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The fractional Fourier domain decomposition

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Abstract

We introduce the fractional Fourier domain decomposition. A procedure called pruning, analogous to truncation of the singular-value decomposition, underlies a number of potential applications, among which we discuss fast implementation of space-variant linear systems. © 1999 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

The singular-value decomposition (SVD) plays a fundamental role in signal and system analysis, representation, and processing. The SVD of an arbitrary $N_r \times N_c$ complex matrix $\boldsymbol{H}$ is

$$\boldsymbol{H}_{N_r\times N_c} = \boldsymbol{U}_{N_r\times N_r}\boldsymbol{\Sigma}_{N_r\times N_c}\boldsymbol{V}^{\dagger}_{N_c\times N_c}, \tag{1}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are unitary matrices. The superscript † denotes Hermitian transpose. $\boldsymbol{\Sigma}$ is a diagonal matrix whose elements σ_j (the singular values) are the nonnegative square roots of the eigenvalues of $\boldsymbol{H}\boldsymbol{H}^\dagger$ and $\boldsymbol{H}^\dagger\boldsymbol{H}$. The number of strictly positive singular values is equal to the rank R of $\boldsymbol{H}$. The SVD can also be written in the form of an outer-product (or spectral) expansion

$$\boldsymbol{H} = \sum_{j=1}^{R} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^\dagger, \tag{2}$$

where $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$ are the columns of $\boldsymbol{U}$ and $\boldsymbol{V}$. It is common to assume that the σ_j are ordered in decreasing value.

In this paper we introduce the *fractional Fourier domain decomposition* (FFDD). While the FFDD may not match the SVD's central importance, we believe it is of fundamental importance in its own right as an alternative which may offer complementary insight and understanding. Although exploring the full range of properties and applications of the FFDD is beyond the scope of this paper, we illustrate its usefulness by showing that it can be used for fast implementation of space-variant linear systems. We believe the FFDD has the potential to become a useful tool in signal and system analysis, representation, and processing (especially in time–frequency space), in some cases in a similar spirit to the SVD.

We refer the reader to [1,11–13] for an introduction to the fractional Fourier transform, here limiting ourselves to a few essential properties of the discrete fractional Fourier transform [2,4,10,15]. The N-dimensional ath-order fractional Fourier transform matrix $\boldsymbol{F}_N^a$ is unitary. $\boldsymbol{F}_N^0$ is the N-dimensional identity matrix and $\boldsymbol{F}_N^1$ is the ordinary N-dimensional discrete Fourier transform (DFT) matrix. $\boldsymbol{F}_N^2$ is the parity matrix and $\boldsymbol{F}_N^{a+4l} = \boldsymbol{F}_N^a$ where l is any integer. Furthermore,

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$F_N^{a_1} F_N^{a_1} = F_N^{a_1+a_2}$ and $(F_N^a)^{-1} = F_N^{-a}$. The ath-order fractional Fourier transform $f_a = F^a f$ of a given time-domain vector f is the representation of f in the *ath fractional Fourier domain* [11]. The ath fractional Fourier domain makes an angle $\alpha = a\pi/2$ with the time domain in the time–frequency plane (Fig. 1(a)) [9,11,12]. The columns of the inverse transform matrix F_N^{-a} constitute an orthonormal basis set for the ath domain, just as the columns of the identity matrix constitute a basis for the time domain and the columns of the ordinary inverse DFT matrix constitute a basis for the frequency domain.

2. The fractional Fourier domain decomposition

Let H be a complex $N_r \times N_c$ matrix and $\{a_1, a_2, \ldots, a_N\}$ a set of $N = \max(N_r, N_c)$ distinct real numbers such that $-1 < a_1 < a_2 < \cdots < a_N \leqslant 1$. For instance, we may take the a_k uniformly spaced in this interval. The corresponding fractional Fourier domains are illustrated in Fig. 1(b). We define the FFDD of H as

$$H_{N_r \times N_c} = \sum_{k=1}^{N} F_{N_r}^{-a_k} [\Lambda_k]_{N_r \times N_c} (F_{N_c}^{-a_k})^{\dagger}, \tag{3}$$

