Bohr phenomena for Laplace–Beltrami operators

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Communicated by Prof. J. Korevaar at the meeting of November 28, 2005

ABSTRACT

We investigate a Bohr phenomenon on the spaces of solutions of weighted Laplace–Beltrami operators associated with the hyperbolic metric of the unit ball in $\mathbb{C}^N$. These solutions do not satisfy the usual maximum principle, and the spaces have natural bases none of whose members is a constant function. We show that these bases exhibit a Bohr phenomenon, define a Bohr radius for them that extends the classical Bohr radius, and compute it exactly. We also compute the classical Bohr radius of the invariant harmonic functions on the real hyperbolic space.

1. INTRODUCTION

It is truly rare that a mathematical paper generates intensive research activity after lying dormant for almost a century following its publication. Yet this is exactly what the short 1914 paper [16] of H. Bohr has accomplished in the last decade.

The only theorem in Bohr's paper involved power series $\sum_m a_m z^m$ mapping the unit disc $\mathbb{D}$ into itself, and stated that there is a radius K such that the sum $\sum_m |a_m z^m|$ of the moduli of the terms of the power series does not exceed 1 for all $|z| < K$. Bohr originally obtained the value $K = 1/6$, but the best value of K was soon shown to be $1/3$, this independently by Riesz, Schur, and Wiener. The

MSC: Primary 31B99, 35C10, 32A05, Secondary 31B10, 31B35, 35B50, 35J99, 33C55

Key words and phrases: Bohr radius, Weighted Laplace–Beltrami operator, Harnack inequality, Maximum principle, Generalized Poisson kernel, Spherical harmonics, Invariant harmonic, Real hyperbolic space

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[1] The research of the author was partially supported by a Fulbright grant.

paper [16], compiled by Hardy from correspondence, contains both Bohr's and Wiener's proofs; see also [28]. It is quite fitting that the best value $1/3$ is called the *Bohr radius* of $\mathbb{D}$, considering that Harald Bohr was the younger brother of Niels Bohr, the physicist after whom another radius was named in the theory of the hydrogen atom. For further historical information on H. Bohr and his work on Dirichlet series that led to his power series theorem, we refer the reader to [12] and [20]. Let us also mention that the Riesz brother cited above was Marcel Riesz, and that Wiener was Friedrich Wilhelm Wiener, whose main contributions to mathematics appear to consist of simpler proofs of some known theorems; see [15].

At least two other proofs of Bohr's theorem were published later (see [34] and [37]), but apart from that, the theorem did not attract much attention for decades. Much later, as a byproduct of a connection between the existence of absolute bases and nuclearity in infinite-dimensional holomorphy, a generalization of the Bohr radius to polydiscs in $\mathbb{C}^N$ was discovered in [21, Theorem 3.2]. But it was the paper [14] that carried Bohr's power series theorem to prominence. Its authors gave upper and lower bounds for the Bohr radius of the unit polydisc $\mathbb{D}^N$ for all N, and in Remark 3, they explained why the result of [21] is only asymptotically true.

A series of papers by several authors followed [14], extending the concept of Bohr radius in several different directions. Two kinds of Bohr radius were defined in [3] for any bounded convergence domain G for power series (a complete Reinhardt domain) in $\mathbb{C}^N$. The classical Bohr radius $K_N(G)$ is the largest r such that if a multiple power series $\sum_\gamma a_\gamma z^\gamma$ is bounded by 1 in G, then $\sum_\gamma |a_\gamma z^\gamma| < 1$ for z in the scaled domain rG. The second Bohr radius $B_N(G)$, which is somewhat easier to work with, is the largest r such that if $\sum_\gamma a_\gamma z^\gamma$ is bounded by 1 in G, then $\sum_\gamma \sup_{rG} |a_\gamma z^\gamma| \leqslant 1$. Estimates for either kind of Bohr radius for a large variety of domains can be found in [4], [13] and [19], among others. Let us note, however, that $\mathbb{D}$ is the only domain in one or several variables for which the exact value of the Bohr radius of either kind is known exactly.

A domain and its Bohr radius are often realized by replacing the usual norm $|z| = |z|_2$ by a new norm $|z|'$ and considering the Bohr radius of the unit ball of the underlying space $\mathbb{C}^N$ with respect to $|\cdot|'$. This idea was treated from an abstract point of view in [18] and [17] using local Banach space theory, and improved estimates were obtained for the classical Bohr radii K_N of the unit balls of rather large classes of finite-dimensional Banach spaces in terms of unconditional basis constants and Banach-Mazur distances between $(\mathbb{C}^N, |\cdot|')$ and $(\mathbb{C}^N, |\cdot|_p)$. The methods in these works as well as those of [13] were probabilistic.

Power series bounded by 1 are members of the unit ball of the function space H^∞ with respect to the norm $\|\cdot\|_\infty$. Bohr properties of the unit balls of other function spaces can be analyzed by supplanting $\|\cdot\|_\infty$ by other norms $\|\cdot\|'$. This course was pursued in [11] on $\mathbb{D}$ with equivalent Hardy norms and yielded a characterization of norms $\|\cdot\|'$ that display a Bohr property, which in turn was applied to a direct proof of the higher order and multivariable Schwarz–Pick estimates of [29]. The Hardy space made another appearance in [32], which also has Bohr-type results for Laurent series on annuli. One can study different function spaces even with

$\|\cdot\|_\infty$; this was done in [8] for power series with initial terms missing, and in [26] for polynomials of a fixed degree. Stressing partial sums of power series instead, a Rogosinski radius was defined and computed in [7].

An interesting application of the Bohr radius to finding a Banach algebra satisfying the nonunital von Neumann inequality and not isomorphic to an operator algebra appeared earlier in [22]. This line of investigation was continued in [30], where operator-theoretic methods led to Bohr-type inequalities for the multiplier algebras of weighted Hardy spaces in several variables, in which $\|f\|_\infty$ is replaced by the multiplier norm of f. More importantly, in this paper and in [33], versions of multivariable Bohr-type inequalities were obtained for several classes of holomorphic and harmonic functions in noncommutative variables of N-tuples of bounded operators. A Bohr phenomenon exists for uniform algebras too; see [31]. The Bohr radius $1/3$ is encountered repeatedly in various contexts in these references.

For the purposes of this paper, the most interesting approach, as initiated in [5] and [6], is the one that views a Bohr phenomenon as a property of a basis for the space of functions under consideration. For holomorphic functions on a complex manifold, a basis having a certain Bohr property must contain a constant function; see [6, Proposition 3.1, Theorem 4.5]. It is easy to construct a basis with no constant element for holomorphic functions on the unit disc without a related Bohr property; see [5, Example 3]. Then it should come as no surprise that in all other literature it is implicit that the bases in question always contain the constant function.

