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Brief paper

PID controller synthesis for a class of unstable MIMO plants with I/O delays☆

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Abstract

Conditions are presented for closed-loop stabilizability of linear time-invariant (LTI) multi-input, multi-output (MIMO) plants with I/O delays (time delays in the input and/or output channels) using PID (Proportional + Integral + Derivative) controllers. We show that systems with at most two unstable poles can be stabilized by PID controllers provided a small gain condition is satisfied. For systems with only one unstable pole, this condition is equivalent to having sufficiently small delay-unstable pole product. Our method of synthesis of such controllers identify some free parameters that can be used to satisfy further design criteria than stability.

Keywords: PID control; Time delay; Unstable systems; Multi-input multi-output systems

1. Introduction

While finite dimensional linear time-invariant (LTI) systems are sufficiently accurate models for a wide range of dynamical phenomena, there are many cases in which delay effects cannot be ignored and have to be included in the model (Gu, Kharitonov, & Chen, 2003). An r input and r output LTI system with I/O delays (time delays in the input and/or output channels) can be represented by $G_\Lambda(s) := \Lambda_o(s)G(s)\Lambda_i(s)$, where $G(s)$ is the finite dimensional part (an $r \times r$ rational matrix), and $\Lambda_\star(s) = \mathrm{diag}[e^{-T_1^\star s}, \ldots, e^{-T_r^\star s}]$ is the delay matrix, where $\star$ stands for i (input delay case) or o (output delay case). This paper considers closed-loop stabilization (see Fig. 1) of such systems using *proper PID-controllers* (Goodwin, Graebe, & Salgado, 2001):

$$C_{pid}(s) = K_p + \frac{K_i}{s} + \frac{K_d s}{\tau_d s + 1}, \tag{1}$$

where K_p, K_i, K_d are real matrices and $\tau_d > 0$.

Stability of delay systems of retarded type, or even neutral type, is extensively investigated and many delay-independent and delay-dependent stability results are available (Gu et al., 2003; Niculescu, 2001). The feedback stabilization of delay systems is also well investigated. Since delay element is an integral part of process control systems, most of the tuning and internal model control techniques used in process control systems apply to delay systems (Aström & Hagglund, 1995). The more special, but practically very relevant (see Goodwin et al., 2001), problem of existence of stabilizing PID-controllers is unfortunately not easy to solve even for the delay-free case. One way of gaining insight into the difficulty of the problem is to note that the existence of a stabilizing PID-controller for a plant of transfer matrix $G(s)$ is equivalent to that of a *constant stabilizing output feedback* for a transformed multi-input multi-output (MIMO) plant. Alternatively, the problem can be posed as determining conditions of existence of a stable and fixed-order controller for the extended plant $G(s)\frac{s+1}{s}$, which is again

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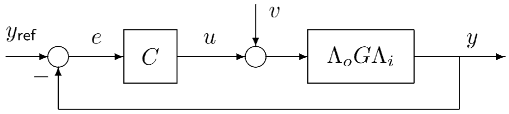

Fig. 1. Unity-feedback system $Sys(G_\Lambda, C)$.

well-known to be a difficult problem (Blondel, Gevers, Mortini, & Rupp, 1994; Vidyasagar, 1985). It should be mentioned that there *are* some computational PID-stabilization methods, which consist of "efficient search" in the parameter space, recently developed for single-input single-output (SISO) delay-free systems (see Saadaoui & Özgüler, 2005 and the references therein). Some of these techniques have been extended to cover scalar, single-delay systems (Silva, Datta, & Bhattacharyya, 2005). Rigorous state-space based methods, which transform the PID design problem to static output feedback control design for an augmented system using LMI approaches, have also been developed for MIMO delay-free systems (see Lin, Wang, & Lee, 2004; Zheng, Wang, & Lee, 2002 and the references therein). A parameter-space approach for finding stability regions of a class of quasi-polynomials is proposed in Hohenbichler and Ackermann (2003). This technique can be used for finding stability regions in the PID controller parameter space for delay systems. For a plant consisting of a chain of integrators, stabilization using multiple delays is studied in Niculescu and Michiels (2004) and Kharitanov, Niculescu, Moreno, and Michiels (2005). Although the motivation of Niculescu and Michiels (2004) and Kharitanov et al. (2005) is to stabilize non-delayed plants using delayed output with static gains, clearly, their problem includes proportional control design for an integrator (and oscillator in the case of Kharitanov et al., 2005) with delay. This is also one of the special cases we study here.

In this paper, making a novel use of the small gain theorem, we obtain two main results: first, for MIMO plants with input and/or output delays, we obtain some sufficient conditions on the existence of stabilizing PID controllers, and second, we explicitly construct PID controllers for plants having only one unstable pole (under the condition that the product of the unstable pole with delay is sufficiently small). This construction is extended to the case of two unstable real or complex poles. As our goal is to establish existence of stabilizing PID controllers at this point, we do not consider performance issues but propose freedom in the design parameters that can be used to satisfy performance criteria.

Notation: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}_-$, $\mathbb{C}_+$ denote real, complex, open left-half plane complex and open right-half plane complex numbers; $\mathscr{U}$ denotes the extended closed right-half plane, i.e., $\mathscr{U} = \{s \in \mathbb{C} \mid \mathscr{Re}(s) \geqslant 0\} \cup \{\infty\}$; $\mathbf{R_p}$ denotes proper rational functions; $\mathbf{S}$ denotes stable proper real rational functions of s. The set of matrices whose entries are in $\mathbf{S}$ is denoted by $\mathscr{M}(\mathbf{S})$. The space $\mathscr{H}_\infty$ is the set of all bounded analytic functions in $\mathbb{C}_+$. For $h \in \mathscr{H}_\infty$, the norm is defined as $\|h\|_\infty = \operatorname{ess\,sup}_{s\in\mathbb{C}_+} |h(s)|$, where ess sup denotes the essential supremum. A matrix-valued function H with all entries in $\mathscr{H}_\infty$ is said to be in $\mathscr{M}(\mathscr{H}_\infty)$, and $\|H\|_\infty = \operatorname{ess\,sup}_{s\in\mathbb{C}_+} \bar{\sigma}(H(s))$, where $\bar{\sigma}$ denotes the maximum singular value. From the induced L^2 gain point of view, a system with transfer matrix H is stable if and only if $H \in \mathscr{M}(\mathscr{H}_\infty)$. For square $H \in \mathscr{M}(\mathscr{H}_\infty)$, H is unimodular if $H^{-1} \in \mathscr{M}(\mathscr{H}_\infty)$. For simplicity, we drop (s) in transfer matrices such as $G(s)$ where this causes no confusion. Since all norms we are interested in are $\mathscr{H}_\infty$ norms, we drop the norm subscript, i.e. $\|\cdot\|_\infty \equiv \|\cdot\|$, whenever this is clear from the context.

