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On the Lévy–Raikov–Marcinkiewicz theorem

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Abstract

Let μ be a finite non-negative Borel measure. The classical Lévy–Raikov–Marcinkiewicz theorem states that if its Fourier transform $\hat{\mu}$ can be analytically continued to some complex half-neighborhood of the origin containing an interval $(0, iR)$ then $\hat{\mu}$ admits analytic continuation into the strip $\{t\colon 0 < \Im t < R\}$. We extend this result to general classes of measures and distributions, assuming non-negativity only on some ray and allowing temperate growth on the whole line.

Keywords: Borel measure; Temperate distribution; Fourier transform; Analytic continuation

1. Introduction

Let μ be a finite Borel measure, and denote by $\hat{\mu}$ its Fourier transform,

$$\hat{\mu}(t) = \int_{\mathbf{R}} e^{-itx}\, d\mu(x). \tag{1}$$

The following principle is well known in harmonic analysis: *Suppose μ is a positive finite Borel measure. If its Fourier transform $\hat{\mu}$ is 'smooth' at the origin then it is 'smooth' on the whole real line.* For example [4, Theorem 2.1.1], if $\hat{\mu}$ is $2n$-times differentiable at the origin, $n \geqslant 1$ being a natural number, then it is $2n$-times differentiable on the whole real

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line and $|\hat{\mu}^{(2n)}(t)| \leqslant |\hat{\mu}^{(2n)}(0)|$, $t \in \mathbf{R}$. For a manifestation of the principle for non-analytic infinite differentiability see [1].

The following result due to P. Lévy and D. Raikov (see, e.g., [4, Theorem 2.2.1, p. 24]) deals with real analyticity: *If the Fourier transform $\hat{\mu}$ coincides in a real neighborhood $(-a, a)$ of the origin with a function analytic in a rectangle $\{t\colon |\Re t| < a,\ -R < \Im t < R\}$, then $\hat{\mu}$ admits analytic continuation to the strip $\{t\colon |\Im t| < R\}$.*

As a generalization of the real analyticity in $(-a, a) \subset \mathbf{R}$, one can consider a weaker property of a function g to be the boundary value of a function analytic in a complex upper half-neighborhood of $(-a, a)$:

(A) g coincides in a real neighborhood $(-a, a)$ of the origin with a function analytic in a rectangle $\{t\colon |\Re t| < a,\ 0 < \Im t < R\}$ and continuous in its closure.

Marcinkiewicz (see, e.g., [4, Theorem 2.2.3, p. 25]) showed that the principle also works with this generalized real analyticity. We state this result in the following form.

Lévy–Raikov–Marcinkiewicz theorem. *Suppose μ is a non-negative finite Borel measure whose Fourier transform satisfies assumption* (A). *Then $\hat{\mu}$ admits analytic continuation into the strip $\{t\colon 0 < \Im t < R\}$ and is representable there by the absolutely convergent integral* (1).

It is convenient for us to consider also the property to be the boundary value of a function analytic in a complex lower half-neighborhood of $(-a, a)$:

(−A) g coincides in a real neighborhood $(-a, a)$ of the origin with a function analytic in a rectangle $\{t\colon |\Re t| < a,\ 0 > \Im t > -R\}$ and continuous in its closure.

One can easily reformulate the Lévy–Raikov–Marcinkiewicz theorem for this case.

This paper is devoted to extensions of the Lévy–Raikov–Marcinkiewicz theorem to some general classes of measures and distributions assuming non-negativity only on a half-line.

2. Statement of results

We show that the assumptions of the Lévy–Raikov–Marcinkiewicz theorem can be substantially relaxed: It is enough to assume non-negativity of μ on some half-line $(b, +\infty)$. Moreover, one can also allow a temperate growth of μ on this half-line:

$$\mu(b, x) \leqslant C|x|^N, \quad x > b, \tag{2}$$

where C and N are some positive constants. Observe that the Fourier transform of measures μ satisfying (2) exists in the sense of distributions.

Theorem 1. *Assume μ is a Borel measure non-negative on some half-line (b, ∞), satisfies* (2), *and is finite on $(-\infty, b]$. If its Fourier transform $\hat{\mu}$ satisfies* (A), *then the conclusion of the Lévy–Raikov–Marcinkiewicz theorem holds.*

Measures satisfying (2) form a subset of the set of temperate distributions (t.d.). We refer the reader to the book [3] for the terminology related to temperate distributions and their basic properties. We shall say that a t.d. g satisfies assumption (A) if the restriction of g to $(-a, a)$ agrees (as a temperate distribution) with a function analytic in the rectangle $\{t\colon |\Re t| < a,\ 0 < \Im t < R\}$ and continuous in its closure. The following theorem extends the Lévy–Raikov–Marcinkiewicz theorem to temperate distributions.