The basis approach proved useful in [27] in obtaining a Bohr radius for certain elliptic domains in the plane, using a basis consisting of Faber polynomials. Similar ideas were applied in [9] to solutions of second-order elliptic partial differential equations that have constants as solutions, and Bohr radii were computed for various classes of harmonic functions.

However, there are second-order elliptic partial differential operators of interest that do not admit constants as solutions. One important class is the weighted Laplace–Beltrami operators associated with the Bergman metric on the unit ball of $\mathbb{C}^N$; see [25] and [1]. The unweighted operator is the invariant Laplacian, and except for this special case, no other operator in this class annihilates constants. The solution spaces of the remaining operators, called the $\alpha\alpha$-harmonic functions, have natural bases none of whose members is a constant function.

Elliptic partial differential operators without constant solutions do not satisfy the usual maximum principle, but some weaker forms of it; see [23, Section 6.4]. However, as remarked in [9, Section 1], the crucial property for a Bohr phenomenon is the Harnack inequality, and it is satisfied by the solutions of all second-order elliptic equations; see [23, Section 6.4.3].

In this work, we show that the solution spaces of the weighted Laplace–Beltrami operators exhibit a Bohr phenomenon. We define and compute the exact value of a Bohr radius for each space of $\alpha\alpha$-harmonic functions. The computations depend on the spherical harmonic expansion of the invariant Poisson kernel obtained in [24], and yield a similar expansion for the more general class of so-called $\alpha\beta$-Poisson kernels in Theorem 3.1. The following combination of Equation (16), Theorem 4.8, and Corollary 4.9 is our main result.

Theorem. *For $N+2\alpha>0$, the Bohr radius $\mathcal{R}_\alpha$ for $\alpha\alpha$-harmonic functions on the unit ball of $\mathbb{C}^N$ is the unique root of the equation*

$$\frac{(1+r)^{N+2\alpha}}{(1-r)^N}=2F\big(-\alpha,-\alpha;N;r^2\big)$$

in the interval $(0,1)$, *where F is the hypergeometric function ${}_2F_1$. In particular, the Bohr radius for invariant harmonic functions on the unit ball of $\mathbb{C}^N$ is*

$$\mathcal{R}_0=\frac{2^{1/N}-1}{2^{1/N}+1}.$$

Almost all known proofs of Bohr theorems rest on the domination of the coefficients of a series expansion by the coefficient of the zeroth term. We obtain such an inequality early in the proof of Theorem 4.8, using properties of some generalized spherical harmonics following [9]; but see also Lemma 4.3. Another key idea in [5] and [9] was to pass to functions with positive real part and obtain a related inequality. Our version for bases without a constant member is Theorem 4.6. There are similarities with the Carathéodory inequality here. In fact, Carathéodory-type inequalities for Laurent series and multiple power series were essential ingredients in establishing Bohr phenomena in [27] and [4]. It is also interesting that we draw results from such diverse areas as harmonic function theory, special functions, and Möbius transformations to achieve our goals.

After a short Section 2 on notation, we set out in Section 3 to develop the basic properties of the general $\alpha\beta$-harmonic functions on the unit ball of $\mathbb{C}^N$ that provide background material for the rest of the paper. We define and compute the Bohr radius for $\alpha\alpha$-harmonic functions in Section 4. It seems that with the present methods the Bohr radius is computable only for real-valued functions, and this is the reason for having $\beta=\alpha$ there. The invariant harmonic functions on the unit ball of $\mathbb{C}^N$ are those that correspond to $\alpha=\beta=0$. We next obtain a Bohr radius for invariant harmonic functions on the unit ball of $\mathbb{R}^n$ in Section 5. In the final Section 6, we compare various Bohr radii and check their asymptotic behavior with respect to N and α.

Added in proof. A recent research announcement [10] brings the work of H. Bohr around a full circle by discovering a Bohr abscissa for Dirichlet series, Bohr's forte.

2. NOTATION

The usual inner product in $\mathbb{C}^N$ is $\langle z,w\rangle=z_1\overline{w}_1+\cdots+z_N\overline{w}_N$, and in $\mathbb{R}^n$ it is $\langle x,y\rangle=x_1y_1+\cdots+x_ny_n$. In either case the norm is $|a|=\sqrt{\langle a,a\rangle}$. We let $\mathbb{B}$ denote the unit ball of $\mathbb{C}^N$ or $\mathbb{R}^n$, $\mathbb{S}$ the sphere bounding $\mathbb{B}$, and σ the Lebesgue measure on $\mathbb{S}$ normalized so that $\sigma(\mathbb{S})=1$. Of course $N\geqslant 1$ and $n\geqslant 2$. We generally use ζ, η, ξ for points in $\mathbb{S}$; z, x for points in $\mathbb{B}$; and often write $z=r\zeta$, $x=r\xi$ with $0\leqslant r\leqslant 1$. The set of all Möbius transformations (automorphisms) of $\mathbb{B}$ is denoted $\mathcal{M}$.

A sum without an initial term is indicated by $\sum'$. The ordinary Laplacian is denoted Δ, the Kronecker delta δ_{ij}, and the gamma function Γ. The Pochhammer symbol is defined as $(a)_b = \Gamma(a+b)/\Gamma(a)$ when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function. For $a>0$, Stirling's formula gives

$$(1)\qquad (a)_b \sim a^b \quad (a\to\infty) \quad \text{and} \quad \frac{\Gamma(a+b)}{\Gamma(a+c)} \sim a^{b-c} \quad (a\to\infty),$$

where $x \sim y$ means that $|x/y|$ is bounded above and below by positive constants.

For $c \notin -\mathbb{N}$, the *hypergeometric function* $F = {}_2F_1$ is

$$F(a,b;c;t) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{t^m}{m!} \qquad (|t|<1).$$

Trivially $F(a,b;c;t) = F(b,a;c;t)$, $F(a,b;c;0) = 1$, and

$$(2)\qquad F(0,b;c;t) \equiv F(a,0;c;t) \equiv 1.$$

If further $\mathrm{Re}(c-a-b) > 0$, then

$$(3)\qquad F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

If $a,b,c>0$, then F is a positive increasing function of t for $0 \leqslant t \leqslant 1$. The hypergeometric function also has the following properties:

$$(4)\qquad \begin{aligned} &F(a,b;c;t) = (1-t)^{c-a-b} F(c-a,c-b;c;t), \\ &\frac{d}{dt} F(a,b;c;t) = \frac{ab}{c} F(a+1,b+1;c+1;t). \end{aligned}$$

3. $\alpha\beta$-HARMONIC FUNCTIONS

We begin with a general family of differential operators to set the stage. Let $D_j = \partial/\partial z_j$ and let $R = \sum_j z_j D_j$ be the *radial derivative*. The linear differential operators

$$\tilde{\Delta}_{\alpha\beta} = -4(1-|z|^2)\left[\sum_{i,j=1}^{N} (\delta_{ij} - z_i \bar{z}_j) D_i \overline{D}_j + \alpha R + \beta \overline{R} - \alpha\beta\right],$$

called the *$\alpha\beta$-Laplacians*, were introduced in [25]. Functions annihilated by $\tilde{\Delta}_{\alpha\beta}$ are called *$\alpha\beta$-harmonic*.

