12}, and Φ_{22}. This way we completely determine the solution Φ of the Lax equations.

(iii) We use Eq. (10) to find F. For our case, A and B are given by Eqs. (20) and (21), which are obtained by the spectral deformation of U and V, respectively. Integrating Eq. (10), we get F.

Now by using a given solution of the mKdV equation, we find the position vector of the mKdV surface. Let $u = k_1 \operatorname{sech} \xi$, $\xi = k_1(k_1^2 t + 4x)/8$, be one soliton solution of the mKdV equation, where $\alpha = k_1^2/4$. By substituting u into the second order equation [Eq. (29)] and using the notation $u_x = k_1 u_\xi/2, (\Phi_{21})_x = k_1(\Phi_{21})_\xi/2$, we find the solution of $\Phi_x = U\Phi$ as follows:

$$\Phi_{21} = iA_1(t)(\tanh \xi + 1)^{i\lambda/2k_1}(\tanh \xi - 1)^{-i\lambda/2k_1} \operatorname{sech} \xi + B_1(t)(k_1 \tanh \xi + 2i\lambda) \times(\tanh \xi - 1)^{i\lambda/2k_1}(\tanh \xi + 1)^{-i\lambda/2k_1}, \tag{34}$$

$$\Phi_{22} = iA_2(t)(\tanh \xi + 1)^{i\lambda/2k_1}(\tanh \xi - 1)^{-i\lambda/2k_1} \operatorname{sech} \xi + B_2(t)(k_1 \tanh \xi + 2i\lambda) \times(\tanh \xi - 1)^{i\lambda/2k_1}(\tanh \xi + 1)^{-i\lambda/2k_1}, \tag{35}$$

$$\Phi_{11} = -\frac{i}{k_1}A_1(t)(2\lambda + ik_1 \tanh \xi)(\tanh \xi + 1)^{i\lambda/2k_1}(\tanh \xi - 1)^{-i\lambda/2k_1} + ik_1B_1(t) \times(\tanh \xi - 1)^{i\lambda/2k_1}(\tanh \xi + 1)^{-i\lambda/2k_1} \operatorname{sech} \xi, \tag{36}$$

$$\Phi_{12} = -\frac{i}{k_1}A_2(t)(2\lambda + ik_1 \tanh \xi)(\tanh \xi + 1)^{i\lambda/2k_1}(\tanh \xi - 1)^{-i\lambda/2k_1} + ik_1B_2(t) \times(\tanh \xi - 1)^{i\lambda/2k_1}(\tanh \xi + 1)^{-i\lambda/2k_1} \operatorname{sech} \xi. \tag{37}$$

Hence one part ($\Phi_x = U\Phi$) of the Lax equations has been solved. By using these solutions in Eqs. (30)–(33) obtained from $\Phi_t = V\Phi$, we find

$$A_1(t) = A_1 e^{i(k_1^2+4\lambda^2)t/8} \quad \text{and} \quad B_1(t) = B_1 e^{-i(k_1^2+4\lambda^2)t/8}, \tag{38}$$

$$A_2(t) = A_2 e^{i(k_1^2+4\lambda^2)t/8} \quad \text{and} \quad B_2(t) = B_2 e^{-i(k_1^2+4\lambda^2)t/8}, \tag{39}$$

where A_1, A_2, B_1, and B_2 are arbitrary constants. We solved the Lax equations for a given U, V and a solution u of the mKdV equation [Eq. (2)]. The components of Φ are

$$\Phi_{11} = -\frac{i}{k_1}A_1 e^{i(k_1^2+4\lambda^2)t/8}(2\lambda + ik_1 \tanh \xi)(\tanh \xi + 1)^{i\lambda/2k_1}(\tanh \xi - 1)^{-i\lambda/2k_1} + ik_1B_1 e^{-i(k_1^2+4\lambda^2)t/8}(\tanh \xi - 1)^{i\lambda/2k_1}(\tanh \xi + 1)^{-i\lambda/2k_1} \operatorname{sech} \xi, \tag{40}$$

$$\Phi_{12} = -\frac{i}{k_1}A_2 e^{i(k_1^2+4\lambda^2)t/8}(2\lambda + ik_1 \tanh \xi)(\tanh \xi + 1)^{i\lambda/2k_1}(\tanh \xi - 1)^{-i\lambda/2k_1} + ik_1B_2 e^{-i(k_1^2+4\lambda^2)t/8}(\tanh \xi - 1)^{i\lambda/2k_1}(\tanh \xi + 1)^{-i\lambda/2k_1} \operatorname{sech} \xi. \tag{41}$$

$$\Phi_{21} = iA_1 e^{i(k_1^2+4\lambda^2)t/8}(\tanh \xi + 1)^{i\lambda/2k_1}(\tanh \xi - 1)^{-i\lambda/2k_1} \operatorname{sech} \xi + B_1 e^{-i(k_1^2+4\lambda^2)t/8}(k_1 \tanh \xi + 2i\lambda) \times(\tanh \xi - 1)^{i\lambda/2k_1}(\tanh \xi + 1)^{-i\lambda/2k_1}, \tag{42}$$