Theorem 2. *Let f be a temperate distribution non-negative on some half-line $(b, +\infty)$. Assume that its Fourier transform $\hat{f}$ satisfies* (A). *Then $\hat{f}$ is the boundary value in $\mathcal{S}'$-topology of a function which is analytic in the strip $\{t\colon 0 < \Im t < R\}$ and $O(|t|^N)$ for some $N > 0$ as $t \to \infty$ in any interior smaller strip.*

Changing roles of f and $\hat{f}$ and using the well-known identity $\hat{\hat{f}} = 2\pi \check{f}$, one can reformulate Theorem 2 as follows.

Theorem 3. *Let f be a t.d. satisfying* $(-\mathrm{A})$. *Assume that its Fourier transform $\hat{f}$ is non-negative on a ray (b, ∞), $b \in \mathbf{R}$. Then f is the boundary value on $\mathbf{R}$ in $\mathcal{S}'$-topology of a function which is analytic in the strip $\{t\colon -R < \Im t < 0\}$ and $O(|t|^N)$ for some $N > 0$ as $t \to \infty$ in any interior smaller strip.*

We also give a variant of Theorem 2 for L_2-functions. In this case the assertion of Theorem 2 can be sharpened:

Theorem 4. *Let f be a function belonging to $L_2(\mathbf{R})$ and non-negative a.e. on a ray $(b, +\infty)$, $b \in \mathbf{R}$. Assume that its L_2-Fourier transform $\hat{f}$ satisfies a.e. in $(-a, a)$ condition* (A). *Then $\hat{f}$ is the angular boundary value of the function analytic in the strip $\{t\colon 0 < \Im t < R\}$ and representable there as the sum $\hat{f} = g_1 + g_2$, where g_1 belongs to the Hardy class H_2 in any strip $\{t\colon 0 < \Im t < r < \infty\}$, and g_2 is analytic in the strip $\{t\colon 0 < \Im t < R\}$, continuous and tending to 0 at ∞ in any strip of kind $\{t\colon 0 \leqslant \Im t \leqslant r < R\}$.*

The following version of Theorem 3 for a finite Borel measure may be of interest because it gives conditions on the Fourier transform of such a measure under which its absolute continuity in a neighborhood of the origin implies its absolute continuity on the whole real line.

Theorem 5. *Let μ be a finite Borel measure on $\mathbf{R}$. Assume that it is absolutely continuous in a neighborhood of the origin $(-a, a)$, and its density satisfies condition* $(-\mathrm{A})$. *If the Fourier transform $\hat{\mu}$ of μ is non-negative on a ray (b, ∞), $b \in \mathbf{R}$, then μ is absolutely continuous on $\mathbf{R}$ and its density is the angular boundary value of a function analytic in the strip $\{t\colon -R < \Im t < 0\}$ and belonging to the Hardy class H_1 in any rectangle $\{t\colon |\Re t| < A,\ -R < \Im t < 0\}$, $A > 0$.*

This paper is an extended version of [5] where the results of this paper were announced without proof. Note that Theorem 2 has important applications to the well-known problem (see [2,7]) of oscillation of real functions having a spectral gap at the origin. These applications are considered in [6].

3. Auxiliary result

In this section we shall prove the following

Proposition 1. *Let f be a t.d. non-negative on $\{x\colon |x| \geqslant \beta\}$, $\beta > 0$. Assume that its Fourier transform $\hat{f}$ coincides on $(-a, a)$ with a function analytic in a rectangle $\{t\colon -r < \Im t < R, -a < \Re t < a\}$. Then $\hat{f}$ coincides on $\mathbf{R}$ with a function analytic in strip $\{t\colon -r < \Im t < R\}$ and being $O(|t|^N)$ for some $N > 0$ as $t \to \infty$ in any interior smaller strip.*

To prove Proposition 1 we need

Lemma 1. *Let f be a non-negative temperate distribution. Assume that its Fourier transform $\hat{f}$ coincides in a neighborhood of the origin with a function analytic in a rectangle $\{t\colon -r < \Im t < R,\ -a < \Re t < a\}$. Then $\hat{f}$ coincides on the whole real axis $\mathbf{R}$ with a function (denote it also by $\hat{f}$) analytic in the strip $\{t\colon -r < \Im t < R\}$ and representable there in the form*

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{-itx}\, d\mu_f(x), \tag{3}$$

where μ_f is a non-negative finite Borel measure on $\mathbf{R}$ and the integral converges absolutely.

