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Regularity and K_0-group of quadric solvable polynomial algebras

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Abstract

Concerning solvable polynomial algebras in the sense of Kandri-Rody and Weispfenning [J. Symbolic Comput. 9 (1990) 1–26], it is shown how to recognize and construct quadric solvable polynomial algebras in an algorithmic way. If $A = k[a_1, \ldots, a_n]$ is a quadric solvable polynomial algebra, it is proved that gl.dim $A \leqslant n$ and $K_0(A) \cong \mathbb{Z}$. If A is a tame quadric solvable polynomial algebra, it is shown that A is completely constructable and Auslander regular.

Keywords: Solvable polynomial algebra; Gröbner basis; $\succeq_{gr}$-filtration; Global dimension; K_0-group

This work is the continuation of [Li] that deals with quadric solvable polynomial algebras. More precisely, the aim of this paper is to study the regularity of general quadric solvable polynomial algebras (at least at the level of having finite global dimension and K_0-group $\mathbb{Z}$). In Section 1, we first note from several known results that quadric solvable polynomial algebras are algorithmically recognizable and constructable by means of the very noncommutative Gröbner bases in the sense of Mora [Mor]. In Section 2, we derive that every *tame* quadric solvable polynomial algebra A (see Section 1 Definition 1.2) is completely constructable (in the sense of Theorem 2.1) and Auslander regular with $K_0(A) \cong \mathbb{Z}$. This is achieved by taking a closer look at the associated graded algebra $G(A)$ of A with respect to the standard filtration FA. After introducing the $\succeq_{gr}$-filtration on modules in Section 3, we prove in Section 4 that every quadric solvable polynomial algebra is of finite global dimension. Returning to the standard filtration again in Section 5, it is proved that $K_0(A) \cong \mathbb{Z}$ holds for every quadric solvable polynomial algebra A. At this

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stage, we may say that every quadric solvable polynomial algebra is regular in the classical sense. However, the author strongly believes that the following conjecture is true, though he himself failed to prove it in general.

Conjecture. *Every quadric solvable polynomial algebra A is Auslander regular.*

Throughout this paper we let k denote a commutative field. All algebras considered are associative k-algebras with 1, and modules are unitary *left* modules. As every solvable polynomial algebra is a left and right Noetherian domain over a field, the invariant basis property holds for such algebras, and consequently, there is no problem to talk about global dimension and K_0-group of such algebras.

1. Quadric solvable polynomial algebras

In this section, after briefly recalling from [K-RW,LW,LWZ] some basic notions and facts concerning noncommutative Gröbner bases and solvable polynomial algebras (but with slight modification), we show how to recognize and construct quadric solvable polynomial algebras in an algorithmic way.

Let $\mathbb{Z}^n_{\geqslant 0}$ be the set of n-tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\leqslant 0}$, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

By a *monomial ordering* on $\mathbb{Z}^n_{\geqslant 0}$ we mean any relation $\succeq$ on $\mathbb{Z}^n_{\geqslant 0}$ satisfying

(1) $\succeq$ is a well-ordering on $\mathbb{Z}^n_{\geqslant 0}$, and
(2) if $\alpha \succ \beta$ and $\gamma \in \mathbb{Z}^n_{\geqslant 0}$, then $\alpha + \gamma \succ \beta + \gamma$.

Any lexicographic ordering on $\mathbb{Z}^n_{\geqslant 0}$, denoted $\geqslant_{lex}$, is a monomial ordering. Another monomial ordering used very often in computational algebra is the graded lexicographic ordering on $\mathbb{Z}^n_{\geqslant 0}$, denoted $\geqslant_{grlex}$, which is defined as follows: for $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, $\alpha >_{grlex} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i, \quad \text{or} \quad |\alpha| = |\beta| \text{ and } \alpha >_{lex} \beta,$$

where $\geqslant_{lex}$ is some lexicographic ordering on $\mathbb{Z}^n_{\geqslant 0}$. More generally, we say that a monomial ordering $\succeq$ on $\mathbb{Z}^n_{\geqslant 0}$ is a *graded monomial ordering*, denoted $\succeq_{gr}$, in case it is defined as: for $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, $\alpha \succ_{gr} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i, \quad \text{or} \quad |\alpha| = |\beta| \text{ and } \alpha \succ \beta.$$

Let $A = k[a_1, \ldots, a_n]$ be a finitely generated k-algebra with generating set $\{a_1, \ldots, a_n\}$. Given any permutation of the generators, say $\{a_{j_1}, a_{j_2}, \ldots, a_{j_n}\}$, we call an element of the form $a_{j_1}^{\alpha_1} \cdots a_{j_n}^{\alpha_n}$ a *standard monomial* in A with respect to the given permutation of generators, where $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}$, and write

$$\mathbf{SM}(A) = \left\{a^\alpha = a_{j_1}^{\alpha_1} \cdots a_{j_n}^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}\right\}$$

for the set of all such standard monomials. It is clear that there is an onto map $\varphi : \mathbb{Z}^n_{\geqslant 0} \to \mathbf{SM}(A)$ with $\varphi(\alpha) = a^\alpha$, in particular,

$$(\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0) = e_i \mapsto a_{j_i}, \quad 1 \leqslant i \leqslant n.$$

If furthermore the map φ is one-to-one and onto, then any ordering $\succeq$ on $\mathbb{Z}^n_{\geqslant 0}$ naturally induces an ordering on $\mathbf{SM}(A)$, also denoted $\succeq$, as follows.

$$a^\alpha \succ a^\beta \quad \text{if and only if} \quad \alpha \succ \beta.$$

Definition 1.1 [K-RW,LW]. Let $A = k[a_1, \ldots, a_n]$ be a finitely generated k-algebra and $\mathbf{SM}(A)$ the set of standard monomials in A with respect to a given permutation $\{a_{j_n}, a_{j_{n-1}}, \ldots, a_{j_1}\}$ of generators of A. Let $\succeq$ be a monomial ordering on $\mathbb{Z}^n_{\geqslant 0}$. A is called a *solvable polynomial algebra* with the monomial ordering $\succeq$ if the following conditions are satisfied.

(S1) $\mathbf{SM}(A)$ forms a k-basis for A (hence there is a one-to-one and onto map $\varphi : \mathbb{Z}^n_{\geqslant 0} \to \mathbf{SM}(A)$ with $\varphi(\alpha) = a^\alpha$), and
(S2) for any a^α, $a^\beta \in \mathbf{SM}(A)$, $a^\alpha a^\beta = \lambda_{\alpha,\beta} a^{\alpha+\beta} + \sum \lambda_\gamma a^\gamma$ with $\lambda_{\alpha,\beta},\ \lambda_\gamma \in k$, $\lambda_{\alpha,\beta} \neq 0$, and $\alpha + \beta \succ \gamma$ (or equivalently, $a^{\alpha+\beta} \succ a^\gamma$) for every a^γ appearing in $\sum \lambda_\gamma a^\gamma$ with $\lambda_\gamma \neq 0$.

It is shown in [K-RW] that every nonzero one-sided ideal L of a solvable polynomial algebra A has a finite Gröbner basis with respect to the given monomial ordering, and hence A is a (left and right) Noetherian domain. If furthermore the ground field is computable, then a Gröbner basis containing a given generating set of L may be computed in terms of the S-polynomials by using a noncommutative version of Buchberger's Algorithm.

($\bullet$) Unless it is otherwise stated, from now on we assume for a solvable polynomial algebra $A = k[a_1, \ldots, a_n]$ that $\mathbf{SM}(A) = \{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \mid (\alpha_1 \ldots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}\}$ (this is always possible by renumbering the generators).

Warning. The convention ($\bullet$) does not necessarily imply that, with respect to the given monomial ordering $\succeq_{gr}$ on $\mathbf{SM}(A)$, there is the ordering $a_n \succ_{gr} a_{n-1} \succ_{gr} \cdots \succ_{gr} a_1$.

Bearing the above convention (•) and the warning in mind, let $A = k[a_1, \ldots, a_n]$ be a solvable polynomial k-algebra with respect to some graded monomial ordering $\succeq_{gr}$ on $\mathbb{Z}^n_{\geqslant 0}$, or equivalently, on the k-basis $\mathbf{SM}(A)$ of A. By the definition of a graded monomial ordering and Definition 1.1(S2), it follows that the generators of A satisfy *only* quadric relations, that is,

$$a_j a_i = \lambda_{ji} a_i a_j + \sum_{k \leqslant \ell} \lambda^{k\ell}_{ji} a_k a_\ell + \sum \lambda_h a_h + c_{ji}, \quad 1 \leqslant i < j \leqslant n, \tag{$*$}$$

where $\lambda_{ji}, \lambda^{k\ell}_{ji}, \lambda_h, c_{ji} \in k$, and $\lambda_{ji} \neq 0$. This leads to the following specific class of solvable polynomial algebras.

Definition 1.2. We call the solvable polynomial algebra A with $\succeq_{gr}$ a *quadric solvable polynomial algebra*. If $k, \ell < j$ in the formula $(*)$ whenever $\lambda^{k\ell}_{ji} \neq 0$, then we call A a *tame* quadric solvable polynomial algebra.

To characterize quadric solvable polynomial algebras in an algorithmic way, we first note an easy fact.

Observation. Let $A = k[a_1, \ldots, a_n]$ be a quadric solvable polynomial k-algebra with $\succeq_{gr}$. Then since

$$a_j a_i = \lambda_{ji} a_i a_j + \sum_{k \leqslant \ell} \lambda^{k\ell}_{ji} a_k a_\ell + \sum \lambda_h a_h + c_{ji}, \quad 1 \leqslant i < j \leqslant n,$$

and since $\mathbf{SM}(A) = \{a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}\}$ forms a k-basis for A, we have $A \cong k\langle X\rangle/I$, where $k\langle X\rangle = k\langle X_1, \ldots, X_n\rangle$ is the free associative k-algebra on $X = \{X_1, \ldots, X_n\}$ and I is the ideal of $k\langle X\rangle$ generated by

$$R_{ji} = X_j X_i - \lambda_{ji} X_i X_j - \sum_{k \leqslant \ell} \lambda^{k\ell}_{ji} X_k X_\ell - \sum \lambda_h X_h - c_{ji}, \quad 1 \leqslant i < j \leqslant n,$$

or in other words, $\{R_{ji} \mid 1 \leqslant i < j \leqslant n\}$ is a set of defining relations for A.

Let W be the multiplicative semigroup of words (including empty word as 1) in the free k-algebra $k\langle X\rangle = k(X_1, \ldots, X_n\rangle$. If $w \in W$, we write $d(w)$ for the length of the word w, where $d(1) = 0$. Recall from [Mor] that a *monomial ordering* on $k\langle X\rangle$ is a well-ordering $\succeq$ on W which is *compatible* with the product:

$$\text{for each } u, v, t_1, t_2 \in W, \quad t_1 \prec t_2 \quad \text{implies} \quad u t_1 v \prec u t_2 v.$$

For example, a *graded lexicographic order* on W, denoted $\geqslant_{grlex}$, is defined as follows. Choose an ordering

$$\mathbf{O}_>: \quad X_{jn} > X_{j_{n-1}} > \cdots > X_{j_1}.$$

For $u, v \in W$, $u >_{grlex} v$ if and only if

either $d(v) < d(u)$ or $d(u) = d(v)$ and v is lexicographically less than u,

where we say that v is lexicographically less than u if

either there is $r \in W$ such that $u = vr$
or there are $w, r_1, r_2 \in W$, and $X_{j_p} < X_{j_q}$ such that $v = wX_{j_p}r_1$, $u = wX_{j_q}r_2$.

Note that the monomial ordering $\geqslant_{grlex}$ defined above yields $X_{j_n} >_{grlex} X_{j_{n-1}} >_{grlex} \cdots >_{grlex} X_{j_1}$ which coincides with the given ordering $\mathbf{O}_>$.

(••) Unless it is otherwise stated, henceforth we assume that a monomial ordering $\succeq$ on the free algebra $k\langle X\rangle = k\langle X_1, \ldots, X_n\rangle$ induces the ordering $\mathbf{O}_{\succeq}$: $X_n \succ X_{n-1} \succ \cdots \succ X_1$ on generators (this is always possible by renumbering the generators, as illustrated by later examples (iv)–(vi)).