2. Problem description

Consider the standard unity-feedback system shown in Fig. 1, where $G \in \mathbf{R}_\mathbf{p}^{r\times r}$ and $C \in \mathbf{R}_\mathbf{p}^{r\times r}$ denote the plant without the time delay term (non-delayed plant, for short) and the controller transfer matrices. It is assumed that the feedback system is well-posed and that the non-delayed plant and the controller have no unstable hidden-modes. It is also assumed that $G \in \mathbf{R}_\mathbf{p}^{r\times r}$ is full normal rank. The delay terms are in the form $\Lambda_\star = \operatorname{diag}[e^{-sT_1^\star}, \ldots, e^{-sT_r^\star}]$, where, for $1 \leqslant j \leqslant r$, we have $T_j^\star \in \Theta_j^\star = [0, T_{j,\max}^\star) \subset \mathbb{R}_+$ and $\star$ stands for i (input channel delays) or o (output channel delays). We assume that the delay upper bound $T_{j,\max}^\star$ is known for all input and output channels $j=1, \ldots, r$. Define $\mathscr{T}^\star := (T_1^\star, \ldots, T_r^\star)$ and $\Theta^\star := (\Theta_1^\star, \ldots, \Theta_r^\star)$. As a shorthand notation we will write $(\mathscr{T}^{\rm i}, \mathscr{T}^{\rm o}) =: \mathscr{T} \in \Theta := (\Theta^{\rm i}, \Theta^{\rm o})$ to represent all possibilities $T_j^\star \in \Theta_j^\star$, $1 \leqslant j \leqslant r$. We denote the delayed plant by $G_\Lambda := \Lambda_{\rm o}(s) G(s) \Lambda_{\rm i}(s)$. The closed-loop transfer matrix $H_{\rm cl}$ from $(y_{\sf ref}, v)$ to (u, y) is

$$H_{\rm cl} = \begin{bmatrix} C(I + G_\Lambda C)^{-1} & -C(I + G_\Lambda C)^{-1} G_\Lambda \\ G_\Lambda C (I + G_\Lambda C)^{-1} & (I + G_\Lambda C)^{-1} G_\Lambda \end{bmatrix}. \tag{2}$$

We consider the proper form of PID-controllers in (1), where the real matrices $K_{\rm p}$, $K_{\rm i}$, $K_{\rm d}$ are called the proportional constant, the integral constant, and the derivative constant, respectively. Due to implementation issues of the derivative action, a pole is typically added to the derivative term (with $\tau_{\rm d} \in \mathbb{R}$, $\tau_{\rm d} > 0$ when $K_{\rm d} \neq 0$) so that the transfer-function $C_{\rm pid}$ in (1) is proper. If one or more of the three terms $K_{\rm p}$, $K_{\rm i}$, $K_{\rm d}$ is zero, then the corresponding subscript is omitted from $C_{\rm pid}$.

Definition 1. (a) The unity-feedback system $Sys(G_\Lambda, C)$, shown in Fig. 1, is said to be stable iff the closed-loop map $H_{\rm cl}$ is in $\mathscr{M}(\mathscr{H}_\infty)$. The set of all controllers stabilizing the feedback system for the plant G_Λ is denoted by $\mathscr{S}(G_\Lambda)$. (b) A delayed plant G_Λ, where $G \in \mathbf{R}_\mathbf{p}^{r\times r}$, is said to admit a PID-controller iff there exists a PID-controller $C = C_{\rm pid}$ as in (1) such that $C_{\rm pid} \in \mathscr{S}(G_\Lambda)$. In this case G_Λ is stabilizable by a PID-controller.

Let $G = Y^{-1}X$ be any left coprime factorization (LCF) of the plant, $C = N_c D_c^{-1}$ be any right coprime factorization (RCF) of the controller, where we use coprime factorizations over $\mathbf{S}$; i.e., for $G, C \in \mathbf{R}_\mathbf{p}^{r\times r}$, $X, Y \in \mathscr{M}(\mathbf{S})$, $\det Y(\infty) \neq 0$, $N_c, D_c \in \mathscr{M}(\mathbf{S})$, $\det D_c(\infty) \neq 0$. Let X_Λ denote the "numerator" matrix of G_Λ, i.e., $X_\Lambda := \Lambda_{\rm o}(s) X(s) \Lambda_{\rm i}(s)$. Now if the "denumerator" matrix Y of $G = Y^{-1}X$ is diagonal, then the delayed plant G_Λ can be expressed as $G_\Lambda = Y^{-1} X_\Lambda$. The controller C stabilizes

G_Λ if and only if $M_\Lambda := YD_c + X_\Lambda N_c \in \mathscr{M}(\mathscr{H}_\infty)$ is unimodular, i.e., $M_\Lambda^{-1} \in \mathscr{M}(\mathscr{H}_\infty)$ (see Smith, 1989).

3. Main results

Throughout the paper we assume that Y^{-1} is diagonal, hence it commutes with Λ_o. Thus $G_\Lambda = Y^{-1}X_\Lambda$ in all cases studied here.

The result in Lemma 1 is used in designing PI or PID controllers from P or PD controllers, i.e., integral action are added once proportional and derivative terms are designed. This result is a slight extension of Theorem 5.3.10 of Vidyasagar (1985) to systems with time delays.

Lemma 1 (*Two-step controller synthesis*). *Let $G \in \mathbf{R}_\mathbf{p}^{r\times r}$. Suppose that $C_g \in \mathscr{S}(G_\Lambda)$, and $C_h \in \mathscr{S}(H_\Lambda)$ where $H_\Lambda := G_\Lambda(I + C_g G_\Lambda)^{-1} \in \mathscr{M}(\mathscr{H}_\infty)$. Then $C = C_g + C_h$ is also in $\mathscr{S}(G_\Lambda)$.*

Although it is obvious that stable plants admit PID-controllers, the freedom in the stabilizing controller parameters is still worth investigating. We propose a PID-controller synthesis for stable plants in Proposition 2, which will be frequently referred to in the sequel.

Proposition 2. *Let $G \in \mathbf{S}^{r\times r}$ and assume (normal) rank $G(s) = r$. (i) PD-design: Choose any $\hat{K}_\mathrm{p}, \hat{K}_\mathrm{d} \in \mathbb{R}^{r\times r}$, $\tau_\mathrm{d} > 0$. Define $\widehat{C}_\mathrm{pd} := \hat{K}_\mathrm{p} + \frac{s\hat{K}_\mathrm{d}}{\tau_\mathrm{d}s+1}$. Then, for any α satisfying $0 < \alpha < \|G\widehat{C}_\mathrm{pd}\|^{-1}$, a PD-controller in $\mathscr{S}(G_\Lambda)$, for all $\mathscr{T} \in \Theta$, is*

$$C_\mathrm{pd}(s) = \alpha\widehat{C}_\mathrm{pd} = \alpha\hat{K}_\mathrm{p} + \frac{\alpha s\hat{K}_\mathrm{d}}{\tau_\mathrm{d}s+1}. \tag{3}$$

(ii) PID-design: Let rank $G(0) = r$. Choose any $\hat{K}_\mathrm{p}, \hat{K}_\mathrm{d} \in \mathbb{R}^{r\times r}$, $\tau_\mathrm{d} > 0$. Define $\widehat{C}_\mathrm{pid} := \hat{K}_\mathrm{p} + \frac{G(0)^{-1}}{s} + \frac{s\hat{K}_\mathrm{d}}{\tau_\mathrm{d}s+1}$. Define $\Psi := s^{-1}[sG_\Lambda(s)\hat{C}_\mathrm{pid} - I]$, $\widetilde{\Psi} := s^{-1}[s\hat{C}_\mathrm{pid}G_\Lambda(s) - I]$. Then a PID-controller stabilizing G_Λ for $\mathscr{T} \in \Theta$ is $C_\mathrm{pid} = \gamma\widehat{C}_\mathrm{pid}$ for any γ satisfying

$$0 < \gamma < \max\left\{\min_{\mathscr{T}\in\Theta} \|\Psi\|^{-1}, \min_{\mathscr{T}\in\Theta} \|\widetilde{\Psi}\|^{-1}\right\}. \tag{4}$$

Proposition 3 gives general existence conditions for stabilizing PID controllers. If a stabilizing P, I, or D-controller exists, then it can be extended to a stabilizing PI, ID, PD, PID-controller:

Proposition 3. *Let $G \in \mathbf{R}_\mathbf{p}^{r\times r}$. Let (normal) rank $G(s) = r$. (a) If G_Λ admits a PID-controller such that the integral constant $K_\mathrm{i} \in \mathbb{R}^{r\times r}$ is nonzero, then G has no transmission-zeros at $s = 0$ and rank $K_\mathrm{i} = r$. (b) If G_Λ admits a PID-controller such that any one of the three constants K_p, K_d, K_i is nonzero, then G_Λ admits a PID-controller such that any two of the three constants is nonzero, and G_Λ admits a PID-controller such that all of the three constants is nonzero. (c) If G_Λ admits a PID-controller such that two of the three constants K_p, K_d, K_i is nonzero, then G_Λ admits a PID-controller such that all of the three constants is nonzero. In (b) and (c), the integral constant $K_\mathrm{i} \neq 0$ only if G has no transmission-zeros at $s = 0$.*

Proposition 3 does not explicitly define plant classes that admit P, I, or D-controllers. We investigate specific classes and propose stabilizing PID-controller design methods next in Section 3.1.