Proof. It is well known (see, e.g., [3, p. 38]) that f is a non-negative locally finite Borel measure (μ_f, say). Let us show that, for each non-negative $\varphi \in \mathcal{S}$, one has

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} \varphi(x)\, d\mu_f(x) < \infty. \tag{4}$$

Let $\psi \in \mathcal{D}$ be non-negative and such that $\psi(0) = 1$, where $\mathcal{D} = C_0^\infty(\mathbf{R})$. It can be readily seen that

$$\psi(hx)\varphi(x) \to \varphi(x) \quad \text{as } h \to 0, \tag{5}$$

in $\mathcal{S}$-topology. Since $\psi(hx)\varphi(x) \in \mathcal{D}$, we have

$$\langle f, \psi(h\cdot)\varphi \rangle = \int_{-\infty}^{\infty} \psi(hx)\varphi(x)\, d\mu_f(x).$$

In virtue of (5), the left-hand side has finite limit $\langle f, \varphi \rangle$ as $h \to 0$. Since $\psi(hx) \to 1$ pointwise, the Fatou lemma implies finiteness of the integral in (4). Using dominated convergence theorem, we get (4).

Now, let φ, $\varphi(0) = (2\pi)^{-1}$, be a non-negative on $\mathbf{R}$ entire function of exponential type 1 belonging to $\mathcal{S}$. Then $\hat{\varphi} \in \mathcal{D}$, $\operatorname{supp} \hat{\varphi} \subseteq [-1, 1]$, and

$$\int_{-\infty}^{\infty} \hat{\varphi}(t)\, dt = 1.$$

Set $\varphi_h := \varphi(h\cdot)$, $h > 0$, and observe that $\operatorname{supp} \hat{\varphi}_h \subseteq [-h, h]$ and $\hat{\varphi}_h \to \delta_0$ in $\mathcal{D}'$-topology as $h \to 0$. Since $\hat{f}$ is an analytic function in a neighborhood of the origin, we have for sufficiently small h,

$$\langle \hat{\varphi}_h, \hat{f} \rangle = \langle \hat{f}, \hat{\varphi}_h \rangle = 2\pi \langle f, \check{\varphi}_h \rangle = 2\pi \int_{-\infty}^{\infty} \check{\varphi}_h(x)\, d\mu_f(x).$$

Since $\check{\varphi}_h \to (2\pi)^{-1}$ pointwise as $h \to 0$, we derive with help of the Fatou lemma that

$$\hat{f}(0) = 2\pi \lim_{h\to 0} \int_{-\infty}^{\infty} \check{\varphi}_h(x)\, d\mu_f(x) \geqslant \int_{-\infty}^{\infty} d\mu_f(x).$$

Thus, the distribution f is a finite non-negative measure μ_f. Applying the Lévy–Raikov theorem, we get assertion of Lemma 1. □

Proof of Proposition 1. Let $0 \leqslant \chi_\delta \leqslant 1$ $(\delta > 0)$ be a function from $\mathcal{D}$ supported by $[-\beta-\delta, \beta+\delta]$ and equal to 1 on $[-\beta, \beta]$. Consider the t.d. $f_1 := \chi_\delta f$ and $f_2 := (1-\chi_\delta) f$. We have

$$\hat{f} = \hat{f}_1 + \hat{f}_2. \tag{6}$$

Since f_1 has a compact support $[-\beta - \delta, \beta + \delta]$, then, by the Paley–Wiener–Schwartz theorem [3, Theorem 7.3.1, p. 181], $\hat{f}_1$ is the restriction to $\mathbf{R}$ of an entire function (we denote it also by $\hat{f}_1$) of exponential type admitting the estimate

$$\left|\hat{f}_1(t)\right| \leqslant C\bigl(1 + |t|\bigr)^N e^{(\beta+\delta)|\Im t|}, \tag{7}$$

where C and N are positive constants. By the assumptions of Proposition 1, the distribution $\hat{f}_2 = \hat{f} - \hat{f}_1$ coincides in a neighborhood of the origin with a function analytic in $\{-r < \Im t < R,\ -a < \Re t < a\}$. Since the distribution f_2 is non-negative, we can apply Lemma 1 and conclude that $\hat{f}_2$ coincides on $\mathbf{R}$ with a function analytic in strip $\{t\colon -r \leqslant \Im t \leqslant R\}$ and representable there by the absolutely convergent integral (3). Therefore (6) and (7) imply the assertion of Proposition 1. □

4. Proofs of Theorems 1–5

Proof of Theorem 2. Without loss of generality we may assume that $b = 0$, so that the t.d. f is non-negative on the positive ray.