Given a monomial ordering $\succeq$ on $k\langle X\rangle$, each element $F \in k\langle X\rangle$ has a unique ordered representation as a linear combination of elements of W:

$$F = \sum_{i=1}^{s} c_i t_i, \quad 0 \neq c_i \in k,\ t_i \in W,\ t_1 \succ t_2 \succ \cdots \succ t_s.$$

So to each nonzero element $F \in \langle X\rangle$ we can associate $\mathbf{LM}(f) = t_1$, the *leading monomial* of f. Let $\mathcal{G} = \{G_j\}_{j\in\Lambda}$ be a nonempty subset of $k\langle X\rangle$ and $I = \langle\mathcal{G}\rangle$ the two-sided ideal of $k\langle X\rangle$ generated by G. $\mathcal{G}$ is called a *Gröbner basis* in $k\langle X\rangle$ with respect to a given monomial ordering $\succeq$ if every element $F \in I$ has a Gröbner representation by $\mathcal{G}$ in the sense of Mora [Mor]: $F = \sum \lambda_j w_j G_j v_j$, where $\lambda_j \in k$ and w_j, v_j are words of $k\langle X\rangle$, such that $\mathbf{LM}(F) \succeq \mathbf{LM}(w_j G_j v_j)$ whenever $\lambda_j \neq 0$.

Proposition 1.3 [LWZ, Theorem 1.2.2]. *With notation and the convention* (••) *as above, let $\succeq$ be a monomial ordering on the free algebra $k\langle X\rangle = k\langle X_1, \ldots, X_n\rangle$. Consider the k-algebra $A = k\langle X\rangle/I$, where I is the ideal of $k\langle X\rangle$ generated by the defining relations*

$$R_{ji} = X_j X_i - \lambda_{ji} X_i X_j - \{X_j, X_i\}, \quad 1 \leqslant i < j \leqslant n,$$

*where $\lambda_{ji} \in k$, $\{X_j, X_i\} = 0$ or $\{X_j, X_i\} \in k\langle X\rangle$—$k$-*span$\{X_j X_i, X_i X_j\}$. *Suppose that*

$$\mathbf{LM}(R_{ji}) = X_j X_i \text{ with respect to } \succeq, \quad 1 \leqslant i < j \leqslant n.$$

The following statements are equivalent.

(i) *Write x_i for the image of X_i in A, $1 \leqslant i \leqslant n$. The set of standard monomials in A, denoted*

$$\mathbf{SM}(A) = \left\{ x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0} \right\},$$

forms a k-basis for A.

(ii) *$\{R_{ji} \mid 1 \leqslant i < j \leqslant n\}$ forms a Gröbner basis in $k\langle X\rangle$ with respect to $\succeq$, as defined above,*

(iii) *For $1 \leqslant i < j < k \leqslant n$, every $R_{kj} X_i - X_k R_{ji}$ has a weak Gröbner representation by $\{R_{ji} \mid 1 \leqslant i < j \leqslant n\}$ in the sense of* [Mor]*:*

$$R_{kj} X_i - X_k R_{ji} = \sum_{p>q} c_{pq} l_{pq} R_{pq} r_{pq} \quad \text{with the property that}$$
$$\mathbf{LM}(R_{kj}) X_i \succ l_{pq} \mathbf{LM}(R_{pq}) r_{pq},$$

where $c_{pq} \in k$ and l_{pq}, r_{pq} are words of $k\langle X\rangle$.

Corollary 1.4. *Consider the k-algebra $A = k\langle X\rangle / I$, where I is the ideal of the free algebra $k\langle X\rangle = k\langle X_1, \ldots, X_n\rangle$ generated by the* quadric *defining relations*

$$R_{ji} = X_j X_i - \lambda_{ji} X_i X_j - \sum \lambda_{ji}^{k\ell} X_k X_\ell - \sum \lambda_h X_h - c_{ji}, \quad l \leqslant i < j \leqslant n,$$

where λ_{ji}, $\lambda_{ji}^{k\ell}$, λ_h, $c_{ji} \in k$. Suppose that

(1) *$\lambda_{ji} \neq 0$, $1 \leqslant i < j \leqslant n$, and*
(2) *one of the following conditions is satisfied whenever $\lambda_{ji}^{k\ell} \neq 0$:*
  (a) *$k = \ell$ and $k, \ell < j$.*
  (b) *$k \neq \ell$ and $k, \ell \leqslant j$, where $k = j$ implies $\ell < i$ and $\ell = j$ implies $k < i$.*

Then A is a quadric solvable polynomial algebra with respect to $x_n >_{grlex} x_{n-1} >_{grlex} \cdots >_{grlex} x_1$, where each x_i is the image of X_i in A, if and only if $\{R_{ji} \mid 1 \leqslant i < j \leqslant n\}$ forms a Gröbner basis in $k\langle X\rangle$ with respect to $X_n >_{grlex} X_{n-1} >_{grlex} \cdots >_{grlex} X_1$.

Proof. Suppose that $\{R_{ji} \mid 1 \leqslant i < j \leqslant n\}$ forms a Gröbner basis in $k\langle X\rangle$ with respect to $X_n >_{grlex} X_{n-1} >_{grlex} \cdots >_{grlex} X_1$. Since by the assumption (2) we have $\mathbf{LM}(R_{ji}) = X_j X_i$, $1 \leqslant i < j \leqslant n$, it follows from Proposition 1.3 that $\mathbf{SM}(A) = \{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}\}$ forms a k-basis for A. Now one checks directly that the assumptions (1)–(2) and the defining relations together make A into a quadric solvable polynomial algebra with respect to $x_n >_{grlex} x_{n-1} >_{grlex} \cdots >_{grlex}$. The converse is clear by Proposition 1.3. □

To realize Proposition 1.3, one may, of course, use the very noncommutative division algorithm and a version of Buchberger algorithm given by Mora [Mor]. However, to avoid large and tedious noncommutative division procedure, it follows from [LWZ] that

Berger's q-Jacobi condition is quite helpful in the case where $\geqslant_{grlex}$ is used (indeed $\geqslant_{grlex}$ is the monomial ordering on a free algebra used most often in practice). To see this, let $k\langle X\rangle = k\langle X_1,\dots,X_n\rangle$ and $A = k\langle X\rangle/I$ be as in Proposition 1.3, where I is generated by the set of defining relations $\{R_{ji} \mid 1 \leqslant i < j \leqslant n\}$. For $1 \leqslant i < j < k \leqslant n$, the *Jacobi sum* $\mathbf{J}(X_k, X_j, X_i)$ (in the sense of [Ber]) is defined as

$$\begin{aligned}\mathbf{J}(X_k, X_j, X_i) = {} & \{X_k, X_j\}X_i - \lambda_{ki}\lambda_{ji}X_i\{X_k, X_j\} \\ & - \lambda_{ji}\{X_k, X_i\}X_j + \lambda_{kj}X_j\{X_k, X_i\} \\ & + \lambda_{kj}\lambda_{ki}\{X_j, X_i\}X_k - X_k\{X_j, X_i\}.\end{aligned}$$

Then, as in the proof of [LWZ, Proposition 1.3.2], we can derive that, for $1 \leqslant i < j < k \leqslant n$,

$$\begin{aligned}R_{kj}X_i - X_kR_{ji} = {} & \lambda_{ji}R_{ki}X_j - \lambda_{kj}X_jR_{ki} - \lambda_{kj}\lambda_{ki}R_{ji}X_k + \lambda_{ki}\lambda_{ji}X_iR_{kj} \\ & - \mathbf{J}(X_k, X_j, X_i).\end{aligned}$$

It follows from Proposition 1.3(iii) that the following proposition holds.

Proposition 1.5. *Let A be as in Proposition* 1.3 *and let $\geqslant_{grlex}$ be the graded lexicographic ordering on $k\langle X\rangle$ such that*

$$\begin{aligned}& X_n >_{grlex} X_{n-1} >_{grlex} \cdots >_{grlex} X_1, \\ & \mathbf{LM}(R_{ji}) = X_jX_i \quad \text{with respect to } \geqslant_{grlex}, \quad 1 \leqslant i < j \leqslant n.\end{aligned}$$

The following statements are equivalent.

(i) $\{R_{ji} \mid 1 \leqslant i < j \leqslant n\}$ *forms a Gröbner basis in $k\langle X\rangle$ with respect to $\geqslant_{grlex}$.*
(ii) *For $1 \leqslant i < j < k \leqslant n$,*

$$\begin{aligned}\mathbf{J}(X_k, X_j, X_i) \in {} & k\text{-span}\{R_{pq} \mid 1 \leqslant q < p \leqslant n\} \\ & + k\text{-span}\left\{\begin{array}{ll} X_hR_{ji}, R_{ji}X_h, R_{ij}X_k, & 1 \leqslant h < k, \\ X_hR_{ki}, R_{ki}X_h, R_{ki}X_k, & 1 \leqslant h < k, \\ X_hR_{kj}, R_{kj}X_m, & 1 \leqslant h < k,\ 1 \leqslant m < i. \end{array}\right\}\end{aligned}$$

Example. In the examples given below, notation is maintained as before. Moreover, by abusing language, some examples will be called "deformations" of certain well-known algebras.

(i) Let $X_2X_1 - qX_1X_2 - aX_1^2 - bX_1 - cX_2 - d = R_{21} \in k\langle X_1, X_2\rangle$, where $q, a, b, c, d \in k$. Then it is easy to know by [Mor] that $\{R_{21}\}$ is a Gröbner basis in $k\langle X_1, X_2\rangle$ with respect to $X_2 >_{grlex} X_1$. Thus, if $q \neq 0$, then the algebra $A = k\langle X_1, X_1\rangle/\langle R_{21}\rangle$ is a tame quadric solvable polynomial algebra with respect to $\geqslant_{grlex}$ (indeed this is a skew polynomial algebra). One sees that we have all 2-dimensional quadric solvable polynomial algebras with respect to $X_2 >_{grlex} X_1$ here.

(ii) *Deformations of* $U(sl_2)$. Let $U(sl_2)$ be the enveloping algebra of the 3-dimensional Lie algebra $\mathbf{g} = k[x, y, z]$ defined by the relations: $[x, y] = z$, $[z, x] = 2x$, $[z, y] = -2y$. This example provides quadric solvable polynomial algebras which are deformations of $U(sl_2)$.

Let $k\langle X_1, X_2, X_3\rangle$ be the free k-algebra on $\{X_1, X_2, X_3\}$ and $A = k\langle X_1, X_2, X_3\rangle/I$ where I is the two-sided ideal generated by the defining relations

$$\begin{aligned} R_{21} &= X_2X_1 - \alpha X_1X_2 - \gamma X_2 - F_{21}, \\ R_{31} &= X_3X_1 - \frac{1}{\alpha}X_1X_3 + \frac{\gamma}{\alpha}X_3 - F_{31}, \\ R_{32} &= X_3X_2 - \beta X_2X_3 - F(X_1) - F_{32}, \end{aligned}$$

where

$$\begin{aligned} &\alpha \neq 0, \qquad \beta, \gamma \in k, \qquad F(X_1) \in k\text{-span}\big\{X_1^2, X_1, 1\big\}, \\ &F_{21}, F_{31}, F_{32} \in k\langle X_1, X_2, X_3\rangle. \end{aligned}$$

If $\alpha = \beta = 1$, $\gamma = 2$, $F(X_1) = X_1$, and $F_{21} = F_{31} = F_{32} = 0$, then $A = U(sl_2)$. Moreover, in the case where $F_{21} = F_{31} = F_{32} = 0$, the family of algebras constructed above includes many well-known deformations of $U(sl_2)$, e.g., Woronowicz's deformation of $U(sl_2)$ [Wor], Witten's deformation of $U(sl_2)$ [Wit], Le Bruyn's conformal sl_2 enveloping algebra [Le], Smith's deformation of $U(sl_2)$ where the dominant polynomial $f(t)$ has degree $\leqslant 2$ [Sm], Benkart–Roby's down-up algebra in which $\beta \neq 0$ (cf. [KMP]).