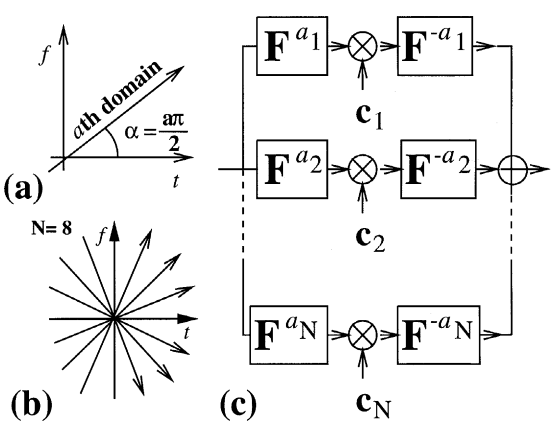

Fig. 1. (a) The ath fractional Fourier domain. The $a = 0$th and $a = 1$st domains are the ordinary time (t) and frequency (f) domains. (b) N equally spaced fractional Fourier domains. (c) Block diagram of the FFDD.

where $\Lambda_1, \Lambda_2, \ldots, \Lambda_N$ are diagonal matrices each of whose $N' = \min(N_r, N_c)$ elements c_{kj}, $j = 1,2,\ldots,N'$, are in general complex numbers. It will sometimes be convenient to represent these diagonal elements $c_{k1}, c_{k2}, \ldots, c_{kN'}$ for any k in the form of a column vector c_k. When H is Hermitian (skew-Hermitian), c_k is real (imaginary). We also note that $(F_{N_c}^{-a_k})^{\dagger} = F_{N_c}^{a_k}$. The FFDD always exists and is unique, as will be discussed below.

Comparing and contrasting the FFDD with the SVD will help gain insight into the FFDD. If we compare one term on the right-hand side of Eq. (3) with the right-hand side of Eq. (1), we see that they are similar in that they both consist of three terms of corresponding dimensionality, the first and third being unitary matrices and the second being a diagonal matrix. But whereas the columns of U and V constitute orthonormal bases specific to H, the columns of $F_{N_r}^{-a_k}$ and $F_{N_c}^{-a_k}$ constitute orthonormal bases for the a_kth fractional Fourier domain. Customization of the decomposition is achieved through the coefficients c_{kj} (and perhaps also the orders a_k).

Denoting the jth columns of $F_{N_r}^{-a_k}$ and $F_{N_c}^{-a_k}$ as $[F_{N_r}^{-a_k}]_j$ and $[F_{N_c}^{-a_k}]_j$, respectively, the kth term of the summation in Eq. (3) can be written as an outer product $\sum_{j=1}^{N'} c_{kj} [F_{N_r}^{-a_k}]_j ([F_{N_c}^{-a_k}]_j)^{\dagger}$ so that Eq. (3) can be rewritten as

$$H = \sum_{k=1}^{N} \sum_{j=1}^{N'} c_{kj} [F_{N_r}^{-a_k}]_j ([F_{N_c}^{-a_k}]_j)^{\dagger}. \tag{4}$$

To a certain extent, the inner summation resembles the outer-product form of the SVD given in Eq. (2). The $N_r \times N_c$ matrices $[F_{N_r}^{-a_k}]_j ([F_{N_c}^{-a_k}]_j)^{\dagger}$ are of unit rank since they are the outer product of vectors. We will denote these matrices by P_{kj} so that

$$H = \sum_{k=1}^{N} \sum_{j=1}^{N'} c_{kj} P_{kj}. \tag{5}$$

This equation is simply an expansion of H in terms of the *basis matrices* P_{kj}, $1 \leqslant k \leqslant N$, $1 \leqslant j \leqslant N'$, where the c_{kj} serve as the weighting coefficients of the expansion.

When H is a square matrix of dimension N, the FFDD takes the simpler form

$$H = \sum_{k=1}^{N} F^{-a_k} \Lambda_k (F^{-a_k})^{\dagger}, \tag{6}$$

where all matrices are $N \times N$. (The continuous counterpart of the FFDD is similar to this equation, with the summation being replaced by an integral over a.)