The proof of Theorem 2 will be divided into several steps.

Step 1. Let $0 \leqslant \chi_\delta \leqslant 1$ $(\delta > 0)$ be a C^∞-function equal to 1 on $(-\infty, 0)$ and 0 on (δ, ∞). Set

$$f_1 = \chi_\delta f, \qquad f_2 = (1 - \chi_\delta) f. \tag{8}$$

Evidently, $f_1, f_2 \in \mathcal{S}'$, $\operatorname{supp} f_1 \subset (-\infty, \delta]$, $\operatorname{supp} f_2 \subset [0, \infty)$, and the t.d. f_2 is non-negative.

Since $\chi_\delta e^{\cdot\eta} \in \mathcal{S}$ for $\eta > 0$ and $(1 - \chi_\delta)e^{\cdot\eta} \in \mathcal{S}$ for $\eta < 0$, we have by Lemma 7.4.1 [3, p.191] that

$$f_1 e^{\cdot\eta} \in \mathcal{S}' \quad \text{for } \eta > 0, \qquad f_2 e^{\cdot\eta} \in \mathcal{S}' \quad \text{for } \eta < 0.$$

By Theorem 7.4.2 [3, p. 192], the Fourier transform of the t.d. $f_1 e^{\cdot\eta}$ $(f_2 e^{\cdot\eta})$ is a function $\hat{f}_1(\xi + i\eta)$ $(\hat{f}_2(\xi + i\eta))$ analytic in $t = \xi + i\eta$ in the upper (lower) half-plane and growing as $O(|\xi|^N)$ for some $N > 0$ as $t \to \infty$ in any interior smaller strip.

Step 2. Let us show that

$$\hat{f}_1(\cdot + i\eta) \to \hat{f}_1 \quad \text{as } \eta \to +0, \qquad \hat{f}_2(\cdot + i\eta) \to \hat{f}_2 \quad \text{as } \eta \to -0, \tag{9}$$

hold in $\mathcal{S}'$-topology.

The proofs of these relations are similar, so we prove only the first relation. For any $\psi \in \mathcal{S}$, we have

$$\langle \hat{f}_1(\cdot + i\eta), \psi \rangle = \langle f_1 e^{\cdot\eta}, \hat{\psi} \rangle = \langle f_1, \chi_\delta e^{\cdot\eta} \hat{\psi} \rangle.$$

Since $\chi_\delta e^{\cdot\eta} \hat{\psi} \to \chi_\delta \hat{\psi}$ as $\eta \to +0$ in $\mathcal{S}$-topology, we see that

$$\lim_{\eta \to +0} \langle \hat{f}_1(\cdot + i\eta), \psi \rangle = \langle f_1, \chi_\delta \hat{\psi} \rangle = \langle \hat{f}_1, \psi \rangle.$$

Step 3. Let φ and φ_h have the same meaning as in the proof of Lemma 1. Since $\hat{\varphi}_h \in \mathcal{D}$, the distributions

$$\hat{f}^h := \hat{f} * \hat{\varphi}_h, \quad \hat{f}_1^h := \hat{f}_1 * \hat{\varphi}_h, \quad \hat{f}_2^h := \hat{f}_2 * \hat{\varphi}_h \tag{10}$$

are C^∞-functions by Theorem 4.1.1 [3, p. 88]. Note that

$$\hat{f}^h = \hat{f}_1^h + \hat{f}_2^h. \tag{11}$$

Step 4. By the assumptions of Theorem 2, the t.d. $\hat{f}$ coincides on $(-a, a)$ with a function analytic in $\{t\colon -a < \Re t < a,\ 0 < \Im t < R\}$. Then, for $0 < h < a$, the t.d. $\hat{f}^h$ coincides with a function analytic in the rectangle $\{t\colon |\Re t| < a - h,\ 0 < \Im t < R\}$ in a neighborhood of the origin.