Set on $k\langle X_1, X_2, X_3\rangle$ the monomial ordering $X_3 >_{grlex} X_2 >_{grlex} X_1$. Then the only Jacobi sum determined by the defining relations of A with respect to the fixed ordering on generators is

$$\begin{aligned} \mathbf{J}(X_3, X_2, X_1) &= \{X_3, X_2\}X_1 - \lambda_{31}\lambda_{21}X_1\{X_3, X_2\} \\ &\quad - \lambda_{21}\{X_3, X_1\}X_2 + \lambda_{32}X_2\{X_3, X_1\} \\ &\quad + \lambda_{32}\lambda_{31}\{X_2, X_1\}X_3 - X_3\{X_2, X_1\} \\ &= \big(f(X_1) + F_{32}\big)X_1 - \frac{1}{\alpha}\cdot\alpha X_1\big(f(X_1) + F_{32}\big) \\ &\quad - \alpha\left(-\frac{\gamma}{\alpha}X_3 + F_{31}\right)X_2 + \beta X_2\left(-\frac{\gamma}{\alpha}X_3 + F_{31}\right) \\ &\quad + \beta\cdot\frac{1}{\alpha}(\gamma X_2 + F_{21})X_3 - X_3(\gamma X_2 + F_{21}) \\ &= F_{32}X_1 - X_1F_{32} - \alpha F_{31}X_2 + \beta X_2F_{31} + \frac{\beta}{\alpha}F_{21}X_3 - X_3F_{21}. \end{aligned}$$

Write

$$\mathcal{F} = F_{32}X_1 - X_1F_{32} - \alpha F_{31}X_2 + \beta X_2F_{31} + \frac{\beta}{\alpha}F_{21}X_3 - X_3F_{21}.$$

By Proposition 1.5, if $\mathbf{LM}(R_{ji}) = X_jX_i$ w.r.t. $\geqslant_{grlex}$, $1 \leqslant i < j \leqslant 3$, and

$$\mathcal{F} \in k\text{-span}\left\{\begin{array}{l} R_{21}, R_{31}, R_{32}, \\ X_1R_{21}, R_{21}X_1, X_2R_{21}, R_{21}X_2, R_{21}X_3, \\ X_1R_{31}, R_{31}X_1, X_2R_{31}, R_{31}X_2, R_{31}X_3, \\ X_1R_{32}, X_2R_{32} \end{array}\right\},$$

then $\{R_{21}, R_{31}, R_{32}\}$ forms a Gröbner basis with respect to $X_3 >_{grlex} X_2 >_{grlex} X_1$. Below we consider two cases:

Case I. Input in the defining relations of A the data

$$\text{(D1)} \qquad \begin{cases} \alpha = \beta \neq 0, \gamma, \mu, q, \varepsilon, \xi, \lambda, \eta_{32} \in k, \\ F(X_1) \in k\text{-span}\{X_1^2, X_1, 1\}, \\ G(X_2) \in k\text{-span}\{X_2^2, X_2, 1\}, \\ H(X_3) \in k\text{-span}\{X_3, 1\}, \\ F_{21} = \mu X_1^2 + qX_1 + H(X_3), \\ F_{31} = \varepsilon(X_1X_2 + X_2X_1) - \xi X_1^2 + \lambda X_1 + G(X_2), \\ F_{32} = (\mu(X_1X_3 + X_3X_1) - \varepsilon\alpha X_2^2) + \xi\alpha(X_1X_2 + X_2X_1) \\ \qquad\qquad - \lambda\alpha X_2 + qX_3 + \eta_{32}. \end{cases}$$

Clearly, in this case we have $\mathbf{LM}(R_{ji}) = X_jX_i$, $1 \leqslant i < j \leqslant 3$, and the conditions of Corollary 1.4 are satisfied. Moreover, a direct verification shows that

$$\begin{aligned} F_{32}X_1 - X_1F_{32} &= \mu\big(X_3X_1^2 - X_1^2X_3\big) + \varepsilon\alpha\big(X_1X_2^2 - X_2^2X_1\big) \\ &\quad + \xi\alpha\big(X_2X_1^2 - X_1^2X_2\big) + \lambda\alpha(X_1X_2 - X_2X_1) \\ &\quad + q(X_3X_1 - X_1X_3), \\ -\alpha F_{31}X_2 + \alpha X_2F_{31} &= \varepsilon\alpha\big(X_2^2X_1 - X_1X_2^2\big) + \xi\alpha\big(X_1^2X_2 - X_2X_1^2\big) \\ &\quad + \lambda\alpha(X_2X_1 - X_1X_2), \\ F_{21}X_3 - X_3F_{21} &= \mu\big(X_1^2X_3 - X_3X_1^2\big) + q(X_1X_3 - X_3X_1), \end{aligned}$$

and consequently, $J(X_3, X_2, X_1) = \mathcal{F} = 0$. By Corollary 1.4, A is a quadric solvable polynomial algebra.

Case II. Input in the defining relations of A the data (D2) which is obtained by setting $\mu = 0$ in the above (D1). As with the data (D1), one checks that in this case we also have $J(X_3, X_2, X_1) = 0$. But now A is a tame quadric solvable polynomial algebra.

Remark. By modulo the ideal I in the above cases, one may indeed obtain a set of $\{F_{21}, F_{31}, F_{32}\}$ in which each member is a linear combination of standard monomials.

(iii) *Non-polynomial central extension of deformations of* $U(sl_2)$. These are the 4-dimensional algebras defined by the relations from the free algebra $k\langle X_1, X_2, X_3, X_4\rangle$

$$\begin{aligned}
R_{21} &= X_2X_1 - \alpha X_1X_2 - \gamma X_2 - F_{21} - K_{21},\\
R_{31} &= X_3X_1 - \frac{1}{\alpha}X_1X_3 + \frac{\gamma}{\alpha}X_3 - F_{31} - K_{31},\\
R_{32} &= X_3X_2 - \alpha X_2X_3 - F(X_1) - F_{32} - K_{32},\\
R_{41} &= X_4X_1 - X_1X_4,\\
R_{42} &= X_4X_2 - X_2X_4,\\
R_{43} &= X_4X_3 - X_3X_4,
\end{aligned}$$

where $K_{21}, K_{31}, K_{32} \in k$-span$\{X_4, 1\}$ and $\{\alpha, \gamma, F(X_1), F_{21}, F_{31}, F_{32}\}$ is taken either from (D1) or from (D2) in example (ii). Since the only possible nonzero Jacobi sums determined by the above relations with respect to $X_4 >_{grlex} X_3 >_{grlex} X_2 >_{grlex} X_1$ are given by

$$\begin{aligned}
\mathbf{J}(X_3, X_2, X_1) &= K_{32}X_1 - X_1K_{32} - \alpha K_{31}X_2 + \alpha X_2K_{31} + K_{21}X_3 - X_3K_{21},\\
\mathbf{J}(X_4, X_3, X_2) &= F(X_1)X_4 - X_4F(X_1) + F_{32}X_4 - X_4F_{32},\\
\mathbf{J}(X_4, X_3, X_1) &= \frac{\gamma}{\alpha}(X_4X_3 - X_3X_4) + F_{31}X_4 - X_4F_{31},\\
\mathbf{J}(X_4, X_2, X_1) &= \gamma(X_2X_4 - X_4X_2) + F_{21}X_4 - X_4F_{21},
\end{aligned}$$

it can be further checked that they have weak Gröbner representations by $\{R_{41}, R_{42}, R_{43}\}$. Thus, Corollary 1.4 and Proposition 1.5 hold. Hence, the algebras defined by the relations given above are quadric solvable polynomial algebras.

(iv) *Deformations of* $A_n(k)$. Let $A_n(k)$ be the nth Weyl algebra over k. This example provides quadric solvable polynomial algebras which are deformations of $A_n(k)$.

Let $k\langle Y \cup X\rangle$ be the free k-algebra on $Y \cup X = \{Y_n, \dots, Y_1, X_n, \dots, X_1\}$, and set on $k\langle Y \cup X\rangle$ the monomial ordering

$$Y_n >_{grlex} X_n >_{grlex} Y_{n-1} >_{grlex} X_{n-1} >_{grlex} \cdots >_{grlex} Y_1 >_{grlex} X_1.$$

Consider the k-algebra $A = k\langle X \cup Y\rangle/I$, where I is the ideal of $k\langle X, Y\rangle$ generated by the defining relations

$$\begin{aligned}
H_{ji} &= X_jX_i - X_iX_j, && 1 \leqslant i < j \leqslant n,\\
\widetilde{H}_{ji} &= X_jY_i - Y_iX_j, && 1 \leqslant i < j \leqslant n,\\
G_{ji} &= Y_jY_i - Y_iY_j, && 1 \leqslant i < j \leqslant n,\\
\widetilde{G}_{ji} &= Y_jX_i - X_iY_j, && 1 \leqslant i < j \leqslant n,\\
R_{jj} &= Y_jX_j - q_jX_jY_j - F_{jj}, && 1 \leqslant j \leqslant n,
\end{aligned}$$

where $q_j \in k, F_{jj} \in k\langle Y \cup X\rangle$. If in the defining relations $q_j \neq 0$ and $F_{jj} = 1$, then A is the additive analogue of $A_n(k)$ introduced and studied in quantum physics [Kur,JBS]; if $q_j = q \neq 0$ and $F_{jj} = 1$, then A is the well-known algebra of q-differential operators with polynomial coefficients. A direct verification shows that the only possible nonzero Jacobi sums determined by the defining relations and the ordering given on generators are

$$\begin{aligned}
&\mathbf{J}(Y_j, X_j, X_i) = F_{jj}X_i - X_iF_{jj}, && 1 \leqslant i < j \leqslant n,\\
&\mathbf{J}(Y_j, X_j, Y_i) = F_{jj}Y_i - Y_iF_{jj}, && 1 \leqslant i < j \leqslant n,\\
&\mathbf{J}(Y_k, Y_j, X_j) = F_{jj}Y_k - Y_kF_{jj}, && 1 \leqslant j < k \leqslant n,\\
&\mathbf{J}(X_k, Y_j, X_j) = F_{jj}X_k - X_kF_{jj}, && 1 \leqslant j < k \leqslant n.
\end{aligned}$$

For $1 \leqslant j \leqslant n$, at least if

$$F_{jj} \in k\text{-span}\{X_j^2, X_j, Y_j, 1\},$$

then all conditions of Corollary 1.4 and Proposition 1.5 are satisfied, and one checks that all Jacobi sums have weak Gröbner representations. It follows that A is a tame quadric solvable polynomial algebra with $\leqslant_{grlex}$ in the case where all $q_j \neq 0$.

(v) *Deformations of Heisenberg enveloping algebra*. Let $k\langle X \cup Z \cup Y\rangle$ be the free k algebra on $X \cup Z \cup Y = \{X_n, \ldots, X_1, Z_n, \ldots, Z_1, Y_n, \ldots, Y_1\}$, $A = k\langle X \cup Z \cup Y\rangle/I$, where I is the ideal generated by the defining relations

$$\begin{aligned}
&R^x_{ji} = X_jX_i - X_iX_j, && 1 \leqslant i < j \leqslant n,\\
&R^y_{ji} = Y_jY_i - Y_iY_j, && 1 \leqslant i < j \leqslant n,\\
&R^z_{ji} = Z_jZ_i - Z_iZ_j, && 1 \leqslant i < j \leqslant n,\\
&R^{zy}_{ji} = Z_jY_i - \lambda_i^{\delta_{ji}}Y_iZ_j, && 1 \leqslant i, j \leqslant n,\\
&R^{xz}_{ji} = X_jZ_i - \mu_i^{\delta_{ji}}Z_iX_j, && 1 \leqslant i, j \leqslant n,\\
&R^{xy}_{ji} = X_jY_i - Y_iX_j, && i \neq j,\\
&R^{xy}_{ji} = X_jY_j - q_jY_jX_j - F_{jj}, && 1 \leqslant i \leqslant n,
\end{aligned}$$

where $\lambda_i, \mu_i, q_j \in k, \delta_{ji}$ is the Kronecker delta, $F_{jj} \in k\langle X \cup Z \cup Y\rangle$. If we take $\lambda_i = \mu_i = g_j = 1$, $Z_j = Z$ and $F_{jj} = Z$, $1 \leqslant j \leqslant n$, then the enveloping algebra of $(2n + 1)$-dimensional Heisenberg Lie algebra is recovered. In the case where $\lambda_i = \mu_i = q \neq 0$, $q_j = q^{-1}$, and $F_{jj} = z_j$, we recover the q-Heisenberg algebra (cf. [Ber,Ros]).