The operator $\tilde{\Delta}_{00} = \tilde{\Delta}$ is called the *invariant Laplacian* and is the Laplace–Beltrami operator associated with the Bergman (hyperbolic) metric of $\mathbb{B}$. The functions annihilated by it are called *$\mathcal{M}$-harmonic* or *invariant harmonic*, because $\tilde{\Delta}(f \circ \psi) = (\tilde{\Delta} f) \circ \psi$ for any $\psi \in \mathcal{M}$.

The $\alpha\beta$-Laplacians are encountered when the Laplace–Beltrami operator is computed on forms; see [25]. They also appear when certain radial derivatives of $\mathcal{M}$-harmonic functions are evaluated; see [2]. Further, $\tilde{\Delta}_{\alpha\alpha}$ is the weighted Laplace–Beltrami operator associated with the Bergman metric of $\mathbb{B}$ with weight $(1-|z|^2)^{-2\alpha}$, and this is the case where our interest mainly lies.

The integral kernel associated with $\tilde{\Delta}_{\alpha\beta}$, the *$\alpha\beta$-Poisson kernel*, is

$$\tilde{\mathcal{P}}_{\alpha\beta}(z,\zeta) = C^{00}_{\alpha\beta}\frac{(1-|z|^2)^{N+\alpha+\beta}}{(1-\langle z,\zeta\rangle)^{N+\alpha}(1-\langle \zeta,z\rangle)^{N+\beta}} \quad (z\in\mathbb{B},\ \zeta\in\mathbb{S}),$$

where

$$(5)\qquad \tilde{\mathcal{P}}_{\alpha\beta}(0,\eta) = C^{00}_{\alpha\beta} = \frac{\Gamma(N+\alpha)\Gamma(N+\beta)}{\Gamma(N)\Gamma(N+\alpha+\beta)} = \frac{1}{F(-\alpha,-\beta;N;1)}.$$

In particular, the kernel $\tilde{\mathcal{P}}_{00} = \tilde{\mathcal{P}}$ is the *Poisson–Szegő kernel.* It is positive and has $C^{00}_{00} = 1$. All of this makes sense when

$$(6)\qquad \operatorname{Re}(N+\alpha+\beta) > 0, \quad N+\alpha \notin -\mathbb{N}, \quad N+\beta \notin -\mathbb{N}.$$

If (6) holds and f is a bounded $\alpha\beta$-harmonic function on $\mathbb{B}$, then there exists a function, again denoted f, defined on $\mathbb{S}$ and in $L^\infty(\sigma)$ such that f is the $\alpha\beta$-Poisson integral of f, that is,

$$(7)\qquad f(z) = \int_{\mathbb{S}} \tilde{\mathcal{P}}_{\alpha\beta}(z,\zeta) f(\zeta)\, d\sigma(\zeta).$$

This is proved by considering $L^\infty(\mathbb{S})$ and changing "weakly" to "weak-$*$" in the proof of [1, Proposition 2.4]. For the special case of the invariant Laplacian, see [35, Theorem 5.8].

For $p,q = 0,1,2,\ldots$, $\mathcal{H}(p,q)$ denotes the space of harmonic polynomials $f(z,\bar{z})$ that are homogeneous of degree p in z and of degree q in $\bar{z}$. Functions in $\mathcal{H}(p,q)$ are uniquely determined by their restrictions to $\mathbb{S}$, and we freely identify $\mathcal{H}(p,q)$ with its restriction to $\mathbb{S}$. Let d_{pq} be the dimension of $\mathcal{H}(p,q)$ and $\{f^{pq}_1, f^{pq}_2, \ldots, f^{pq}_{d_{pq}}\}$ a basis for $\mathcal{H}(p,q)$ orthonormal with respect to the inner product of $L^2(\sigma)$. Put

$$(8)\qquad H^{pq}(\zeta,\eta) = \sum_{j=1}^{d_{pq}} f^{pq}_j(\zeta)\overline{f^{pq}_j(\eta)}.$$

The reproducing kernels H^{pq} of $\mathcal{H}(p,q)$ can be expressed in terms of Jacobi polynomials, and thus have the properties $H^{pq}(\zeta,\eta) = \overline{H^{pq}(\eta,\zeta)} = \overline{H^{qp}(\zeta,\eta)}$ and

$$(9)\qquad \sum_{j=1}^{d_{pq}} \left|f^{pq}_j(\zeta)\right|^2 = H^{pq}(\zeta,\zeta) = d_{pq} \leqslant k(p+q)^{4N-3}$$

for some constant k. The space $\mathcal{H}(0,0)$ consists of constants with orthonormal basis $\{f^{00}_1\} = \{1\}$; hence $H^{00}(\zeta,\eta) \equiv 1$. These results can be found in [24].

Let again (6) hold. By [1, Theorem 2.2], if $g \in \mathcal{H}(p,q)$, then the unique solution to the Dirichlet problem $\tilde{\Delta}_{\alpha\beta} f = 0$ on $\mathbb{B}$ with boundary data $f|_{\mathbb{S}} = g$ has the form $f(r\zeta) = r^{p+q} S^{pq}_{\alpha\beta}(r) g(\zeta)$, where

$$S^{pq}_{\alpha\beta}(r) = C^{pq}_{\alpha\beta} F\big(p-\beta, q-\alpha; N+p+q; r^2\big)$$

and

$$C_{\alpha\beta}^{pq} = \frac{1}{F(p-\beta, q-\alpha; N+p+q; 1)} = \frac{\Gamma(N+p+\alpha)\Gamma(N+q+\beta)}{\Gamma(N+p+q)\Gamma(N+\alpha+\beta)}.$$

Theorem 3.1. *If α, β satisfy (6), then the $\alpha\beta$-Poisson kernel has the spherical-harmonic expansion*

$$\tilde{\mathcal{P}}_{\alpha\beta}(r\zeta, \eta) = \sum_{p,q=0}^{\infty} r^{p+q} S_{\alpha\beta}^{pq}(r) H^{pq}(\zeta, \eta).$$

The series converges absolutely, and uniformly for $r\zeta$ on a compact subset of $\mathbb{B}$.