Let us show that C^∞-function $\hat{f}_1^h$ $(\hat{f}_2^h)$ is the boundary value on $\mathbf{R}$ of a function analytic in the upper (lower) half-plane and continuous in its closure. Again, we can restrict ourselves by the function $\hat{f}_1^h$. Let us *define* $\hat{f}_1^h$ in the upper half-plane in the following way:

$$\hat{f}_1^h(t) = \langle \hat{f}_1(t - \cdot), \hat{\varphi}_h \rangle = \langle \hat{f}_1(\cdot + i\eta), \hat{\varphi}_h(\xi - \cdot) \rangle, \quad t = \xi + i\eta,\ \eta > 0.$$

This function is also analytic in the upper half-plane and according to (9) we have

$$\lim_{\eta \to +0} \hat{f}_1^h(\xi + i\eta) = \langle \hat{f}_1, \hat{\varphi}_h(\xi - \cdot) \rangle = (\hat{f}_1 * \hat{\varphi}_h)(\xi). \tag{12}$$

The limit in (12) is uniform in ξ on each finite interval. Indeed,

$$\hat{f}_1^h(\xi + i\eta) = \big\langle \hat{f}_1(\cdot + i\eta), \hat{\varphi}_h(\xi - \cdot)\big\rangle = \langle f_1, 2\pi\chi_\delta e^{-i(\xi+i\eta)\cdot}\varphi_h\rangle.$$

It remains to observe that

$$\chi_\delta e^{-i(\xi+i\eta)\cdot}\varphi_h \to \chi_\delta e^{-i\xi\cdot}\varphi_h \quad \text{as } \eta \to +0,$$

in any seminorm of the space $\mathcal{S}$ uniformly in ξ on each finite interval.

Remark. In a similar way it can be shown that each derivative of $\hat{f}_1^h(\xi + i\eta)$ tends as $\eta \to +0$ to the corresponding derivative of $(\hat{f}_1 * \hat{\varphi}_h)(\xi)$ uniformly in ξ on each finite interval. So in fact $\hat{f}_1^h$ is the boundary value on $\mathbf{R}$ of a function analytic in the upper half-plane and infinitely differentiable in its closure. We will not use this fact.

Step 5. From (11) we derive that

$$\hat{f}_2^h(t) = \hat{f}^h(t) - \hat{f}_1^h(t), \quad t \in \mathbf{R}. \tag{13}$$

The function in the left-hand side is analytic in the lower half-plane and continuous in its closure. The function in the right-hand side is analytic in rectangle

$$\big\{t\colon |\Re t| < a - h,\ 0 < \Im t < R\big\} \tag{14}$$

and continuous in its closure. Therefore $\hat{f}_2^h$ admits analytic continuation into (14). Theorem 7.1.15 [3, p. 166] implies that the distribution $\hat{f}_2^h$ $(= \hat{f}_2 * \hat{\varphi}_h)$ is the Fourier transform of the distribution $2\pi f_2\varphi_h$ which is non-negative by the construction of f_2. Therefore we may apply Lemma 1 to $\hat{f}_2^h$. We see that this function admits analytic continuation into strip $\{t\colon 0 < \Im t < R\}$ (and, hence, half-plane $\{t\colon \Im t < R\}$) and is representable there in the form

$$\hat{f}_2^h(t) = 2\pi \int_0^\infty e^{-ixt}\varphi_h(x)\, d\mu_{f_2}(x), \quad \Im t < R, \tag{15}$$

where μ_{f_2} is the non-negative measure representing f_2 and the integral converges absolutely.

Step 6. We have just proved that all members of equality (13) can be considered as analytic in the region (14). Therefore this equality holds true in (14). Let us consider it at point $t = i\eta$, $0 < \eta < R$. Taking into account (15), we can write it in the form

$$2\pi \int_0^\infty e^{x\eta}\varphi_h(x)\, d\mu_{f_2}(x) = \hat{f}^h(i\eta) - \hat{f}_1^h(i\eta). \tag{16}$$

It was mentioned in Step 1 that $\hat{f}_1$ is analytic in the upper half-plane. By the assumptions of the theorem, $\hat{f}$ is analytic in $\{t\colon -a < \Re t < a,\ 0 < \Im t < R\}$. Therefore the limit as $h \to +0$ of the right-hand side of (16) exists (and is equal to $\hat{f}(i\eta) - \hat{f}_1(i\eta)$). Since $\varphi_h \to (2\pi)^{-1}$ pointwise, we conclude by the Fatou lemma that

$$\int_0^\infty e^{x\eta}\, d\mu_{f_2}(x) < \infty \tag{17}$$

for $0 < \eta < R$ and, hence, for $-\infty < \eta < R$.