Set the monomial ordering

$$X_n >_{grlex} \cdots >_{grlex} X_1 >_{grlex} Z_n >_{grlex} \cdots >_{grlex} Z_1 >_{grlex} Y_n >_{grlex} \cdots >_{grlex} Y_1.$$

Then a direct verification shows that the only possible nonzero Jacobi sums determined by the defining relations and the ordering given on generators are

$$\begin{aligned}
&\mathbf{J}(X_k, X_j, Y_j) = F_{jj}X_k - X_kF_{jj}, && 1 \leqslant j < k \leqslant n,\\
&\mathbf{J}(X_k, X_j, Y_k) = -F_{kk}X_j + X_jF_{kk}, && 1 \leqslant j < k \leqslant n,\\
&\mathbf{J}(X_k, Z_j, Y_k) = -F_{kk}Z_j + Z_jF_{kk}, && 1 \leqslant k, j \leqslant n,\\
&\mathbf{J}(X_k, Y_k, Y_j) = F_{kk}Y_j - Y_jF_{kk}, && 1 \leqslant k < j \leqslant n,\\
&\mathbf{J}(X_j, Y_k, Y_j) = -F_{jj}Y_k + Y_kF_{jj}, && 1 \leqslant j < k \leqslant n,
\end{aligned}$$

It can be further checked that, for $1 \leqslant j \leqslant n$, at least if

$$F_{jj} \in k\text{-}\operatorname{span}\{Z_j^2, Z_j, Y_j^2, Y_j, X_j, 1\},$$

then all conditions of Corollary 1.4 and Proposition 1.5 are satisfied, and all Jacobi sums have weak Gröbner representations by the defining relations. It follows that A is a tame quadric solvable polynomial algebra with $\geqslant_{grlex}$ in the case where all $q_j \neq 0$.

(vi) *Berger's q-enveloping algebras*. Recall from [Ber] that a q-algebra $A = k[x_1, \ldots, x_n]$ over a *commutative ring k* is defined by the quadric relations

$$\begin{aligned}
R_{ji} = X_jX_i - q_{ji}X_iX_j - \{X_j, X_i\},\quad 1 \leqslant i < j \leqslant n,\ \text{where } q_{ji} \in k,\\
\text{and}\quad \{X_j, X_i\} = \sum \alpha_{ji}^{k\ell} X_kX_\ell + \sum \alpha_h X_h + c_{ji},\quad \alpha_{ji}^{k\ell}, \alpha_h, c_{ji} \in k,\\
\text{satisfying if}\quad \alpha_{ji}^{k\ell} \neq 0,\quad \text{then}\quad i < k \leqslant \ell < j,\ \text{and } k - i = j - \ell.
\end{aligned}$$

Define two k-subspaces of the free algebra $k\langle X_1, \ldots, X_n\rangle$

$$\begin{aligned}
\mathcal{E}_1 &= k\text{-Span}\{R_{ji} \mid n \geqslant j > i \geqslant 1\},\\
\mathcal{E}_2 &= k\text{-Span}\{X_iR_{ji}, R_{ji}X_i, X_jR_{ji}, R_{ji}X_j \mid n \geqslant j > i \geqslant 1\}.
\end{aligned}$$

For $1 \leqslant i < j < k \leqslant n$, if every Jacobi sum $\mathbf{J}(X_k, X_j, X_1)$ is contained in $\mathcal{E}_1 + \mathcal{E}_2$, then A is called a *q-enveloping* algebra with respect to the natural total ordering $x_n > x_{n-1} > \cdots > x_1$. A q-enveloping algebra is said to be *invertible* if in the defining relations all coefficients q_{ji} are invertible, $1 \leqslant i < j \leqslant n$. In [Ber] the q-PBW theorem was obtained for q-enveloping algebras, that is, the set of standard monomials $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}\}$ forms a k-basis for a q-enveloping algebra A. Clearly, if we set the monomial ordering $X_n >_{grlex} X_{n-1} >_{grlex} \cdots >_{grlex} X_1$, then the defining relations of a q-algebra A satisfy

$$\begin{aligned}
&\mathbf{LM}(R_{ji}) = X_jX_i,\quad 1 \leqslant i < j \leqslant n,\quad \text{and}\\
&k, \ell < j\quad \text{in } \sum \alpha_{ji}^{k\ell} X_kX_\ell\quad \text{whenever } \alpha_{ji}^{k\ell} \neq 0.
\end{aligned}$$

Hence, by Proposition 1.3 and Corollary 1.4, the defining relations of a q-enveloping algebra form a Gröbner basis in $k\langle X_1, \ldots, X_n\rangle$; if furthermore A is an invertible q-enveloping algebra then A is a tame quadric solvable polynomial algebra.

We observe that the conditions $i < k \leqslant \ell < j$ and $k - i = j - \ell$ the definition of a q-algebra are not necessarily satisfied by a quadric solvable polynomial algebra, or more generally, a quadric algebra characterized by Corollary 1.4 is not necessarily a q-enveloping algebra in the sense of [Ber].

Remark. In the end of first part of [LWZ], it was pointed out that a q-enveloping algebra over a field k is generally not a solvable polynomial algebra with respect to $\geqslant_{grlex}$. This is, of course, not true for invertible q-enveloping algebras, as argued in the above example (vi). The author takes this place to correct that incorrect remark.

We finish this section with more quadratic solvable polynomial algebras associated to given quadric solvable polynomial algebras.

Let $A = k[a_1, \ldots, a_n]$ be a finitely generated algebra. Consider the *standard filtration* FA on A:

$$k = F_0A \subset F_1A \subset F_2A \subset \cdots \subset F_pA \subset \cdots$$

where for each $p \in \mathbb{Z}_{\geqslant 0}$, $F_p = k\text{-span}\{a_{i_1}^{\alpha_1} a_{i_2}^{\alpha_2} \cdots a_{i_n}^{\alpha_n} \mid \alpha_1 + \cdots + \alpha_n \leqslant p\}$. With respect to FA, the associated graded algebra of A is defined as $G(A) = \bigoplus_{p \in \mathbb{Z}_{\geqslant 0}} G(A)_p$ with $G(A)_p = F_pA/F_{p-1}A$, and the (graded) Rees algebra of A is defined as $\widetilde{A} = \bigoplus_{p\in\mathbb{Z}_{\geqslant 0}} \widetilde{A}_p$ with $\widetilde{A}_p = F_pA$. If $a \in F_pA - F_{p-1}A$, then we say that a has *degree* p and write $\sigma(a)$, respectively $\tilde{a}$, for the image of a in $G(A)_p$, respectively the homogeneous element of degree p in $\widetilde{A}_p = F_pA$ represented by a. Moreover, we let X stand for the homogeneous element of degree 1 in $\widetilde{A} = F_1A$ represented by 1. Then $A \cong \widetilde{A}/(1-X)\widetilde{A}$, $G(A) = \widetilde{A}/X\widetilde{A}$. The results presented in the next proposition are modifications of [LW, Theorems 3.1 and 3.5] and [LWZ, Theorems 2.3.1 and 2.3.3].

Proposition 1.6. *Let $A = k[a_1, \ldots, a_n]$ be a quadric solvable polynomial algebra with $\succeq_{gr}$, and let FA, $G(A)$, and $\widetilde{A}$ be as defined above. The following holds.*

(i) *$G(A) = k[\sigma(a_1), \ldots, \sigma(a_n)]$ with the k-basis*

$$\mathbf{SM}\big(G(A)\big) = \big\{\sigma(a_1)^{\alpha_1}\sigma(a_2)^{\alpha_2}\cdots\sigma(a_n)^{\alpha_n} \mid (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 0}\big\}$$

is a quadratic solvable polynomial algebra. $\widetilde{A} = k[\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n, X]$ with the k-basis $\mathbf{SM}(A) = \{\tilde{a}_1^{\alpha_1}\tilde{a}_2^{\alpha_2}\cdots\tilde{a}_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}\}$ is a quadratic solvable polynomial algebra. In particular, if A is tame then so is $G(A)$.

(ii) *If A is defined by the defining relations*

$$R_{ji} = X_jX_i - \lambda_{ji}X_iX_j - \sum \lambda_{ji}^{k\ell}X_kX_\ell - \sum \lambda_hX_h - c_{ji}, \quad 1 \leqslant i < j \leqslant n,$$

with respect to the graded lex ordering $X_n >_{grlex} \cdots >_{grlex} X_1$ such that $\mathbf{LM}(R_{ji}) = X_j X_i$, $1 \leqslant i < j \leqslant n$, then $G(A)$ has the quadratic defining relations

$$\sigma(R_{ji}) = X_j X_i - \lambda_{ji} X_i X_j - \sum \lambda_{ji}^{k\ell} X_k X_\ell, \quad 1 \leqslant i < j \leqslant n,$$

which form a Gröbner basis in the free algebra $k\langle X_1, \ldots, X_n\rangle$ with respect to $X_n >_{grlex} \cdots >_{grlex} X_1$; and $\widetilde{A}$ has the quadratic defining relations

$$TX_i - X_i T, \quad 1 \leqslant i \leqslant n,$$
$$\widetilde{R}_{ji} = X_j X_i - \lambda_{ji} X_i X_j - \sum \lambda_{ji}^{k\ell} X_k X_\ell - \sum \lambda_h X_h T - c_{ji} T^2, \quad 1 \leqslant i < j \leqslant n,$$

which form a Gröbner basis in the free algebra $k\langle X_1, \ldots, X_n, T\rangle$ with respect to $X_n >_{grlex} \cdots >_{grlex} X_1 >_{grlex} T$.

Remark. One may see that some of the quadric solvable polynomial algebras constructed in this section are tame and some of them are iterated skew polynomial algebras starting with the ground field. From the presentation that we give it appears that other examples above are not tame, and they are not iterated skew polynomial extensions over k. However our methods do not rule out the possibility that some other presentation might show that these algebras are tame or iterated skew polynomial algebras over k.

2. Tame case: A is completely constructable and Auslander regular with $K_0(A) \cong \mathbb{Z}$

In this section we derive that every tame quadric solvable polynomial algebra (Definition 1.2) is completely constructable (in the sense of Theorem 2.1 below) and Auslander regular with K_0-group $\mathbb{Z}$. Notation is maintained as in Section 1.

Theorem 2.1. *Let $k\langle X\rangle = k\langle X_1, \ldots, X_n\rangle$ be the free k-algebra on $X = \{X_1, \ldots, X_n\}$. Set on $k\langle X\rangle$ the graded lexicographic monomial ordering $X_n >_{grlex} X_{n-1} >_{grlex} \cdots >_{grlex} X_1$, and let I be the ideal of $k\langle X\rangle$ generated by the R_{ji}, where*

$$R_{ji} = X_j X_i - \lambda_{ji} X_i X_j - \sum_{k \leqslant \ell < j} \lambda_{ji}^{k\ell} X_k X_\ell - \sum \lambda_h X_h - c_{ji}, \quad 1 \leqslant i < j \leqslant n.$$

Suppose that $\{R_{ji} \mid 1 \leqslant i < j \leqslant n\}$ forms a Gröbner basis in $k\langle X\rangle$ with respect to $>_{grlex}$. Then $B = k\langle X\rangle/I$ is a tame quadric solvable polynomial algebra with respect to the graded lexicographic monomial ordering $x_n >_{grlex} x_{n-1} >_{grlex} \cdots >_{grlex} x_1$, where each x_i is the image of X_i in B.