Proof. We simply combine [1, Theorem 2.2, Proposition 2.4] with the proof of [24, Theorem]. $\square$

The results for $\alpha = \beta = 0$ were first obtained in [24]. For comparison with ordinary harmonicity, let us note that the members of $\mathcal{H}(p, q)$ and in particular the functions $f_1^{pq}, \ldots, f_{d_{pq}}^{pq}$ are the counterparts of spherical harmonics, and the functions H^{pq} correspond to zonal harmonics.

If (6) holds and f is a bounded $\alpha\beta$-harmonic function, (7) evaluated at $z = 0$ along with (5) yields the weighted mean value equality

$$f(0) = C_{\alpha\beta}^{00} \int_{\mathbb{S}} f \, d\sigma \tag{10}$$

at 0. If f is not bounded, we can integrate on a smaller sphere; see [25, Theorem 1.1]. Obvious estimates of (7) lead to weighted maximum properties for $\alpha\beta$-harmonic functions. Equality (10) is a true mean value property for $\mathcal{M}$-harmonic functions, which in fact satisfy the stronger invariant mean value property, namely

$$f(a) = \int_{\mathbb{S}} f\big(\varphi_a(r\zeta)\big) \, d\sigma(\zeta) \quad (a \in \mathbb{B},\ 0 \leqslant r < 1),$$

where $\varphi_a \in \mathcal{M}$ is an involution exchanging 0 and a. Consequently, $\mathcal{M}$-harmonic functions have the true maximum and minimum principles.

Suppose again (6) and that f is a bounded $\alpha\beta$-harmonic function. Then by (7), Theorem 3.1, and (8), we have

$$\begin{aligned} f(z) &= \int_{\mathbb{S}} \sum_{p,q=0}^{\infty} r^{p+q} S_{\alpha\beta}^{pq}(r) H^{pq}(\zeta, \eta) f(\eta) \, d\sigma(\eta) \\ &= \sum_{p,q=0}^{\infty} r^{p+q} S_{\alpha\beta}^{pq}(r) \sum_{j=1}^{d_{pq}} c_j^{pq} f_j^{pq}(\zeta), \end{aligned} \tag{11}$$

where

$$c_j^{pq} = \int_{\mathbb{S}} f(\eta)\overline{f_j^{pq}(\eta)}\, d\sigma(\eta). \tag{12}$$

In particular, by (10),

$$c_1^{00} = \frac{f(0)}{C_{\alpha\beta}^{00}}. \tag{13}$$

Thus the set of functions

$$\mathcal{B}_{\alpha\beta} = \{h_j^{pq}(z) = r^{p+q} S_{\alpha\beta}^{pq}(r) f_j^{pq}(\zeta)\colon\ j = 1, \ldots, d_{pq},\ p, q = 0, 1, 2, \ldots\}$$

forms a basis for the space of bounded $\alpha\beta$-harmonic functions on $\mathbb{B}$ in $\mathbb{C}^N$, where α, β satisfy (6). Because $r^{p+q} S_{\alpha\beta}^{pq}(r) = 1$ when $r = 1$, $\mathcal{B}_{\alpha\beta}$ is an orthonormal basis with respect to the inner product of $L^2(\sigma)$.

4. BOHR RADIUS FOR LAPLACE–BELTRAMI OPERATORS

As mentioned in the Introduction, when it comes to a Bohr radius, we consider only real-valued functions. A real bounded $\alpha\beta$-harmonic function has real boundary values on $\mathbb{S}$. Thus in order to satisfy (7), $\tilde{\mathcal{P}}_{\alpha\beta}$ must be real-valued too. This requires $\alpha = \beta \in \mathbb{R}$. So from now on we consider only the weighted Laplace–Beltrami operator $\tilde{\Delta}_{\alpha\alpha}$, and conditions (6) reduce to our standing assumptions

$$\alpha \in \mathbb{R} \quad \text{and} \quad N + 2\alpha > 0. \tag{14}$$

Note that (14) implies $N + \alpha > 0$. Rewriting $\tilde{\Delta}_{\alpha\alpha}$ in terms of real coordinates, it can be checked that it is elliptic with $c(x) = \alpha^2 \geqslant 0$, where $c(x)$ is the coefficient of an elliptic operator as used in [23, Chapter 6]. It is because of this term that a general $\tilde{\Delta}_{\alpha\alpha}$ has no constant solutions and does not satisfy the usual maximum principle. Also, since $\tilde{\Delta}_{\alpha\alpha}$ has real-analytic coefficients, its solutions are also real-analytic.

Remark 4.1. With α satisfying (14), $\tilde{\mathcal{P}}_{\alpha\alpha}$ is not only real, but is in fact positive. Then (7) yields that if $f > 0$ on $\mathbb{S}$, then $f > 0$ on $\mathbb{B}$ as well. This is a weak minimum principle for $\alpha\alpha$-harmonic functions.

Let us have a closer look at the functions of the basis $\mathcal{B}_{\alpha\alpha}$ in the light of (14). Note that $h_1^{00} = S_{\alpha\alpha}^{00}$. First $S_{\alpha\alpha}^{pq}(0) = C_{\alpha\alpha}^{pq} > 0$ and $S_{\alpha\alpha}^{pq}(1) = 1$ by (3) for any p, q. Next by (4),

$$\begin{aligned} \frac{d}{dr} S_{\alpha\alpha}^{pq}(r) &= C_{\alpha\alpha}^{pq} \frac{(p-\alpha)(q-\alpha)}{N+p+q} 2r \\ &\quad \times F(1+p-\alpha, 1+q-\alpha; N+1+p+q; r^2) \\ &= C_{\alpha\alpha}^{pq} \frac{(p-\alpha)(q-\alpha)}{N+p+q} 2r (1-r^2)^{N-1+2\alpha} \\ &\quad \times F(N+q+\alpha, N+p+\alpha; N+1+p+q; r^2). \end{aligned} \tag{15}$$

The derivative of each $S_{\alpha\alpha}^{pq}(r)$ has a constant sign that depends on p, q for $0 \leqslant r \leqslant 1$ by the properties of F. Hence $S_{\alpha\alpha}^{pq}(r)$ is either a decreasing or an increasing function for $0 \leqslant r \leqslant 1$. This implies that either $C_{\alpha\alpha}^{pq} \leqslant S_{\alpha\alpha}^{pq}(r) \leqslant 1$ or $1 \leqslant S_{\alpha\alpha}^{pq}(r) \leqslant C_{\alpha\alpha}^{pq}$ by (3). By (1), the constants $C_{\alpha\alpha}^{pq}$ are bounded above and below by positive constants independent of α, p, q.