3.1. Delayed plants that admit PID-controllers

Lemma 2 (*Strong stabilizability is a necessary condition for PID stabilization*). *Let $G \in \mathbf{R}_\mathbf{p}^{r\times r}$. Let rank $G(s) = r$. If G_Λ admits a PID-controller for any $\mathscr{T} \in \Theta$, then G is strongly stabilizable.*

We now consider plants with a limited number of $\mathscr{U}$-poles, including $s = 0$. Limitations on the number of $\mathscr{U}$-poles are not surprising. Plants with an odd number of positive real-axis poles are not even strongly stabilizable if there are two or more positive real-axis zeros (including infinity). But even when the parity-interlacing-property is satisfied, plants that have more than two $\mathscr{U}$-poles do not necessarily admit PID-controllers. For example, the Routh–Hurwitz test shows that $G = (s - p)^{-3}$ does not admit a stabilizing PID controller for $p \geqslant 0$.

3.1.1. Plants with only one unstable real-axis pole

We consider transfer matrices $G \in \mathbf{R}_\mathbf{p}^{r\times r}$ in the form

$$G = Y^{-1}X = \left[\frac{(s-p)}{as+1}I\right]^{-1}\left[\frac{(s-p)}{as+1}G\right], \tag{5}$$

where $p \in \mathbb{R}$, $p \geqslant 0$ and $a \in \mathbb{R}$, $a > 0$, and rank $X(p) = \mathrm{rank}(s - p)G(s)|_{s=p} = r$. Since G has no transmission-zeros at $s = 0$, rank $X(0) = \mathrm{rank}(s-p)G(s)|_{s=0} = r$. By a slight abuse of notation, we say that G has only one unstable pole if $Y(s)$ in (5) is identity times a scalar transfer function with a single finite $\mathscr{U}$-zero.

Proposition 4. *Let G be as in (5), with $X = \frac{(s-p)}{as+1}G \in \mathscr{M}(\mathbf{S})$, rank $X(p) = r$. Let $X(0)$ be nonsingular, $G^{-1}(0) = -pX(0)^{-1}$. (i) PD-design: Choose any $\tau_\mathrm{d} > 0$, $\hat{K}_\mathrm{d} \in \mathbb{R}^{r\times r}$. Define $\widehat{C}_\mathrm{pd} := X(0)^{-1} + \frac{s\hat{K}_\mathrm{d}}{\tau_\mathrm{d}s+1}$ and $\Phi_\Lambda := s^{-1}[(s-p)G_\Lambda(s)\hat{C}_\mathrm{pd}(s) - I]$, $\widetilde{\Phi}_\Lambda := s^{-1}[\hat{C}_\mathrm{pd}(s)(s-p)G_\Lambda(s) - I]$. If $0 \leqslant p < \max\{\min_{\mathscr{T}\in\Theta}\|\Phi_\Lambda\|^{-1}, \min_{\mathscr{T}\in\Theta}\|\widetilde{\Phi}_\Lambda\|^{-1}\}$, then for any positive $\alpha \in \mathbb{R}$ satisfying (6), a PD-controller that stabilizes G_Λ for $\mathscr{T} \in \Theta$ is given by (7); if $\hat{K}_\mathrm{d} = 0$, (7) is a P-controller:*

$$p < \alpha + p < \max\left\{\min_{\mathscr{T}\in\Theta}\|\Phi_\Lambda\|^{-1}, \min_{\mathscr{T}\in\Theta}\|\widetilde{\Phi}_\Lambda\|^{-1}\right\}, \tag{6}$$

$$C_\mathrm{pd}(s) = (\alpha + p)\widehat{C}_\mathrm{pd}(s). \tag{7}$$

(ii) PID-design: Let C_pd be as (7) and $H_\mathrm{pd} := G_\Lambda(I + C_\mathrm{pd}G_\Lambda)^{-1}$. Define $\Upsilon := s^{-1}[H_\mathrm{pd}(s)H_\mathrm{pd}(0)^{-1} - I]$, $\widetilde{\Upsilon} := s^{-1}[H_\mathrm{pd}(0)^{-1}H_\mathrm{pd}(s) - I]$. Then with $H_\mathrm{pd}(0)^{-1} = \alpha X(0)^{-1}$, for

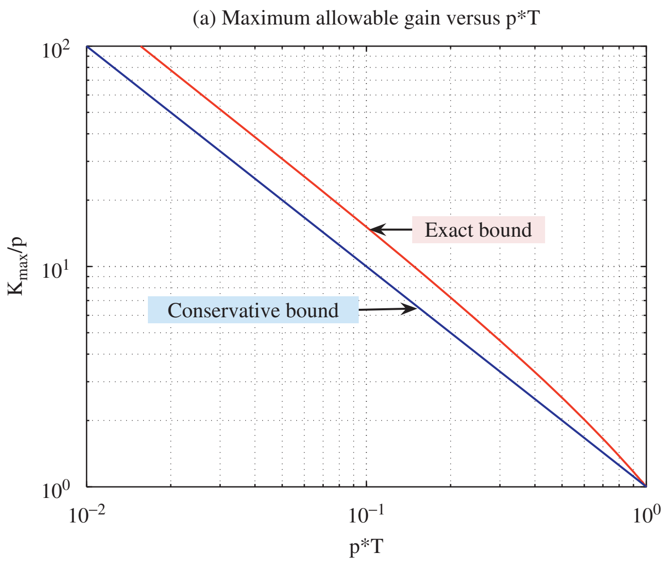

Fig. 2. Maximum K_{p} versus pT.

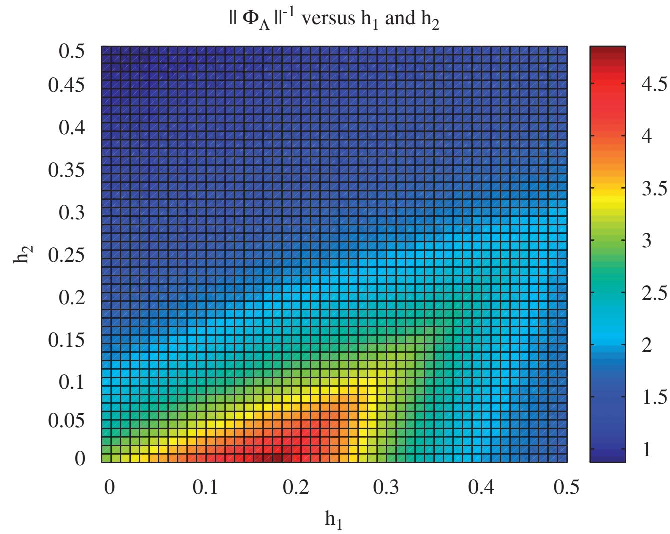

Fig. 3. Maximum $\|\Phi_\Lambda\|^{-1}$ versus h_1 and h_2.