Conversely, let $A = k[a_1, \ldots, a_n]$ be a tame quadric solvable polynomial algebra with some graded monomial ordering $\succeq_{gr}$. Then A is isomorphic to a k-algebra of type B with $>_{grlex}$, as described above.

Thus, we may say that tame quadric solvable polynomial algebras are completely constructable.

Proof. That $B = k\langle X\rangle/I$ is a tame quadric solvable polynomial algebra with respect to the graded lexicographic monomial ordering $x_n >_{grlex} x_{n-1} >_{grlex} \cdots >_{grlex} x_1$ follows from the given defining relations R_{ji}, Proposition 1.3 and Corollary 1.4 immediately.

The converse follows from the definition of a tame quadric solvable polynomial algebra, the observation made after Definition 1.2, Proposition 1.3 and Corollary 1.4. $\square$

Let B be a k-algebra. Recall that B is said to be *Auslander regular* if B is left and right Noetherian with finite global dimension, and for every finitely generated B-module M, every $i \geqslant 0$, and every submodule $N \subset \mathrm{Ext}^i_B(M,B)$, the inequality $j_B(N) \geqslant i$ holds, where $j_B(N)$ is the smallest integer k such that $\mathrm{Ext}^k_B(N,B) \neq 0$. Also recall that if B has K_0-group $\mathbb{Z}$ then every finitely generated (left) B-module has a finite free resolution.

Theorem 2.2. *Let $A = k[a_1,\ldots,a_n]$ be a tame quadric solvable polynomial algebra with some graded monomial ordering $\succeq_{gr}$ and the k-basis $\mathbf{SM}(A) = \{a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_n^{\alpha_n} \mid (a_1,\ldots,a_n) \in \mathbb{Z}^n_{\geqslant 0}\}$. Let $G(A)$ and $\widetilde{A}$ be the associated graded algebra and Rees algebra of A with respect to the standard filtration FA on A as defined in Section* 1. *Then A, $G(A)$, and $\widetilde{A}$ are Auslander regular domains with K_0-group $\mathbb{Z}$.*

Proof. By the definition of a tame quadric solvable polynomial algebra and Proposition 1.6, the generators of $G(A) = k[\sigma(a_1),\ldots,\sigma(a_n)]$ satisfy the quadratic relations

$$\sigma(a_j)\sigma(a_i) = \lambda_{ji}\sigma(a_i)\sigma(a_j) + \sum_{k,\ell<j}\lambda_{ji}^{k\ell}\sigma(a_k)\sigma(a_\ell), \quad 1 \leqslant i < j \leqslant n,$$

and $G(A)$ has the k-basis

$$\mathbf{SM}\big(G(A)\big) = \big\{\sigma(a_1)^{\alpha_1}\sigma(a_2)^{\alpha_2}\cdots\sigma(a_n)^{\alpha_n} \mid (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n_{\geqslant 0}\big\}.$$

Consequently, the above defining relations determine an iterated skew polynomial algebra structure starting with the polynomial algebra $k[\sigma(x_1)]$. Therefore, $G(A)$ is an Auslander regular domain. It follows from [Li1,LV01,LV02] that A and $\widetilde{A}$ are Auslander regular domains, and it follows from the K_0-part of Quillen's theorem [Qui, Theorem 7] that

$$\mathbb{Z} \xrightarrow{\cong} K_0(k) \begin{array}{l} \overset{=}{\nearrow} \\ \xrightarrow{=} \\ \overset{=}{\searrow} \end{array} \begin{array}{l} K_0(F_0A) \cong K_0(A) \\ K_0\big(G(A)_0\big) \cong K_0\big(G(A)\big), \\ K_0(\widetilde{A}_0) \cong K_0(\widetilde{A}_0) \end{array}$$

as desired. $\square$

3. The $\succeq_{gr}$-filtration on modules

As remarked in the end of Section 1, it seems very hard to know whether every quadric solvable polynomial algebra could be tame or not. To study the regularity and K_0-group of an arbitrary quadric solvable polynomial algebra $A = k[a_1, \ldots, a_n]$ with respect to a graded monomial ordering $\succeq_{gr}$, in this section we introduce the $\succeq_{gr}$-filtration on A-modules and discuss the $\succeq_{gr}$-filtered homomorphisms and the associated $\mathbb{Z}^n_{\geqslant 0}$-graded homomorphisms.

First recall from [Li] the definition and some basic properties of the $\succeq_{gr}$-filtration $\mathcal{F}A$ on A. Let $\mathbf{SM}(A) = \{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}\}$ the standard k-basis of A. For each $\alpha \in \mathbb{Z}^n_{\geqslant 0}$, construct the k-subspace

$$\mathcal{F}_\alpha A = k\text{-span}\bigl\{a^\beta \in \mathbf{SM}(A) \mid \alpha \succeq_{gr} \beta\bigr\}.$$

Clearly, if $\alpha \succ_{gr} \gamma$, then $\mathcal{F}_\gamma A \subset \mathcal{F}_\alpha A$. Thus, since $\succeq_{gr}$ is a graded monomial ordering, we have a $\mathbb{Z}^n_{\geqslant 0}$-filtration of k-subspaces on A satisfying

(1) $1 \in \mathcal{F}_0 A = k$,
(2) every $\mathcal{F}_\alpha A$ is a *finite* dimensional k-space, and $A = \bigcup_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} \mathcal{F}_\alpha A$,
(3) $\mathcal{F}_\alpha A \cdot \mathcal{F}_\beta A \subset \mathcal{F}_{\alpha+\beta} A$.

To emphasize the role of $\succeq_{gr}$ in our discussion, this filtration $\mathcal{F}A$ is called the $\succeq_{gr}$-*filtration*. Note that $\alpha \succeq_{gr} 0 = (0, \ldots, 0)$ for all $\alpha \in \mathbb{Z}^n_{\geqslant 0}$, and the feature of a graded monomial ordering yields that, for each $\alpha \in \mathbb{Z}^n_{\geqslant 0}$, there exists

$$\alpha^* = \max\bigl\{\gamma \in \mathbb{Z}^n_{\geqslant 0} \mid \alpha \succ_{gr} \gamma\bigr\}.$$

Then we have a well-defined $\mathbb{Z}^n_{\geqslant 0}$-graded algebra

$$G^{\mathcal{F}}(A) = \bigoplus_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} G^{\mathcal{F}}(A)_\alpha \quad \text{with } G^{\mathcal{F}}(A)_\alpha = \mathcal{F}_\alpha A / \mathcal{F}_{\alpha^*} A,$$

where the addition is given by the componentwise addition and the multiplication is given by

$$\begin{aligned} G^{\mathcal{F}}(A)_\alpha \times G^{\mathcal{F}}(A)_\beta &\longrightarrow G^{\mathcal{F}}(A)_{\alpha+\beta} \\ (\bar{f}, \bar{g}) &\longmapsto \overline{fg} \end{aligned}$$

where, if $f \in \mathcal{F}_\alpha A$, then $\bar{f}$ stands for the image of f in $G^{\mathcal{F}}(A)_\alpha = \mathcal{F}_\alpha A / \mathcal{F}_{\alpha^*} A$. $G^{\mathcal{F}}(A)$ is called the *associated graded algebra* of A with respect to $\mathcal{F}A$.

For an element $f \in \mathcal{F}_\alpha A - F_{\alpha^*} A$, we say that f has *degree* α and write $\sigma(f)$ for the image of f in $G^{\mathcal{F}}(A)_\alpha$. Recalling the conventional correspondence made in Section 1:

$$a_i \leftrightarrow (\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0) = e_i \in \mathbb{Z}^n_{\geqslant 0}, \quad 1 \leqslant i \leqslant n,$$

we see that $0 \neq \sigma(a_i) \in G^{\mathcal{F}}(A)_{e_i}$, and it is not hard to see that, for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}$ and $a^\alpha \in \mathbf{SM}(A)$,

$$\sigma(a_1)^{\alpha_1} \cdots \sigma(a_n)^{\alpha_n} = \sigma\left(a_1^{\alpha_1} \cdots a_n^{\alpha_n}\right) = \sigma\left(a^\alpha\right).$$

Hence, for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}$,

$$G^{\mathcal{F}}(A)_\alpha = k\text{-span}\left\{\sigma(a_1)^{\alpha_1} \cdots \sigma(a_n)^{\alpha_n}\right\} \quad \text{(i.e., a 1-dimensional space)}.$$

If the quadric relations satisfied by the generators of A are

$$a_j a_i = \lambda_{ji} a_i a_j + \sum_{k \leqslant \ell} \lambda_{ji}^{k\ell} a_k a_\ell + \sum \lambda_h a_h + c_{ji}, \quad 1 \leqslant i < j \leqslant n, \tag{$*$}$$

where λ_{ji}, $\lambda_{ji}^{k\ell}$, λ_h, $c_{ji} \in k$, and $\lambda_{ji} \neq 0$, then we obtain the following basic properties of $G^{\mathcal{F}}(A)$.

Proposition 3.1. *With notation as above, the following holds.*

(i) *$G^{\mathcal{F}}(A)$ is a $\mathbb{Z}^n_{\geqslant 0}$-graded k-algebra generated by $\sigma(a_1), \dots, \sigma(a_n)$, i.e., $G^{\mathcal{F}}(A) = k[\sigma(a_1), \dots, \sigma(a_n)]$, and the generators of $G^{\mathcal{F}}(A)$ satisfy*

$$\sigma(a_j)\sigma(a_i) = \lambda_{ji}\sigma(a_i)\sigma(a_j), \quad \lambda_{ji} \neq 0, \quad 1 \leqslant i < j \leqslant n.$$

(ii) *The set of homogeneous elements* (*monomials*)

$$\sigma\left(\mathbf{SM}(A)\right) = \left\{\sigma(a_1)^{\alpha_1} \cdots \sigma(a_n)^{\alpha_n} \mid (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}\right\}$$

forms a k-basis for $G^{\mathcal{F}}A$.

(iii) *$G^{\mathcal{F}}(A)$ is an iterated skew polynomial algebra starting with the ground field k. Consequently, $G^{\mathcal{F}}(A)$ is an Auslander regular domain of global dimension n and $K_0(G^{\mathcal{F}}(A)) \cong \mathbb{Z}$.*

Proof. (i) and (ii) follow from [Li, Proposition 2.1]. That $G^{\mathcal{F}}(A)$ is an iterated skew polynomial algebra starting with the ground field k follows from parts (i)–(ii), and the rest of (iii) have been well-known facts about an iterated skew polynomial algebra. $\square$

Now, we turn to modules.

Definition 3.2. Let M be a (left) A-module. M is said to be a *$\succeq_{gr}$-filtered A-module* if there is a family $\mathcal{F}M = \{\mathcal{F}_\alpha M\}_{\alpha \in \mathbb{Z}^n_{\geqslant 0}}$ consisting of k-subspaces $\mathcal{F}_\alpha M$ of M such that

(a) $\bigcup_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} \mathcal{F}_\alpha M = M$,
(b) $F_\beta M \subset \mathcal{F}_\alpha M$ if $\alpha >_{gr} \beta$, and
(c) $\mathcal{F}_\alpha A \mathcal{F}_\beta M \subset \mathcal{F}_{\alpha+\beta} M$ for all $\alpha, \beta \in \mathbb{Z}^n_{\geqslant 0}$.

$\mathcal{F}M$ is called a $\succeq_{gr}$-*filtration* on M.

Since for each $\alpha \in \mathbb{Z}^n_{\geqslant 0}$ there is $\alpha^* = \max\{\gamma \in \mathbb{Z}^n_{\geqslant 0} \mid \alpha \succ_{gr} \gamma\}$, to be convenient, for the least element $0 = (0, \ldots, 0) \in \mathbb{Z}^n_{\geqslant 0}$, we set $F_{0^*}M = \{0\}$ in every $\mathcal{F}M$.