Eq. (5) is a linear relation between the matrices $\boldsymbol{H}$ and c_{kj} with the four-dimensional tensor $\boldsymbol{P}_{kj}$ representing the transformation between them. Let $\mathscr{H}$ denote a column ordering of the matrix $\boldsymbol{H}$, with dimensions $N_c N_r \times 1$. Also let C denote the $NN' \times 1$ column vector obtained by stacking the column vectors $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_N$ on top of each other. Notice that we always have $NN' = \max(N_r, N_c)\min(N_r, N_c) = N_r N_c$. These column orderings determine a corresponding ordering which converts the four-dimensional tensor (or two-dimensional array of matrices) $\boldsymbol{P}_{kj}$ into a square matrix $\mathscr{P}$ of dimensions $N_c N_r \times N_c N_r$. (The vector obtained as the column ordering of the matrix $\boldsymbol{P}_{kj}$ for a specific kj, goes into the $[(k-1)N' + j]$th column of the matrix P.) Now, we can write Eq. (5) as the linear square matrix equation $\mathscr{H} = \mathscr{P}\mathscr{C}$. This equation will have a unique solution for $\mathscr{C}$ and thus c_{kj} if and only if the columns of $\mathscr{P}$ are linearly independent. Since the columns of $\mathscr{P}$ are merely column orderings of the basis matrices $\boldsymbol{P}_{kj}$, this is the same as linear independence of these basis matrices. Recalling the definition of these matrices (just before Eq. (5)), their linear independence follows from the fact that the inner product of any column of $\boldsymbol{F}^a$ with any column of $\boldsymbol{F}^{a'}$ ($a' \neq a$) is always nonzero. Thus *the FFDD always exists and is unique* (for given a_k).

3. Applications

Let $\boldsymbol{H}$ denote a linear matrix operator. Eq. (3) represents a decomposition of this operator into N terms. Each term, taken by itself, corresponds to filtering in the a_kth fractional Fourier domain [8,12], where an a_kth-order forward transform is followed by multiplication with a filter function $\boldsymbol{c}_k$ and concluded with an inverse a_kth-order transform. If $a_k = 1$, this corresponds to ordinary Fourier domain filtering. If $a_k = 0$, this corresponds to multiplication of a signal with a filter function directly in the time domain. All terms taken together, the FFDD can be represented by the block diagram shown in Fig. 1(c) and interpreted as the decomposition of an operator into fractional Fourier domain filters of different orders. An arbitrary linear system $\boldsymbol{H}$ will in general not correspond to multiplicative filtering in the time or frequency domain or in any other single fractional Fourier domain. However, $\boldsymbol{H}$ can always be expressed as a combination of filtering operations in different fractional domains. *A sufficient number of different-ordered fractional Fourier domain filtering operations "span" the space of all linear operations*. The fundamental importance of the FFDD is that it shows how an arbitrary linear system can be decomposed into this complete set of domains in the time–frequency plane.

If $\boldsymbol{H}$ represents a time-invariant system, all filter coefficients except those corresponding to $a_k = 1$ will be zero. More generally, different domains will make varying contributions to the decomposition. By eliminating domains for which the coefficients $c_{k1}, c_{k2}, \ldots, c_{kN'}$ are small, significant savings in storing and implementing $\boldsymbol{H}$ becomes possible. This procedure, which we refer to as *pruning* the FFDD, is the counterpart of truncating the SVD. An alternative to this selective elimination procedure will be referred to as *sparsening*, in which we simply work with a more coarsely spaced set of domains.

Remembering that the $\boldsymbol{P}_{kj}$ are not orthogonal, we will in general have $\|\boldsymbol{H}\|_2 \leqslant \sum_{k=1}^{N}\sum_{j=1}^{N'}|c_{kj}|$, where $\|\boldsymbol{H}\|_2$ denotes the Frobenious norm of $\boldsymbol{H}$. Let $\hat{\boldsymbol{H}}$ denote the approximation to $\boldsymbol{H}$ obtained by pruning or sparsening certain orders. Then the approximation error $\|\boldsymbol{H} - \hat{\boldsymbol{H}}\|_2$ will likewise be less than or equal to the sum of the absolute values of the coefficients c_{kj} of the terms omitted from the expansion. This bound on the error indicates that we should eliminate orders whose associated coefficients are small in absolute value. One strategy for advantageously selecting the orders a_k would be to initially calculate the full decomposition for an interpolated version of $\boldsymbol{H}$ with larger N_r, N_c. By examining the decompositions of representative members of the set of matrices we are dealing with, we can determine the terms which have stronger coefficients and hence the values of a_k to be used in the actual decomposition.