$$\Phi_{22} = iA_2 e^{i(k_1^2+4\lambda^2)t/8}(\tanh \xi + 1)^{i\lambda/2k_1}(\tanh \xi - 1)^{-i\lambda/2k_1} \operatorname{sech} \xi + B_2 e^{-i(k_1^2+4\lambda^2)t/8}(k_1 \tanh \xi + 2i\lambda) \times(\tanh \xi - 1)^{i\lambda/2k_1}(\tanh \xi + 1)^{-i\lambda/2k_1}. \tag{43}$$

Here we find that $\det(\Phi) = [(k_1^2+4\lambda^2)/k_1](A_1B_2 - A_2B_1) \neq 0$.

Inserting A, B, and Φ in Eq. (10), and solving the resultant equation and letting $A_1 = A_2$, $B_1 = (A_1 e^{\pi\lambda/k_1})/k_1$, and $B_2 = -B_1$, we obtain a three parameter (λ, k_1, μ) family of surfaces parametrized by

$$y_1 = -\frac{1}{4k_1(e^{2\xi}+1)}R_1(E(e^{2\xi}+1) + 32k_1), \tag{44}$$

$$y_2 = -4R_1 \cos G \operatorname{sech} \xi, \tag{45}$$

$$y_3 = 4R_1 \sin G \operatorname{sech} \xi, \tag{46}$$

where

$$R_1 = -\frac{\mu k_1}{2(k_1^2 + 4\lambda^2)}, \tag{47}$$

$$G = t\left(\lambda^2 + \frac{1}{4}k_1^2[1+\lambda]\right) + x\lambda, \tag{48}$$

$$E = (t[8\lambda + k_1^2] + 4x)(k_1^2 + 4\lambda^2), \tag{49}$$

$$\xi = \frac{k_1^3}{8}\left(t + \frac{4x}{k_1^2}\right). \tag{50}$$

This surface has the following first and second fundamental forms:

$$(\mathrm{d}s_{\mathrm{I}})^2 = \frac{1}{4}\mu^2\left[\left(\mathrm{d}x + \left[\frac{1}{4}k_1^2 + 2\lambda\right]\mathrm{d}t\right)^2 + k_1^2 \operatorname{sech}^2 \xi \mathrm{d}t^2\right],$$

$$(\mathrm{d}s_{\mathrm{II}})^2 = \frac{1}{2}\mu k_1 \operatorname{sech} \xi\left[\mathrm{d}x + \left(\frac{1}{4}k_1^2 + \lambda\right)\mathrm{d}t\right]^2 + \frac{1}{8}\mu k_1^3 \operatorname{sech} \xi[2\operatorname{sech}^2 \xi - 1]\mathrm{d}t^2, \tag{51}$$

and the Gaussian and mean curvatures, respectively, are

$$K = \frac{k_1^2}{\mu^2}(2\operatorname{sech}^2 \xi - 1), \tag{52}$$

$$H = \frac{1}{4\mu k_1 \operatorname{sech} \xi}(6k_1^2 \operatorname{sech}^2 \xi + (4\lambda^2 - k_1^2)). \tag{53}$$

Proposition 4: The surface which is parametrized by Eqs. (44)–(46) is a cubic Weingarten surface, i.e.,

$$4\mu^2 H^2(2[\mu^2 K + k_1^2]) - 9\mu^4 K^2 - 12\mu^2(k_1^2 + 2\lambda^2)K - (k_1^2 + 2\lambda^2)^2 = 0. \tag{54}$$

When $k_1 = 2\lambda$ *in Eqs. (52) and (53), it reduces to a quadratic Weingarten surface, i.e.*,

$$8\mu^2 H^2 - 9\mu^2 K - 36\lambda^2 = 0. \tag{55}$$

B. The analyses of the three parameter family of mKdV surfaces

In general, y_2 and y_3 are asymptotically decaying functions, and y_1 approaches $\pm\infty$ as ξ tends to $\pm\infty$. For some small intervals of x and t, we plot some of the three parameter family of surfaces for some special values of the parameters k_1, λ, and μ in Figs. 1–4.