If M is a $\succeq_{gr}$-filtered A-module with $\succeq_{gr}$-filtration $\mathcal{F}M$, then the *associated graded* $G^{\mathcal{F}}(A)$-modules of M is defined as the $\mathbb{Z}^n_{\geqslant 0}$-graded additive group

$$G^{\mathcal{F}}(M) = \bigoplus_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} G^{\mathcal{F}}(M)_\alpha \quad \text{with } G^{\mathcal{F}}(M)_\alpha = F_\alpha M / F_{\alpha^*} M$$

on which the module action of $G^{\mathcal{F}}(A)$ is given by

$$\begin{aligned} G^{\mathcal{F}}(A)_\alpha \times G^{\mathcal{F}}(M)_\beta &\longrightarrow G^{\mathcal{F}}(M)_{\alpha+\beta} \\ (\bar{f}, \overline{m}) &\longmapsto \overline{fm} \end{aligned}$$

where, if $f \in \mathcal{F}_\alpha A$, respectively if $m \in \mathcal{F}_\beta M$, then $\bar{f}$ stands for the image of $f \in G^{\mathcal{F}}(A)_\alpha = \mathcal{F}_\alpha A / \mathcal{F}_{\alpha^*} A$, respectively $\overline{m}$ stands for the image of m in $G^{\mathcal{F}}(M)_\beta = \mathcal{F}_\beta M / \mathcal{F}_{\beta^*} M$. A $\succeq_{gr}$-filtration $\mathcal{F}M$ has the property that if $0 \neq m \in M$, then there is $\alpha \in \mathbb{Z}^n_{\geqslant 0}$ such that $m \in \mathcal{F}_\alpha M - \mathcal{F}_{\alpha^*} M$. In this case we call α the *degree* of m and write $\sigma(m)$ for its corresponding homogeneous element in $G^{\mathcal{F}}(M)_\alpha$.

Before dealing with the associated $\mathbb{Z}^n_{\geqslant 0}$-graded $G^{\mathcal{F}}(A)$-module $G^{\mathcal{F}}(M)$ of a $\succeq_{gr}$-filtered A-module M with $\succeq_{gr}$-filtration $\mathcal{F}M$, we first note that, for $\alpha, \beta \in \mathbb{Z}^n_{\geqslant 0}$ with $\alpha \succ_{gr} \beta$, the equation $\alpha = \beta + x$ does not necessarily have a solution in $\mathbb{Z}^n_{\geqslant 0}$. In particular, even if for $\alpha \succ_{gr} \beta$ and $\alpha^* \succ_{gr} \beta$, by the definition of α^*, the equations

$$\alpha = \beta + x \quad \text{and} \quad \alpha^* = \beta + y$$

may not have solutions in $\mathbb{Z}^n_{\geqslant 0}$ simultaneously. This makes the $\succeq_{gr}$-filtrations behave quite different from $\mathbb{Z}$-filtrations. To remedy this defect, let us put

$$[0, \alpha] = \left\{\gamma \in \mathbb{Z}^n_{\geqslant 0} \mid \alpha \succeq_{gr} \gamma\right\}.$$

Then clearly, $\alpha^* = \max\{[0, \alpha] - \{\alpha\}\}$.

Lemma 3.3. *Let $\alpha, \eta \in \mathbb{Z}^n_{\geqslant 0}$ be such that $\alpha = \eta + \gamma$ for some $\gamma \in \mathbb{Z}^n_{\geqslant 0}$. For any $\beta \in [0, \alpha^*]$, if $\beta = \eta + \delta$ for some $\delta \in \mathbb{Z}^n_{\geqslant 0}$, then $\gamma^* \succeq_{gr} \delta$; if $\beta = \alpha^*$, then $\delta = \gamma^*$.*

Proof. Note that $\succeq_{gr}$ is a monomial ordering. The first conclusion is then clear by the definition of a $*$-element. Suppose $\alpha^* = \eta + \delta$. Then, $\gamma \succ_{gr} \gamma^*$ implies $\alpha = \eta + \gamma \succeq_{gr} \eta + \gamma^*$. This, in turn, implies $\eta + \delta = \alpha^* \succeq_{gr} \eta + \gamma^*$, and hence $\delta \succeq_{gr} \gamma^*$. Combining the first conclusion, we conclude that $\delta = \gamma^*$. $\square$

Proposition 3.4. *Let M be an A-module.*

(i) *If M has a $\succeq_{gr}$-filtration $\mathcal{F}M$ such that $G^{\mathcal{F}}(M) = \sum_{i\in J} G^{\mathcal{F}}(A)\sigma(\xi_i)$ with $\xi_i \in M$ and $\deg\sigma(\xi_i) = \alpha(i) \in \mathbb{Z}^n_{\geqslant 0}$, then $M = \sum_{i\in J} A\xi_i$. In particular, if $G^{\mathcal{F}}(M)$ is finitely generated then so is M.*

(ii) *If M is finitely generated, then M has a $\succeq_{gr}$-filtration $\mathcal{F}M$ such that $G^{\mathcal{F}}(M)$ is finitely generated over $G^{\mathcal{F}}(A)$.*

Proof. (i) Since $G^{\mathcal{F}}(M) = \sum_{i\in J} G^{\mathcal{F}}(A)\sigma(\xi_i)$ with $\xi_i \in M$ and $\deg\sigma(\xi_i) = \alpha(i) \in \mathbb{Z}^n_{\geqslant 0}$, we have

$$G^{\mathcal{F}}(M)_\alpha = \sum_{\substack{i\in J\\ \beta(i)+\alpha(i)=\alpha}} G^{\mathcal{F}}(A)_{\beta(i)}\sigma(\xi_i), \quad \alpha \in \mathbb{Z}^n_{\geqslant 0}.$$

Thus, for any $m \in \mathcal{F}_\alpha M$, $m = \sum a_{\beta(i)}\xi_i + m'$, where $a_{\beta(i)} \in \mathcal{F}_{\beta(i)}A$ with $\beta(i) + \alpha(i) = \alpha$, $m' \in \mathcal{F}_{\alpha^*}M$. Similarly we have $m' = \sum a'_{\gamma(i)}\xi + m''$, where $\alpha'_{\gamma(i)} \in \mathcal{F}_{\gamma(i)}A$ with $\gamma + \alpha(i) = \alpha^*$, $m'' \in \mathcal{F}_{\alpha^{**}}M$. Since $\alpha \succeq_{gr} \alpha^* \succeq_{gr} \alpha^{**}$ and $\succeq_{gr}$ is a graded monomial ordering, after a finite number of repetition of the above procedure, we arrive at

$$m \in \sum_{i\in J}\Biggl(\sum_{\substack{\gamma\in[0,\alpha]\\ \gamma(i)+\alpha(i)=\gamma}} \mathcal{F}_{\gamma(i)}A\Biggr)\xi_i$$

and it follows that

$$\mathcal{F}_\alpha M = \sum_{i\in J}\Biggl(\sum_{\substack{\gamma\in[0,\alpha]\\ \gamma(i)+\alpha(i)=\gamma}} \mathcal{F}_{\gamma(i)}A\Biggr)\xi_i$$

because $\xi_i \in \mathcal{F}_{\alpha(i)}M$, $i \in J$. Hence $M = \sum_{i\in J} A\xi_i$.

(ii) Suppose $M = \sum_{i=1}^{s} A\xi_i$ and $\{\xi_1, \ldots, \xi_s\}$ is a *minimal* set of generators for M. Choose $\alpha(1), \ldots, \alpha(s) \in \mathbb{Z}^n_{\geqslant 0}$ arbitrarily and set

$$\sum_{\substack{\gamma\in[0,\alpha]\\ \gamma(i)+\alpha(i)=\gamma}} \mathcal{F}_{\gamma(i)}A = \{0\}$$

if $\gamma = \alpha(i) + x$ has no solution for any $\gamma \in [0, \alpha]$. Then, it is easy to see that the family $\mathcal{F}M$ consisting of

$$\mathcal{F}_\alpha M = \sum_{i=1}^{s}\Biggl(\sum_{\substack{\gamma\in[0,\alpha]\\ \gamma(i)+\alpha(i)=\gamma}} \mathcal{F}_{\gamma(i)}A\Biggr)\xi_i, \quad \alpha \in \mathbb{Z}^n_{\geqslant 0},$$

is a $\succeq_{gr}$-filtration on M, where $\xi_i \in \mathcal{F}_{\alpha(i)}M - \mathcal{F}_{\alpha(i)^*}M$, i.e., $\deg \xi_i = \alpha(i)$, $i = 1, \ldots, s$. And by Lemma 3.3, it can be verified directly that $G^{\mathcal{F}}(M) = \bigoplus_{i=1}^{s} G^{\mathcal{F}}(A)\sigma(\xi_i)$ with

$$G^{\mathcal{F}}(M)_\alpha = \sum_{\substack{1\leqslant i\leqslant s\\ \gamma(i)+\alpha(i)=\alpha}} G^{\mathcal{F}}(A)_{\gamma(i)}\sigma(\xi_i), \quad \alpha \in \mathbb{Z}^n_{\geqslant 0}. \qquad \Box$$

Let M and N be $\succeq_{gr}$-filtered A-modules with $\succeq_{gr}$-filtrations $\mathcal{F}M$ and $\mathcal{F}N$, respectively. An A-module homomorphism $\varphi : M \to N$ is said to be a $\succeq_{gr}$-filtered homomorphism, if $\varphi(\mathcal{F}_\alpha M) \subset \mathcal{F}_\alpha N$ for all $\alpha \in \mathbb{Z}^n_{\geqslant 0}$. A $\succeq_{gr}$-filtered homomorphism φ is said to be *strict* if

$$\varphi(\mathcal{F}_\alpha M) = \varphi(M) \cap \mathcal{F}_\alpha N, \quad \alpha \in \mathbb{Z}^n_{\geqslant 0}.$$

If M is a $\succeq_{gr}$-filtered A-module with $\succeq_{gr}$-filtration $\mathcal{F}M$, and if $N \subset M$ is an A-submodule of M, then N has the $\succeq_{gr}$-filtration $\mathcal{F}N$ consisting of

$$\mathcal{F}_\alpha N = \mathcal{F}_\alpha M \cap N, \quad \alpha \in \mathbb{Z}^n_{\leqslant 0},$$

and the quotient A-module M/N has the $\succeq_{gr}$-filtration $\mathcal{F}(M/N)$ consisting of

$$\mathcal{F}_\alpha(M/N) = (\mathcal{F}_\alpha M + N)/N, \quad \alpha \in \mathbb{Z}^n_{\geqslant 0}.$$

The $\succeq_{gr}$-filtrations $\mathcal{F}N$ and $\mathcal{F}(M/N)$ defined above are called the *induced $\succeq_{gr}$-filtration* on N and M/N, respectively. With respect to the induced filtration on N and M/N, the inclusion map $N \to M$ and the natural map $M \to M/N$ are strict $\succeq_{gr}$-filtered homomorphisms.

If $\varphi : M \to N$ is $\succeq_{gr}$-filtered A-homomorphism, then φ induces naturally a $\mathbb{Z}^n_{\geqslant 0}$ graded $G^{\mathcal{F}}(A)$-module homomorphism:

$$G^{\mathcal{F}}(\varphi):\ G^{\mathcal{F}}(M) = \bigoplus_{\alpha\in\mathbb{Z}^n_{\geqslant 0}} G^{\mathcal{F}}(M)_\alpha \longrightarrow \bigoplus_{\alpha\in\mathbb{Z}^n_{\geqslant 0}} G^{\mathcal{F}}(N)_\alpha = G^{\mathcal{F}}(N)$$
$$\sum \overline{m} \longmapsto \sum \overline{\varphi(m)}.$$

Proposition 3.5. *Let*

$$K \xrightarrow{\varphi} M \xrightarrow{\psi} N \tag{$*$}$$

be a sequence of $\succeq_{gr}$-filtered A-modules and $\succeq_{gr}$-filtered homomorphisms such that $\psi \circ \varphi = 0$. Then

$$G^{\mathcal{F}}(K) \xrightarrow{G^{\mathcal{F}}(\varphi)} G^{\mathcal{F}}(M) \xrightarrow{G^{\mathcal{F}}(\psi)} G^{\mathcal{F}}(N) \tag{$G^{\mathcal{F}}(*)$}$$

is an exact sequence of $\mathbb{Z}^n_{\geqslant 0}$-graded $G^{\mathcal{F}}(K)$-modules and $\mathbb{Z}^n_{\geqslant 0}$-graded homomorphisms if and only if $(*)$ is exact and φ, ψ are strict.