In any event, the resulting smaller number of domains will be denoted by $M < N$. The upper limit of the summation in Eq. (3) is replaced by M and the equality is replaced by approximate equality, leading us to $\mathscr{H} \approx \mathscr{P}\mathscr{C}$. If we solve this in the least-squares sense, minimizing $\|\mathscr{H} - \mathscr{P}\mathscr{C}\|$, we can find the coefficients resulting in the best *M-domain approximation* to $\boldsymbol{H}$. This procedure amounts to projecting $\boldsymbol{H}$ onto the subspace spanned by the M basis matrices, which now do not span the whole space.

Since the fractional Fourier transform can be computed in $O(N \log N)$ time, implementation of the pruned version of Fig. 1(c) takes $O(MN \log N)$ time. If an acceptable approximation to $\boldsymbol{H}$ can be found with a relatively small value of M, this can be much smaller than the time $O(N_r N_c)$ associated with direct implementation of the linear system. Likewise, optical implementation requires a space-bandwidth product of $O(MN)$, as opposed to $O(N_r N_c)$ for direct implementation [14]. In passing, we note that the pruned FFDD is directly related to the concept of *parallel filtering* [6,7], which together with its dual *repeated filtering* [5] constitute a general framework for synthesizing linear systems.

As an example, we consider the problem of recovering a signal consisting of multiple chirp-like components, which is buried in white Gaussian noise such that the signal-to-noise ratio is 0.1. We assume the signal consists of six chirps with uniformly distributed random amplitudes and time shifts, and that the chirp rates are known with a $\pm 5\%$ accuracy. We find that the general linear optimal Wiener filter $\boldsymbol{H}$ for this problem can be approximated with a mean-square error of 5.2% by using only $M = 6$ domains. $\boldsymbol{H}$ can also be approximated by truncating Eq. (2) to M terms, leading to an implementation time of $O(MN)$. For the present example, $M = 6$ results in an error of 20%, demonstrating an instance where the FFDD yields better accuracy than the SVD.

Next, we consider restoration of images blurred by a space-varying point-spread function whose diameter increases linearly with position. This time we use the M-domain expansion as a constraint on the linear recovery filter and optimize directly over the coefficients c_{kj} to minimize the mean-square estimation error. The error is found to be 7% for $M = 5$. One may construct a similar constrained optimization problem by using the truncated SVD. However, this leads to a much more difficult nonlinear optimization problem because $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$ in Eq. (2) are also unknowns, whereas the only unknowns in Eq. (3) are the $\boldsymbol{\Lambda}_k$, leading to a linear optimization problem.

Other potential applications other than fast implementation of linear systems include data compression, statistically optimum filtering, and regularization of ill-posed inverse problems, all of which may be based on the same basic idea of appropriately pruning or weighting the different domains.

The optimal choice of the transform orders a_k and hence the basis matrices is an issue requiring further exploration. When $M = N$, the basis matrices form a complete set and any choice is acceptable. However, certain choices may offer better numerical stability. When $M < N$, the choice of a_k may reflect our knowledge about the ensemble of matrices $\boldsymbol{H}$ we wish to approximate. This prior knowledge of the structure of the matrices we are dealing with may be statistical or in the form of restrictions on the set of matrices possible, and might allow judicious choice of the orders so that a better approximation can be obtained by retaining fewer terms in the decomposition. In the absence of such knowledge, the natural strategy would be to choose the transform orders uniformly. It is in principle also possible to attempt to optimally choose the orders for each given matrix. However, fixing the orders beforehand for a given set of matrices has the advantage of allowing one to determine the coefficients easily by precomputing the set of matrices biorthonormal to $\boldsymbol{P}_{kj}$.

A natural extension of the FFDD would be the linear canonical domain decomposition (LCDD) based on linear canonical transforms [3].

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