Example 2: By taking $k_1 = 2, \lambda = 1$, and $\mu = -8$ in Eqs. (44)–(46), we get the surface (Fig. 1). The components of the position vector of the surface are

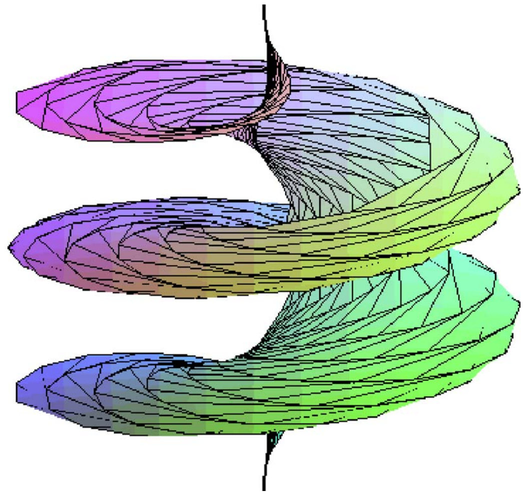

FIG. 1. $(x,t) \in [-3,3] \times [-3,3]$.

$$y_1 = -E_1 - 8/(e^{2\xi}+1), \quad y_2 = -4\cos G \operatorname{sech} \xi, \quad y_3 = -4\sin G \operatorname{sech} \xi, \tag{56}$$

where $E_1 = 4(x+3t)$, $G = x+3t$, and $\xi = x+t$. As ξ tends to $\pm\infty$, y_1 approaches $\pm\infty$, and y_2 and y_3 approach zero. This can also be seen in Fig. 1. For small values of x and t, the surface has a twisted shape.

Example 3: By taking $k_1 = 2$, $\lambda = 0$, and $\mu = -4$ in Eqs. (44)–(46), we get the surface (Fig. 2). The components of the position vector of the surface are

$$y_1 = -E_1 - 8/(e^{2\xi}+1), \quad y_2 = -4\cos G \operatorname{sech} \xi, \quad y_3 = -4\sin G \operatorname{sech} \xi, \tag{57}$$

where $E_1 = 2(x+3t)$, $G = t$, and $\xi = x+t$. As ξ tends to $\pm\infty$, y_1 approaches $\pm\infty$, and y_2 and y_3 tend to zero. This can also be seen in Fig. 2. Asymptotically, this surface and the surface given in Example 2 are the same. However, for small values of x and t, they are different.

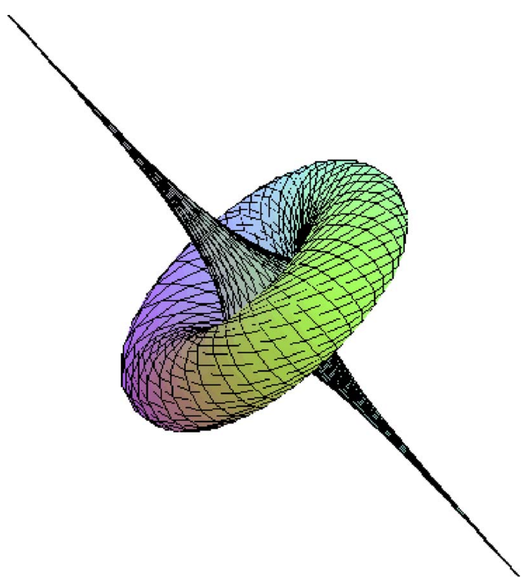

FIG. 2. $(x,t) \in [-6,6] \times [-6,6]$.

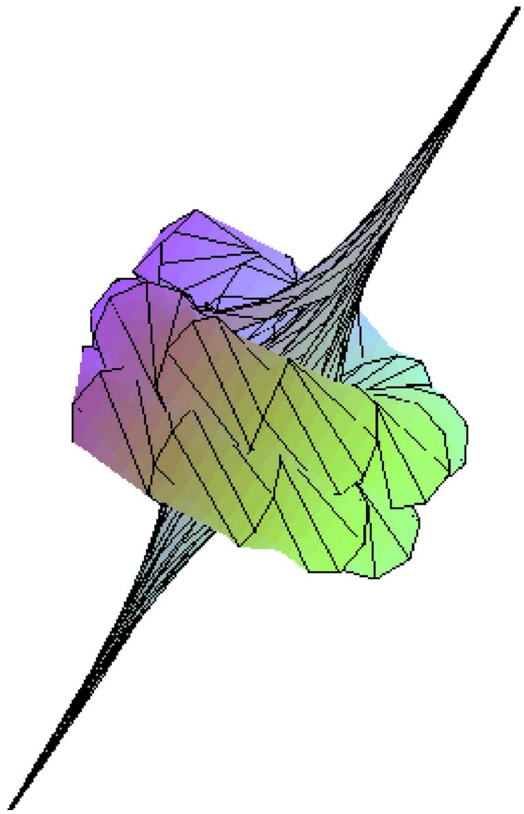

FIG. 3. $(x,t) \in [-6,6] \times [-6,6]$.

Example 4: By taking $k_1=3$, $\lambda=1/10$, and $\mu=-452/75$ in Eqs. (44)–(46), we get the surface (Fig. 3). The components of the position vector of the surface are

$$y_1 = -E_1 - 8/(e^{2\xi}+1), \quad y_2 = -4\cos G \operatorname{sech}\xi, \quad y_3 = -4\sin G \operatorname{sech}\xi, \tag{58}$$

where $E_1=-(5537t+2260x)/750$, $G=17(497t+20x)/200$, and $\xi=(12x+27t)/8$. Asymptotically, this surface is similar to the previous two surfaces. For small values of x and t, the surface looks like a shell.