Let us write down the equality (13) for $0 < h < a/2$ at an arbitrary point of the rectangle $\{t\colon |\Re t| < a/2,\ 0 < \Im t < R\}$,

$$2\pi \int_0^\infty e^{-itx} \varphi_h(x)\, d\mu_{f_2}(x) = \hat{f}^h(t) - \hat{f}_1^h(t). \tag{18}$$

Now, let $h \to +0$. The limit of the right-hand side evidently is $\hat{f}(t) - \hat{f}_1(t)$. Using (17) and dominated convergence theorem, we can take limit under the integral sign in the left-hand side, and we obtain the following equality in the mentioned rectangle:

$$\int_0^\infty e^{-itx}\, d\mu_{f_2}(x) = \hat{f}(t) - \hat{f}_1(t). \tag{19}$$

Step 7. The equality (19) can be rewritten in the form

$$\hat{f}(t) = \hat{f}_1(t) + \int_0^\infty e^{-itx}\, d\mu_{f_2}(x). \tag{20}$$

It has been shown that it holds true in the rectangle $\{t\colon |\Re t| < a/2,\ 0 < \Im t < R\}$. But the first term of right-hand side is (see Step 1) analytic in the upper half-plane and has growth not exceeding a power of $|t|$ in any strip of kind $\{t\colon 0 < r_1 \leqslant \Im t \leqslant r_2 < \infty\}$. The second term of right-hand side is analytic and bounded in the half-plane $\{t\colon \Im t \leqslant r_2 < R\}$ because the integral converges there absolutely and uniformly by virtue (17). Taking into account that $\hat{f}_1(\cdot + i\eta) \to \hat{f}_1$ in $\mathcal{S}'$-topology (see (9)), we get the assertion of Theorem 2. $\square$

Proof of Theorem 4. If in the proof of Theorem 2 one assumes that $f \in L_2(\mathbf{R})$, then we have

$$f_1, f_2, \hat{f}, \hat{f}_1, \hat{f}_2 \in L_2(\mathbf{R}),$$

and $\hat{f}_1$ belongs to the Hardy class H_2 in any strip of kind $\{t\colon 0 < \Im t < r < \infty\}$. Since the measure μ_{f_2} coincides (in the distributional sense) with the function f_2, we conclude that $d\mu_{f_2} = f_2\, dx$. Therefore (17) is equivalent to

$$\int_0^\infty e^{x\eta} f_2(x)\, dx < \infty, \quad -\infty < \eta < R,$$

and, in particular, we get $f_2 \in L_1(\mathbf{R})$. Hence

$$\hat{f}_2(t) = \int_0^\infty e^{-itx} f_2(x)\, dx$$

admits the analytic continuation from $\mathbf{R}$ to the half-plane $\{t\colon \Im t < R\}$ and tends to 0 at ∞. Since $\hat{f} = \hat{f}_1 + \hat{f}_2$, we obtain the desired conclusion. $\square$

Remark. The question arises whether $\hat{f}$ belongs to H_2 in $\{t\colon 0 < \Im t < R\}$. In general, the answer is negative, moreover, $\hat{f}$ may not belong to H_2 even in any smaller strip. A counterexample can be constructed in the following way.

It suffices to construct a non-negative function $f \in L_2(\mathbf{R}_+)$ satisfying conditions:

$$\text{(i)} \qquad \int_0^\infty e^{x\eta} f(x)\,dx < \infty, \quad \forall \eta > 0,$$

$$\text{(ii)} \qquad \int_0^\infty e^{2x\eta} f^2(x)\,dx = \infty, \quad \forall \eta > 0.$$

Indeed, (i) implies that $\hat{f}$ can be analytically extended to the whole plane, and, hence, the conditions of Theorem 6 are satisfied for f. On the other hand, (ii) implies (by the Parseval equality) that $\hat{f}(\cdot + i\eta) \notin L_2(\mathbf{R})$ for any $\eta > 0$ and, hence, $\hat{f}$ does not belong to H_2 in any parallel strip lying in the upper half-plane.

A function f satisfying (i) and (ii) can be taken in form

$$f(x) = \begin{cases} 1/|x-k|^{\alpha_k} & \text{for } |x-k| < \delta_k,\ k = 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

where the parameters $0 < \delta_k < 1/2$ and $0 < \alpha_k < 1/2$ are defined by the equations

$$\delta_k = e^{-k^2}, \quad \delta_k^{1-2\alpha_k}/(1-2\alpha_k) = k^{-2}, \quad k = 1, 2, \ldots. \tag{21}$$