The function

$$S_{\alpha\alpha}^{00}(r) = C_{\alpha\alpha}^{00} F\big(-\alpha, -\alpha; N; r^2\big)$$

deserves separate mention. By (15) and (2), its derivative vanishes identically and thus $S_{\alpha\alpha}^{00} \equiv C_{\alpha\alpha}^{00} = 1$ if and only if $\alpha = 0$. This is expected, because constants are $\alpha\alpha$-harmonic if and only if $\alpha = 0$. Otherwise, $C_{\alpha\alpha}^{00} < 1$ and $S_{\alpha\alpha}^{00}(r)$ is increasing with a local minimum of $C_{\alpha\alpha}^{00}$ at 0 by (15). Hence the mean value property and the usual minimum principle fail spectacularly for $\alpha\alpha$-harmonic functions if $\alpha \neq 0$, but this does not contradict the weaker maximum principles in [23, Section 6.4]. Let us mention in passing that if α is a positive integer, then $S_{\alpha\alpha}^{00}$ is a polynomial of degree 2α.

Although the initial element $h_1^{00} = S_{\alpha\alpha}^{00}$ of $\mathcal{B}_{\alpha\alpha}$ is not constant, all its other elements vanish at 0. By the properties of $S_{\alpha\alpha}^{00}$ developed above and (9), we have $|h_j^{pq}(z)| \leqslant k r^{p+q}\sqrt{d_{pq}} \leqslant k a^{p+q} r^{p+q}$ for some fixed constants k and a. Thus the natural basis $\mathcal{B}_{\alpha\alpha}$ satisfies property (2) of bases given on [9, p. 388] with $x^0 = 0$, but does not satisfy property (1) there. Instead, $\mathcal{B}_{\alpha\alpha}$ satisfies

(1′) $0 < C_{\alpha\alpha}^{00} \leqslant h_1^{00} \leqslant 1 < \infty$.

Yet the work below shows that $\mathcal{B}_{\alpha\alpha}$ exhibits a Bohr phenomenon.

Notation 4.2. If f is a bounded $\alpha\alpha$-harmonic function on $\mathbb{B}$, we let it have the expansion (11) with (12) in mind, that is, we let $f(z) = \sum c_j^{pq} h_j^{pq}(z)$. Then we set

$$A_f(z) = \sum \big|c_j^{pq} h_j^{pq}(z)\big| = \sum_{p,q=0}^{\infty} r^{p+q} S_{\alpha\alpha}^{pq}(r) \sum_{j=1}^{d_{pq}} \big|c_j^{pq}\big| \, \big|f_j^{pq}(\zeta)\big|.$$

Lemma 4.3. *Assume* (14). *If f is a bounded $\alpha\alpha$-harmonic function on $\mathbb{B}$, $f \leqslant B$ on $\mathbb{S}$, and $f = \sum c_j^{pq} h_j^{pq}$, then $c_1^{00} \leqslant B$ and*

$$\big|c_j^{pq}\big| \leqslant \sup_{\mathbb{S}} \big|f_j^{pq}\big| \left(B - \frac{f(0)}{C_{\alpha\alpha}^{00}}\right) = \sup_{\mathbb{S}} \big|h_j^{pq}\big| \big(B - c_1^{00}\big) \quad \big((p,q) \neq (0,0)\big).$$

Proof. The statement about c_1^{00} follows from (10) and (13). For the other statement, we follow the proof of [9, Lemma 2.1] by considering $B S_{\alpha\alpha}^{00} - f$ as in Section 2, which is equal to $B - f$ on $\mathbb{S}$. We are done by (10) and (13). □

Now we show that there are bases without a constant element that exhibit a Bohr phenomenon. The next result can be called a specialized Harnack inequality. It is true also on the general domains described in [9, Section 2].

Theorem 4.4. *Let α satisfying* (14) *be given. There is a positive $R < 1$ such that if f is a bounded $\alpha\alpha$-harmonic function satisfying $|f| < 1$ on $\mathbb{S}$, then we have $A_f(z) < h_1^{00}(z) = S_{\alpha\alpha}^{00}(r) \leqslant 1$ for $|z| = r < R$.*

Proof. Assuming, without loss of generality, $f(0) \geqslant 0$, we obtain $0 \leqslant c_1^{00} < 1$ by Lemma 4.3. Then again by Lemma 4.3,

$$\begin{aligned} A_f(z) &< c_1^{00} S_{\alpha\alpha}^{00}(r) + (1 - c_1^{00}) \sum' |h_j^{pq}(z)| \sup_{\mathbb{S}} |h_j^{pq}| \\ &= c_1^{00} S_{\alpha\alpha}^{00}(r) + \big(1 - c_1^{00}\big) T(z). \end{aligned}$$

In the proof of [9, Theorem 2.2], which covers the case $\alpha = 0$, it is shown that T can be made as small as one wishes by taking r small enough. We choose R so that $T(z) < C_{\alpha\alpha}^{00} \leqslant S_{\alpha\alpha}^{00}(r)$ for $r < R$. $\square$

Remark 4.5. Theorem 4.4 is in fact stronger than the usual Bohr property. Indeed, $A_f(z)$ is not only less than 1, but also less than $S_{\alpha\alpha}^{00}(r) < 1$ for $|z| = r < R$. The hypothesis $|f| < 1$ on $\mathbb{S}$ is satisfied, in particular, if $|f(z)| < S_{\alpha\alpha}^{00}(z)$ or $|f(z)| < 1$ for all $z \in \mathbb{B}$.

Theorem 4.4 coupled with Remark 4.5 yields a method for computing a Bohr radius for $\alpha\alpha$-harmonic functions.

Theorem 4.6. *Let α satisfy* (14). *Given a neighborhood $U \subset \mathbb{B}$ of 0, the following are equivalent for a bounded $\alpha\alpha$-harmonic function f.*

(i) *If $f > 0$ on $\mathbb{B}$, then $A_f < 2c_1^{00} S_{\alpha\alpha}^{00} = 2f(0) S_{\alpha\alpha}^{00} / C_{\alpha\alpha}^{00}$ in U.*
(ii) *If $|f| < S_{\alpha\alpha}^{00}$ on $\mathbb{B}$, then $A_f < S_{\alpha\alpha}^{00}$ in U.*

Condition (i) implies that $A_f < 2f(0)$ in U, and further condition (ii) implies that $A_f < 1$ on $\mathbb{S}$, but the latter two conditions are not equivalent.