Example 5: By taking $k_1=1$, $\lambda=-1/10$, and $\mu=-52/25$ in Eqs. (44)–(46), we get the surface (Fig. 4). The components of the position vector of the surface are

$$y_1 = -E_1 - 8/(e^{2\xi}+1), \quad y_2 = -4\cos G \operatorname{sech}\xi, \quad y_3 = -4\sin G \operatorname{sech}\xi, \tag{59}$$

where $E_1=-13(20x+t)/250$, $G=(47t-20x)/200$, and $\xi=(4x+t)/8$. Asymptotically, this surface is similar to the previous three surfaces.

III. mKdV SURFACES FROM SPECTRAL-GAUGE DEFORMATIONS

In this section, we find surfaces arising from a combination of the spectral and gauge deformations of the Lax pair for the mKdV equation.

Proposition 5: Let u satisfy (which describes a traveling mKdV wave) Eq. (2). The corresponding su(2) valued Lax pair U and V of the mKdV equation are given by Eqs. (3) and (4), respectively. The su(2) valued matrices A and B are

$$A = i\begin{pmatrix} \left(\frac{1}{2}\mu - \nu u\right) & -\nu\lambda \\ -\nu\lambda & -\left(\frac{1}{2}\mu - \nu u\right) \end{pmatrix}, \tag{60}$$

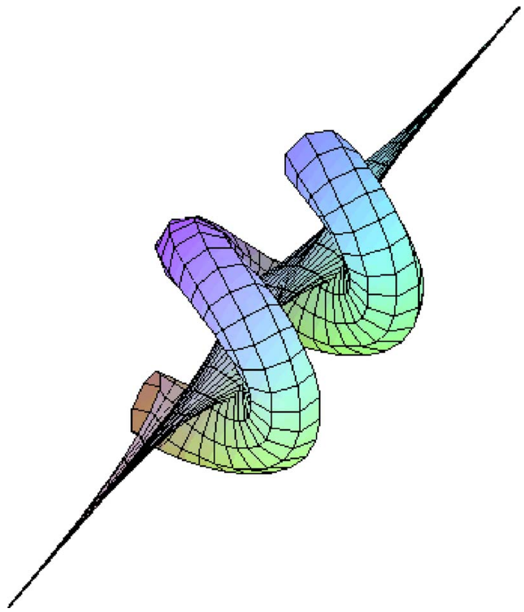

FIG. 4. $(x,t) \in [-20,20] \times [-20,20]$.

$$B = i\begin{pmatrix} \frac{1}{2}\mu(\alpha+2\lambda) - \nu(\alpha+\lambda)u & -\frac{1}{2}\mu u + \nu\left(\frac{1}{2}u^2 - \alpha - \alpha\lambda - \lambda^2\right) \\ -\frac{1}{2}\mu u + \nu\left(\frac{1}{2}u^2 - \alpha - \alpha\lambda - \lambda^2\right) & -\frac{1}{2}\mu(\alpha+2\lambda) + \nu(\alpha+\lambda)u \end{pmatrix}, \tag{61}$$

where $A=\mu U_\lambda + \nu[\sigma_2, U]$, $B=\mu V_\lambda + \nu[\sigma_2, V]$, λ *is the spectral parameter,* μ *and* ν *are constants, and* σ_2 *is the Pauli sigma matrix. The surface S , generated by* U, V, A, *and* B, *has the following first and second fundamental forms* $(j,k=1,2)$:

$$(\mathrm{d}s_{\mathrm{I}})^2 = g_{jk}\mathrm{d}x^j\mathrm{d}x^k, \tag{62}$$

$$(\mathrm{d}s_{\mathrm{II}})^2 = h_{jk}\mathrm{d}x^j\mathrm{d}x^k, \tag{63}$$

where

$$g_{11} = \frac{1}{4}\mu^2 + \nu(\nu[u^2+\lambda^2] - \mu u), \tag{64}$$

$$g_{12} = \frac{1}{4}(\alpha+2\lambda)\mu^2 + \frac{1}{4}\nu(\nu[2(\lambda+2\alpha)u^2 + 4(\lambda^3+\alpha\lambda+\lambda^2\alpha)] - 4\mu(\alpha+\lambda)u), \tag{65}$$

$$g_{22} = \frac{1}{4}(u^2+(2\lambda+\alpha)^2)\mu^2 + \nu\left(\nu\left[\frac{1}{4}u^4 + \alpha(\alpha-1+\lambda)u^2 + ((1+\lambda)\alpha+\lambda^2)^2\right] - \frac{1}{2}\mu u^3 - \mu(\alpha^2+(2\lambda-1)\alpha+\lambda^2)u\right), \tag{66}$$