Indeed, for $\eta > 0$,

$$\int_0^\infty e^{x\eta} f(x)\,dx = \sum_1^\infty \int_{k-\delta_k}^{k+\delta_k} \frac{e^{x\eta}\,dx}{|x-k|^{\alpha_k}} < 2\sum_1^\infty e^{(k+1)\eta} \int_0^{\delta_k} \frac{du}{u^{\alpha_k}}$$
$$= 2\sum_1^\infty e^{(k+1)\eta} \frac{\delta_k^{1-\alpha_k}}{1-\alpha_k} < 4e^{\eta} \sum_1^\infty e^{k\eta}\sqrt{\delta_k} < \infty$$

by the first of conditions (21). Further

$$\int_0^\infty f^2(x)\,dx = \sum_1^\infty \int_{k-\delta_k}^{k+\delta_k} \frac{dx}{|x-k|^{2\alpha_k}} = 2\sum_1^\infty \frac{\delta_k^{1-2\alpha_k}}{1-2\alpha_k} < \infty$$

by the second of the conditions (21). Finally, for $\eta > 0$,

$$\int_0^\infty e^{2x\eta} f^2(x)\,dx = \sum_1^\infty \int_{k-\delta_k}^{k+\delta_k} \frac{e^{2x\eta}\,dx}{|x-k|^{2\alpha_k}} > \sum_1^\infty e^{(2k-1)\eta} \frac{2\delta_k^{1-2\alpha_k}}{1-2\alpha_k} = \infty$$

also by the second of conditions (21).

Proof of Theorem 1. If in the proof of Theorem 2 one assumes that f is a temperate measure μ, then f_1 and f_2 are also temperate measures, μ_1 and μ_2 say, defined by $\mu_1 =$

$\chi_\delta\mu$, $\mu_2 = (1-\chi_\delta)\mu$. The proof of Theorem 2 shows that μ_2 ($\equiv \mu_{f_2}$) is a non-negative finite measure on $\mathbf{R}_+$ satisfying (17). Therefore $|\mu|(\mathbf{R}_+) \leqslant |\mu|((0,\delta)) + \mu_2(\mathbf{R}_+) < \infty$ and hence $|\mu|(\mathbf{R}) < \infty$. Moreover, $\hat{\mu}_2$ is analytic in $\{t\colon \Im t < R\}$, continuous and bounded in $\{t\colon \Im t \leqslant R_1 < R\}$. Since $|\mu|(\mathbf{R}_-) < \infty$, the function $\hat{\mu}_1(t)$ is analytic in the upper half-plane and continuous in its closure. In the proof of Theorem 2 (Step 7) it had been shown that $\hat{\mu}(t) = \hat{\mu}_1(t) + \hat{\mu}_2(t)$ for $0 < \Im t < R$. Since $\hat{\mu}$ is the boundary value of $\hat{\mu}(t)$ on $\mathbf{R}$ in $\mathcal{S}'$-topology, we get the assertion of the theorem. □

Proof of Theorem 5. Let a measure μ satisfy conditions of Theorem 5. Set $f = \hat{\mu}$. This is a continuous bounded function on $\mathbf{R}$, and we shall consider it as a distribution from $\mathcal{S}'$. The conditions of Theorem 5 imply that f is non-negative on the positive ray, and the distribution $\hat{f} = 2\pi\check{\mu}$ coincides in $(-a,a)$ with a function analytic in $\{0 < \Im t < R,\ -a < \Re t < a\}$. Hence, f satisfies the conditions of Theorem 2 and therefore all arguments from its proof are applicable.

The distributions f_1, f_2 defined by (8) are now continuous bounded functions on $\mathbf{R}$ with $\operatorname{supp} f_1 \subset (-\infty,\delta]$, $\operatorname{supp} f_2 \subset [0,\infty)$. Functions $\hat{f}^h$, $\hat{f}_1^h$, $\hat{f}_2^h$ are C^∞-functions on $\mathbf{R}$ and, moreover, $\hat{f}_1^h$ ($\hat{f}_2^h$) is the boundary value of a function $\hat{f}_1^h(t)$ ($\hat{f}_2^h(t)$) analytic in the upper (lower) half-plane. Further $\hat{f}_2^h(t)$ admits analytic continuation to the half-plane $\{t\colon \Im t < R\}$, is representable there by formula (15), and

$$\hat{f}^h(t) = \hat{f}_1^h(t) + \hat{f}_2^h(t), \quad 0 \leqslant \Im t < R.$$

Let us show that the following estimate holds for any $0 < h < 1$:

$$\left|\hat{f}^h(t)\right| \leqslant \frac{C}{\Im t}, \quad 0 < \Im t < R, \tag{22}$$

where C is a positive constant depending on neither t nor h.