Proof. Suppose (i) holds. Let f be $\alpha\alpha$-harmonic with expansion as in Notation 4.2 and satisfying $|f| < S_{\alpha\alpha}^{00}$ on $\mathbb{B}$. Then $|f(0)| = |c_1^{00}| S_{\alpha\alpha}^{00}(0) < C_{\alpha\alpha}^{00}$, implying that $|c_1^{00}| < 1$. Without loss of generality, $f(0) \geqslant 0$. Then $0 \leqslant c_1^{00} < 1$. Put $g = S_{\alpha\alpha}^{00} - f$; then $0 < g < 2S_{\alpha\alpha}^{00}$ and is bounded. By (i),

$$A_g = \big(1 - c_1^{00}\big) S_{\alpha\alpha}^{00} + \sum' |c_j^{pq} h_j^{pq}| < 2\big(1 - c_1^{00}\big) S_{\alpha\alpha}^{00}$$

on U. Then

$$A_f = c_1^{00} S_{\alpha\alpha}^{00} + \Big(\sum{}' |c_j^{pq} h_j^{pq}| + (1 - c_1^{00}) S_{\alpha\alpha}^{00}\Big) - (1 - c_1^{00}) S_{\alpha\alpha}^{00}$$
$$< c_1^{00} S_{\alpha\alpha}^{00} + 2(1 - c_1^{00}) S_{\alpha\alpha}^{00} - (1 - c_1^{00}) S_{\alpha\alpha}^{00} = S_{\alpha\alpha}^{00}$$

on U.

Now suppose (ii) holds. Let f be a bounded positive $\alpha\alpha$-harmonic function with expansion as in Notation 4.2. Choose a constant $k > 0$ so that $0 < kf < 2S_{\alpha\alpha}^{00}$ on $\mathbb{B}$. Put $g = S_{\alpha\alpha}^{00} - kf$; then $|g| < S_{\alpha\alpha}^{00}$. By (ii),

$$A_g = |1 - kc_1^{00}| S_{\alpha\alpha}^{00} + k \sum{}' |c_j^{pq} h_j^{pq}| < S_{\alpha\alpha}^{00}$$

on U. Then

$$A_f = \frac{1}{k}\Big[kc_1^{00} S_{\alpha\alpha}^{00} + \Big(k \sum{}' |c_j^{pq} h_j^{pq}| + |1 - kc_1^{00}| S_{\alpha\alpha}^{00}\Big) - |1 - kc_1^{00}| S_{\alpha\alpha}^{00}\Big]$$
$$< \frac{1}{k}\Big[kc_1^{00} S_{\alpha\alpha}^{00} + S_{\alpha\alpha}^{00} - (1 - kc_1^{00}) S_{\alpha\alpha}^{00}\Big] = 2c_1^{00} S_{\alpha\alpha}^{00}$$

on U. $\square$

Definition 4.7. Consider an α satisfying (14). The *Bohr radius* $\mathcal{R}_\alpha$ for $\alpha\alpha$-harmonic functions on the unit ball of $\mathbb{C}^N$ is the largest of the positive numbers $R \leqslant 1$ such that if f is $\alpha\alpha$-harmonic and $|f| < S_{\alpha\alpha}^{00}$ on $\mathbb{B}$, then $A_f(z) < S_{\alpha\alpha}^{00}(z)$ for all z with $|z| = r < R$.

Now for an α satisfying (14), consider the equation

$$\frac{(1+r)^{N+2\alpha}}{(1-r)^N} = 2F(-\alpha, -\alpha; N; r^2). \tag{16}$$

At $r = 0$, its left-hand side is 1 and its right-hand side is $2 > 1$; as $r \to 1$, its left-hand side tends to ∞ while its right-hand side tends to $2/C_{\alpha\alpha}^{00} < \infty$. Thus (16) has at least one root ρ_α in the interval $(0, 1)$. Computing the first and second derivatives of the two sides of (16) shows that both are increasing functions of r with no change in concavity on the interval $(0, 1)$. Thus ρ_α is unique. When $\alpha = 0$, (16) takes the simple form $(1 + r)^N = 2(1 - r)^N$ by (2).

Theorem 4.8. *For α satisfying* (14), *the Bohr radius $\mathcal{R}_\alpha$ equals ρ_α.*

Proof. Let f be a positive bounded $\alpha\alpha$-harmonic function on $\mathbb{B}$. By (12), Hölder's inequality, (9), (10), and (13), we have

$$\sum_{j=1}^{d_{pq}} |c_j^{pq}|^2 = \sum_{j=1}^{d_{pq}} \Big| \int_{\mathbb{S}} f \overline{f_j^{pq}}\, d\sigma \Big|^2 \leqslant \Big(\int_{\mathbb{S}} f\, d\sigma\Big)\Big(\int_{\mathbb{S}} f \sum_{j=1}^{d_{pq}} |f_j^{pq}|^2\, d\sigma\Big)$$
$$= \Big(\int_{\mathbb{S}} f\, d\sigma\Big)\Big(\int_{\mathbb{S}} f(\zeta) H^{pq}(\zeta, \zeta)\, d\sigma(\zeta)\Big)$$

$$= d_{pq}\Big(\int_{\mathbb{S}} f\,d\sigma\Big)^2 = d_{pq}\frac{f(0)^2}{(C^{00}_{\alpha\alpha})^2} = d_{pq}\big(c_1^{00}\big)^2.$$

Then by Hölder's inequality, (9), and Theorem 3.1, we obtain

$$\begin{aligned}
A_f(r\zeta) &\leqslant \sum_{p,q=0}^{\infty} r^{p+q} S^{pq}_{\alpha\alpha}(r)\sqrt{\sum_{j=1}^{d_{pq}}|c_j^{pq}|^2}\sqrt{\sum_{j=1}^{d_{pq}}|f_j^{pq}(\zeta)|^2}\\
&\leqslant \sum_{p,q=0}^{\infty} r^{p+q} S^{pq}_{\alpha\alpha}(r)\sqrt{d_{pq}}c_1^{00}\sqrt{d_{pq}} = c_1^{00}\sum_{p,q=0}^{\infty} r^{p+q} S^{pq}_{\alpha\alpha}(r)H^{pq}(\zeta,\zeta)\\
&= c_1^{00}\tilde{\mathcal{P}}_{\alpha\alpha}(r\zeta,\zeta) = c_1^{00}C^{00}_{\alpha\alpha}\frac{(1-r^2)^{N+2\alpha}}{(1-r)^{2N+2\alpha}} = c_1^{00}C^{00}_{\alpha\alpha}\frac{(1+r)^{N+2\alpha}}{(1-r)^{N}}.
\end{aligned}$$

Thus $A_f(r\zeta) < 2c_1^{00}S^{00}_{\alpha\alpha}(r) = 2c_1^{00}C^{00}_{\alpha\alpha}F(-\alpha,-\alpha;N;r^2)$ if $r<\rho_\alpha$. It follows that $\mathcal{R}_\alpha \geqslant \rho_\alpha$ by Theorem 4.6.