any $\gamma \in \mathbb{R}$ *satisfying* (8), *a PID-controller that stabilizes* G_Λ *for* $\mathcal{T} \in \Theta$ *is given by* (9); *if* $\hat{K}_{\mathrm{d}} = 0$, (9) *is a PI-controller*:

$$0 < \gamma < \max \left\{ \min_{\mathcal{T} \in \Theta} \|\Upsilon\|^{-1}, \min_{\mathcal{T} \in \Theta} \|\widetilde{\Upsilon}\|^{-1} \right\}, \tag{8}$$

$$C_{\mathrm{pid}}(s) = C_{\mathrm{pd}}(s) + \frac{\gamma \alpha X(0)^{-1}}{s}. \tag{9}$$

Example 1. Consider the delayed plant $G_\Lambda(s) = \frac{\mathrm{e}^{-sT}}{s-p}$, where $p > 0$. Then for $a > 0$, $X := 1/(as+1)$, $X(0) = 1$. Choose any $\hat{K}_{\mathrm{d}} \in \mathbb{R}$, $\tau_{\mathrm{d}} > 0$. By Proposition 4, if $p < \min_{\mathcal{T} \in \Theta} \|\Phi_\Lambda\|^{-1} = \min_{\mathcal{T} \in \Theta} \| \frac{\mathrm{e}^{-sT}-1}{s} + \frac{\mathrm{e}^{-sT}\hat{K}_{\mathrm{d}}}{\tau_{\mathrm{d}}s+1} \|^{-1}$, then for any α as (6), $C_{\mathrm{pd}}(s) = (p+\alpha) + \frac{(p+\alpha)s\hat{K}_{\mathrm{d}}}{\tau_{\mathrm{d}}s+1}$ is a stabilizing PD-controller for G_Λ. For SISO plants, $\Phi_\Lambda = \tilde{\Phi}_\Lambda$. Now consider proportional controller design for a fixed T and p in this example. It is easy to show that a stabilizing P-controller exists if and only if $pT < 1$. Moreover, for any fixed $pT < 1$, there is a maximum allowable gain $K_{\max}$ for the proportional controller; this is shown in Fig. 2 as the exact bound. On the other hand, our approach uses the small gain argument and leads to $C_{\mathrm{p}} = (p+\alpha)$ as the controller gain. With $\|\Phi_\Lambda\| = \|T \frac{(\mathrm{e}^{-sT}-1)}{sT}\| = T$, the condition $p < \|\Phi_\Lambda\|^{-1}$ is the same as $pT < 1$. From the bound given in (6), $\alpha < T^{-1} - p$; the largest controller gain we can use in our case is $1/T$. This bound is also shown in Fig. 2, which illustrates that the approach used here is not too conservative. Fig. 2 also demonstrates the difficulty of controlling this plant using a proportional controller when the product of the unstable pole with delay is relatively large. Other fundamental performance limitations can also be quantified in terms of smallest achievable sensitivity level (Stein, 2003), or mixed sensitivity $\mathscr{H}_\infty$ cost (Enns, Özbay, & Tannenbaum, 1992). Clearly, by using the derivative term we can improve the bound on largest allowable pT. The largest pole delay product for which we can find a PD-controller is $1.38 = 1/0.725$, and that corresponds to $\tau_{\mathrm{d}} \to 0$ and $\hat{K}_{\mathrm{d}}/T = 0.31$.

Example 2. Consider the transfer matrix $G(s)$ of a distillation column (Friedland, 1986), where $G(s) = \frac{1}{s} G_{\mathrm{o}} G_1(s)$ with $G_{\mathrm{o}} = \begin{bmatrix} 3.04 & -278.2/180 \\ 0.052 & 206.6/180 \end{bmatrix}$, $G_1(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{180}{(s+6)(s+30)} \end{bmatrix}$. An LCF is $G(s) = Y(s)^{-1}X(s)$, with $X(s) = \frac{1}{as+1} G_{\mathrm{o}} G_1(s)$, $Y(s) = \frac{s}{as+1} I$, $a > 0$. Let the delays in the input channels be h_1 and h_2, and consider proportional control only. In this case we have $\widehat{C}_{\mathrm{p}} = X(0)^{-1} = G_{\mathrm{o}}^{-1}$, $C_{\mathrm{p}} = \alpha X(0)^{-1} = \alpha G_{\mathrm{o}}^{-1}$, and $\Phi_\Lambda(s) = s^{-1}[G_{\mathrm{o}} G_1(s) \Lambda_{\mathrm{i}}(s) G_{\mathrm{o}}^{-1} - I]$. Fig. 3 shows $\|\Phi_\Lambda\|^{-1}$ versus h_1 and h_2; the largest value 4.86 is obtained for $h_1 = 0.18$, $h_2 = 0$. A delay of 0.18 s is needed in the first channel to equalize the phase lag in the input channels of $G_1 \Lambda_{\mathrm{i}}$. In this case stability is guaranteed if $\alpha < \|\tilde{\Phi}_\Lambda\|^{-1}$, where $\|\tilde{\Phi}_\Lambda\| = \max\{h_1, h_2 + 0.2\}$. Clearly, the largest gain allowable is $\alpha_{\max} = 5$, for $h_2 = 0$ and $0 \leqslant h_1 < 0.2$. This result is less conservative than the one obtained using the bound $\alpha < \|\Phi_\Lambda\|^{-1}$. For $h_2 = 0$, $h_1 > 0.2$ we have $\alpha_{\max} = 1/h_1$. But when $C(s) = \alpha G_{\mathrm{o}}^{-1}$, the characteristic equation of this system is $(1 + \frac{\alpha \mathrm{e}^{-h_1 s}}{s})(1 + \frac{\alpha 180 \mathrm{e}^{-h_2 s}}{s(s+6)(s+30)}) = 0$. When $h_2 = 0$, actual largest gain we can use is $\alpha_{\max,\mathrm{act}} = \min\{\alpha_{\max,1}, 36\}$, where $\alpha_{\max,1} = \frac{\pi}{2h_1}$; for $h_1 > 0.2$, $\alpha_{\max,\mathrm{act}} = \frac{\pi}{2h_1} \approx \frac{1.57}{h_1} > \alpha_{\max} = \frac{1}{h_1}$, which illustrates the level of conservatism in this example. Now consider the PD-controller $C_{\mathrm{pd}} = \alpha(I + \frac{s\tilde{K}_{\mathrm{d}}}{\tau_{\mathrm{d}}s+1}) G_{\mathrm{o}}^{-1}$ in (7), where $\hat{K}_{\mathrm{d}} =: \tilde{K}_{\mathrm{d}} G_{\mathrm{o}}^{-1}$. The optimal derivative gain matrix $\hat{K}_{\mathrm{d}} = \tilde{K}_{\mathrm{d}} G_{\mathrm{o}}^{-1}$ is the one that minimizes $\|\tilde{\Phi}_\Lambda\|$. Since $\tilde{\Phi}_\Lambda$ is diagonal, we restrict $\tilde{K}_{\mathrm{d}}$ to be in the form $\mathrm{diag}(K_{\mathrm{d},1}, K_{\mathrm{d},2})$. Fig. 4 shows optimal $K_{\mathrm{d},1}$ (resp. $K_{\mathrm{d},2}$) versus h_1 (resp. h_2).