$$h_{11} = \frac{1}{2}\mu u - \nu(u^2+\lambda^2), \tag{67}$$

$$h_{12} = \frac{1}{2}\mu(\alpha+\lambda)u - \nu\left(\lambda(\lambda^2+\alpha\lambda+\alpha) + \frac{1}{2}(\lambda+2\alpha)u^2\right), \tag{68}$$

$$h_{22} = \frac{1}{4}\mu(u^3 + 2[\alpha^2 + (2\lambda-1)\alpha + \lambda^2]u) - \nu\left(\frac{1}{4}u^4 + \alpha(\alpha-1+\lambda)u^2 + ((1+\lambda)\alpha+\lambda^2)^2\right), \tag{69}$$

and the corresponding Gaussian and mean curvatures are

$$K = \frac{2u(u^2-2\alpha)}{\nu(2\nu u[u^2-2\alpha] - 3\mu u^2 - 2\mu(\lambda^2-\alpha)) + \mu^2 u}, \tag{70}$$

$$H = \frac{\mu(3u^2 + 2(\lambda^2-\alpha)) - 4u\nu(u^2-2\alpha)}{2\nu(2\nu u[u^2-2\alpha] - 3\mu u^2 - 2\mu(\lambda^2-\alpha)) + 2\mu^2 u}, \tag{71}$$

where $x^1=x$ *and* $x^2=t$.

A. The parametrized form of the four parameter family of mKdV surfaces

We apply the same technique that we used in Sec. II to find the position vector of the corresponding surface. Let $u=k_1 \operatorname{sech}\xi$, $\xi=k_1(k_1^2 t+4x)/8$, be the one soliton solution of the mKdV equation, where $\alpha=k_1^2/4$. The Lax pair U and V are given by Eqs. (3) and (4), respectively, which is the same as in the spectral deformation case. So we can use the solution of the Lax equation [Eq. (5)] that we found in the spectral deformation case. By solving Eq. (10), we obtain the position vector, where the components of Φ are given by Eqs. (40)–(43) and A, B are given by Eqs. (60) and (61), respectively. Here we choose $A_1=A_2, B_1=(A_1 e^{\pi\lambda/k_1})/k_1$, and $B_2=-B_1$ to write F in the form $F=i(\sigma_1 y_1+\sigma_2 y_2+\sigma_3 y_3)$. Hence we obtain a four parameter (λ, k_1, μ, ν) family of surfaces parametrized by

$$y_1 = R_2\frac{e^{2\xi}-1}{(e^{2\xi}+1)}\operatorname{sech}\xi + R_3\tilde{E} + R_4\frac{1}{e^{2\xi}+1},$$

$$y_2 = \left[\frac{1}{2}R_4\operatorname{sech}\xi + R_5\frac{(e^{4\xi}+1)}{(e^{2\xi}+1)^2} - R_6\operatorname{sech}^2\xi\right]\cos G + R_7\frac{(e^{2\xi}-1)}{(e^{2\xi}+1)}\sin G, \tag{72}$$

$$y_3 = \left[-\frac{1}{2}R_4\operatorname{sech}\xi - R_5\frac{(e^{4\xi}+1)}{(e^{2\xi}+1)^2} + R_6\operatorname{sech}^2\xi\right]\sin G + R_7\frac{(e^{2\xi}-1)}{(e^{2\xi}+1)}\cos G,$$

where

$$R_2 = \frac{2k_1^2\nu}{k_1^2+4\lambda^2}, \quad R_3 = \frac{\mu}{8}, \tag{73}$$

$$R_4 = \frac{4\mu k_1^2}{k_1^2+4\lambda^2}, \quad R_5 = \frac{\nu(k_1^2-4\lambda^2)}{k_1^2+4\lambda^2}, \tag{74}$$

$$R_6 = \frac{\nu(4\lambda^2+3k_1^2)}{2(k_1^2+4\lambda^2)}, \quad R_7 = \frac{4\lambda k_1^2\nu}{k_1^2+4\lambda^2}, \tag{75}$$

$$G = t\left(\lambda^2 + \frac{1}{4}k_1^2[1+\lambda]\right) + x\lambda, \tag{76}$$

$$\tilde{E} = (t[8\lambda + k_1^2] + 4x), \tag{77}$$

$$\xi = \frac{k_1^3}{8}\left(t + \frac{4x}{k_1^2}\right). \tag{78}$$

Thus the position vector $\mathbf{y}=(y_1(x,t),y_2(x,t),y_3(x,t))$ of the surface is given by Eq. (72). This surface has the following first and second fundamental forms $(j,k=1,2)$:

$$(\mathrm{d}s_{\mathrm{I}})^2 = g_{jk}\mathrm{d}x^j\mathrm{d}x^k, \tag{79}$$

$$(\mathrm{d}s_{\mathrm{II}})^2 = h_{jk}\mathrm{d}x^j\mathrm{d}x^k, \tag{80}$$

where

$$g_{11} = \frac{1}{4}\mu^2 + \nu(\nu[k_1^2 \operatorname{sech}^2 \xi + \lambda^2] - \mu k_1 \operatorname{sech} \xi),$$

$$g_{12} = \frac{1}{4}(\alpha + 2\lambda)\mu^2 + \frac{1}{4}\nu(\nu[2k_1^2(\lambda + 2\alpha)\operatorname{sech}^2 \xi + (4\lambda^3 + 4\alpha\lambda + 4\lambda^2\alpha)] - 4\mu(\alpha+\lambda)k_1 \operatorname{sech} \xi),$$

$$g_{22} = \frac{1}{4}(k_1^2 \operatorname{sech}^2 \xi + (2\lambda+\alpha)^2)\mu^2 + \nu\left(\nu\left[\frac{1}{4}k_1^4 \operatorname{sech}^4 \xi + \alpha k_1^2(\alpha - 1 + \lambda)\operatorname{sech}^2 \xi + ((1+\lambda)\alpha + \lambda^2)^2\right]\right.$$
$$\left. - \frac{1}{2}\mu k_1^3 \operatorname{sech}^3 \xi - \mu k_1(\alpha^2 + (2\lambda - 1)\alpha + \lambda^2)\operatorname{sech} \xi\right),$$

$$h_{11} = \frac{1}{2}\mu k_1 \operatorname{sech} \xi - \nu(k_1^2 \operatorname{sech}^2 \xi + \lambda^2),$$

$$h_{12} = \frac{1}{2}\mu(\alpha+\lambda)k_1 \operatorname{sech} \xi - \nu\left(\lambda(\lambda^2 + \alpha\lambda + \alpha) + \frac{1}{2}k_1^2(\lambda + 2\alpha)\operatorname{sech}^2 \xi\right),$$

$$h_{22} = \frac{1}{4}\mu(k_1^3 \operatorname{sech}^3 \xi + 2k_1^2[\alpha^2 + (2\lambda - 1)\alpha + \lambda^2]\operatorname{sech}^2 \xi)$$
$$- \nu\left(\frac{1}{4}k_1^4 \operatorname{sech}^4 \xi + \alpha k_1^2(\alpha - 1 + \lambda)\operatorname{sech}^2 \xi + ((1+\lambda)\alpha + \lambda^2)^2\right),$$

and the corresponding Gaussian and mean curvatures are

$$K = \frac{2k_1 \operatorname{sech} \xi(k_1^2 \operatorname{sech}^2 \xi - 2\alpha)}{\nu(2\nu k_1 \operatorname{sech} \xi[k_1^2 \operatorname{sech}^2 \xi - 2\alpha] - 3\mu k_1^2 \operatorname{sech}^2 \xi - 2\mu(\lambda^2 - \alpha)) + \mu^2 k_1 \operatorname{sech} \xi},$$

$$H = \frac{\mu(3k_1^2 \operatorname{sech}^2 \xi + 2(\lambda^2 - \alpha)) - 4\nu k_1 \operatorname{sech} \xi(k_1^2 \operatorname{sech}^2 \xi - 2\alpha)}{2\nu(2\nu k_1 \operatorname{sech} \xi[k_1^2 \operatorname{sech}^2 \xi - 2\alpha] - 3\mu k_1^2 \operatorname{sech}^2 \xi - 2\mu(\lambda^2 - \alpha)) + 2\mu^2 k_1 \operatorname{sech} \xi},$$

where $x^1=x$, $x^2=t$, and $\alpha=\frac{1}{4}k_1^2$.

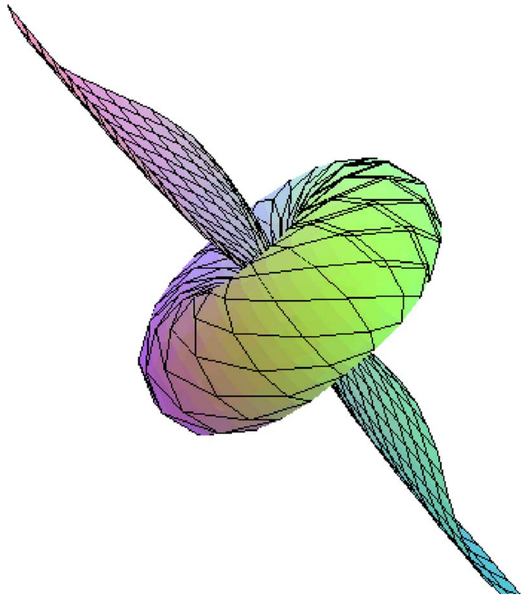

FIG. 5. $(x,t) \in [-4,4] \times [-4,4]$.