The expansion in Theorem 3.1 of the $\alpha\alpha$-Poisson kernel provides the required reverse inequality. We fix $\eta = (1/\sqrt{N},\dots,1/\sqrt{N}) \in \mathbb{S}$, let $0<u<1$, and consider the functions

$$f_u(z) = \sum_{p,q=0}^{\infty} u^{p+q} r^{p+q} S^{pq}_{\alpha\alpha}(r) H^{pq}(\zeta,\eta) = \sum_{p,q=0}^{\infty} r^{p+q} S^{pq}_{\alpha\alpha}(r)\sum_{j=1}^{d_{pq}} c_j^{pq} f_j^{pq}(\zeta),$$

where $c_j^{pq} = u^{p+q}\overline{f_j^{pq}(\eta)}$ and hence $c_1^{00}=1$. As in the proof of the Theorem of [24], the series defining f_u is uniformly convergent for $z=r\zeta\in\mathbb{S}\cup\mathbb{B}$; hence f_u is bounded on $\mathbb{B}$. Each term of the series is $\alpha\alpha$-harmonic by (8) and the solution to the Dirichlet problem; thus f_u is $\alpha\alpha$-harmonic in $\mathbb{B}$. Recalling that $d_{pq}=d_{qp}$ and comparing H^{pq} with H^{qp}, we see that f_u is real-valued. Setting $r=1$, we have

$$f_u(\zeta) = \tilde{\mathcal{P}}_{\alpha\alpha}(u\zeta,\eta) = C^{00}_{\alpha\alpha}\frac{(1-u^2)^{N+2\alpha}}{|1-u\langle\zeta,\eta\rangle|^{2N+2\alpha}} \geqslant C^{00}_{\alpha\alpha}\frac{(1-u^2)^{N+2\alpha}}{2^{2N+2\alpha}} > 0$$

for $\zeta\in\mathbb{S}$. Then by Remark 4.1, $f_u(z)>0$ for all $z\in\mathbb{B}$. For each u, there is a radius $0<R_u<1$ such that

$$A_{f_u}(r\eta) \geqslant 2c_1^{00}S^{00}_{\alpha\alpha}(r) = 2C^{00}_{\alpha\alpha}F\big(-\alpha,-\alpha;N;r^2\big)$$

if $r\geqslant R_u$. The radii R_u decrease to a limit ρ'_α as $u\to 1$. Then letting $u\to 1$ and using (9) and Theorem 3.1, we see that

$$\begin{aligned}
\lim_{u\to 1} A_{f_u}(r\eta) &= \sum_{p,q=0}^{\infty}\sum_{j=1}^{d_{pq}} r^{p+q}S^{pq}_{\alpha\alpha}(r)\big|f_j^{pq}(\eta)\big|^2 = \sum_{p,q=0}^{\infty} r^{p+q}S^{pq}_{\alpha\alpha}(r)H^{pq}(\eta,\eta)\\
&= \tilde{\mathcal{P}}_{\alpha\alpha}(r\eta,\eta) = C^{00}_{\alpha\alpha}\frac{(1+r)^{N+2\alpha}}{(1-r)^{N}} \geqslant 2C^{00}_{\alpha\alpha}F\big(-\alpha,-\alpha;N;r^2\big)
\end{aligned}$$

if $r \geqslant \rho'_\alpha$. Therefore $\mathcal{R}_\alpha \leqslant \rho'_\alpha = \rho_\alpha$ by Theorem 4.6. □

Corollary 4.9. *The Bohr radius for $\mathcal{M}$-harmonic functions on $\mathbb{B} \subset \mathbb{C}^N$ is*

$$\mathcal{R}_0 = \frac{2^{1/N}-1}{2^{1/N}+1}.$$

5. BOHR RADIUS FOR $\mathcal{M}$-HARMONIC FUNCTIONS ON $\mathbb{B} \subset \mathbb{R}^N$

We compute one more Bohr radius, both to show the wide applicability of the methods and for comparison purposes. The theory of $\mathcal{M}$-harmonic functions on $\mathbb{B} \subset \mathbb{R}^n$ was developed in [36], and we start by summarizing the essential information.

The *invariant Laplacian* $\tilde{\Delta}$ is the Laplace–Beltrami operator associated with the hyperbolic metric of $\mathbb{B} \subset \mathbb{R}^n$ and is defined by

$$\begin{aligned}(\tilde{\Delta} f)(a) = \Delta(f \circ \varphi_a)(0) &= 4\big(1-|a|^2\big)^2(\Delta f)(a) \\ &\quad + 8(n-2)\big(1-|a|^2\big)\big\langle a, (\nabla f)(a)\big\rangle,\end{aligned}$$

where ∇ is the ordinary gradient, and $\varphi_a \in \mathcal{M}$ is an involution that exchanges 0 and $a \in \mathbb{B}$. It satisfies $\tilde{\Delta}(f \circ \psi) = (\tilde{\Delta} f) \circ \psi$ for any $\psi \in \mathcal{M}$. A function that is annihilated by $\tilde{\Delta}$ is called *$\mathcal{M}$-harmonic* or *invariant harmonic*. Such a function has the invariant mean value property and satisfies the maximum principle.

The integral kernel associated with $\tilde{\Delta}$ is called the *invariant Poisson kernel* and is the function

$$\tilde{\mathcal{P}}(x,\xi) = \left(\frac{1-|x|^2}{|x-\xi|^2}\right)^{n-1},$$

which is the ordinary Poisson kernel only for $n=2$. This kernel has the expansion

$$\tilde{\mathcal{P}}(r\xi,\eta) = \sum_{m=0}^{\infty} r^m S_m(r) Z_m(\xi,\eta), \tag{17}$$

where

$$\begin{aligned} S_m(r) &= C_m F\big(m, -n/2+1; n/2+m; r^2\big), \\ C_m &= \frac{1}{F(m,-n/2+1;n/2+m;1)} = \frac{\Gamma(n/2)\Gamma(n-1+m)}{\Gamma(n/2+m)\Gamma(n-1)}, \end{aligned}$$

and the $Z_m(\xi,\eta)$ are the *zonal harmonics* of degree $m = 0,1,2,\ldots$; see [36, Theorems 5.7, 5.10]. The series (17) converges absolutely, and uniformly for $r\xi$ on a compact subset of $\mathbb{B}$. The expansion of the ordinary Poisson kernel does not have the extra factor $S_m(r)$. The zonal harmonics have the further expansion

$$Z_m(\xi,\eta) = \sum_{j=1}^{d_m} f_j^m(\xi)\overline{f_j^m(\eta)}, \tag{18}$$

where $\{f_1^m, f_2^m, \dots, f_{d_m}^m\}$ is an orthonormal basis (with respect to the inner product of $L^2(\sigma)$) for the space of harmonic polynomials that are homogeneous of degree m.