3.1.2. Plants with two unstable poles

Let $G \in \mathbf{R}_{\mathbf{p}}^{r \times r}$ have full (normal) rank. Let G have no transmission zeros at $s = 0$. Define $d := (a_1 s + 1)(a_2 s + 1)$ and $n := (s - p_1)(s - p_2)$, where $p_1, p_2 \in \mathscr{U}$, $a_1, a_2 \in \mathbb{R}$, $a_1, a_2 > 0$,

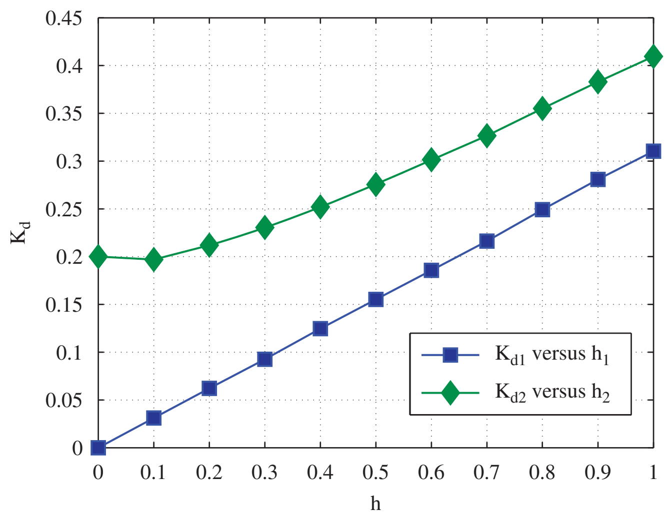

Fig. 4. Optimal $K_{\mathrm{d},1}$ and $K_{\mathrm{d},2}$.

and let G have an LCF:

$$G = Y^{-1}X = \left[\frac{n}{d}I\right]^{-1}\left[\frac{n}{d}G\right], \tag{10}$$

where rank $X(p_j) = \operatorname{rank} nG(s)|_{s=p_j} = r$, $j = 1, 2$. Since G has no transmission-zeros at $s = 0$, rank $X(0) = \operatorname{rank} nG(s)|_{s=0} = r$. We consider real and complex-conjugate pairs of poles as two separate cases:

Case (a): The two unstable poles are real: $p_1, p_2 \in \mathbb{R}_+$. Proposition 5(a) shows that under certain assumptions, the delayed plant G_Λ admits PD and PID-controllers. Some plants in this class (for example, $G = \frac{1}{(s-p_1)(s-p_2)}$, $p_1 \geqslant 0$, $p_2 \geqslant 0$) do not admit P, D, or I-controllers.

Case (b): The two poles are a complex-conjugate pair, i.e., $p_1 = \bar{p}_2$, $n = s^2 - (p_1 + p_2)s + p_1 p_2 = s^2 - 2fs + g^2$, $f \geqslant 0$, $g > 0$, $f < g$. In this case, $X(0) = g^2 G(0)$. Proposition 5(b) shows that under certain assumptions, the delayed plant G_Λ admits D, PD, ID, PID-controllers. Some plants in this class (for example, $G = \frac{1}{s^2+g^2}$, $g \geqslant 0$) do not admit P-controllers or I-controllers.

Proposition 5. *Let G be as* (10), *with $X = \frac{n}{d}G \in \mathbf{S}^{r\times r}$, rank $X(p_j) = r$, $j = 1, 2$. Let $X(0)$ be nonsingular. Choose any $\tau_\mathrm{d} > 0$. Define $\Phi_\Lambda := s^{-1}[\frac{n}{(\tau_\mathrm{d}s+1)}G_\Lambda(s)X(0)^{-1} - I]$ and $\tilde{\Phi}_\Lambda := s^{-1}[\frac{n}{(\tau_\mathrm{d}s+1)}X(0)^{-1}G_\Lambda(s) - I]$.*

(a) *Let $p_1, p_2 \in \mathbb{R}_+$.* (i) *PD-design: If $0 \leqslant p_1 < \Omega$ where $\Omega := \max\{\min_{\mathscr{T}\in\Theta}\|\Phi_\Lambda\|^{-1}, \min_{\mathscr{T}\in\Theta}\|\widetilde{\Phi}_\Lambda\|^{-1}\}$, then choose any $\alpha_1 \in \mathbb{R}$ satisfying*

$$p_1 < \alpha_1 + p_1 < \Omega. \tag{11}$$

Let $W := (s - p_2)G_\Lambda X(0)^{-1}$, $\widetilde{W} = (s - p_2)X(0)^{-1}G_\Lambda$. Define $\Phi_{2\Lambda} := s^{-1}[\alpha_1(I + \frac{(\alpha_1+p_1)}{\tau_\mathrm{d}s+1}W)^{-1}W - I]$, $\widetilde{\Phi}_{2\Lambda} := s^{-1}[\alpha_1(I + \frac{(\alpha_1+p_1)}{\tau_\mathrm{d}s+1}\widetilde{W})^{-1}\widetilde{W} - I]$. If $0 \leqslant p_2 < \Omega_2$, where $\Omega_2 :=$ $\max\{\min_{\mathscr{T}\in\Theta}\|\Phi_{2\Lambda}\|^{-1}, \min_{\mathscr{T}\in\Theta}\|\widetilde{\Phi}_{2\Lambda}\|^{-1}\}$, *then choose any $\alpha_2 \in \mathbb{R}$ satisfying*

$$p_2 < \alpha_2 + p_2 < \Omega_2. \tag{12}$$

Let $K_\mathrm{p} = (\alpha_1\alpha_2 - p_1p_2)X(0)^{-1}$, $K_\mathrm{d} = (\alpha_1 + p_1)(1 + \tau_\mathrm{d}p_2)X(0)^{-1}$; then a PD-controller that stabilizes G_Λ for $\mathscr{T} \in \Theta$ is given by $C_{\mathrm{pd}}(s) = K_\mathrm{p} + \frac{sK_\mathrm{d}}{\tau_\mathrm{d}s+1}$. (ii) *PID design: Let C_{pd} be as above. Then for any $\gamma \in \mathbb{R}$ satisfying* (8) *with $H_{\mathrm{pd}}(0)^{-1} = \alpha_1\alpha_2X(0)^{-1}$, a PID-controller that stabilizes G_Λ for $\mathscr{T} \in \Theta$ is given by*

$$C_{\mathrm{pid}}(s) = C_{\mathrm{pd}}(s) + \frac{\gamma\alpha_1\alpha_2X(0)^{-1}}{s}. \tag{13}$$

(b) *Let $p_1 = \bar{p}_2 \in \mathbb{C}$, $n = s^2 - (p_1+p_2)s + p_1p_2 = s^2 - 2fs + g^2$, $f \geqslant 0$, $g > 0$, $f < g$.* (i) *PD-design: If $f + 2g < \Omega$, then choose any $\beta_1, \beta_2 \in \mathbb{R}$, $\beta_1, \beta_2 \geqslant 0$, satisfying*

$$\beta_1 + \beta_2 + (f + 2g) < \Omega. \tag{14}$$

Let $K_\mathrm{p} = [\beta_1\beta_2 + \beta_1(g - f) + \beta_2 g - fg]X(0)^{-1}$, $K_\mathrm{d} = (\beta_1 + \beta_2 + f + 2g)X(0)^{-1} - \tau_\mathrm{d}K_\mathrm{p}$; then a PD-controller that stabilizes G_Λ for $\mathscr{T} \in \Theta$ is

$$C_{\mathrm{pd}}(s) = K_\mathrm{p} + \frac{K_\mathrm{d}s}{\tau_\mathrm{d}s+1} = \frac{\vartheta}{\tau_\mathrm{d}s+1}\frac{G(0)^{-1}}{g^2}, \tag{15}$$

$$\vartheta := (\beta_1 + \beta_2 + f + 2g)s + \beta_1(\beta_2 + g - f) + \beta_2 g - fg.$$

If $2(f + g) < \Omega$, let $K_\mathrm{d} = 2(f + g)X(0)^{-1}$; then a D-controller that stabilizes G_Λ is

$$C_\mathrm{d}(s) = \frac{K_\mathrm{d}s}{\tau_\mathrm{d}s+1} = \frac{2(f+g)G(0)^{-1}s}{g^2(\tau_\mathrm{d}s+1)}. \tag{16}$$

(ii) *PID-design: Let C_{pd} be as* (15). *Then with $H_{\mathrm{pd}}(0)^{-1} = (\beta_1 + g)(\beta_2 + g - f)X(0)^{-1}$, for any $\gamma \in \mathbb{R}$ satisfying* (8), *a PID-controller that stabilizes G_Λ for $\mathscr{T} \in \Theta$ is*

$$C_{\mathrm{pid}}(s) = C_{\mathrm{pd}}(s) + \frac{\gamma(\beta_1 + g)(\beta_2 + g - f)}{s}\frac{G(0)^{-1}}{g^2}. \tag{17}$$

Let C_d be as (16). *Then with $H_\mathrm{d}(0)^{-1} = g^2X(0)^{-1} = G^{-1}(0)$, for any $\gamma \in \mathbb{R}$ satisfying* (8), *an ID-controller that stabilizes G_Λ for $\mathscr{T} \in \Theta$ is*

$$C_{\mathrm{id}}(s) = C_\mathrm{d}(s) + \frac{\gamma G(0)^{-1}}{s}. \tag{18}$$

4. Conclusions

We showed existence of stabilizing PID-controllers for a class of LTI, MIMO plants with delays in the input and/or output channels. Moreover, for plants with only one or two unstable poles (and finitely many $\mathbb{C}_-$ poles) we gave explicit formulae for PID controller parameters. These results are obtained from a small gain based argument. Therefore, they are conservative. We were able to quantify the level of conservatism on the examples given.