Proof. First suppose that $(*)$ is exact and φ, ψ are strict. If $G^F(\psi)(\overline{m}) = 0$ with $m \in \mathcal{F}_\alpha M - \mathcal{F}_{\alpha^*}M$, then $0 = \overline{\psi(m)} \in G^{\mathcal{F}}(N)_\alpha$, i.e., $\psi(m) \in \mathcal{F}_{\alpha^*}N \cap \psi(M) = \psi(\mathcal{F}_{\alpha^*}M)$. Thus, $\psi(m) = \psi(m')$ for some $m' \in \mathcal{F}_{\alpha^*}M$, and hence $m - m' \in \operatorname{Ker}\psi \cap \mathcal{F}_\alpha M = \varphi(K) \cap F_\alpha M = \varphi(\mathcal{F}_\alpha K)$. Let $m = m' = \varphi(k)$ for some $k \in \mathcal{F}_\alpha K$. Then $\overline{m} = \overline{m - m'} = \overline{\varphi(k)} = G^{\mathcal{F}}(\varphi)(k)$. This shows that $\operatorname{Ker} G^{\mathcal{F}}(\psi) = G^{\mathcal{F}}(\varphi)(G^{\mathcal{F}}(K))$, i.e., the graded sequence is exact.

Conversely, suppose that the graded sequence $G^{\mathcal{F}}(*)$ is exact. To show the strictness of ψ, let $f \in \mathcal{F}_\alpha N \cap \psi(M)$ and $f \notin \mathcal{F}_{\alpha^*}N$. Then $f = \psi(m)$ for some $m \in \mathcal{F}_\beta M$ where $\beta \succeq_{gr} \alpha$. If $\beta = \alpha$, then $f = \psi(m) \in \psi(\mathcal{F}_\alpha M)$. If $\beta \succeq_{gr} \alpha$, then since $f \in \mathcal{F}_\alpha N$, we have $G^{\mathcal{F}}(\psi)(\overline{m}) = \overline{\psi(m)} = 0$ in $G^{\mathcal{F}}(N)$. By the exactness, $\overline{m} = G^{\mathcal{F}}(\varphi)(\bar{k}) = \overline{\varphi(k)}$ for some $k \in \mathcal{F}_\beta K$. Put $m' = m - \varphi(k)$. Then $m' \in \mathcal{F}_{\beta^*}M$, and $\psi(m') = \psi(m - \varphi(k)) = \psi(m) = f$. Note that the chain

$$\beta \succ_{gr} \beta^* \succ_{gr} \beta^{**} \succ_{gr} \cdots \succ_{gr} \alpha$$

has finite length in $\mathbb{Z}^n_{\geqslant 0}$. It follows that, after a finite number of repetition of the above procedure, we have $f = \psi(m_\alpha) \in \psi(\mathcal{F}_\alpha M)$. This shows that $\mathcal{F}_\alpha N \cap \psi(M) \subset \psi(\mathcal{F}_\alpha M)$, i.e., ψ is strict. A similar argument do reach the strictness of φ and the exactness of $(*)$. □

Corollary 3.6. *Let $\varphi : M \to N$ be a $\succeq_{gr}$-filtered A-homomorphism. Then $G^{\mathcal{F}}(\varphi)$ is injective, respectively surjective, if and only if (φ) is injective, respectively surjective, and φ is strict.*

4. General case: gl. dim $A \leqslant n$

Let $A = k[a_1, \dots, a_n]$ be an arbitrary quadric solvable polynomial algebra with $\succeq_{gr}$ as defined in Section 1, and let $\mathcal{F}A$ be the $\succeq_{gr}$-filtration on A as defined in Section 3. With the preparation made in Section 3, we proceed to show gl.dim $A \leqslant n$ in the present section.

First recall a well-known result concerning graded projective modules over a (semi) group-graded ring (e.g., [NVO]). Let G be an additive (semi)group and $S = \bigoplus_{g\in G} S_g$ a G-graded ring. A *graded free* A-module is a free S-module $T = \bigoplus_{i\in J} Se_i$ on the basis $\{e_i\}_{i\in J}$, which is also G-graded such that each e_i is homogeneous, i.e., if $\deg(e_i) = g_i$, then $T = \bigoplus_{g\in G} T_g$ with $T_g = \bigoplus_{h_i+g_i=g} S_{h_i}e_i$. For any graded S-module $M = \bigoplus_{g\in G} M_g$, there is a graded free S-module $T = \bigoplus_{g\in G} T_g$ and a graded surjective S-homomorphism $\varphi : T \to M$. If T is a graded free S-module and P is a graded S-module such that $T = P \oplus Q$ for some graded S-module Q with the property that $T_g = P_g + Q_g$ for all $g \in G$, then P is called a *graded projective* S-module.

Proposition 4.1. *Let G be a* (*semi*)*group, S a G-graded ring and P a graded* (*left*) *S-module. The following statements are equivalent.*

(i) *P is a graded projective S-module.*

(ii) *Given any exact sequence of graded S-modules and graded S-homomorphisms* $M \xrightarrow{\psi} N \to 0$, *if* $P \xrightarrow{\alpha} N$ *is a graded S-homomorphism, then there exists a unique graded homomorphism* $P \xrightarrow{\varphi} M$ *making the following diagram commute*:

$$\begin{tikzcd} & P \arrow[dl, "\varphi"'] \arrow[d, "\alpha"] & \\ M \arrow[r, "\psi"'] & N \arrow[r] & 0 \end{tikzcd}$$

(iii) *P is projective as an* (*ungraded*) *S-module.*

Return to modules over the quadric solvable polynomial algebra A. Let $L = \bigoplus_{i\in J} Ae_i$ be a free A-module on the basis $\{e_i\}_{i\in J}$. In view of Lemma 3.3 and the proof of Proposition 3.4, if $\alpha(i) \in \mathbb{Z}^n_{\geqslant 0}$ are arbitrarily chosen for $i \in J$, we may define a $\succeq_{gr}$-filtration $\mathcal{F}L$ on L:

$$\mathcal{F}_\alpha L = \bigoplus_{i\in J}\Biggl(\sum_{\substack{\gamma\in[0,\alpha]\\ \gamma(i)+\alpha(i)=\gamma}} \mathcal{F}_{\gamma(i)}A\Biggr)e_i, \quad \alpha \in \mathbb{Z}^n_{\geqslant 0},$$

where $[0,\alpha] = \{\gamma \in \mathbb{Z}^n_{\geqslant 0} \mid \alpha \succeq_{gr} \gamma\}$ as defined before Lemma 3.3, and

$$\sum_{\substack{\gamma\in[0,\alpha]\\ \gamma(i)+\alpha(i)=\gamma}} \mathcal{F}_{\gamma(i)}A = \{0\}$$

if $\gamma = \alpha(i) + x$ has no solution in $\mathbb{Z}^n_{\geqslant 0}$ for any $\gamma \in [0,\alpha]$.

Observation. In the construction of $\mathcal{F}L$ made above, the following properties may be verified directly by using the monomial ordering $\succeq_{gr}$.

(i) For each $\alpha \in \mathbb{Z}^n_{\geqslant 0}$ and each $i \in J$,

$$\text{either} \quad \sum_{\substack{\gamma\in[0,\alpha]\\ \gamma(i)+\alpha(i)=\gamma}} \mathcal{F}_{\gamma(i)}A = \{0\} \quad \text{or} \quad \sum_{\substack{\gamma\in[0,\alpha]\\ \gamma(i)+\alpha(i)=\gamma}} \mathcal{F}_{\gamma(i)}A = \mathcal{F}_{\tilde{\gamma}(i)}A$$

where $\tilde{\gamma}(i) = \max\{\gamma(i) \in \mathbb{Z}^n_{\geqslant 0} \mid \gamma(i) + \alpha(i) = \gamma \text{ for some } \gamma \in [0,\alpha]\}$,

(ii) For each $i \in J$, $e_i \in \mathcal{F}_{\alpha(i)}L - \mathcal{F}_{\alpha(i)^*}$, i.e., each e_i is of degree $\alpha(i)$.

Definition 4.2. Write $\mathcal{F}L = \{\mathcal{F}_\alpha L;\, \alpha(i), i \in J\}_{\alpha\in\mathbb{Z}^n_{\geqslant 0}}$ for the $\succeq_{gr}$-filtration on L as defined above. L is called a $\succeq_{gr}$*-filtered free* A-module with the $\succeq_{gr}$-filtration $\mathcal{F}L$.

Proposition 4.3. *With notation as above, the following holds.*

(i) *If L is a $\succeq_{gr}$-filtered free A-module with the $\succeq_{gr}$-filtration $\mathcal{F}L$, then $G^{\mathcal{F}}(L)$ is a $\mathbb{Z}^n_{\geqslant 0}$-graded free $G^{\mathcal{F}}(A)$-module.*
(ii) *If L' is a $\mathbb{Z}^n_{\geqslant 0}$-graded free $G^{\mathcal{F}}(A)$-module, then $L' \cong G^{\mathcal{F}}(L)$ for some $\succeq_{gr}$-filtered free A-module L.*
(iii) *If L is a $\succeq_{gr}$-filtered free A-module with the $\succeq_{gr}$-filtration $\mathcal{F}L$, N is a $\succeq_{gr}$-filtered A-module with $\succeq_{gr}$-filtration $\mathcal{F}N$ and $\varphi : G^{\mathcal{F}}(L) \to G^{\mathcal{F}}(N)$ is a graded surjection, then $\varphi = G^{\mathcal{F}}(\psi)$ for some strict $\succeq_{gr}$-filtered surjection $\psi : L \to N$.*

Proof. Let $\mathcal{F}L = \{\mathcal{F}_\alpha L; \alpha(i), i \in J\}_{\alpha \in \mathbb{Z}^n_{\geqslant 0}}$ be the $\succeq_{gr}$-filtration on the free A-module $L = \bigoplus_{i \in J} Ae_i$. By Lemma 3.3, it can be verified directly that, for $\alpha \in \mathbb{Z}^n_{\geqslant 0}$,

$$G^{\mathcal{F}}(L)_\alpha = \bigoplus_{\substack{i \in J \\ \beta(i)+\alpha(i)=\alpha}} G^{\mathcal{F}}(A)_{\beta(i)}\sigma(e_i),$$

where each $\sigma(e_i)$ is a homogeneous element of degree $\alpha(i)$ and $\{\sigma(e_i)\}_{i \in J}$ forms a free $G^{\mathcal{F}}(A)$-basis for $G^{\mathcal{F}}(L)$. This proves (i), and then (ii) follows immediately.