If f is a bounded $\mathcal{M}$-harmonic function on $\mathbb{B}$, then there exists a function, again denoted f, defined on $\mathbb{S}$ and in $L^\infty(\sigma)$, such that f is the invariant Poisson integral of f, that is,

$$f(x) = \int_{\mathbb{S}} \tilde{\mathcal{P}}(x, \xi) f(\xi)\, d\sigma(\xi).$$

Then by (17) and (18), we have

$$f(x) = \sum_{m=0}^{\infty} r^m S_m(r) \sum_{j=1}^{d_m} c_j^m f_j^m(\xi). \tag{19}$$

Thus the set of functions

$$\left\{r^m S_m(r) f_j^m(\xi)\colon\ j = 1, \dots, d_m,\ m = 0, 1, 2, \dots\right\} \tag{20}$$

forms a basis (orthonormal with respect to the inner product of $L^2(\sigma)$) for the space of bounded $\mathcal{M}$-harmonic functions on $\mathbb{B}$ in $\mathbb{R}^n$. The difference between (20) and the well-known basis of the ordinary harmonic functions on $\mathbb{B}$ is the presence of the factors $S_m(r)$. The initial element of (20) is $r^0 S_0(r) f_1^0(\xi) \equiv 1$, and all its other elements vanish at 0. Then (20) satisfies the properties (1) and (2) of bases given on [9, p. 388] with $x^0 = 0$. Therefore, by [9, Theorem 2.2], (20) exhibits a Bohr phenomenon, and we make the following definition.

Definition 5.1. The *Bohr radius* $R_{\mathbb{R}}$ for $\mathcal{M}$-harmonic functions on the unit ball of $\mathbb{R}^n$ is the largest of the positive numbers $R \leqslant 1$ such that if f is $\mathcal{M}$-harmonic with expansion (19) and $|f| < 1$, then

$$A_f(x) = \sum_{m=0}^{\infty} r^m S_m(r) \sum_{j=1}^{d_m} |c_j^m|\, |f_j^m(\xi)| < 1$$

for all $x = r\xi \in \mathbb{B}$ with $r < R$.

Moreover, [9, Lemma 3.1] applies to the basis (20). So equivalently, $R_{\mathbb{R}}$ is the largest of the positive numbers $R \leqslant 1$ such that if f is a positive bounded $\mathcal{M}$-harmonic function with expansion (19), then

$$A_f(x) < 2f(0)$$

for all $x = r\xi \in \mathbb{B}$ with $r < R$.

Theorem 5.2. *The Bohr radius for $\mathcal{M}$-harmonic functions on $\mathbb{B} \subset \mathbb{R}^n$ is*

$$R_{\mathbb{R}} = \frac{2^{1/(n-1)} - 1}{2^{1/(n-1)} + 1}.$$

Proof. This is no different from the case $\alpha = 0$ of the proof of Theorem 4.8 and is therefore omitted. $\square$

6. ASYMPTOTICS

We take $n = 2N$ and compare $\mathcal{R}_0$ and $R_{\mathbb{R}}$ to two other Bohr radii computed in [9], the Bohr radius R_{Har} for ordinary harmonic functions on the unit ball of $\mathbb{R}^n$ and the Bohr radius R_{SHar} for separately harmonic functions on the unit polydisc in $\mathbb{C}^N$. The radius R_{Har} is the root of the equation $1 + r = 2(1 - r)^{n-1}$ lying in $(0, 1)$. Incidentally, $\mathcal{R}_0 = R_{\text{SHar}}$ although they are for different kinds of functions on different domains. These four Bohr radii are equal to $1/3$ when $N = 1$, since these four types of harmonicity coincide on the unit disc in $\mathbb{C}$. The value $1/3$ is also the Bohr radius for analytic functions on the unit disc.

Asymptotically, on $\mathbb{C}^N = \mathbb{R}^{2N}$,

$$R_{\text{Har}} \sim R_{\text{SHar}} \sim \mathcal{R}_0 = \frac{\log 2}{2N} + O\left(\frac{1}{N^2}\right) \quad (N \to \infty),$$

whereas

$$R_{\mathbb{R}} = \frac{\log 2}{4N} + O\left(\frac{1}{N^2}\right) \quad (N \to \infty).$$

On the other hand, for a given α satisfying (14), the right-hand side of (16) evaluated at $r = 1$ is bounded independently of N because of (1). Hence it is bounded for all $r \in [0, 1]$, say, by a constant k_α, since it is increasing. Then similarly to the above, we obtain

$$\mathcal{R}_\alpha = \frac{\log(2k_\alpha)}{2(N+\alpha)} + O\left(\frac{1}{N^2}\right) \quad (N \to \infty).$$

We also check how $\mathcal{R}_\alpha$ behaves as $\alpha > -N/2$ varies. As $\alpha \to -(N/2)^+$, (16) takes the form $(1-r)^{-N} = 2F(N/2, N/2; N; r^2)$, and as this happens, its right-hand side increases and its left-hand side decreases. Then the solution $\mathcal{R}^N$ to this equation is the terminal value that the Bohr radii $\mathcal{R}_\alpha$ increase to as $\alpha \to -(N/2)^+$. This equation can be solved with computer algebra software for each value of N. The first few solutions are $\mathcal{R}^1 = 0.54052\ldots$, $\mathcal{R}^2 = 0.31043\ldots$, $\mathcal{R}^3 = 0.21565\ldots$.

To see what happens at the other extreme, let us first write (16) in the form

$$\frac{1}{(1-r)^N} = \frac{2F(-\alpha, -\alpha; N; r^2)}{(1+r)^{N+2\alpha}}. \tag{21}$$

The left-hand side behaves as the left-hand side of (16). For any given r in $(0, 1)$, it is easy to see by (3) and (1) that the ratio $F(-\alpha, -\alpha; N; 1)/(1+r)^{N+2\alpha}$ tends to 0 as $\alpha \to \infty$. Then since $F(-\alpha, -\alpha; N; r^2)$ is increasing in r on $(0, 1)$, the same limit is attained for the right-hand side of (21). Thus $\mathcal{R}_\alpha$ tends to 0 as $\alpha \to \infty$.

ACKNOWLEDGEMENTS

The author thanks M. Stoll of the University of South Carolina for a valuable comment and for making the manuscript [36] available to him. He thanks H. Taşeli and S.Ö. Kaptanoğlu too, both of Middle East Technical University, for useful discussions.

Thanks are due also to the referees and the editor for suggestions that led to a more readable paper.

Parts of this work were done while the author's host institution was Middle East Technical University and during his visit to the University of Virginia. He thanks the former for granting a sabbatical leave, the Department of Mathematics of the latter and the operator theory group for their hospitality.

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(Received May 2005)