In the light of inequality conditions (6) and (8) of Proposition 4, an interesting problem to study is the computation of optimal $\widehat{K}_\mathrm{d}$ which minimizes $\|\Phi\|$ or $\|\widetilde{\Phi}\|$, and optimal $(\alpha, \widehat{K}_\mathrm{d})$ minimizing $\|\Upsilon\|$ or $\|\widetilde{\Upsilon}\|$. Fig. 4 answers this question partially for the specific example considered. The numerical values in this figure are computed from a brute-force search. An analytic solution is possible, see Özbay and Gündeş (2006) for further details.

Appendix A. Proofs

Proof of Lemma 1. Let $G=Y^{-1}X$ be an LCF. The controller C_g, which admits an RCF in the form $C_g=N_gD_g^{-1}$, stabilizes $G_\Lambda=Y^{-1}X_\Lambda$ if and only if $M_\Lambda:=YD_g+X_\Lambda N_g$ is unimodular. Since C_g stabilizes G_Λ, $H_\Lambda=G_\Lambda(I+C_gG_\Lambda)^{-1}$ and $I-C_gH_\Lambda=(I+C_gG_\Lambda)^{-1}$ are stable. Now $C_h\in\mathscr{S}(H_\Lambda)$ if and only if $C_h(I+H_\Lambda C_h)^{-1}\in\mathscr{M}(\mathscr{H}_\infty)$, and $(I+H_\Lambda C_h)^{-1}\in\mathscr{M}(\mathscr{H}_\infty)$. Write $C=C_g+C_h=[N_g+(I-C_gH_\Lambda)C_h(I+H_\Lambda C_h)^{-1}D_g][(I+H_\Lambda C_h)^{-1}D_g]^{-1}$. Define $N_c:=[N_g+(I-C_gH_\Lambda)C_h(I+H_\Lambda C_h)^{-1}D_g]\in\mathscr{M}(\mathscr{H}_\infty)$, $D_c:=(I+H_\Lambda C_h)^{-1}D_g\in\mathscr{M}(\mathscr{H}_\infty)$.Then $YD_c+X\Lambda N_c=Y[(I+H_\Lambda C_h)^{-1}+H_\Lambda C_h(I+H_\Lambda C_h)^{-1}]D_g+X_\Lambda N_g=M_\Lambda$ is unimodular. Therefore, $C=N_cD_c^{-1}\in\mathscr{S}(G_\Lambda)$. □

Proof of Proposition 2. (i) Let $M_{\rm pd}:=I+G_\Lambda C_{\rm pd}$, where $C_{\rm pd}=\alpha\widehat{C}_{\rm pd}$; $\alpha\|G_\Lambda\widehat{C}_{\rm pd}\|=\alpha\|G\widehat{C}_{\rm pd}\|<1$ implies $M_{\rm pd}$ is unimodular. Therefore, $C_{\rm pd}$ stabilizes G_Λ. (ii) The controller $C_{\rm pid}$ stabilizes G_Λ if and only if $M_{\rm pid}:=\frac{s}{s+\gamma}I+G_\Lambda\frac{s}{s+\gamma}C_{\rm pid}$ (equivalently $\widetilde{M}_{\rm pid}:=\frac{s}{s+\gamma}I+\frac{s}{s+\gamma}C_{\rm pid}G_\Lambda$) is unimodular. Writing $M_{\rm pid}=I+\frac{\gamma s}{s+\gamma}\frac{(sG_\Lambda(s)\widehat{C}_{\rm pid}-I)}{s}$, and $\widetilde{M}_{\rm pid}=I+\frac{\gamma s}{s+\gamma}\frac{(s\widehat{C}_{\rm pid}G_\Lambda(s)-I)}{s}$, a sufficient condition for unimodularity of $M_{\rm pid}$ is that γ satisfies the first upper bound in (4) and for $\widetilde{M}_{\rm pid}$ is that γ satisfies the second upper bound in (4).Since $M_{\rm pid}$ is unimodular if and only if $\widetilde{M}_{\rm pid}$ is, the less conservative one of these bounds suffices and hence $C_{\rm pid}:=\gamma\widehat{C}_{\rm pid}\in\mathscr{S}(G_\Lambda)$ for $\gamma\in\mathbb{R}$ satisfying (4). □