(iii) Let $L = \bigoplus_{i \in J} Ae_i$ with the $\succeq_{gr}$-filtration $\mathcal{F}L = \{\mathcal{F}_\alpha L; \alpha(i), i \in J\}_{\alpha \in \mathbb{Z}^n_{\geqslant 0}}$. For each i, choose $x_i \in \mathcal{F}_{\alpha(i)} N$ such that $\varphi(\sigma(e_i)) = \bar{x}_i$, where $\bar{x}_i$, is the homogeneous element in represented by x_i. Now $\psi : L \to N$ may be constructed by putting

$$\psi\Bigl(\sum a_i e_i\Bigr) = \sum a_i x_i, \quad \text{where } \sum a_i e_i \in L.$$

Clearly, ψ is a $\succeq_{gr}$-filtered homomorphism and $G^{\mathcal{F}}(\psi) = \varphi$ since they agree on generators. By Corollary 3.6, ψ is a strict $\succeq_{gr}$-filtered surjection. □

Proposition 4.4. *Let P be $\succeq_{gr}$-filtered A-module with $\succeq_{gr}$-filtration $\mathcal{F}P$. The following holds.*

(i) *If $G^{\mathcal{F}}(P)$ is a projective $G^{\mathcal{F}}(A)$-module, then P is a projective A-module.*
(ii) *If $G^{\mathcal{F}}(P)$ is $\mathbb{Z}^n_{\geqslant 0}$-graded free $G^{\mathcal{F}}(A)$-module, then P is a free A-module.*

Proof. (i) By Proposition 4.3, let $\varphi : (L)G^{\mathcal{F}}(L) \to G^{\mathcal{F}}(P)$ be a graded surjection, where L is a $\succeq_{gr}$-filtered free A-module and hence $G^{\mathcal{F}}(L)$ is a graded free $G^{\mathcal{F}}(A)$-module. Again by Proposition 4.3, $\varphi = G^{\mathcal{F}}(\psi)$ for some strict $\succeq_{gr}$-filtered surjection $\psi : L \to P$. Let $K = \operatorname{Ker}\psi$ and $\mathcal{F}K$ the $\succeq_{gr}$-filtration on K induced by $\mathcal{F}L : \mathcal{F}_\alpha K = K \cap \mathcal{F}_\alpha L$, $\alpha \in \mathbb{Z}^n_{\geqslant 0}$. There is the short exact sequence

$$0 \to K \xrightarrow{\ell} L \xrightarrow{\psi} P \to 0$$

and it follows from Proposition 3.5 and Corollary 3.6 that the sequence

$$0 \to G^{\mathcal{F}}(K) \xrightarrow{G^{\mathcal{F}}(\ell)} G^{\mathcal{F}}(L) \xrightarrow{G^{\mathcal{F}}(\psi)} G^{\mathcal{F}}(P) \to 0$$

is exact. By Proposition 4.1, this sequence splits by graded $G^{\mathcal{F}}(A)$-homomorphisms. Consequently, $G^{\mathcal{F}}(L) = G^{\mathcal{F}}(P) \oplus G^{\mathcal{F}}(K)$ with $G^{\mathcal{F}}(L)_\alpha = G^{\mathcal{F}}(P)_\alpha \oplus G^{\mathcal{F}}(K)_\alpha$, $\alpha \in \mathbb{Z}^n_{\geqslant 0}$, and there is a graded surjection $\gamma : G^{\mathcal{F}}(L) \to G^{\mathcal{F}}(K)$ such that $\gamma \circ G^{\mathcal{F}}(\ell) = 1_{G^{\mathcal{F}}(K)}$. By Proposition 4.3(iii), $\gamma = G^{\mathcal{F}}(\beta)$ for some strict $\succeq_{gr}$-filtered surjection $\beta : L \to K$. Note that $G^{\mathcal{F}}(\beta) \circ G^{\mathcal{F}}(\ell) = G^{\mathcal{F}}(\beta \circ \ell) = 1_{G^{\mathcal{F}}(k)}$. It follows from Corollary 3.6 that $\beta \circ \ell$ is an automorphism of K, and hence $L \cong K \oplus P$. This shows that P is projective.

(ii) Suppose $G^{\mathcal{F}}(P) = \bigoplus_{i\in J} G^{\mathcal{F}}(A)\sigma(\xi_i)$, where each $\xi_i \in P$ has degree $\alpha(i)$ and $\{\sigma(\xi_i)\}_{i\in J}$ is a $\mathbb{Z}^n_{\geqslant 0}$-graded free basis for $G^{\mathcal{F}}(P)$ over $G^{\mathcal{F}}(A)$. Then, by Proposition 3.4, $P = \sum_{i\in J} A\xi_i$ with

$$\mathcal{F}_\alpha P = \bigoplus_{i\in J}\Bigg(\sum_{\substack{\gamma\in[0,\alpha]\\ \gamma(i)+\alpha(i)=\gamma}} \mathcal{F}_{\gamma(i)}A\Bigg)\xi_i, \quad \alpha \in \mathbb{Z}^n_{\geqslant 0}.$$

We claim that $\{\xi_i\}_{i\in J}$ is a free basis for P over A. To see this, construct the $\succeq_{gr}$-filtered free A-module $L = \bigoplus_{i\in J} Ae_i$ with the $\succeq_{gr}$-filtration $\mathcal{F}L = \{\mathcal{F}_\alpha L; \alpha(i), i \in J\}_{\alpha\in\mathbb{Z}^n_{\geqslant 0}}$ as before, such that each e_i has the same degree $\alpha(i)$ as ξ_i does. Then we have an exact sequence of $\succeq_{gr}$-filtered A-modules and strict $\succeq_{gr}$-filtered A-homomorphisms

$$0 \longrightarrow K \longrightarrow L \xrightarrow{\varphi} P \longrightarrow 0$$

where K has the $\succeq_{gr}$-filtration induced by $\mathcal{F}L$, and it follows from Proposition 3.5 that this sequence yields an exact sequence

$$0 \longrightarrow G^{\mathcal{F}}(K) \longrightarrow G^{\mathcal{F}}(L) \xrightarrow{G^{\mathcal{F}}(\varphi)} G^{\mathcal{F}}(P) \longrightarrow 0$$

However, $G^{\mathcal{F}(\varphi)}$ is an isomorphism. Hence $G^{\mathcal{F}}(K) = \{0\}$ and then $K = \{0\}$. This proves that φ is an isomorphism, or in other words, P is free. □

Proposition 4.5. *Let M be a $\succeq_{gr}$-filtered A-module with $\succeq_{gr}$-filtration $\mathcal{F}M$, and let*

$$0 \to K' \to L'_n \to \cdots \to L'_0 \to G^{\mathcal{F}}(M) \to 0 \qquad (*)$$

be an exact sequence of $\mathbb{Z}^n_{\geqslant 0}$-graded $G^{\mathcal{F}}(A)$-modules and graded homomorphisms, where the L'_i are graded free $G^{\mathcal{F}}(A)$-modules. The following holds.

(i) *There exists a corresponding exact sequence of $\succeq_{gr}$-filtered A-modules and strict $\succeq_{gr}$-filtered homomorphisms*

$$0 \to K \to L_n \to \cdots \to L_0 \to M \to 0 \qquad (**)$$

in which the L_i are $\succeq_{gr}$-filtered free A-modules. Moreover, we have the isomorphism of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, "\cong"'] & L'_n \arrow[r] \arrow[d, "\cong"'] & \cdots \arrow[r] & L'_0 \arrow[r] \arrow[d, "\cong"'] & G^{\mathcal{F}}(M) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & G^{\mathcal{F}}(K) \arrow[r] & G^{\mathcal{F}}(L_n) \arrow[r] & \cdots \arrow[r] & G^{\mathcal{F}}(L_0) \arrow[r] & G^{\mathcal{F}}(M) \arrow[r] & 0
\end{tikzcd}$$

(ii) *If K' is a projective $G^{\mathcal{F}}(A)$-module, then K is a projective A-module; If K' is a $\mathbb{Z}^n_{\geqslant 0}$-graded free $G^{\mathcal{F}}(A)$-module, then K is a free A-module.*

(iii) *If the modules in $(*)$ are finitely generated over $G^{\mathcal{F}}(A)$ then the modules in $(**)$ are finitely generated over A.*

Proof. (i) By Proposition 4.3, the homomorphism $L'_0 \to G^{\mathcal{F}}(M)$ in $(*)$ has the form $G^{\mathcal{F}}(\beta)$ for some strict $\succeq_{gr}$-filtered surjection $\beta : L_0 \to M$, where $L'_0 \cong G^{\mathcal{F}}(L_0)$ and L_0 is a $\succeq_{gr}$-filtered free A-module. Let $K_0 = \operatorname{Ker}\beta$ with the $\succeq_{gr}$-filtration $\mathcal{F}K_0$ induced by $\mathcal{F}L_0$. Then we have the exact diagram of graded $G^{\mathcal{F}}(A)$-modules and graded homomorphisms

$$\begin{tikzcd}
\cdots \arrow[r] & L'_2 \arrow[r] & L'_1 \arrow[r] & L'_0 \arrow[r] \arrow[d, "\cong"'] & G^{\mathcal{F}}(M) \arrow[r] \arrow[d, "="'] & 0 \\
 & 0 \arrow[r] & G^{\mathcal{F}}(K_0) \arrow[r] & G^{\mathcal{F}}(L_0) \arrow[r] & G^{\mathcal{F}}(M) \arrow[r] & 0
\end{tikzcd}$$

Note that the square involved in the above diagram commutes. Hence the homomorphism $L'_1 \to L'_0$ factors through $G^{\mathcal{F}}(K_0)$, i.e., there is the graded exact sequence $L'_1 \to G^{\mathcal{F}}(K_0) \to 0$. Starting with $G^{\mathcal{F}}(K_0)$, the foregoing construction can be repeated for finishing the proof of (i). (ii) and (iii) follow immediately from Proposition 4.4 and Proposition 4.5, respectively. $\square$

We are ready to mention the finiteness of global dimension for A.

Theorem 4.6. *Let $A = k[a_1, \ldots, a_n]$ be a quadric solvable polynomial algebra with the $\succeq_{gr}$-filtration $\mathcal{F}A$. Write* p. dim *for projective dimension and write* gl. dim *for global dimension. The following holds.*

(i) *If M is a $\succeq_{gr}$-filtered A-module with $\succeq_{gr}$-filtration $\mathcal{F}M$, then* $\operatorname{p.dim} M \leqslant \operatorname{p.dim} G^{\mathcal{F}}(M) \leqslant n$.

(ii) $\operatorname{gl.dim} A \leqslant \operatorname{gl.dim} G^{\mathcal{F}}(A) = n$.

Proof. Note that every A-module M has a $\succeq_{gr}$-filtration $\mathcal{F}M$. (i) and (ii) follow from Propositions 4.5 and 3.1. $\square$

5. General case: $K_0(A) = \mathbb{Z}$

We put the result as stated by the above title in this separate and final section just for emphasizing that we are returning to use the standard filtration again.

Let $A = k[a_1, \dots, a_n]$ be an arbitrary quadric solvable polynomial algebra with $\succeq_{gr}$ as defined in Section 1. Going back to the *standard filtration* $\mathcal{F}A$ on A (see Section 1):

$$\{0\} \subset k = F_0A \subset F_1A \subset \cdots \subset F_pA \subset \cdots$$

where $F_pA = k\text{-span}\{a_{i_1}^{\alpha_1} a_{i_2}^{\alpha_2} \cdots a_{i_n}^{\alpha_n} \mid \alpha_1 + \cdots + \alpha_n \leqslant p\}$, $p \in \mathbb{Z}_{\geqslant 0}$ then we have the associated $\mathbb{Z}_{\geqslant 0}$-graded algebra $G(A) = \bigoplus_{p \in \mathbb{Z}_{\geqslant 0}} F_pA/F_{p-1}A$ and the $\mathbb{Z}_{\geqslant 0}$-graded Rees algebra $\widetilde{A} = \bigoplus_{p \in \mathbb{Z}_{\geqslant 0}} F_pA$ for A, respectively.

Theorem 5.1. *Let A, $G(A)$, and $\widetilde{A}$ be as above. Then*

$$\mathbb{Z} \cong K_0(A) = K_0\big(G(A)\big) = K_0(\widetilde{A}).$$

Proof. By Proposition 1.6(i), $G(A) = k[\sigma(a_1), \dots, \sigma(a_n)]$ and $\widetilde{A} = k[\tilde{a}_1, \dots, \tilde{a}_n, X]$ are quadric solvable polynomial algebras with respect to some $\succeq_{gr}$, respectively. It follows from Theorem 4.6 that $\text{gl.dim}\, G(A) \leqslant n$ and $\text{gl.dim}\, \widetilde{A} \leqslant n + 1$. Now, it follows from the K_0-part of Quillen's theorem [Qui, Theorem 7] that

$$\mathbb{Z} \xrightarrow{\cong} K_0(k) \begin{array}{l} \xrightarrow{=} \nearrow\; K_0(F_0A) \cong K_0(A) \\ \xrightarrow{=}\; K_0\big(G(A)_0\big) \cong K_0\big(G(A)\big). \\ \xrightarrow{=} \searrow\; K_0(\widetilde{A}_0) \cong K_0(\widetilde{A}_0) \end{array} \qquad \square$$

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