Since f_1 is a continuous bounded function on $\mathbf{R}$ with $\operatorname{supp} f_1 \subset (-\infty,\delta]$, then its Fourier transform $\hat{f}_1$ is analytic in the upper half-plane and satisfies the inequality

$$\left|\hat{f}_1(t)\right| \leqslant \frac{C_1}{\Im t}, \quad \Im t > 0. \tag{23}$$

Noting that

$$\hat{f}_1^h(t) = (\hat{f}_1 * \hat{\varphi})(t) = \int_{-\infty}^{\infty} \hat{f}_1(x+i\eta)\hat{\varphi}_h(\xi - x)\,dx, \quad t = \xi + i\eta,$$

we get from (23),

$$\left|\hat{f}_1^h(t)\right| \leqslant \frac{C_1}{\eta}\int_{-\infty}^{\infty}\left|\hat{\varphi}_h(x)\right|dx = \frac{C_1}{\eta}\int_{-\infty}^{\infty}\left|\hat{\varphi}(x)\right|dx = \frac{C}{\eta}. \tag{24}$$

It follows from (15) and (17) that the functions $\hat{f}_2^h$, $0 < h < 1$, are uniformly bounded in the half-plane $\{t\colon \Im t \leqslant R\}$. Hence by (24) we obtain (22).

Let us fix any rectangle

$$\Pi_A = \left\{t\colon |\Re t| \leqslant A,\ 0 \leqslant \Im t \leqslant R/2\right\}.$$

Functions

$$g^h(t) := \hat{f}^h(t)(A^2 - t^2), \quad t \in \Pi_A,\ 0 < h < 1,$$

are analytic in $\operatorname{int} \Pi_A$, continuous in Π_A and, by (22), uniformly bounded on $\partial \Pi_A$. By the maximum modulus theorem, these functions are uniformly bounded in Π_A. The compactness principle says that any sequence of values of h tending to 0 contains a subsequence $h_k \to 0$ such that g^{h_k} converges uniformly on compacta in $\operatorname{int} \Pi_A$ to a function g analytic in $\operatorname{int} \Pi_A$.

By the Cauchy integral formula we have

$$\frac{1}{2\pi i} \int\limits_{\partial \Pi_A} \frac{g^h(\zeta)\, d\zeta}{\zeta - t} = \begin{cases} g^h(t) & \text{for } t \in \operatorname{int} \Pi_A, \\ 0 & \text{for } t \notin \Pi_A. \end{cases} \tag{25}$$

Observe that, for $t \in \partial \Pi_A \setminus [-A, A]$, we have $\hat{f}^h(\zeta) \to \hat{f}(\zeta)$ and hence $g^h(\zeta) \to \hat{f}(\zeta)(A^2 - \zeta^2)$ boundedly as $h \to +0$. Since $\hat{f} = 2\pi \check{\mu}$ is a finite measure, we have for $\zeta \in [-A, A]$, $h \to 0$,

$$\hat{f}^h(\zeta)\, d\zeta = 2\pi(\check{\mu} * \hat{\varphi}_h)(\zeta)\, d\zeta \to 2\pi\, d\check{\mu}(\zeta),$$

in the sense of weak-star convergence. Therefore we can pass to the limit as $h = h_k \to 0$ under the integral sign in (25) and obtain

$$\frac{1}{2\pi i} \int\limits_{\partial \Pi_A} \frac{d\nu(\zeta)}{\zeta - t} = \begin{cases} (A^2 - t^2) f(t) & \text{for } t \in \operatorname{int} \Pi_A, \\ 0 & \text{for } t \notin \Pi_A, \end{cases} \tag{26}$$

where ν is a measure on $\partial \Pi_A$ defined by

$$d\nu(\zeta) = \begin{cases} (A^2 - t^2) f(\zeta)\, d\zeta & \text{for } \zeta \in \partial \Pi_A \setminus [-A, A], \\ (A^2 - t^2) 2\pi\, d\check{\mu}(\zeta) & \text{for } \zeta \in [-A, A]. \end{cases}$$

The equality (26) means that the integral in its right-hand side is a Cauchy integral of measure. By the well-known theorem of brothers Riesz, the measure ν is absolutely continuous with respect to the Lebesgue measure on $\partial \Pi_A$ and its density coincides with the angular boundary values of $(A^2 - t^2) f(t)$, $t \in \operatorname{int} \Pi_A$. Moreover, the latter function belongs to H_1 in $\operatorname{int} \Pi_A$. Using the arbitrariness of choice of A, we get the assertion of Theorem 5. □

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