Proof of Proposition 3. : (a) Let $G=Y^{-1}X$ be an LCF of G. Let $C_{\rm pid}=K_{\rm p}+\frac{K_{\rm i}}{s}+\frac{K_{\rm d}\,s}{\tau_{\rm d}s+1}$ be in $\mathscr{S}(G_\Lambda)$. An RCF $C_{\rm pid}=N_cD_c^{-1}=[(K_{\rm p}+\frac{K_{\rm d}s}{\tau_{\rm d}s+1})\frac{s}{s+a}+\frac{K_{\rm i}}{s+a}][\frac{s}{s+a}I_r]^{-1}$, for any $a\in\mathbb{R}, a>0$. Since $C_{\rm pid}$ stabilizes G_Λ, $M_\Lambda=YD_c+X_\Lambda N_c$ is unimodular, which implies rank $M_\Lambda(0)=r=$ rank $X(0)K_{\rm i}$. Therefore, rank $X(0)=r$, equivalently, G has no transmission-zeros at $s=0$, and rank $K_{\rm i}=r$. (b) Suppose that G_Λ is stabilized by $C_{\rm p}$, equivalently $H_{\rm p}=G_\Lambda(I+C_{\rm p}G_\Lambda)^{-1}\in\mathscr{M}(\mathscr{H}_\infty)$; or by $C_{\rm d}$, equivalently $H_{\rm d}=G_\Lambda(I+C_{\rm d}G_\Lambda)^{-1}\in\mathscr{M}(\mathscr{H}_\infty)$; or by $C_{\rm i}$, which implies $H_{\rm i}=G_\Lambda(I+C_{\rm i}G_\Lambda)^{-1}\in\mathscr{M}(\mathscr{H}_\infty)$. The (normal) ranks of $H_{\rm p}$, $H_{\rm d}$, $H_{\rm i}$ are equal to rank $G=r$. By Proposition 2(i), there exists a P-controller for $H_{\rm d}$, for $H_{\rm i}$, and for $H_{\rm id}$; there exists a D-controller for $H_{\rm p}$, for $H_{\rm i}$, and for $H_{\rm pi}$. By Proposition 2(ii), there exists an I-controller for $H_{\rm p}$, for $H_{\rm d}$, and for $H_{\rm pd}$. Consider $H_{\rm p}\in\mathscr{M}(\mathscr{H}_\infty)$: if G has no transmission-zeros at $s=0$, then rank $H_{\rm p}(0)=$rank$(Y+X_\Lambda C_{\rm p})^{-1}(0)X_\Lambda(0)=$ rank $X(0)=r$. Let $C_{{\rm d}h}$ be a D-controller and $C_{{\rm i}h}$ be an I-controller for $H_{\rm p}$. By Lemma 1, the PD-controller $C_{\rm pd}=C_{\rm p}+C_{{\rm d}h}$ and the PI-controller $C_{\rm pi}=C_{\rm p}+C_{{\rm i}h}$ stabilize G_Λ. Similarly, consider $H_{\rm d}\in\mathscr{M}(\mathscr{H}_\infty)$: Since $M_{{\rm d}\Lambda}:=(Y+X_\Lambda C_{\rm d})$ is unimodular, rank $M_{{\rm d}\Lambda}(0)=$ rank $Y(0)=r$; i.e., G has no poles at $s=0$. If G has no transmission-zeros at $s=0$, then rank $H_{\rm d}(0)=$ rank $M_{{\rm d}\Lambda}^{-1}(0)X_\Lambda(0)=$ rank $X(0)=r$. Let $C_{{\rm p}h}$ be a P-controller and $C_{{\rm i}h}$ be an I-controller for $H_{\rm d}$. By Lemma 1, the PD-controller $C_{\rm dp}=C_{\rm d}+C_{{\rm p}h}$ and the ID-controller $C_{\rm di}=C_{\rm d}+C_{{\rm i}h}$ stabilize G_Λ. Consider $H_{\rm i}\in\mathscr{M}(\mathscr{H}_\infty)$: let $C_{{\rm p}h}$ be a P-controller and $C_{{\rm d}h}$ be a D-controller for $H_{\rm i}$. By Lemma 1, the PI-controller $C_{\rm ip}=C_{\rm i}+C_{{\rm p}h}$ and the ID-controller $C_{\rm id}=C_{\rm i}+C_{{\rm d}h}$ stabilize G_Λ. (c) Suppose that G_Λ is stabilized by $C_{\rm pd}$, equivalently $H_{\rm pd}=G_\Lambda(I+C_{\rm pd}G_\Lambda)^{-1}\in\mathscr{M}(\mathscr{H}_\infty)$; or by $C_{\rm pi}$, which implies $H_{\rm pi}=G_\Lambda(I+C_{\rm pi}G_\Lambda)^{-1}\in\mathscr{M}(\mathscr{H}_\infty)$; or by $C_{\rm id}$, which implies $H_{\rm id}=G_\Lambda(I+C_{\rm id}G_\Lambda)^{-1}\in\mathscr{M}(\mathscr{H}_\infty)$. The (normal) ranks of $H_{\rm pd}$, $H_{\rm pi}$, $H_{\rm id}$ are equal to rank $G=r$. Consider $H_{\rm pd}\in\mathscr{M}(\mathscr{H}_\infty)$: if G has no transmission-zeros at $s=0$, then rank $H_{\rm pd}(0)=$ rank$(Y+X_\Lambda C_{\rm pd})^{-1}(0)X_\Lambda(0)=$ rank $X(0)=r$. Let $C_{{\rm i}h}$ be an I-controller for $H_{\rm pd}$. Let $C_{{\rm d}h}$ be a D-controller for $H_{\rm pi}$. Let $C_{{\rm p}h}$ be a P-controller for $H_{\rm id}$.By Lemma 1, each of the PID-controllers $C_{\rm pdi}=C_{\rm pd}+C_{{\rm i}h}$, $C_{\rm pid}=C_{\rm pi}+C_{{\rm d}h}$, and $C_{\rm idp}=C_{\rm id}+C_{{\rm p}h}$ stabilize G_Λ. □

Proof of Lemma 2. Let $G=Y^{-1}X$ be an LCF of G. Let $C_{\rm pid}\in\mathscr{S}(G_\Lambda)$. An RCF $C_{\rm pid}=N_cD_c^{-1}$ is given in Proposition 3. Then $\det D_c(z_{\rm i})=\det\frac{z_{\rm i}}{z_{\rm i}+a}I_r>0$ for all $z_{\rm i}>0$. If $C_{\rm pid}\in\mathscr{S}(G_\Lambda)$, then $M_\Lambda=YD_c+X_\Lambda N_c$ is unimodular, which implies $\det M_\Lambda(z_{\rm i})=\det Y(z_{\rm i})\det D_c(z_{\rm i})$ has the same sign for all $z_{\rm i}\in\mathscr{U}$ such that $X(z_{\rm i})=0$; equivalently, $\det Y(z_{\rm i})$ has the same sign at all blocking-zeros of G. Therefore, G has the parity-interlacing-property; hence, it is strongly stabilizable (Vidyasagar, 1985). □

Proof of Proposition 4. (i) $C_{\rm pd}\in\mathscr{S}(G_\Lambda)$ if and only if $M_{\rm pd}:=Y+X_\Lambda C_{\rm pd}=\frac{(s-p)}{as+1}(I+G_\Lambda C_{\rm pd})$ is unimodular, equivalently, $\det\frac{(s-p)}{as+1}(I+G_\Lambda C_{\rm pd})=\det\frac{(s-p)}{as+1}$ $\det(I+C_{\rm pd}G_\Lambda)$ is a unit in $\mathscr{H}_\infty$; equivalently, $\widetilde{M}_{\rm pd}:=\frac{(s-p)}{as+1}(I+C_{\rm pd}G_\Lambda)=Y+C_{\rm pd}X_\Lambda$ is unimodular. Writing $M_{\rm pd}=\frac{(s-p)}{as+1}(I+(\alpha+p)G_\Lambda\widehat{C}_{\rm pd})=(I+\frac{(\alpha+p)s}{s+\alpha}\Phi_\Lambda)\frac{(s+\alpha)}{(as+1)}$, and $\widetilde{M}_{\rm pd}=(I+\frac{(\alpha+p)s}{s+\alpha}\widetilde{\Phi}_\Lambda)\frac{(s+\alpha)}{(as+1)}$, a sufficient condition for unimodularity of $M_{\rm pd}$ is $(\alpha+p)<\min_{\mathscr{T}\in\Theta}\|\Phi_\Lambda\|^{-1}$and for $\widetilde{M}_{\rm pd}$ is $(\alpha+p)<\min_{\mathscr{T}\in\Theta}\|\widetilde{\Phi}_\Lambda\|^{-1}$. Since $M_{\rm pd}$ is unimodular if and only if $\widetilde{M}_{\rm pd}$ is, the less conservative one of these bounds suffices and hence, $C_{\rm pd}$ in (7) stabilizes G_Λ for α satisfying (6). (ii) Since $C_{\rm pd}$ stabilizes G_Λ, $H_{\rm pd}:=M_{\rm pd}^{-1}X_\Lambda=G_\Lambda(I+C_{\rm pd}G_\Lambda)^{-1}\in\mathscr{M}(\mathscr{H}_\infty)$, where $H_{\rm pd}(0)^{-1}=G^{-1}(0)+K_{\rm p}=X(0)^{-1}Y(0)+(\alpha+p)X(0)^{-1}=\alpha X(0)^{-1}$. Using similar steps as in the proof of Proposition 2, the I-controller $K_{\rm i}/s=\gamma H_{\rm pd}(0)^{-1}/s$ stabilizes $H_{\rm pd}$ for any $\gamma\in\mathbb{R}$ satisfying (8). So, $C_{\rm pid}$ in (9) stabilizes G_Λ. □