B. The analyses of the four parameter family of surfaces

Asymptotically, y_1 approaches $\pm\infty$, y_2 approaches $R_5 \cos G \pm R_7 \sin G$, and y_3 approaches $-R_5 \sin G \pm R_7 \cos G$ as ξ tends to $\pm\infty$. For some small intervals of x and t, we plot some of the four parameter family of surfaces for some special values of the parameters k_1, λ, μ, and ν in Figs. 5–7.

Example 6: By taking $k_1=2$, $\lambda=0$, $\mu=-4$, and $\nu=1$ in Eq. (72), we get the surface (Fig. 5). The components of the position vector are

$$y_1 = 2 \operatorname{sech} \xi(e^{2\xi}-1)/(e^{2\xi}+1) - E_2 - 8/(e^{2\xi}+1), \tag{81}$$

$$y_2 = [-4 \operatorname{sech} \xi + (e^{4\xi}+1)/(e^{2\xi}+1)^2 - (3/2)\operatorname{sech}^2 \xi]\cos G, \tag{82}$$

$$y_3 = [4 \operatorname{sech} \xi - (e^{4\xi}+1)/(e^{2\xi}+1)^2 + (3/2)\operatorname{sech}^2 \xi]\sin G, \tag{83}$$

where $E_2=-2(x+t), G=t$ and $\xi=x+t$. As ξ tends to $\pm\infty$, y_1 approaches $\pm\infty$, y_2 approaches $\cos t$, and y_3 approaches $-\sin t$.

Example 7: By taking $k_1=2$, $\lambda=1$, $\mu=1/10$, and $\nu=1$ in Eq. (72), we get the surface (Fig. 6). The components of the position vector are

$$y_1 = \operatorname{sech} \xi(e^{2\xi}-1)/(e^{2\xi}+1) + E_2 + 1/(10(e^{2\xi}+1)), \tag{84}$$

$$y_2 = [(1/20)\operatorname{sech} \xi - \operatorname{sech}^2 \xi]\cos G + \sin G(e^{2\xi}-1)/(e^{2\xi}+1), \tag{85}$$

$$y_3 = [-(1/20)\operatorname{sech} \xi + \operatorname{sech}^2 \xi]\sin G + \cos G(e^{2\xi}-1)/(e^{2\xi}+1), \tag{86}$$

where $E_2=(x+3t)/20$, $G=(x+3t)$, and $\xi=x+t$. As ξ tends to $\pm\infty$, y_1 tends to $\pm\infty$, y_2 approaches $\sin(x+3t)$, and y_3 approaches $\cos(x+3t)$.

Example 8: By taking $k_1=1$, $\lambda=-1/10$, $\mu=-52/25$, and $\nu=-1$ in Eq. (72), we get the surface (Fig. 7). The components of the position vector are

$$y_1 = -(25/13)\operatorname{sech} \xi(e^{2\xi}-1)/(e^{2\xi}+1) - E_2 - 8/(e^{2\xi}+1), \tag{87}$$

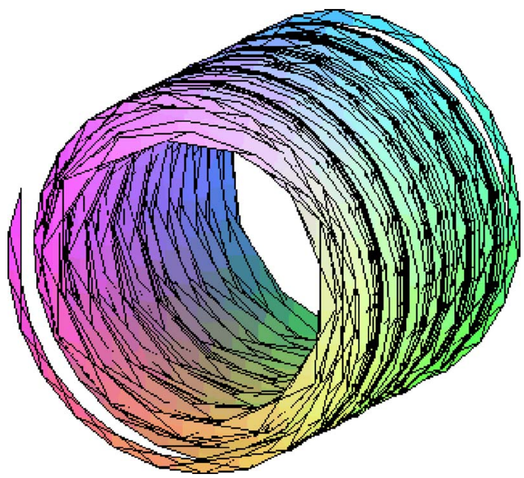

FIG. 6. $(x,t) \in [-6,6] \times [-6,6]$.

$$y_2 = [-4 \operatorname{sech} \xi - (12/13)(e^{4\xi}+1)/(e^{2\xi}+1)^2 + (19/13)\operatorname{sech}^2 \xi]\cos G + (5/13)\sin G(e^{2\xi}-1)/(e^{2\xi}+1), \tag{88}$$

$$y_3 = [4 \operatorname{sech} \xi + (12/13)(e^{4\xi}+1)/(e^{2\xi}+1)^2 - (19/13)\operatorname{sech}^2 \xi]\sin G + (5/13)\cos G(e^{2\xi}-1)/(e^{2\xi}+1), \tag{89}$$

where $E_2 = 13(20x+t)/250$, $G = (47t-20x)/200$, and $\xi = (4x+t)/8$. As ξ tends to $\pm\infty$, y_1 tends to $\pm\infty$, y_2 approaches $\cos t$, and y_3 approaches $-\sin t$.