Proof of Proposition 5. (a) (i) Let $M_1:=\frac{(s-p_1)}{a_1s+1}I+(\alpha_1+p_1)\frac{(a_2s+1)}{\tau_{\rm d}s+1}X_\Lambda(s)X(0)^{-1}=\frac{(s-p_1)}{a_1s+1}(I+\frac{(\alpha_1+p_1)}{\tau_{\rm d}s+1}W)$; M_1 is unimodular if and only if $\widetilde{M}_1:=\frac{(s-p_1)}{a_1s+1}(I+\frac{(\alpha_1+p_1)}{\tau_{\rm d}s+1}\widetilde{W})$ is unimodular. Writing $M_1=(I+\frac{(\alpha_1+p_1)s}{s+\alpha_1}\Phi_\Lambda)\frac{(s+\alpha_1)}{a_1s+1}$, a sufficient condition for unimodularity of M_1 is $(\alpha_1+p_1)<\min_{\mathscr{T}\in\Theta}\|\Phi_\Lambda\|^{-1}$; similarly for $\widetilde{M}_1$ is $(\alpha_1+p_1)<\min_{\mathscr{T}\in\Theta}\|\widetilde{\Phi}_\Lambda\|^{-1}$. Since M_1

is unimodular if and only if $\widetilde{M}_1$ is, the less conservative upper bound suffices and hence, M_1 is unimodular if α_1 satisfies (11). Since $C_{\rm pd} = (\alpha_1 + p_1)\frac{(s-p_2)}{\tau_{\rm d}s+1}X(0)^{-1} + \alpha_1(\alpha_2 + p_2)X(0)^{-1}$. Let $M_{\rm pd} := Y + X_\Lambda C_{\rm pd} = \frac{(s-p_2)}{a_2s+1}[\frac{(s-p_1)}{a_1s+1}I + (\alpha_1 + p_1)\frac{(a_2s+1)}{\tau_{\rm d}s+1}X_\Lambda(s)X(0)^{-1}] + \alpha_1(\alpha_2 + p_2)X_\Lambda(s)X(0)^{-1} = M_1[\frac{(s-p_2)}{a_2s+1}I + \alpha_1(\alpha_2 + p_2)M_1^{-1}X_\Lambda(s)X(0)^{-1}] =: M_1M_2$. Since M_1 is unimodular, $M_{\rm pd}$ is unimodular if and only if $M_2 = \frac{(s-p_2)}{a_2s+1}[I + \frac{(a_2s+1)}{s-p_2}\alpha_1(\alpha_2 + p_2)M_1^{-1}X_\Lambda(s)X(0)^{-1}] = [I + \frac{(\alpha_2+p_2)s}{s+\alpha_2}\Phi_{2\Lambda}]\frac{(s+\alpha_2)}{a_2s+1}$ is unimodular, equivalently, $\widetilde{M}_2 = \frac{(s-p_2)}{a_2s+1}[I + \alpha_1(\alpha_2 + p_2)X(0)^{-1}(I + \frac{(a_1s+1)}{\tau_{\rm d}s+1}W)^{-1}G_\Lambda(s)] = [I + \frac{(\alpha_2+p_2)s}{s+\alpha_2}\widetilde{\Phi}_{2\Lambda}]\frac{(s+\alpha_2)}{a_2s+1}$ is unimodular. A sufficient condition for unimodularity of M_2 is $(\alpha_2 + p_2) < \min_{\mathcal{T}\in\Theta}\|\Phi_{2\Lambda}\|^{-1}$ and for $\widetilde{M}_2$ is $(\alpha_2 + p_2) < \min_{\mathcal{T}\in\Theta}\|\widetilde{\Phi}_{2\Lambda}\|^{-1}$. Since M_2 is unimodular if and only if $\widetilde{M}_2$ is, the less conservative upper bound suffices and hence, $M_{\rm pd}$ is unimodular if α_2 satisfies (12). Therefore, $C_{\rm pd}$ stabilizes G_Λ. (ii) Since $C_{\rm pd}$ stabilizes G_Λ, $H_{\rm pd} := M_{\rm pd}^{-1}X_\Lambda = G_\Lambda(I + C_{\rm pd}G_\Lambda)^{-1} \in \mathscr{M}(\mathscr{H}_\infty)$; $H_{\rm pd}(0)^{-1} = K_{\rm p} + X(0)^{-1}Y(0) = \alpha_1\alpha_2X(0)^{-1}$. For any $\gamma \in \mathbb{R}$ satisfying (8), the I-controller $K_{\rm i}/s = \gamma H_{\rm pd}(0)^{-1}/s$ stabilizes $H_{\rm pd}$. By Lemma 1, $C_{\rm pid} = C_{\rm pd} + K_{\rm i}/s$ in (13) stabilizes G_Λ. (b) Define $y := (s + \beta_1 + g)(s + \beta_2 + g - f)$, where $g - f > 0$ by assumption. Let $x := y - n = (\beta_1 + \beta_2 + f + 2g)s + \beta_1\beta_2 + \beta_1(g - f) + \beta_2 g - fg$. Then $\|\frac{sx}{y}\| \leqslant (\beta_1 + \beta_2 + f + 2g)$, where $\frac{p_1+p_2}{2} + 2\sqrt{p_1p_2} = f + 2g$. If $\beta_1 + \beta_2 < \min_{\mathcal{T}\in\Theta}\|\Phi_\Lambda\|^{-1} - (f + 2g)$, then $\|\frac{sx}{y}\Phi_\Lambda\| \leqslant (\beta_1 + \beta_2 + f + 2g)\|\Phi_\Lambda\| < 1$ implies $M_{\rm pd} := Y + X_\Lambda C_{\rm pd} = \frac{n}{d}(I + G_\Lambda C_{\rm pd}) = \frac{y}{d}(I + \frac{x}{y}\Phi_\Lambda)$ is unimodular, equivalently $\widetilde{M}_{\rm pd} := \frac{n}{d}(I + C_{\rm pd}G_\Lambda) = \frac{y}{d}(I + \frac{x}{y}\widetilde{\Phi}_\Lambda)$ is unimodular (a sufficient condition is $\beta_1 + \beta_2 + (f + 2g) < \min_{\mathcal{T}\in\Theta}\|\widetilde{\Phi}_\Lambda\|^{-1}$). Since $M_{\rm pd}$ is unimodular if and only if $\widetilde{M}_{\rm pd}$ is, the less conservative upper bound suffices and hence, $C_{\rm pd}$ in (15) stabilizes G_Λ for β_1, β_2 satisfying (14). Similarly, let $u := (s+g)^2$; then $u - n = 2(f+g)s$. Since $2(f+g)\|\Phi_\Lambda\| < 1$ implies $M_{\rm d} := Y + X_\Lambda C_{\rm d} = \frac{n}{d}[I + G_\Lambda C_{\rm d}] = \frac{u}{d}[I + \frac{2(f+g)s^2}{u}\Phi_\Lambda]$ is unimodular, $C_{\rm d}$ in (16) stabilizes G_Λ. (ii) Since $C_{\rm pd}$ in (15) stabilizes G_Λ, $H_{\rm pd} := M_{\rm pd}^{-1}X_\Lambda = G_\Lambda(I + C_{\rm pd}G_\Lambda)^{-1} \in \mathscr{M}(\mathscr{H}_\infty)$; $H_{\rm pd}(0)^{-1} = G^{-1}(0) + K_{\rm p} = Y(0)X(0)^{-1}$. For any $\gamma \in \mathbb{R}$ satisfying (8), the I-controller $K_{\rm i}/s = \gamma H_{\rm pd}(0)^{-1}/s$ stabilizes $H_{\rm pd}$. By Lemma 1, $C_{\rm pid} = C_{\rm pd} + K_{\rm i}/s$ in (17) stabilizes G_Λ.Similarly, starting with $C_{\rm d}$ in (16), $H_{\rm d} = M_{\rm d}^{-1}X_\Lambda = G_\Lambda(I + C_{\rm d}G_\Lambda)^{-1} \in \mathscr{M}(\mathscr{H}_\infty)$; $H_{\rm d}(0)^{-1} = G^{-1}(0)$ and the conclusion follows. □

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