IV. CONCLUSION

In this work, we considered mKdV 2-surfaces by using two deformations, spectral deformation and a combination of gauge and spectral deformations of mKdV equation and its Lax pair. We found the first and second fundamental forms, and the Gaussian and mean curvatures of the

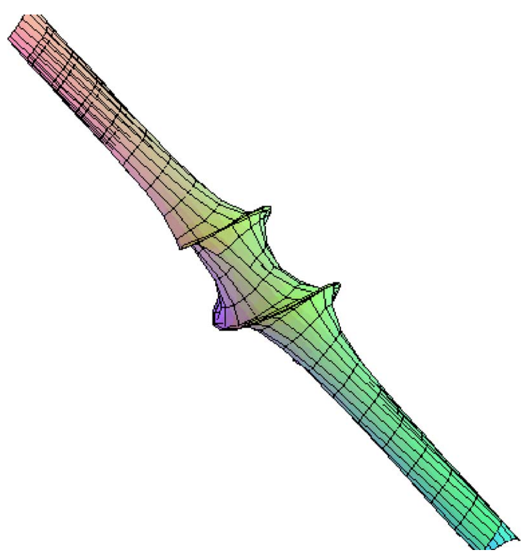

FIG. 7. $(x,t) \in [-20,20] \times [-20,20]$.

corresponding surfaces. By solving the Lax equation for a given solution of the mKdV equation and the corresponding Lax pair, we also found position vectors of these surfaces.

The surfaces arising from the spectral deformation are Weingarten and Willmore-like surfaces. We also obtained some mKdV surfaces from the variational principle for the Lagrange function, that is, a polynomial of the Gaussian and mean curvatures of the surfaces corresponding to the spectral deformations of the Lax pair of the mKdV equation. For some special values of parameters, we plotted these three parameter family of surfaces in Examples 2—5.

In the case of the gauge-spectral parameter deformations, we obtained a four parameter family of mKdV surfaces. For some special values of the parameters in the position vectors [Eq. (72)] of these surfaces, we plotted them in Examples 6—8.

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[1] A. Sym, Lett. Nuovo Cimento Soc. Ital. Fis. **33**, 394 (1982).
[2] A. Sym, in *Geometrical Aspects of the Einstein Equations and Integrable Systems*, Lecture Notes in Physics Vol. 239, edited by R. Martini (Springer, Berlin, 1985), p. 154.
[3] A. S. Fokas and I. M. Gelfand, Commun. Math. Phys. **177**, 203 (1996).
[4] A. S. Fokas, I. M. Gelfand, F. Finkel, and Q. M. Liu, Selecta Math., New Ser. **6**, 347 (2000).
[5] Ö. Ceyhan, A. S. Fokas, and M. Gürses, J. Math. Phys. **41**, 2251 (2000).
[6] M. Gürses, J. Nonlinear Math. Phys. **9**, 59 (2002).
[7] M. Gürses and S. Tek, Report No. nlin.SI/0511049 (unpublished).
[8] B. G. Konopelchenko, Phys. Lett. A **183**, 153 (1993).
[9] B. G. Konopelchenko and I. A. Taimanov, Report No. dg-ga/9506011 (unpublished).
[10] J. Cieslinski, P. Goldstein, and A. Sym, Phys. Lett. A **205**, 37 (1995).
[11] J. Cieslinski, J. Math. Phys. **38**, 4255 (1997).
[12] M. P. Do Carmo, *Differential Geometry of Curves and Surfaces* (Prentice-Hall, Englewood Cliffs, NJ, 1976).
[13] Z. C. Tu and Z. C. Ou-Yang, J. Phys. A **37**, 11407 (2004).
[14] Z. C. Tu and Z. C. Ou-Yang, Phys. Rev. E **68**, 061915 (2003).
[15] Z. C. Ou-Yang, J. Liu, and Y. Xie, *Geometric Methods in the Elastic Theory of Membranes in Liquid Crystal Phases* (World Scientific, Singapore, 1999).
[16] Z. C. Tu and Z. C. Ou-Yang, Proceeding of the Seventh International Conference on Geometry, Integrability and Quantization, Varna, Bulgaria, 2005, edited by I. M. Mladenov and M. de León, SOFTEX, Sofia, 2005, p. 237, Report No. math-ph/0506055.
[17] T. J. Willmore, *Total Curvature in Riemannian Geometry* (Wiley, New York, 1982).
[18] T. J. Willmore, *Annals of Global Analysis and Geometry* (Springer, New York, 2000), Vol. 18, pp. 255–264.
[19] Z. C. Ou-Yang and W. Helfrich, Phys. Rev. Lett. **59**, 2486 (1987).
[20] Z. C. Ou-Yang and W. Helfrich, Phys. Rev. A **39**, 5280 (1989).
[21] I. M. Mladenov, Eur. Phys. J. B **29**, 327 (2002).